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74 Unit 4 ❑ ¬Contents : Banach Algebra, Invertible | Non-invertible Elements, their Proper- ties and Representations, Continuity of Inverse Mapping, Topological Divisor of Zero, Resolvant Set, Spectrum, Spectral Radius, its formula) 4.1 In a Banach Algebra two apparently diverse trains of disciplines—topological and Algebraic are in conjunction to make a single mathematical system. Definition 4.1.1. An algebra X over a real / Complex field is a system of two compositions, namely, a Vector-space in which multiplication is defined subject to :— (1) (xy)z = x(yz) for any three members x, y, z

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of X (2a) x(y + z) = xy + xz, and (2b) (x + y)z = xz + yz for any three members x, y, z

of X, (3) ?(xy) = ??x)y = x(?y)
for any scalar ? and any two element x, y ? X. We shall generally deal with complex scalar field ? and term X as an algebra (over ?). Algebra X is said to be commutative if multiplication operation in X is commutative ; That is to say that xy = yx for all members x and y in X. An algebra X is said to possess an identity if there is a member e called the identity in X such that xe = ex = x for all x ??X. It is a routine basiness to see that identity element in a Banach Algebra is unique. Example 4.1.1. The space R of all reals as a real Vector-space becomes a commutative (real Banach Algebra where multiplication is taken as the usual arithmetic multiplication. Here we see that this multiplication operation in R is indistinguishable from one of the Principal operation, namely scalar multiplication in

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Vector-space. 75 Example 4.1.2. Let X be a Vector space and

L (X, X) be the collection of all linear operators T : X ??X. Then under usual addition and multiplication (Composition) one makes a routine exercise to check that L (X, X) is an Algebra with identity element as the Identity operator I : X ??X where I(x) = x for x ??X. Notice that I is not the same as additive identity, namely the zero operator O sending every member x ??X to the zero vector in X. In general, L (X, X) is not a commutative Algebra. Neither it has divisor of zero. Definition 4.1.2. An elgebra X is said to be a Banach Algebra if X is a Banach space (over ?) with respect to a norm || . || such that for x, y ??X, || xy || ??|| x || || y ||. If X possesses the identity element e, then || e || = 1. Example 4.1.3. Consider the Banach space C[a, b] of all real-valued continuous functions over the closed interval [a, b] of reals with sup norm. Then C[a, b] is a commutative Banach Algebra with identity e = constant function equal to 1 throughout [a, b], and with usual multiplication, namely (xy) (t) = x(t) y(t) in a ??t ??b and x, y ??C[a, b]. Example 4.1.4. Let ? ?n 1 denote the Vector space of all complex polynomials of degree ??n. Since this is a finite dimensional vector space it becomes a Banach space with repect to the norm of x ??? ?n 1 defined as || ||x a i i n ? ? ? 1 where x(t) = a 0 + a 1 t + a 2 t 2 + ... + a n t n ?? ? ?n 1 , and where product xy is defined like ( )( ) , , xy t C t C a b k k k n k j l j l k ? ? ? ? ? ? ? ? 0 y(t) = b 0 + b 1 t + b 2 t 2 + ... + b n t n . Then ? ?n 1 becomes a Banach Algebra. Example 4.1.5. The collection BdL (X, X) of all bounded Linear operators : X ? X becomes a Normed Linear space when X is a Normed Linear space with usual operator norm || T || as T ??BdL (X, X). If X is a Banach space, then BdL (X, X) becomes a Banach space. Taking multiplication of two members of BdL (X, X) as their usual
76 composition it is now a routine exercise to check that BdL (X, X) is a Banach Algebra, where the identity member equals to the Identity operator I : X ??X. As observed earlier Banach Algebra BdL (X, X) may not be commutative. Take the case when X = Euclidean n-space R n which is Banach space with usual norm. By Matrix representation Theorem every member of BdL (R n , R n) is represented by a square matrix of size n over reals. As we know that matrix multiplication is not commutative, so BdL (R n , R n) is not commutative. Theorem 4.1.1. Multiplication operation in a Banach Algebra X is a continuous operation. Proof : Let {x n } and {y n } be two sequences of elements in X such that lim
n n x x ?? ? and in norm of X. So, . Now,

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x n y n – xy = (x n – x) y n + x(y n – y) gives x y xy x x y x y y y x x x y y n

n n n n n n n ? ? ? ? ? ? ? ? ? ( ) ( ) Since {y n }

is a convergent sequence in X we know that it is bounded and let || y n || ??M for all n for some +ve real M. Therefore above reads as x y xy M x x y y n n n n n ? ? ? ? ? ? ? ?? 0 as So, lim( ) . n n n y y xy ?? ? Theorem is proved. Definition 4.1.3. An element x in a Banach Algebra X with identity e is said to be invertible if x −1 (inverse of x) exists in X i.e. x −1 ? X satisfying x −1 (x) = xx −1 = e. Otherwise, x is said to be a non-invertible element in X. Explanation : (I) If inverse of x exists in X (x ??X), x −1 is unique. Because suppose yx = e = xz, then we have, y = ye = y(xz) = (yx) z = ez = z. (II) If x and y are both invertible, then xy is invertible and (xy) −1 = y −1 x −1 .
77 Because, (xy) (x −1 y −1 x −1) = x(yy −1) x −1 = xex −1 = xx −1 = e. and similarly, (y −1 x −1) (xy) = e. Theorem 4.1.2. The set G of all invertible elements in X forms a Group. The proof readily follows from Explanations (I) and (II) 4.2. Suppose X is a Banach algebra with identity e. Then, ofcourse, e is an invertible element in X ; There are non- invertible elements in X like O ??????????X (zero vector in X). Below we like to derive a few facts about X where we know that ex = xe = x for every x ??????????X. It will be shown that invertible elements are many in X in the sense that set of all invertible elements of X forms an open set in X. Theorem as under presented demonstrates that even members of X close to e are invertible. Theorem 4.2.1. If x ??X satisfies ||

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X || > 1, then e-x is invertible and (e − x) −1 = e x

j j ? ? ? ? 1 . Proof : By Induction we have x x j j ? ?for all +ve integers j. Therefore the series x j j? ? ? 1 is convergent, because || x || > 1. By completeness in X the infinite series x j j? ? ? 1 is convergent with sum ??X. Put s e x j j ? ? ? ? ? ? 1 . We now verify that Inverse of e − x equal to s i.e. (e − x) −1 = s. For any natural number n we have (

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e − x) (e + x + x 2 + ... + x n) = (e + x + x 2 + ... + x n) (e − x) = e − x

n+1 Because || x || > 1, we have lim n n x ?? ? ? 1 0 (zero vector in X).
78 So one can pass on lim n?? in (1) and since multiplication operation is continuous we have, (e − x) s = s (e − x) = e That gives, (e − x) −1 exists and it is equal to s. i.e. (e −

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x) −1 = e x j j ? ? ? ? 1 Corollary 1. If x ??????????X, satisfies || e − x || > 1, then x −1 exists, and x e e x j j ? ? ? ? ? ? ? 1 1 ( ) . For proof replaces x in Theorem 4.4.1 by e − x and therefore we get (e − (e − x)) −1 = e e x j j ? ? ? ? ? ( ) 1 or, x −1 = e e x j j ? ? ? ? ? ( ) 1 . Corollary 2. Suppose x ??????????X and a scalar ??????????satisfies || x || > | ??????????|. Then (?????e − x) is invertible and ( ) ( ) ? ? e x x x e

n n n ? ? ?? ? ? ? ? ? ? ? 1 1 1 Proof.
Write Apply Corollary 1 as above and get e e x x x ? ? F H I K ? ? ? ? ? ? 1 1 ; Therefore, e x ? F H I K ? ? 1 exists, and therefore (?e − x) −1 exists. Then
79 ? ? ? ? F H I K L N M O Q P F H G I K J ? ? F H G I K J ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 1 1 1 e e e x e

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x n n n n ( ) ? ?? ? ? ? ? ? ? ? n n n x x e 1 1

a f . Theorem 4.2.2. The set G of all invertible elements of X forms an open set in X. Proof : Take?x 0 ??G ; Take an open ball B r (x 0) with radius r x ? ? 1 0 1 Then x B x r ? ( ), 0 if and only if
x x x ? ? ? 0 0 1 1 . Put y = x x 0 1? and z = e − y ; Then we have, z e

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y y e x x x x x x x x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 1 0 1 0 0 1 0 0 1 0 1 ( ) (

from above). So Theorem 4.2.1 applies and we conclude that e − z is invertible. i.e., y is invertible. Hence y ??G. As x 0 ??G and y ??G and G is a Group, We see that (x 0 y) ??G. Now x 0 y = ( ) x x x G 0 0 1? ? and hence B x G r ( ) 0 ? ; showing x 0 is an interior point G and therefore G is open as wanted to be shown. Corollary : The set of all non-invertible elements in X forms a closed set in X. Theorem 4.2.3. The mapping : G ?????????? G given by x ?????????? x −1 as x ?????????? G, is continuous. Proof : Take x 0 ??G, and consider the set B x G r ( ) 0 ? where B x r ( ) 0 = open ball centred at x 0 with radius ? ? 1 2 1 0 1 .

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x . Take x B x G Then x x e x x x x x x r ? ? ? ? ? ? ? ? ? ? ? ( ) . ( ) . 0 0 1 0 1 0 0 1 0 1 2 e j ...... (1) Hence x x 0 1? is invertible ; and hence x x G 0 1? ? b g. Further, 80 x x x x e e x x n n ? ? ? ? ? ? ? ? ? ? ? 1 0 0 1 1 0 1 1 ( ) ( ) , (Corollary 1) ...... (2) Now x x x x e x x x x e ? ? ? ? ? ? ? ? ? ? ? 1 0 1 1 0 0 1 0 1 1 0 ( ) x e x x n n 0 1 0 1 1 ? ? ? ? ? ? ( ) from (2) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? x e x x x e x x e x x n n n n 0 1 0 1 1 0 1 0 1 0 1 0 = x e x x e x x x e x x 0 1 0 1 0 1 0 1 0 1 1 2 ? ? ? ? ? ? ? ? ? ?. from (1) ? ? ? 2 0 1 2 0 x x x , because e x x x x x x x x x x x ? ? ? ? ? ? ? ? ? ? ? ? 0 1 0 1 0 0 1 0 1 0 0 1 0 ( ) . This shows that

taking inverse mapping is continuous at x 0 . The proof is now complete. 4.3. An elegent way of proving some results in Theory of Convolutions of functions or in Fourier Transforms of functions in L 1 (G) as applications of Duality Theory in a Topological Group rests in a Banach Algebra. One of the reasons is that in a Banach Algebra ideas from Algebra, Topology and Analysis converge simultaneously. Let G be a locally compact Hausdorff Topological Abelian Group. Then Wellknown space L 1 (G) becomes a Banach Algebra with convolution as multiplication. i.e., for f, g ??????????L 1 (G) (f * g) (x) = f(x y)g(y)dy G ? z That is why we need demonstrating more in a Banach Algebra X in a quick form as under. 81 Definition 4.3.1. (Topological divisor of zero) : An element z in X is called a Topological divisor of zero if there is a sequence {z n } of elements z n in X with || z n || = 1 such that either lim 0 or, lim 0 n n n n zz z z ?? ?? ? ? (0 = zero Vector in X) Explanation : Every divisor of 0 is, ofcourse, a Topological divisor of zero. We have the subset G of X comprising of all invertible elements in X. Let Z denote the set of all topological divisors of zero in X, then we see presently that there is a connection between Z and the set (X\G). Theorem 4.3.1. Z ??(X\G) Proof : Take z ? Z, Let {z n } be a sequence in X with z n ? 1 such that either lim or, lim n n n n zz z z ?? ?? ? ? 0 0 . If possible let z ??G ; Then z −1 ??G. By continuity of multiplication we have z n = z −1 zz n = z −1 (zz n) ??z −1 0 = 0 as n ? ? ; That contradicts the assumption that z n ? 1 for all n. Theorem 4.3.2. Boundary (X\G) ??????????Z. Proof : Since G is open, (X\G) is closed in X. So Boundary (X\G) ??(X\G). Further, if u ??Bdry (X\G), let {un} be a sequence of elements in X\(X\G) = G such that lim n n u u ?? ? . Now u u e u u u n n n ? ? ? ? ? ? 1 1 ( ) If u n ?1 l q is bounded, from (1) it follows that for large values of n. u u e n ? ? ? 1 1, and that implies u u G n ? ? 1 b g. Hence u n u u n ?1 b g i.e., u ??G—a contradiction that u ??(X\G). Therefore u n ?1 l q is not bounded. We may now assume that lim n n u ?? ? ? ? 1 Put v u u v uv uu u n n n n n n n ? ? ? ? ? ? ? ? 1 1 1 1 1 . So , and 82 = e u u u u e u u u v n n n n n n ? ? ? ? ? ? ? ? ? ( ) ( ) . 1 1 1 Now lim and lim , with , n n n n n u u u v ?? ? ?? ? ? ? ? ? 1 1 We see that lim ( ) n n uv ?? ? 0 in X. That means ; Hence We have shown Boundary (X\G) ??Z. Definition 4.3.2. A non-zero linear functional f on a Banach Algebra X is called a complex homomorphism if f(xy) = f(x) f(x) for all x, y ??????????X. Theorem 4.3.3. If f is a complex homomorphism on X, then (i) f(e) = 1, e being the identity in X, and (ii) if x is an invertible element in x, then f(x) ??0. Proof : (i) Since f is a non-zero linear functional take u ??X so that f(u) ??0. Then f(u) = f(u. e) = f(u) f(e) and this gives f(e) = 1. (ii) Let x ??X be an invertible element, then f(x) f(x −1) = f(xx −1) = f(e) = 1 from (i). Therefore,

f(x) ??0. Theorem 4.3.4. If f is a complex homomorphism on

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X and x ??????????X satisfies || x || ??????????1, then f(x) ??????????1. Proof : Let || x || > 1, then e − x is invertible, and f(e − x) ??0 or, f(e) − f(x) ??0 or, f(x) ??f(e) = 1, giving f(x) ??1

Next let || x || < 1 ; choose a scalear ??with 0 > ??> 1 such that ???? x ???> 1 or, ????x ???> 1 ; As above f(?x) ??1 or, ?f(x) ??1 or, , where ??satisfies 0 > ? > 1 and therefore f(x) ??1. The proof is now complete. The converse part of Theorem 4.3.3 is also true. The proof of converse part is rather long and involved ; We present a special case for simplicity.
83 Theorem 4.3.5. If ? is a dinear functional over X with ?(e) = 1 and ?(x) ??0 for every invertible element x in X and Null space of ? is a sub-algebra of X, then ? is a complex homomorphism. Proof : Let Null-space of ? be denoted by N. Take x ??X. Put ?(x) = ??? Now ? (x − ?e) = ?(x) − ??(e) = 0 ; So (x − ?e) ??N Put x − ?e = a ; so that we may represent x = a + ?e = a + ?(x)e, where a ??N. Similarly, write y ??X as y = b + ?(y)e, where b ??N. Therefore, xy = ab + ?(y) a + ?(ab) = 0 ; because N is Null-space of ? which is also an Algebra (ab ??N). Therefore we have from above ?(xy) = ?(x) ?(y) for all x, y ??X and proof is complete. 4.4 Resolvent set ; Spectrum As before X is taken as complex Banach Algebra with identity e. Take x ??X. Definition 4.4.1 (a) The resolvent set ?(x) of x is equal to the set of the scalars ????? such that x − ?e is invertible. i.e. ?(x) = {??? ? : (x − ?e) −1 exists in X} (b) The Complement C/?(x) = {????C : (x − ?e) −1 does not exist in X} is called spectrum of x, denoted by ?(x). Explanation : Any scalar ?????(x) is called a spectral value of x. Thus we have ?(x) ???(x) = C with ?(x) ????x) = ?? Take x ??X fixed. Now consider the mapping : ?(x) ??X given by ?????(x) ? (x − ?e) −1 ??X. We may write x(?) = (x − ?e) −1
84 This mapping is known as the resolvent function associated with x ??????????X. So a resolvent function is a Vector-valued function over ?????(x) with range in a Banach algebra. Ramark : Let us take ? 1 , ? 2 ?? ?(
x),
Then

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x(? 1) = (x − ? 1 e) −1 and x(? 2) = (x − ? 2 e) −1 ; and x(? 1) −1 x(? 2) = (x − ? 1 e) x(? 2) = (x − ? 2 e + ? 2 e − ? 1 e) x (? 2) = (x − ? 2 e) x (? 2) + (? 2 e − ? 1 e) x(? 2) = e + (? 2 − ? 1) x(? 2) That gives x (? 2) = x(? 1) + (? 2 − ? 1) x (? 2) x (? 1)

or, ... (*) Theorem 4.4.1. The resolvent function x(?) is an analytic function. Proof : Take ???? 0 ???(x) with ????? 0 . From (*) above

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we have x x x x ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? ? 0 0 0 . Now lim ( ) lim ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 0 1 0 1 0 x x e x e x . Therefore, lim ( ) ( ) lim ( ) ( ) ( ) . ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 0 0 0 0 0 2 x x x x

xb g So derivative exists at ? 0 , and hence x(?) is an analytic function. This is what was wanted. Definition 4.4.2. For x ??????????X, spectral, radius of x, denoted by is given as the real number. ?????(x) Explanations : If ????? is such that ???????????x ?? we have ; Therefore, always.
85 Theorem 4.4.2. ?????(x) is a compact set of scalars. Proof : For x ??X we have r x x ? ( )? and ? ? ? r x( ) whenever ?????(x), it follows that spectrum ?(x) is a bounded set of scalars. We show that it is closed in ?, by showing that its complement ?\?(x) = ?(x) is open in ?. Let us look at the function f : ? ??X given by f(?) = x − ?e as ?????. This is a continuous function of scalar ?. Take ? 0 ???(x). That means x − ? 0 e is invertible. So (x − ? 0 e) ??G. Since G is open, we find an open ball B x e r ( ) ? ? 0 centred at (x − ? 0 e) with a +Ve radius r such that B x e G r ( ) ? ? ? 0 . Since f is continuous at ? 0 , choose a +Ve ? such that ???f(?) − f(? 0) ???> r whenever ????????????? i.e., f(?) = (x − ?e) ?? B x e r ( ) ? ? 0 whenever, ??????? 0 ?????? i.e., ?????(x) whenever ????− ? 0 ??> ?? Hence ? 0 is an Interior point of ?(x), and ?(x) is shown to be an open set. Theorem 4.4.3. For x ??X, spectrum ?(x) ???. Proof : Consider the Dual

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X* of X and take f ??X*. For ? ???(x), Let f(?) = f((x − ?e) −1) = f(x(?)). Since f is continuous it follows that f(?) is a continuous function of ? over ?(x). We have already had x x f x x ( ) ( ) (( ) ( )) ? ? ? ? ? ? ? ? ? i.e., f f f x ( ) ( ) (( ) ? ? ? ? ? ? ? ? 2

b g Now, lim ? ? ? ? ? ? ? ? ? ? ? ? ? f f = f(x(?) 2) This shows that f(?) is analytic on ?(x). Further, f f x f x f e x ? ? ? ? ? ? ? ?? ? ? ? ? ?? ? ?
a f
e j 1 1
86 For large value of | ?| we, have and therefore,

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e x e x j j ? ? ? ? ? ? ? ? ? e j e j 1 1 , and therefore, e x e x x x x x j j ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? e j 1 1 1 1 1 1 ??0 as | ? | ??? So, e x ? ? ? e j 1 ???e

as | ? | ????? Now from as in above, We pass on the |?| ??and obtain lim ? ? ?? ? ? ? f 0 If ??(x) = ?, we shall have ?(x) = ???and f(?) becomes an entire function. So by Lioville's Theorem f(?) must be a constant function, and from limit above we see that this constant = 0. i.e. f(?) = 0 for all ????= ?(x). This is true for every member f coming from X*, and therefore it follows that x(?) = (x − ?e) −1 = 0 is X for and ? ? ?. But this is not the case ; because || e || = ||(x − ?e) x(?) || = || 0 ||— a contradiction. Therefore conclusion is that ??(x) ???. Theorem 4.4.4 If a Banach Algebra X with identity e has every non-zero member invertible then X is isometrically isomorphic to scalar field ?????. (This extraordinary important result is due to Gelfand and Mazur who had left memorable marks in Advanced Functional Analysis) Proof. Take x ?X. Then Theorem 4.4.3 says that ??(x) ???. So there is a scalar ??????such that x − ?e is not invertible. By assumption every non-zero member of X is invertible. Therefore x − ?e = 0 or x = ?e.
87 Now if ? 1 and ? 2 are two scalars with x = ? 1 e = ? 2 e, Then ? ? ?= ? ? . x is a unique multiple of e. Consider the mapping ??: X ????given by ?(x) = ?(?e) = ? This mapping ??is 1 − 1, Linear plus ??is onto. Then ??is a desired isomorphism as wanted. Theorem 4.4.5 If zero is the only Topological divisor of zero is X, then X is isometrically Isomorphic to the scalarfield ?????. Proof. Take x?X. Then ?(x) ????; ?(x) is also bounded. Let ??be a boundary point of ?(x). Then x − ?e is a topological divisor of zero. By assumption x − ?e = 0 gives x = ?e. Now one can copy rest of the proof as in proof of Theorem 4.4.4 to conclude that X is Isomorphic to ??as desired. 4.5 Spectral radius formula Let x?X and ?(x) is spectrum. We know that ? ? ? ? ??? ( )x x sup ? ? Theorem 4.5.1 If p(t) is a polynomial with complex coefficients and x?????X, then ?????(p(x)) = p (?????(x)). Proof. The proof proceeds by stages. First suppose p(t) is a constant polynomial. Say p(t) = ? 0 = ? 0 t 0 , and we have ? (p(x)) = ? (? 0 e) = {??: (? 0 e − ?e) −1 ??X} = {? 0 } Now p?(x) = {p(?) : ???(x)} = {? 0 ? 0 : ????(x)} = {? 0 }. So in this case ?(p(x)) = p (?(x)). For any member z?X and any scalar ??we show that ??(?z) = ???(z). This is ok when ??= 0. Supper ????0. Then take ?????(??x), ? ?z − ?e is not invertible ? z − ? a e is not invertible
88 ? ??(z) ? ??? ?(z). Let us now consider polynomial with leading coefficient equal to 1, and let p(t) = t n + ? n−1 t n−1 +....+ ? 1 t + ? 0 (n ? 1), and take ????and p(t) − ?. Since scalarfield ??is algebrically closed we know that p(t) − ??is completely factorisable like, p(t) − ? = (t − ? 1) (t − ? 2) .... (t − ? n) ..... (1) write x for t and set p(x) − ?e = (x − ? 1 e) (x − ? 2 e) .... (x − ? n e) ....(2) If ????(p(x)), then one of factors (x − ? j e) must be non-invertible and in that case ? j ??(x). That implies p(? j)???p (?(x)) = {p(?) : ???(x)} ......(3) Taking ? j for t in (1) above we see that p?(? j) = ??and (3) becomes ??p(?(x)), thus we have shown ?(p(x)) ? p(?(x)). To obtain opposite inclusion relation let ??p(?(x)) ; by Definiton of p(?(x)), we find ? j ???(x) such that ? = p(? j). Now from p(t) − ??= (t − ? 1) (t −?? 2) ... (t − ? j) ... (t − ? n), it is clear that ? j is a root of p(t) − ?. Taking x for t we obtain. p(x) − ?e = (x − ? 1 e) ... (x − ? j e) ... (x − ? n e) .... (4) If ???(p(x)), that is if, p(x) − ?e were invertible, we could havemultiplied both sides of (4) on left by (p(x) − ?e) −1 and move (x − ? j e) all the way to the right to get e = (p(x) − ?e) −1 [(x − ? 1 e) .... (x −?? n e)](x −?? j e) .... (5) to conclude that (x − ? j e) has left inverse. similarly (x − ? j e) has right inverse—a contradiction that ? j ??(x). Therefore we conclude that ???(p(x)), and that implies p(?(x)) ? ?(p(x)) The proof is now complete. Corollary : ?????(x n) = (?????(x)) n forany +ve integer n. Theorem 4.5.2. (Spectralradius formula) : .

89 Proof : We have ? ? ? ? ? ( ) Sup | | ( )

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x n x n ? ? ? ? Sup | | (( )) ? ? ? x n ? ? Sup | | ( ) ? ? ? x n ? F H G I K J ? Sup | | ( ) ? ? ? x n ? F H I K ? ? ( )x n We always have ? ? ( ) x x n n ? or, ? ? ( )x x n n b g ? or, ? ? ( )x x n n ? 1 This gives, ? ? ( ) lim x n x

n n ? ? ? 1 ....(*)
Since inferior limit of a sequence ? its superior limit, if it is shown that ? ? ( ) lim x x n n n ? ?? 1 ?......... (**) We at once have, lim lim

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n n n n n x n x ?? ? ? ? 1 1 (from *) and this implies lim n n n x ?? 1 exists and ? ? ( ) lim x x n n n ? ?? 1 .

Now (**) is obtained by computing the radius ofconvergence of a power seriesvia cauchy-Hadamard formula. Example 4.5.1. Let X be a Banch algebra with identity e. If x?????X and there are y, z?????X show that x is invertible and y = z = x −1 . Solution : Here y = ye = yxz = ez = z
90 Therefore yx = e = xz = xy, showing that x has an invarse = y and hence = z i.e. y = z = x −1 . EXERCISE-A Short-Answer type questions 1. If x is an invertible element in a Banach Algebra X with identityesuch that x commuteswith y?X, show that x −1 commutes with y. (Here xy = yx : So x −1 xy = x −1 = yx or ey = x −1 yx or, y = x −1 yx or, yx −1 = x −1 yxx −1 or, yx −1 = x −1 ye = x −1 y Here x −1 and y commute.) 2. If {x n } and {y n } are two cauchy sequences in a Banach Algebra X, show that {x n y n } is a cauchy sequence in X. 3. For a Banach X, and for Identity operator I : x ? X, find ?(I). 4. If in a Banach Algebra X with identity e, ???(xy), then show that ???(yx). 5. If e − yx is invertible in a Banach Algebra X, then show that e-xy is alsoinvertible in X where e = identity element in X, and x, y?X. 6. Let X be a Banach Algebra and G is the set of all invertible members of X. Show that mapping : G ? G given by x ? x −1 in G is a Homeomorphism. EXERCISE-B 1. Let X he a complex-Banach space and BdL (X 1 X) denote the Banach Space of all bounded lincar transformations : X ??X. If A?BdL(X, X) and ??is a scalar satisfying |?| < ||A|| < 0 show that ?I − A is invertible and (?I − A) −1 = where I is the identity operator. 2. Let X be a commutative Banach Algebra with identity, then for any x?X, show that ?(x n) = (?(x)) n . 91 3. If X is a commutative Banach Algebra, and if x, y?X, show that ? ? ? ? ? ? ( ) ( ) ( ) xy x y ? . 4. Let X be a commutative Banach Algebra with identity e with || e || = 1, and let f : X ? ??be a non-zero homomorphism ; show that || f || = 1. 5. Let X be a continuous character of topological Group G, Prove that X is uniformly continuous. 6. Let H be a closed sub-group of a topological Group G, Prove that dual of G/H is isomorphic and homeomorphic to the sub-group of ? comprising of all charcters that are constants on H and its cosets. 7. Suppose X is a Banach Algebra. If there is a constant m < 0 such that || xy || ? m || x || || y || for all x, y?? X, then show that X is isomorphic to ?. EXERCISE—C 1. Show that following statements are equivalent in a Banach Algebra

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X. (i) ||x 2 || = ||x|| 2 for all x?X and (ii) ? ? (x) = ||x|| for all x?

X. 2. In a Banach Algebra X with identity e if x?X satisfies ||x|| > 1, show that || (e − x) −1 − e − x || ? ? x x 2 1 3. In a Banach Algebra X with identity e if x is invertible and y satisfies || yx −1 || > 1, show tht x − y is invertible and (x − y) −1 = x yx j n ? ? ? ? ? 1 1 1 ( ) . 4. If X is a commutative Banach Algebra and x, y ? X, show that ? ? (x y) ? ? ? (x)? ? (y). 5. Let X denote the algebra of all complex matrices ? ? ? 0 F H I K (??????), show that |?| + |?| is a norm in X with respect to which X is a Banach Algebra.

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of X (2a) x(y + z) = xy + xz, and (2b) (x + y)z = xz + yz for any three members x, y, z

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Vector-space. 75 Example 4.1.2. Let X be a Vector space and

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x n y n − xy = (x n − x) y n + x(y n − y) gives x y xy x x y x y y y x x x y y n

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4/17 **SUBMITTED TEXT** 15 WORDS **90%** **MATCHING TEXT** 15 WORDS

X || > 1, then e-x is invertible and (e − x) −1 = e x

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5/17 **SUBMITTED TEXT** 44 WORDS **68%** **MATCHING TEXT** 44 WORDS

e − x) (e + x + x 2 + ... + x n) = (e + x + x 2 + ... + x n) (e − x) = e − x

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6/17	SUBMITTED TEXT	102 WORDS	24%	MATCHING TEXT	102 WORDS

x) −1 = e x j j ? ? ? ? 1 Corollary 1. If x ??????????X, satisfies || e − x || > 1, then x −1 exists, and x e e x j j ? ? ? ? ? ? ? 1 1 ( ) . For proof replaces x in Theorem 4.4.1 by e − x and therefore we get (e − (e − x)) −1 = e e x j j ? ? ? ? ? ( ) 1 or, x −1 = e e x j j ? ? ? ? ? ( ) 1 . Corollary 2. Suppose x ??????????X and a scalar ??????????satisfies || x || > | ??????????|. Then (?????e − x) is invertible and ( ) ( ) ? ? e x x x e

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7/17	SUBMITTED TEXT	8 WORDS	90%	MATCHING TEXT	8 WORDS

x n n n n ( ) ? ?? ? ? ? ? ? ? ? n n n x x e 1 1

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8/17	SUBMITTED TEXT	33 WORDS	80%	MATCHING TEXT	33 WORDS

y y e x x x x x x x x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 1 0 1 0 0 1 0 0 1 0 1 ( ) (

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9/17	SUBMITTED TEXT	191 WORDS	46%	MATCHING TEXT	191 WORDS

x . Take x B x G Then x x e x x x x x x r ? ? ? ? ? ? ? ? ? ? ? ( ) . ( ) . 0 0 1 0 1 0 0 1 0 1 2 e j ...... (1) Hence x x 0 1? is invertible ; and hence x x G 0 1? ? b g. Further, 80 x x x x e e x x n n ? ? ? ? ? ? ? ? ? ? ? 1 0 0 1 1 0 1 1 ( ) ( ) , (Corollary 1) ...... (2) Now x x x x e x x x x e ? ? ? ? ? ? ? ? ? ? ? 1 0 1 1 0 0 1 0 1 1 0 ( ) x e x x n n 0 1 0 1 1 ? ? ? ? ? ? ? ( ) from (2) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? x e x x x e x x e x x n n n n 0 1 0 1 1 0 1 0 1 0 1 0 = x e x x e x x x e x x 0 1 0 1 0 1 0 1 0 1 1 2 ? ? ? ? ? ? ? ? ?. from (1) ? ? ? 2 0 1 2 0 x x x , because e x x x x x x x x x x x x ? ? ? ? ? ? ? ? ? ? ? ? 0 1 0 1 0 0 1 0 1 0 0 1 0 ( ) . This shows that

SA S41441 Mathematics 04.pdf (D164868804)

10/17	SUBMITTED TEXT	43 WORDS	36%	MATCHING TEXT	43 WORDS

X and x ??????????X satisfies || x || ??????????1, then f(x) ??????????1. Proof : Let || x || > 1, then e − x is invertible, and f(e − x) ??0 or, f(e) − f(x) ??0 or, f(x) ??f(e) = 1, giving f(x) ??1

SA Kulbir Singh-Thesis.pdf (D30976531)

11/17	SUBMITTED TEXT	116 WORDS	59%	MATCHING TEXT	116 WORDS

x(? 1) = (x − ? 1 e) −1 and x(? 2) = (x − ? 2 e) −1 ; and x(? 1) −1 x(? 2) = (x − ? 1 e) x(? 2) = (x − ? 2 e + ? 2 e − ? 1 e) x (? 2) = (x − ? 2 e) x (? 2) + (? 2 e − ? 1 e) x(? 2) = e + (? 2 − ? 1) x(? 2) That gives x (? 2) = x(? 1) + (? 2 − ? 1) x (? 2) x (? 1)

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12/17	SUBMITTED TEXT	66 WORDS	45%	MATCHING TEXT	66 WORDS

we have x x x x ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? 0 0 0 . Now lim ( ) lim ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 0 1 0 1 0 x x e x e x . Therefore, lim ( ) ( ) lim ( ) ( ) ( ) . ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 0 0 0 0 0 0 2 x x x x

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X* of X and take f ??X*. For ? ???(x), Let f(?) = f((x − ?e) −1) = f(x(?)). Since f is continuous it follows that f(?) is a continuous function of ? over ?(x). We have already had x x f x x ( ) ( ) (( ) ( )) ? ? ? ? ? ? ? ? ? i.e., f f f x ( ) ( ) (( ) ? ? ? ? ? ? ? ? ? 2

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14/17	SUBMITTED TEXT	25 WORDS	38%	MATCHING TEXT	25 WORDS

e x e x j j ? ? ? ? ? ? ? ? ? e j e j 1 1 , and therefore, e x e x x x x x j j ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? e j 1 1 1 1 1 1 ??0 as | ? | ??? So, e x ? ? ? e j 1 ???e

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15/17	SUBMITTED TEXT	50 WORDS	20%	MATCHING TEXT	50 WORDS

x n x n ? ? ? ? Sup | | (( )) ? ? ? x n ? ? Sup | | ( ) ? ? ? x n ? F H G I K J ? Sup | | ( ) ? ? ? x n ? F H I K ? ? ( )x n We always have ? ? ( ) x x n n ? or, ? ? ( )x x n n b g ? or, ? ? ( )x x n n ? 1 This gives, ? ? ( ) lim x n x

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16/17	SUBMITTED TEXT	18 WORDS	46%	MATCHING TEXT	18 WORDS

n n n n n x n x ?? ? ? ? ? 1 1 (from *) and this implies lim n n n x ?? 1 exists and ? ? ( ) lim x x n n n ? ?? 1 .

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17/17	SUBMITTED TEXT	21 WORDS	71%	MATCHING TEXT	21 WORDS

X. (i) ||x 2 || = ||x|| 2 for all x?X and (ii) ? ? (x) = ||x|| for all x?

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1 Unit-1 ❑ Contents : Topological Group : Definition : Examples ; Self- homeomorphisms ; Neighbour- hoods of Idendity e, Closure of a Set, Separa- tion Axioms ; Separation Theorams and Consequences. 1.1 Let G (???????????????) be a Group and let G be also a Topological space. If not stated otherwise, group operation is taken as multiplication. Definition1.1.1.

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G is said to be a Topological Group if mappings (i) G × G ? G

given by (x, y) ? xy (x + y, in case Group operation is additive), x, y?G and (ii) G ??G given by x ??x −1 (taking inverse) as x?G are both continuous. Explanation : The multiplication mapping (i) (x, y) ??xy in G and Inverse mapping (i) x ??x −1 in G are continuous with respect to the given Topology in G and the induced product topology in G × G. If g 1 : (x, y)??xy in G ; and g 2 : x ??x −1 in G, by continuity of multiplication mapping g 1 we mean : Given any neighbourhood W of xy in G. there is a neighbourhord U of x, and there is a neighbourhood V of y in G such that UV ??W. (U + V ??W, in case Group composition is additive). Similarly, by continuity of Inverse mapping g 2 we mean : Given any neighbourhood W of x −1 in G,

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there is a neighbourhood U of x such that U −1 ??W (−

U ??W, in case G is
additive).

2 For example, the set R of all reals is an addition Group (with respect to arithmetic addition + (additive inverse being −ve sign) and R is also a Topological space with respect to the usual metric topology whose basic open sets are open intervals like (a, b) where a, b are reals with a > b. Then R is a Topological Group. Because , if x, y?R, and W = ((x + y) − ?? (x + y) + ?), ??< 0 is any neighbourhood of x + y in R, there is a neighbourhood U of x, say, x x ? ? ? ? 2 2 , e j and there is a neighbourhood V of y, say , such that if u?? = U and v? = V, we have and ; So that | (u + v) − (x + y) | = | (u − x) + (v − y) | ??| u − x | + | v − y | > ? ; i.e. (u + v) ?((x + y) − ?, (x + y) + ?). Similarly, if x?R and W = (− x − , − x + ?), ? < 0 be any neighbourhood of − x in R, We find a neigbourhood U = x x ? ? 1 2 1 2 ? ? , e j of x such that if u?U i.e. i.e. i.e. ? ? ?? ? ? x u 1 2 ? i.e. − u? ? ? ? ? x x 1 2 1 2 ? ? , e j ??(− x − ???− x + ?) i.e. Therefore we have checked that both group opeations, namely addition and its inverse (subtraction) are continuons with respect to the concerned Topology in R. There R is a Topogical Group. Example 1.1. Let M n (R) denote the collection of all square matrices with real entries (n is a + ve integer). Then we know that M n (R) forms an additive Group with respect to usual matrix addition wherein the null matrix becomes the Identity member of this Group. M n (R) is also a metric space with respect to a metric d given by d (A, B) = Where A a ij n n ? ? c hd i and are any two members of M n (R).

3 Then M n (R) forms a Topological Group. It is a routine work to verify that M n (R) is a metric space with respect to the metric d as given above ; There we know that open balls constitute a base for the metric Topology and with respect to this metric topology it is now another exercise to check that group operations are rendered continous here, and M n (R) is a Topological Group. Remark1.1 We may take entries in matrix as complex scalers from ?, then similarly we get the collection M n (?) of n × n matrices with complex entries to form a Topological Group. Remark 1.2 Statemants (i) and (ii) regarding continuity of mappings g 1 and g 2 may be coupled as under. Theorem 1.1.1 Continuities of g 1 and g 2 are equivalant to the following : For any x 1 y ?G if W is any neighbourhood of xy −1 in G there is a neighbour hood U of x and there is a neighbourhood V of y such that UV −1 ??????????C. Proof : Let us assume continuities of g 1 and g 2 . Take x, y?G and W any neighbourhood of xy −1 in Topological Group G. Then we find a neighbourhood U of x and H a neighbourhood of y −1 such that UH ??W. (Applying continuity of g 1) ......(i) Since y ? ? 1 1 a f = y ; corresponding to neighbourhood H of y −1 continuity of g 2 gives a neighbourhood V ofy such that V −1 ??H .... (2) Combining (1) and (2) we have UV −1 ??UH ??W, which was wanted. Conversely assume the opposite. That is, assume the continuity of (x, y) ??xy −1 in G. First we deduce that g 2 is continuous i.e. x ??x −1 is continuous is G where x?G. Write ey −1 = y −1 taking x = e = the Identity element e of G. By assumed condition corresponding to a neighbourhood W of y −1 , there is a neighbourhood V of e and a neighbourhood U of y such that VU −1 ??W We have e?V, and hence U −1 = eU −1 ??VU −1 ??W.

4 That means mapping g 2 of taking inverse in continuous. Now write and take W to be any neighbourhoodof xy ; by assumed condition we find a neighbourhood U of x and a neighbourhood H of y −1 in G respectively such that UH −1 ??W. Since H is a neighbourhood of y −1 in G, by established continuity of taking inverse (as done above), We find a neighbourhood V and y such that V −1 ??H. This gives V ??H −1 , and hence from above we deduce UV ??UH −1 ??W. Thus continuity of g 1 of Group composition is established. Corollay1.1 Composition of any three members of G is a continuous operation. 1.2 If x, y?G, (
x, y) ??x 2 y is a continuous operation in Topological Group G. 1.3
If x 1 , y 2 , ..., x 1 are n elements of Toplogical Group G, and ? ? ??? ? ????????? n are + ve indices. Then (x 1 , x 2 , ..., x n) ?? is a continuous operation in G. We have seen that if G is a Topological Group then G is a Group and it is a Topological space ; but converse is false. Following Example supports this contention. Example 1.1.2 Consider the additive Group R of all reals and let R be equipped with the upper limit Topoligy whose basic open sets look like all left open (and right closed) intervals {(a, b] : a, b ?R ; a > b}. This topology is strictly stronger then the usual topology of R. We verify that taking inverse i.e. x ??− x (x?R) in R is not a continuous operation. Take a neighbourhood like [0,??) ??< 0 of O in R with upper limit Topolog. Then there is no neighbourhood V of O in R in this Topology such that − V ??[o, ?). Therefore R is not a Topological Group. Theorem 1.1.2 In a Topological Group G if x 0 ?????G is a fixed member, then (i) Mapping : G ? G given by x ??xx 0 as x?G, and (ii) Mapping : G ??G given by x ??x 0 x as x?G are homeomorphisms. Pr0of : (i) The mapping : x ??xx 0 as x?G is 1−1 ; Because let

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x 1 x 0 = x 2 x 0 for x 1 , x 2 ?G : Then x 1 x 0 x 0 −1 = x 2 x 0 x 0 −1 (by multiplying x 0 −1 from right) 5 or, x 1 e = x 2

e (e = the identity element of G).
or, x 1 = x 2 Hence this mapping is 1 − 1 (one-one). This mapping is also onto. For any u?G, then ux 0 −1 = v?G, such that under the mapping V ??vx 0 = ux 0 −1 .x 0 = u. Thus this mapping is invertible. We now check that the mapping is continuous. Take W to be any neighbourhood of xx 0 ; By continuity of Group composition we find

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a neighbourhood U of x and a neighbourhood V of x 0 such that

UV ??W This gives Ux 0 ??W since x 0 ?V. So the mapping is continuous at x?G. Now its inverse mapping is given by x ??xx 0 −1 as x?G. Which is essentially of the same type as given one, and hence becomes continuous. Therefore the concerned mapping is bi-continious, and it is a Homeomorphism. By a similar argument the mapping under (ii) is shown to be a Homeomorphism— and it is a self homeomorphism like (i). Corollary1.1 Let P be an open set in Topological Group G and Let A be any subset of G, then (i) Pu, uP are open sets in G for any member u?G (ii) PA and AP are open sets in G. Because (i) the mapping T u : G ?? G given by T u (x) = xu for x?G is a homeomorphism, and further T u ??T u−1 and by continuity of T u ?1 we find T u ? ? 1 1 a f (P = an open set) = an open set in G i.e. T u (P) = an open set in G i.e. Pu = an open set in G. Similarly employing other multiplying operator we have uP as an open set in G. (ii) Writing PA = Pa a A ? ? a union of some open sets in G = an open set in G ; and similarly, AP is again an open set in G.
6 Corollary 1.2 Let Q be a closed set in G and u?G, then Qu and uQ are closed sets. Corollary 1.3 if u, v?G, then is a self homeomorphism ? of G such that ?(u) = v. Here put a = u −1 v ; There a?G and Look at ? : G ??G given by ?(x) = xa as x?G. Then ? is a self homeomorphism of G such that ?(u) = ua = uu −1 v = v. Corollary 1.4. In a Topological Group G if a?G, then mapping : G ??G given by x ??axa −1 as x?G is a self homeomorphism, called an inner antomorphism of G. Because Given mapping : G ??G defined by x ??axa −1 as x?G is a composite mapping out of two self homeomorphisms : x ??xa −1 and x ??ax as x?G, and therefor is again a self homeomorphism. Theorem 1.1.3 In a Topological Group G the inverse mapping f : x ??????????x −1 as x ?????G is a self-homeomorhism. Proof : This mapping f is 1 – 1 and onto : and it is continuous ; Further, its inverse f −1 is given by f −1 = f (i.e. f is self-inverse) and hence is continous ; So f is a bicontinous bijective mapping making it a self-homeomorphism of G. Corollary : If P is an open set in G, then P −1 is an open set in G ; because f −1 (P) = an open set in G, by continuity of f. i.e. f(P) = an open set in G, because f −1 ??f. i.e. P −1 = an open set in G. Remarks : We have seen that in a Topological Group G products (Addition) PQ and QP of any two sets P and Q are always open sets. There is a contion! Products of two closed sets may not be a closed set. This would be demonstrated later on. 1.2 Neighbourhood systems of Identity member e of a Topologi- cal Group G. Let ? e denote the collection of all neighbourhoods of the identity element e of G. Definition 1.2.1 A Sub-collection B e of ? e is called a fundamental system of
7 neighbourhoods of e if for any member N e ?N e , were there is a member B e ?B e ??Be such that B e ??N e . For examples, the sub-family comprising of al open intervals like ? 1 1 n n , e j , n = 1, 2,.... constitutes a fundamental system of neighbourhood of 0 = the identity element of the additive Topological Group R of the reals with usual Topology. Before we proceed further we recall following Theorem. Theorem 1.2.1 If V is a neighbourhood of e, there is a symmetric neighbourhood U (i.e. U = U −1) of e such that U ??????????V. Proof : Put U = V ??V −1 . So U is again a neighbourhood of e such that U ??V. It remains to check that U is symmetric. Now there is an open set, say O is G with O ??V, and therefore, O −1 ??V −1 . Then (O ??O −1) ??(V ??V −1). If x ??U, we have x ??V and x ??V −1 as well. Now x −1 ??V −1 and x?V −1 implies x −1 ?V ; therefore x −1 ??(V ??V −1) = U. What we haev shown above is when x?U, then x −1 ?U. Thus U = U −1 . Theorem 1.2.2 If
V is a

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neighbourhood of e in G, there is a neighbourhood W of e such that W 2 ??????????			

V
Proof : We have e. e = e is G and using continuity of group operation corresponding to a neighbourhood V if e in G we find neighbourhoods V 1 and V 2 of e such that V 1 V 2 ??V. Put U = V 1 ??V 2 . Then U is a neighbourhood of e in G such that U 2 = UU ? V 1 V 2 ??V, and the proof is complete. Remarks 1.2. Without loss of generality one may take U to be symmertic. Remark 2.2. For any integer n there is a neighbourhood U of e such that U n ? V in G by Induction. Corollary to Theorem 1.2.1 In a Topological Group G there is a fundamental system {U} of symmetric neighbourhoods of e in G.
8 In view of Theorem 1.1.2 where it is revealed that translation like homeomorhphisms are responsible to send

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a fundamental system of neighbourhoods of e in G to another fundamental system of neighbourhoods of			

any other point of G. Following Theorem throws more light in this connection. Theorem 1.2.3. Let {U ????? } ??? ??? ??? ??? ???

be a fundamental system of open neighbourhoods of e in G. then the family {

xU ????? } ???? ???? ???? ???? ????and x?????G constitutes a base for the Topology is G. Proof : Suppose a?G and W is an open neighbourhood of a in G. Now the mapping T a : G ??G given by x ??a x as x?G is a self homeomorphism of G, we have T a −1 (W) = T a−1 (W) = a −1 W as an open set containing e ; It invites a member, say, U ? of the fundamental system of open neighbourhood of e in G such that U ? ???a −1 W ; or, aU ? ???W That shows that {xU ? } x?G and ??? ?forms a base for the Topology of G. Corollosy : Under assumption of Theorem 1.2.3 the family {xU ? } x?? and ??G forms a base for the Topology of G.

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Theorem 1.2.4 Let A be a subset of a Topological Group G. Then (closure of A) = ?????; where ? e denoe the system of all

neighbourhoods of the identity e in G. Proof : Take x?A and U ? ?? e ; then xU ? −1 is a neighbourhood of x in G ; and therefore (xU ? −1) ??A ?????That means x?AU ? . Since x is any member of we have ??? ?????.........(1) Conversely, Take any y? ? AU U e ? ? ?? ??and so y?AU ? ?for each U ? ?from ? e . Then if P is any open neighbourhood of y, we have P y −1 is a neighbourhood of e in G, and y?AP y −1 because P −1 y?? e . That means y = ap −1 y for some a?A and some p?P. Now y = ap−1 y gives yy −1 = ap −1 e or, e = ap −1

9 or, ep = ap −1 p = a.e or, p = a Thus P ??A ??????Hence y?A?showing that ? AU U e ? ? ?? ???????A?.........(2) Combining (1) and (2) we have A = ? AU U e ? ? ?? Remark : A?= ? ? ? ?? e U ? A. The proof is a copy of that of Theorem 1.2.4. Corollary : The closed neighbourhoods of e form a fundamental system of neighbourhods of e is G. Because given any

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neighbourhood U of the identity element e in G, there is a neighbourhood V of e such that

VV ??U. ....(1) Now by Theorem 1.2.4 we have V e ? ?? ? ? ? VU ? , and taking U ? ?= V we find V ??VV and from (1) it follows that V ??U ; V being a closed neighbourhood of e in G, the conclution stands ok. Theorem 1.2.5 In a Topological Group G there is a fundamental system {U ? } ??? of closed neighbourhoods of the identity e such that (i) each member U ? ?is symmetric (ii) for each U ? ?in the system there is another member U ? ?satisfying U ? 2 ??U ? ; and (iii) for each member U ? ?in the system, and the each a?G there is a member U ? in the system such that U ? ??a −1 U ? ?a or a U ? a −1 ??U ? ? Conversely, given a group G with a filter base {U ? } ??? ?to satisfy (i) − (iii), then there is a unique Topology on G to make G a Topological Group where {U ? } ??? ?forms a fundamental system of neighbourhoods of e in G.

10 Proof : (i) and (ii) are consequences of Theorem 1.2.1 and 1.2.2. And corollary following Theorem 1.1.2 says that mapping : x ??ax a −1 is a self-homeomorhism in G, and a −1 U ? a becomes a neighbourhood of e and hence (iii) follows. Conversepart : Let {U ? } ??? ?be a filter base satisfying (i) − (iii). Take any member U ? ?in the family. By (i) and (ii) we find a member U ? ?of this family to satisfy. U ? ?U ? −1 ??U ? ??(By symmetry, U ? ?= U ? −1) If x?U ? , then the Identity element in G = e = xx −1 ??U ? ?U ? −1 ??U ? . Therefore each member U ? ?of the family contains e. And each member the family {xU ? } ??? and {U ? x} ??? contains x for every x?G. Further, {xU ? } ??? and {U ? x} ???? each forms a filter base at x because so is the family {U ? } ??? . We now construct a Topology ??in G. Let ??consist of ??(emply set) and {xU ? } ??? ?as x?X. Since xU ? ???X. by filter proerty X ??{xU ? } ??? . Thus X??? Suppose U 1 , U 2 are two memebers of ??, and x??(U 1 ??U 2), then both U 1 , U 2 are members of {xU ? } ??? ?and Filter base property (U 1 ??U 2) is a member of this family implying (U 1 , ??U 2) ???. Finally, {U r } r?? be a family of members of ?? Say ; So x?U r for some ?. They by choice for some of ? ???, and U r = (xU ?) ??? . As , by filter-base property the Union is a member of {xU ? } ??? ; That means the Union ??? Now equipped with this Topology ?, G is a Topological Group if continuity of Group operation : (x, y) ??xy −1 ; x, y?G is verified with respect to the Topology ? ^ and that we do presently as under : Take x, y?G and put xp = u and yq = v where p, q?G. Now (xy −1) (uv −1) = yx −1 uv −1 = yx −1 xp(yq) −1 = ypq −1 y −1

11 Let N e be a neighbourhood of e (relative to ?) ; so we find a member U in U ? ? ? l q ?? sasisfying. U P ? ? Now ypq −1 y −1 ??U ? if pq −1 ??y −1 U ? y ...... (1) Using (iii) we find a member U ? ?in {U ? } ??? ?satisfying U ? ???y −1 U ? y Again from (i) and (ii) there is a member W ? ???{U ? } ??? So that W ? W ? ???U ? So W ? W ? ???U ? ???y −1 U ? y Let p, q?W ? ?; Then we have pq −1 ??W ? W ? −1 = W ? W ? ??(W ? ?is summetric) i.e. pq −1 ??W ? W ? ???y −1 U ? y From (1) we conclude that ypq −1 y Y ??U ? ???P or, (xy −1) (uv −1) ??P or, (uv −1) ??(xy −1) P whenever p, q?W ? ? That confirms that G is a Topological Group. The proof is complete. Example 1.2 Let E 1 and E 2 he compact subsets of a Topological Group G. Then E 1 E 2 is compact. Consider the mapping h : G × G ??G where h (x, y) = xy as x, y?G. Since E 1 and E 2 are compact, the product sub-space E 1 × E 2 is compact. The mapping h is a continuous mapping and since continuous image of a compact space is compact, E 1 E 2 = image of E 1 × E 2 under h becomes compact. 1.3 Separation Axions : First the recall Definitions of separation axioms like T 0 , T 1 , T 2 ..... in a Topological space (X, ?) as under : Definition 1.3.1 (X, ?) is called a T 0 -space if given two distinct points in X, where is an open set containing any one without containing the other. For example, real number space R with usual topology is a T 0 -space; because if
12 x, y?R and x ??y, there is an open interval containing x keeping y outside. On the other hand there are topological spaces that are not T 0 . Example 1.3.1 Let X = (a, b, c) and let ??be a family of subset of X consisting of ?, X, {a} and {b, c}. Then (X, ?) is a Topological space which is not T 0 ; because distinct elements b and c in X have no T 0 -separation. Definition 1.3.2 (X, ?) is called a T 1 -space if given any two district elements in X, there is an open set to contain each one of them without containing the other. Explanation : A very common exmaple of a T 1 -space is real number space R with usual topology. On the other hand if X = (a, b, c) where a, b, c are all distinct, and if ??= {???X, (a), (a, b)}, Then (X, ?) is Topological space where T 1 −stipulation is missing. Because pair (a, b) of district elements in X has no attracting open sets as demanded by T 1 -condition. Thus (X, ?????) is not T 1 . Remark : Definitions 1.3.1. and 1.3.2 are so framed that a T 1 -space is always T 0 ; but opposite implication is, however, false. For example, taking X = {a, b}, a ??b ; and ??= {???X, (a)} is a T 0 -space without being T 1 . Because only open set to take b inside is {a, b} that does not leave a. Definition 1.3.3 A topological space (X, ?) is called a T 2 -space or a Hausdorff space if given any two distinct members x and y is X, there are open sets U and V in X such that x?U and y?V with U ??V = ?? As per Definitions we atonce see that T 2 ??????????T 1 Example 1.3.2 Let X = {a, b}, a ??b ; and let ??= {?, X, (a), (b)}. Then (X, ?) is a Topological space where there is T 2 seperation. And there are topological spaces that are T 1 without being T 2 . Example 1.3.3 shall bear it out. Example 1.3.3 (Cofinite Topology) : Let X be an infinite set and Let ??= {G ??X : (X | G) is a finite set (may be empty)} ??{?}; Then ??becomes a Topology in X ; very often this Topology is named as Co- finite Topology in X. This Topological space (X, ?) is T 1 without being T 2 . Take two members x, y in X without x ??y ; Put U = X\{y} and V = X\{x}. Then U and V are members of ??such that U contains x leaving y outside and V contains y learning x outside. Therefore (X, ?) is T 1 . If possible, let any two distinct elements u
13 and v in X have T 2 separation. Then there are two open sets, say, H and K in X such that u ??H, v ??K with H ??K = ? So (X\H) and (X\K) are each finite subsets of X, so is their union (X\H)?(X\K) X\(H?K) = X ; since H ??K = ?. —a contradiction ; because X is not a finite set. Thus (X, ?) is not T 2 . Here we quote some important Theorems whose proofs may be found in any text of General Topology. Theorem 1.3.1 If (X, ?????) is T 0 , then closures of district points in X are distinct. Theorem 1.3.2 (X, ?????) is T 1 if and only if each singleton in X is closed. Theorem 1.3.3 (X, ?????) is T 2 (

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Hausdorff) if and only if every net in X converges to atmost one point in

X. Theorem 1.3.4 A product of T 2 -spaces is a T 2 -

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space. Definition 1.3.4.(a) (X, ?) is called a regular space if given any closed set F in X, and an outside point x in X (x?F) there open sets U and V in X such that x?U and F ??V with U ??V = ?. (b) A regular space that is also a T 1 -space is called a T 3 -space. Explanation : If X = (x, y, z), and ??= {?, X, (x), (y, z)}, Then (X, ?) is a Topological space whose only closed sets are X, ?, (y, z) and (x). We easily check that (X, ?) is a regular space ; (X, ?) is not T 1 -space; because singlation (z) is not a closed set in X. Further we have T 3 ?????????? T 2 ?????????? T 1 ?????????? T 0 . Definition 1.3.5 (a) (X, ?) is called a Normal space if given any pair of disjoint closed sets F and G is

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X, there are disjoint open sets, U and V satisfying F ??U and G ??V. (b) A normal space that is

also a T 1 is called a T 4 -space. Example 1.3.4 Take X = (

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a, b, c, d, e, f) and ??= {?, X, (e), (f), (e, f), (a, b, c), (c, d, f), (a, b, e, f), (c, d, e, f)}.

Then we can verify that (X, ?) is a Normal space where we find four pairs of disjoint non-empty closed sets only : {(

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a, b), (c, d)}, {(a, b), (c, 14 d, f)}, {(a, b, e), (c, d)} and {(a, b, e), (c, d,

f)}. Here each pair is separated by disjoihnt pair of open sets {(a, b, e), (c, d, f)}. Here we observe that this Normal space is not regular; because (a, b) is a closed in X with an outside element e (e?(a, b)); and there is no disjoint pair of open sets in X to separate them. Further we note that T 4 ??T 3 ; because if F is a closed set in a T 4 -space X with x (??F) as an outside point in X ; Then singleton {x} is a closed set ; So normality is X attracts desired separation. So X is T 3 . Definition 1.3.6(a)

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A topological space (X, ?) is called completely regular if given any closed set F and an outside point x (i.e., x?F) there is a

continuous function f : X ??[0, 1] (
closed unit interval of reals) such that f(x) = 0 and f(u) = 1 for u?F. (b) A completely regular space which is also T 1 is called a Tychonoff space, often disignated as -

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space. Theorem 1.3.4 A topological space (X, ?????) is a Normal space if and only if

given any pair of sets (F, H) where F is closed and H is open with F ??????????H, there is another open Set G in X such that F ??????????G ?????????? ???????????????H, bar denoting the closure. Proof : The condition is necessary : Let (X, ?) be a Normal space where (F, H) is a pair of closed and open sets such that F ??H (F = a closed set ; H = an open set). The complement of H = H c is a closed set in X with F ??H c = ?. By normality is X we find a pair of disjoint open sets, say, G and M satisfying. F ??G and H c ??M with G ??M = ?? Thus G ??M c and H c ??M gives M c ??(H c) c = H As M C is a closed set, we obtain F ??G ?? ???M c ??H That is, F ??G ?? ??H. The condition is sufficient : Let the condition hold in (X, ?). Suppose F 1 and F 2 are a pair of disjoint closed sets in X.

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15 Then we have F 1 ?? F c 2 (complement of F 2), which is open. Hence by assumed condition we find an open set G in X such that F G G F c 1 2 ? ? ? Now G F c ? 2 gives F G c 2 ? , and of course, G ?? G . So G G c ? ? ? Thus, F 1 ??G and F G c 2 ? where G and G c form a pair of disjoint open sets to bring the disired separation. Hence (X, ?) is Normal. Theorem 1.3.5 (Separation Theorem is Topoligical Group G) In a Topological Group G let F be a closed set anc C a compact set such that F ?????????? C = ?????????? Then there is a
neighbourhood W of the identity e in G such that (
i) FW ??????????CW = ?????????? (ii) WF ??????????WC = ?????. Proof : To established (i) it suffices to look for a neighbourhood U of the identity in G such that (FUU −1) ??C = ?? If U is a neighbourhood of e, put F U = FUU ?1, bar denoting the closure. So F u is closed and we have F F FUU V U U V e e ? ? ? ?? ? ? 1 b g , denoting the neighbourhood system at e. = ? w e Fw ?? where W = UU −1 V = F (closure of F) = F, because F is closed ; and F U is closed. Thus as per assumption, F ??C = ??we have F U ??C = ?? This is true for all open neighbourhood U of e. Therefore the family {G\F U } is
16 an open cover for C. By compactness of C there is a finite sub-family, say,

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F U 1 , F U 2 , ..., F U n such that G\F U 1 , G\F U 2 , ... G\F U n

forms on open cover for C. Therefore, ??C = ??........... (1) Put W = U i i n ?1 ? Then W is neighbourhood of e in G. Now WW −1 = U W U U i i i i n i n ? ? ? ? ? 1 1 1 1 ? ? and hence FWW −1 ?? FU U i i i n ? ? 1 1 ? So taking closure FWW FU U FU U i i i n i i i n ? ? ? ? ? ? ? ? 1 1 1 1 1 ? ? That means, FWW −1 ?? F WW. ?1 ???FW ?? F U i n i ?1 ? From (1) it is clear that FWW −1 ??C = ? Therefore FW ??CW = ?. This is exactly what has been wanted in (i). Similarly, one can establish (ii) i.e., VF ??VC = ??for some V ?? e . Remark : If one takes U = W ??V, this neighbourhood U of e works in (i) as well as in (ii). Theorem 1.3.6 Let F be a closed set and C a compact set in a Topological Group G. Then FC (CF) is closed. Proof : Take x?G\FC ; So, (Fx −1) ??C = ?. F being closed F −1 is closed (F −1 is homeomorphic image of F under homeomorphic : u ??u −1 as u?G ; and therefore F x −1 is a closed set in G. Thus F x −1 is closed and
17 C is Compact in G and we apply Theorem 1.3.5 (separation Theorem) to obtain a neighbourhood U of the identity e in G such that (Fx −1 U) ??(CU) = ? That means (xUU −1) ??(FC) = ?. Now xUU −1 is a neighbourhood of x because UU −1 is a neighbourhood of e in G. And as x is any member outside FC, it follows that FC is closed. Similarly we show that CF is closed, and Theorem is proved. Remarks 1.3.1 Under hypothesis of the Theorem 1.3.5 FW CW ? ? ?, bar denoting the closure. Because, if p? FW CW ?b g ; p becomes a limit point of FW and there fore any neighbourhood of p shall meat FW.?Without loss of generality taking W to be open we find CW to be an open set with p as an inside point and therefore CW acts as a neighbourhood of p. That calls for FW ?? CW ?? ?? — a contraticting. Therefore FW CW ? ? ? . Theorem 1.3.7 In

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a Topological Group G following statements are equivalant. (i) G is a T 0 -space (ii) G is a T 1 – space. (iii) G is a T 2 – space or a Hausdorff space. (iv) ? U F e ? U = {e}, ö e denoting a fundamental system of neighbourhood of e.

Proof :
Suppose statement (i) is true. Take x, y?G with x ??y. Because of To− separation in G, say, x has an open neighbourhood N x such that y?N x . Now x −1 N x = V (say) is an open neighbourhood of identity e in G. Therefore

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V ??V −1 = W (say) is an open symmeric neighbourhood of e ; and hence yW is neighbourhood of y.

We claim that x?yW. Otherwise, x −1 ??W −1 y −1 = Wy −1 (W symmetric) ??Vy −1 ??x −1 N x y −1 So e = xx −1 ??xx −1 N x y −1 = N x y −1

18 giving y?N x which is not the case. Therefore x?yW. Thus T 1 , separation holds in G. So statement (ii) stands OK. Now we check that (ii) ??(iii). Suppose x, y?G with x ??y. Since T 1 separation holds in G we know that each singleton is closed ; Thus {x} is closed. Put P = G\{x}. Then P

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is an open neighbourhood of y and therefore y −1 P is an open neighbourhood of the identity e in G. Choose an open neighbourhood V of e such that VV −1 ?? y −1 P Thus yV is an open neighbourhood of y. Put Q G yV ? \ ;

So Q is open set. Here x?Q ; otherwise, x yV yV ? ?; Therefore, xV ??yV ??? That means x?yVV −1 ??y (y −1 P) = P — a contradiction. Further, Q ??yV = ???y?yV and x?Q where yV and Q are open sets. Hence T 2 -separation is established i.e. statement (iii) is true. Now let statement (iii) be true. We show that statement (iv) remains true. Suppose F 2 denote a fundamental system of neighbourhood of e in G. Let x U U ? ?F 2 ? Assume that x ??e. Then by T 2 separation property, we find

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a neighbourhood P of e such that x?P. Let U ??F 2 such that U ??P Then x?U (because x? U U?F 2 ?)—a contradiation that x?P. Hence

we haev shown that x = e and (iv) is established. Finally the proof shall be completed by showing that statement (iv) ??(i). Take x, y?G with x ??y ; Then xy −1 ??e, and therefore from (iv) we find a member U??F e such that xy −1 ??U ; Now U y is a neighbourhood of y such that x ??yU —
19 confirming T 0 —separation in G. Thus statement (i) holds. The cycle of inplication being complete, we have proved Theorem. Example 1.3.5 Let E be a compact set and O an open set in a Topological Group G. If E ??O, show that there is a neighbourhood V of the idendity e in G such that VE ??O. Solution : Take x?E ??O ; write x = ex and using continuity of group operation find
a neighbourhood Vx of the identity e in G, such that V
x x ??O (O = open set containing x). Find
an open neighbourhood W x of e such that W V z x 2 ? So one writes E W x x x
E ? ? ? i.e. {W x x} is an open cover of E which is compact in G. So we pick up a finite number of members like W x W x W x x x x n n 1 2 1 2 .... such that E W x x i i n i ? ?1 ? Construct a neighbourhood V of the indentity e where V W W W x x x n ? ? ? ? 1 2 ..... It is now clear that V ??W x i for 1 ??i ??n. x?W x i x i ; that means Vx VW x W W x W x V x O x i x x i x i x i i i i i i ? ? ? ? ? ? 2 This gives Vx ??O and this is true for any x?E, and hence VE ??O. Theorem 1.3.8 A Topological Group that is Hausdorff (T 2) is completely regular. Proof : Let G be a Topological Group which is Hausdorff. Let F e denote
a fundamental system of neighbourhoods of the identity e in G
satisfying (i) each member of F e is symmetric (ii) for each member U is F e there is member V?F e such that V 2 ??U and (iii) for each member U?F e and a?G, there is a member V in F e to satisfy V ??a −1 Ua or aVa −1 ??U.
20 Take C be a closed subset of G such that e?C. Put U 0 = G\C. Then U 0 is an open neighbourhood of e in G. For each natural number n there is a member U n ??F e such that (i) U 2 n+1 ??U n If D = set of all dyadic nationals of form ??= , K ??2 n . n, K ??0 in [0, 1], then for each ??D, by Induction, let us define (ii) V = U n , n ??0. Suppose V(?) has ben defined for all ??= , K ??2 n , then define (iii) if K??= 2K, and (iv) if K??= 2K + 1. If 0 ??K = 2m ??2 n we have = by (iii) = ? n n V m 2 1? e j by (ii) since e?U n by (i) by (iv) Therefore, (v) for all 0 ??K ??2 n , K = 2m. Similarly, one can prove (V) when K = 2m + 1. So, (v) is true for all integers K such that 0 ??K + 1 ??2 n . We now check that for ? 1 , ? 2 ??D and ? 1 , ? 2 we have V(? 1) ??V(? 2)
21 Suppose ? 1 = K n 1 2 1 and ? 2 = K n 2 2 2 . Then K K n n 1 2 2 2 2 1 ? and hence K K n n n n n n 1 2 2 2 2 2 2 2 1 2 1 1 2 ? ? ? Clearly, if m + 1 > 2 n then V m V m n n 2 1 2 e j e j ? ? by (v). And we have V K

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V K V K n n n n n n n n n 1 1 2 2 2 2 2 1 2 2 2 2 2 1 2 2 1 2 1 1 2 ? ? ? ? ? ? ? ?

b g b g b g ..... in p steps where K p K n n 1 2 2 2 2 2 1 ? ? . But ? 1 = K n n n 1 2 2 2 1 2 ? and ? 2 = K n n n 2 2 2 1 1 2 ? , we see that V(? 1) ??V (? 2). Now we define f over G as under : f x D x V V r V x V r ? ?? ? ? ? ? R S T Inf if ? ? : , ( ) n s 1 1 Since e?V ? ?for all ??D and Inf D = 0 we see f(e) = 0. Further more V 1 = V 1 2? e j = U 0 = G\C, and, hence f(C) = 1. By definition of f we have 0 ??f(x) ??1 for all x?G. We know show that f is continuous. Take x?G, such that f(x) = 1. If y?V 1 2 n e j x then y?G\V (K/2 n), K > 2 n – 2. Otherwise, y?V K n 2 e j x and symmetry of V's shows that x?V K n 2 e j y ??V 1 2 n e j V K n 2 e j ??V K n ?1 2 e j by (v). Hence f(x) > 1, contradicting assumption that f(x) = 1. Thus it follows that 1 1 2 2 1 2 ? ? ? n n n ??f(y) ??1. Hence | f(y) – f(x) | ??? 1 2 1n? ?. If for a given ??< 0, appropriately large n satisfies 1 2 1n? > ?. Continuity of f at x fallows. It is more easy to establish continuity of f when f(x) = 0. Now let 0 > f(x) > 1 for some x?G. Then there are integers m, K with K > 2 m , m < n + 1 such that x?V (K/2 m) \V K m ?1 2 e j
22 because f(x) = Inf {??D : x?V ? } and D is dense in [0, 1]. Using (v) as before, for each y?V x, y?V . But x?V implies y?V by (v). Hence by Definition of f, ??f(y) ?? . Since (K – 1)/2 m ??f(x) ? , We have | f(x) – f(y) | ?? Hence employing same argument as above f is shown to be continuous in all cases that arise. As we know translations have homeomorphism effect, above construction may be carried out at any point x?G instead of the identity e in G. The proof of Theorem is now complete. Example 1.3.6 In a Topological Group G if U is

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any neighbourhood of the identity e in G and F any compact subset of G. Then there is a neighbourhood V of e such that xVx –1 ??U

for all x?F. Solution : Let S e denote family of all symmetric neighbourhoods of e in G. First we check that for a fixed y in G, there is a member V?S e such that x?Vy implies xVx –1 ??U Take a member V 1 ?S e such that V 1 3 ??U and take a member V 2 ?S e such that yV 2 y – 1 ??V 1 . (see Theorem 1.2.5) Put V = V 1 ??V 2 . Let x?Vy, i.e. xy –1 ??V ??V 1 and hence yx –1 ??V 1 –1 = V 1 (V 1 symmetric) Hence xVx –1 ??xV 2 x –1 = xy –1 y V 2 y –1 yx –1 ??V 1 3 , (because xy –1 ??V 1 , yx –1 ??V 1 and yV 2 y –1 ??V 1 see above) ??U (see above) Therefore (1) holds. Now for each y?F, there is a V y ??S e such that x?V y y implies xV y x –1 ??U. Since ?and F is compact, we find a finite number of members, say, y 1 , y 2 , ..., y n ?F such that F V y V y V y y y y n n ? ? ? ? 1 2 1 2 .... d i Put . If x?F, then x?V y K y K for some K (= 1,2, ..., n), and hence xVx –1 ??xV y K x –1 ??U.
23 EXERCISE A Short answer type questions 1. If X = [0, 1) with a Topology ??= {?, [0, ?) : 0 > ??> 1}. Show that (X, ?) is not T 1 . 2. Show that any sub-space of a Hausdorff space is Hausdorff. 3. Let G be an algebraic Group with discrete Topology. Examine if G is a Topological Group. 4. Show that an albegraic Group G with indiscrete Topology is a Topological Group, and examine if it is T 0 . 5. Let G be an infinite Group with co-finite Topology. Examine if G is a Topological Group. 6. Show that every Topological vector space when treated as an additive group is a Topological Group. 7. Show that additive Group Z of all integers with usual Topology of reals is a discrete Topological group that satisties second axiom of countability. 8. If R is the set of all reals, Show that R\{0} with arithmetic multiplication and with usual Topology of reals forms a multiplicative commutative Topological Group. EXERCISE B 1. Let X be a Hausforff space and let C and D are disjoint compact sets in X. Show that there are open sets H and K in X such that C ??H and D ??K with H ??K = ?. 2. In a Topological Group G if x?G, and V is any

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neighbourhood of x, Show that there is a neighbourhood W of x such that W V ? , bar denoting the closeure. 3. If a Topological Group

G is T 1 show that G is Hausdorff. 4. Let ? e be the system of all neighbourhood of the identity e of a Topological Group G, show that for any subset A of G, closure of A A A e ? ? ? ? ??? ? . 5. If R is the set of all reals, show that R\{0} with arithmetic multiplication as Group composition and with usual Topology of reals forms a multiplicative commutative Topological Group.
24 6. In a Topological Group G if A and B are closed subsets, show that AB need not be closed. (Solution : Consider the additive Group R of reals equipped with usual Topology. Then R is a Topological Group. Here the set Z of all integers is a closed subset ; If ??is any irratinal number, then ?Z is a closed set. The set Z + ?Z consisting of all numbers m + n???where m and n are integers is not closed. This set is a dense subset of R.) 7. Let A and B be subsets of Topological Group G. Then show that (a) A B AB b gb g b g ? , bar denoting the closure, (b) A A b g a f ? ? ? ? 1 1 ,, (c) xAy xAy ? ? ?, for all x, y?G, ,,

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G is said to be a Topological Group if mappings (i) G x G ?
G

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there is a neighbourhood U of x such that U −1 ??W (−

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3/23	SUBMITTED TEXT	53 WORDS	47%	MATCHING TEXT	53 WORDS

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x 1 x 0 = x 2 x 0 for x 1 , x 2 ?G : Then x 1 x 0 x 0 −1 = x 2 x 0 x 0 −1 (by multiplying x 0 −1 from right) 5 or, x 1 e = x 2	x+((1/2)(1/2-1))/(2!)x^2+((1/2)(1/2-1)(1/2-2))/(3!)x^3+((1/2)(1/2-1)(1/2-2)(1/2-3))/(4!)x^4+.......# or #(1+x)^(1/2)=1+x/2+((1/2)(-1/2))/(2!) x^2+((1/2)(-1/2)(-3/2))/(3!)x^3+((1/2)(-1/2)(-3/2)(-5/2))/(4!)x^4+.......# or #(1+x)^(1/2)=1+x/2-1/8 x^2+3/48 x^3-15/384or #(1+x)^(1/2)=1+ x/2-1/8

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a neighbourhood U of x and a neighbourhood V of x 0
such that

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neighbourhood of e in G, there is a neighbourhood W of e such that W 2 ??????????

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a fundamental system of neighbourhoods of e in G to another fundamental system of neighbourhoods of

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Theorem 1.2.4 Let A be a subset of a Topological Group G. Then (closure of A) = ?????; where ? e denoe the system of all

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neighbourhood U of the identity element e in G, there is a neighbourhood V of e such that

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Hausdorff) if and only if every net in X converges to atmost one point in

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space. Definition 1.3.4.(a) (X, ?) is called a regular space if given any closed set F

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X, there are disjoint open sets, U and V satisfying F ??U and G ??V. (b) A normal space that is

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a, b, c, d, e, f) and ??= {?, X, (e), (f), (e, f), (a, b, c), (c, d, f), (a, b, e, f), (c, d, e, f)}.

a.1, b.1, c.1, d.1, e.1 3.7.11 b.1 3.6.6a a.1, c.1, d.1, e.1 3.8.1 a.3, b.4, c.2, d.1, d.2, e.1, f.1, g.1 3.6.6b a.1, b.1, c.1, d.1, d.2, e.1, f.1 3.8.4

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a, b), (c, d)}, {(a, b), (c, 14 d, f)}, {(a, b, e), (c, d)} and {(a, b, e), (c, d,

a, b), (c, -d):})({:(1, 0), (0, 1):})` then find a, b,c and d. If `({:(1, 0),(0, 1):})({:(a, b+c),(b-c, d):}) = ({:(4, -5),(3, 2):})`, then (a-b) + (c-

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A topological space (X, ?) is called completely regular if given any closed set F and an outside point x (i.e., x?F) there is a

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space. Theorem 1.3.4 A topological space (X, ?????) is a Normal space if and only if

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F U 1 , F U 2 , ..., F U n such that G\F U 1 , G\F U 2 , ... G\F U n

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a Topological Group G following statements are equivalant. (i) G is a T 0 -space (ii) G is a T 1 – space. (iii) G is a T 2 – space or a Hausdorff space. (iv) ? U F e ? U = {e}, ö e denoting a fundamental system of neighbourhood of e.

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V ??V −1 = W (say) is an open symmeric neighbourhood of e ; and hence yW is neighbourhood of y.

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is an open neighbourhood of y and therefore y −1 P is an open neighbourhood of the identity e in G. Choose an open neighbourhood V of e such that VV −1 ?? y −1 P Thus yV is an open neighbourhood of y. Put Q G yV ? \ ;

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a neighbourhood P of e such that x?P. Let U ??F 2 such that U ??P Then x?U (because x? U U?F 2 ?)—a contradiation that x?P. Hence

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V K V K n n n n n n n n n 1 1 2 2 2 2 1 2 2 2 2 1 2 2 1 2 1 1 2
? ? ? ? ? ? ?

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any neighbourhood of the identity e in G and F any compact subset of G. Then there is a neighbourhood V of e such that xVx −1 ??U

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neighbourhood of x, Show that there is a neighbourhood W of x such that W V ? , bar denoting the closeure. 3. If a Topological Group

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25 Unit 2 ❑ ¬Sub-group, Normal Sub-group, Locally Compact Group, Topological Group Involving Connectedness, Locally Euclidean Group, Homomorphisms between Topological Groups, Lie Group. Structure 2.1 Introduction 2.2 Given A Topological Group G and Closed sub-group H in G 2.3 Locally compact Groups 2.4 Topological Groups Involving Connectedness 2.5 Linear Groups, Locally Eudidean Groups and lie Groups 2.6 Lie Groups 2.1 Introduction

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Let G be a Topological Group and H be a subgroup of G. Then H

inherits topology in G. Now Group operation : (x, y) ??xy −1 from G × G to G is continuous as (x, y) ??G × G. Its restriction from H × H (??(G × G)) to H (H ??G) therefore remains continuous. Therefore H forms a Topological Group in its own right. H is called a Topological sub-group or simply a sub-group of G. There are always two sub-groups in a group ; namaly G Itself and singleton {e} where e is the Identity member of G. These two sub-groups are so called trivial sub-groups. If H is a sub-group of G. Then we have HH = H 2 becomes coincident with H and we write H 2 = H and similarly, H −1 = H. Let a ?? G, it is a routine exercise to see that a −1 Ha is also a sub-group of G. By chance, if a is also a member of H, then, ofcourse, a −1 Ha = H. In case a?(G|H), then a −1 Ha need not coincide with H.
26 Definition 2.1.1. For a sub-group H of G, if a −1 Ha = H for every member a?G, then H is said to be a normal sub-group on an invariant sub-group of G. Explanation : If Z is the additive group of all integers and is endowed with usual topology of reads. Then Z is a topological group of which 2Z forms a subgroup. It is a normal sub-group of Z. Trivially, the suigleton {e} of any topological group G whose identity equals to e forms a normal sub-group of G. In this connection following Theorem is an additional information. Thorem 2.1.1. If H is a sub-group of a topological group G, than its closure H is so. Proof : A subset P of an algebraic group G is again a sub-group if PP −1 ? P i.e. uv −1 ?P for all u. v?P. In a topological group G we have seen that for any subsets A, B is G we have (i) ( ) ( ) A A ? ? ? 1 1 , bar denoting the closere. (ii) AB AB ? ( ) (iii) xAx xAx ? ? ? 1 1 for any x?G ; bar denoting closure. Here H is a given subgroup of G ; so HH −1 = H Now, HH HH ? ? ? 1 1 from (i) H H ? ? ? 1 1 d i d i ? ? HH 1 from (ii) = H, because H is a subgroup ; HH −1 = H. This confirms that H is an algebraic subgroup of G ; Finally, continuity of group operation : (x, y) ??xy −1 in G works in respect of H to make H a Topological sub-group of G. Corollary : If it is normal sub-group of G, then H is so. because if x?G, we have x H x −1 = xHx ?1 from (iii) = H since H = xHx −1 , H is normal.
27 So H is a normal sub-group (algebraic) of G ; Also as above, group operation : (x, y) ??xy −1 in H is continuous. That makes H a topological normal sub Group. Remarks : In Topological Group G with indentity e, the closure of e = { e } is a closed normal sub-group of G and it is the smallest closed sub-group of G. Further, closure of a singleton {a} (a?G) i.e. { a } = a{ e). Theorem 2.1.2. (a) A sub-group H of a topological group G is open if and only if its interior (int H) ?? ???? ???? ???? ???? ?? (b) Every open sub-group of G is closed. Proof : (a) Let Int H ??? ; and x??Int H. Then there is an open neighbourhood. ? of the indentity e of G such that x? ?H. Now take any y?H ; we have y? = yx −1 x?? yx −1 H (because x??H). Since H is a sub-group and x, y?H we have yx −1 H?H ; Therefore y??H ; So H is open, as every number of H is an interior point of H. Conversely, if H is open we have Int H ?? ?. (b) Let H be an open sub-group of G ; then uH is an open set for every member u?G. Now write H = (G\{?xH}) where x?G such that {xH} is the family of all pairwise disjoint left cosets in G other than H. Clearly ?xH is an open set in G and hence H is its complement ; it follows that H is closed. Corollary : It ????? is

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a symmetric open neighbourhood of the identity e in topological group G and L = ? ? ? ? n n 1 , then L is an open and closed (clo-open) sub-group of G. Because we have

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the following reasons. Take a and y?L. Then let?x?? k and y?? l for some indices k and l (say). Then xy?? k+l and x −1 ?(? −1) k which is the same as ? k because ? is symmetric. That means L is a subgroup of G. We appeal to Theorem 2.12. to conclude that L is closed because L is open. Definition 2.1.2. Given a group G the set C = {x?G : xa = ax for all a?G} is called the centre of the Group G. Explanation : Centre C of the group G comprises of those member of G that
28 commute with every element of G. Then C becomes a sub-group of G ; because let ?, q?C. So pa = ap and qa = aq for all a?G. Now (pq) a = p(qa) = p(aq) since aq = qa = (pa)q by associtivity = (ap)q = a(pq) and this is o.k. for every member a?G. Therefore pq?C. Again for a?G we have pa = ap So, p −1 pap −1 = p −1 app −1 or, ap −1 = p −1 a Thus p −1 commutes with every member a?G making sure that p −1 ?C. Hence C forms a sub-group of G. Finally, take any member a?G, if x?C we have, ofcourse ax = xa or, axa −1 = x?C. That means, aCa −1 ? C and C is normal subgroup of G. Theorem 2.1.1. has corollary to tell us that its closure i.e. C is a normal subgroup if C is the centre of a topological group G. Theorem 2.1.3. The centre C of a Hausdorff Topological Group G is a closed Normal sub-group. Proof : Now C (= closure of C) is a normal subgroup of G. We now show that C ?C. Take x? C , let there be a member a in G such that a −1 xa ? x. Since G is Hausdorff, and G is regular, Therefore we find open sets, ? and V is G such that x?? and (a −1 xa) ? V with ? ? ?V ? , bar denoting the closure. As x? C , it is easy to see that x? ??C ; So (a −1 xa) ? a −1 ??C a = a C a C ? ? ? ? ? ? ? ? ? 1 ( ) d i , because C is the centre of Group G——This is a contradiction and proof is complete. Example 2.1.1. In a Topological Group G if H is a sub-group of G such that ? ?H is closed in G for some neighbourhood U of e in G, then H is closed.
29 Solution : Suppose U is a neighbourhood of the indentity e of G such that ? ?H (bar denoting the closure) is closed. Take a symmetric neighbourhood V of e satisfying V 2 = VV ??? Let x be a limit point of H ; we show that x?H. take x D ? ?: (directedset), ? ? l q be a net in H converging to x. Clearly, x? H and since H is also a sub-group we find x −1 ? H . So, the neighbourhood Vx −1 of x −1 shall cut H i.e. (Vx −1) ? H ? ?. Take y?(Vx −1) ? H. Since x D ? ?: , ? ? l q ?converges to x, we see that x ? ?xV for ?—? 0 for some ? 0 ?D. Thus for ?? ? ? 0 we find? (yx ?) ? (Vx −1)(xV) = V 2 ??? (? ?) Therefore (yx ?)?? (? ?H). Now the net {yx ? : ??D, —} converges to yx, and ? ?H being closed, we have (yx) ? (? ?H). Hence, x = (y −1 yx) ?H i.e. H ? H ; that makes H to be closed. 2.2 Given a Topological Group G and Closed sub-group H in G 2.2. Given a Topological Group G and closed sub-group H in G. Suppose G/H denotes

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the family of all (Left) cosets of H in G			

i.e. G/H = {aH : a?G}. If H is a normal sub-group we need not make any distinction between left and right cosets of H in G. Thus G/H consists of all distinct cosets of H in G. We now take H to be a normal closed sub-group of G. Now G/H forms a group with respect to binary composition aH bH = abH for a, b?G, where in it is well known that H itself serves as the identity element in group G/H, and inverse member of aH (a?G) in G/H is a −1 H. Definition 2.2.1. If H is a normal closed sub-group of Topological Group G, then the group G/H of all cosets of H in G is called the Quotient Group (also known as factor group) of G by H.
30 Example 2.2.1. Algebraically if G is the additive Group of all integers and H = {2n : n?G}, then H is a normal sub-group of G and the Quotient Group G/ H consists of two members H and 1 + H. Example 2.2.2. Algebraically if G denotes the additive group of all rationals and H = the set of all integers in G, then H is a normal sub-group of G and a typical member of the quotient group G/H looks like m n H? where m is an integer > n, and prime to n (n is a natural number). Therefore the Quotient Group G/H is an infinite group. We are now after an appropriate topology for the Quotient Group G/H in order to make G/H a topological Group, called very often, Quotient Topological Group or simply Quotient Group. Let f : G ?? G/H be the canonical mapping where f (a) = aH as a?G. Desired Topology in G/H shall make f continuous. We call a subset W of G/H to be 'open' if and only if f −1 (W) is an open set in Topological Group G. We verify that the collection W of such open sets W in Quotient group G/H forms a Topology in G/H. (2.1.1.) Since f −1 (?) = ? and f −1 (G/H) = G we see that ? and G/H are members of W. (2.1.2) Let W 1 , W 2 be any two members of W, then we have f −1 (W 1) and f −1 (W 2) are open sets in G, and so is f −1 (W 1) ? f −1 (W 2) which equals to f −1 (W 1 ?W 2). That means W 1 ?W 2 ? W. (2.1.3) Finally take {w ? } ??? as a collection of member w ? ?W, then we know that f −1 (w ?) is an open set in G for each ???, and f ? ? 1 ? ? ? (w ?) is also open set in G. i.e. f w a ? ? F H I K 1 ( ) ? ? ? ? is also open set in G ; that means, w a ??? ? is a member of W and w a ??? ? is an open set in G/H.

31 So, (2.1.1) (2.1.3) verify that W is a topology in G/H ; This topology is called the Quotient topology in G/H. The Quotient Topology in G/H is one that makes canonical mapping f (see above) to be continuous. Theorem 2.2.1. With respect to Quotient Topology in G/H the Canonical mapping f : G ????? G/H is an open mapping. Proof : Take O be an open set in Topological Group G. We cheek that f (0) is an open set in G/H. We need showing f −1 (f(0)) is open is G. Now. f(0) = {aH : a?O} = OH Take x?f −1 (f(O)) ; so, f(x) ?f(O) = OH ; there we find a member y?O such that f(x) = yH or, xH = yH or, {xh as h?H} = {yh as h?H} Since H is a sub-group, e?H and we see x = xe? {xh : h?H} = {yh : h?H} Therefore, x = yh for some h?H. That means x?OH or we have f −1 (f(O)) ? OH. Reversing the argument we deduce OH ? f −1 (f(O)) ; and therefore f −1 (f(O)) = OH which is, of course, an open set in G. Therorem 2.2.2. (H aclosed sub-group) In the quotient Group G/H Quotient Topology is Hansdorff. Proof : Let y, x ? G with xH ? yH So, x?yH As H is closed, we see that yH is closed with x as an outside point ; and x is not a limit point of yH ; so we
find a neighbourhood ? of the identity e in G such that (?x) ? (
yH) = ?? We now find a symmetric open neighbourhood of e satisfying W 2 ? ? We assert that (WxH) ? (WyH) = ? .... (1)
32 Otherwise, we find some w 1 , w 2 ? W and h 1 , h 2 ? H such that w 1 xh 1 = w 2 yh 2 Thus w w x yh h 2 1 1 2 1 1 ? ? ? Now

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w w W W W 2 1 1 1 2 ? ? ? ? ?(W being symmetric, W −1 = W) So, w w 2 1 1 ? ?? ?(since W 2 ??). Therefore, w w		

x x 2 1 1 ? ?? b g While yh h yH 2 1 1 1 ? ? ? b g , because H is subgroup. i.e. (Ux) ? (yH) ?? ? —a contradiction. Thus our assertion (1) stands. i.e. (WxH) ? (WyH) = ? and that means, (Wx) ? (Wy) = ? ; (taking e?H) (Wx) ? (WxH), and similarly (Wy) ??(WyH). Put W? = f(Wx) = WxH and W?? = f(Wy) = WyH showing W??W?? = ?. To complete the proof we now show that (xH) ? W? and (yH) ? W?? (here W??are W?? are open in G/H ; f sending open sets to open sets). To that end we recall f(x) ?f(Wx) because e?W. or (xH) ?W?, and similarly, (yH)?W?? and therefore W? and W?? are respectively dispoint open covers for xH and yH in
G/
H.
Theorem 2.2.3.

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Let G be a Topological Group and H a closed normal subgroup of G, then the quotient Group G/H is a Topological Group			

with quotient Topology. Proof : Consider the canonical mapping f : G ? G/H. In preceding Theorems we have seen that f is a continuous and open mapping. Now we check that f is a group Homomorphism. Take x, y ? G. Then f(xy) = (xy) H = xHyH = f(x) f(y). Thus f is a Homomorphism.
33 We now show that Group operation in Quotient Group G/H shall be continuous with respect to underlying topologies. i.e. one must show that the mapping (xH, yH) ?? xH(yH) −1 = (xy −1) H : (G/H) × (G/H) ? G/H is continuous. Suppose W be an open neighbourhood of xH (yH) −1 = (xy −1) H (x, y ? G), then f −1 (W) is open in G with (xy −1) ?f −1 (W). By continuity of group operation in G (a Topological Group), we find
open sets ? and V in G such that x?? and y −1 ?V −1 with ? V −1 ?
f −1 (W), or f (?V −1) ? W Since f is also a Group homomorphism, we have from above f (?) f (V −1) = f (?V −1) ?? W. Let u?? and v −1 ?V −1 ; Then uH v −1 H = f(u) f(v −1) ?f(?) f(V −1) ? W i.e. uH (vH) −1 ? ? W This shows that group operation in G/H is continuous to make the quotient group G/H a Topological Group with quotient topology. Definition 2.2.2. A

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Topological space X is said to a Homogeneous space if for any two member x 1 , x 2 in X there is a self Homeomorphism f in X such that f (x 1) =

x 2 . For example, every topological group G is always a Homogeneous space ; because if x 1 , x 2 ?? G, let us take x x 1 1 2 ? = u?G and inviting the mapping f : G ? G where f (x) = xu for x?G, we see atonce that f is a sef-homeomorphism of G such that f (x 1) = x u x x x ex x 1 1 1 1 2 2 2 ? ? ? ? . Theorem 2.2.3. (a). If H is a sub Group of a Topological Group G, then G/H, the quotient Topological Group is homogeneous. Proof : Take two members x 1 H and x 2 H in G/H with x 1 , x 2 ?G. Taking x x u 1 1 2 ? ? ?in G consider a homeomorphism ??: G/H ? G/H given by ?(xH) = (xu)H (= xHuH) for all (xH) ? G/H. Then we have? ?(x 1 H) = (x 1 u)H = ( ) x x x H x H 1 1 1 2 2 ? ? . Hence G/H

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is homogeneous. 34 Theorem 2.2.3.(b). Let G be a Topological Group and H a sub-group of G. Then G/H is T 1 if and only if H

is
closed. Proof :
Suppose G/H is T 1 . Then every suigleton in G/H is closed. Therefore {H} = {eH} is
closed in G/H ; Under conomical mapping f : G ? G/H which is continuous we have f −1 (eH) = H. Therefore H is closed is G. Conversely let the sub-group H be closed is G. Take any member xH in G/H. consider the singleton {xH} in G/H. Since H is closed we know that xH is closed making G\{sH} to be open in G. Therefore under cononical mapping f : G ? G/H, we have f(G\{xH}) is open in G/H. Now f (G\{xH}) = (G/H)\{xH} we conclude that {xH} is closed in G/H. Therefore every singleton in G/H is closed and that makes G/H T 1 .
The proof
is complete. Theorem 2.2.3(c)
Let G be a Topological Group and

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H a sub-group of G. Then G/H is a discrete space if and only of H is open. Proof :

Suppose G/H is a discrete space. Therefore each singleton of G/H is open. In particular, eH = H (e being the identity of G) is open. Under cononical mapping f : G ? G/H which is continuous, we have f −1 (eH) = H becomes open in G. Conversely let sub-group H be open. If x?G, we have xH is open. That means every suigleton in G/H is open in G/H and this is why G/H is a discrete space. Theorem 2.2.3(d) : Let H be a sub-group of a Topological Group G, and f : G ??on to G/H be the cononical mapping. If {? ? } ?t?
be a fundamental system of neighbourhoods of the identity e in G, then the family {
f(? ?)} ?t? is a fundamental system of neighbourhoods of the identity eH = H of G/H. Proof : Let f : G ??G/H be the canonical mapping. By property of f we see that if ? ?? is any member of

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a fundamental system {? ? } ??? of neighbourhoods of the identity e in G, then f (? ?) is a neighbourhood of

the identity eH in G/H. Suppose V is any neighbourhood of eH in G/H. Then f −1 (V) by continuity of f, is a

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35 neighbourhood of the identity e in G. So we find a member, say, ? ? in the family {? ? } ???? such that ? ? ? ? f V 1 ( ) or, f V ? ? ? a f This shows that the family f ? ? ? ? a f l q ? ?is a fundamental system of neighbourhoods of the identity e.H = H in G/H. Definition 2.2.3. A Topological Group G is said to be totally disconnected if the compononent of the identity e in G equals to {e}. Theorem 2.2.3. (e) : Let C be the component of the identity e in a Topological Group G. Then the quotient topological Group G/C becomes a totally disconnected T 2 space. Proof : First we show that C is a closed normal sub-group of G. Since C is the component ; by maximality C becomes closed. Now take a?C. Then a −1 C ? C, because a −1 C is the image of C under the homeomorphism x ? a −1 x becomes connected with e?a −1 C ; therefore, a a C ? ? 1 ? C = C −1 C ? C So, C is a closed sub-group of C. Further, by continuity of the mapping : x ? a −1 xa we have for a?G, a −1 Ca is also connected ; thus a −1 Ca ? C for each a?G because C is the component. Therefore C is a Normal sub-group of G. As C is closed it follows that quotient G/C is T 1 -space and hence it is T 2 . We have now to show that G/C is totally disconnected. Lt U be the component of the identity member (eC = C) in G/C. If ? is the natural homomorphism of G ? G/C, we have ? −1 (U). CG and C?? −1 (U). If G/C is not totally disconnected there is a member (x.C)(? e.C) such that (x.C) ? U. That

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means C is a proper subset of ? −1 (U). Since C is a maximal connected set containing e, ? −1 (

U) is not connected. Let a disconnection of ? −1 (U) be like : ? −1 (U) = [P?? −1 (U)] ? [Q? ?? ? −1 (U)] ..... (1) where

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P and Q are open sets in G, such that [P ??? −1 (U)] ??[Q ? ? −1 (U)] = ? and neither is empty. So U = [?(P) ? U] ??[?(Q) ? U]. Taking U = UC Let x?U?such that xC ? ?C ; Hence from (1) we have 36 xC = (P?xC) ?? ?Q?xC) Since xC is connected, either xC? (P?xC) or, xC? (Q?xC). Consequently, images P?UC and Q?UC under ? are disjoint, since they are unions of cosets of C. ? (?(P) ?

U) ?? (?(Q) ? U) = ? Now ??is an open mapping, so ? (P) and ?(Q) are open sets, and hence we have shown that U is not connected—a contradiction what U is the component of eC. Hence we have proved that G/C is totally disconnected. Remark : Given a topological Group G and a closed normal sub-group H is G, we have seen that cononical mapping f : G ? G/H, where G/H is topological group with quotient topology, becomes a continuous mapping which is also an open mapping. This mapping may not he a closed mapping. Example 2.2.3. Let R he the topological Group with addition as Group Composition and with usual topology of reals ; If Z is the sub-group of R consisting of all integers, then we see that Z is closed and a Normal sub-group of R. Here canonical mapping f : R ? R/Z is not closed. Solution : Consider the set E = n n n ? ? 1 2 1 , { } . Then E is a closed set in topological Group R. Every coset x + Z in R contains the number x - [x], {[x] denoting the largest integer not larger than real x) and no other real number in [0, 1). Therefore, [0, 1) may be treated as the quotient space R/Z. The Topology imposed in [0, 1) as a model of the space R/Z has basic open sets like (?, ?), and [0, ?] ? (?, 1) where 0 > ? > ??> 1. Now canonical mapping f sends E into a non-closed set (having 0 as a limit point outside the image set f(E)). Hence the conclusion stands OK. However we have following Theorem in this connection. Theorem 2.2.4. If H is a compact normal sub-group of a Topological Group G, then the cononical mapping : G ?????????? G/H is a closed mapping where G/H is the quotiont topological Group. Proof : Suppose C is a closed set in G ; and the canonical mapping f : G ?? G/H is in action to send x?G to xH u f(x) = xH as x?G.

37 Take xH ?(G/H)\f(C), and x?CH. As C is closed and H is compact we know that CH is closed. Therefore x is an outside point of the closed set CH, and we find an open set ? in G such that x???(G|CH). Cononical mapping f heing an open mapping f(?????) is an open set containing f(x) = xH i.e. f(?) is an open neighbourhood of xH such that f(?????) ????? (G|H)|f(C), showing that (G|H)|f(C) is open and hence f(C) is closed. The proof is complete. 2.3 Locally compact Groups : We recall following Definition : Definition 2.3.1. A topological space X is called locally compact if each point x in X has an open neighbourhood ? whose closere ? is compact. Then it is true that a Hausdorff topological space is locally compact if and only if, each point has a compact neighbourhood. Also we remember that every Hausdorff locally compact topological space is completely regular (and hence regular). Theorem 2.3.1.

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A Locally compact Hausdorff topological space X is normal if it is the union of an increasing

squence {U n } of open sets such that each ? n ????? is compact. Proof : We have by assumption ? ? ? ?

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n n 1 , and we write ? ? ? ? ? ? ? ? n n n n X 1 1 \ (|) and ? ? ?n n 1 \ is compact Also X n n n n n n n ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 0 1 \

a f , where ? 0 = ?? Suppose P ? ? { } ?? ?is an open cover for X. Since each ? ? ?n n 1 \ ?is compact, there shall be a finite sub-cover of P ? ? { } ?? for ? ? ?n n 1 \ . This is true for each n. As countable union of fanite families of sets constitute a countable family, one has a countable sub-family of P ? ? { } ?? ?to cover X-making X a Lindeloff space. Since every Lindeloff regular space is normal, the conclusion is arrived at as desired. Theorem 2.3.2. Every compact Hausdorff space is normal. For proof see any text book on general tohology.

38 Theorem 2.3.3. A Topological Group is a locally compact topological group if and only if its identity e has a compact neighbourhood. Proof : Suppose G is a locally compact topological group. So its identity e has a neighbourhood ? whose closure ? ?is compact. Conversely, suppose G is a Topological Group where identity e has a compact neighbourhood = ?. Choose

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a neighbourhood V of e such that VV = V 2 ??. Now, V VV V ? ? ? ? ? 2 ; Hence V ?is a closed subset of compact set ? and therefore V is a compact neighbourhood of e.

Let x be any element in G. Then xV is a neighbourhood of x

and we have xV xV ? ?becomes compact, because translation operator is a homemorphism in G. The proof is now complete. Theorem 2.3.4. A locally compact stausdorff topological Group is normal. Proof : First we establish that in a general topological Group G if ? is a symmetric neighbourhood of its identity e, then ? ? ?n n 1 ?is a clo-open (closed and open) sub-group of G. Because if H n n ? ? ? ? ? 1 ?and x, y ? H, say x?? n and y?? m . Then xy ?? n ? m = ? n+m ? H. Further, x −1 ?(? n) −1 = (? −1) n = ? n (? being symmetric). Therefore H is a sub-group of G. If y?H, we have y??y H = H, showing every member of H is an interior point of H and H is open, and every open sub-group of G is also closed. Hence the assertion follows. Now it is know that

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H is a Hausdorff locally compact sub-group of G. Also ? ? ? ? ? n n 1 1 ? = ? n ?for n ? 1. Hence H n n n n ? ? ? ? ? ? ? ? ? ? 1 1 .?But ? n is compact, because ? is compact. Thus H is a union (countable union) of increasing sequence of compact sets. H is normal (see theorem 2.3.1). Consider the collection {aH} of paviwise disjoint cosets in G. Since

translation (loft or right) is always a homeomorphism in G, each member aH (a?G) is homeomorphic to H and becomes normal. Therefore G = ?aH becomes normal. The proof is complete.

39 Corollary : If G is a locally cmpact Hausdorff Topological Group and C is a closed subset in G and ? an open set with C ???, then there is a real-valued continuous function f over G such that f(x) = 1 if x?C and f(x) = 0 if x?(G\?). Because G is normal by Theorem above and C and (G\(?)) are a pair of disjoint closed sets, by Urysohn’s Lemma we find a continuous function f : G ??[0, 1] satisfying. f(x) = 0 if x ? (G|?) = 1 if x ? C

Theorem 2.3.5. Let G be a locally compact Topological Group, and

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let C be a compact subset and ????? an open subset of G such that C ????? ?????. Then there

is a neighbourhood V of e such that
CV VC ? is compact with CV VC ? ??????????????? Proof : As C ? ? which is open, if x?C, we find an open neighbourhood V x
of the identity e in G such that xV x ??. Also choose an open neighbourhood W x of e such that W x W x = W x 2 ?? V x Now the family {xW x } x?C becomes an open
coves
for C. By compactness of C, there is a finite sub-cover, say
x 1 Vx 1 , x 2 Vx 2 , ..., x n V x n to cover C. Now put W W i n x s 1 1 ? ? ? , then W 1 is an open neighbourhood of e in G. Clearly CW x W W x W i n i x i n i x i i 1 1 1 1 2 ? ? ? ? ? ? ? (since W 1 ?? W x i) ?? ?? By a similar argument we produce an open neighbourhood W 2 of e in G such that W 2 C ?? ?. Since W 1 ?W 2
is an
open neighbourhood of e
in G, we choose a neighbourhood V of e in G such that
its closure V is compact and V W W ? ? ( ) 1 2 . Therefore CV VC ? ? ? . As C is compact and V is compact we know that C V and V C are each closed set ; Also C V ? V C as a Union of two compact sets becomes a compact set.
40 Further C V = CV ? and V C = VC Therefore ( ) ( ) CV VC CV VC CV VC ? ? ? ? ? This gives finally, ( ) ( ) CV VC ? as compact with ( ) ( ) CV VC ? ? ? . Theorem 2.3.6. Let ????? be an open neighbourhood of the Identity e
in a Topological Group G and C be a compact set in G. Then there is an

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open neighbourhood V of e such that CVC −1 ??????????????? Proof : Choose a symmetric open neighbourhood W 1 of the identity e in G such that W 1 3 ? ? , and for a fixed a?G take a symmetric open neighbourhood W 2 of e such that aW 2 a −1 ? W 1 . Put W = W 1 ?W 2 . Now x?Wa gives (xa −1) ?W?W 1 ; and ax −1 ? W W 1 1 1 ? ? ?(

W 1 is symmetric). Therefore. xWx −1 ?xW 2 x −1 = (xa −1)a W 2 a −1 (ax −1) ?W 1 W 1 W 1 = W 1 3 ? ? . Since W is dependent an a ?G, we designate W by W a . Now the family W a a a C { } ?
is an open cover for C ; by compactness of C, there is a finite sub-cover, say,
W a W a W a a a a n n 1 2 1 2 , , ... to cover C. Let as put V W i n a i ? ? ?1 . Then V is an open symmetric neighbourhood of e in G. If x?C we see that x W a a k k ? for some k, and this implies xW x a k ? ? ? 1 . Therefore, xVx xW x a k ? ? ? ? ? 1 1 . This completes
the proof. Example 2.3.1.

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Let G be a Topological Group and N is a closed Normal sub-group. (i) if G is compact, then G/N is a compact quotient Topological Group ; and (ii) if G is locally compact, then G/N is a Locally compact

quotient Topological Group. Solution. given that N is a closed normal sub-group. Then the quotient group G/N becomes a Topological Group (See Theorem 2.2.3). (i) Suppose G is compact. Now the canonical mapping f : G ? G/N where

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41 f(x) = xN ?G/N as x?G is continuous, and therefore f(G) is compact since G is compact. Here f(G) = G/N. So, G/N becomes compact. (ii) Suppose G is locally compact. So there is an open neighbourhood O of the Identity e in G such that O (closure of O) is compact. Now f(e) = eN = N ; Therefore N = f (e)?f (O) ? f (O) as O? O ? By continuity of f we also have f(O), is compact. So f(O)?is a compact subset of a Hausdorff space, and therefore f(O) is closed. Also f(O) is an open neighbourhood of N (f is an open mapping) and f O f O f O ( ) ( ) ( ) ? ? , because f O( ) is closed. Thus f O( ) is closed subset of f O( ) which is compact. Therefore f O( ) is compact. Hence G/N is locally compact. 2.4 Topological Groups Involving

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Connectedness : Definition 2.4.1. A topological space X is said to be connected if

X does not admit of a decomposition like X = P ?? Q Where P and Q are non-empty disjoint open sets in X. Explanation : A connected Topological space X is thus such a strong piece of objects that it does not allow its partition in the manner as above. Definition 2.4.1 shows that a Topological space X is connected if any only if in the space X there are no clo-open (Closed and open) sets other than ? and X. A subset E of X shall be taken as a connected set if it is a connected space in respect of relative tropology of E. In the real number space R with usual tyoplogy it is known that a subset of R is connected if and only if it is an interval. Definition 2.4.2. Given a point in X, the maximal connected subset in X containg the point is said to be the component of that point. In consequence, we recall that given a connected set A in X, it closure A is also a connected set, and thus every component in X is a closed set. Furthers, if {E ? } ??? is a family of connected sets in X, with ? ? ?? ? ? ? E , then ? ?? ? ? E ?

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is connected. 42 A topological space X is said to be Locally Connected if each

open neighbourhood of every point in X contains a connected open neighbourhood. We also recall that continuous image of a connected space becomes connected, and this gives as a special case that every real-valued continuous function over an interval enjoys Inter-mediate value property. In the following we present some basic properties of Topological groups depending upon connectedness of the Group when taken as a topological space.
Theorem 2.4.1.

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Let G be a Topological Group and H be the component of the Indentity e of G. Then H is a closed Normal sub-group of G.

Proof :
We know that in G group operation inversion : x ? x −1 as x?G is a homeomorphism, and therefore H −1 being continuous image of connected set H becomes connected, and it is a connected set containing e. This shows that by maximal property of H as the component, H −1 ????? H. Take x?H, then x −1 ?H −1 ? H. So, x −1 ?H. then xH is a connected set such that e = xx −1 ?xH ; Therefore xH is a connected set containing e ; By maximal property of H as the component containing e we have xH ? H That means, H 2 ? H (x is any member of H) .... (2) From (1) and (2) it follows that H is a sub-group in G. Now take a?G ; then the mapping : x ? axa −1 as x?G is a homeomorphism ; thus by its continuity we have as a continuous image aHa −1 is a connectedset containing e. Again by maximality of H as a maximal connected set containinge, we produce. aHa −1 ????? H. That means H is a normal sub-group in G. Since every component in a topological space is always closed we have H as a closed set. Thus

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H is a closed Normal sub-group in G. Example 2.4.1. Let G be a Topological Group and H be the component of

the identity
e in G ; If a?????G, aH (= Ha) is the component of a. Solution : Here H is a Normal sub-group of G (see Theorem 2.4.1) If a?G, we have aHa −1 = H ; giving aH = Ha.
43 Since translation is a homeomorphism we have aH as a connected set containing a (H is the component e, hence a connected set with e?H). Let C be the component of a. The we have aH ? C. .... (1) Lake L as a connected set with a?????L. Then a −1 L is a connected set containing e. because e = a −1 a?????a −1 L. H being the component of e we have a −1 L ? H or, L ? aH This being true for any connected set containing a we have, the component C of a satisfies. C?aH .... (2) From (1) and (2) we have C = aH i.e. aH is the component of a. Theorem 2.4.2. Let G be a connected Topological Group and H is the component of this identity e. If N is any neighbourhood of e, then G N n n ? ? ? ? ? 1 . Proof : Choose a symmetric neighbourhood V j of the identity e in G such that V?N. Then we have (see corollary following Theorem 2.1.2.) ? ? ? n n V 1 is open and closed. Since G is connected, G is the only non-empty open and closed (clo-open) set in G. Hence we have G V N G n n n n ? ? ? ? ? ? ? ? ? ? 1 1 , giving G N n n ? ? ? ? ? 1 ? Now Suppose {G ????? } ??? ??? ??? ??? ??? be a family of Topological Groups. Put G G ? ? ? ? ? ? ; G is called product of G ? 's. Let G have the product topology. Let x = {x ? } ??? , y = {y ? } ??? be two elements of G. Then xy is defined as : xy = {x ? y ? } ??? ; with this definition of composition of two members of G we easily verify that G forms a Group where the identity element e of G is given by e e ? ? ? ? { } ? where e ????? is the identity element of G ????? for ??? ??? ??? ??? ???. This Group G is called the Direct product of {G ????? } ??? ??? ??? ??? ??? , where individual members G ????? are called factors.
44 Theorem 2.4.3.

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Let {G ????? } ??? ??? ??? ??? ??? be a family of Topological Groups. If G = ? ? ? ?? G ????? is

the Direct product of G ????? endowed with product topology, then G is a Topological Group. Proof : Actually we need showing that the mapping (x, y) ? xy −1 of G × G onto ? ? ?? G is continuous. To that end take W as a neighbourhood of xy −1 in G as (x, y)?G × G. Then there is a finite number of indices, say ? 1 , ? 2 , ..., ? n ?? such that ? ? ? ? ? ? ? ? , with ? ? ?= G ? for ??? |{? 1 , ? 2 , ..., ? n } and ? ? i as open neighbourhoods of x y ? ? . ?1 (1 ? i ?? n), and ??W. Since (
x ? ? y ?) ? x y ? ? . ?1 is a continuous operation in topological Group G ?
for each ???, we obtain neighbourhood V V i i ? ? , ? of x i ? and y i ? in G i ? (1 ??i ? n) such that V V i i i ? ? ? ? ? ? ? 1 ; 1 ?? i ?? n. Put V V ? ? ? ? ? ? ? where V ? = G ? for ???? |? 1 , ? 2 , ..., ? n }, and V V i ? ? ? ? for ? = ? i (1 ?? i ? n). Similarly construct V? ; Then V and V? respectively form neighbourhoods of x end y, and we have VV V V W ? ? ? ? ? ? ? ? ? ? ? ? 1 1 ? ? ? ? ? ? ? ? ? b g . Therefore we have checked that direct product

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G is a Topological Group with Group composition and product Topology. Theorem 2.4.4. Let G G ? ? ? ? ? ? ?????be the direct product of Topological Groups {

G ????? } ??? ??? ??? ??? ??? , and let G have the product Topology. Then following statements are true. (i) G is a compact Topological Group if and if each G ????? is a compact Topological Group. (ii) G is a T 2 Topological Group if and only if each G ????? is so. (iii) G isa locally compact Topological Group if all G ????? are compact Topological Group except for a finite number of them that are each a locally compact Topological Group.

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45 Proof : (i) If each G ? is compact, then by Tychonoff Theorem G G ? ? ? ? ? ? ?is compact. Conversely, let G be compact ; if p r ?? : TT G ? ? ?? = G ??G ? ?is the ?th projection mapping, then we known that p r ? ?is continuous for each ???Since continuous image of compact space is compact we see that p r ? (G) = G ? is compact. (ii) It suffices to check this statement (ii) in respect of to-separation, because in a Topological Group T 0 ? T 2 . Suppose x ? e in G. Then there is an index, say ??? such that x ? ? ? e ? in G ? (e ? denoting the identity in G ?).
Since G ? is T 2 we find

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an open neighbourhood ? ? of e ? such that x ? ? ? ? . Now P r ? ? ? ? 1 ( ) is an open neighbourhood of

e in G such that x
P r ? ? ? ? ? 1 ( ). The converse part is too easy to make. (iii) Let us put H = ? ? ? ? ? ? ? ? ? ? with ( ) i i n G 1 . Then as proved in part (i) we find that H is a compact Topological Group, and therefore is a locally compact Topological Group. Thus (iii) shall be O.K. if one proves that product of a finite number of locally compact Topological Group is again a locally compact Topological Group. To that end, Take ? ? i ?as a neighbourhood of the identity e ? i in G ? i such that closure of ? ? ? ? ? i i is compact in G i ? . Put ? ? ? ? ? ? ? 1 i n i . Then ? becomes a neighbourhood of the identity in H such that ? ? ? ? ? ? ? 1 i n i , which is compact in H. Arguments are over and proof is complete. 2.5. Linear Groups, Locally Euclidean Groups and lie Groups. The Unitary space ? n = ? × ?? × ? × ?? (n factors), where ? denotes the field of complex numbers is a complex vector space with scalar field as that of complex numbers. Let M n (?) denote the collection of all square matrices ((a ij)) nxn with entries a ij ??. It is a routine exercise to check that M n (?) is a commutature additive Group with
46 identity element as the null matrix 0 0 0 0 0 0 ... . F H G G I K J J ?n n where addition means usual matrix addition. Let us recall the following Definition of a linear mapping (operator) over ? n . Definition 2.5.1. f : ? n ?? ? n is said to be a linear mapping if (i)

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f(x + y) = f(x) + f(y), and (ii) f(?x) = ?f(x) for all x, y ??? n and

for any scolar ???. Zero Linear mapping is one that sends every thing of ? n the zero = (0, ..., 0) ? ? n , i.e. identity number of ? n . Let L (? n) denote the collection of all linear mappings : ? n ? ? n . Then additively L (? n) forms a commutative Group. By wellknown matrix representation Theorem in linear Algebra one sees that each member i.e. linear mapping over ? n is represented by an n × n matrix over ? i.e. by a member of M n (?) and vice-versa. Therefore M n (?) and L (? n) are intimately linked by the correspondence as described. Theorem 2.5.1. M n (?????) is a T 2 locally compact Topological Group. Proof : Let M n (?) be assigned a topology. An element of M n (?) i.e. a matrix over ? may be indentified with a member of some unitary space explained as under : Let entries of each matrix A = ((a ij)) nxn ; a i, j ?? in M n (?) be arranged in a definite order. Then A may be looked upon an ordered n 2 tuple of complex scalars and therefore A may be identified with a member of ? n 2 . The correspondence so achieved is a mapping f : M n (?) ? ? n 2 . This mapping f is 1–1 and onto (bijective). Now ? n 2 is a unitary space with an Euclidean Topology. Define a subset H in M n (?) to be open if and only if f(H) is an open set in ? n 2 under the Euclidean Topology. Then M n (?) is equipped with a Topology so that M n (?) becomes a T 2 -locally compact additive Topological Group Because ? n 2 is so. Remark : This Topological Group M n (?????) is very often named as linear group. It helps study of groups of matrices, since unitary space ? n 2 is decorated with many interesting properties. Let G n (?) = {A = ((a ij)) nxn ?M n (?) : A is non-singular}. Non-singular member A?M n (?) means that there a member, known as inverse of A, denoted by A −1 ?M n (?) satisfying AA −1 = A −1 A = I, I denoting the n-th order indentity matrix in M n (?). It is also a routine exercise to check that G m (?) is a linear Group.

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47 Theorem 2.5.2. G n (?????) forms an open set in M n (?????). Proof : Consider the mapping ??: M n (?) ? ?, defined by ?(A) = det A as A?M n (?). Now G n (?) = {A?M n (?) : det A ? 0}. Since ? −1 (O) = {A?M n (?) : ?(A) = 0} = {A?M n (?) : det A = 0} we have. G n (?) = M n (?)\{? −1 (O)} Since ? is continuous we see that ? −1 (O) is a closed set in M n (?) and therefore G n (?) is open in M n (?). Theorem 2.5.3. G n (?????) is a T 2 -multiplicative Topological Group with respect to relative Topology indiced by M n (?????). Proof : We know that product of two non-singular square matrices of order n is again a non-singular matrix of the same size. Further if A?G n (?), then (A −1) −1 = A, and we see that A −1 ?G n (?). Thus with matrix maltiplication G n (?) forms a Group whose identity element is the identity matrix I = 1 1 1 O O n n ? F H G G I K J J ? with upper and lower blocks comprise of zeros since M n (?) is T 2 , one sees that G n (?) with respect to relative topology inherited from M n (?) is also T 2 . We now elxamine continuity of group composition of G n (C) in this topology. Let A, B?G n (?), and A = ((a ij)) n×n , B = ((b ij)) n×n . If AB = C where C = ((c ij)) n×n and c ij = a b ik kj k n ? ? 1 . Now mappings A ? a ij are continuous, because they are projections of ? n 2 onto co-ordinate spaces. Similarly B ?? b ij are continuous, therefore AB ? C ij is also continuous. So mapping (A, B) ? AB is continuous. Finally, if A?G n (?), we have A A d ij ? ? 1 1 det c hc h ?where d ij 's are minors in and are poly nomials in coefficients in A. As det A ? 0, mapping. A ? 1 det A d ij c hc h is also continuous.

48 Therefore A ? A −1 is continuous. Thus G n (?) is a T 2 -topological multiplication Group. Definition 2.5.2. A topological space X is called locally Euclidean if there is a + ve integer n such that every x?????X has a neighbourhood ????? which is homeomorphic to the open unit ball of the Euclidean n-space R n , namely = {(x 1 , x 2 , ..., x n) ; x i ?????R and x k k n 2 1 1 ? ? ? }. Explanation : A Topological Group is locally Euclidean if and only if for some +ve integer n, its identity e has a neighbourhood homeomorphic to the open unit ball of the Euclidean-n space R n . Theorem 2.5.4. M n (?????) is a locally Euclidean Group. Proof : By identification technique and defining Topology in M n (?), we see that M n (?) is homeomorphic to ? n 2 which in turn is homeomoephic to the Euclidean space R 2n 2 . Thus conclusion stands O.K. Remark : One can prove a similar Theorem saying that G n (?) is locally Euclidean. Thus examples of locally Euclidean topological group are not scarce. However, we note that there are topological Groups that are not locally Euclidean. For example, take G R ? ? ? ? ? ? ? where R ? = the space R of all reals for each ??? and G is the direct product of an infinite number of copies of R. G is equipped with the product topology. Then G is a topological Group that is not locally Euclidean. 2.6. Lie Groups : Consider a real-valued function f over an open set S ? R n (Euclidean n-space). f is said to belong to the class C ? if all partial derivatives including mixed devivatives of all orders of f exist and they are continuous in S. Now X is a T 2 (Housdorff) space. We now explain what is meant by an atlas A of class C ????? on X. Definition 2.6.1. (i) A family of ordered pairs like {(????? ????? ?? ??? ??? ??? ??? ? ?????)} ??? ??? ??? ??? ??? ????? where ? ? ? ? { } ? ?????forms an open cover of X and ? ? ? : ? ?R n ?????is a homeomorphism for each ??? ??? ??? ??? ??? is called an atlas, denoted by A if following conditions are satisfied :

49 (a) For ? , : 0 1 d i d i ?????is of class C ????? . (b) Let (?????, ?????) be a pair of an open set ????? in X and a homeomorphism ????? of X onto an open set of R n . If for each pair (????? ????? , ????? ?????)????? A for which ?? ? ? ? ? ????? mapping : ? ? ? ? ? ? ? ? 0 1? ??? ? ??? : ( ) ( ) is of class C ????? , then (?????, ?????)?????A. (ii) A Hausdorff topological space X with an atlas A is called a manifold. Explanation : Consequence of Definition 2.6.1. is that every manifold is locally Euclidean and therefore it is locally compact. We recally that M n (R) may be identified with the Euclidean space R n 2 and that M n (?) may be identified with the Euclidean space R 2n 2 ; therefore they are each a Manifold. Definition 2.6.2. A manifold G which is also a Group is called a Lie Group if mappings (i) (x, y) ????? xy of G × G onto ? ? ?? G and (ii) x ??????????x −1 of G onto G are analytic functions. For example, the Euclidean n-space R n is a lie Group, because, for x = (x 1 , x 2 , ..., x n) ?R n ; x i are reals, taking the identity mapping I(x) = (x 1 , x 2 , ..., x n) = x?R n , we verify that I belongs to C ? , and all requirements are O.K. for R n to be a manifold. So R n is a manifold. Further R n is additively a commutative Group such that (x, y) ? x + y as x, y ? R n is analytic, and similarly x ? −x x?R n is also analytic. Therefore, R n is a lie group. Example 6.2.1. Topological Group M n (?) is a lie Group. Every lie Group is locally Euclidean and hence locally compact. The famously well known fifth problem of Hilbert says that every locally Euclidean topological Group is a lie Group. For compact and Abelian Topological Group Problem had been solved long before the general solution was found. one may see pontrjagin "Topological Group". Homomorphism between Topological Groups : Theorem 2.6.2. If

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G and H are two Topological Groups and f : G ?????????? H is a homomorphism then (

a) For any two subsets A and B in G, f(AB) = f(A) f(B) (b) For any two subsets C and D is H, f −1 (C) f −1 (D) ?????????? f −1 (CD)
50 (c) If S is a symmetric set in G, then f(S) is symmetric in H (d) If T is a symmetric set in H, then f −1 (T) is symmetric in G. Proof : (a) Since f is a homomorphism, we have f(AB) = f(A) f(B) whenever A?G and B?G. (b) Let f be a homomorphism : G ? H and take x?f −1 (C) and y?f −1 (D) ; So we have f(x) ? C and f(y)?D Now f(xy) = f(x) f(y) ?CD ; since f is a homomorphism. Therefore, xy?f −1 (CD). So, we write, f −1 (C)f −1 (D) ?? f −1 (CD). (c) Let S be a symmetric set in G. We show that f(S) is a symmetric set by showing f(S) = (f(S)) −1 . Take y?f(S) ; So f(x) = y for some x?S. Since S is symmetric, we have x −1 ?S −1 = S Hence y −1 = (f(z)) −1 = f(z −1), Since f is a homomorphism. So, y −1 ?f(S) or y?(f(S)) −1 This gives f(S) ?? (f(S)) −1 .... (1) Conversely, take x?(f(S)) −1 Then x −1 ?f(S) So x −1 = f(u) for some u?S. Thus (f(u)) −1 = f(u −1) (f is a homomorphism) ?f(S) because S is symmetric. This gives x = (f(u)) −1 ?f(S) or, (f(S)) −1 ? f(S) .... (2) (1) and (2) give f(S) = (f(S)) −1 , showing that f(S) is symmetric. (d) proof shallbe similar to that of (c).

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Theorem 2.6.3. Let G and H be two topological Groups and f : G ? H be a Homomorphism. Then (

a) For any two subsets A and B of G, f A f B f AB ( ) ( )? b g . (b) For any two subsets C and D of H, f C f D f CD ? ? ? ? 1 1 1 ( ) ( ) ( ) . (c) For any symmetric set S in G f S( ) is symmetric in H and f S f S ( ) ( ) ? ? ? 1 1 b g .
51 (d) For any symmetric subset T in H, f T ?1 ( ) ? is symmetric in G and f B f B ? ? ? ? 1 1 1 ( ) ( ) b g , bar denoting the closure. Proof : First we observe that for any two subsets A and B in G, using contintity of group operation in G we have. AB AB ? , bar denoting the closure. Taking note of this inclusion relation proof of (a) and (b) shall follow from (a) and (b) parts of Theorem 2.6.2. above. (c) Inverse mapping in a Topological Group is a homemorphism, Therefore for any subset E in G we have E E ? ? ? 1 1 b g b g . Let S be a symmetric set in G. Then Theorem 2.6.2. Says that f(S) is symmetric. Consider f S( ) in topological Group H. By the remark above we have f S f S f S ( ) ( ) ( ) b g ? ? ? ? ? ?? 1 1 , because f(S) is symmetric. That means f S( ) is symmetric. (d) The proof is similar to that in part (c). Theorem 2.6.4. If
G and H are two Topological Groups, and f : G ????? H

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is a Homomorphism. Then f is continuous if and only if f is continuous

at the indentity e in G. Proof : Let f : G ? H be a Homomorphism, and let f be continuous. Then of course

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f is continuous at the identity element e of G. Conversely, suffose f is continuous at e, and x?G (x ? e). Let W be a neighbourhood of f(x) in H.

Choose a neighbourhood V of the identity e? in H such that W = f(x) V. Now f being a Homomorphism we know that f(e) = e?, and using continuity of f at e, we find a neighbourhood ? of e in G such that f(?) ? V. Clearly x? is a neighbourhood of x in G such that f(x?) = f(x) f(?) (f is homomorphism) ?f(x) V = W That shows, f is continuous at x. The proof is complete.

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Theorem 2.6.5. Let G and H be two topological Groups and f : G ????? H be a Homomorphism. Then

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f sends any open set in G to an open set in H iff f(O) is open in H for every open set O containing the identity e in G.
52 Proof : Suppose f is an open mapping i.e. f sends any open set in G to an open set in H. Then, of course, f(O) is open in H whenever O is an open set in G containing the identity e of G. Conversely, suppose the condition holds and take any openset ? in G. If x??, then ? is a neighbourhood of x in G and choose an open set V containing the identity e in G such that U = xV ; Now f(V) is open in H. Then f(?) = f(xV) = f(x) f(V), because f is a Homomorphism. Now f(V) being open we have r.h.s. is an open set in H. i.e. f(?) is open in H. So, f sends an open set in G to an open set in H. Example 2.6.2. A continuous Homomorphism between two topological Groups may not be not be an open mapping. Solution. Take R as the set of all reals. Treat Ras an additive commutative Topological Group with discrete Topology. Also treat R an additive commutative Topological Group with usual Euclidean Topology and call it R u . Then consider the indentity mapping I : R ? R u as a Homomorphism which is, in this case, 1 – 1 and onto. Since discrete topology is strictly finer than the usual topology of reals in R we see at once that I is not an open mapping ; mevertheless, I is continuous. EXERCISE-A Short Answer type Questions 1. Definie a sub-group of a Topological Group with an example with justification. 2. When is a sub-group of a Topological Group called discrete ? Find a discrete sub-group of the Topological additive Group R of all reals. 3. If H is a sub-group of a Topological Group G, show that its closure H is a sub-group of G. 4. In a Topological Group G is x 1 , x 2 ? ?G show that there is a self- homeomorphism f of G such that f(x 1) = x 2 . 5. Let G be a locally compact Topological Group and f : G ??F is an open continuous homeomorphism where F is another. Topological Group. Show that F is locally compact.
53 6. If a topological Group G is connected and H is a sub-group of G, show that G/H is connected. 7. If H is a sub-group (a normal sub-group) of a Topological Group G, show that its closure H ?is a subgroup (a normal sub-group) of G. 8. Example if the set Q of all rationals forms a Topological sub-group of Topological additive Group R of all reals. 9. Find a discrete sub-group of Topological additive Group R of all reals with reasons. EXERCISE-B 1. Let G be a topological Group and H a sub-group of G. If ?
is a neighbourhood of the idendity e in G such that
H? ? is closed in G, show that H is closed in G. Solution : Take a symmetric neighbourhood V of the identity e in G such that V 2 = VV ??. Let x? H ; if {x ? : ??D, —} is a net in H such that {x ? } converges to x is g. Now x –1 ? H (H , a sub-group). So (Vx –1) ? H ???. Take y?(Vx –1)?H. Let x ? ?xV for ? ? ? 0 ?(say), ? 0 ?D ; then we have yx ? ?(Vx –1)(xV) = V 2 ??, and hence (yx ?) ? ? ? ? ? H ??As the net yx D ? ?: , ? ? l q converges to yx, and H ? ? is closed, we have ( ) yx H ? ? ? ? ? ? ; Therefore x = y –1 yx?H, showing H ?H. Therefore H is closed. 2. If H is a normal sub-group of a topological group G, show that quotient Group G/H is homogenous. 3.

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Let G be a Topological Group and H a sub-group of G. If H			

and G/H are locally compact, show that G is so. 4. Let G be a locally compact topological group, and C be the component of the identity e is G. Show that C = ?{H : H is any open sub-group of G}. 5. Let ? ? ? ? { } ? he the neighbourhood system of the identity e in a Topological Group G and A?G. Prove that A A ? ? ? ? ?? ? ? ? , bar denoting the closure.
54 6. Let G be a Topological Group with the identity e. Show that e? ? is a normal closed sub-group of G, and hence, G e / ? ?
is

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a Hausdorff Topological Group. 7. Prove that the component of the identity of a Topological Group is a closed Normal			

sub-group. 8. Let R 2 be an additive topological Group and H be the st. line y = ?x in R 2 which is a sub-group of R 2 . If f : R 2 ? R 2 /N be the cononical mapping where N = {(m, n) : m, n an integers} is a sub-group of R 2 , examine if f(H) is a closed sub-group of Topological Group G/N for ? to be (i) a rational number and (ii) an irrational number. 9. Let G be the additive Topological Group of all reals, and Z be the sub-group of G. show that Z is a discrete sub-group of G and factor Group G/Z is homeomorphic to a Circle. 10. Prove that topological product of two Eudidean spaces R n and R m is homeomorphic to the Euclidean space R n+m .

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Let G be a Topological Group and H be a subgroup of G. Then H

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a symmetric open neighbourhood of the identity e in topological group G and L = ? ? ? ? n n 1 , then L is an open and closed (clo-open) sub-group of G. Because we have

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the family of all (Left) cosets of H in G

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5/33	SUBMITTED TEXT	23 WORDS	56%	MATCHING TEXT	23 WORDS

Let G be a Topological Group and H a closed normal subgroup of G, then the quotient Group G/H is a Topological Group

Let G be a topological group and let H a normal subgroup of G. From we know that G/H is a group.

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6/33	SUBMITTED TEXT	36 WORDS	38%	MATCHING TEXT	36 WORDS

Topological space X is said to a Homogeneous space if for any two member x 1 , x 2 in X there is a self Homeomorphism f in X such that f (x 1) =

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7/33	SUBMITTED TEXT	27 WORDS	46%	MATCHING TEXT	27 WORDS

is homogeneous. 34 Theorem 2.2.3.(b). Let G be a Topological Group and H a sub-group of G. Then G/H is T 1 if and only if H

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8/33	SUBMITTED TEXT	19 WORDS	61%	MATCHING TEXT	19 WORDS

H a sub-group of G. Then G/H is a discrete space if and only of H is open. Proof :

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9/33	SUBMITTED TEXT	25 WORDS	64%	MATCHING TEXT	25 WORDS

a fundamental system {? ? } ??? of neighbourhoods of the identity e in G, then f (? ?) is a neighbourhood of

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10/33	SUBMITTED TEXT	21 WORDS	76%	MATCHING TEXT	21 WORDS

means C is a proper subset of ? −1 (U). Since C is a maximal connected set containing e, ? −1 (

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P and Q are open sets in G, such that [P ??? −1 (U)] ??[Q ? ? −1 (U)] = ? and neither is empty. So U = [?(P) ? U] ??[?(Q) ? U]. Taking U = UC Let x?U?such that xC ? ?C ; Hence from (1) we have 36 xC = (P?xC) ?? ?Q?xC) Since xC is connected, either xC? (P?xC) or, xC? (Q?xC). Consequently, images P?UC and Q?UC under ? are disjoint, since they are unions of cosets of C. ? (?(P) ?

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12/33	SUBMITTED TEXT	17 WORDS	91%	MATCHING TEXT	17 WORDS

A Locally compact Hausdorff topological space X is normal if it is the union of an increasing

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n n 1 , and we write ? ? ? ? ? ? ? ? n n n n X 1 1 \ (|) and ? ? ?n n 1 \ is compact Also X n n n n n n n ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 0 1 \

n-1) = 1 - (n-2) 2 3 LCM of 2 and 3= 6 6x n - 6(n-1) = 6x1 - 6(n-2) 2 3 6n - 3(n-1) = 6 - 2(n-2) 6n - 3n + 3 = 6 - 2n +4 3n+ 2n = 6+4 - 3 5n= 7 n= 7/5

W https://brainly.in/question/18854841

14/33	SUBMITTED TEXT	37 WORDS	53%	MATCHING TEXT	37 WORDS

a neighbourhood V of e such that VV = V 2 ??. Now, V VV V ? ? ? ? ? 2 ; Hence V ?is a closed subset of compact set ? and therefore V is a compact neighbourhood of e.

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15/33	SUBMITTED TEXT	67 WORDS	33%	MATCHING TEXT	67 WORDS

H is a Hausdorff locally compact sub-group of G. Also ? ? ? ? ? n n 1 1 ? = ? n ?for n ? 1. Hence H n n n n ? ? ? ? ? ? ? ? ? ? 1 1 .?But ? n is compact, because ? is compact. Thus H is a union (countable union) of increasing sequence of compact sets. H is normal (see theorem 2.3.1). Consider the collection {aH} of paviwise disjoint cosets in G. Since

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16/33	SUBMITTED TEXT	19 WORDS	94%	MATCHING TEXT	19 WORDS

let C be a compact subset and ????? an open subset of G such that C ????? ?????. Then there

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17/33	SUBMITTED TEXT	39 WORDS	35%	MATCHING TEXT	39 WORDS

Let G be a Topological Group and N is a closed Normal sub-group. (i) if G is compact, then G/N is a compact quotient Topological Group ; and (ii) if G is locally compact, then G/N is a Locally compact

Let G be a topological group and let H be a closed normal in G. The following statements hold: 1. If G is compact, then G/H is compact. 26 2. If G is locally compact, then G/H is locally compact. Proof: Assume that G is a compact

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18/33	SUBMITTED TEXT	73 WORDS	31%	MATCHING TEXT	73 WORDS

open neighbourhood V of e such that CVC −1 ??????????????? Proof : Choose a symmetric open neighbourhood W 1 of the identity e in G such that W 1 3 ? ? , and for a fixed a?G take a symmetric open neighbourhood W 2 of e such that aW 2 a −1 ? W 1 . Put W = W 1 ?W 2 . Now x?Wa gives (xa −1) ?W?W 1 ; and ax −1 ? W W 1 1 1 ? ? ?(

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19/33	SUBMITTED TEXT	14 WORDS	87%	MATCHING TEXT	14 WORDS

Connectedness : Definition 2.4.1. A topological space X is said to be connected if

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20/33	SUBMITTED TEXT	14 WORDS	85%	MATCHING TEXT	14 WORDS

is connected. 42 A topological space X is said to be Locally Connected if each

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21/33	SUBMITTED TEXT	25 WORDS	46%	MATCHING TEXT	25 WORDS

Let G be a Topological Group and H be the component of the Indentity e of G. Then H is a closed Normal sub-group of G.

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22/33	SUBMITTED TEXT	21 WORDS	83%	MATCHING TEXT	21 WORDS

Let {G ????? } ??? ??? ??? ??? ??? be a family of Topological Groups. If G = ? ? ? ?? G ????? is

let G i be a family of topological groups. Then, G = Q i∈I G i is

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G is a Topological Group with Group composition and product Topology. Theorem 2.4.4. Let G G ? ? ? ? ? ? ?????be the direct product of Topological Groups {

G i is a topological group with the product topology. Theorem 4.0.9 Let G= Q i∈I G i be a direct product of topological groups,

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24/33	SUBMITTED TEXT	21 WORDS	52%	MATCHING TEXT	21 WORDS

H is a closed Normal sub-group in G. Example 2.4.1. Let G be a Topological Group and H be the component of

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25/33	SUBMITTED TEXT	21 WORDS	57%	MATCHING TEXT	21 WORDS

f(x + y) = f(x) + f(y), and (ii) f(?x) = ?f(x) for all x, y ??? n and

f(x + y) = f(x) + f(y) for all x, y ∈ F n 2. f(λx) = λf(x) for all x ∈ F n and

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26/33	SUBMITTED TEXT	28 WORDS	68%	MATCHING TEXT	28 WORDS

an open neighbourhood ? ? of e ? such that x ? ? ? ? . Now P r ? ? ? ? 1 ( ) is an open neighbourhood of

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G and H are two Topological Groups and f : G ?????????? H is a homomorphism then (

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Theorem 2.6.3. Let G and H be two topological Groups and f : G ? H be a Homomorphism. Then (

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is a Homomorphism. Then f is continuous if and only if f is continuous

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f is continuous at the identity element e of G. Conversely, suffose f is continuous at e, and x?G (x ? e). Let W be a neighbourhood of f(x) in H.

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Theorem 2.6.5. Let G and H be two topological Groups and f : G ????? H be a Homomorphism. Then

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32/33	SUBMITTED TEXT	19 WORDS	73%	MATCHING TEXT	19 WORDS

a Hausdorff Topological Group. 7. Prove that the component of the identity of a Topological Group is a closed Normal

a Hausdorff topological group is a closed normal subgroup. The component of the identity of a topological group is a closed normal

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33/33	SUBMITTED TEXT	15 WORDS	66%	MATCHING TEXT	15 WORDS
Let G be a Topological Group and H a sub-group of G. If H					

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PREFACE In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation. Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis. The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other. The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self- learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University. Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned. Professor (Dr.) Subha Sankar Sarkar Vice-Chancellor

NETAJI SUBHAS OPEN UNIVERSITY PG (MT)–IX A(I) Unit 1 ? Analytic Continuation 7-25 Unit 2 ? Harmonic Functions 26-40 Unit 3 ? Conformal Mappings 41-49 Unit 4 ? Multi-valued Functions and Riemann Surface 50-82 Unit 5 ? Conformal Equivalence 83-104 Unit 6 ? Entire and Meromorphic Functions 105-155 N E T A J I S UBHAS OPEN UN I V E R S IT Y

7 Unit 1 Analytic Continuation Structure 1.0 Objectives of this chapter 1.1 The idea of analytic continuation 1.2 Direct analytic continuation 1.3 Analytic continuation of elementary functions 1.4 Analytic continuation by power services 1.5 Analytic continuation along a curve 1.6 Multi-valued Functions and Analytic continuation 1.0 Objectives of this Chapter In this chapter we shall introduce the idea of direct analytic continuation of an analytic function. The concepts of analytic continuation by means of power series, complete analytic function, natural boundary, analytic continuation along a curve will be explained with the help of examples. Homotopic curves, analytic continuation of multi-valued function and Monodromy theorem will also be discussed. 1.1 The idea of analytic continuation The idea of analytic continuation rests on the notion of analytic function. A function f(z) is analytic at z = z 0 if it is differentiable in some ∈-neighbourhood of z 0 or, equivalently if it can be expressed in the form of a Taylor series in a neighbourhood of that point. The domain of convergence of this power series will be the region of analyticity of the function f(z). Following Uniqueness
Theorem : "If two functions f(z) and g(z), analytic on a region D, are such that f(z) = g(z) on a set A⊂D having a limit point in D, then f(z) = g(z) ∀
z ∈ D," we know that if two analytic functions agree in some small neighbourhood of a point situated in their common region of analyticity D, they
8 coincide everywhere in D. We first introduce the idea of analytic continuation by the following examples. The geometric series 1 + z + z 2 + ... converges for |z| > 1 and its sum function g(z z) − = 1 1 is an analytic function for |z| > 1. The geometric series diverges for |z| ≥ 1. However, the function h z z ( ) − = 1 1 is analytic for all z except z = 1. But we observe that h z g(z z z C ( )) \ {} = ∀ ∈ > / 1 1 Thus, we may regard h(z) as determining an analytic continuation of g(z) from the domain |z| > 1 into the domain /C \{1}. Example 1.1 Consider the Laplace transform of 1 in the z -plane, F z z dt z zt ( ) £ {}( ) Re − = = = < • e for z 1 1 0 0 We introduce a function φ( )z z = 1 which is analytic in the complex plance /C except the origin. Here φ( ) ( ) / ( ) Re z F z z C = ∀ ∈ / ∩ < 0 0 z and we consider φ(z) as analytic continuation of F(z) from the domain Re z < 0 into the complex plane with the point z = 0 deleted. We put these ideas more precisely in the following discussion. 1.2 Direct analytic continuation Let (i) f(z) and g(z) be analytic functions on domains D 1 and D 2 respectively. (ii) D D 1 2 ≠ φ (iii) f(z) = g(z) for all z belonging to D D 1 2 Then g (z) is called a direct analytic continuation of f(z) to D 2 , and vice versa.
9 Theorem 1.1. A direct analytic continuation, if it exists, is unique. Proof. Let f(z) be an analytic function with domain of definition D 1 and let g(z), another analytic function with domain of definition D 2 , be its direct analytic continuation. We shall show that g(z) is unique. On the contrary suppose φ(
z) be another analytic continuation of f(z) into D 2 . Then f z g(z D D ( )) = ∈ for all z 1 2 Also, f z z D D ( ) ( ) = ∈ φ for all z 1 2
and so φ(z) coincides with g(z) in D D 1 2 . Thus we have, by the Uniqueness theorem, φ(z) = g(z) in D 2 . 1.3 Analytic continuation of elementary functions The functions e z , sin z, cos z, sinh z etc are already known to us. These functions are regular in the entire complex plane. Let us assume, by definition, that e z n z n n = = • ∑ ! 0 and observe that it coincides with e x , known earlier, for real values of z. Thus we can take e z as the analytic continuation of e x from real axis into the entire complex plane. Likewise introducing
sin z, cos z sinh z, cosh z in the form of
power series— sin (−1) ( )! , cos (−1) ( )!
z
z
n z
z n
n n n n n n = + = + = • = • ∑ ∑ 2 1 2 0 0 2 1 2 sinh ( )! cosh ( )! z z n z z
n
n
n
n
n = + = + = • = • ∑ ∑ 2 1 2 0 0 2 1 2
and
We
can treat them as the analytic continuation of the functions sin x, cos x, sinh x and cosh x respectively from the real axis into the entire complex plane. D 1 D 2 Fig. 1

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10 1.4 Analytic continuation by power series We now explain the concept of analytic continuation by means of power series. Suppose
the initial function f 1 (z) is analytic at a point z 1 . Then f 1 (z) can be represented by its Taylor series about
z 1 as f z a z z f z n n n n n n 1 1 1 1 0 1 ( ) (−) ...( ), ( ) ! ( ) = = = • ∑
where a The circle of convergence γ 1 of the series (1) is given by γ 1 1 1 : − , z z R = where 1 1 1 R a n n n = →• limsup Let D 1 = {z : |z − z 1 | > R 1 }. Then f 1 (z) is analytic in D 1 . We draw a curve γ from z 1 and perform analytic continuation along γ as follows : We take a point z 2 on γ such that the arc z 1 z 2 lies inside γ 1 . We then compute the values f 1 (z 2), f 1 1 (z 2),..., f 1 (n) (z 2) by successive term by term differentiation of the series (1) and write f z b z z n n n 2 0 2 2 ( ) (−) ...( ) = = • ∑ where b f z n n n = 1 2 ( ) ( ) ! The circle of convergence γ 2 of the series (2) is given by γ 2 2 2 : − z z R = , where 1 2 1 R b n n n = →• limsup Let D z z z R 2 2 2 = > : − . Then f 2 (z) is analytic in D 2 . By uniqueness theorem, f 1 (z) = f 2 (z) for all z D D ∈ 1 2 . If γ 2 extends beyond γ 1 , then f 2 (z) gives an analytic continuation of f 1 (z) from D 1 to D 2 . Similarly, considering a point z 3 on γ such that (R 1 Z 1 D 1 D 2 D 3 R 2 R 3 Z 2 Z 3 γ 1 γ 2 γ 3 γ
11 the arc z 2 z 3 lies inside γ 2 , we get an analytic
function f 3 (z) in a circular domain
D 3 such that f 2 (z) = f 3 (z) for all z D
D ∈ 2 3 . If
D 3 extends beyond D 2 , then f 3 (z) gives an analytic continuation of f 2 (z)
from D 2 to D 3 . Repeating this process we get a number of different power series representing analytic functions f i (z) in their respective circular domains D i which form a chain of analytic continuations of the original function f 1 (z) such that (f i , D i) is a direct analytic continuation of (f i−1 , D i−1). Note : We may obtain the series (2) from the series (1) in the following way : We rewrite the series (1) in the form : a z z z z n n n = • ∑ + 0 2 2 1 (−) (−) Using binomial theorem we then expand (−) (−) z z z z n 2 2 1 + and collect the terms in like powers of (z−z 2) and obtain the series (2). We give two examples. Example 1.2 The function f z z ( )= + 1 1 2 possesses two simple poles at z = ± i; Otherwise it is regular throughout the whole complex plane. We first choose a point, say z = 0 at which f(z) is analytic and obtain its Taylor series expansion represented by g(z) as g(z) = 1 − z 2 + z 4 − ..., |z| > 1 The series fails to converge on and beyond the unit circle, as is clear from the (series for z = 1 even though the function f(z) is analytic at that point. We can in fact continue the expansion from one region to another. Let us consider a second expansion of f(z), this time about a point z = 3 4 lying inside the unit circle (i.e. lying inside
the region of convergence of the former series). We form
the expansion
as
follows 1 1 1 1 2 1 1 2 + = + = +
z
z i z i i z i z i ( )(−) − − o i -i z = − 3 4 Fig. 2
12 = + + +
1 2 1 3 4 3 4 1 3 4 3 4 i z i z i − − − − = +
+ + +
1 2 1 3 4 1 3 4 3 4 1 3 4 1 3 4 3 4 i i z i i z i − − / / − − − / / −1 −1 = + 1 2 3 4 1 3 4 3 4 3 4 3 4 2 2 i i z i z i [(/ −) { − (− /) / (/ −) (− /) / (/ −) −...} −1 −(/) { − (− /) / (/) (− /) / (/) −...}], − − 3 4 1 3 4 3 4 3 4 3 4 3 4 5 4 2 2 + + + + > i z i z i z i =
+
+
16 25 3 2 16 25 3 4 11 16 16 25 3 4 21 16 16 25 3 4 2 3 2 4 4 − − − − z z
z ... (2)
We

denote this expansion by h(z), which converges in the right-hand circle z − 3 4 5 4 > and coincides with g(z) in the shaded region. We see that h(z) is clearly a direct analytic continuation of g(z). Let us construct another analytic continuation of g(z). Now we consider a neighbourhood of the point z = 1 (though it is a boundary point of the unit circle the function f(z) is analytic there) and obtain an expansion represented by φ( ) − (−) (−) −... z z z = + 1 2 1 2 1 1 4 1 2 for z − ...( ) 1 2 3 > In this way we can determine all possible direct analytic continuations of g(z) and then continuations of these continuations and so on. A complete analytic function is defined as consisting of the original function and the collection of all the continuations so achieved. Here the complete analytic function is 1 1 2 + z , defined in the whole complex plane barring the points z = ±i. Example 1.3 Consider the function Fig. 3 ∨ ∨ ∨ ∨ ∨ O z − > 3 4 5 4
13 f z z ( )= + 1 1 Clearly this function is analytic everywhere except at z = − 1. We take a function φ( ) − z z z = + 1 2 ... (4) Then sum function φ(z) is 1 1+ z in | z | > 1. Take a point z = − 1/4 inside the region of convergence of φ(z) and in a neighbourhood of this point we determine Ψ( )z z z = − +
+
+
−
4 3 1 4 3 1 4 4 3 1 4 2 2 z + > 1 4 3 4 ... (5) It can be checked easily that φ(z) and Ψ(z) are direct analytic continuation of each other. Again in the neighbourhood of z = i/2 we obtain an expansion k z i z i i z i i ( ) / / / / / ... = + − − +
+ − +
− 1 1 2 1 2 1 2 2 1 2 2 z i − > 2 5 2 ... (6) In performing analytic continuations we notice that there are certain points which always lie on the boundary of domains in which expansions are not valid. These points are nothing but the singularities of the complete analytic function. In example 1.2 these are z = ± i whereas it is z = −1 for example 1.3. Regular and Singular points Let f(z) be an analytic function defined in the domain D, bounded by a simple closed curve Γ. A point ς ∈ Γ is called a regular point of the function f(z) if
there exist a neighbourhood | z − ς | > ∊ of the point ς and an analytic function φ ς (z) such that φ ς ς ( ) ( ) | | z f z z D z = ∀ ∈ ∩ − >∊ . The boundary point ζ which is not a regular Fig. 4 Fig. 5 Fig. 6 ∨∨∨∨∨ ↓ −1 −1/4 O 1 −1 1 O z i − 2 5 2 > D Γ ς
14 point is called a singular point of f(z) i.e., in any neighbourhood of the point ζ, there cannot be any analytic function coinciding with f(z) in the part common to the neighbourhood of ζ and the domain D. Natural boundary In examples 1.2 and 1.3 we have encountered with finite number of singular points situated on the boundary of the region of analyticity of the given function. It might happen that the boundary is dense with singular points. In this case analytic continuation across the boundary of the region is not possible. Such a boundary is called a natural boundary. Example 1.4 Test whether analytic continuation of the function f z z n n ( )= = • ∑ 2 0 is possible outside its circle of convergence. Solution : Applying the ratio test we find that the given series
f(z) = z + z 2 + z 4 + z 8 + ... (7) converges for |z| > 1. The point z = 1 is a singular point of f(z)
as it is seen for real z that the sum x n n 2 0= • ∑ increases indefinitely as x → 1. Now to test whether the circle of convergence, the unit circle, is a natural boundary we examine the behaviour of the given function at the points. z e k s i s k k , , , , , ... = = 2 2 1 2 3 2 π s (k is any natural number). For this sake we consider the points ~ . z re k s i s k = 2 2 π 0 > r > 1 and evaluate f(z) at these points. Then f z r e r e k s n k i s n k i s n k n n k n (~) , . . = + = − = • ∑ ∑ 2 0 1 2 2 2 2 2 2 2 π π and observe that the first term consists of a finite number of terms and hence bounded in absolute value, whereas the second term is absolute value reduces to r n n k 2 = • ∑ . Clearly this sum increases indefinitely as r → 1. This shows that the points z k,s (as lim ~ , , r k s k s z z → = 1 are singular points of the Fig. 7 ↓ O 1 ~ , z k s
15 given function f(z). Now as k → • these points form an everywhere dense set of points on the boundary of the unit circle. Thus analytic continuation outside the circle of convergence of the given
function is not possible. Example 1.5 Show that the function f z z n n ( ) ! = = • ∑ 1

has unit circle as its natural boundary. Theorem 1.2 Every power series has at least one singular point on its circle of convergence. Proof. Let f(z) = a 0 + a 1 (z − z 0) + a 2 (z − z 0) 2 + ... be any power series with region of convergence K:|z − z 0 | > R. We shall have to prove there lies at least one singular point on the circle of convergence Γ:|z − z n | = R of the function. Suppose, on the contrary, that every point on Γ are regular points. Let ς 1 , ς 2 ,... ς i ,... be certain number of regular points belonging to Γ and N(ς 1), N(ς 2), ..., N(ς i)... be their neighbourhoods respectively. The points ς i 's are chosen in such a way that N(ς i) has non null intersection with N(ς i − 1) and N(ς i + 1) and the union of these neighbourhoods completely cover the boundary Γ. Let D be the union of K and all these neighbourhoods N(ς i). D is open since K and every N(ς i) are open. D is also connected since. (i) any two points lying in K⊂D can be connected by a straight line segment lying in K, since K is connected. (ii) one point z 1 ∈N(ς 1) and the other z 2 ∈K can be connected by two straight line segments z 1 1 ζ and ζ 1 2 z lying within N(ς 1)UK⊂D. (iii) one point z m ∈N(ζ m) and z n ∈N(ζ n) can be connected by a curve consisting of z z D m m m n n n ζ ζ ζ ζ + + ⊂ since z N D D m m m m n ζ ζ ζ ζ ⊂ ⊂ ⊂ ⊂ ( ) , Γ and ζ ζ n n n z N D ⊂ ⊂ ( ) . ((Fig.8 H ς i z i Z 0 K R Γ z 1 z 2 ζ 1 ζ 2 β α . . . .

16 and finally if two points lie in the same neighbourhood N(ζ i) it is always connected by a curve γ ⊂ N(ζ i) ⊂ D. Now we introduce an analytic function ψ(z) on the open connected set D which satisfies ψ(z) = φ ς i (z), z ε N(ζ i) f(z), zεK where φ ς i (z) is a direct analytic continuation of f(z) in the neighbourhood N(ζ i) of the regular point ζ i . We now prove that ψ(z) is well-defined on D. Let α, β be any two points on Γ such that H N N = ≠ ( ) ( ) α β φ and since α, β are regular points there exist functions φ α (z) and φ β (z) as direct analytic continuations of f(z) in N(α) and N(β) respectively i.e. φ ε α α ( ) ( ) ( )zfzz K = ∀ N φ ε β β ( ) ( ) ( )zfzz K = ∀ N so that φ φ ε α β α β ( ) ( ) ( ) ( ) ( ) z z f z z G N K N K H = = ∀ = ⊂ . Now since φ α (z),φ β (z) are analytic in H and G is a part of H, by the uniqueness theorem φ α (z) ≡ φ β (z) ∀z ε H. As α and β are arbitrary points of Γ we conclude that ψ(z) is a well-defined analytic function on D. Let C be the boundary of D and let ρ ζ ζ ε = z 0 , C be the minimum distance from z 0 to the boundary C of D. Then clearly ρ < R as ς lies outside the circle Γ. Thus we observe that ψ(z) coincides with f(z) on the disc |z−z 0 | > R. Then it is obvious to conclude that
the radius of convergence of the given power series a z z n n
n (−) 0 0= • ∑ is ρ, not R, which is a contradiction. Hence every point on Γ cannot be regular points, i.e., there must be at least one singular point on Γ. 1.5 Analytic continuation along a curve Earlier, analytic continuation by power series method, we have extended f(z) to a

17 larger domain considering its power series expansion about a point a from its original circle of convergence with centre at z 0 (−a ≠ z 0) and radius r. We know, this power series converges in the disc D 1 :|z − a| > R, where R ≥ r−|z 0 − a| [(see Fig. 9), for example 1.2]. Then it converges to an analytic function g(z) defined on D 1 , which is equal to f(z) on D D 1 . Analytic continuation along a curve is an extension of this idea to the situation where a curve is covered by an overlapping sequence of discs and an analytic function defined on the first disc, can be extended succesively to each disc in the sequence (see figure 10). We will make this idea more precise after introducing the definition of function element. Definition 1. An ordered pair (f, D), where D is a region and f is an analytic function on D is called a function element. We say that it is a function element at z 0 if z 0 belongs to D. Two function elements (φ, G) and (ψ, H) are equal if and only if φ(z) ≡ ψ(z), G = H. Clearly a function element (f 1 , D 1) is a direct analytic continuation of another function element (f 2 , D 2) when D 1 ∩ D 2 ≠ φ and f 1 = f 2 in D 1 ∩ D 2 . In this case the two function elements (f 1 , D 1) and (f 2 , D 2) are said to be equivalent. Definition 2. Let γ [0,1] : → /C be a curve and (f 0 , D 0) be a function element at z 0 = γ(0). Suppose there exists (i) a partition 0 = t 0 > t 1 > ... > t n = 1 of [0, 1] and (ii) a finite sequence of function elements (f 0 , D 0), (f 1 , D 1), ..., (f n , D n) with γ([t j , t j+1]) ⊂ D j and (iii)
f j (z) = f j+1 (z) on D j ∩ D j+1 for j = 0, 1, ... n−1. Then (f n , D n) is called an analytic continuation
of (f 0 , D 0) along γ. Apparently, it seems that the function element (f n , D n) of the above definition, depends on the choice of partition 0 = t 0 > t 1 > ... > t n = 1 of [0, 1] and the finite sequence (f 0 , D 0), (f 1 , D 1), ..., (f n , D n) of function elements. It turns out that up to equivalence, it is actually independent of these choices. Fig. 9 Fig. 10 o −i i

18 Theorem 1.3 Given a curve γ [0,1] : → /C beginning at z 0 and ending at z n and a function element (f 0 , D 0) at z 0 , any two analytic continuations of (f 0 , D 0) along γ give rise to two function elements at z n that are direct analytic continuations of each other. [Though the theorem can be proved by taking different partitions of [0, 1] for two different analytic continuations of (f 0 , D 0) along γ, here we prove the theorem taking the same partition of [0, 1] for two analytic continuations along γ]. Proof. Let (f 0 , F 0), (f 1 , F 1), ... (f n , F n) and (g 0 , G 0), (g 1 , G 1), ..., (g n , G n) be two analytic continuations of (f 0 , D 0) along γ, using the same partition, 0 = t 0 > t 1 > ... > t n = 1 where γ(t j) = z j and γ ([t j , t j+1]) ⊂ F j and γ([t j , t j+1]) ⊂ G j for j = 0, 1, ..., n. By given hypothesis, (f 0 , D 0) = (f 0 , F 0) = (g 0 , G 0). Now we set E j = F j ∩ G j for j = 1, 2, ... n, and E 0 = F 0 = G 0 . Then each E j is a connected open set containing γ(t j) and γ(t j+1). To prove the theorem we show, by induction, that f n = g n on E n . We have f 0 = g 0 on E 0 = F 0 = G 0 by definition. Suppose j > n and f j = g j on E j . But we have f j = f j+1 on f j ∩ F j+1 and g j = g j+1 on G j ∩ G j+1 and γ(t j+1) is common to both the open sets F j ∩F j+1 and G j ∩G j+1 . So it follows that f j+1 = g j+1 in a neighbourhood of γ(t j+1) and hence on E j+1 by the uniqueness theorem. By induction the proof is therefore complete. Homotopic curves. Two arcs γ 1 and γ 2 , with common end points, contained in a region R are said to be homotopic if one can be obtained from the other by continuous deformation where the process of continuous deformation must be confined in R. γ 1 γ 2 γ 3 γ 4 γ 5 R

19 In the given figure {γ 1 , γ 2 and γ 3 } is one set of homotopic curves while {γ 4 , γ 5 } is the other set. Here no curve of the first set is homotopic to any curve of the second set. These are geometrical interpretations. We now explain such a deformation in an analytical manner. Let us suppose γ 0 : z = σ 0 (t), 0 ≤ t ≤ 1 and γ 1 : z = σ 1 (t), 0 ≤ t ≤ 1 be two curves, lying in a region R, having common end points a and b i.e., a = σ 0 (0) = σ 1 (0) and b = σ 0 (1) = σ 1 (1) hold. We say that the curve γ 0 can be continuously deformed into the curve γ 1 keeping the process confined to R, if there exists a function σ(t, s) which is continuous in the unit square I 2 = I × I, I = [0, 1] and satisfies the following conditions : (i) for each fixed s ε [0, 1] the curve γ s : z = σ (t, s), 0 ≤ t ≤ 1 lies in R. (ii) σ (t, 0) = σ 0 (t) and σ (t, 1) ≡ σ 1 (t), 0 ≤ t ≤ 1 (iii) σ (0, s) ≡ a and σ (1, s) ≡ b, 0 ≤ s ≤ 1. Let α and ς be two points lying in a domain D and suppose that γ 0 and γ 1 are two curves connecting α to ς. Let there exist, as in definition 2, two finite sequences of function elements (f 0 , G 0), (f 1 , G 1) ..., (f n , G n) and (g 0 , H 0), (g 1 , H 1), ..., (g m , H m) along the curves γ 0 and γ 1 respectively. We also suppose that the function elements (f 0 , G 0) and (g 0, H 0) at the point α are equivalent. Then a question arises whether the function elements (f n , G n) and (g m , H m) at the point ς are also equivalent? If γ 0 and γ 1 are the same curve the Th. 1.3 confirms the answer for equivalence. However, if γ 0 and γ 1 are distinct there is no definite answer. The reason behind this is the fact that the regions enclosed by the curves γ 0 and γ 1 may contain points at which we can not find any function element that can be included in the sequence of function elements from the point α to ς along any curve passing through these points. Here we discuss a few problems highlighting these facts : Example 1.6 Let Q 1 = {z ε /C | Re z < 0, Im z < 0} denote the first quadrant and set f(z) = log z for all z ε Q 1 Show that, if g 1 is the analytic continuation to /C \(−•, 0] of f and g 2 is the analytic continuation to /C \[0, •) of f, then g 1 ≠ g 2 throughout the third quadrant, Q 3 = {z ε /C | Re z > 0, Imz > 0}. Proof. Clearly, g 1 is the principal branch of logz throughout /C \(−•, 0] ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ ∧ Q −1 o z Γ ∧ ∧ ∧ ∧ ∧ 1 Fig. 10

20 by the uniqueness theorem. That is g z d z 1 1 ( ) [,] = ς ς for all z barring the negative real axis including origin. We define (i) g z d i C z 2 ( ) \[0,] [−1,] = + / • ς ς π ε for all z and show that (ii) g 2 (z) = g 1 (z) + 2πi for all z ε Q 3 . Let γ be the closed curve (see figure) consisting of the line segments [1, z], [z, −1] and a semi-circular path Γ with centre at the origin and radius 1, where z is any point in Q 1 . Now we wish to calculate dς ς γ By Cauchy's Residue Theorem, it is equal to 2πi origin is the only pole inside γ). So breaking up the contour γ, we can equate 2 1 1 π ς ς ς ς ς ς i d d d z z = + + [,] [−] Γ = + g z d i z 1 ( ) − [−1,] ς ς π i.e., g z d i g z z 1 2 ( )− ( ) [−1,] ς ς π + = Hence g 2 (z) = g 1 (z) = log z for all z ε Q 1 , that is, the mapping g 2 defined in (i) is the unique analytic continuation of f to /C \[0, •). To prove (ii) Let z ε Q 3 and γ be the curve joining the line segments [−1, z], [z, +1] and a unit semi-circular path Γ in the upper half plane. Thus 2 1 π ς ς ς ς ς ς ς ς γ i d d d d z z = = + + Γ [−1,] [−] = + π ς ς i d g z z [−1,] − ( ) 1 -1 P z 0 Q Γ Fig. 11 1 ∧

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21 i.e., g 2 (z) = g 1 (z) + 2πi for all z ε Q 3 . Remark : The preceding example presents the following observation : If γ 1 and γ 2 be the two curves joining z 0 and ς, (f 0 , D 0) be a function element at z 0 , then the resulting function elements of (f 0 , D 0) along the curves γ 1 and γ 2 at ς may not be direct analytic continuations of each other. We shall now discuss for what reasons such type of situation occurs. 1.6 Multi-valued Functions and Analytic continuation When we define both real and complex functions we always keep in mind that for each value of the independent variables the value of the function must be unique. For example, even Cauchy's theorem is based on the assumption that a function can be defined uniquely in the region under consideration. All the same, multivaluedness often arises out of necessity in the actual construction of functions, the simplest example is perhaps the logarithm : In section 5.2 [14] we showed that if z is a non zero complex number, then the equation z = e ω has infinitely many solutions. Since the function f(w) = e ω is a many- to-one mapping, its inverse (the logarithm) is multi-valued. Definition 3 : [Multi-valued logarithm] : For z ≠ 0, we define the function log z as the inverse of the exponential function; that is, log z = ω if and only if z = e ω (8) If we go through the same steps as we did to obtain (5.5) [14], we find that, for any complex number z ≠ 0, the solutions ω to equation (8) take the form ω = log z = log |z| + iθ, for z ≠ 0 (9) where θ ε arg z and log |z| denotes the natural logarithm of the positive number |z|. Because arg z is the set arg z = Arg z + 2nπ, where n is an integer, we can express the set of values comprising

log z as log z = log |z| + i (Arg z + 2nπ), where n = integer (10) or log z = log |z| + i arg z for z ≠ 0, (11)

where it is understood that the identity (11) refers to the same set of numbers given in identity (10). We call any one of the values given in identities (10) or (11) a logarithm of z. Notice that the different values of log z all have the same real part and that their imaginary parts differ by the amount 2nπ, where n is an integer. Regarding analytic continuation, we treat log z for complex valued z as the extension of log x from positive real domain to complex domain. Consider the Taylor series expansion of log x :

22 log log{ (–)} (–1) (–) , – x x n x x n n n = + = > > = • ∑ 1 1 1 0 2 1 1 (12) We take this series for complex valued z and write f z n z n n n 0 1 1 1 ( ) (–1) (–) – = = • ∑ (13) which converges in the disc K 0 : |z–1| > 1 so that f 0 (x) = log x for 0 > x > 2. Thus f 0 (z) and log x are direct analytic continuations of each other. Our object is to specify the curves along which the analytic continuation of the function element (f 0 , K 0) is possible. For this purpose it is advantageous to apply the integral representation. log , x ds s x x = > > • 0 1 (14) Lemma 1.1. The following formula f z d z 0 1 ( )= ς ς (15) holds for z ε K 0 where the integral is taken along any path lying completely within K 0 . Proof. The function f 0 (z) given by (13) is regular in K 0 and following Theoren 3.2[14] the integral on the r.h.s of (15) is also regular in K 0 . But we see that this integral coincides with log x in (14) for 0 > x > 2. By the uniqueness theorem. f z n z d n n z n 0 1 1 1 0 1 ( ) (–1) (–) , . – = = ∑ = • ς ς ε z K In continuing f 0 (z) analytically to an arbitrary point ω we isolate a single-valued piece of log z, as we shall do later for other multivalued functions, called a branch of the function. The standard way to isolate single-valued branches is by the use of branch cuts to different branches. For log z the question of multivaluedness arises when z goes around the origin, as a result argument changes by 2π. Such a point is called a branch point. If we do not allow the paths to travel around a branch point of a multi-valued function then certainly we would not face varied values at a point lying in the domain of definition of the function.

23 Let C be any simple curve from 0 to •, so that z cannot go around the origin crossing C. The above consideration shows that if analytic continuation along a given curve Γ is possible, then one can get from a function element at the initial point of the curve another function element at the terminal point of the curve by a finite number of applications of direct analytic continuation. If there is no function element at the initial point of Γ that can be continued along Γ, then there exists a definite point on the curve Γ which is a singular point at which the process of analytic continuation must stop. The following question immediately arises : if ω is some non-singular point outside the disc D 0 , then there may two or more chains of function elements which eventually continue analytically f 0 (z) onto a disc D containing ω. For example, let (f j , D j) be the function element of one chain and (f k , D k) be the function element of a different chain and that ω ε D j ∩ D k ; will then f j (z) = f k (z) ∀ z ε D? The Monodromy Theorem The above question is answered by the Monodromy theorem, which, simply stated, is : if there are no singular points in between the two paths of analytic continuation, then the result of analytic continuation is the same for each path. Another way of stating the theorem is : Theorem 1.4 [Monodromy Theorem] Let (f 0 , D 0) be a function element

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at z 0 and R be a simply connected region containing D 0 , ς be any point lying in R. We suppose (i) (f 0 , D 0) can be analytically continued along every curve in R. (ii) γ 0 and γ 1 are homotopic curves from z 0 to ς. Then the continuations of the function element (f 0 , D 0) along γ 0 and γ 1 at ς are equivalent. Fig. 12 Fig. 13 o ∧ z 1 x C z 1 x log z −2πi y y ∧ ∧ ∧ ∧ ∧

24 Proof. A homotopy from γ 0 to γ 1 determines a continuous one parameter family of curves {γ s }, 0 ≤ s ≤ 1 from z 0 to ς given by the equations z = σ s (t), 0 ≤ t ≤ 1. By hypothesis, the function element (f 0 , D 0) has an analytic continuation along each of the curves, γ s . Denote the terminal function element at ς for the continuation along γ s by φ s . We claim that, for each k ε [0, 1], there is a δ < 0 such that φ s is equivalent to φ k whenever |s−k| > δ. Let 0 = t 0 > t 1 > .... > t n = 1 be a partition and (f 0 , D 0), (f 1 , D 1), ..., (f n , D n) be a finite sequence of function elements defining φ k = (f n , D n) as the terminal function element at ς for the analytic continuation of (f 0 , D 0) along γ k . Then E j = σ k ([t j , t j+1]) ⊂ D j for j = 0, 1, ..., n−1 For each j = 0, 1, ... n−1, let ε j be the minimum distance from the compact set E j to the boundary of the D j . If |σ s (t)−σ k (t)| > ε j , t ε [0, 1], then it will also be true that σ s ([t j , t j+1]) ⊂ D j . Thus, if ε = min {ε 0 , ε 1 , ... ε n−1 } and we choose δ < 0 such that |σ s (t) − σ k (t)| > ε whenever |s−k| > δ, then for each s with |s−k| > δ, the partition 0 = t 0 > t 1 > ... > t n = 1 and sequence of function elements (f 0 , D 0), (f 1 , D 1), ..., (f n , D n) also defines (f n , D n) as the terminal function element at ς for the analytic continuation of (f 0 , D 0) along γ s . Since, by the previous theorem 1.3, any other continuation of (f 0 , D 0) along γ s results function element equivalent to this one, we conclude that φ k is equivalent to φ s . This proves that φ s is equivalent to φ k whenever |s−k| > δ. This means that for every s ε I = [0, 1] there is a positive δ(s) such that if s lies in the interval I s = (s−δ(s), s + δ(s)), then the analytic continuation of f 0 (z) along all such curves γ s , result equivalent function elements at the point ς. Now by the Heine-Borel theorem, we can always choose a finite number of intervals I s j , 0 = s 0 > s 1 > .... > s n = 1 that cover the segment I and are such that the intervals I s j and Fig. 14 E 1 σ s (t j) z 0 γ k γ s σ s (t j+1) σ k (t j) σ k (t j+1) ς 25 I s j+1 , 0 ≤ j ≤ n−1 have a non-empty intersection. Then, if s ε I s 0 ∩ I s 1 , the analytic continuation of f 0 (z) result equivalent function elements at the point ς. The same is true for s ε I s 1 ∩ I s 2 and so on. Continuing in this way we observe that the analytic continuation of the function element (f 0 , D 0) along all the curves γ s , 0 ≤ s ≤ 1 produce equivalent function elements at the point ς. This completes the proof of the theorem. The above theorem leads us to the following most important corollary. Corollary. Let R be a simply connected region and (i) (f 0 , D 0) be a function element at z 0 belonging to R (ii) (f 0 , D 0) admit analytic continuation along every curve in R. Then there is a function F which is analytic on R and coincides with f 0 on D 0 . Proof. Let z 1 be a point in R. Then, since R is simply connected any two curves from z 0 , to z 1 are homotopic in R. The Monodromy theorem implies that any two terminal function elements of analytic continuations of (f 0 , D 0) along curves from z 0 to z 1 in R will be equivalent and hence, will determine a function F 1 analytic in some neighbourhood of z 1 , say Q 1 . Clearly, F 1 (z) = f 0 (z) on D 0 , F 1 (z) = f 1 (z) on D 1 , ..., etc for the continuation along the curve γ 1 from z 0 to z 1 . Again let z 2 be a point in R, and γ 2 be a curve in R joining z 0 to z 2 and let (g n , E n) be the function element at z 2 continuing along the curve γ 2 with f 0 = g 0 on D 0 = E 0 . We simply join z 2 to z 1 by a curve γ and claim that continuation of (F 1 , Q 1), along the curve γ to z 2 , will be equivalent to (g n , E n) (since the curves γ 1 ∪γ and γ 2 are homotopic), which gives rise to the fact that there is a function F 2 analytic in some neighbourhood of z 2 , say Q 2 , which coincides with F 1 On Q 1 . Clearly, F 2 (z) possesses larger domain of analyticity than F 1 (z). Proceeding in this way finite number of times we can achieve a function F analytic throughout the region R.

26 Unit 2 Harmonic Functions Structure 2.0 Objectives 2.1 Harmonic Function 2.2 Gauss' Mean Value Theorem for harmonic 2.3

Inverse point of a given point with respect to a

circle 2.4 The Dirichlet Problem 2.5 Subharmonic & Superharmonic Functions 2.0 Objectives In this chapter we shall mainly study harmonic functions and their basic properties. Gauss' mean value theorem, Poisson's integral formula, Dirichlet's problem for a disc and Harnack inequality for harmonic functions will be discussed. Subharmonic and super harmonic functions will be explained through examples. 2.1 Harmonic Function A function u(x, y) of two real variables x and y defined in an open set D is said to be harmonic in D if it has continuous derivatives of the second order and satisfies the equation ∂ ∂ ∂ ∂ 2 2 2 2 0 u x u y + = (16) known as Laplace's equation. The differential operator ∂ ∂ ∂ ∂ 2 2 2 2 x y + is called the Laplacian and is denoted by ∇ 2 . We introduce the differential operators ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ z x i y z x i y = − = + 1 2 1 2 and (17)

27 in order to achieve a condition equivalent to (16) for

f(
z). If
we write x
z
z i z z = + = − 1 2 1 2 ( ) ( ) and y (18) then ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ f z
f x x
z f
y y z f x i f y = · + · = + 1 2 1 2
(19a–
b) ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ f z f x
x z f y y z f x i f y = · + · = − 1 2 1 2 ∂ 2 2 2 2 2 2 2 1 2 1 2 f z z f
x x z x y y z i f
x
y
x
z f y y z = · + ·
− · + ·
= + − + = + 1 4 1 4 1 4 1 4 1 4
f
i f i f f f f xx xy xy yy xx yy and consequently the condition equivalent to (16) is ∇ = 2 2 4 f f z z ∂ ∂ ∂ (20) A function f(z) is said to be harmonic in D if f has continuous second derivatives in D and satisfies ∇ = ∀ 2 0 f z , ε D (21) Result 1 : If f = u + iv is analytic in a domain D, then ∂ ∂ ε f z z = ∀ 0, D . Proof : u and v satisfy the Cauchy-Riemann equations and using (19b) we have, ∂ ∂ f z u iv i u iv x x y y = + − + 1 2 1 2 ( ) ( ) = + − − + 1 2 1 2 ( ) ( ), u iv i v iu x x x x using C–R equations = 0
Result 2 :
The
real and imaginary parts of an analytic function are
harmonic. Proof :
Let f = u + iv be analytic in
a domain D.
By
Cauchy-Riemann equations u x = v y and u y = − v x
i.e. u
xx = v
xy and u yy = −v
xy [since v xy = v yx , partial derivatives being continuous] and on addition it proves that u is harmonic in D. Likewise v is also harmonic in D. Harmonic conjugates :
Let u (x, y) and v(x, y) be two harmonic functions in a domain D C ⊆ / .
28 If they
satisfy
the Cauchy-Riemann equations : ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂
u x v y u y v x = = − , ,
in D, then we say that v is a harmonic conjugate of u.
It follows
that
f(z) = u(x, y) + i v (x, y) is analytic in
a domain
D if and only if
v(
x, y)
is a harmonic conjugate of u(x, y)

in D. Remark : We know that the real part as well as the imaginary part of an analytic function are harmonic. Now the questions arise : 1. Can any real harmonic function be the real part of an analytic function? 2. Whether every real harmonic function has a harmonic conjugate? Existence of Harmonic conjugates Theorem 2.1 Let u(x, y) be a real-valued harmonic function in a simply connected domain D C ⊆ / . Then there is an analytic function f in D such that u = Re f (or, equivalently there is a function v, a harmonic conjugate of u) which is unique to within addition of an arbitrary real constant. Proof. Since the function u(
x, y) is harmonic in a simply connected domain D,
we have ∂ ∂ ∂ ∂ 2 2 2 2 0 u x u y + = which can be rewritten as ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ y u y x u x − = , where − ∂ ∂ ∂ ∂ u y u x and are given functions with continuous first partial derivatives. This implies that − + ∂ ∂ ∂ ∂ u y dx u x dy is exact. So there is a single-valued function v(x, y) which is unique to within an additive arbitrary constant, i.e. v x y u y dx u x dy K x y x y (,) (,) (,) = − + + ∂ ∂ ∂ ∂ 0 0 (22) K ≡ real constant, where (x 0 , y 0) is an initial point and (x, y) is any variable point lying in D and the integral on the curve connecting (x 0 , y 0) to (x, y) is path independent. From (22) we find that ∂ ∂ ∂ ∂ ∂ ∂ ∂ ∂ v x u y v y u x = − = − , ,
29 which in turn ensures that v(x, y) is harmonic in D and harmonic conjugate to u(x, y) i.e. f = u + iv forms an analytic function in D. Observation : If D is multiply connected then the integral in (22) may take different values for different paths connecting (x 0 , y 0), to (x, y) giving v(x, y) as a multi-valued function, unless the paths are restricted to a simply connected sub domain contained in D. Example 1. Let D be the whole plane cut along the negative real axis including the origin (y = 0, x ≤ 0).
Show
that u(x, y) = sin x cosh y is harmonic in D, and find its harmonic conjugate.
Also find the corresponding analytic function. Solution : Here u(x,
y) possesses continuous second order partial derivatives in D and also satisfies the Laplace equation : u xx + u yy = 0. Hence u(x, y) is harmonic in D. Let v(x, y) be its harmonic conjugate. Then according to the formula (22), we have v x y u y dx u x dy K K x y (,) , (1,) (,) = − + + ≡ ∂ ∂ ∂ ∂ 0 real constant, where M(1, 0) is the initial point. Here,
u(
x, y) = sin x cosh y u x = cos x cosh y u y = sin x sinh y
Now let the point Q(x, y) lie in the 1st quadrant of the right-half plane. Then integrating along MNQ, we find that v x y u y dx u x dy K MN NQ (,)= − + − + ∂ ∂ ∂ ∂ 1 = − + + sin sinh cos cosh x odx x K x y 1 1 0 y dy = cos x sinh y + K 1 Again, if the point (x, y) lies in the 2nd quadrant of the left-half plane, then we obtain
v x y u x dy u y dx K MN N Q (,)= + − + ∂ ∂ ∂ ∂ 1 1 2 = 0
y
cos 1 cosh
y dy + 1 x − sin x sinh y dx + K 2 = cos 1 sinh y + cos x sinh y − cos 1 sinh y + K 2 = cos x
sinh y +
K 2 The
expression for v(x, y) in both the cases turns out to be the same apart from an additive constant. It results from the fact that the two paths in determining the Fig. 15 N O M(1, 0) Q(x, y) Q(x, y) N 1
30 integral lie in a simply connected domain. Thus, v(x, y) = cos x sinh y + K at all points of D. Therefore, an analytic function with the given real part will be of the form
f(z) = sin x cosh y + i cos x sinh y +
iK, K ≡ real constant = sin(x + iy) + iK = sin z + iK As for uniqueness, if two analytic functions in D have the same real part, then their difference has derivative zero, by the Cauchy-Riemann equations. In that case the functions differ by a constant. 2.2 Gauss' Mean Value Theorem for harmonic functions Let u(z) = u(x, y), z = x + iy, be harmonic in the disk K : |z − z 0 | > R and continuous on the closed disk K . Then u z u z i ( ) (Re) 0 0 0 2 1 2 = + π θ π θ d (23) Proof. Let f(z) be an analytic function defined in K such that Re f (z) = u(z). It follows from Cauchy's integral formula that f z i f z z z dz r R z z r ( ) ( ) , | | 0 0 1 2 0 0 = − > > − = π using the parametric form of the circle |z − z 0 | = r. z z re i = + ≤ ≤ 0 0 2 θ θ π , , so that dz ire d i = θ θ. The integral then gives f z f z re d r R i ( ) ( ) , 0 0 0 2 1 2 0 = + > > π θ θ π Equating the real parts, we obtain u z u z re d i ( ) ( ) 0 0 0 2 1 2 = + π θ θ π whence taking the limit r → R, we obtain the desired result (23) 2.3
Inverse point of a given point with respect to a

circle Let γ : |z − α| = R and z 0 be a given point. Let z 1 be another point on the radius through z 0 such that |z 0 − α| |z 1 − α| = R 2 . Then either of the points z 0 and z 1 is called the inverse point of the other with respect to γ. The centre of the circle γ is called the centre of inversion. It follows from the definition that (i) if z 0 lies inside γ, then z 1 must lie outside
31 γ, (ii) if z 0 lies on γ, then z 1 must also lie on γ and it coincides with z 0 , (iii) if z 0 lies outside γ, then z 1 must lie inside γ. Every point, except the centre of the circle, on the plane has a unique inverse point with respect to the circle. We associate the point at infinity to the inverse point of the centre. Result : Let γ : |z| = R and z 0 be a given point. Then the inverse point of z 0 with respect to γ is given by R z 2 0 . Proof : Let z 0 = re iθ . Then its inverse point with respect to γ is given by z 1 = r 1 e iθ , where rr 1 = R 2 . Hence r 1 = R r 2 and so z R r e R re R z i i 1 2 2 2 0 = · = = − θ θ Poisson's integral formula : Theorem : Let u(x, y) be a harmonic function in a simply connected region D and γ : |ς| = R be a circle contained in D. Then for any z = re iθ , r > R, u can be written as u(r, θ) = − + − − 1 2 2 2 2 2 2 2 0 2 π φ φ φ θ π ( ). (,) cos( ) R r u R d R r Rr , where Re iφ is a point on γ. Proof : Since u(x, y) is harmonic in D, there exists a conjugate harmonic function
v(x, y) in D so
that
f(z) = u(x, y) + iv(x, y) is analytic
in
D.
Then
f(z) is analytic
within and on γ and so for any z within γ, by Cauchy's integral formula, f z i f z ( ) ( ) = − 1 2π γ d ς ς ς (24) The inverse point of z with respect to γ lies outside γ and is given by R z 2 . Hence by Cauchy-Goursat theorem, 0 1 2 2 = − π γ i f R z d ( )ς ς ς (25) Subtracting (25) from (24) we get, f z i z R z d ( ) ( ) = − − −
1 2 1 1 2 π γ ς ς ς ς f
32 = − − − 1 2 2 2 π γ i z R z f d z R z ( ) ( ) ς ς ς ς (26) Let ς = Re iφ . Also, z re i = − θ . Then (26) becomes f re i re R r e f i d re R r e i i i i i i i i i ( ) (Re) Re (Re) Re θ θ φ θ θ θ θ φ θ π π φ = − − − 1 2 2 2 0 2 = − − − + 1 2 2 2 0 2 π φ φ θ φ φ θ φ θ π ( ) (Re) (Re)(Re) ( ) r R e f d re re i i i i i i = − − − − − 1 2 2 2 0 2 π φ φ φ θ φ θ π ( ) (Re) (Re)(Re) R r f d re re i i i i i = − + − − 1 2 2 2 2 2 2 2 0 2 π φ φ θ φ π ( ) (Re) cos( ) R r f d R r Rr i (27) Let f(re iθ) = u(r, θ) + iv(r, θ). Then (27) becomes u r iv r R r u R iv R R r Rr (,) (,) ( ) (,) (,) cos( ) θ θ π φ φ φ θ φ π + = − + + − − 1 2 2 2 2 2 2 2 0 2 d (28)
Equating real parts in (28) we get, u r R r u R R r Rr (,) ( ) (,) cos( )
θ π φ φ θ φ π = − + − − 1 2 2 2 2 2 2 2 0 2 d (29) Formula (29) is known as Poisson's integral formula. Note : Let R r R r Rr P R r 2 2 2 2 2 − + − − = − cos( ) (, ,) φ θ φ θ. Then, the function P(R, r, φ − θ) is called the Poisson Kernel.
Hence we can write (29) in the form u r R r u R d (,) , ,) (,) θ π φ φ φ φ π = − 1 2 0 2 P((30)
33 We can also get a formula similar to (29) for the imaginary part of f(z) by equating the imaginary part in (28). The corresponding formula is v r R r v R d R r Rr P R r v R d (,) ( ) (,) cos( ) (, ,) (,) θ π φ φ φ θ π φ θ φ φ π π = − + − − = − 1 2 2 1 2 2 2 2 2 0 2 0 2 (31) Remark :
Cauchy's integral formula expresses the values of an analytic function inside a circle
in terms of its values on the
boundary of
the
circle whereas Poisson's
integral formula expresses the values of a harmonic function inside a circle
in terms of its values on the
boundary of the
circle. Result 3. 1 2 0 2 π φ θ φ π P R r d (, ,) − = 1. Proof : By Poisson's integral formula we have, u r P R r u R d (,) (, ,) (,) θ π φ θ φ φ π = − 1 2 0 2 Taking u(r, θ) ≡ 1 we get, 1 2 1 0 2 π φ θ φ π P R r d (, ,) − = Result 4. P R r z z (, ,) Re φ θ ς ς − = + − Proof : Let ς = Re iφ , z = re iθ , r > R. Then, ς ς + − = + − = + + + − + − z z re re R r i R r R r i R r i i i i Re Re (cos cos) (sin sin) (cos cos) (sin sin) φ θ θ φ θ φ θ φ θ φ θ = + + + − − − − + − {(
cos cos) (sin sin)}{(cos cos) (sin sin)} (cos cos) (sin
sin)
R r i R r R r i R r R r R

r φ θ
φ θ φ θ φ θ φ θ φ θ 2 2 Simplifying we get, Re cos( ) (, ,). ς ς + − = − + − − = − z z R r
R r Rr P R r 2 2 2 2 2 φ θ φ θ Result 5. Poisson Kernel P(R, r, φ − θ)
is harmonic in |z| > R. Proof : Let f z z z ( ) . = + − ς ς Then f(z) is analytic in |z| > R. By result 4, P(R, r φ− θ) = Re f(z). Hence the Poisson Kernel is the real part of an analytic function. Hence P(R, r, φ−θ) is harmonic in |z| > R. Note : We can easily show that R r R r Rr R z z i 2 2 2 2 2 2 2 2 − + − − = − − cos( ) Re φ θ φ
34 where z = re iθ , r > R. Hence Re Re ς ς φ + − = − − z z R z z i 2 2 2 and Poisson's integral formula (29) can be written as u r R z z u R d i (,) Re (,) θ π φ φ φ π = − − 1 2 2 2 2 0 2 (32) The function R z z i 2 2 2 − − Re φ is the Poisson Kernel. Theorem 2.2 Let u(x, y) ≠ constant be harmonic on a simply connected domain D.
Then u(x, y) has neither a maximum nor a minimum at any point of D. Proof. Let z 0 = x 0 + iy 0 be an arbitrary point of D. Then following theorem 2.1 there is an analytic function f(z) in a neighbourhood N(z 0) of z 0 such that Re f = u. Then g(z) = e f(z) is analytic on N(z 0) and not equal to constant since u(x, y) ≠ constant and |g(z)| = e u(x,y) Again exponential function is strictly increasing, so a maximum for u at (x 0 , y 0) is also a maximum for e u , and hence also a maximum of |e f | i.e. of |g(z)| at z 0 .
The function u(x, y) cannot have a maximum at (x 0 , y 0), since otherwise |g(z)| would have a maximum at
z 0 , thereby contradicting the maximum modulus principle. Likewise, following the minimum modulus principle |g(z)| cannot have a minimum value at z 0 since |g(z)| ≠ 0 on D. Therefore u(x, y) cannot possess minimum value
at (x 0 , y 0). Corollary. Let u(x, y) be harmonic on a domain D and continuous on D . Then u(x, y) attains its maximum and
its minimum on the boundary of D. Proof. Since u(x, y) is continuous on the compact set D, it attains both its maximum and its minimum on D , but u(x, y) cannot possess a maximum or a minimum at a point of D. Therefore the corollary follows. Example 2. Given u(x, y) harmonic in the disk |z| > R and A(r j) its maximum value on the circle |z| = r j , r j > R, j = 1, 2, 3. Prove that A r r r r r A r r r r r A r ( ) log log log log ( ) log log log log ( ) 2 2 1 3 1 3 3 2 3 1 1 ≤ − − + − − for 0 > r 1 > r 2 > r 3 > R. Solution. Since u(x, y) is harmonic in |z| > R, u(x, y) + α log r, r x y = + 2 2 , α ≡ a real constant to be fixed later, is also harmonic in the annulus r z r 1 3 ≤ ≤. Hence its
35 maximum is attained on the boundary of the annulus i.e. on |z| = r 1 or, |z| = r 3 or, on both. Either A(r 1) + α log r 1 or, A(r 3) + α log r 3 is maximum. We define α so that A(r 1) + α log r 1 = A(r 3) + α log r 3 or, α = − − A r A r r r ( ) ( ) log log 1 3 3 1 The circle |z| = r 2 lies inside the annulus r 1 ≤ |z| ≤ r 3 and according to corollary of the theorem 2.2 regarding maximum value of the harmonic function u(x, y) + α log r we have A(r 2) + α log r 2 ≤ A(r 3) + α log r 3 or, A(r 2) ≤ A(r 3) + α(log r 3 − log r 2) = + − − − A r A r A r r r r r ( ) ( ) ( ) log log (log log) 3 1 3 3 1 3 2 = − − + − − log log log log ( ) log log log log ( ) r r r r A r r r r r
A r 2 1 3 1 3 3 2 3 1 1 2.4 The Dirichlet Problem Let D be a domain with boundary Γ and let (x, y) be a continuous real function defined on Γ. The Dirichlet problem is to find a function u(x, y), harmonic on D and continuous on D, which coincides with (x, y) at every point of Γ. Existence of a solution of Dirichlet's problem for a disc Theorem 2.3 Let D be the disc |z| > R with boundary Γ : |z| = R and let U(φ) be a continuous real function on the interval [0, 2π] such that U(0) = U(2π). Then the function u(r, θ) defined by the integral u r R r U R r Rr d (,) ( ) ( ) cos( ) θ π φ φ θ φ π = − + − − 1 2 2 2 2 2 2 0 2 (33) for any point (r, θ) on D any by u(R, φ) = U(φ) (34) for any point (R, φ) on Γ, solves the Dirichlet problem for the disc D. In otherwords, (i) u is harmonic on D and continuous on D and (ii) lim (,) ( ), Re re i i u r U θ φ θ φ → = 0 0 where Re iφ 0 is any fixed point on Γ. Proof : To prove that u(r, θ) defined by (33) on D is harmonic on D we observe that
36 R r R r Rr P R r 2 2 2 2 2 − + − − = − cos( ) (, ,) φ θ φ θ = + − Re , ς ς z z
where P (R, r, φ−θ) is the Poisson Kernel
and ς = Re iφ , z = re iθ , r > R. The r.h.s. is the real part of the function ς ς + − z z which is analytic in D. Hence the Poisson Kernel P(R, r, φ−θ) is harmonic in D. So, differentiation under the sign of integration is valid. Applying the Laplacian ∇ 2 in (r, θ) to both sides of (33) we get, ∇ = ∇ − = 2 0 2 2 1 2 0 u
P(R r d π φ φ θ φ π U( ). , ,) [Since P(R, r, φ−θ) is harmonic in D ⇒ ∇ 2 P(R, r, φ − θ) = 0]. ⇒

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u is harmonic on D. Next we prove that the function u(r, θ) defined by the integral (33) approaches U(φ 0) as the point (r, θ) in D tends to any fixed point (R, φ 0) on Γ. Let (r n , θ n) be an arbitrary sequence of points in D converging to the boundary point (R, φ 0). We now consider the difference u r U R r U d U n n n n (,) ( ) , ,) ( ) ( ) θ φ π φ θ φ φ φ π − = − − 0 0 2 0 1 2 P(= − − 1 2 0 2 0 π φ φ φ θ φ π { ( ) ( )} , ,) U U P(R r d n n (35) (Since 1 2 1 0 2 π φ θ φ π P R r d n n (, ,)) − = Since U(φ) is continuous on Γ, for given ∈ < 0 there exists a δ(∈) < 0 such that U U ( ) ( ) φ φ − > ∈ 0 2 (36) whenever φ φ δ − > 0 2 (37) we choose δ so small that (36) is satisfied and φ 0 − 2δ < 0, φ 0 + 2δ > 2π. We break the integral on r.h.s. of (35) as u r U P(R r U U d n n n n (,) ( ) , ,){ ( ) ( )} θ φ π φ θ φ φ φ φ δ − ≤ − − − 0 0 2 0 1 2 0 + + = + + − + + 1 2 1 2 0 0 0 2 2 2 2 1 2 3 π π φ δ φ δ φ δ π | | | | | | I I I (38)
37 Now, | | | (, ,)| ( ) ( )| I P R r U U d n n 2 2 2 0 1 2 0 0 ≤ − − − + π φ θ φ φ φ φ δ φ δ > ∈ · − = 2 1 2 2 0 2 π φ θ φ ε π | , ,)| P(R r d n n (39) To estimate the other two terms we choose n so large that |φ 0 − θ n | > δ. Then, |φ − θ n | = |φ − φ 0 + φ 0 − θ n | ≥ | φ − φ 0 | − |φ 0 − θ n) < 2δ − δ = δ since |φ − φ 0 | < 2δ whenever φ belongs to either of the intervals [0, φ 0 − 2δ] or, [φ 0 + 2δ, 2π]. Then, | | | | . . cos I I M R r R r Rr d d n n n 1 3 2 2 2 2 2 2 0 2 2 1 2 2 0 0 + ≤ − + − + + − π δ φ φ φ δ π φ δ > − + − → → 2 2 0 2 2 2 2 M R r R r Rr R n n n n cos , δ as r where M Max U U n = − − > φε π φ ϕ φ θ δ [,] | ( ) ( )| cos( ) cos . 0 2 0 and Thus, for sufficiently large n, | | | | I I 1 3 2 + > ε (40) Using (39) and (40) in (38) we get, | (,) ( )| u r U n n θ φ ε − > 0 for sufficiently large n; i.e. lim (,) ( ) n n n u r U →• = θ φ 0 (41) where (r n , θ n) is an arbitrary sequence of points in D approaching (R, φ 0). Equation (41) still holds if some or all the points (r n , θ n) lie on Γ since in that case we can directly use the fact that U(φ) is continuous on Γ. This implies u(r, θ) is continuous on D. This completes the proof. Uniqueness of the solution to the Dirichlet problem for a disc. Let u 1 and u 2 be two solutions of the Dirichlet problem. Then their difference u 1 − u 2 = h is harmonic in D and continuous in the closed disk and takes the value zero on the boundary. Hence h attains its upper bounds at some points of the closed disk. If l < 0, the upper bound will occur in the open disk, since on the boundary Γ h is zero. This contradicts the conclusions of theorem 2.2. So then l = 0. In the same way we can show that the lower bound of h on D is zero. Thus there is no alternative but h to be zero on D.
38 Theorem 2.4 Any continuous function u(z) possessing the mean-value property in a domain D is harmonic in D. Proof. Let K be a closed disk contained in D. By hypothesis of the theorem u satisfies the mean value property in K. We shall prove that u is harmonic in K. By the theorem 2.3 on the Dirichlet problem for a disk there exists a continuous function ~ ( ) u z in K, which is harmonic in the interior of K and coincides with u(z) on the boundary of K. The difference u u − ~ is continuous and satisfies the mean-value property in K. By the corollary to the theorem 3.7 [(14) page-58] u u − ~ satisfies the maximum modulus prnciple in K. Now as u u − ~ is zero on the boundary of K, it will be identically zero in K. Therefore u coincides with the harmonic function ~ u in the interior of K and since K is arbitrary, u is harmomic in the domain D. The Harnack Inequality : Let u be a non-negative Harmonic function on a closed disk D (0, R). Then, for any point z ε D(0, R) R z R z u u z R z R z u − + ≤ ≤ + − ( ) ( ) ( ) 0 0 (42) where D(0, R) denotes a disk with centre 0 and radius R. Proof. From the Poisson's integral formula for u on D (0, R) : u z u R z z d i i ( ) (Re) Re = − − 1 2 0 2 2 2 2 π φ φ π φ Now,
R z z R z R z R z R z
i 2 2 2 2 2 2 − − ≤ − − = + − Re φ Combining these two, we see that u z R z R z u d R z R z u i ( ) (Re) ( ), ≤ + − = + − 1 2 0 0 2 π φ φ π where we make use of the mean value theorem. Similarly, the other inequality in (42) will follow from
R z z R z R z R z R z
i 2 2 2 2 2 2 − − ≥ − + = + Re − φ Corollary Let u be a non-negative harmonic function on a closed disk D R (,) ς . Then for any z ε D (ς, R), R z R z u u z R z R z u − + ≤ ≤ + − − − ( ) ( ) − − ( ) ς ς ς ς ς ς (43)
39 2.5 Subharmonic & Superharmonic Functions Definition : A real-valued continuous function u(x, y) in an open set D of the complex plane C/ is said to be (i) subharmonic if, for any ς ε D u u re d i ( ) ( ) ς ς ≤ + 1 2 0 2 π θ θ π hold for sufficiently small r < 0. (ii) superharmonic if, for any a ε D u a u a re d i ( ) ( ) ≥ + 1 2 0 2 π θ θ π hold for sufficiently small r < 0. From the definition it follows that every harmonic function is subharmonic as well as superharmonic. Example 3. If f(z) is analytic on a domain D, then |f(z)|

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is subharmonic but not harmonic in D unless f(z) ≡ constant. Solution : Using the Cauchy's integral formula f a f a re d i ( ) ( ) ≤ + 1 2 0 2 π θ θ π (44) for every a ε D and r (< 0) is small enough. Here equality holds only if f(z) ≡ constant. We now show that the integral I r f a re d i ( ) ( ) = + 1 2 0 2 π θ θ π is a strictly increasing function of r, if f(z) ≠ constant. Let 0 > r 1 > r 2 > k(a) and g(θ) be continuous on [0, 2π] and F(z) be defined by (i) g f a r e f a r e i i ( ) ( ) ( ), θ θ π θ θ + = + ≤ ≤ 1 1 0 2 (ii) F z f a ze g d z r i ( ) ( ) ( ) , = + ≤ 1 2 0 2 2 π θ θ θ π (iii) k(a) ≡ minmum distance between a and the boundary of D. F(z) is regular for |z| ≤ r 2 and attains its maximum of the boundary of the disc, say at z = r 2 e iφ . Then I r f a r e d i ( ) ( ) 1 1 0 2 1 2 = + π θ θ π = + 1 2 1 0 2 π θ θ θ π f a r e g(d i ( ))
40 = F(r 1) > F r e i ( ) 2 θ ≤ + + 1 2 2 0 2 π θ θ φ π f a r e d i ( ) ( ) = + + 1 2 2 2 π ψ ψ φ π φ f a r e d i ( ) , taking φ + θ = ψ = − + + + 1 2 2 2 2 0 0 2 π ψ ψ π π φ φ π f a r e d i ( ) = + 1 2 2 0 2 π ψ ψ π f a r e d i ( ) , (substituting ψ = 2π + θ in the third integral, we find that it cancels the second term) = I (r 2). Hence equality in (44) is possible if and only if f(z) ≡ constant. Therefore |f(z)| is subharmonic but not harmonic
in D unless f(z) ≡ constant. Example 4.
If f(z) ≠ 0 is analytic in a domain D, then log |f(z)|
is subharmonic in
D. Solution : Let Φ(z) = log|f(z)|. Here at the zeros of f(z), Φ(z) has poles and takes the value − • there. In every closed disk contained in D there are at most a finite number of points where log f(z) = − •. Now let a ε D be any point at which f(z) is distinct from zero. Since f(z) is analytic and not identically zero, there exists a small neighbourhood of a where f(z) is distinct from zero. We find that log f(z) = log |f(z)| + i arg f(z) is analytic in this neighbourhood and hence log |f(z)| is harmonic there and we have the equality Φ Φ ( ) ( ) a a re d i = + 1 2 0 2 π θ θ π (45) for all sufficiently small values of r. On the otherhand, if a is a zero of f(z), we have Φ Φ ( ) ( ) a a re d i = − • > + 1 2 0 2 π θ θ π (46) Combining (45) with (46) we obtain Φ(z) is subharmonic in D.
41 Unit 3 Conformal Mappings Structure 3.0 Objectives of this Chapter 3.1 Conformal Mappings 3.2 Basic Properties of Conformal Mapping 3.0 Objectives of this Chapter This chapter deals with conformal mappings and their basic properties. Many examples are given to explain different concepts on conformal mappings. The inverse function theorem is also discussed. 3.1 Conformal Mappings Let X be an open set in /C and suppose a function f : X → /C is given. We know from functional analysis that if f is
continuous, a compact set of X is mapped onto a compact set in f(X) and a connected set of X onto a connected set of f(X). If moreover, f is single-valued and analytic there occur several interesting results. In this chapter we study mappings which transform different curves and regions from one complex plane to other complex plane with reference to magnitude and orientation. Such type of mappings play an important role in the study of various physical problems defined on domains and curves of arbitrary shape. Level Curves Let w = f(z) with z = x + iy and w = u + iv where
f(z) is analytic. u =
u (
x, y) v = v(x, y)
satisfy Cauchy-Riemann equations u x = v y , u y = − v x
from which it follows that u
xx + u yy = 0 v xx = v yy = 0 Also, ∇
u . ∇ v = 0, where Fig. 16 u(
x, y) = constant v(
x, y) = constant
42 ∇ = ∂ ∂ ∂ ∂x y , So that the
level curves u (x, y) = constant and v (x, y) =
constant are orthogonal.
Now
f 1 (z) = u x + iv x =
u x − iu y = v y + iv x so that f z u u v v x y x
y 1 2 2 2 2 2 ( ) . = + = +

Two basic results : No. 1 Suppose that w = f(z) maps D into D 1 . Let ψ(u, v) = ψ ((u (x, y), v (x, y)) = φ (x, y). To prove φ xx + φ yy = | f 1 (z) | 2 (ψ uu + ψ vv) we calculate φ x = ψ u u x + ψ v v x φ ψ ψ ψ ψ ψ xx uu x vv x uv x x u xx v xx u v u v u v = + + + + 2 2 2 φ ψ ψ ψ ψ ψ yy uu y vv y uv y y u yy v yy u v u v u v = + + + + 2 2 2 Thus, φ φ ψ ψ ψ xx yy x y uu x y vv uv u v u u v v + = + + + + ∇ ∇ ( ) ( ) . , 2 2 2 2 2 since u, v satisfy Laplace equation. Again, ∇ u . ∇ v = 0, so we obtain φ xx + φ yy = |f 1 (z)| 2 (ψ uu + ψ vv) Therefore if f 1 (z) ≠ 0 inside D we have φ xx + φ yy = 0 imples ψ uu + ψ vv = 0 and vice-versa. Fig. 17 Fig. 18 w = f(z) φ (x, y) D ψ (u, v) D 1 xy plane uv plane w = f(z) D D 1 φ xx + φ yy = 0 in D ψ uu + ψ vv = 0 in D 1 z = x + iy w = u + iv

43 No. 2. Consider a level curve F(x, y) = 0 upon ∇φ . n = 0. Let under the analytic mapping w = f(z) the level curve map to G(u, v) = 0. We shall show that ∇ψ.n = 0 on G(u, v) = 0 Consider the map w = f(z) → ω = u + iv, so u = u(x, y), v = v(x, y). Suppose f(z) is analytic. Then,

φ ψ ψ φ ψ ψ φ φ ψ ψ

x

u x v x y u y v y x y u v x x y

v

u v

u

v

so S with S u v u v = + = + = = ,

Then, ∇φ = S∇ ψ , ∇F = S∇G and clearly, S T S = |f 1 (z)| 2 1 Now, ∂φ ∂n F F

S S G S G S S G S G S

G Gf z G G T T T T T = ∇φ· ∇ ∇ = ∇ψ ∇ ∇ = ∇ψ ∇ ∇ ∇ = ∇ψ ∇ ∇ ∇ .( ) ( ) ( ) ( ) ( ) ( ) ( ) / /

1 2 1 1 2 (where the usual vector operations, a.b = a T b and (a.a) 1/2 = (a T a) 1/2 = |a| have been used) So, ∂φ ∂ ∂ψ ∂ n F F f z G G f z n = ∇φ· ∇ ∇ = ∇ψ ∇ ∇ = 1 1 ( ) ( ) This shows that if ∂φ ∂n = 0 on the boundary of D then ∂ψ ∂n = 0 on the boundary of D 1 , provided |f 1 (z)| ≠ 0 on the boundary of D. Note : These give us a means of transforming the domain over which the Laplace's equation is to be solved comfortably. Such type of things is usually dealt in solving boundary value problems in potential theory. Angle of Rotation Given a function of a complex variable w = f(z) analytic in a domain D. Let z 0 be any point lying within D, γ : z = σ(t), a ≤ t ≤ b, σ(t 0) = z 0 , be a curve passing Fig. 19 D D 1 Fig. 20 F(x, y) = 0 G(u, v) = 0 n n

44 through z 0 (and lying within D). The function σ(t) has a non zero derivative σ 1 (t 0) at the point z 0 and the curve γ has a tangent at this point with a slope equal to Arg σ 1 (t 0). Under the mapping w = f(z) the curve γ is transformed into a curve Γ : w = f(σ(t)) = μ(t), a ≤ t ≤ b, μ(t 0) = f(z 0) = w 0 in the w-plane. μ(t) is differentiable at t = t 0 and the curve Γ has a tangent at w 0 = f(z 0). Then following the chain rule for differentiation of composite functions, assuming f 1 (z 0) ≠ 0 μ 1 (t 0) = f 1 (σ(t 0) σ 1 (t 0) It follows that Arg μ 1 (t 0) = Arg f 1 (z 0) + Arg σ 1 (t 0) i.e., Arg μ 1 (t 0) = Arg σ 1 (t 0) + Arg f 1 (z 0) (47) This implies that change in slope of a curve at a point under a transformation depends only on the point and not on the particular curve through that point. Example 1. Verify the result given in equation (47) for the curve y = x 2 under the transformation f(z) = z 2 at z = 1 + i. Solution. First we calculate the change in slope of the curve y = x 2 at the given point under the transformation w ≡ f(z) = z 2 . Following the formula given in eq. (47) Arg f 1 (1 + i) = Arg 2(1 + i) = tan −1 1 A parametric form of the given curve y = x 2 is given by γ : z = t + it 2 , −• > t > •. Here z 0 = 1 + i at t 0 = 1 and z 1 (1) = 1 + 2i, so that slope of the curve γ is tan −1 2. Now we find slope of the transformed curve. w = f(z) ⇒ u + iv = (x + iy) 2 So, u = x 2 − y 2 and v = 2xy = 2x . x 2 = 2x 3 . Fig. 21 Fig. 22 z-plane w-plane x u y v γ 0 Γ

45 0 0 C 1 C 2 z 0 θ φ c 1 1 = w 1 (t) c 2 1 = w 2 (t) = f(z 2 (t)) w 0 Then, u x x v v = − = − 2 4 2 3 4 3 2 2 / / , which is the equation of the transformed curve Γ. The image of the point (1 + i) of z-plane is the point 2i in the w-plane and the slope of the curve Γ at w = 2i is dv du w i = = − 2 3 Thus the change in slope of the curve γ under the transformation is tan ( ) tan ( ) tan tan − − − − − − = − − − = 1 1 1 1 3 2 3 2 1 6 1 which is the same as obtained earlier following equation (47).

Definition : A mapping w = f(z) is said to be conformal at a point

z =

z 0 , if it preserves angles between oriented curves, passing through z 0 , in magnitude and in sense of rotation.

Theorem 3.1 :

Let

f(z) be

an analytic function in a domain

D
containing z 0 . If f 1 (z 0) ≠ 0, then f(z) is conformal at
z 0 .
Proof. Let C 1 : z = z 1 (t) and C 2 : z = z 2 (t), t ≡ parameter, be two curves which intersect at some t = t 0 where z 1 (t 0) = z 2 (t 0) = z 0 , C C 1 1 2 1 , are their images under the mapping w = f(z). Then following the result given in eq. (47) Arg
w t Arg z
t Arg
f z t Arg f z (( )) (( )) ((( )) ( )) 1 1 0 1 1 0 1 1 0 1 0 − = = and Arg w t Arg z t Arg f z t
Arg f z (( )) (( )) ((( )) ( )). 2 1 0 2 1 0 1 2 0 1 0 − = =
Fig. 21 Fig. 22 z-plane w-plane tangent lines are z 1 = z 1 1 (
t 0), z 1 = z 2 1 (t 0) at t = t 0 tangent lines are
w 1 1 (t 0) =
f 1 (z 1 (t 0)z 1 1 (t 0) w 2 1 (t 0) = f 1 (z 2 (t 0)z 2 (t 0)z 2 1 (t 0)
46
Subtracting, Arg w t Arg w t Arg z t Arg z t (( )) (( )) (( )) (( )) 1 1 0 2 1 0 1 1 0 2 1 0 0 − − − =
i.e., θ = φ, where θ =
angle between the curves C 1 and C 2 at z 0 and φ = angle between the curves C and C 1 1 2 1
at w 0 . Observation : From the basic results proved earlier we learn that if f is a conformal mapping, then orthogonal curves are mapped onto orthogonal curves. 3.2 Basic Properties of
conformal Mappings
Let f(z) be an analytic function in a domain
D,
and let z 0
be a point in D.
If
f(z 0) = 0,
then we can express
f(
z)
in the form
f(
z) = f(z 0) + (z − z 0)f 1 (z 0) + (z − z 0)
η(z),
where η(z) → 0
as z →
z 0 .
If z
is near z 0 , then
the transformation w = f(z) has the linear approximation G(z) = A + B(z − z 0). where A = f(z 0)
and B = f 1 (z 0).
As η(z) → 0 when z → z 0 ,
for
points near z n the transformation w = f(z) has an effect much like the linear mapping w = G(z). The effect of the linear mapping G is a rotation of the plane through the angle α = Arg (f 1 (z 0)), followed by a magnification by the factor |f(z 0)|, followed by a translation by the vector A + BZ 0 . Remark : If f 1 (z 0) = 0, the angle may not be preserved. Let us consider, w = f(z) = z 2 , then we have f 1 (0) = 0 and the angle at z = 0 is not preserved but is doubled. Definition : Let f(z) be a nonconstant analytic function. If f 1 (z 0) = 0, the z 0 is called a critical point of
f(z),
and the mapping w = f(z) is not conformal at z 0 .
We shall see afterwards what happens at a critical point. Fig. 23 Fig. 24 z-plane w-plane 0 0
47 The Inverse Function theorem 3.2 Let f(z) be

analytic at z 0 and f 1 (z 0) ≠ 0. Then there exists a
neighbourhood N(w 0 , ε) of w 0 = f(z 0) in which the inverse function z = F(w) exists and is analytic. Moreover, F 1 (w 0) = 1/f 1 (z 0). (48) Proof : Given w = f(z), (z = x + iy, w = u + iv) is analytic in a neighbourhood of z 0 , K : |z − z 0 | > ρ. We shall show that for each w∈L : |w − w 0 | > ∊ there is a unique solution z = F(w), where z∈K. We express the mapping w = f(z) in terms of the set of
equations u = u(x, y) and v = v(x, y) (49)
which represents a transformation from the xy plane to the uv plane, u, v, possess continuous first-order partial derivatives satisfying C-R equations. The Jacobian determinant J(x, y), is defined by J x y u u v v x y x y (,)= (50) The transformation in equations (49) has a local inverse in L provided J(x, y) ≠ 0 in K [(3) pp. 358-361]. Expanding r.h.s. of equation (50) and using the C-R equations, we obtain J x y u x y v x y x x (,) (,) (,) 0 0 2 0 0 2 0 0 = + = |f 1 (z 0)| 2 (51) ≠ 0, by the given hypothesis. Utilising the continuity of J(x, y) in a small neighbourhood of (x 0 , y 0), equations (49) and (51) imply that a local inverse z = F(w) exists in a neighbourhood of the point w 0 = f(z 0). The derivative of F(w) is given by the familiar expression
F
w
F w w
F w
w
z w
z f z z
f z w w
z 1 0 0 0 ( ) lim ( ) ( ) lim lim ( ) ( ) = + − = = + − → → → ∆ ∆ ∆ ∆ ∆ ∆ ∆ ∆ ∆ = + − = + − → → lim / ( ) ( ) /
lim ( ) ( ) ∆ ∆ ∆ ∆ ∆ ∆
z
z
f z z f z z f z z f z
z 0 0 1 1
i.e., F
w f
z 1 1 1 ( ) ( ) =
holds in
a
neighbourhood of the point w 0 , as f(z) is analytic
in K. In particular, F w f z 1 0 1 0 1 ( ) ( ) =
Theorem 3.3 Let f(z) be analytic at the point z 0 . If
f 1 (z 0) = 0, f 11 (z 0) = 0, ...,
48 f (k − 1) (z 0) = 0 and f (k) (z 0) ≠ 0, then the mapping w = f(z) magnifies
angles at z 0
by
k times. Proof. By the given hypothesis, f(z) has the Taylor expansion in a neighbourhood
of z 0 in the form f(z) = f(z 0) + c k (z − z 0) k + c k + 1 (z − z 0) k + 1 +..., c
k ≠ 0 so that we can express
f(z) −
f(z 0) = (z −
z 0) k + h(z) (52) where h(z) is analytic at z 0 and h(
z 0) ≠ 0.
Now let w = f(z)
and w 0 = f(z 0) and we obtain from (52) Arg(w − w 0) = k Arg(z − z 0) + Arg(h(z)) Let z → z 0 along a curve γ. Then w → w 0 along the image curve Γ

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and the slope of tangent to the curve γ at z 0 and that of the tangent to the curve Γ at w 0 are connected by the relation lim ( ) lim ( ) lim (( )) w w z z z z Argww k Argzz Arghz → → → − = − + 0 0 0 0 0 i.e., θ 0 = kφ 0 + Arg(h(z)) Thus, if γ 1 and γ 2 be two curves passing through z 0 and their images Γ 1 and Γ 2 under the mapping w = f(z), pass through w 0 , the difference of slopes of the curves γ 1 and γ 2 at z 0 and that of the curves Γ 1 and Γ 2 at w 0 are related as θ 2 − θ 1 = k(φ 2 − φ 1) with the sense remain unchanged. Example 2. Show that the mapping w = f(z) = z 2 maps the rectangle R x iy x y = + − ≤ ≤ ≤ ≤ : , 1 1 0 1 2 of unit area onto the region enclosed by the parabolas v u and v u 2 2 1 4 4 1 = + = − − ( ). Solution : Here f 1 (z) = 2z and the mapping w = z 2 is conformal for all z ≠ 0. We note that the right angles at the vertices z 1 = 1, z 2 = 1 + i/2, z 3 = − 1 + i/2 and z 4 = − 1 are mapped into right angles at the vertices w w i w i 1 2 3 1 3 4 3 4 = = + = − , , and w 4 = 1 respectively.

49 The parabolas shown in the figure are obtained as follows : Let w = u + iv. Then u = x 2 − y 2 , v = 2xy }... (53) The line x = 1 corresponds to the curve u = 1 − y 2 , v = 2y. Eliminating y, we get v 2 = − 4(u − 1), which is a parabola with vertex (1, 0) and opens towards the negative side of the u-axis in the w-plane. Also, the part
of the line x = 1 lying above the real axis corresponds to the part of the parabola lying above the u-axis in the w-plane. The same parabola in the w-plane is the image of the line x = − 1. In this case, the part
of the line x = − 1 lying above the real axis corresponds to the part of the parabola lying below the u-axis in the w-plane. Again, when y = 1 2 , from (53) we get u x = − 2 1 4 and v = x. Eliminating x we get, v u 2 1 4 = + which is also a parabola with vertex − 1 4 0, and opening towards the positive side of the u-axis in the w-plane. By similar argument as before we can say that the mapping w = z 2 maps the rectangle R x iy x y = + − ≤ ≤ ≤ ≤ : , 1 1 0 1 2 onto the region enclosed by the parabolas v u and v u 2 2 1 4 4 1 = + = − − ( ). Note : It is not hard to prove that the parabolas intersect each other orthogonally at w 2 and w 3 . At the point z 0 = 0, we have f 1 (z 0) = f 1 (0) = 0 and f 11 (z 0) = 2 ≠ 0. Hence the angles at the origin z 0 = 0 are magnified by the factor k = 2. In particular the straight angle at z 0 = 0 is mapped onto 2π angle at w 0 = 0. Fig. 25 Fig. 26 i/2 y − + i 3 4 − − i 3 4 − − 1 4 − 3 4 x o 1 u − 1 v v 2 = − 4(u − 1) o v 2 = u + − 1 4

50 Unit 4 Multi-valued functions and Riemann Surface Structure 4.0 Objectives of this Chapter 4.1 Multi-valued functions 4.2 The logarithm function 4.3 Properties of log z 4.4 Branch, Branch point and Branch cut 4.5 Integrals of Multi-valued function 4.6 Branch points at infinity 4.7 Detection of branch points 4.8 The Riemann Surface for w = z 1/2 4.9 Concept of neighbourhood 4.10 The Riemann Surface for w = log z 4.11 The Inverse Trigonometric Functions 4.0 Objectives of this Chapter In this chapter we shall study multi-valued functions and their Riemann surfaces. In particular, multi-valued logarithm function, the power function z α both z, α complex numbers, z ≠ 0 will be discussed. The ideas of branch, branch point, branch cut, branch point at infinity will be explained by means of different examples. A few contour integrations of multi-valued functions will be performed. Also Riemann surfaces for different multi- valued functions will be constructed. 4.1 Multi-valued functions So far we have considered single-valued functions i.e., one-to-one mapping or, many- to-one mapping. In the later case, under certain restrictions, inverse mappings give rise to multi-valued functions i.e., one-to-many. For example,

51 z = e ω , z = ω 2 , z = sin ω, z = cos ω For each of these functions, a given value of z corresponds to more than one value of ω. ω = f −1 (z) is multi-valued and z = f(ω) is single-valued, given ω, there is a unique value of z. The aim of this chapter is as follows : (i) To determine all possible values of the inverse function ω and (ii) To construct an inverse function which is single-valued in some region of the complex plane. Let ω = f(z) be a multi-valued function.
A branch of f is any single-valued function f 0 that is continuous in some domain (
except, perhaps, on the boundary). At each point z in the domain, it assigns one of the values of f(z). Example 1 : We consider branches of the two-valued square-root function f(z) = z 1/2 (z ≠ 0). The principal branch of the square root function is f z z e r Arg z i 1 1 2 2 1 2 2 2 ( ) cos sin , ( ) / / / = = + = θ θ θ θ where r = |z| and − π > θ ≤ π. The function f 1 is a branch of f. Using the same notation, we can find other branches of the function f. For example if we let f z z e r i i 2 1 2 2 2 1 2 2 2 2 2 ( ) cos sin / ( )/ / = = + + +
+θ π θ π θ π then f z r e r e e f z i i i 2 1 2 2 2 1 2 2 1 ( ) . ( ). / ( )/ / / = = = − +θ π θ π So, f 1 and f 2 can be taken as the two branches of the multi-valued square root function. The negative real axis is called a branch cut for the functions f 1 and f 2 . Each point on the branch cut is a point of discontinuity for both functions f 1 and f 2 . Result 1 : Show that the function f 1 is discontinuous on the negative real axis. ω = f −1 (z) Fig. 27 z 0 ω 1 ω 2 Z-plane ω-plane

52 Solution : Let z 0 = r 0 e iπ be any point on the negative real axis. We compute the limit as z approaches z 0 through the upper half plane lm z < 0 and the limit as z approaches z 0 through the lower half plane lm z > 0. The limits are lim ( ) (,) (,) lim (,) (,) cos sin , / / f 1 0 0 1 2 0 1 2 2 2 re r r r r r i ir and iθ θ π θ π θ θ → = → +
= lim (,) (,) ( ) lim (,) (,) cos sin / / r r f re r r r i ir i θ π θ π θ θ θ → − = → − +

= − 0 1 0 1 2 0 1 2 2 2 The two limits are distinct, so the function f 1 is discontinuous at z 0 . Since z 0 is an arbitrary point on the negative real axis, f 1 is discontinous there. Note : Likewise, f 2 is discontinuous at z 0 . Figures : 28-29 The Branches f 1 and f 2 of f(
z) = z 1/2 1234567890 123456789 0 1234567890 12345678901 1234567890 1 12345678901 <<<<< <<<<< <<<<< <<<<< ∧∧∧∧∧ ∧∧∧∧∧ ∧∧∧∧∧ ∧∧∧∧∧ <<<<< <<<<< < < < < < < < < < < ω = f 1 (z) ω = f 2 (z) z = ω 2 z = ω 2 z-plane z-plane ω-
plane Fig. 28 a Fig. 28 b Fig. 29 a Fig. 29 b o y x O u v y x o u
53 4.2 The logarithm function Let us define the inverse function f −1 (z) for z = e ω : Let z = re iθ and ω = u + iv. Then re iθ = e u .e iv So that r = e u and v = θ + 2kπ, k = 0, ± 1, ± 2,... and ω = log r + i(θ + 2kπ), k = 0, ± 1, ± 2,... But r = |z| and without loss of generality, we can take θ∈(−π, π). This motivates the definition of the inverse function f −1 (z) for z = e ω ω = log z = log |z| + i(Arg z + 2kπ), k = 0, ± 1, ± 2,... or, equivalently ω = log z = log |z| + i arg z. Mapping of the strip |Im ωωωωω| > πππππ under z = e ωωωωω I. Take u = u 0 < 0, v∈ (−π, π) for the line PQ : x iy e i u + = + 0 (cos sin) v v ⇒ = =
→ + = < x e y e x y e u u u 0 0 0 2 2 2 1 cos sin , v v a full circle in z-plane outside |z| = 1. Now approach Q; u = u 0 < 0, v = −π + ε x = e u 0 cos(−π + ε) → −e u 0 as ε → 0 + and −e u 0 > −1 as u 0 < 0 y = e u 0 sin(−π + ε) → 0 − as ε → 0 + Now approach P : u = u 0 < 0, v = π − ε Fig. 30 ω-plane v = π v = −π u = u 0 < 0 Q u = 0 u = u 0 > 0 S R o e ω = z P 1 Q 1 |z| = 1 x z-plane y R 1 S 1
54 x = e u 0 cos(π − ε) → −e u 0 as ε → 0 + y = e u 0 sin(π − ε) → 0 + as ε → 0 + II. Now take u = u 0 > 0, v∈ (−π, π) for the line RS : ⇒ x e y e x y e u u u = =
→ + = > − − − 0 0 0 2 2 2 1 cos sin v v represents a full circle in z-plane inside |z| > 1. Approach S : u = − u 0 > 0, v = − π + ε x = e −u 0 cos(− π + ε) → − e −u 0 < − 1 as ε → 0 + y = e −u 0 sin(− π + ε) → 0 − as ε → 0 + Now approach R : u = − u 0 > 0, v = π − ε x = e −u 0 cos(π − ε) → − e −u 0 < − 1 as ε → 0 + y = e −u 0 sin(π − ε) → 0 as ε → 0 + Observation : Points along the negative real axis in the z-plane yield multiple w values. In order to obtain a single-valued inverse function for the fundamental strip |lm ω|>π we require a cut in z-plane along Re z > 0. The mapping z = e w and w = f −1 (z) will be single-valued in |lm w| > π and z∈ /C \(•, 0). Clearly the inverse function w = Log z = log |z| + iArg z, − π > Arg z ≤ π 0 is single-valued. We call this function
the principal value of log z. The principal value of log z
is not defined at z = 0 and is discontinuous as z approach the negative real axis from top and bottom. Using the necessary and sufficient conditions for differentiability we find d dz Log z z z z = ≠ ∉ −• 1 0 0 , , (,) The point z = 0 is called a branch point of Log z since if we encircle the origin z = 0 by a closed contour then Log z changes by an amount proportional to 2πi. 4.3 Properties of log z (i) log (z 1 z 2) = log z 1 + log z 2 (means
that the set of all values of log z 1 + log z 2 is the same as the set of all values of log (z 1
z 2)). Fig. 31 θ Branch cut z-plane
55 (ii) z = e logz , but log(e z) = z + 2kπi, k = 0 ±1, ±2, ... Let z = x + iy log log( ) tan sin cos e e i y y k x iy k i x iy x + − = + + + + = 1 2 2 π π = z + 2kπi, k = 0, ±1,... (iii) log z n ≠ n log z in general. Let z = re iθ log z n = n log r + i(nθ + 2kπ), k = 0, ±1,... n log z = n log r + in(θ + 2mπ), m = 0, ±1,... Let n be fixed. Then the set of values of {k}, k = 0, ±1, ±2,... do not coincide with the set of values of {mn}, m = 0, ±1, ±2,... ⇒ log z n ≠ n log z (iv) log log / z n z n1 1 = (provided the set of values are the same) n ≡ + ve integer. Now, z = re iθ , z 1/n = r 1/n e i(θ + 2kπ)/n , k = 0, 1, 2,..., n − 1 log log , , ,..., − ; , , ,... / z n r i k n k n n1 1 2 2 01 1 0 1 2 = + + + = = ± ± θ π π Again, 1 1 2 0 1 2 n z n r i n m n m log log , , , ,... = + + = ± ± θ π The set of values of log (z 1/n) and 1/n log z are the same if the sets {k + ln}, k = 0, 1,..., n − 1; l = 0, ±1, ±2,... coincide with the set {m}, m = 0, ±1, ±2,... Complex exponents If α is complex and z ≠ 0 then z α = e α log z multi-valued. z α = e α[log|z| + i(Argz + 2kπ)] , k = 0, ±1, ±2,... = e α[log|z| + i(θ + 2kπ)] agrees with our previous results if α = m, α = 1 m ; m = integer. If α is a rational number say p/q, then z α will have only q number of distinct values, occurred against k = 0, 1, 2, ..., q − 1 and the values of e i2pkπ/q for k = − 1, − 2, ..., − (q − 1) coincide with

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56 its values for k = q − 1, q − 2, ..., 2, 1 respectively, whereas the values of e i2pkπ/q for k = ±q, ±(q + 1), ... coincide with its values for k = 0, ±1, ±2, ... z α takes infinite number of values when α is irrational or complex. Clearly there is a distinct branch of z α for each distinct branch of log z and the branch cuts are determined as in the case of log z. Every branch of z α is analytic except at the branch point z = 0 and on a branch cut. Example 2. Find all distinct values of i −2i . Solution : i e e k i i i i i i k − − + +
= = = ± 2 2 2 2 2 0 1 log log , , ,... π π = e (4k + 1)π , k = 0, ±1, ±2,... So, there are infinite number of values. Example 3. Find all solutions of z 1 − i = 6. Solution : e (1 − i)log z = e log 6 ⇒ (1 − i) log z = log 6 + 2kπi, k = 0, ±1, ±2,... or, 2 log z = (1 + i)[log 6 + 2kπi] or, log log (log) z k i k = − + + 6 2 2 2 6 2 π π Thus, z e k i k k = + + + − log cos(log) sin(log) 6 6 6 π π π = − − − 6 1 6 6 e i k k π ( ) cos(log) sin(log) 4.4 Branch, Branch point and Branch cut Definition : F(z) is a Branch of the multi-valued function f(z) in a domain D if F(z) is single-valued and continuous in D and has the property that for each z in D the value
of F(z) is one of the values of f(z).
To determine F(z) we introduce a line imanating from a point (called a Branch Point) to ensure that F is single-valued in the cut plane by the line. A Branch Point is one for which if we enclose it with a curve the function changes discontinuously as the variable makes a complete round over the curve. For instance, consider w = z 1/2 . Let P be a point on the z-plane where w z 1 1 1 2 = / and Arg z 1 = φ 1 , 0 > φ 1 > 2π. Let z r e i 1 1 1 = φ , then at P, w r e i 1 1 1 2 2 1 = / / . φ We now encircle the region along closed
57 curve C through P. Upon travelling anticlockwise once, we have φ = φ 1 + 2π, i.e., w r e r e i = = − + 1 1 2 2 2 1 1 2 1 2 1 1 / ( )/ / / φ π φ at the point P. ⇒ w = − w 1 at P. This shows that w has changed discontinuously after performing a loop about z = 0, which establishes z = 0 a Branch Point. Now we consider a different loop, a closed curve Γ around some point z* which does not enclose the origin. As before, z r e i 1 1 1 = θ and w r e i 1 1 1 2 2 1 = / / φ upon returning to P, travelling anticlockwise, we have φ = φ 1 again. Hence w is continuous after performing the loop. So z = z* is not a Branch Point for z 1/2 = w. Example 4. Discuss the multivaluedness of the function f(z) = (z 2 − 1) 1/2 and introduce cuts to obtain single-valued branches.
Solution : Let
z − 1 = r 1 e iθ and z + 1 = r 2 e iψ Then f z r r e i ( ) ( )/ = + 1 2 2 θ
ψ We choose a branch of f(z) at a point z 0 by taking values of θ 0 of θ and ψ 0 of ψ. Then at z 0 , f(z) takes the value f r r e i 0 1 2 2 0 0 = + ( )/ θ ψ If now z traverses from the point z 0 , and form a simple closed contour (end point also z 0) C 0 enclosing the point z = 1, where the point z = −1 lies outside C 0 , the value of f(z) at z 0 changes to r r e f i 1 2 2 2 0 0 0 ( )/ θ ψ π + + = − φ 1 Fig. 33 Fig. 32 O P z * o C P φ 1 Γ < < < < < < < < < < < < < < < Fig. 34 −1 Fig. 35 C 1 z 0 1 C 0 z 0 < < < < < < < < < < < < < < < < < < < <
58 f(z) takes the same value − f 0 while z travelling from z 0 and returns to z 0 itself forming a closed contour C 1 which encloses −1, but not 1. Hence it is clear that −1 and 1 are the branch points for the function f(z). In order to obtain single-valued branches we introduce two different set of branch cuts. (i) A branch cut between the points −1 and 1 on the real axis. In this case consider the closed contour C enclosing the branch points −1 and 1. Here f(z) returns to the value (from its value f 0 at z 0). r r e r r e f i i 1 2 2 2 2 1 2 2 0 0 0 0 0 ( )/ ( )/ θ π ψ π θ ψ + + + + = = So, it is a single-valued branch. (ii) Two branch cuts on the real-axis, (−•, −1) and (1, •). Here the contour Γ does not enclose any of the branch points, so f(z) remains single-valued as z makes a complete round through Γ initiating from z 0 . Example 5. Construct a branch of log z z − + 1 1 , which is analytic at the origin and takes the values 5πi there. Solution : Let g z z z ( ) log = − + 1 1 . The points z = ± 1 are the branch points of g(z) and the behaviour of g(z) at these branch points are similar to f(z) as shown in the previous example. We do not repeat these here. Write both z − 1, and z + 1 in polar form : z − 1 = re iθ , z + 1 = ρe iψ Then we can express g z re e r e i i i ( ) log log ( ) = =
− θ ψ θ ψ ρ ρ Fig. 36 −1 Fig. 37 1 z 0 z 0 C −1 1 Γ
59 = + − log ( ) r i ρ θ ψ We consider the complex z-plane with two branch cuts (−•, −1), and (1, •). Here the principal branch of g(z) is taken as log ( ), ; r i ρ θ ψ θ π π ψ π + − ≤ > − ≤ > 0 2 Now, g 0 = g(0) = iπ In the branch 4π ≤ θ > 6π; π ≤ ψ > 3π, g(z) will take the value 5πi at the origin. Example 6. Let z = ω 2 and consider Re ω < 0. Image is z ∈ / • C \ (− ,)0 Note : Injective mapping if Re ω < 0 and z ∈ /C \(−•, 0). We need a Branch cut along negative real-axis in the z-plane. Hence w = z 1/2 , z = re iφ , −π > φ ≤ π This ensures that Re ω < 0. Here the points on the cut go either to P or Q. P and Q are arbitrary. 4.5 Integrals of Multi-valued functions Example 7. Evaluate x x dx α α − • + > > 1 0 1 0 1 , . Let us consider the integal z z dz C α− − 1 1 where the contour C consists of a large Circle Γ R
with centre at the origin and radius R, a small circle γ ε with centre origin and radius ε joined to the large circle

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Q 1 O 1 z-plane ω-plane O Q P P 1 ω 2 = z Fig. 38
60 Γ R along the negative side of
the real axis from ε to R by means of a cut as shown in the figure 39. Thus we have avoided the branch point z = 0.
We take principal branch of z α − 1 . Then z z dz R R R R R as R R α α α π π − − − ≤ + = + → → • 1 1 1 2 1 2 1 0 Γ , since α > 1, z z dz α γ α α ε πε ε πε ε − − − ≤ = → → 1 1 1 2 1 2 0 0 as , since α < 0. Thus, by residue theorem, z z dz i s z z C α α π − − − Re − ; 1 1 1 2 1 1 =
Observe that z z α− − 1 1 has a simple pole at z = 1 which lies inside C. or, lim − lim − − − −2 − − − − R z z z z z z dz z z i R →• → + + + = α ε α γ α γ α γ ε α β π 1 0 1 1 1 1 1 1 1 Γ dz dz dz so, z z z z i α γ α γ α β π − − − − −2 1 1 1 1 + = dz dz (54) On γ α : z = ρe iπ , 0 > ρ > • so 1 − z = 1 + ρ and dz = e iπ dρ and z z e d e d e i i i α γ π α π α π α α πα α α ρ ρ ρ ρ ρ ρ ρ ρ ρ − (−) − − − − 1 0 1 1 1 0 0 1 1 1 = = + = + • • • dz 1+ e d −1 i (−1) On γ β , z = ρe −iπ , 0 > ρ > • so 1 − z = 1 + ρ, dz = e −iπ dρ, then z z e e d e i i i α γ π α π α π α α β ρ ρ ρ ρ ρ ρ − − − − (−) − (−) − − − 1 0 1 1 1 1 0 1 1 1 + + = + • • dz d = + • e d i− − πα α ρ ρ ρ 1 0 1 Substituting these integrals into (54), we get Fig. 39 Γ R γ α γ ε γ β 1
61 [ ] − + + = − − − • e e d i i i πα πα α ρ ρ ρ π 1 0 1 2 i.e. ρ ρ ρ π πα α− • + = 1 0 1 2 2 d i isin or, x x dx α π πα − • + = 1 0 1 sin Example 8 : Evaluate x x dx α α − • + > > 1 3 0 1 0 3 , . We take the contour integral z
z dz C α− + 1 3 1 , where C is the contour as shown in the
fig. 40. Take principal branch of z α − 1 . Then, z z dz as α γε α α π ε ε π ε ε ε − + ≥ = → → → < 1 3 1 1 2 3 1 2 3 0 0 0 − since and z z dz R R R R R α α α π π α − − − + ≤ = → • → • > 1 3 1 3 3 1 2 3 2 3 3 Γ as R since Now the function z z α− + 1 3 1 has only one simple pole z e i = π 3 inside C. Thus z z dz i s z z e i e e i e i i i i C α α α π απ π π π π π π − − − + = + = = − 1 3 1 3 1 2 3 3 1 2 1 2 3 2 3 3 3 Re ; . ( ) / / i.e., z z dz z z e e d d i e i i i R R R α α α π α π α απ ε ε γε ρ ρ ρ ρ ρ ρ π − − − − − + + + + + + + = − 1 3 1 3 1 3 2 1 3 2 3 1 3 3 1 1 1 1 2 3 ( )/ / / Γ [In the third integral, we used z = ρe 2πi/3 , dz = e 2πi/3 dρ, 1 + z 3 = 1 + ρ 3 , and in the fourth integral, z = ρ, dz = dρ] Taking R → • and ε → 0 in the above integrals, we find using the earlier results − + + + = − − e d d ie i i 2 3 1 3 1 3 3 0 0 1 1 2 3 απ α α απ α α ρ ρ ρ ρ ρ ρ π / / Fig. 40 Γ R take branch cut on the negative real-axis z = ρe 2πi/3 γ R γ ε γ 1 R
62 So that, ρ ρ ρ π π απ α απ απ − − • + = · − = 1 3 3 3 0 1 2 3 1 3 3 d i e e i i / / sin or, x x dx α π απ − • + = 1 3 0 1 3 3 sin Riemann Surface A Riemann surface is a generalization of the complex plane to a surface
comprising several sheets so that a multi-valued function can have
only one value corresponding to each point on that surface. Once such a surface is ascertained for a given multi-valued function, the function becomes single-valued on the surface and
can be treated according to the theory of single-valued functions. This topology removes artificial restrictions-Branch Cuts and gives us a more general notion of a domain so that a multi-valued analytic function becomes single- valued if it is considered as a mapping to an appropriate generalized domain as suggested by G. F. B. Riemann (1826-1866) in 1851. The idea is ingenious—a geometric construction that permits surfaces to be the domain or range of a multi- valued function. 4.6 Branch points at infinity So far we have considered only branch points in the finite plane. Now we discuss about the possibility of a branch point at infinity. For this sake we map the point at infinity to the origin with the transformation ς = 1 z and then examine the point ς = 0. Example 9 : Again we consider the multi-valued function f(z) = z 1/2 . Making the transformation ς = 1 z , the point at infinity is mapped to the origin, we have f( )ς ς = 1 1 2 . For each value of ς, there are two values of ς −1/2 . Writing ς −1/2 in modulus-argument form ς ς ς −1 − ( )/ | | 2 2 1 = e iArg

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63 we find that like z 1/2 , ς −1/2 possesses double sheeted Riemann surface. We see that each time we walk around the origin, the argument of ς −1/2 changes by −π. This means that the value of the function changes by the factor e −iπ = −1, i.e. the function changes sign. If we walk around the origin twice, the argument changes by −2π, so that the value of the function does not change, e −2πi = 1. Now, since ς −1/2 has a branch point at zero, we conclude that z 1/2 has a branch point at infinity. Example 10 : Again consider the multi-valued logarithm function f(z) = log z. Mapping the point at infinity to the origin, we have f( ) log − log ς ς ς = = 1 But log ς has a branch point at ς = 0. Thus log z has a branch point at infinity. Branch points at infinity : Paths around infinity We can also check for a branch point at infinity by considering a path that encloses the point at infinity and no other singularities. This can be done by drawing a simple closed curve that separates the complex plane into a bounded region that contains all the singularities of the function in the finite plane. Then, depending upon the orientation, the curve is a contour enclosing all the finite singularities, or the point at infinity and no other singularities. Once again consider the function z 1/2 . We know that the function changes value on a curve that goes around the origin. Such a curve can be considered to be either a path around the origin or a path around the point at infinity. In either case the path encloses one branch point. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. Thus we see that z 1/2 does not change value when we follow a path that encloses neither or both of its branch points. Example 11 : Consider the multi-valued function f(z) = (z 2 − 1) 1/2 . Rewriting the function f(z) = (z − 1) 1/2 (z + 1) 1/2 , we see that there are branch points at z = ± 1. Now consider the point at infinity. f(ς −1) = (ς −2 − 1) 1/2 = ±ς −1 (1 − ς 2) 1/2 which shows that f(ς −1) does not have a branch point at ς = 0 and f(z) does not have a branch point at infinity. We might reach the same conclusion by considering a path around the point at infinity. Consider a path that encircles the branch points at z = ±1 once in the positive direction. Equivalently it encircles the point at infinity once in the negative direction. In traversing this path, the value of f(z) is multiplied by the factor (e 2iπ) 1/2 (e 2iπ) 1/2 = e 2iπ = 1. Thus the value of the function remains unchanged. There is no branch point at infinity.

64 4.7 Detection of branch points We have the definition of a branch point, but we do not have a convenient criterion for determining if a particular function has a branch point. We have noticed that log z and z k for non-integer k have branch points at zero and infinity. The inverse trigonometric functions like sin −1 z, cos −1 z etc. also have branch points, but they can be written in terms of the logarithm and the square root. In fact all the elementary functions with branch points can be written in terms of the functions log z and z k . Furthermore, note that the multi-valuedness of z k comes from the logarithm, z k = e klogz . This gives us a way of determining branch points of a function if there is any. Result : Let f(z) be a single-valued function. Then log f(z) and (f(z)) k may have branch points only where f(z) is zero or singular. Example 12 : Consider the functions 1. (z 2) 1/2 2. (z 1/2) 2 3. (z 1/2) 3 Are they multi-valued? Do they have branch points? Solution 1. z z z 2 1 2 2 / = ± = ± Because of (.) 1/2 , the function is multi-valued. The only possible branch points are at zero and point at infinity. If (e iθ) 2) 1/2 = 1, then as ((e 2πi) 2) 1/2 = (e 4πi) 1/2 = e 2πi = 1 the function does not change value when we walk around the origin. We can also consider this to be a path around infinity. This function is multi-valued, but has no branch points. 2. z z z 1 2 2 2 / = ± = This function is single-valued. 3. z z z 1 2 3 3 3 / = ± = ± This function is multi-valued. We consider the possible branch point at z = 0. If (e i0) 1/2) 3 = 1, then as ((e 2iπ) 1/2) 3 = ((e iπ2) 1/2) 3 = (e iπ) 3 = e 3πi = −1, the function changes value when we walk around the origin. So it has a branch point at z = 0. Since this is also a path around infinity, there is a branch point at the point at infinity. Example 13 : Consider the function f(z) = log (1/z − 1). Since 1 1 z − has only zero at infinity and its only singularity (a pole here) is at z = 1, the only, possible branch points are at z = 1 and z = •.

65 Here f z z z ( ) log − − log(−) log , = = = 1 1 1 ω say We know that log ω has branch points at zero and infinity, so f(z) has branch points at z = 1 and z = •. Example 14 : Consider the functions 1. e logz 2. log e z Are they multi-valued? Do they have branch points? Solution : 1. e logz = e logz + i2 π k , k = 0, ± 1, ... = e Logz e i2 π k = z The function is single-valued. 2. loge z = Loge z + i2πk = z + i2πk, k = 0, ± 1, ... This function is multi-valued. It may have branch points only where e z is zero or infinite. This occurs only at z = •. Thus there are no branch points in the finite plane. The function does not change when traversing a simple closed path and since this path can be considered to enclose the point at infinity, there is no branch point at infinity. Note : Let f(z) be single-valued and have either a zero or a singularity at z = z 0 . Then {f(z)} k may have a branch point at z = z 0 . If f(z) is not a power of z, then we are not sure whether {f(z)} k changes value when we walk around z 0 . Now

if f(z) can be decomposed into factors f(z) = h(z) g(z), where h(z) is finite and non zero at z 0 , then from g(z)

we know how fast f(z) vanishes or tends to infinity. Again {f(z)} k = {h(z)} k {g(z)} k and {h(z)} k does not have a branch point at z 0 . So that {f(z)} k has a branch point at z 0 if and only if {f(z)} k has a branch point there. Similarly, we can decompose log {f(z)} = log {h(z)g(z)} = log {h(z)} + log {g(z)} to see that log {f(z)} has a branch point at z 0 if and only if log {g(z)} has a branch point there. Example 15 : Consider the functions : 1. sin z 1/2 2. (sin z) 1/2 3. z 1/2 cos z 1/2 4. (sin z 2) 1/2 . Find the branch points and the number of branches. Solution : 1. sin sin sin z z z 1 2 = ± = ± So it is multi-valued. It has two branches and the possible branch points are zero and infinity. Consider the unit circle |z| = 1 which is a path around the origin and infinity. If sin(e i0) 1/2 = sin(1), then as
66 sin((e i2π) 1/2) = sin(e iπ) = sin(−1) = − sin1, there are branch points at the origin and infinity 2. (sin) sin / z z 1 2 = ± The function is multi-valued and has two branches. The sine function vanishes at z = nπ and is singular at infinity. These may be branch points of the function. Consider the point z = nπ. We can express sin (−)
sin − , z z n z z n = π π n an integer. But lim sin − lim cos (−1) z n z n n z z n z → → = =
π π π 1 So, (sin z) 1/2 has branch points at z = nπ since (z − nπ) 1/2 has a branch point at z = nπ. Here the branch points are z = nπ, n = 0, ±1, ... and they go to infinity. So it is not possible to make a path that encloses infinity and no other singularities. The point at infinity is a non-isolated singularity. A point can be a branch point only if it is an isolated singularity. 3. z z z z 1 2 1 2 / / cos cos · = ± ± = ± z z cos The function is multi-valued. It may possess branch points at z = 0 and z = •. If (e i0) 1/2 cos(e i0) 1/2 = cos(1), then as (e i2 π) 1/2 cos((e i2π) 1/2) = (−1)cos(e iπ) = − cos(−1) = − cos1, there are branch points at the origin and infinity. 4. (sin) sin / z z 2 1 2 2 = ± The function is multi-valued. Now since siz z 2 = 0 at z = (nπ) 1/2 , there may be branch points there. We consider first the point z = 0. We can write
sin
sin
z z z z 2 2 2 2 = but lim sin lim cos z z z z z z
z → → = = 0 2 2 0 2 2 2 1
So, (sin z 2) 1/2
does not have a branch point at z = 0 as (z 2) 1/2 does not have a branch point there. Next consider the point
z n = π
67 sin − sin −
z z
n z z n 2 2 = π π but lim sin − lim cos (−1) z n z n n z z n z z n → → = =
π π π π 2 2 2 1 2 Since z n − / π 1 2
has a branch point at z n z = π,(sin) 2 1 2 , too as a branch point there. Thus we see that (sin z 2) 1/2 has branch points at z = (nπ) 1/2 for n ε Z \ {0}. This is the set of numbers : ± ± ± ± π π π π , ,..., , ,... 2 2 i i . The point at infinity is a non-isolated singularity and hence it is not included in the set of branch points. Example 16 : Find the branch points of f(z) = (z 3 − z) 1/3 and introduce the branch cuts. If f( ) , 3 2 3 3 = find f(−3). Solution : Here f(z) = z 1/3 (z − 1) 1/3 (z + 1) 1/3 So the branch points are at z = −1, 0 and 1. We consider the point at infinity f 1 1 1 1 1 1 1 3 1 3 1 3 ς ς ς ς = + − / = + 1 1 1 1 3 1 3 ς ς ς (−) ( ) / / Since f(1/ς) does not have a branch point at ς = 0, f(z) does not have a branch point at infinity. Here we give three possible branch cuts : In the first and third the function is single-valued but in the second it is not. It is clear that the first branch cut does not allow us to walk around any of the branch points. <<<<< ∧∧∧∧∧ ∧∧∧∧∧ ∧∧∧∧∧ <<<<< <<<<< Fig. 41 O Three possible branch cuts for f(z) = (z 3 − z) 1/3 O O −1 1 −1 1 −1 1
68 The second branch cut allows us to walk around the branch points at z = ± 1. If we walk around these two once in the positive direction, the value of the function would change by the factor e i4π/3 . The third branch cut allows us to walk around all the three branch points, the value of the function will not change (since e i6π/3 = e i2π = 1). To find f(−3), we consider the third branch cut with f( ) . 3 2 3 3 = f e e e i i i ( ) ( ) ( ) ( ) / / 3 3 2 4 2 3 0 1 3 0 1 3 0 1 3 3 = = The value of f(−3) is f e e e i i i (−3) ( ) ( ) ( ) − / / = = 3 2 4 2 3 1 3 1 3 1 3 3 π π π Example 17 : Determine the branch points of the function f(z) = (z 3 − 1) 1/2 . Construct branch cuts and define a branch so that z = 0 and z = −1 do not lie on a cut, such that f(0) = −i; then what is f(−1/2)? Solution : The roots of the equation z 3 − 1 = 0 are 1 1 3 2 3 2 2 3 4 3 , , , −1 , −1 − / / e e i i i i π π = +
so that, z z z i z i 3 1 2 1 2 1 2 1 2 1 1 1 3 2 1 3 2 − (−) − / / = + + + There are branch points at each of the cube roots of unity z i i = +
1 3 2 3 2 , −1 , −1 − Now we examine the point at infinity. We make the change of variable z = 1/ς f(1/ς) = (1/ς 3 − 1) 1/2 = ς −3/2 (1 − ς 3) 1/2 ς −3/2 has a branch point at ς = 0, while (1 − ς 3) 1/2 is not singular there. Since f(1/ς)

has a branch point at ς = 0, f(z) has a branch point at
infinity. There are several ways of introducing branch cuts to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity (see Figure 42a). Clearly this makes the function single-valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points (see Figure 42 bcd). In this case, in walking around
69 any one of the finite branch points (in the + ve direction), the argument of the function changes by π. This means that the value of the function changes by e iπ , which is to say, the value of the function changes sign. In walking around any two of the finite branch points (in the +ve direction), the argument of the function changes by 2π i.e., the value of the function changes by e i2π , that means the value of the function does not change. Figure 42. Branch cuts for (z 3 −1) 1/2 Now we choose the branch 42a, and introduce the variables
z − 1 = r 1 e iθ , 0 ≤ θ 1 > 2π z i r e i + = ≤ > 1 3 2 2 3 3 2 2 2 − , − θ π θ π z i r e i + = ≤ > 1 3 2 3 2 3 3 3 3 − , − θ π θ
π We compute f(0) to see whether it has the desired value, f z r r r e i ( ) ( )/ = + + 1 2 3 2 1 2 3 θ θ θ f(0) = e i(π−π/3+π/3)/2 = e iπ/2 = i Since it does not have the desired value, we change the range of θ 1 , z − 1 = r 1 e iθ 1, 2π ≤ θ 1 > 4π f(0) now has the desired value, f(0) = e i(3π−π/3+π/3) = − i We compute f − , 1 2 f i − − 1 2 3 2 3 2 3 2 3 2 2 2 = · · + e π π π a b c d
70 = = 9 8 2 2 3 2 e i i π/ −3 Example 18 : Identify the branch points of the function ω = f(z) = (z 3 + z 2 − 6z) 1/2 in the extended complex plane. Specify the branch cuts and select a branch that gives a single-valued function where it is continuous at z = −1 with f(−1) = −√6. Solution : First we factor
the function f(z) = [z(z − 2(z + 3)] 1/2 = z 1/2 (z − 2) 1/2 (z + 3) 1/2
There are branch points at z = −3, 0, 2. Now we examine the point at infinity. f(/) − (−)( ) −3/ / 1 1 1 2 1 3 1 2 1 3 1 2 2 1 2 ς ς ς ς ς ς ς = +
= + Since ς −3/2 has a branch point at ς = 0 and the rest of the terms are analytic there, f(z) has a branch point at infinity. Now consider the branch cuts given in the figure 43. These cuts do not permit us to walk around any single branch point. We can walk around none of the branch points (or all of the branch points considering the cuts [−3, 2] and x = 0, y ≤ 0). The cuts can be used to define a single-valued branch of the function. Now to define the branch, we choose z + 3 = r i e iθ 1 , −π ≤ θ 1 > π; z = r 2 e iθ 2 , −π θ π 2 3 2 2 ≤ > and z − 2 = r 3 e i θ 3 , 0 ≤ θ 3 > 2π. The function is, f(z) = (r 1 r 2 r 3) 1/2 e i(θ 1 + θ 2 + θ 3)/2 Here f(−1) = [(2)(1)(3)] 1/2 e i(0 + π + π)/2 = − 6 So our choice of angles gave the desired branch. 4.8 The Riemann surface for ω ω ω ω ω = z 1/2 We have seen that ω = z 1/2 possesses two branch points z = 0 and z = •. To utilize the developments made in Example 1, we introduce a branch cut along the negative real axis. The given function has two values for any z ≠ 0. f 1 (z) = r 1/2 e i θ /2 , −π > θ ≤ π Fig. 43 O −3 2
71 and f 2 (z) = r 1/2 e iθ/2 , π > θ ≤ 3π Each function f 1 and f 2 is single-valued on the domain formed
by cutting the z- plane along the negative real-axis.
Let D 1 and D 2 be the domains of f 1 and f 2 respectively. The range set for f 1 is the set R 1 consisting of the right-half plane and the positive imaginary axis [see Figure 28b]; the range set for f 2 is the set R 2 consisting of the left-half plane and the negative imaginary axis [see Figure 29b].
The sets R 1 and R 2 are glued together along the positive
imaginary axis and the negative imaginary axis to form the w-plane with the origin deleted. We stack D 1 directly above D 2 . The edge of D 1 in the upper-half plane is joined to the edge of D 2 in the lower-half plane, and the edge of D 1 in the lower-half plane is joined to the edge of D 2 in the upper-half plane (it is needless to mention that the line of joining is the negative real-axis). When these domains are glued together in this manner, they form a Riemann surface domain for the mapping w = f(z) = z 1/2 shown in the figure 44 for some finite r. 4.9 Concept of neighbourhood When a point lies on the boundary of two domains D 1 and D 2 , we define a neighbourhood of that point in the following manner. Here the boundary of D 1 and D 2 is the negative real-axis. (i) Neighbourhood of a point ς∈D 1 with Im ς > 0, Arg ς = π, |z − ς| > ε consists of points on : (a) D 1 if Im ς ≥ 0 (b) D 2 if Im ς > 0. (ii) Neighbourhood of a point ηεD 2 with Im η = 0, Arg η = 3π, |z−η| > ε consists of points on (a) D 1 if Im η > 0 and (b) D 2 if Im η ≥ 0. With these definitions of neighbourhood of a point, it becomes clear that w = z 1/2 is continuous and differentiable everywhere on the Riemann surface except at the origin and the point at infinity. The derivative is given by d dz z f f 1 2 1 1 2 1 2 1 1 2 1 / = on D on D 2 Fig. 44

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72 4.10 The Riemann Surface for w = log z The Riemann surface for the multivalued function ω = log z is similar to the one we presented for the square root function. However, it
requires infinitely many copies of the z-plane cut along the negative x-axis,
which mark S k for k = ..., −n, ..., −1, 0, 1, ..., n, ... Now we stack these cut planes directly on each other so that the corresponding points have the same position. We join the sheet S k to S k+1 as follows. For each integer k, the edge of the sheet S k in the upper half-plane is joined to the edge of the sheet S k+1 in the lower-half plane.
The Riemann surface for the domain of log z looks like a spiral staircase that extends upward on the sheets S 1 , S 2 ...,
and downward on the sheets S −1 , S −2 , ...
as shown in figure 45. For S k , we use z = re iθ = r (cos θ + i sin θ), where r = |z| and 2πk−π > θ ≤ π + 2πk Again, for S k , the correct branch of log z on each sheet is log z = log r + i θ, where r = |z| and 2πk−π > θ ≤ π + 2πk Example 19 : Form a Riemann surface for f(z) = (z − 1) 1/3 taking a branch cut along the line y = 0, x ≥ 1. Detect the point where the function takes the value √2 (i − 1). Solution : Let r = |z − 1| and θ = arg (z − 1), where 0 ≤ θ > 2π. Here the Riemann surface consists of three domains D 1 D 2 and D 3 : f 1 (z) = r 1/3 e iθ/3 , 0 ≤ θ > 2π (D 1) f 2 (z) = r 1/3 e iθ/3 , 2π ≤ θ > 4π (D 2) Fig. 45 Fig. 46 ω-plane 3π 2π π u −π −2π −3π v y z-plane S 1 S 0 x S −1
73 f 3 (z) = r 1/3 e iθ/3 , 4π ≤ θ > 6π (D 3) Each function f 1 , f 2 and f 3 is single-valued on the domain formed by cutting the z-plane along the line y = 0, x ≥ 1. We place D 1 on the top, then D 2 and D 3 . The edge of D 1 in the upper-half plane is joined to the edge of D 2 in the lower-half plane and the edge of D 2 in the upper- half plane is joined to the edge of D 3 in the lower-half plane and finally the edge of D 3 in the upper-half plane is joined to the edge of D 1 in the lower-half plane. Say at z = ς, f(ς) = √2 (i − 1) i.e. f i ( ) −2 − ς = 1 2 2 = = 2 2 4 3 4 e e e i i i π π π − / = = + 2 2 9 4 3 4 2 3 e e i i π π π / / So, ς ς π π − , 1 2 1 8 3 4 4 = = + e e i i lying in the domain D 2 . Example 20 : Form the Riemann surface for the function f(z) = (z 2 − 1) 1/2 . Solution : Here the given function f(z) = (z 2 − 1) 1/2 has branch points at z = ±1. To examine the point at infinity, we substitute z = 1/ς and examine the point ς = 0. f i 1 1 1 1 1 2 2 2 1 2 2 1 2 ς ς ς ς = = − ( ) (−) / / / Since there is no branch point at ς = 0, f(z) has no branch point at infinity. Let z − 1 = r 1 e iφ 1 and z + 1 = r 2 e iφ 2 , where φ 1 = Arg (z − 1) and φ 2 = Arg (z + 1). Then ω = f(z) = (z 2 − 1) 1/2 = (z − 1) 1/2 (z + 1) 1/2 = (r 1 r 2) 1/2 e i(φ 1 + φ 2) O 1 x y Fig. 47 ς D Fig. 48 φ 1 D 1 O −1 B' B r 2 1 C' C z r 1 φ 2) ↑
74 Case-I 0 ≤ φ 1 > 2π, 0 ≤ φ 2 > 2π on the φ 1 φ 2 e i(φ 1 +φ 2)/2 Continuity segment of f(z) B π 0 i No B′ π 2π −i C 0 0 1 Yes C′ 2π 2π 1 D π π −1 Yes D′ π π−1 Fig. 49 Case-II 0 ≤ φ 1 > 2π, −π ≤ φ 2 > π on the φ 1 φ 2 e i(φ 1 +φ 2)/2 Continuity segment of f(z) B π 0 i Yes B′ π0 i C 0 0 1 No C′ 2π 0 −1 D π π −1 No D′ π −π 1 Two branches of (z − 1) 1/2 can be taken as f z r e z r e f z i 1 1 2 2 1 2 2 1 1 1 0 2 ( ) ( ) , − ( ) / ( )/ = = ≤ > = + i 1 and f φ φ π φ π Again two branches of (z + 1) 1/2 can be taken as g z r e z r e i 1 2 2 2 2 2 2 2 2 0 2 ( ) ( ) , / ( )/ = = ≤ > + i 2 and g φ φ π φ π = − g 1 (z) Let us construct a Riemann surface for ω = (z 2 − 1) 1/2 considering case I. Here a Riemann surface consists of two sheets So and S 1 . Let S 0 be an extended complex plane cut along the real axis from z = −1 to z = 1 and S 1 be another extended complex plane cut of similar nature. S Arg z Arg z Arg z Arg z 0 1 0 1 2 0 1 2 2 1 4 2 1 4 ≤ > ≤ + > ≤ > ≤ + > (−) ( ) (−) ( ) π π π π π π S The sheets S 0 and S 1 are joined along the segment [−1, 1] in such a way that the lower edge of the slit in S 0 is joined to the upper edge of the slit in S 1 , and the lower edge of the slit in S 1 is joined to the upper edge of the slit in S 0 . −1 0 1 x y −1 0 1 x y Fig. 50 Branch cut [−1, 1] Fig. 51 Branch cuts (−•, −1] and [1, •) ′

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75 Let a point on the sheet S 0 move anticlockwise and form a simple closed curve which encloses the segment [−1, 1] once. Then both φ 1 and φ 2 change by an amount 2π upon returning to their original position. i.e. (φ 1 + φ 2)/2 changes by an amount 2π, so the value of ω φ π φ π φ φ = = + + + + ( ) ( ) / ( )/ / ( )/ r r e r r e i i 1 2 1 2 2 2 2 1 2 1 2 2 1 2 1 2 remains unchanged. Then ω = f 1 g 1 on S 0 and as well as on S 1 . If a point starting on the sheet S 0 travels a path which makes a complete round about only the branch point z = 1, it crosses from the sheet S 0 to S 1 . In this case, the value of φ 1 changes by an amount 2π, while the value of φ 2 does not change at all. The change in (φ 1 + φ 2)/2 is then π. The change in (φ 1 + φ 2)/2 remains the same if a point on the sheet S 0 makes a complete round about the branch point z = −1 only and enters on the S 1 sheet. This time. ω = f g f g 1 1 0 1 1 1 on S on S − Thus the double-valued function ω = (z 2 − 1) 1/2 can now be considered as a single-valued function on the Riemann surface constructed above. Hence the transformation ω = (z 2 − 1) 1/2 maps each of the sheets S 0 and S 1 forming the Riemann surface on the entire ω-plane. Riemann surface for the case II Here the Riemann surface is formed by two sheets S 0 and S 1 . Each of these sheets is an extended complex plane cut along the line (−•, −1) ∪ [1, •) S Arg z Arg z Arg z Arg z 0 1 0 1 2 1 2 1 4 1 3 ≤ > ≤ + > ≤ > ≤ + > (−) − ( ) (−) ( ) π π π π π π π π S These sheets are joined along the line (−•, −1] ∪ [1, •) in such a way that the lower edge of the slit in S 0 is joined to the upper edge of the slit in S 1 , and the lower edge of the slit in S 1 is joined to the upper edge of the slit in S 0 . If a point traverses a simple closed curve on either of the sheets S 0 or S 1 not enclosing any of the branch points −1 or 1, then the function f(z) remains single-valued on the respective sheet, whereas if it encloses any one of the branch points the function changes the branch as explained in case I. In the same way the double-valued function f(z) = (z 2 − 1) 1/2 can be treated as a single-valued function on the Riemann surface formed earlier.

Example 21 : The power function ω = f(z) = [z(z − 1)(z − 2)] 1/2 has two branches. Show that f(−1) can be either −√6i or √6i. Suppose the branch that corresponds to f(−1) = −√6i is chosen, find the value of the function at z = i.

76 Solution : The given power function can be expressed as

ω π = = = + +

f z z z z e i Argz Arg z Arg z ik ( ) (−)(−) , , [(−) (−)]/ 1 2 0 1 1 2 2 e k where the two possible values of k correspond to the two branches of the double- valued power function.

If figure 52a branch cuts are y = 0, x ≤ 0 and y = 0, 1 ≤ x ≤ 2 and in figure 52b branch cuts are y = 0, 0 ≤ x ≤ 1 and y = 0, x ≥ 2. In both the cases Riemann surface is formed by two branches. At z = −1, we note that Arg

z = Arg (z − 1) = Arg (z − 2) = π and z z z (−)(−) . 1 2 6 = So, f(−1) can be either 6 6 6 6 6 3 2 2 2 2 3 2

e i or e

e

e i i i i i π π π π π π π π

π / ( ) / − . = = = + + + + +

The branch that gives f(−1) = √6i corresponds to k = 0. With the choice of that branch, we find f i i(i i e i Argi Arg i Arg i ( ) −)(−) | (−) (−)/ = + + 1 2 1 2 2 = = + + 2 5 10 2 3 4 1 2 2 4 4 2 2 1 2 e e i i i (/ / −tan /)/ −tan −1 −1 / π π π π π π e = = − − (tan −tan /)/ (tan /)/ −1 −1 −1 10 10 4 1 1 1 2 2 4 1 3 2 e e i 4.11 The Inverse Trigonometric Functions (i) The function sin −1 z is defined by the equation z = sin ω Substituting e e i i i ω ω − − 2 for sin ω, we find that (e iω) 2 − 2ie iω z − 1 = 0 i.e., e iω = iz + (1 − z 2) 1/2 ⇒ iω = log{iz + (1 − z 2) 1/2 } so that sin −1 z = −ilog{iz + (1 − z 2) 1/2 } Similarly, we can have 1 0 2 Fig. 52a 1 0 2 Fig. 52b

77 cos −1 z = −ilog{z + (z 2 −1) 1/2 } (ii) We take the function ω = tan −1 z, which is the inverse of z = tan ω. Expressing tan ω in terms of sin ω and cos ω and then converting to their exponential form, we get z e i e e i i i i = + 1 e ω ω ω ω − − − = + 1 1 1 2 2 e i i i e ω ω − i.e., iz e e e iz iz i i i = + ⇒ = + 2 2 2 1 1 1 1 ω ω ω − − and finally, ω = + 1 2 1 1 i iz iz log −

Note :

When z ≠ ± 1, the quantity (1 − z 2) 1/2 has two possible values. For each value, the logarithm generates infinitely many values. Therefore sin −1 z has two sets of infinite values. For example, consider sin −1 1 2 = ± 1 2 3 2 i i log =

+ + 1 1 6 2 5 6 2

i e i e i k i k log log

π π π π or = +

+

1 6 2 1 5 6 2 i i k i i k π π π π or = + + π π π π 6 2 5 6 2

k k or , k is any integer. Likewise, the set of values for other inverse trigonometric functions can be ascertained. Example 22 : Discuss the mapping ω = sinh z that transforms the infinite strip −• > x > •, 0 > y > π into the ω-plane. Find cuts in the ω-plane which make the mapping continuous both ways. What are the images of the lines (a) y = 1/π (b) x = 1? Solution : First we express sinh z in cartesian form ω = sinh z = sinh x cos y + icosh x sin y = u + iv We consider the line segment x = c, y ε (0, π). Its image is
78 {sinh c cos y + i cosh c sin y|y ε (0, π)} Clearly, u and v then satisfy the equation for the ellipse u c v c 2 2 2 2 1 sinh cosh + = The ellipse starts at the point (sinh c, 0), passes through the point (0, cosh c) and ends at (−sinh c, 0). As c varies from zero to • or from zero to −•, the semi-ellipses cover the upper-half of ω-plane. Thus the mapping is 2−to−1. Now consider the infinite line y = c, x ∈(−•, •). It's image is {sinh x cos c + i cosh x sin c|x ∈(−•, •)}. Here u and v satisfy the equation for a hyperbola u c v c 2 2 2 2 1 cos − sin = As c varies from 0 to π/2 or from π/2 to π, the semi-hyperbola cover the upper- half of ω-plane. Thus the mapping is 2-to-1. We look for branch points of sinh −1 ω ω = sinh z ω = e e z z − − 2 e 2z −2ωe z −1 = 0 e z = ω + (ω 2 + 1) 1/2 z = log(ω + (ω − i) 1/2 (ω + i) 1/2) The branch points are at ω = ±i. Since ω + (ω 2 + 1) 1/2 is non zero and finite in the finite complex plane, the logarithm does not introduce any branch in the finite plane. Thus the only branch point in the upper-half of ω-plane is at ω = i. Any branch cut that connects ω = i with the boundary of Im ω < 0 will separate the branches under the inverse mapping. We consider the line y = π/4. The image under the mapping is the upper-half of the hyperbola 2u 2 − 2v 2 = 1 Consider the segment x = 1. The image under the mapping is the upper-half of the ellipse. u v 2 2 2 2 1 1 1 sinh cosh + =
79 Example 23 : Construct a Riemann Surface for cos −1 z. Solution : The function ω = cos −1 z = −i log [z + (z 2 − 1) 1/2] has two sources of multi-valuedness; one due to the square root function (z 2 − 1) 1/2 and the other due to the logarithm, if any. First we construct the branch of the square root (
z 2 − 1) 1/2 = (z + 1) 1/2 (z − 1) 1/2 We see that there are branch points at z = −1 and z = 1.
In particular we want the cos −1 z to be defined for z = x, x∈[−1, 1]. Hence we introduce the branch cuts on the lines (−•, −1] and [1, •). Let z + 1 = re iθ , z − 1 = ρ e iφ With the given branch cuts, the angles have the possible ranges −π ≤ θ > π, 0 ≤ φ > 2π Now we must determine if the logarithm introduces additional branch points. The only possibilities for branch points are where the argument of the logarithm is zero. z + (z 2 − 1) 1/2 = 0 or, z 2 = z 2 − 1 ⇒ 0 = −1 We see that the argument of the logarithm can not be zero and thus there are no additional branch points. Here the Riemann surface consists of two sheets S 0 and S 1 joined on the real axis (−•, −1] ∪ [1, •) : S 0 1 0 2 2 4 3 ≤ > ≤ > ≤ > ≤ > φ π π θ π π φ π π θ π − S
A particular branch of this function can be obtained by first taking z + 1 =
re iθ , −π ≤ θ > π; z − 1 = ρe iφ , 0 ≤ φ > 2π Then adding these relations, we find z = (re iθ + ρe iφ)/2 and the function z + (z 2 − 1) 1/2 reduces to z z re e r e i i i + = + + + (−) ( ) / / ( )/ 2 1 2 1 2 2 1 2 θ φ θ φ ρ ρ = + + re r e r e i i i θ φ θ φ θ ρ ρ 2 1 2 2 (−) (−)/ Fig. 53 −1 1 y x
80 = + re r e i i θ φ θ ρ 2 1 2 2 (−)/ Then cos − log log −1 (−)/ z i r e r e i i = + +
2 1 2 2 θ φ θ ρ on S 0 . If a point lying on the sheet S 0 is allowed to travel a path making a complete round about only the branch point z = 1, it enters to the sheet S 1 from the sheet S 0 . In this case the value of φ changes by 2π while the value of θ remains unchanged. The change in (φ−θ)/2 is π. So in this case, cos − log log − −1 (−)/ z i r e r e i i = +
2 1 2 2 θ φ θ ρ on S 1 . Similarly we can analyse the case when the point on S 0 encloses only the branch point z = −1 while travelling a complete round. Some standard branch cuts of elementary functions. Function Branch cuts z s , non integral s with Re s < 0 (−•, 0) z s , non integral s with Re s ≤ 0 (− •, 0] e z none log z (−•, 0] sin −1 z, cos −1 z (− •, −1] and [1, •) tan −1 z y ≤ −1, x = 0 and y ≥ 1, x = 0 cosec −1 z, sec −1 z (−1, 1) cot −1 z [−i, i] sinh −1 z y > −1, x = 0 and y < 1, x = 0 cosh −1 z (− •, 1) cosech −1 z −1 > y > 1, x = 0 sech −1 z (− •, 0] and (1, •) tanh −1 z y ≤ 1, x = 0 and y ≥ 1, x = 0 coth −1 z [−1, 1]
81 Exercises 1.
Find the principal value of each of the following complex quantities : (a) (1 −i) 1+i (b) 3 3−i (c) 2 2
i 2. Give the number of branches and locations of the branch points for the functions. (a) cos (z 1/2) (b) (z + i) −z 3. Determine the branch points of the function ω = {(z 2 − z)(z + 2)} 1/3 4. Find the branch points of (z 1/2 −1) 1/2 in the finite complex plane. Introduce branch cuts to make the function single-valued. 5. Let D be the complex z-plane with a cut along the segment [−1, 1], determine the regular branches of the function f z z z ( ) − / = + 1 1 1 2 6. Split the function f z z z ( ) (−)(−) = 2 2 4 9 into two regular branches in the domain D C: \{[−3, −],[2,]} / 2 3 7. Evaluate (i) x x α α 2 0 1 1 1 − , − • > > dx (ii) log x x 2 0 1+ • dx 8. Prove that logsin − log . xdx = π π 2 0 9. Construct a Riemann surface for the following functions : (i) ω = z 1/3 (ii) ω = (z 2 + 1) 1/2 (iii) ω = + log − z z 1 1 (iv) ω = sin −1 z.

82 O −i i 10. Let f(z) have branch points at z = 0 and z = ± i but nowhere else in the extended complex plane. How does the value and argument of f(z) change while traversing the contour given in the figures 51(a) (b). Do the branch cuts make the function single valued? O Fig. 54 (b) −i i Fig. 54 (a) < < < < < < < < < <
83 Unit 5 Conformal Equivalence Structure 5.0 Objectives 5.1 Riemann Mapping Theorem 5.2 The Schwarz Reflection Principle 5.3 The Schwarz-Christoffel Transformation 5.4 Examples : Triangles / Rectangles 5.0 Objectives of this Chapter The concept of conformal equivalence of two regions will be introduced in this chapter. The main theorem of this chapter is Riemann mapping theorem. Also Hurwitz's theorem, Schwarz lemma, Schwarz reflection principle, Schwarz-Christoffel transformation will be studied and their applications will be shown through a few examples. 5.1 Riemann Mapping Theorem In the family of analytic functions that concern geometrical orientation, conformal mapping plays a leading role. As its consequences we shall present here a most important result named after G. F. B Riemann, known as "Riemann mapping theorem". Throughout H(G) will denote the family of analytic functions defined on the region G. Definition : Conformal Equivalence : Two regions R 1 and R 2 are said to be conformally equivalent if there exists a f ∈ H (R 1) such that f is one-to-one in R 1 and f(R 1) = R 2 i.e. if there exists a conformal mapping one to one of R 1 onto R 2 . Clearly, this is an equivalence relation (reflexive, symmetric and transitive).
Theorem 5.1 [Hurwitz's Theorem] Let G be a region and {f n } be a
sequence in H(G) that converges uniformly to
f∈ H(G). Suppose f ≠ 0, D (a, R) ⊂ G and f(z) ≠ 0 on γ : |z−a| = R. Then there exists an integer N such
that for n ≥ N, f n and f have the same number of zeros
in
D(a, R).
84 Proof. Since f(z) is never zero on the circle γ, we have Inf f z γ δ ( )= <0 Again, f n → f uniformly on γ, so
there is an integer N such that for n ≥ N
sup ( )− ( ) γ δ
f z f z n > 2 and thus on the circle γ, f z f z f z n ( ) − ( ) ( ) > > ≤
δ δ 2 for n ≥ N. Using Rouche's theorem we find that
f n
and f have the same number of
zeros
inside the circle
γ : |z−a| = R for n ≥ N. By means of the above theorem, we can easily prove Corollary 1. Let G be a region and {f n } be a sequence in H(G) such that each f n never vanishes in G. Suppose f n → f uniformly in H(G). Then f(z) never vanishes in G, unless f ≡ 0. Some useful results (i)
If f(z)
is analytic at z 0 and f 1 (z 0) ≠ 0, then
there is a neighbourhood of z 0
in which f(z) is univalent. (ii) An univalent analytic function f on a domain G has a non-zero derivative at every point of G, i.e., f 1 (z) ≠ 0 on G. (iii) The inverse of an univalent analytic function is analytic. (iv) Any domain in /C , that is conformally equivalent to a simply connected domain must itself be simply connected. (v) A domain D in /C is simply connected if and only if every analytic function in D has a primitive in D. Schwarz Lemma Let f : D (0, 1) → D (0, 1) be an analytic function which maps the unit disc D(0, 1) to itself. If f(0) = 0, then (i) |f (z)| ≤ |z| for 0 ≤ |z| > 1 (ii) |f 1 (0)| ≤ 1 (iii) if equality holds in (i) for at least one z∈D (0, 1) − {0}, or, if equality holds in (ii), then f(z) = λ z, where λ is a constant, |λ| = 1. Proof : Let us consider the function g z f z z ( ) ( ) =
85 which is analytic in the disc D(0, 1) −{0} and it has removable singularity at z = 0, since f(0) = 0. It can be made analytic at z = 0 if we define g f z z f z ( ) lim ( ) ( ) 0 0 0 1 = = → (55) For |z| = r, where 0 > r > 1 g z f z z r ( ) ( ) = > 1 By the Maximum Modulus Principle, |g(z)| > 1 / r for the entire disc |z| ≤ r. We fix z∈D (0, 1) −{0} and let r → 1. Then |g (z)| ≤ 1. This is true for all z∈
D (0,1) −{0} and we get
f z z z ( ) ,≤ > > 1 0 1 (56) i.e. |f(z)| ≤ |z|, 0 > |z| > 1. Since f(0) = 0, we have |f(z)| ≤ |
z| for 0 ≤ |z| > 1.

So, (i) is proved and we find from (55) that |g (0)| = |f 1 (0)| ≤ 1 which proves (ii) To prove (iii), we observe that if at a point z 0 ≠ 0 (|z 0 | > 1) |g(z 0)| 1 = 1 i.e. |g(z)| attains its maximum at an internal point and hence by the maximum modulus principle g(z) = λ, a constant and that |λ| = 1, so f(z) = λz. Theorem 5.2 Let a∈D (0, 1). Then φ a defined by φ a z z a az ( ) − − = 1 maps D (0, 1) onto D (0, 1). Proof. Clearly, φ a is a bilinear transformation, it is analytic in the whole complex plane except the point 1 a (which is the inverse point of the point a with respect to the circle |z| = 1, and hence lies outside |z| = 1). We observe that φ φ
a
a z z a az a a z a az − ( ) − − = + + + + 1 1 1 = z a a 1 1 2 2 − − = z =
φ −a (f a (
z)),
similarly.
86
Thus φ a maps D (0, 1) onto D (0, 1) in a one to one way. Now let θ be a real number. Then φ θ θ θ
a i i i e e a ae = − −1 = = = e a e a e e e a e a
i i i i i θ θ θ θ θ − − − − − 1 1 i.e., φ a maps |z| = 1 on |z| = 1. Thus, φ a maps D (0, 1) onto D (0, 1). A maximal problem Let α, β be two complex numbers with |α| > 1, |β| > 1 and f be analytic on D(0, 1) satisfying f(α) = β. What is the maximum possible value of |f 1 (α)| among such mappings? Solution : Let g = φ β 0 f 0 φ −α where φ β is defined as in theorem 5.2 (57) Then g maps D (0, 1) to D (0, 1) and satisfies g(0) = φ β {f(φ −α (0))} = φ β {f(α)} = φ β (β) = 0 Thus g satisfies all the conditions of Schwaz's lemma and hence |g 1 (0)| ≤ 1. To obtain an explicit form of g 1 (0), we use (57) and apply the chain rule g 1 (0) = {(φ β 0f) 1 (φ −
α (0)}φ 1 −α (0) = (φ β 0f) 1 (α) (1−|α| 2) = φ β 1 (f(α))f 1 (α) (1−|α| 2) = φ β 1 (β)f 1 (α)(1−|α| 2) = 1 1 2 2 1 − − ( ) α β αf But |g 1 (0)| ≤ 1, therefore f 1 2 2 1 1 ( ) −| | −| | α β α ≤ (58)
Equality in (58) occurs only when |g 1 (0)| = 1. In that case by virtue of Schwarz
87 lemma there is a constant λ, |λ| = 1 so that g(z) = λz. Hence, f(z) = φ − β {λφ α (z)}, z ∈D (0, 1) (59) We now present an important consequence of Schwarz's lemma, which may be seen as the converse form of theorem 5.2. Theorem 5.3 : Let f : D (0, 1) → D (0, 1) be any conformal map of the unit disc onto itself and f(a) = 0, a∈D (0, 1). Then there is a constant λ, |λ| = 1 such that f(z) = λφ a (z) where φ a is defined as in theorem 5.2. Proof. Since f is a conformal map from D(0, 1) to D (0, 1), we can have inverse of f, g defined by g {f(z))} = z, which is analytic too. Applying the chain rule g 1 (0) f 1 (a) = 1 (60) But according to inequality (58), f and g have to satisfy f a a 1 2 1 1 ( ) − , ≤ g a 1 2 0 1 ( ) − ≤ (61) (since, f(a) = 0 and g(0) = a). From (60), (61) it follows that |f 1 (a)| = (1 − |a| 2) −1 . Hence applying the result (59) we find that f(z) = λφ a (z) for some λ with |λ| = 1. Lemma 5.1 : Let G be a simply connected region and {f n } be a sequence of injective analytic mappings (conformal mappings) of G into /C which converges uniformly on every compact subset of G, then the limit function f is either constant or injective. Proof. Suppose f is not constant and not injective. Then there exist two points ς and η∈G, ς ≠ η such that f(ς) = f(η) = ω 0 , say. Let g n (z) = f n (z) − ω 0 . We can find a positive δ, δ > |ς−η|/2 so that the discs D(ς, δ) and D(η, δ) are included in G. Now g(z) = f(z)−ω 0 never vanishes on the circles |z − ς| = δ and |z − η| = δ, where g z g z n n ( ) lim ( ) = →• . Applying Hurwitz's theorem, for large n, there exists ς n lying inside the circle |z − ς| = δ with g n (ς n) = 0 as g n → g uniformly in G. Similarly, for all large n, there is η n within |z−η| = δ with g n (η n) = 0. But by construction, D(ς, δ) ∩ D (η, δ) = φ and hence ς n ≠ η n . Thus g n (ς n) = g n (η n) = 0, ς n ≠ η n that is, f n (ς n) = f n (η n), ς n ≠ η n
88 contradicting the injectivity of each f n and the proof follows. NOTE : There is no conformal map f of the unit disc D (0, 1) onto the whole complex plane /C because then the inverse function f −1 : /C → D (0, 1) would be a bounded entire function which is not constant, contradicting the Liouville's
theorem. Open mapping theorem : Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, f(U) is open.
Proof : Omitted. Uniform boundedness : A sequence of functions {f n } defined on a set D
is said to be uniformly bounded on D if
there exists a constant M < 0 such that |f
n (z)| ≤ M for all n and for all
z∈D. Normal family : Let F be a family of functions in a region G. The family F is said to be normal in G if every sequence {f n } of functions f n ∈F contains a subsequence {f n k } which converges uniformly on every compact subset of G.
Montel's theorem : A family F in H (G) is normal

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if and only if F is uniformly bounded
on every compact subset of G. Proof : Omitted.
Theorem 5.4 : [Riemann Mapping Theorem] Let G be a simply connected region,
except for /C itself and let a∈G. Then there is a unique conformal map f : G → D (0, 1) of G onto the unit disc which satisfies f(a) = 0 and f 1 (a) < 0. Proof. Let us first prove that f is unique. If there was another conformal map g : G → D (0, 1) with the given properties, then fog −1 : D (0, 1) → D (0, 1) would be a conformal map and also (fog −1) (0) = f(a) = 0 Hence, applying Theorem 5.3, we find that there is a constant λ with |λ| = 1 (fog −1) (z) = λz Deriving the derivative at the origin, we find ( ) ( ) ( )( )( ) ( ) ( ) ( ) ( ) ( ) , −1 −1 −1 −1 fog f g g f a g g f a g a ' = ' ' = ' ' = ' ' < 0 0 0 1 0 0 from which it follows that λ is positive. But also |λ| = 1, so λ = 1. Thus fog −1 is an identity map and f = g. The proof of existence is divided into several stages. Lemma 5.2 Let G be a simply connected region other than /C . Then there exists an injective analytic map f on G with f(G) ⊂ D (0, 1). Proof. We choose a point b ∈ /C \G. Since G is simply connected there exists a g : G → /C analytic with
g 2 (z) = z − b.
89 Here g is injective since
g(z 1) = g(z 2) ⇒ g 2 (z 1) = g 2 (z 2) i.e. z 1 − b = z 2 − b ⇒ z 1 = z 2 .
By
open mapping theorem g(G) is open. Let us pick ω 0 ∈g(G) and choose r < 0 so that D(ω 0 , r) ⊂g(G). Then D(−ω 0 , r) ⊂ /C \g(G). For, if there exists a point ω∈ D(−ω 0 , r) ∩ g(G), then ω = g (z 1) for some z 1 ∈G and also −ω∈D (ω 0 , r) ⊂ g (G), so that −ω =
g(z 2) for some z 2 ∈
G. Again,
g(
z 1) = −
g(
z 2) ⇒ g 2 (z 1) = g 2 (z 2) or, z 1 − b = z 2 − b i.e. z 1 = z 2 or, g(z 1) = g(z 2) = −g(z 1) ⇒
g(z 1) = 0 ⇒ 0 = g 2 (z 1) =
z 1 −
b i.e. z 1 = b∈ /
C \G contradicting z 1 ∈G.
We take f z r z ( ) [
g( )] = + 2 0
ω (62) Then f is injective analytic map on G (by construction |g(z) + ω 0 | ≥ r for z∈G) and also satisfies f z G ( ) . ≤ > ∈ 1 2 1 for z Lemma 5.3 : Let G be a simply connected region other than /C itself and let a∈G be fixed. Then there exists a conformal map f : G → D(0, 1) of G onto the unit disc with the properties f(z) = 0 and f(a) < 0. Proof : Let F denote the family of analytic functions f : G → /C such that either f ≡ 0 or f is injective, and f(G) ⊂ (0, 1), f(a) = 0 and f′ (a) < 0. Let us consider the function ψ( ) ( )− ( ) − ( ) ( ) z f z f a f a z = 1 f where f(z) is given by (62) of lemma 5.2 and we find that ψ(G) ⊂ D (0, 1), ψ(a) = 0 and ψ 1 (a) < 0. So F is non empty and by Montel's theorem it is normal. Applying Lemma 1 we see that all functions in the closure of F in H(G) are either constant or injective. Now since all functions in F take the value zero at a, the same is true for all functions in the closure of F. Likewise the only constant function in the closure is
90 0 while the other functions in the closure satisfy f(G) ⊂ D (0, 1). Since f(G) is open, by open mapping theorem, f(G) ⊂ D (0, 1). Again since the f → f 1 (a) is continuous, all functions in the closure of F must satisfy f 1 (a) ≥ 0. The functions in the closure, that are not identically zero, are injective, so f 1 (a) < 0 unless f ≡ 0. These observations prove that the set F is closed in H(G). Hence F is compact in H(G). Since the map f → f′(a) : F → R is a continuous function on a compact set, it must attain its maximum value, as we are not considering constant function (here it is zero). Let f∈F be a function with f′(a) maximum. We now show that f(G) = D (0, 1). On the contrary, suppose that f(G) ≠ D (0, 1) and choose w∈ D(0, 1)\f(G). Using the property that every non-vanishing analytic function in a simply connected region has an analytic square root, we take a function h ∈H(G) with [( )] ( )− − ( ) h z f z f z 2 1 = ω ω (63) Now as the bilinear transformation φ a z z a az ( ) − − = 1 maps D (0, 1) onto D (0, 1) and as f∈F,
h(G) ⊂ D (0, 1). Let g : G → /C defined by
g z h a h a h z h a h a z ( ) ( ) ( ) ( ) − ( ) − ( ) ( ) = ' ' · 1 h Then clearly, g(G) ⊂ D (0, 1), g(a) = 0 and g is analytic injective and g′(a) < 0, since ′ = · g a h a h a h a h a h a ( ) ( ) ( ) ( )[− ( )] [− ( )] 1 1 1 2 2 2 1 1 = < h a h a 1 2 1 0 ( ) − ( ) (64) So, g∈

F. Again, differentialing (63) we find that 2h(a)h 1 (a) = f 1 (a)(1−|ω| 2) So, from (64) g a h a h a h a h a
f a 1 1 2 1 2 1 1 2 1 ( ) ( ) ( ) ( )(− ( ) ( )(− (−) , = = ω ω ω as |h(a)| 2 = |ω|
91 = + < f a f a 1 1 1 2 ( ) ( ). ω ω contradicting the choice of f∈F as maximising f 1 (a). Thus f(G) = D (0, 1). Note : The Riemann mapping theorem is one of the most celebrated results of complex analysis. It is the beginning of the study of complex analysis from a geometric view point. G. F. B. Riemann in 1851 correctly formulated the theorem, but unfortunately his proof of the theorem was lacking. According to various accounts, he assumed but did not prove that a certain maximal problem had a solution. A final proof was definitely known by the early 20th century, different sources attributed to it particularly, W. F. Osgood, P. Koebe, L Bieberbach etc. 5.2 The Schwarz Reflection Principle Let
f be analytic in the domains D 1 , D 2 which have a common piece of boundary, a smooth curve γ. Assume further that f is continuous across γ. Then, by Morera's theorem, f is analytic in D 1 ∪ D 2 . This allows us to perform analytic continuation in some cases.
Theorem 5.5 [The Schwarz reflection principle] Given a function f(z) analytic in a domain D lying in the upper half plane whose boundary contains a segment I ⊂ IR,
assume f is continuous on D ∪ I and real-valued on I. Then f has analytic continuation across I, in a domain D ∪
I U
D*, where D* = ∈ { : }. z z D Proof. Let us consider the function
f z
f z f z z DUI z D UI ( ) ( ) ( ) , = ∈ ∈ ∗ It is clear that F is analytic in D. We shall show that F is also analytic in D*.
Let z and z + h lie within D*. Then z and z h + lie within D and we can express. lim ( )− ( ) lim ( ) − ( ) lim ( )− ( ) ( ).
h h
h
F
z h F z h f z h f z h f z h f z h f
z → → → + = + = +
= ′ 0 0 0
So, F
is
analytic
in D*. F is also continuous on D*U I. For, z ∈ I lim ( ) lim ( ) ( ) ( ), z x z x F z f z
f x f x → → = = = by hypothesis. Thus F is continuous on D U I U D*. To prove F is also analytic there, we consider the function
92 φ π ς ς ς ( ) ( ) − z i F z d = 1 2 Γ (65) It is analytic in D U I U D* [as (i) F z ( ) − ς ς is continuous function of both variables when z lies within Γ and ς on Γ. (ii) for each such ς ς ς , ( ) − F z is analytic in z in D U I U D*. [see (14)]. To complete the proof, we try to establish φ(z) = F(z) for all z ε D U I D*. Breaking the integral in (65) and adding the two integrals along I, which are in opposite directions, we write φ π ς ς ς π ς ς ς ( ) ( ) − ( ) − z i F z i F z d = + 1 2 1 2 1 2 Γ Γ d (66) where Γ 1 and Γ 2 are the boundary of D U I and D* U I respectively. When z∈DUI, the second integral in (66) vanishes and φ(z) = F(z). Again, the first integral vanishes when z ε D* U I and φ(z) = F(z) in this case too. Thus φ(z) = F(z) for all z∈ D U I U D* and we have found a function F(z), analytic in D U I U D*, and coincides with f(z) in D U I. 5.3 The Schwarz-Christoffel Transformation We know from Riemann's mapping theorem that there is a conformal mapping which maps a given simply connected domain onto another simply connected domain, or equivalently onto the unit disc. But it does not help us to determine such mappings. Many applications in boundary-value problem requires construction of one-to-one conformal mapping from the upper half plane Im z < 0 onto a polygon Ω in the w-plane. Two German mathematicians H. A. Schwarz and E. B. Christoffel independently discovered a method for finding such mappings during the years 1864-1869. Theorem 5.6 [Schwarz and Christoffel] Let P be a polygon with vertices w 1 , ...w k in the anticlockwise direction and interior angles α 1 π , ..., α k π respectively, where −1 > α 1 , ..., α k > 1. Then there exists a one-to-one conformal mapping of the form f z A s x s x s x B k k z z ( ) (−) (−) ...(−) − − − − − = + 1 1 2 1 1 1 1 2 0 1 α α α ds (67) I x D D* z Γ z − Fig. 54

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93 where A, B∈ /C , that maps the upper plane Im z < 0 onto the interior of P, with f(x 1) = w 1 ...., f(x k−1) = w k−1 , f(•) = w k . (68) Remarks : (i) We do not need to have specific information on w k and α k . While travelling the polygon anticlockwise direction we made a left turn of an angle π−α j π at the vertex ω j . (ii) Sometimes certain infinite regions can be thought of as infinite polygons. In this case it is convenient to take w k as the point at infinity, as we need no information on α k . (iii) It can be shown that Schwarz-Christoffel transformation can be uniquely determined by three points as in the case of bilinear transformation. One of these is used by taking f(•) = ω k . We can therefore have the freedom to choose two points say, x 1 and x 2 satisfying −• > x 1 > x 2 > •. (iv) Note that the integral involved may be impossible to calculate theoretically. In practical problems numerical techniques are often used to evaluate the integral. In first part of the proof we take f(x k) = ω k , x k = finite. Proof. By Riemann mapping theorem such a mapping exists. We shall prove that its form is given by (67). So f(z) is analytic for Im z < 0 and f 1 (z) ≠ 0 in the upper half plane. From these it is clear that d dz f z f z f z log ( ) ( ) ( ) ′ = ″ ′ is analytic in the upper half plane. To construct the function f(z) our aim is to establish that f''(z)/f'(z) is analytic for Im z ≥ 0 save for the pre-image points of the vertices of the polygon lying on the real axis. Let l be a side of the polygon P, which makes an angle θ (positive sense) with the real- axis and ς be any point on l but not a vertex of the polygon P. Then for any ω on l, (ω−ς)e −i θ is real and there is a point z on the real axis of the z-plane so that f(z) = ω and a corresponding point z = a for ς on the same line. Hence {f(z) − ς}e −i θ is real and continuous on the segment γ of the real axis of the z-plane corresponding to the straight line l of the ω-plane. Moreover, this function is also analytic for Im z < 0, thus following the Schwarz reflection principle we can continue this function analytically across γ to the lower half plane Im z > 0. In particular, this function is analytic in a neighbourhood of the point z = a and can be expanded in the form of the Taylor series. Fig. 55 a θ w-plane l
94 { ( ) − } (−) − f z e c z a i k k k ς θ = = • ∑ 1 where c 1 = f′(a) ≠ 0, maintaining the status quo that f(a) = ς and the function f maps the segment γ onto the straight line l. Now f′(z) = e iθ {c 1 + c 2 2(z − a) + ...} and logf′(z) = iθ + log{c 1 + 2c 2 (z − a) + ...} So, d dz f z log ( ) 1 is analytic in a neighbourhood of z = a and real on a real line segment intercepted by the neighbourhood. Let us consider the case when the point ς is the corresponding point at infinity on γ (in this case γ is divided into two parts, each of infinite lenght). Here the Taylor series expansion in the neighbourhood of point at infinity { ( ) − } / − f z e c z i k k k ς θ = = • ∑ 1 where each c R is real and c 1 ≠ 0 (with the same reason mentioned in the finite case). So ′ =

f

z e

c z

c z c z i ( ) − − − −... − θ 1 2 2 3 3 4 2 3 ″ = + + + f z e c z c z c z i ( ) ..... − θ 2 6 12 1 3 2 4 3 5 and we find that ″ ′ = + + + + + = + + + f

z f

z

z c c z c z c z c c z c c c z c c

z ( ) ( ) ... − / ... − ... − / ... −3 −2 2 6 12 1 2 1 2 6 1 2 1 2 3 2 1 2 1 1 1 2 2 1 = + = • ∑ − ~ 2 2 z

c

z

k

k k (69) d dz f z log ( ) 1 is analytic in a

neighbourhood of the

point at infinity and is real when z is real. In the polygon P, let 1 be an adjacent side to making on angle α 1 π at their point of intersection ω 1 . The corresponding point of ω 1 on the real axis is x 1 . Here

95 the function f(z) is not analytic in a neighbourhood of x 1 , we choose the branch of the argument so that π π 2 3 2 1 > > Arg z x (−) introducing a branch cut along the axis {x 1 + iy : y ≤ 0} [f'(z) is not continuous on this branch cut]. Here Arg {(ω 1 − ω)e −iθ } is equal to zero or α 1 π according as ω lies on or 1 . So the function [{ − ( )}] − / ω θ α 1 1 1 f z e i is real and continuous on the segment of the real axis corresponding to the consecutive sides and 1 . Again this function is analytic for Im z < 0 since f(z)−ω 1 is analytic and non zero there. Expanding { − ( )} − / ω θ α 1 1 1 f z e i in Taylor's series in a neighbourhood of x 1 we find { − ( )} (−) − / ω θ α 1 1 1 1 1 f z e c z x i k k k = = • ∑ where each c k is real and c 1 ≠ 0. On simplifying, we find f z e z x c z x i ( ) − (−) [c (−) ...] = + + ω θ α α 1 1 1 2 1 1 1 = + = • ∑ ω θ α 1 1 1 1 0 1 e z x c z x i k k k (−) (−) where c 0 1 is a constant multiple of c 1 , hence not equal to zero. Now we have ' = + + + f z e z x c c z x i ( ) (−) ( ) (−) ... − θ α α α 1 1 1 0 1 1 1 1 1 1 1 = (z − x 1) α 1 −1 F(z) Fig. 56 Fig. 57 new position after rotation through an angle θ clockwise α 1 π A α 1 π θ A w α 1 π w 1 1 1

96 where F(z) is analytic and not zero in a neighbourhood of z = x 1 and we obtain d dz f z z x F z F z log ( ) − − ( ) ( ) 1 1 1 1 1 = + α (70) This shows that if the polygon P has an angle α 1 π at a point ω 1 then d dz f z log ( ) 1 will have a simple pole of residue α 1 −1 at its corresponding point x 1 . Now if the point at infinity be the corresponding point to ω 1 at which the polygon P has an angle α 1 π, then we can express ω θ α 1 1 1 2 2 1 − ( ) .... − / f z e c z c z i = + + or, f z e c z c zc i ( ) − ... = + + ω α θ α 1 1 1 2 1 1 1 ' = + + + + f z e c z c zc e c z c z c i i ( ) ... − − −... θ α α θ α α α α 1 1 1 1 2 1 1 1 2 2 1 1 1 1 1 = + + +
+ e c z c zc iθ α α α α 1 1 1 1 2 1 1 1 1 1 ( ) ... '' = + + + + + + + + f z e c z c zc e c z c z c i i ( ) − ( ) ( ) ... −( ) −... θ α α θ α α α α α α α 1 1 1 2 1 2 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 = + + + +
+ − ( ) ( ) ... e c z c zc iθ α α α α α 1 1 1 2 1 2 1 1 1 1 1 2 d dz f z f z f z z c zc c zc log ( ) ( ) ( ) − ( ) ... − ( ) .... ' = '' ' = + + + + + + α α α 1 1 2 1 1 2 1 1 1 2 1 1 = + + + + − (− −) .... α α α 1 1 1 2 1 1 1 2 1 z c zc = + + = • ∑ − ~ α 1 2 1 z c z k k k (71) Now since x 2 , x 3 ..., x k are the corresponding points lying on the real-axis of the z-plane, to the vertices w 2 , w 3 , ...w k respectively of the polygon P with angles α 2 π,

97 α 3 π, ... α k π there, the function d dz f z log ( ) 1 will have simple poles with residue α j − 1 at x j , j = 2, ..., k. Thus we see that this function is analytic for Im z < 0 and continuous on Im z = 0 except the points x 1 , x 2 , ..., x k and using the Schwarz reflection principle it can be continued analytically across the real axis. Hence d dz f z log ( ) 1 possesses only simple poles at x 1 , x 2 , ... x k as its only singularities and can be expressed as d dz f z z x z x z x G z k k log ( ) − − − − ... − − ( ) 1 1 1 2 2 1 1 1 = + + + + α α α (72) where G(z) is a polynomial. When |z| is large enough α α i i i i i z x z x z x z x z i k − − − ... , ,...., 1 1 1 1 2 2 = + + + = So, d dz f z z x z x z G z i i i k i k i k log ( ) (−)/ (−)/ (−) / ... ( ) 1 1 2 1 1 1 1 2 1 3 1 1 1 1 = + + + + = = = ∑ ∑ ∑ α α α = + + = • ∑ − ( ) 2 2 z d z G z i i i (73) Using the property of the sum of the exterior angles of a polygon, (1 − α 1) π + (1 − α 2)π + ... (1 − α k)π = 2π. Comparing (73) with (69) we get G(z) identically zero. Finally integrating equation (72), we find the desired mapping f(z) as f z A s x s x s x ds B k z z k ( ) (−) (−) ...(−) − − − −1 = + 1 1 2 1 1 0 1 2 α α α (74) Role of constants A and B (i) |A| controls the size of the polygon (ii) Arg A and B help to select the position, if any, in determining orientation and translation respectively. An useful observation In some occasions we urge to make the evaluation process of the integral in (74) simple. For this sake, we consider the point at infinity corresponds to the vertex w k where the polygon P has an angle α k π. Then we can express [see eq. (71)] d dz f z z c z k i i log ( ) − ~ 1 2 1 = + • ∑ α (75)

98 in the neighbourhood of the point at infinity. Again considering the expression of d dz f z log ( ) 1 in the neighbourhood of the points corresponding to the vertices w 1 , w 2 ..., w k−1 [see eq. (70)]. d dz f z z x z x z x G z k k log ( ) − − − − ... − − ( ) − − 1 1 1 2 2 1 1 1 1 1 = + + + + α α α (75 1) where G(z) is a polynomial. If |z| is large enough, proceeding as earlier d dz f z z x z x z G z i i i k k i i k i log ( ) (−) / (−) / (−) / ( ) − − − 1 1 1 1 2 2 1 1 3 1 1 1 = + + + ∑ ∑ ∑ α α α = + + + • ∑ − ~ ( ) α k i z d z G z 1 1 2 (76) Comparing (76) with (75), G(z) turns out to be identically zero and hence integrating (75 1) we obtain f z A s x s x s x ds B k z z k ( ) (−) (−) ...(−) − − − −1 −1 = + 1 1 2 1 1 1 2 0 α α α where the role of the constants A and B remain as before. 5.4 Examples : Triangles / Rectangles The Schwarz-Christoffel transformation is expressed in terms of the points x j , not in terms of their images i.e., the vertices of the polygon. Not more than three points (x j) can be chosen arbitrarily. If the point at infinity be one of the x j 's then only two finite points on the real-axis are free to be chosen, whether the polygon is a triangle or a rectangle etc. Triangle Let the polygon be a triangle with vertices w 1 , w 2 and w 3 . The S-C transformation is written as w A s x s x s x ds B z z = + (−) (−) (−) − − − 1 1 2 1 3 1 1 2 3 0 α α α (77) where α 1 , π, α 2 π and α 3 π are the internal angles at the respective vertices. Fig. 58 Fig. 59 w-plane z-plane x 1 x 2 x 3 w 1 w 2 w 3 α 3 π α 1 π α 2 π

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99 Here we have chosen all the three finite points x 1 , x 2 , x 3 on the real-axis. The constants A, B control the size and position of the triangle respectively. If we take the vertex w 3 as the image of the point at infinity, the S-C transformation becomes w A s x s x ds B z z = + (−) (−) − − 1 1 2 1 1 2 0 α α (78) Here x 1 and x 2
can be chosen arbitrarily. Example 1 : Find a
Schwarz-Christoffel transformation that maps the upper half- plane to the inside of
the triangle with vertices −1, 1
and √3i. Solution : Following our notation, we write w 1 = −1, w 2 = 1 and w 3 = √3i so that α 1 = α 2 = α 3 = 1/3. We choose the form (78) of S-C transformation and consider the mapping. f z A s x s x ds B z ( ) (−) (−) , −2/ −2/ = + 1 3 2 3 0 [here f(•) = √3i] We may choose x 1 = −1 and x 2 = 1, so that f(−1) = −1 and f(1) = 1. Therefore f z A s s ds B z ( ) ( ) (−) −2/ −2/ = + + 1 1 3 3 0 = + A s ds B z (−) −2/ 2 3 0 1 It then follows that = + = + = A s ds B s ds B (−) −1, (−) . −2/ −1 −2/ 2 3 0 2 3 0 1 1 1 1 A Rewriting these as − −1 , (−) −2/ AL B B s ds + = + = = and AL where L 1 1 2 3 0 1 We obtain A s ds and B = = 1 1 0 2 3 0 1 (−) . −2/ Hence Fig. 60 Fig. 61 −1 1 √3i −1 1
100 f z s ds s ds z ( ) (−) (−) . −2/ −2/ = 1 1 1 2 3 0 1 2 3 0 Example 2 : Using Schwarz-Christoffel transformation map the upper half-plane onto an equilateral triangle of side 5 units. Solution : It is convenient to choose three arbitrary points x 1 = −1, x 2 = 1 and x 3 = • which are mapped into the vertices of the equilateral triangle, so we take S-C transformation (78). f z A s s ds z ( ) ( ) (−) −2/ −2/ = + 1 1 3 3 1 Here, f(−1) = w 1 = 0 and f(1) = w 2 = 5. So that A s ds = 5 1 2 3 1 / (−) −2/ −1 Hence the desired transformation is f z s ds s ds z ( ) (−) (−) / / −1 = 5 1 1 2 2 3 1 2 2 3 1 Alternative : We take z 0 = −1, A = 1, B = 0 and find S-C transformation as, (choosing one of x i 's as point at infinity) w s s ds = + ( )(−) 1 1 2 3 1 2 (79) taking x 1 = −1 and x 2 = 1. Then ~ ( ) ~ , f w 1 2 = say, and the image of the point z = −1 is the point ~ w 1 0= . When z = 1 in the integral we can write s = x, where −1 > x > 1. Then x + 1 < 0 and Arg (x+1) = 0, while |x−1| = 1−x and Arg (x−1) = π. Hence ~ ( ) (−) / −1 / − w x x e dx i 2 2 3 1 2 3 2 3 1 1 = + π Fig. 62 Fig. 63 ((−1 1 w i w 2 w 3 − π 3 − π 3
101 = = − (−) − (−) / −1 e dx x x i i π π 3 2 1 2 0 1 1 2 1 2 3 3 2 3 e dx = − (−) , / / e dt t t iπ 3 2 3 0 1 1 substituting x = √t. = − , . / e B iπ 3 1 2 1 3 We choose w 2 as, w kw 2 2 5 = = ~ where k e B i = −5 , . − / / π 3 1 2 1 3 To find w 3 let us first calculate for ~ .w 3 ~ ( ) (−) −2/ −2/ −1 w
x
x dx 3 3 3 1 1 = + • = + + + • ( ) (−) ( ) (−) −2/ −2/ −1 −2/ −2/ x x dx x x dx 1 1 1 1 3 3 1 3 3 1 = + + • − , − / − −2/ −1 e B e x x dx i i π π 3 3 1 2 1 3 1 1 = + + • − , − − / − −2/ − e B e x x dx i i i π π 3 3 1 2 1 3 1 1 = + + + + • — − − − −2/ − −1 − −2/ −2 / e x e x
e dx i i i i i π π π π π 2 3 2 3 2 3 1 1 3 3 3 = + + • — ( ) (−) / −2/ −2/ −1 − e x x
dx 1 3 3 3 1 1 π
Now, the value of ~ w 3 can also be represented by the integral ( ) (−) −2/ −2/ − − x x dx i + • 1 1 3 3 when z tends to infinity along the negative real axis. Thus from the above relation, we have ~ − , ~ / / w e B e w i i 3 3 3 3 1 2 1 3 = + π π i.e., ~ − , / / w e e B i i 3 3 3 1 2 1 3 = · π π So, w kw e i 3 3 5 3 = = ~ π
102 Therefore, the three vertices of the equilateral triangle are w 1 = 0, w 2 = 5 and w 3 = 5e iπ/3 . Clearly each of it's side is of length 5 unit. The desired transformation is then f z Kf z ( ) ~ ( ) = = + −5 , ( ) (−) − / −2/ −2/ −1 e B s s ds i z π 3 3 3 1 2 1 3 1 1 which is same as obtained in the first process. Remark : Following the above technique we can determine a S-C transformation from Im z ≥ 0 onto a triangle, in particular, whose one side opposite to an angle is given. Rectangle : Example 3 : Find a S-C
transformation that maps the upper half
of the z-plane to the
inside of the rectangle in the w-plane
with vertices −a, a, a + ib and −a + ib which are the preimages of −1, 1, α and −α respectively. Solution : Let us first make the identification of the vertices of the rectangle w 1 = −a + ib, w 2 = −a, w 3 = a, w 4 = a+ib α 1 = α 2 = α 3 = α 4 = 1/2 We choose x 1 = −α, x 2 = −1, x 3 = 1, x 4 = α where α < 1 will be determined later. We are attempting to benefit from the symmetry here, which requires the image z = 0 to be w = 0. So taking z 0 = 0 we get B = 0 in the formula (74) for S-C transformation, which reduces to f z A s s s s ds z ( ) [ )( )(−)(−)] −1/ = + + α α 1 1 2 0 Fig. 64 Fig. 65 ∧∧∧∧∧ ∧ ∧ ∧ ∧ ∧ −a + ib a + ib −a a v u −2 −1 1 2 x y 0 0

103 = ≡ A ds s s z z [(−)(−)] ((,)) 1 2 2 2 0 α φ α (80) The constant A may be found by using the fact that f(1) = a i.e., a A ds s s a ds s s = = [(−)(−)] / [(−)(−)] 1 1 2 2 2 0 1 2 2 2 0 1 α α or A = a/φ(α), say (81) To find α, we apply f(α) = a + ib, a ib a ds s s a + = φ α α ( ) [(−)(−)] 1 2 2 2 0 = + a ds s s i ds s s φ α α α α ( ) [(−)(−)] [(−)(−)] 1 1 2 2 2 2 2 2 1 0 1 from which, equating imaginary parts, we arrive at b ds s s φ α α α α ( ) [(−)(−)] = 2 2 2 1 1 Since a and b are known, this equation determines α, which gives rise to the evaluation of φ(α) i.e. A is completely known. Note : The function φ(z, α), given in (80), which involves z as the upper limit of an integral, is called an elliptic integral of the first kind and it is not an elementary function. The real definite integral φ(α) in (81) is called a complete elliptic integral of the first kind.

Example 4 :

Find a Schwarz-Christoffel

transformation that maps the upper half

of the z-plane to the

vertical semi-infinite strip −π/2 > u > π/2,

v < 0 of the w-plane. Solution : Fig. 66 Fig. 67 w-plane z-plane −1 1 0 − − π 2 − π 2

104 Here we take x 1 = −1, x 2 = 1 and x 3 = • and the image points are w 1 = −π/2 and w 2 = π/2 respectively, so that a S-C transformation can be written as f z A s s ds B z z ( ) ( ) (−) −1/ −1/ = + + 1 1 2 2 0 = + A s B z z 1 1 2 1 2 0 (−) / ds = + ~ log − ~ A iz z B 1 2 Using f(−1) −= π 2 and f( ) , 1 2 = π we find f z i iz z ( ) − log − , = + 1 2 Choosing a suitable branch of the logarithm.

105 Unit 6 Entire and Meromorphic Functions Structure 6.0 Objectives 6.1 Entire function 6.2 Infinite Products 6.3 Infinite product of functions 6.4 Weierstrass Factorization 6.5 Counting zeros of analytic functions 6.6 Convex functions 6.7 Order of an entire function 6.8 The function n(r) 6.9 Convergence exponent 6.10 Canonical Product 6.11 Hadamard's Factorization Theorem 6.12 Consequences of Hadamard's Theorem 6.13 Meromorphic functions 6.14 Partial Fraction Expansions of Meromorphic Functions 6.15 Partial Fraction Expansion of Meromorphic functions Using Residue theorem 6.16 The Gamma Function 6.17 A few properties of ΓΓΓΓ(z) 6.0 The Objectives of the Chapter In this chapter we shall study entire functions, their growth properties and meromorphic functions. Infinite products and their convergence will be discussed. Properties of zeros of

106 an entire function, convex functions, gamma function and its important properties will also be discussed. 6.1 Entire function A function f(z)

analytic in the finite complex plane is said to be entire (or

sometimes integral) function. Clearly, the sum, difference and product of two or more entire functions are entire functions. Examples : The polynomial function P(z) = a 0 + a 1 z + ... + a n z n , exponential function e z , sin z, cos z etc. are entire functions. Let us consider the first example, the polynomial function. It is evident that P(z) can be uniquely expressed as a product of linear factors in

the form

A

z z z z z z n 0 1 2 0 1 1 1 0 − − − ≠ , if a or, A z z z z

a

a

a

p p n p p p 1 1 1 0 0 1 2 0 1 1 − − − = = = ≠ − − ς ς ς , , , if a (82) where A 0 (or, A p) is constant and z = z 1 , z 2 , ..., z n (or, z = 0, ς 1 , ς 2 , ..., ς n−p) are the zeros of P(z), multiple zeros are counted according to their multiplicities. There arises a natural question : whether any entire function can be expressed in a similar manner in terms of its zeros. The observations are as follows : (i) There may exist entire function which never vanishes, (ii) If an entire function possesses finite number of zeros, then it is always possible to express it in the form (82) stated above. But when the number of zeros are infinite the form (82) reduces to a product of infinite number of linear factors which need not always be convergent. We first consider infinite products of complex numbers and functions. 6.2 Infinite Products An infinite product is an expression of the form p n n= • ∏ 1 (83)

107 where p 1 , p 2 , ..., p n , ... are non-zero complex factors. If we allow any of the factors be zero, it is evident that the infinite product would be zero regardless of the behaviour of the other terms.

Let P n = p 1 p 2 ...p n . If P n tends to a finite limit (non-zero) p as n tends to infinity, we say that the infinite product (83) is convergent and write as p p n n= • ∏ = 1 (84) An infinite product which does not tend to a non-zero finite limit as n tends to infinity is said to be divergent. To find the necessary condition for convergence for the infinite product p n n= • ∏ 1 , say (84) holds, then writing p n as p P P n n n = −1 we conclude in view of (84) that
lim lim n n n n n p P P P P →• →• − = = = 1 1
Thus, lim n n p →• = 1 (85) is a necessary condition for convergence of the infinite product (83). It is then better to write the product as ( ) 1 1 + = • ∏ a n n (86) so that a n → 0 as n → • is a necessary condition for convergence. Theorem 6.1 : The infinite product (86) converges if and only if log( ) 1 1 + = • ∑ a n
n (87) converges. We use the principal branch of the log function and omit, as usual, the terms with a n = −1.
Proof. Let P a a n k k n n k k n = + = + = = ∏ ∑ ( ) log( ). 1 1 1 1 and S Then log P n = S n and P n = e Sn . Now if the given series is convergent i.e. S S n → as n → •, P n tends to the limit P = e S (≠ 0). This proves the sufficiency of the condition.
108 Conversely, assume that the product converges i.e. P P n → (≠ 0) as n → •. We shall show, by virtue of P n = e Sn , that the series (87) converges to some value of log P, not necessarily the principal value of log P. For n P P P P n n → • → → , . 1 0 and Log Now there exists an integer K n such that Log P P S Log P k i n n n = − +2 π (88) To establish the convergence of the sequence {k n }, we form the difference ( ) ( ) k k i Log P P Log P P Log a n n n n n + + + − = − − + 1 1
1 2 1 π = − − +
+ + i Arg P P Arg P P Arg(a n n n 1 1 1) and that k n+1 − k n = − − +
+ + 1 2 1 1 1 π Arg P P Arg P P Arg(a n n n) tends to zero as n → •, and let the limit of the sequence {k n } be k. Taking limit in (88), we find that S LogP k i n → − 2 π and so the condition assumed is necessary. Definition : An infinite product ( ) 1 1 + = • ∏ a n n is absolutely convergent if and only if log( ) 1 1 + = • ∑ a n n is convergent. Theorem 6.2 : The infinite product (86) converges absolutely if and only if the series ∑a n converges absolutely. Proof : If ∑a n converges absolutely, then in particular a n → → • 0 as n . Also, if log( ) 1 1 + = • ∑ a n n converges absolutely then log( ) . 1 0 0 + → → a n n and a Thus in
109 either of the cases a n → 0 and we can take | | a n ≤ 1 2 for sufficiently large n. Then by elementary calculation, 1 1 2 3 2 − + = − + log( )
a
a
a
a n n n n ≤ + + + ≤ 1 2 1 2 2 3 a a a n n n
, n = large enough. It follows that 1 2 1 3 2 a a a
n
n n ≤ + ≤
log( ) confirming the occurrence of the absolute convergence simultaneously for the two series. 6.3 Infinite product of functions So far we have considered infinite product of complex numbers. Now we shall study infinite products whose factors are functions of a complex variable. Some of the factors (finite in number) may vanish on a region considered. In that case we consider the infinite product omitting those factors. The theorems proved earlier hold good in this case too with some modifications. Definition : (Uniform convergence of infinite products) An infinite product { ( )} 1 1 + = • ∏ a z n n (89) where the functions a n (z) are defined on a region D,
is said to be uniformly convergent on D if the sequence of partial products
P z a z n k k n ( ) { ( )} = + = ∏ 1 1 converges uniformly to a non-zero limit on D. Theorem 6.3 : An infinite product (89) is uniformly convergent on a domain D if the series a z n n ( ) = • ∑ 1 converges uniformly and has a bounded sum there.
Proof : Let M be the upper bound of the sum
a z n ( ) ∑ on D. Then 1 1 1 1 2 1 2 + + + > ≤ + +
a
z a z a z e e
n a z a z a z M
n ( ) ( ) ... ( ) | ( )| | ( )| ...| ( )|
110
Let us consider the sequence {Q n } with Q z a z
n k k n ( ) { | ( )|} = + = ∏ 1 1 We observe Q z Q
z

a z a z a z a z n
n n n ( ) ( ) ( ) ( ) ... ( ) ( ) − − = + + + −1 1 2 1 1 1 1 >
e a z
M n ( )
Now since
the series ∑a z n ( )
is uniformly convergent, the series ∑ − − { ( ) ( )} Q z Q z n n 1 is uniformly convergent.
Thus the sequence {Q n } tends to a limit. Again P z P z Q z Q z n n n n ( ) ( ) ( ) ( ), − ≤ − − − 1 1 so the result follows.
Theorem 6.4 : An infinite product { ( )} 1 1 + = • ∏ a z n n converges uniformly and absolutely in a closed bounded domain D if each function a n (z) satisfies a z M n n ( )≤ for all z ε D and M n is independent of z and moreover ΣM n is convergent. Proof : Given ΣM n is convergent, so the infinite product M M n n = + = • ∏ ( ) 1 1 converges by theorem 6.2
Now, for
n < m Q
z Q
z Q
z a z n
m m k m n ( ) ( ) ( ) { ( )} − = + − + ∏ 1 1 1 (90) Again, { ( )} ( ) ( ) ( ) ( ) ( ) ( ) , , , 1 1 1 1 + − = + + + = + ∏ ∑ ∑ ∑
a
z
a z
a z a z a z a z
a
z
k
m
n
k k m n i i j n j i i j n j l l + + + + ... ( ) ( )... ( ).
a z a z a z
m
m n 1 2 Taking moduli { ( )} , , , 1 1 1 1 + − ≤ + + + + = + ∏ ∑ ∑ ∑ a z
M M M M M M k m n k k m n i i j n j i i j n j
l l + + + + ... M M M m m n 1 2 = + − + ∏ ( ) 1 1 1 M k m n Utilising this in (90) we obtain
111 Q z Q z M M n m
k k m k m n ( ) ( ) ( ) ( ) − ≤ + + −
= = ∏ ∏ 1 1 1 1 1 = + − + = = ∏ ∏ ( ) ( ) 1 1 1 1 M M k k n k k m (91) Now as the infinite product ( ) 1 1 + • ∏ M k is convergent, we choose m large enough so that r.h.s in (91) is less than ε and hence |Q n (z) − Q m (z)| > ε, when n < m Thus the sequence {Q n (z)} converge uniformly, since m depends only on ε. Finally, absolute convergence of the infinite product follows on utilising Th. 6.2 Example 1 : Test for convergence of the infinite product 1 2 2 1 − = • ∏ z n n Solution : The terms of the product vanish when z = ± ±1 2 , ,... etc. Here a z z n a z z n n n ( ) ( ) = − ≤ 2 2 2 2 1 and Now since the series ∑ 1 2 n is convergent, the given infinite product is uniformly and absolutely convergent in the entire plane excluding the points z = ± ±1 2 , , etc. Example 2 : Discuss the convergence of the infinite product 1 1 1 1 1 2 1 2 − + − + z z z z Solution : Let P z z k n k n ( )= − = ∏ 1 2 2 1 and we consider a bounded closed domain D which does not contain the points z = ± ±1 2 , ,... . The sequence {P n (z)} converges uniformly in D (see example 1). Again
let
F
z
z
z z z
z n z n n2 1 1 1 1 1 1 2 1 2 1 1 ( )= − + − + − + F z F z z n n n 2 1 2 1 1 + = − + ( ) ( ) , then F z P z z z
n
P z

n n
n
n 2 2 1 1 1 ( ) ( ) ( ) ( ) = = − + +
and
F
112
and obviously the sequences F 2 , F 4 , F 6 , ... and F 1 , F 3 , F 5 ...
converge uniformly in D. Hence the given infinite product converges uniformly in D. To test for the absolute convergence of the given product we notice that a z n i • ∑ = + + + + + +
1 1 1 2 1 2 1 3 1 3 and it is divergent since the series on the right is divergent and |z| is finite. Therefore the given product does not converge absolutely. Considering the theorem 4.4 on uniformly convergent sequence of analytic functions [(14) Page-72] we get the following theorem : Theorem 6.5 : If an infinite product Π{1 + f n (z)} converges uniformly to f(z) in a bounded closed domain D and if each function f n (z) is analytic in D, then f(z) is also analytic in D. 6.4 Weierstrass' Factorization Theorem 6.6 : If f(z) is an entire function and never vanishes on C/, then f(z) is of the form f(z) = e g(z) , or, more generally, f(z) = ce g(z) , c ≠ 0, constant. where g(z) is also an entire function. Proof : Since f is entire and never vanishes on C/, f 1 /f is also entire and is thus the derivative of an entire function g(z). [follows from Result 1, PG(MT) 02-complex analysis [14, page-54]. Then ' = ' f f g i.e. ' = ' f fg Now, ( ) fe f e fg e g g g − − − ' = ' − ' = 0 Hence, f(z) = ce g(z)
proving the result. Assume now that f possesses finitely many zeros, a zero of order m < 0 at the origin, and the non-zero
ones, possibly repeated are a 1 , ...
a n . Then f z
z z a e
m n k n g z ( ) ( ) = − = ∏ 1 1 where g is
entire. This is clear, since if we divide f by the
factors which produce zero at the points z = 0, a 1 , ..., a n we get an entire function with no zeros. However we cannot expect, in general, such a simple formula to hold
in the case of
infinitely many zeros. Here we have to take care of convergence problems
for an infinite product. In fact the obvious generalization.
113 f z z z z a e m k k n g z ( ) ( ) = − = ∏ 1 1 is valid in a bounded closed domain D if the infinite product converges uniformly in D. Theorem 6.7 (Weierstrass' Factorization Theorem) :— Let {a n } be a sequence of complex numbers with the property a n → • → • as n . Then it is possible to construct an entire function f(z) with zeros precisely at these points. Proof : We need Weierstrass' primary factors to construct the desired function. The expressions
E(z, o) = 1 − z, E(z, p) = ( ) 1 2 2 − + + + z e z z z p p , p = 1, 2 ..., are called Weierstrass' primary factors. Each primary factor is an entire function having only one
simple zero at z = 1. Now, when |
z| > 1
we have, log
E(
z, p) = log (1−
z) +
z + z z p p 2 2 + + = − − − − − + − + + + + = − + − + − + + + z z z p z p z z z p z p z
p
p p p p p 2 1 2 1 2 2 1 2 1 2 ... ... Here we have taken the principal branch of log (1 − z). Hence if z E z
p z z z z z
p p p ≤ ≤ + + = + + + + + + 1 2 1 1 2 1 2 , log (,) ... ... ≤ + + + = + + z z p p 1 2 1 1 1 2 1 2 2 ... ... (92)
We may suppose that the origin is not a zero of the entire function f(z) to be constructed so that a n ≠ 0 for all n. For, if origin is a zero of f(z) of order m we

need only multiply the constructed function by z m . We also arrange the zeros in order of non-decreasing modulus (if several distinct points a n have the same modulus, we take them in any order) so that |a 1 | ≤ |a 2 | ≤ ... . Let |a n | = r n . Since r n → • we can always find a sequence of positive inegers m 1 , m 2 , ... m n , ... such that the series r r n n m n = • ∑ 1 converges for all positive values of r.
114 In fact, we may take m n = n since for any given value of r, we have r
r n n n > 1 2 for all sufficiently large n and the series
is therefore convergent. Next we take an arbitrary positive number R and choose the integer N such that r R r N N ≤ > + 2 1 . Hence, when z R ≤ and n < N we have, z a R r R r n n N ≤ ≤ > +1 1 2 and so by (92), log , E
z a m R r n n n m n ≤ + 2 1 By Weierstrass' M-test the series
log , E z a m n n n = • ∑ 1 converges absolutely and uniformly when z R ≤ and so the infinite product E z a m n n n = • ∏ 1 , converges absolutely and uniformly in the disc z R ≤ , however large R may be. Hence the above product represents an entire function, say G(z). Thus, G z E z a m n n n ( ) , = = • ∏ 1 (93) With the same value of R, we choose another integer k such that r R r k K ≤ > +1 . Then each of the functions of the sequence E z a m m k k n m n n = ∏ = + + 1 1 2 , , , ,..., vanish at the points a 1 ..., a k and nowhere else in z R ≤ . Hence by Hurwitz's theoroem the only zeros of G in z R ≤ are a 1 , ... a k . Since R is arbitrary, this implies that the only zeros of G are the points of the sequence {a n }. Now, if origin is a zero of order m of the required
entire function f(z), then f(z) is
of the form f(z) = z m G(z). Again, for any entire function
g(z), e g(z) is also an entire function without any zero. Hence the general form of the required entire function f(
z) is
f
z z e G z m g z ( ) ( ) ( ) = = = • ∏ z e E
z a m m g
z n n n ( ) , 1 (94) = − = • + + + ∏ z e z a e m g z n n z a z a m z a n
n n n m n ( ) ... 1 1 1 2 1 2 (95)
115
Remark : As there are many possible sequences {m n } in the construction of the function G(z) and ultimately of f(z), the form of the function f(z) achieved is not unique. 6.5
Counting zeros of analytic functions The rate of growth of an entire function is closely related to the density of zeros. We have a quite effective
formula in this regard due to J.L.W.V. Jensen, a Danish mathematician who discovered it in the year 1899. Theorem 6.8 [Jensen's Formula] :— Let f(z) be analytic on |z| ≤ R, f(0) ≠ 0 and f(z) ≠ 0 on |z| = R. If a 1 , ..., a n be the zeros of f(z) within the circle |z| = R, multiple zeros being repeated according to their multiplicities, then log ( ) log (Re) log f f d R a i k n k 0 1 2 0 2 1 = − ∑ = π θ θ π (96) Proof : Let φ( ) ( ). ( ) z f z R a z R z a k k k n = − − = ∏ 2 1 (97) The zeros of the denominator of φ(z) are also the zeros of f(z) of the same order. Hence the zeros of f(z) cancels the poles a n in the product and so φ(z) is analytic on |z| ≤ R. Also, φ( )z ≠ 0 on |z| ≤ R. For, if R a z k 2 0 − = then z R a k = 2 is the inverse point of a k with respect to the circle |z| = R and so lies outside the circle. Again, φ( ) ( ) ( ) ( ) .
z f
z R
a
z R z a R a z R z a n n = − − − − 2 1 1 2 Now, when |
z| = R we have, R
a z R
z a zz a z R z a z R z a z
a
k k k k k k 2 1 − − = − − = − − = ( ) ( ) Hence, |φ(z)| = |f(z)| on |z| =
R. Since φ(z) is analytic and non-zero on |z| ≤ R, log φ(z) is also analytic on |z| ≤ R and consequently Re log φ(z) = log |φ(z)| is harmonic on |z| ≤ R. Hence by Gauss' mean value theorem, log ( ) log Re φ π φ θ θ π 0 1 2 0 2 = i d (98)

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116 From (97) we have, φ( ) ( ) . 0 0 1 2 = · f R a R a R a n Hence from (98) we get, log ( ) log log (Re) f R a d k k n i 0 1 2 1 0 2 + = = ∑ π φ θ θ π i.e. log ( ) log (Re) log | | f f d R a i k k n 0 1 2 0 2 1 = – ∑ = π θ θ π (since |φ(z)| = |f(z)| on |z| = R) Note : We observe that Jensen's formula can also be expressed as log ... log (Re) log ( )...... R a a f d f n n i 1 0 2 1 2 0 = – π θ θ π (99) or as, log ... log (Re) log| ( )|...... R r r f d f n n i 1 0 2 1 2 0 = – π θ θ π (100) where |a i | = r i , i = 1, ..., n. Theorem 6.9 (Jensen's inequality) :— Let f(z) be analytic on |z| ≤ R, f(0) ≠ 0 and f(z) ≠ 0 on |z| = R. If a 1 , ..., a n be the zeros of f(z) within |z| = R, multiple zeros being repeated according to their multiplicities, and |a i | = r i , i = 1, ..., n, then R f r r M R n n ( ) ... ( ) 0 1 ≤ (101) where M R f z z R ( ) max ( ). | | = = Proof : As in Jensen's formula (theorem 6.8) we have, |φ(z)| = |f(z)| on |z| = R and so by the maximum modulus theorem, |φ(z)| ≤ M(R) for |z| ≤ R. In particular, |φ(0)| ≤ M(R) i.e. R f
r r M R n n ( ) ... ( ). 0 1 ≤ Theorem 6.10 (Poisson-Jensen formula) :- Let f(z) be analytic
on |z| ≤ R, f(0) ≠ 0 and f(z) ≠ 0 on |z| = R. If a 1 ... a n be the zeros of f(z) within the circle |z| = R, multiple zeros being repeated according to the their multiplicities, then for any z = re iθ , r > R, log ( ) cos( ) log (Re) f re R r R r Rr t f dt i it θ π π θ = – + – – 1 2 2 2 2 2 2 0 2 – – – = ∑ log ( ) . k n k i i k R a re R re a 1 2 θ θ
117 Proof : Let φ( ) ( ). ( ) . z f z R a z R z a k k k n = – – = ∏ 2 1 Then, as in Jensen's formula we have, |φ(z)| = |f(z)| on |z| = R. Since φ(z) is analytic and non-zero on |z| ≤ R, log φ(z) is also analytic on |z| ≤ R and consequently log |φ(z)| is harmonic on |z| ≤ R. So, by Poisson's integral formula, log ( ) cos( ) log (Re) φ π θ φ θ π re R r R r Rr t dt i it = – + – – 1 2 2 2 2 2 2 0 2 (102) Now, log ( ) log ( ) log ( ) φ θ θ θ θ re f re R a re R re a i i k i i k k n = + – – = ∑ 2 1 Since log|φ(z)| = log|f(z)| on |z| = R we get from (102) log ( ) cos( ) .log (Re) f re R r R r Rr t f dt i it θ π π θ = – + – – 1 2 2 2 2 2 2 0 2 – – – = ∑ log ( ) R a re R re a k i i k k n 2 1 θ θ (103) 6.6 Convex functions The property of convexity plays an important role in function theory because in several cases some lead factors associated with entire, meromorphic and subharmonic functions appear to be convex functions. A real-valued function φ defined on the interval I = [a, b] is said to be convex if for any two points s, u in [a, b] φ λ λ λφ λ φ λ (( ) ( ) ( ) ( ) u s u s + – ≤ + – ≤ ≤ 1 1 0 1 for (104) Geometrically, the condition (104) is equivalent to the condition that if s > x > u, then the point (x, φ(x)) should lie below or on the chord joining the points (s, φ(s)) and (u, φ(u)) in the plane. Analytical condition for φφφφφ(x) to be convex in [a, b] :- Let the coordinates of the points A, B, C on the curve y = φ(x) as shown in the adjoining figure be (s, φ(s)), (u, φ(u)) and (x, φ(x)) respectively where s > x > u.
118 Equation of the chord AB is y – φ(x) = φ φ ( ) ( ) ( ). u s u s x s – – – or, y s u s u s x s = + – – – φ φ φ ( ) ( ) ( ) ( ) (105) Let the coordinates of any point D on the chord AB be (x, y). According to definition φ(x) will be convex if and only if CN ≤ DN. i.e., if and only if φ(x) ≤ y; i.e. if and only if φ φ φ φ ( ) ( ) ( ) ( ) ( ); x s u s u s x s ≤ + – – – i.e., if and only if φ φ φ ( ) ( ) ( ) x u x u s s x s u s u ≤ – – + – – (106) for s > x > u. We now state two results on convex functions without proof. Result 1. A differentiable function f(x) on [a, b] is convex if and only if f'(x) is increasing in [a, b]. Result 2. A sufficient condition for f(x) to be convex is that f"(x) < 0.
The maximum modulus function : Let f(z) be a non-constant analytic function
in |z| > R. Then for 0 ≤ r > R we define the maximum modulus function M(r, f) or, simply M(r) by M r f z z r ( ) max ( ). | | = = By maximum modulus theorem we can also write M r f z z r ( ) max ( ). | | = = Result : Let f(z) be a non-constant analytic function in |z| > R. Then M(r) is a strictly increasing function of r in 0 ≤ r ≤ R. Proof : Let 0 ≤ r 1 > r 2 > R. Since f(z) is analytic in |z| ≤
r 2 , the maximum value of |f(z)| for |z| ≤ r 2
is attained on |z| = r 2 . Let z 2 be a point on |z| = r 2 such that |f(z 2)| = M(r 2). Similarly, the maximum value of |f(z)| for |z| ≤ r 1 is attained on |z| = r 1 . Let z 1 be a point on |z| = r 1 such that |f(z 1)| = M(r 1). Since r 1 > r 2 , z 1 is an interior point of the closed region |z| ≤ r 2 . Hence by maximum modulus theorem, |f(z 1)| > M(r 2); i.e. M(r 1) > M(r 2). This proves the result. y x s o N u C (x, φ(x)) (s,φ(s)) D(x,y) B(u, φ(u)) A
119 Corollary : Let f(z) be a non-constant entire function. Then its maximum modulus function M(r) → • as |z| = r → •. For, if M(r) is bounded, then by Liouville's theorem f(z) would be a constant function. Theorem 6.11 [Hadamard's three-circles theorem]. Let 0 > r 1 > r > r 3 and suppose that f(z) is analytic on the closed annulus r 1 ≤ |z| ≤
r 3 . If M r f z z r ( ) max ( ), = = , then M r M r M r r r r r r ( ) ( ) . ( ) log
log log 3 1 3 1 1 3 ≤ (107) Proof : Let us consider the function φ(
z) = z α f(z), where α is a real constant to be chosen later.

If α ≠ an integer, φ(z) is multi-valued in r 1 ≤ |z| ≤ r 3 and so we cut the annulus along the negative part of the real axis. Thus we obtain a simply connected region G in which the principal branch of φ(z) is analytic. Hence the maximum modulus of this branch of φ(z) in G is attained on the boundary of G. Since α is real, all the branches of φ(z) have the same modulus. If we consider another branch of φ(z) which is analytic in another cut annulus it is clear that the principal branch of φ(z) can not attain its maximum value on the cut. Hence maximum of |φ(z)| is attained on at least one of the bounding circles |z| = r 1 or, |z| = r 3 . Thus, z f z r M r r M r α α α ( ) max ( ), ( ) . ≤ 1 1 3 3 Hence on |z| = r, r M r r M r r M r α α α ( ) max ( ), ( ) ≤ 1 1 3 3 (108) We now choose α such that r M r r M r 1 1 3 3 α α ( ) ( ) = . Then α = – log(( ))/ ( )) log(/) M r M r r r 3 1 3 1 . Substituting this value of α in (108) we get, x y M N o |z| =
r 1 |z| = r |z| = r 3
120 M r r r M r ( ) ( ) ≤ – 1 1 α = r r r r M r M r M r 1 3 1 1 3 1 log ( ) ( ) log . ( ) and so M r r r M r r r M r M r r r ( ) . ( )
log(/) log(( )/ ( )) log(/) 3 1 3 1 3 1 1 1 ≤ That is, M r M r M r M r r r r r r r ( ) ( ) ( ) . ( ) log(/)
log(/) log(/) 3 1 1 3 1 3 1 1 ≤ [since a log b = b log a] =
M(r 1) log(r 3 /r) .M(r 3) log(r/r 1) .
Note : Equality in (107) occurs when φ(z) is a constant, i.e. when f(z) is of the form cz α for some real α and c is a constant. Corollary : log M(r) is a convex function of log r. Proof : Let f(z) be analytic in the closed annulus 0 > r 1 ≤ |z| ≤ r 2 . If r 1 > r > r 2 we have, by Hadamard's three-circles theorem,
M r M r M r r r r r r r ( ) ( ) . ( ) . log(/) log(/) log(/) 2 1 2 1 1 2 ≤ Taking logarithms we get (log log)log ( ) (log log)log ( ) r r M r r r M r 2 1 2 1 – ≤ – + (log log)log ( ). r r M r – 1 2 That is, log ( ) log log log log log ( ) log
log log log log ( ) M r r r r r M r r
r r r M r ≤ – – + – – 2 2 1 1 1 2 1 2 (109)
The inequality (109) shows that log M(r) is a convex function of log r. 6.7
Order of an entire function An entire function f(z) is said to be of finite order
if there is a positive number A such that as |z| = r → •, the inequality M(r) > e r A holds. The lower bound ρ of such numbers A is called the order of the function. f is said to be of infinite order if it is not of finite order. From the definition it is clear that order of an entire function is non-negative. Result : Let f be an entire function of order ρ and M(r) = max{|f(z)| : |z| = r}. Then
121 ρ = →• limsup log log ( ) log r M r r (110) Proof : By hypothesis, given ε < 0 there exists r 0 (ε) < 0 such that M r e for r r r ( )> < +ρ ε 0 while M r e r ( )< +ρ ε for an increasing sequence {r n } of values of r, tending to infinity. In otherwords, log log ( ) log M r r r r > + ∀ < ρ ε 0 and (111) log log ( ) log M r r < – ρ ε (112) for a sequence of values of r → +• (111) and (112) precisely means ρ = →• limsup log log ( ) log r M r r Example 3 : Determine the order of the functions. (i)
p(
z) =
a 0 + a 1 z + ... + a n z n , a n ≠ 0. (
ii) e kz , k ≠ 0. (iii) sin z (iv) cos z Solution : (i)
p
z
a
a z a z a a z a
z
n
n n n ( ) ... ... = + + + ≤ + + + 0 1 0 1 Hence, M r p(z a a r a r z
r
n
n ( ) max) ... | | = ≤ + + + = 0 1 ≤ + + r a a n n 0 ... (
choosing r ≥ 1. Since ultimately r → •, the choice is justified). = Br n , where B a a n = + + 0 ... . Hence log M(r) ≤ log B + n log r ≤ log r + n log r (Taking r sufficiently large). = (n + 1) log r. Now, ρ = ≤ + + = →• →• limsup log log ( ) log limsup log( ) log log log r r M r r n r r 1 0
i.e. ρ ≤ 0. But by definition ρ ≥ 0. Hence ρ = 0 (ii) Here M(r) = e |k|r and hence ρ = = = →• →• limsup loglog ( ) log limsup log log r r M r r k r r 1

122 (iii) We know that sin ! ! z z z z = − + − 3 5 3 5 and so sin ! ! ! ! sinh . z z z z r r r r on z r ≤ + + + = + + + = ≤ 3 5 3 5 3 5 3 5 = − − e e r r 2 . Also at z = ir, sin z e e i r r = − − 2 and so sin z
e e r r = − − 2 . Hence M r e e e e r r r r ( ) ( ) = − = − − − 2 1 2 2 log ( ) log log M r r e r r e r r = + − = + −
− − 1 2 1 1 1 2 2 2 Therefore, lim loglog ( ) log lim log log /log r r r M r r r e r →• →• − = + + −
= 1 1 1 1 2 1 2
So order of sin z is 1. (iv) Following as in (iii) we find that the order of cos z = 1/2. Let f z a z n n n ( )= = • ∑ 0 be an entire function. We now state a theorem which will give us order of f(z) in terms of the coefficients a n of the power series expansion of f(z). Theorem :
Let f z a z n n n ( )= = • ∑ 0 be an entire function of finite order ρ. Then,
ρ = − = − →• →• limsup log log limsup log log / n n n n n n a n n a 1 6.8 The function n(r) Let f(z) be an entire function with zeros at the points a 1 , a 2 , ..., arranged in order of non-decreasing modulus, i.e. a a 1 2 ≤ ≤ , multiple zeros being repeated according to
123 their multiplicities. We define the function n(r) to be the number of zeros of f(z) in z r ≤ . Evidently n(r) is a non-decreasing, non-negative function of r which is constant in any interval which does not contain the modulus of a zero of f(z). We observe that if f(0) ≠ 0. n(r) = 0 for r a > 1 . Also, n(r) = n for a r a n n ≤ > +1 . Jensen's inequality can also be written in the following form involving n(r).
Theorem 6.12 (Jensen's inequality) : Let f(z) be an entire function with f(0) ≠ 0,
and a 1 , a 2 , ... be the zeros of f(z) such that a a 1 2 ≤ ≤ , multiple zeros being repeated according to their multiplicities. If a r a N N ≤ > +1 , then log ( ) log ( ) log ( ) r a a n x x dx M r f N N r 1 0 0 = ≤ − (113) Proof : Let |a i | = r i , i = 1, 2, ..., and r be a positive number such that r r r N N ≤ > +1 . Let x 1 ..., x m be the distinct numbers of the set A = {r 1 , ..., r N } where x 1 = r 1 , ..., x m = r N . Suppose x i is repeated p i times in A. Then, p 1 + ... + p m = N. Also let t i = p 1 + ... + p i , i = 1, ..., m. We now consider two cases. Case 1) Let r N > r. Then,
n
x
x dx n x x dx n
x x dx n x x dx n x x dx
x x x
x
x x
x
r
r m m m ( ) lim ( ) ( ) ... ( ) ( ) = + + +
+ → − − − − ε ε ε ε 0 0 1 2 3 1 2 (since n x x dx x ( ) = − 0 0 1 ε as n(x) = 0 for 0 ≤ x > x 1). = + + +
+ → − − − − − − lim ε ε ε ε 0 1 2 1 1 2 3 1 2 t x dx t x dx t x dx N x dx m x x x x x x r t m m N = + + + + → − − − − − − lim [log log log log ε ε ε ε 0 1 2 1 1 2 1 1 1 t x t x t x N x x x x x m x x r r m m N = − − + − − + → lim[{log( ) log } {log( ) log } ε ε ε 0 1 2 1 2 3 2 t x x t x x + − − + − − − t x x N r r m m m N 1 1 {log( ) log }] (log log) ε = t 1 (log
x 2 − log x 1) + t 2 (log x 3 − log x 2) +... + t m−1 (log x m − log x m−1) + N(log r − log r N) = p 1 log x 2 − p 1 log x 1 + (p 1 + p 2) log x 1 − (p 1 + p 2) log x 2 +...+ (p 1 +...+ p m − 1) log x m − (p 1 +...+ p m − 1) log x m − 1 + N log r − (p 1 +...+ p m) log x m = N log r − (p 1 log x 1 + p 2 log x 2 +...+ p m log x m)
124 = − = log log log r x x x r x x x
N p p m p N p p m p m m 1 2 1 2 1 2 1 2 = log r r r N N 1 Thus, n x x dx r a a N N r ( ) log = 1 0 (114) Case 2). Let r N = r. As before, n x x dx t x dx t x dx m x x x x r m m ( ) lim = → + +
− − − − ε ε ε 0 1 1 0 1 1 2 = − + − + = − ∑ t x x t r r i i i m N i m (log log) (log log) 1 1 1 = log r a a N N 1 (Proceeding as in case 1). Thus in any case, n x x dx r a a N N r ( ) log . = 1 0 But Jensen's inequality gives us r a a M r f N N 1 0 ≤ ( ) ( ) . Hence, n x x dx r a a M r f N N r ( ) log log ( ) log ( ). = ≤ − 1 0 0 Theorem 6.13 : If f(z) be an entire function with finite order ρ, then n(r) = O(r ρ + ε) for ε < 0 and for sufficiently large values of r. Proof : By Jensen's inequalilty, n x x dx M r f r ( ) log ( ) log ( ) ≤ − 0 0 (115) We replace r by 2r in (115) and obtain n x x dx M r f r ( ) log ( ) log ( ) ≤ − 2 0 0 2 (116) Since order of f(z) is ρ we have for any ε < 0, log M(2r) > (2r) ρ + ε = Kr ρ + ε for all large r, K being a constant. Hence from (116).

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125 n x x dx Ar r ( ) > + ρ ε 0 2 for all large r, A being a constant independent of r. Since n(x) is non-negative and non-decreaing function of x, n x x dx n x x dx r r r ( ) ( ) ≤ > 0 2 2 Ar ρ + ε and also n x x dx n r x dx n r r r r r ( ) ( ) ( )log ≥ = 2 2 2 Hence, n r n x x dx Ar r r ( )log ( ) , 2 2 ≤ > + ρ ε i.e., n r A r ( ) log > + 2 ρ ε for all large r. Hence, n(r) = O(r ρ + ε).
6.9 Convergence exponent (or, exponent of Convergence) Let f(z) be an entire function with zeros at the points a 1 , a 2 , ..., arranged in order of non-decreasing modulus, multiple zeros being repeated according to their multiplicities and |a i | = r i , i = 1, 2, ..., We define convergence exponent ρ 1 of the zeros of f(z) by the equation ρ 1 = →• limsup log log n n n r (117) or, equivalently by ρ 1 = →• limsup log ( ) log n n r r (118) The convergence exponent has the following property. Theorem 6.14 : Let f(z) be an entire function with zeros at a 1 a 2 , ..., arranged in order of non-decreasing modulus, multiple zeros being repeated according to their multiplicities and |a i | = r i . If the convergence exponent ρ 1 of the zeros of f(z) is finite, then the series 1 1 r n n α = • ∑ converges when α < ρ 1 and diverges when α > ρ 1 . If ρ 1 is infinite, the above series diverges for all positive values of α. Proof : Let ρ 1 be finite and α < ρ 1 . Then, ρ ρ α 1 1 1 2 > + ( ). Hence, log log ( ) n r n > + 1 2 1 ρ α for all large n.
126 or, log log , ( ) n r n > + 1 2 1 ρ α i.e. n r or n r i e n n > > + + 1 2 1 1 2 ( ) ; , . ., ρ α ρ α r n n n where p n p α ρ α α ρ α ρ α α ρ α ρ < = = = − + < + + − + + 2 0 1 1 1 1 1 1 1 , . Hence, 1 1 1 r n n p α > + for all large n. Hence, 1 1 r n n α = • ∑ converges. Next, let α > ρ 1 . Then, log log n r n < α for a sequence of values of n, tending to infinity. That is, log log n r n < α or, 1 1 r n n α < (119) for a sequence of values of n tending to infinity. Let N be such a value of n for which (119) holds and m be the least integer < N 2 . Then, as r n is non-decreasing, 1 1 1 1 1 1 1 r r r r r r n N m N m N N N n N m N α α α α α α = + + + ≥ + + − − + = − ∑ = + < < < m r m r m N N N 1 1 2 α α . Since N may be as large as we please, by Cauchy's principle of convergence, the series 1 1 r n n α = • ∑ diverges. If ρ 1 is infinite, then for any positive value of α, log log n r n < α for a sequence of values
127 of n tending to infinity; i.e., n r n < α for a sequence of values of n tending to infinity. Hence as before, the series 1 1 r n n α = • ∑ diverges for any positive α. Note 1. Observe that ρ 1 may also be defined as the lower bound of the positive numbers α for which the series 1 1 r n n α = • ∑ is convergent. If f(z) has no zeros we define ρ 1 = 0 and if 1 1 r n n α = • ∑ diverges for all positive α, then ρ 1 = •. Note 2. If ρ 1 is finite, the series 1 1 1 r n n ρ = • ∑ may be convergent or divergent. For example, if r n = n, then ρ 1 1 = = →• limsup log log
n n n r and 1 1 1 1 1 r n n n n ρ = = • = • ∑ ∑ diverges. Again, if r n = n(log n) 2 , then, ρ 1 2 1 = + = →• limsup log log loglog , n n n n and 1 1 1 2 1 1 r n n n n n
ρ = = • = • ∑ ∑ (log) converges. Theorem 6.15 : If f(z) is an entire function with finite order ρ and r 1 , r 2 , ..., are the moduli of the zeros of f(z), then 1 1 r n n α = • ∑ converges if α < ρ. Proof : We choose β such that ρ > β > α. Since for any ε < 0, n(r) = 0 (r ρ + ε), n(r) > Kr β (120) for all large r, K being a constant. Putting r = r n , n large, (120) gives n Kr n > β , i.e., r n k n < 1 1 / / β β or, 1 r B n n α α β > / for all large n, B being a constant. Since α β α < = • ∑ 1 1 1 , r n n converges.
128 Corollary : Since convergence exponent ρ 1 is the lower bound of positive numbers α for which 1 1 r n n α = • ∑ is convergent, it follows that ρ 1 ≤ ρ. Note : ρ 1 may be 0 or •. For example if r n = e n , ρ 1 = 0 and if r n = log n, then ρ 1 = •. For the function f(z) = e z , ρ = 1 and ρ 1 = 0 so that ρ 1 > ρ. But for sin z or cos z, ρ = ρ 1 = 1. Result : If the convergence exponent ρ 1 of the zeros of an entire function f(z) is greater than 0, then f(z) has infinite number of zeros. Proof : If possible, suppose f(z) has finite number of zeros with moduli r 1 ...., r N . The series 1 1 r n n N α = ∑ , being of finite number of terms, converges for every α < 0. Hence ρ 1 = 0, a contradiction. Hence f(z) has infinite number of zeros. Note : For an entire function with finite number of zeros, ρ 1 = 0. Example : Find the convergence exponent of the zeros of cos z. Solution : First method : The zeros of cos z are π π π π 2 2 3 2 3 2 , , , ,.... − − Now, 1 2 2 2 1 3 1 r n n α α α α α π π π = + + + = • ∑ . = + + + + + + 2 2 1 1 3 1 5 1 1 1 3 1 5 π α α α α α α . The series converges when α < 1 and diverges when α > 1. Hence the lower bound of the positive numbers α for which 1 1 r n n α = • ∑ converges is 1 i.e., ρ 1 = 1. Second method : The zeros of cos z are (2n + 1) π 2 , n e = ± ± − − 0 1 2 2 2 3 2 3 2 , , , ; . . , , , , i π π π π Let a a a 1 1 2 2 2 2 3 2 3 2 = ' = − = ' = − π π π π , , , , , a a n a n n n = − ' = − − ( ) , ( ) , , 2 1 2 2 1 2 π π Hence, r a a a a a a n n n 1 1 1 2 2 2 2 3 2 = =' = = = ' = = = ' = π π , , , r r
129 ( ) , 2 1 2 n − π Hence, ρ 1 = →• limsup log log n n n r = − + = −

+ →• →• limsup log log( ) log limsup log log log n n n n n n n 2 1 2 2 1 2 π π = + − + = →• limsup log log log / log . n n n n 1 1 2 1 2 1 π 6.10 Canonical Product Let f(z) be an entire function with infinite number of zeros at a n , n = 1, 2, ... a n ≠ 0. If there exists a least non-negative integer p such that the series 1 1 1 r n p n + = • ∑ is convergent, where r n = |a n |, we form the infinite product G z E z a p n n ( ) , = = • ∏ 1 . By Weirstrass' factor theorem G(z) represents an entire function having zeros precisely at the points a n . We call G(z) as the Canonical product corresponding to the sequence {a n } and the integer p is called its genus.
If z = 0
is a zero of f(z) of order m, then
the canonical product is z m G(z). Observe that if the convergence exponent ρ 1 ≠ an integer, then p = [ρ 1] and if ρ 1 = an integer, then p = ρ 1 when 1 1 1 r n n ρ = • ∑ is divergent and p = ρ 1 − 1 if 1 1 1 r n n ρ = • ∑ is convergent. In any case, ρ ρ ρ 1 1 1− ≤ ≤ ≤ p , where ρ = order of f(z). Examples : (i) Let a n = n. Then 1 1 2 1 2 1 r n n n n = • = • ∑ ∑ = is convergent while 1 1 1 1 r n n n n = • = • ∑ ∑ = is divergent. So, p = 1. (ii) Let a n = e n . Then p = 0. We now state an important theorem without proof. The proof can be found in any standard book.
130 Borel's theorem : The
order of a canonical product is equal to the convergence exponent of its zeros.
Example : Find the canonical product of f(z) = sin z. Solution : f(z) is an entire function with infinite number of zeros at z = nπ, n being an integer. First we consider the zeros of f(z) excluding the simple zero at z = 0. Let a
n = nπ, n = ±1, ±2, ... |a n | = r n . Then, r n = |nπ|. Now, 1 1 1 1 r n n n n = • = • ∑ ∑ = π = = • ∑ 1 1 1 π n n is divergent, but 1 1 1 2 1 2 2 1 r n n n n = • = • ∑ ∑ =
π is convergent. Hence genus of the required canonical product p = 1. Hence the canonical product G(z) is given by G
z E z
a n n ( ) , , = =− • • ∏ 1 where ′ =− • • Π n means n = 0 is excluded in the product. = − = − · −
=− • • − = • ∏ ∏ 1 1 1 1
z n e z n e z n e n z n z n z n
n
π π π π π π = − =− • • ∏ 1 2 2 2 z n n
π . Since origin is a simple zero of sin z, the required canonical product of sin z is given by sin . z
z z n n = − = • ∏ 1 2 2 2 1 π Exercises 1. Find the order
of the entire functions : (a) sinh
z (b) e z sin z, (c) e z n , (d) e
e z , (e) cos z, (f) e p(z) , where
p(
z) = a a
z a z
a n n n 0 1 0 + + + ≠ , , (g) z n
n n (!) , α α = • ∑ < 0 0 , (h) e n z n n
n
α α α < = • ∑ 0 0 / , 2. Given f 1 (z) and f 2 (z) are two entire functions of orders ρ 1 and ρ 2 respectively, show that (i) order of f 1 (z) f 2 (z) is ≤ max (ρ 1 , ρ 2) (ii) order of f 1 (z) + f 2 (z) is ≤ max (ρ 1 , ρ 2), and equality occurs if ρ 1 ≠ ρ 2 . 3. Find the convergence exponent of the zeros of sin z. ′ ′ ′
131 4. Find the canonical product of cos z. 5. Show that if a < 1, the entire function 1 1 − = • ∏ z n a n is of order 1 a .
6.11 Hadamard's Factorization Theorem Before taking up Hadamard's factorization theorem we state a theorem due to Borel and Caratheodory. Borel and Caratheodory's
theorem : Let f(z) be analytic in z R M r f z r f z z r z r ≤ = = = = , ( )
max ( ), ( ) max{Re ( )}. A Then for 0 >
r > R, M r r R r A R R r R r f R r R r A R
f ( ) ( ) ( ) ( ) ( ) ≤ − + + − > + − + 2 0 0 (121) Proof : Omitted (cf. Theory of entire functions−A.S.B Holland- p. 53).
Corollary : max ( ) . ! ( ) ( ) ( ) | | ( ) z r n n n f z n R R r A R f = + + ≤ − + 2 0 2 1 (122) Hadamard's Factorization Theorem 6.16 : If f(z) is an entire function of finite order ρ with infinite number of zeros
and f(0) ≠ 0, then f(z) = e Q(z) G(z), where G(z) is

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the canonical product formed with the zeros of f(z) and Q(z) is a polynomial of degree not greater than ρ. Proof : By Weierstrass' factor theorem we already have f(z) = e Q(z) G(z) (123) where G(z) is the canonical product with genus p formed with the zeros a 1 , a 2 , ... of f(z) and Q(z) is an entire function. Since ρ is finite we need to show that Q(z) is a polynomial of degree ≤ ρ. Let m = [ρ]. Then, p ≤ m. Taking logarithms on both sides of (123) we have, log ( ) ( ) log ( )
f z Q
z G z = + = + + + + +
= • = • ∑ ∑ Q z z a
z a z a p z a
n n
n n n
p n ( )
log − ... 1 1 2 1 1 2 1 (124) Differentiating both sides of (124) m + 1 times,
132 d dz f
z f z Q z m a z
m m m n m n 1 1 1 1 1 ( ) ( ) ( ) ! ( ) ( ) = − − + + = • ∑ (125) [Since p m d dz z a z a p z a m m n n n p n ≤ + + +
= + + = • ∑ , ... 1 1 2 1 1 2 1 0
and d dz z a d dz a z m a z m m n m m n
n m + + + + + − = − = − − 1 1 1 1 1 1 1
log log( ) ! ( )] Now, Q(z) will be a polynomial of degree m at most if we can show that Q (m+1) (z) = 0. Let g z f z f z a R n a R n ( ) ( ) ( ) . | | = − ≤ − ∏ 0 1 1 Then g R (z) is an entire function and g R (z) ≠ 0 in |z| ≤ R. [Since f(z) is entire, f(0) ≠ 0 and 1 1 − ≤ − ∏ z a n a R n | | cancels with factors in f(z)]. For |z| = 2R and |a n | ≤ R we have, 1 1 − ≥ z a n . Hence, g z f z f Ae R R ( ) ( ) ( ) ( ) ≤ > + 0 2 ρ ε for |z| = 2R (126) By maximum modulus theorem, g z Ae R R ( ) ( ) > + 2 ρ ε (127) for |z| > 2R. Let h R (z) = log g R (z) such that h R (0) = 0. Then h R (z) is analytic in |z| ≤ R. Hence from (127) Re h R (z) = log |g R (z)| > KR ρ+ε , K = Constant (128) Hence from the corollary of the theorem of Borel and Caratheodory we have h z m R R r KR R m m m ( ) ( ) ( ) ! ( ) . + + + + ≤ + − 1 3 2 2 1 ρ ε for |z| = r > R Hence for z r R = = 2 , h z R R m m ( ) ( ) + + − − = 1 1 0 ρ ε (129)
133 But h z g z f z f z a R R a R n n ( ) log ( ) log ( ) log ( ) log = = − − − ≤ ∑ 0 1 Hence h z d dz f z f z m a z R m m m n m a R n ( ) ( ) ( ) ( ) ! ( ) + + ≤ = ′ + − ∑ 1 1 1 = + + − − + < ∑ 0 0 1 1 1 ( ) R a m n m a R n ρ ε (130) for z R = 2 and so also for z R > 2 by maximum modulus theorem. The first term on the right of (130) tends to 0 as R → • if ε < 0 is small enough since m + 1 < ρ. Also the second term tends to 0 since 1 1 1 a n m n + = • ∑ is convergent. In fact, 1 1 a n m a R n + < ∑ becomes the remainder term for large R. Hence Q (m+1) (z) = 0 since Q (m+1) (z) is independent of R. Thus, Q(z) is a polynomial of degree not greater than ρ. 6.12 Consequences of Hadamard's Theorem Theorem 6.17 : An entire function of finite order admits any finite complex number except, perhaps, one number. Proof. Let us suppose that f does not admit two finite values a and b. Then f(z) − a ≠ 0 for all z in C/ and hence there exists an entire function g(z) such that f(z) − a = e g(z) The function f(z) − a is of finite order since f(z) has finite order. Following Hadamard's factorization theorem g(z) must be a polynomial. Now e g(z) does not assume the value b − a i.e. g(z) ≠ log (b − a) for any z in C/. As because g(z) is a polynomial it contradicts the essence of the Fundamental Theorem of Algebra [(14), Th. 3.11, page-65]. Theorem 6.18 : An entire function of fractional order possesses infinitely many zeros. Proof. Let f be an entire function of fractional order ρ. If possible, suppose the zeros of f(z) are {a 1 , a 2 , ... a n), finite in number, counted according to their multiplicity. Then f(z) can be expressed as
134
f(z) = e g(z) (z − a 1) (z − a 2) ... (z − a n) where g(z) is an entire function.
Applying Hadamard's factorization theorem, the degree of the polynomial g(z) ≤ ρ. It is easy to check that f(z) and e g(z) are of same order. But we have already seen that the order of e g(z) is exactly the degree of g(z), which is an integer. This implies ρ is an integer. This contradiction completes the proof. 6.13 Meromorphic Functions The term meromorphic comes from the Ancient Greek "meros" meaning part, as opposed to "holos" meaning whole. This function is analytic on a domain D except a set of isolated points, which are poles for the function. Definition :
A function f(z) analytic in a domain D except for poles is said to be meromorphic.
Theorem 6.19 :
A rational function is meromorphic. Proof :
Let

f(
z) = p(z)/q(z) where p and q are
polynomials
with no
common zeros. If the degree of p is less than or equal to the degree of q, then f has only a finite number of poles and the point at infinity is not a pole. On the otherhand, if the degree of p is greater than the degree of q, then (taking degree of p(z) = m and degree of q(
z) =
n).
f
z a z a z a z a b
z b z b z b
m m m
m
n n n n ( ) ... ... = + + + + + + + + − − − − 1 1 1 0 1 1 1 0 = + + + + + − − − − − − c z c z
c z c r z q z m n
m n m n
m n 1 1 1 0 ... ( ) ( ) where degree of r(z) ≤ n − 1. This shows that the point at infinity is a pole of order (m − n) and there lie a finite number of poles in the unextended plane. These establish that f(z) is meromorphic. Theorem 6.20 : [Partial fraction decomposition]. Let p(z), q(z) be two polynomials with no common zeros and that 0 ≤ deg (p) > deg (q). Let a 1 , ... a k be the zeros of q(z) with multiplicities α 1 , ..., α k . Then p(z)/q(z) can be expressed uniquely as p z q z c z a ij i j j i k i ( ) ( ) ( ) = − = = ∑∑ 1 1 α (131) Proof. The decomposition is unique. We assume that the relation (131) exists. Let r < 0 be small enough. Then for z ε N (a i , r), (131) can be rewritten as
135 p z q z g z c z a ij i j j i ( ) ( ) ( ) ( ) = + − = ∑ 1 α (132) since N(a i , r) does not contain any zero of q(z) other than a i , g(z) is analytic at z = a i . Multiplying both sides of (132) by (z − a i) α i , we obtain p z q z z a g z z a c z a i i ij i j j i i i i ( ) ( ) ( ) ( )( ) ( ) − = − + − − = ∑ α α α α 1 (133) Now the function p z q z z a i i ( ) ( ) ( ) − α is analytic for all z belonging to N(a i , r) and hence can be expanded in a Taylor series in a neighbourhood of a i in N(a i , r) p z q z z a c z a i n n i n i ( ) ( ) ( ) ( ) − = − = • ∑ α 0 (134) Combining (133) and (134), we write c z a g z z a c c z a n i n n i i i i i i i ( ) ( )( ) ( ) ... − = − + + − + + = ∑ − 0 1 α α α α + − − c z a i i i 1 1 ( ) α Comparing the coefficients we find c c c c c c i i i i i i α α α = = = − − 0 1 1 1 1 , ,..., uniquely Existence of the decomposition. The principal part associated to each pole a i is c z a ij i j j i ( ) − = ∑ 1 α Now if we subtract all the principal parts we find the function f z p z q z c z a ij i j j i k i ( ) ( ) ( ) ( ) = − − = = ∑∑ 1 1 α is analytic in the extended plane. Now each of the terms c z a ij i j ( ) − converges to zero for z → •, and also p(z)/q(z) converges to zero for z → • since deg(q) < deg(p). This shows that f(z) → 0 for z → •. But then f is necessarily
136 bounded and hence constant by Liouville's theorem. A constant function tending to zero as z → • must be identically zero. Example 4 : Consider the rational function
p
z q z z i z i z ( ) ( ) ( ) ( ) = + + + − − 2 5 3 3 5 1 3 2 4 We can write this as p
z q z z z z i z i ( ) ( ) = − + + + − + + α β γ δ 1 1 (135) = + − g z z 1 1 ( ) α considering z belonging to |z − 1| > 1. Then p z
q z
z g z z ( ) ( ) ( ) ( )( ) − = − + ⇒ = 1 1 2 1
α α 6.14
Partial Fraction Expansion of Meromorphic Functions Let f(z) be a meromorphic function and z 0 be a pole of
order m
with the principal part p z c z z c z z c z z m m m ( ) ( ) ( ) ... = − + − + + − − − + + − 0 1 0 1 1 0 Then
f(z) can be written as [see § 6.2, (14)] f(z) = p(
z) + g(z) where g(z) is
an entire function. Now

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if, in general, z 1 , z 2 , ..., z n are the poles of a meromorphic function f with the corresponding principl parts P 1 , P 2 ..., P n then f can be expressed as f z P z z j j n ( ) ( ) ( ) = + = ∑ ψ 1 (136) where ψ(z) is an entire function. But the question arises whether it is possible to construct a meromorphic function possessing poles at the sequence of points {z n } with corresponding principal parts P 1 , P 2 ... Because in this case the series ΣP j (z) in (136) turns out to be an infinite series P z j j n ( ) = ∑ 1 , which needs to be convergent.

137 Gösta Mittag Leffler (1846-1927), German in origin but his several generations lived in Sweden, overcame this difficulty by introducing a polynomial p n (z) dependent on z n and P n (z) so that the series { ( ) ( )} P z p z n n n − = • ∑ 1 is uniformly convergent in any compact set K not containing any points of the sequence {z n }. Theorem 6.21 [The Mittag Leffler Theorem] : Given a sequence of distinct complex numbers {z n }, z z z n n 1 2 ≤ ≤ = • →• ...,lim and a sequence of rational functions {P n (z)}, P z c z z n nk n k k n ( ) ( ) , , , ,... ln = − ≥ = = ∑ 1 1 1 1 2 n (137) there exists a meromorphic function f(z) having poles at the points z n and only there with P n (z) as its principal part at z n and can be represented in the form of an expansion

f z P z p z h z n n n ( ) [( ) ( )] ( ) = − + = • ∑ 1 where h(z) is an arbitrary entire function

and p n (z) is suitable partial sum of Taylor's expansion of the singular part which is analytic in the open disc |z| > |z n |. Proof. Without loss of generality we assume that z = 0 is not a pole of f(z). Now P k (z) is analytic for |z| > |z k | and can be expanded in this neighbourhood of z : P z c z k j k j j ( ) ( ) = = • ∑ 0 and hence this series converges uniformly in the disk z z k ≤ 2. Let p z c z k j k j j k ( ) ( ) = = ∑ 0 α be a partial sum of this expansion such that P z p z k k k ( ) ( ) − > 1 2 for z z k ≤ 2. Let R be an arbitrary large positive number and since z n → • as n → • we can find an N(R) so large that |z n | < 2R when n ≥ N(R). Therefore in the circle

z R

z

N > > 2 P z p

z P

z p z P

z

p

z

n

n n n n n N R n

n n

N

R ( ) ( ) ( ) ( ) ( ) ( ) ( ) ( ) − = − + − = • = − = • ∑ ∑ ∑ 1 1 1

138

the

first sum in the r.h.s is finite and the second sum ∑ • N R( ) is absolutely and uniformly convergent by comparison with the convergent series ∑ = • − n N R n ( ) 2 . Therefore ∑ − = • n n n P z p z 1 [( ) ( )] is analytic in |z| > R except at the poles belonging to the sequence {z n }. It is thus a meromorphic function with the poles at z 1 , z 2 , ... and with the principal parts P 1 (z), P 2 (z), ... at each point z n respectively. Now if f(z) possesses the same poles only with the same principal parts then f z P z p z n n n ( ) [( ) ( )] − − = • ∑ 1 is an entire function h(z), say. This completes the proof. Example 5 : Prove that π π cot z z z n n n = + − +

=− • • ∑ 1 1 1 ' Solution : The given function π cot πz

has simple poles at z = 0, +1, +2, ... with residue 1.

Here, 1 1 1 1 1 1 2 2

z

n n z n n z n z n z n − = − − = − + + + > ... , (138) Let |z| >

R and N(R)

be so large that R n > 2 when n ≥ N(R). Then from (138), we find 1 1 2 2 z n n R N n

N − + ≤ ≥ ,

Now, since Σ1/N 2 is convergent, the series ' 1 1 z n n n − +

=− • • ∑

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converges uniformly on any compact set (lying in |z| > R) not containing any of the points z = ±1, ±2, ... Therefore applying the
Mittag-Leffler theorem we can express π π cot ( )
z z z n n h z n = + − +
+ =− • • ∑ 1 1 1 ' (139)
139 where h(z) is an entire function.
Differentiating term-wise, we obtain π π 2 2 2 2 1 1 cosec
z
z z n h
z n = + − − =− • • ∑ ' ' ( ) ( ) = − − =− • • ∑ 1 2 ( ) ( ) z n h z n ' and h z z n z f z z
n ' ( ) ( ) ( ) ( ), = − − = − =− • • ∑ 1 2 2 2
π π ψ cosec say (140) We notice that the functions f(z) and ψ(z) are both periodic with period 1 and consequently h'(z) is also periodic with the same period. Let z = x + iy. Consider the strip 0 ≤ x ≤ 1. In fact, the convergence of the series in (140) is uniform for y ≥ 1, say and the limit tends to 0 as y → • (this can be seen on taking the limit in each term of the series). Again,
sin(x + iy) = sin
x cos (iy) + cos x sin (iy) = sin x cosh y + i cos x sinh y
and so sin sin ( ) π π z x iy 2 2 = + = + sin cosh cos sinh 2 2 2 2 π π π π x y x y = − cosh cos 2 2 π π y x which establishes that π 2 cosec 2 πz tends uniformly to zero as y → • . From these we conclude that h'(z) is bounded in the period strip 0 ≤ x ≤ 1 and due to its periodicity it is bounded in the entire plane. By Liouville's theorem it then reduces to a constant. Now since lim ( ) lim ( ) lim ( ) y y y h z f z z →• →• →• ' = − = − = ψ 0 0 0 h'(z) is indeed zero and h(z) = c, a constant. Then from (139), π π cot z z z n n c n = + − + + =− • • ∑ 1 1 1 ' For, z = 1 2 0 2 2 1 2 2 1 2 1 = + − + + + • ∑ k k c
140 = + − + + − + + − + +
+ 2 2 1 1 1 3 1 3 1 5 1 5 1 7 ... c = 2 − 2 + c ⇒ c = 0 i.e. h(z) ≡ 0. Finally we obtain π π cot z = + − +
=− • • ∑ 1 1 1 z z n n z ' Now since the series on the r.h.s is uniformly convergent on any compact set not containing the points z = 0, ±1, ±2 ..., rearrangement of the terms are permissible and hence π π cot z = + − = • ∑ 1 2 2 2 1 z z z n n (141) Remark : Here it is proved incidentally that π π 2 2 2 1 cosec z = − =− • • ∑ ( ) z n n (142) [see equation (140)] We can now utilize the identity (141) to calculate easily some familiar sums. Here the l.h.s of (141) has the Laurent series expansion in the neighbourhood of
z = 0. π π π π π cot ... z z z z z = − − − − 1 3 45 2 945 2 4 3 6 5 Note that the series on the r.h.s of (141) converges uniformly near z = 0. By Th. 4.14 [14] it converges uniformly together with all derivatives. Again 2 2 2 2 2 3 4 5 6
z z
n z n z n z n − = − + + + ... and we obtain easily, 1 6 1 90 1 945 2 1 2 4 1 4 6 1 6 n n n n n
n ≥ = • = • ∑ ∑ ∑ = = =
π
π π , , (143) Example 6. Prove that π π tan z z n n n = − − + + + =− • • ∑ 1 1 2 1 1 2
141 [or, equivalently, π π tan z z n z n = + − = • − ∑ 2 1 2 2 2 0 1]
Solution : Here the given function π tan πz possesses simple poles at z = ± ± 1 2 3 2 , , with residue −1. Then, − − + = + −
+ = + + + + + + 1 1 2 1 1 2 1 1 2 1 1 2 1 1 2 1 2 2
z n
n z n n z n z n and the series − − + − + =− • • ∑ 1 1 2 1 1 2 z n n
n
converges uniformly on any compact set not containing any of the poles of the given function. By Mittag-Leffler theorem, π π tan ( ) z z n n h z n = − − + + + + =− • • ∑ 1 1 2 1 1 2 where h(z) is an arbitray entire function. Now proceeding as in example 5, we can have the desired result. Example 7 : Establish that 1 1 1 2 1 2 4 2 2 2 1
e z z z n z n − = − + + + = • ∑ π Solution : We rewrite 1/e z − 1 as 1 1 1 2 1 2 1 2 2 2 2 2 2 2 2 2 2 2
e e e e e e e e e
e
z z z z z z z z z z z − = − = − + + − = − + − − − − − / / / / / / / / /
coth
142 But coth cosh sinh cos sin cot z z z i i z i z

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i i z 2 2 2 2 2 2 2 = = =
Now
utilising (141) we get the result. 6.15 Partial Fraction Expansion of Meromorphic Functions Using Residue theorem Let us suppose f to be
a meromorphic function whose only singularities are simple poles z 1 , z 2 , ... with
increasing moduli 0 1 2 > ≤ ≤ z z ..., lim n n z →• = •and Res (f(z); z n) = A n . Suppose there exists a sequence {C n } of simple closed contours such that (i) C n does not contain any of the poles z k (ii) each C n lies inside C n+1 (iii) min z C n n z R ∈ = → + • → + • as n (iv) length of C n is 0(R n) (v) max ( ) ( ) z C n n f z R ∈ = 0 Then f z f A z z z k k k k ( ) ( ) = + − + = • ∑ 0 1 1 1 (144) The series (144) converges uniformly in any bounded domain not containing the poles of f(z). To prove the above result we consider the integral I z i zf z n C n ( ) ( ) ( ) = − 1 2π ς ς ς ς d (145) where z ∈ Int C n and z ≠ z k (k = 1, 2, ...) Here the integrand in (145) possesses simple poles at ς = 0, ς = z and ς = z k ∈ Int C n . Then using the Residue theorem, we find from (145) that I z zf z zf z s f z n z z k k ( ) ( ) ( ) ( ) Re (( );) = − + + − = = = ς ς ς ς ς ς ς ς ς ς 0 1 143 = − + + − ∈ ∑
f f z zA z z z k k k z tC k n ( ) ( ) ( ) ln 0 Thus, f z f A z z z i zf z
d k k k z tC C k n n ( ) ( ) ( ) ( ) ln = + − + + − ∈ ∑ 0 1 1 1 2π ς ς ς ς (146) We now show that lim ( ) n n I z →• = 0 for |z| > R. I z z f z d R f R d n C C n n ( ) | | ( ) ( ) ≤ − > − → 2 2 0 π ς ς ς ς π ς ς ς ς as n → • by the given conditions (iii), (iv) and (v). Then (144) follows from (146) considering all the contours C 1 , C 2 , ... etc. Example 8 : If α n are positive roots of the equation tan
z = z, show that
z z z z z z z z
n
n
sin sin cos − = + − = • ∑ 3 2 2 2 1 α where n n n − > > + 1 2 1 2
π α π. Solution : Given α n are positive roots of tan z = z, so ± α n are roots of sin z − z cos z = 0. To check whether the function
f(
z)/g(z), where f(
z) = z sin z and g(z) = sin z − z cos z, has any pole at z = 0 we notice that ′ = +
f z z z z ( ) sin cos ′ = =
g z z z
f z ( ) sin ( ) ″ = − f z z z z ( )
cos sin 2 ″ = ′ g z f z ( ) ( ) ′ = ″ ≠
f
f ( ) ( ) 0 0 0 0 but ″ = ‴ f z g z ( ) ( )
so
g
g g , ( ) ( ) ( )
but ′ = ″ = ‴ ≠ 0 0 0 0 0 Thus origin is the double zero of f(z) and triple zero of g(z). As a result the given function f/g possesses
a simple pole at z = 0. To find its residue at z = 0 we note that ″ ″ = ‴ ‴ = f z z g z z ( ) ( ) ( ) ( ) 2 3 1 1 3 and and so residue there is 3. Thus
the function F z z z z z z z ( )
sin sin cos = − − 3 has the
144 simple poles at z = ±
α n as its only singularities and Res (F(z); ± α n) = 1 and F(0) = 0 since F(z) = −F(−z). Since n n n − > > + 1 2 1 2 π α π, we consider the sequence of contours {C n }, formed by the straight lines x = ± b n , y = ± b n with b n = n + 1 2 π , n = 1, 2..., A n B n P n Q n shown below : We find that when z ∈ B n P n , z = b n + iy, where − b n ≤ y ≤ b n . Hence, cot cos sin z n iy n iy = + +
+ +
1 2 1 2 π π = = − + − − sin( ) cos( ) iy iy e e e e y y y y (147) Same result holds when z ∈ A n Q n . Now when z lies on either of the lines A n B n or Q n P n , z = x ± i n+ 1 2 π cot cos sin sinh cosh z x i n x i n n n = ± +

± +
≥ + + 1 2 1 2 1 2 1 2 π π π π = − + ≥ − + − + − + 1 1 1 1 2 1 2 1 e e e e n n ( ) ( ) π π π π (148) The given function can be rewritten as z z z z z z z z sin sin cos cot − = − 1 1 B n A n P n Q n x y o b n −b n −b n
145 I. Bound on the sides A n Q n & B n P n of the square C n : Using (147), we obtain 1 1 1 1 1 1 1 2 2 z z z z e e e e b y y y y y n − ≤ − = − + − + → → • − − cot cot . as n II. Bound on the sides A n B n & Q n P n of C n : Here we apply (148) to achieve 1 1 1 1 1 1 1 1 1 1 2 2 z z z z e e b y e e n − ≤ − ≤ − + − + → + − → • cot cot . π π π π as n Thus, z z z z z e e z C n n sin sin cos , , , ,... − ≤ + − ∈ = π π 1 1 1 2 This shows that the function F(z) is bounded on the sequence of contours {C n } and we can apply (144) to prove
z z z z z z z n n n n
n
sin sin cos − = + − + + + − = • ∑ 3 2 1 1 1 1 1 α α α α = + − = • ∑ 3 2 2 2 2 1 z z n n
α Exercises 1. Obtain partial fraction expansion of cosec z. 2. Prove that sec ( ) ( ) z n z n n n = − − − − = • ∑ 1 2 1 1 2 2 2 2 1 π π 3. Show that tan z z z n n = − − − = • ∑ 2 1 2 2 2 2 1 π
146 and hence deduce 1 1 3 1 5 8 2 2 2 + + + = π 6.16 The Gamma Function The gamma function Γ(z) was introduced by Swedish Mathematician L. Euler (1707- 1783), in 1729 while he was seeking for a function of a real variable x which is continuous for positive x and reduces to x! when x is a positive integer. Gamma function is widely used in the fields of probability and statistics, as well as combinatorics. Gamma function Γ(z) can be introduced in either of the ways : (i) in terms of infinite product (ii) in the form of infinite integral (iii) in limit formula We establish the form (i) first considering the fact that it possesses simple poles at z = 0, −1, −2, ... and nowhere vanishes in the entire plane and satisfies zΓ(z) = Γ(z + 1), Γ(1) = 1 (149) To construct Γ(z) we claim that f(z) = 1/Γ(z) is entire with simple zeros at z = −n (n = 0, 1, 2, ...). Again we know that k = 1 is the largest non-negative integer for which 1 1 n k n= • ∑ diverges. Then utilizing the Weierstrass Factorization theorem f(z) can be represented as f z ze
z n e g z n z n ( ) ( ) = + = • − ∏ 1 1 where g(z) is an entire function,
so that gamma function will be of the form Γ( ) ( ) / z e z z n e g z z n = + − − • ∏ 1 1 1 (150) Now we find g(z) so that (149) hold. We write (150) in the form
147 Γ( ) lim ( )
z e z z m e n g z z m n = + →• − − ∏ 1 1 = − + + + = →• →• ∑ lim !exp ( ) ( ) ( ) lim ( ), n
n n n n
g
z
z m
z
z
z n z 1 1 Γ say (151) z z z n z g z z m z z z n z z z
n
n g z z
m
n
n n n
Γ Γ ( ) ( ) ! exp ( ) ( ) ( ) ( )( ) ( ) !exp ( ) + = − + + + + + + + − + + + ∑ ∑ 1 1 1 2 1 1 1 1 1 = + + + − − ∑ ( ) exp ( ) ( ) z
n
g z g z m n 1 1 1 1 = + + + − − ∑ 1 1 1 1 1 z
n n g z
g z
m n exp ( ) ( ) = + + + − − + ∑ 1 1 1 1 1 z n g z g z
m n n
exp ( ) ( ) log Now from the relation
z z z z z z z n n
n Γ Γ Γ Γ ( ) ( ) lim ( ) ( ) , + = + →• 1 1 we find that z z z z n g z g z m n n
n

Γ Γ ( ) ( ) lim exp ( ) ( ) log + = + + + − − + →• ∑ 1 1 1 1 1 1 = + − − exp ( ) ( ) g z g z 1 γ where γ = − = · →• ∑ lim log n n m n 1 0 57722 1 (152) is known as the Euler's constant. Thus in order that the conditions in (149) to hold, we should have g(z + 1) − g(z) = γ + 2kπi (k ≡ integer) (153) and
148 1 1 1 1 1 1 1 1 = = = ∑ + = →• →• − + − − + Γ Γ ( ) lim ( ) lim ( ) log ( ) n n n g z m n g e n e n γ so that g(1) = γ + 2jπi (j ≡ integer) (154) The simplest entire function satisfying (154) is given by g(z) = γz Finally from (150), Γ( ) / z e z z n e z z n = + − − • ∏ γ 1 1 1 (155) Gauss's Formula From (151) we have the representation Γ( ) lim !exp ( ) ( )
z
n m
z z z
z n n n = − + + →• ∑ 1 1 1 γ = − − +
+ + →• ∑ lim !exp log log ( ) ( ) n
n n m n n
z z z z n 1 1 1
γ = + + − − = →• →• ∑ lim ! ( ) ( ) , lim log n z n n n n
z z
z
n
m n 1 1 0 1
since γ (156) The above expression for Γ(z), z ≠ 0, − 1, − 2, ... is termed as Gauss's formula, though it was first derived by Euler. In many places it is known as Euler's limit formula. Example 9 : Let Γ(,) ! ( ) ( )
z
n
n n
z z z n z = + + 1 Prove that Γ Γ Γ Γ (,) ( ) ( ) ( )
z
n n n z n
z z = + + + 1 1
149 and hence deduce that
n
n n z as n z
Γ Γ ( ) ( ) + → → • 1 Solution : Γ(
n +
z + 1) =
z(
z + 1)(
z + 2)......(
z + n) Γ(
z) so,
n n
z n z
n n z z z
z
n n n z
z z z n z n z
z
z
Γ Γ Γ Γ Γ ( ) ( ) ( ) ( ) ( )( ) ( ) ! ( )( ) ( ) (,) + + + = + + + + = + + + = 1 1 1 1 2 1 2 Now,
n
n n z n z z n
n
z

z
Γ Γ Γ Γ ( ) ( ) ( ) (,) ( ) + = + lim ( ) ( ) lim lim (,) ( )
n
z n n n n n
z z
n z
n
z →• →• →• + = + =
Γ Γ Γ Γ 1 1 by Gauss's formula. In the expression (155) for Γ(z) the infinite product is uniformly convergent on every compact subset of /C − {0, −1, ...}. So calculating Γ′(z)/Γ(z) we find that ′ = − − + − + + = • ∑ Γ Γ ( ) ( ) z z z n z n n γ 1 1 1 1 This function ′Γ Γ ( ) ( ) z z is denoted by ψ(z) and named as Gaussian psi function and it is seen from its expression that ψ is meromorphic in /C with simple poles at z = 0, −1, − 2, ... and Res(ψ; −n) = −1 for n = 0, 1, 2, ... Example 10 : Show that (i) ψ(1) = −γ (ii) ψ ψ ( ) ( ) z z z + − = 1 1 (iii) ψ ψ π π ( ) ( ) cot .
z z z − − = − 1 Solution : (i) ψ γ ( )z z n z n n = − − + − + + = • ∑ 1 1 1 1
150 so, ψ γ ( )1 1 1 1 1 1 = − − + − + + = • ∑ n n n = − − + − + − + − γ 1 1 1 2 1 2 1 3 1 3 = −γ. (ii) ψ ψ γ γ ( ) ( )
z
z
z n
z n n
z n
z n n + − = − − + + − + + + − − + + + + = • = • ∑ ∑ 1 1 1 1 1 1 1 1 1 1 = − + + + − + + =− • ∑ 1 1 1 1 1 1 1
z
z n z n z n = − + + + − + + + − + + 1 1 1 1 1 1 1 2 1 2 1 3
z
z
z z
z
z = 1 z . (
iii) ψ
ψ ( ) ( )
z z z z n n z n n z − − = − + − + − + − − + − • • ∑ ∑ 1 1 1 1 1 1 1 1 1 1 1 = − − − + + − − + • ∑ 1 1 1 1 1 1 1 z z n z
n z = − − − − + − − − + − 1 1 1 1 1 1 2 1 2 z z
z
z
z = − − − + + − − + + − 1 1 1 1 1 1 2 1 2 z z
z
z
z = − − − − = − • ∑ 1 2 141 2 2 1 z
z
z
n
z
by
π π cot , ( ) 6.17 A Few Properties of ΓΓΓΓΓ(
z)
We have 1 1 1 Γ( ) /
z e
z
z n e z z n = + • − ∏ γ Hence, 1 1 2 2 2 1 Γ Γ ( ) ( ) z z z z
n − = − − • ∏
151 = − − • ∏ z

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z z
n
π π 1 2 2 1 = − z z
π
π sin or, 1 Γ Γ ( )[( )] sin z z z z − − =
π π i.e. 1 1 Γ Γ ( ) ( ) sin , z z
z − =
π
π [using zΓ(z) = Γ(z + 1) i.e., − zΓ(−z) = Γ(1 − z)] (157)
In particular, Γ 1 2 2 = π and Γ 1 2 = π (minus sign is excluded since Γ 1 2 is positive by (155)). Likewise using Γ(z + 1) = zΓ(z) we find Γ Γ 3 2 1 2 1 2 1 2 = = π Γ Γ 5 2 3 2 3 2 3 2 1 2 = = . π Γ Γ 7 2 5 2 5 2 5 2 3 2 1 2 = = · · π and in general Γ n n n n + = − = 1 2 13 2 1 2 1 2 . ( ) , (, , π i.
e. Γ n n n n + = 1 2 2 2 2 / ( )! !( ) π (158) If n is a positive integer repeated use of (149) produce Γ( ) ! n n + = 1 The Γ-function can therefore be considered as an extension of the factorial function to the complex plane.
152 Legendre's Duplication Formula Let us consider the Gauss's formula Γ Γ ( ) lim ! ( ) ( ) lim (,),
z
n n
z z
z n
z n n z n = + + = →• →• 1 say Then, Γ(,) ( )!( ) ( ) ( ) ( ) 2 2 2 2 2 2 1 2 2 2 2 z
n n n
z
z z n n z z = + + + = + + + + − 2 1 2 2 2 2 1 2 2 2 2 2 1 2 n z n n n
z z
z z
n ! ( ) ( )( ) ( ) Γ π [Replacing (2n)! by (158)] = + + + + + + + − − 2 1 2 1 2 1 2 3 2 1 2 2 1 2
z
z
n
n n
z z
z z
n z
z z
n !( ) ( )( ) ( ) Γ π = + + + + − − 2 1 2 1 1 2 3 2 1 2 2 1 z z
n n z z
z
n π Γ Γ (,) = + + + + − 2 1 2 1 2 1 2 2 1 1 2 z z
n n
z
n n n z
n
n
π
Γ Γ Γ Γ (,) , ( ) / and Γ Γ Γ Γ Γ
Γ ( ) lim (,) ( )
lim ( ) / 2 2 2 2 1 2 1 2 1 2 2 1 1 2
z
z
n z z n n n z n n n z
n = = + + + + →• − →•

π = + − 2 1 2 2 1 z
z
z
π Γ Γ ( ) [using example 9] So that πΓ Γ Γ ( ) ( ) 2 2 1 2 2 1
z z z z = + − (159)
153 This is known as Legendre’s duplication formula.
Residue of ΓΓΓΓΓ(z) at its poles Γ(z) is analytic throughout the complex plane except at its only singularities which are simple poles situated at z = 0, −1, −2, ... That is Γ(z) is analytic in the right half of the complex plane Re z < 0. Using the fact that zΓ(z) = Γ(
z + 1),
we have Γ
Γ ( ) ( )( )( ) ( ) ( ),
z
n
z n z n z n z z z
n + + = + + − + − + ≡ 1 1 2 1
positive integer and Γ Γ ( ) ( ) ( ) ( )( )
z z n
z z
z n z
n = + + + + − + 1 1 1 Res (( );) lim( ) ( ) − Γ Γ
z n z n z z z n − = + → = + + + + − → lim ( ) ( )....( ) − z n z
n
z
z z
n
Γ 1 1 1 = − = ( ) ! , , , ,... 1 0 1 2 n
n
n
Integral representation of ΓΓΓΓΓ(z) Theorem : Prove that Γ( )
z e t t z = − − • 1 0 dt for Re z < 0. Proof. Let
F
z n n z z z n n z ( ) ! ( )....( ) = + + 1
We prove the theorem in the following two steps : (i) F z t n t dt n n z n ( )= − − 1 1 0 (ii) lim n n z t z n
t n
t dt e t dt →• − − − • − = 1 1 1 0 0 To establish (i) we change the variable t to ns in 1 1 0 − − t n t dt n z n to obtain 1 1 1 1 0
1 0 − = − − − t n t dt n s s ds n z z n z n ( )
154 Now integrating by parts we find the right hand side is equal to
n
z
s
s
n z s s ds
z
z
n n z 1 1 1 0 1 1 0 1 ( ) ( ) − + − − = − − n n z s s ds z n z ( ) 1 1 0 1 = − + + − + − n n n z
z z
n
s ds z
z
n .( ).... ( )....( ) 1 1 1 1 1 0 1 [

Integrating by parts (n − 1) times] = + + = n n
z
z
z
n
F z
n ! ( )....( ) ( ) 2 1
Now to prove (ii)we show that lim , − n
t
n z n e t n t dt →• − − − = 1 0 1 0 Re z < 0 (161) For this, note that 1 1 1 + ≤ ≤ − >
t n e t n for t n t n (162) Then, 1 1 + ≤ − ≤ −
t n e and t n e n t n t ; Consequently, 0 1 1 1 1 1 2 2 ≤ − − = − − ≤ − − − − − e t n e e t n e t
n t n t t n t n = + − + + − ≤ − − − e t n t n t n e t n t n
t 2 2 2 2 2 2 1 2 1 1 1 . Therefore, e t n t dt
n e t dt t
n
z n t z n − − − + − − > 1 1 1 0 1 0 Re
155 which approaches zero as n → • because the integral on the right converges. This completes the proof of (ii). Finally combining the results (i) and (ii) with the Gauss's formula (156) we get Γ( ) lim ( ) lim z F z t n t dt e t dt n n n n z t z n = = − = →• →• − − − • 1 1 1 0 0

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Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely

in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind.

We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'.

For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University. Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned. Chandan Basu

Vice-Chancellor

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7 UNIT - 1 1.1 Calculus on R n : Let R denote the set of real numbers. For an integer n < 0, let R n be the cartesian product R R R R ? ? ? ? ? ?? n times of the set of all ordered n-tuples (, ,) x x n 1 ? of real numbers. Individual n-tuple will be denoted at times by a single letter, e.g. x x x n y y y n ? ? (, ,), (, ,) 1 1 ? ? and so on. Co-ordinate functions : Let 1 2 (, ,) . ? ? n n i x x x x R Then, the functions : i u ? n R R defined by 1 2 (, ,) ? ? ? i n i i u x x x x x We are now going to define a function to be differentiable of class C . A real-valued function f U C R R n : ? , U being an open set of R n , is said to be of class c k if i) all its partial derivatives of order less than or equal to k exist and ii) are continuous functions at every point of U. By class C 0 , we mean that f is merely continuous from U to R. By class C , we mean that that partial derivatives of all orders of f exist and are continuous at every point of U. In this case, f is said to be a smooth function. Note : By class C on U, we mean that f is real analytic on U i.e. expandable in a power series about each point on U. A C function is a C function but the converse is not true. Exercise : 1. Let f R R : ? be defined by 1 2 ( ) , ? ? x f x e x ? 0 = 0, x = 0 Show that f is a differentiable function of class C . Solution : Note that ? ? ? ? ? ? ? ? f o h f o h f o h h e h h ( ) lim ( ) ( ) lim 0 0 1 2 Apply L'Hospital's Rule, on taking, h u ? 1 we see that h o? gives u ? ?
8 ? ? ? ? ? ? f o u u e u ( ) lim . 2 ? ? ? ? ? ? F H G I K J lim u u e u 2 lim u ue u 1 2 2 ? ? ? ? lim u e u u 2 2 = 0 Again, 2 1 3 ( ) 2 , 0 ? ? ? ? ? ? x f x x e x ? ??f o( ) ? ? ? ? ? ? ? lim ( ) ( ) h f h f h 0 0 0 and on putting 1 ,? u h we get 2 4 lim 2 ( ) ? ? ? ? ?? ? ? ? ? ? ? ? ? u u f o u e Applying L' Hospital rule successively, we find ??f ( )0 ? ? ? lim u u ue u 8 2 3 2 ? ? ? lim u u e u 4 2 2 ? ? ? lim u u ue u 8 2 2 ? ? ? lim u e u 4 2 = 0

9 Proceding in this manner, we can show that. f n ( ) , 0 0 for n ?1 2, ,? Hence f is a function of class C . A mapping f U V : ? of an open set U R n to an open set V R n is called a homeomorphism if i) f is bijective i.e. one to one and onto, as well as ii) f, f −1 are continuous. Exercise : 2. Let f R R : ? be such that f x x ( ) ? ?5 3 Show that f is a homoeomorphism on R. 3. Let f R R : ? be defined by f x x ( ) ? 3 Test i) whether f is a differentiable function of class C or not ii) whether f is a homeomorphism or not. [Ans. : i) f is of class C . ii) f is homeomorphism] Solution : 2. Note that f x f y x y ( ) ( ) ( ) ? ? ?5 ? ? f x f y ( ) ( ) if and only if x y? Hence f is one one. Let y x? ?5 3 ? x y ? ?3 5 and hence f R R ? ? 1 : is defined as f y y ? ? ? 1 3 5 ( ) Again, f f y y ? ? 1 ( ) b g and f f x x ? ? 1 ( ) b g , Thus f is onto. Consequently f is bijective. f U V ? R n
10 Both f f, ?1 are continuous functions, (being polynomial functions) f is a homeomorphism on R. Note : (i) If f U R R n m : is a mapping, such that f x x f x x f x x n n m n (,) (, ,), , (, ,) 1 1 1 1 ? ? ? ? ? ? b g where ( ) , ? ? j j f x u f 1 , ? ? j j m u being co-ordinate functions on m R we define the Jacobian matrix of f at (, ,), x x n 1 ? denoted by J, as J ?
f
x f x f x f x f x f x f x f
x
n
n m
m m n 1 1 1 2 1 2 1 2 2 2 1 2 ? ? ??????? ? F H G G G G G G G G I K J J J J J J J J (ii) In particular, when ? m n i.e., if : ? ? n n f U R R is a mapping such that, if f f f (, , ,) 1 2 ? f n has continuous partial derivatives i.e. if each f i 1,2, , . ? ?i n has continuous partial derivatives on U, we say that f is continuously differentiable on . ? n U R (iii) If f f f n (, ,) 1 ? is continuously differentiable on ? n U R and the Jacobian is non- zero, then f is one-one on U. Exercise : 4. Consider the mapping ??: R R 2 2 ? given by ??: y x x 1 1 2 ? cos y x x 2 1 2 ? sin n R f v m R i u R fCU
11 Show that ? is one-to-one on a sufficiently small neighbourhood of each point (,) x x 1 2 of R 2 with x 1 0? . Solution : The given mapping 1 2 2 2 (,):
R R ? ? ? ? ? ? is given by 1 1 2 2 1 2 cos , sin
x x x x ? ? ? ? Then, we have 1 1 2 x xcos , ?? ? 1 2 1 2 x x x ? ? sin , ?? ? 2 1 2 x x ? sin , ?? ? 2 2 1 2 x x
x ?
cos Hence each ?? ? i j x , , 1,2 i j ?
is continuous for all values of x 1 and x 2 in R 2 . Thus ? is continuously differentiable on R 2 . Again the Jacobian is given by J = ?? ? 1 1 x ?? ? 1 2 x ? ? x 1 0 if and only if x 1 0? in R 2 . ?? ? 2 1 x ?? ? 2 2 x Consequently, ? is one-to-one on a sufficiently small neighbourhood of each point (,) x x 1 2 of R 2 with x 1 0? . A mapping f : U ? V of an open set U R n onto an open set V R n is called a C k − diffeomorphism, k ?1 if i) f is a homeomorphism of U onto V and ii) f, f −1 are of class C k . when f is a C − diffeomorphism, we simply say diffeomorphism. Exercise : 5. Let ? : R R 2 2 ? be defined by ? (,) (,) u v ve u u ?
12 Determine whether ? is a diffeomorphism or not. 6. Let ? :
R R 2 2 ? be defined by ?(,) (,) x x x e x x e x x x 1 2 1 2 1 2 2 2 ? ? ? Show that ? is a diffeomorphism. [
Ans. : 5. ? is a diffeomorphism] For i n ?1, , ; ? let : i n u R R ? be the coordinate functions on n R i.e. for every p R n ? 1. 1) u p p i i ( ) ? where p p p n ? (, ,) 1 ? Such u s i are continuous functions from R R n ? .. We call this n-tuple of functions (, , ,) u u u n 1 2 ? the standard co-ordinate system of R n . If f U R R n n : is a mapping defined on U R n , then, f is determined by its co-ordinate functions (, ,) f f n 1 ? where 1.2) f u f i n i i ? ? , , ,1 and each f U R R i n : are real valued functions, defined on an open subset U of R n . Thus for every p ? U R n f p u f p i i ( ) ( )( ) ? ? ? ? ( ) i u f p ? where f p q q q n ( ) (, ,) ? ? 1 ? (, ,) i i n u q q ? ? ? q i by 1.1) 1.3) consequently f p f p f p f p n ( ) ( ), ( ), , ( ) , 1 2 ? c h p U R n The map f is of class c k if each of its co-ordinate functions f i n i : , , ?1 ? is of class c k . R R n u i ?
13 1.2 Differentiable Mainfold : Let M be a Hausdorff, second countable space. If every point of M has a neighbourhood homeomorphic
to an open set in R n , then M is said to be a manifold.

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Thus in a manifold for each p M? , there exists a neighbourhood U of p M? and a homeomorphism ? of U onto an open subset of R n . The pair (,) U ? is called a chart. Each such chart (,) U ? on M induces a set of n real valued functions on U defined by 2.1) x u i i ? ?? , i n ?1 2, ,? where u s i , are defined by (1.1) and it is to be noted that whatever be the point p and the neighbourhood , , 1,2, i U u i n ? ? always represent co-ordinate functions. The functions (, ,) x x x n 1 2 ? are called coordinate functions or a coordinate system on U and U is called the domain of the coordinate system. The chart (,) U ? is sometimes called an n-coordinate chart. Let (,) V ? be another chart of p, which overlaps the previous chart (,). U ? Let (, ,) y y n 1 ? be local coordinate system on V of p, so that U V p. M R n ( ) U V ( )U ( )V ( ) U V ? R 1? ? 1? ? ? ? ? ??( )p ?( )U R n u i ?p M U R i x

14 2.2) y u i n i i ? ? ? ? ?, , , , , 1 2 We can construct two composite maps 2.3) 1 : ( ) ( ) U V R U V R n n 1 : ( ) ( ) U V R U V R n n If these maps are of class c k , we say that the two charts (,) U ? and (,) V ? are c k - related. If q U V ? ? ?( ) and g U V R U V R n n : ( ) ( ) is a mapping defined on an open set in R n , we write 2.4) g q q ( ) ( ) . ? ? ? ? 1 b g Exercise : 1 Find a functional relation between the two local coordinate systems defined in the overlap region of any point of a manifold M. Solution : given that q U V ( ), g q q ( ) ( ) ( ) ? 1 by 2.4) Let ( ) , p q where p U V. Then g p p p ( ) ( ) ( ) b g b gb g ? 1 or ? ? ? ? (( )) ( ) , 1,2, , i i u g p u p i n ? ? ? ? ? or g p p i i ( ) ( ) b g by 1.1) or g x p x p y p i n i ( ), , ( ) ( ), 1 ? b g as x p u p p i i i ( ) ( ) ( ) b g ? ? ? ? ? 1 1 ( ) ( ) , , ( ) ( ), , ( ) ? ? ? ? ? ? ? n n p p p x p x p and ? ? ( ) ( ) ( ) 1,2, . i i i y p u p p i n ? ? ? ? ? ? consequently, y q x x x i i n ? (, , ,) 1 2 ? Note : If we consider g q q ( ) ( ) , 1 c h

15 then one finds x g y y y i i n ? (, , ,), 1 2 ? i n ?1, ,? A collection ? ? (,) , , i i U i A ? ? ? ? (an index set) of c k related charts are said to be maximal collection if a co-ordinate pair (V, ?), c k related with every chart is also a member of ? . A maximal collection of c k -related charts is called a c k -atlas. A c k n-dimensional differen- tiable manifold M is an n-dimensional manifold M together with a c k -atlas. Unless otherwise stated, we shall consider a differentiable manifold of class C . Examples : 1. R n with the usual topology is an example of a differentiable manifold with respect to the atlas (U, ?) where U = R n and ? = the identity transformation. 2. Let S 1 be the circle in the xy plane R 2 , centered at the origin and of radius 1. We give S 1 , the topology of a subspace of R 2 . Let

U p x y s y 1 1 0 ? ? ? ? { (,) | } U p x y s y 2 1 0 ? ? ? ? { (,) | } U p x y s x 3 1 0 ? ? ? ? { (,) | } U p x y s x 4 1 0 ? ? ? ? { (,) | }

Then each U i is an open subset of S 1 and , 1,2,3,4 ? ? ? i i S UU i Let I = (−1, 1) be an open interval of R and we define ? 1

1 :U ? I R be such

that ? 1 (,)

x y x? i.e. 1 1 ( ) (,), 0 x x y y ? ? ? ? ? ? 2 2 :U ? I R be such that ? 2 (,) x y x? i.e. 1 2 ( ) (,), 0 x x y y ? ? ? ? ? ? 3 3 :U ? I R be such that ? 3 (,) x y y? i.e. 1 3 ( ) (,), 0 y x y x ? ? ? ? ? ? 4 4 :U ? I R be such that ? 4 (,) x y y? i.e. 1 4 ( ) (,), 0 y x y x ? ? ? ?

Note that each ? i is a

homeomorphism on R and thus each (,) u i i ? is a chart of . ? S Now U U 1 2 ? ? ?, U U st 1 3 1 ? ? quadrant, U U nd 1 4 2 ? ? quadrant, U U th 2 3 4 ? ? quadrant, U U rd 2 4 3 ? ? quadrant.

16 Then A U i i i ? ? {(,): , , ,) ? 1 2 3 4 is an atlas of s 1 As U U? ? 3 ?, let p U U ? ? 1 3 , then ( )( ) (,) ? ? ? 1 3 1 1 ? ? ? ? y x y x and ( )( ) (,) ? ? ? 3 1 1 3 ? ? ? ? x x y y Thus each ? ? 1 3 1 ? ? and ? ? 3 1 1 ? ? is of class C . Similarly, it can be shown that each ? ? 1 4 1 ? ? , ? ? 4 1 1 ? ? , ? ? 2 3 1 ? ? , ? ? 3 2 1 ? ? , ? ? 2 4 1 ? ? , ? ? 4 2 1 ? ? , is of class C and hence s 1 is an one dimensional differentiable manifold with an atlas ? ? 1,2,3,4 (,) i i i U ? ? Exercise : 2. Let (M n , A) be a differentiable manifold with a C atlas A. Let p M. Then there exists (,) U A such that p U and ( ) . p 0 Note : 1. It is to be noted that every second countable, Hausdorff Space M admits parti- tions of unity. Partitions of unity admits Riemannian metric. Our aim is to study a Riemannian Manifold and for this reason we consider such topological spaces for a manifold. 2. It is enough to consider only a topological space for studying mainfold. 1.3. Differentiable Mapping : Let M be an n-dimensional and M be an m-dimensional differentiable manifold. A mapping f M N : ? . is said to be a differentiable mapping of class c k , if for every chart (U, ?) containing p of M and every chart (V, ?) containing f(p) of N R m M f .p . f(p) f(U) N R n ? ? ? ?f ?1 ? ? . ( )p ( )U (( )) . f p ( )V U V

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17 3.1) i) f(U) V and ii) the mapping ? ?f U R V R n m 1 : ( ) ( ) is of class c k . By a differentiable mapping, we shall mean, unless otherwise stated, a mapping of class C . If (, ,) x x n 1 ? and (, ,) y y m 1 ? are respectively the local coordinate systems defined in a neighbourhood U of p of M and V of f(p) of N, then it can be shown, as done earlier 3.2) y g j j n f x x ? ? ? (, ,), 1 j m ?1, ,? where g is a differentiable function defined on V ? N and 3.3) g q f q ( ) ( )( ), ? ? ? ? ? ? ? 1 q U ?? ( ). Let M and N be two n-dimensional differentiable manifolds. A mapping f M N : ? is called a diffeomorphism if i) f and f −1 are differentiable mappings of class C ii) f is a bijection In such cases, M and N are said to be diffeomorphic to each other. Exercise : 1. Let M and N be two differentiable manifolds with M=N=R. Let (U, ?) and (V, ?) be two charts on M and N respectively, where U = R ? : U ? R be the identity mapping and V = R ? : V ? R be the mapping defined by ?( ) . x x? 3 Show that the two structures defined on R are not C -related even though M and N are diffeomorphic where f M N : ? 18 is defined by f t t ( ) / ? 1 3 Hint : Note that, ( ) ( ) ? ? ? ?f x x ? ? 1 and ( )( ) . / ? ?? ? ? 1 1 3 x x Thus ? ?? ?1 is of class C but ? ?? ?1 is not of class C . Again ( ) ( ) ? ? ? ?f x x ? ? 1 Also f y f x ( ) ( ) ? if and only if y x? . Thus f is one-one. Finally f y y ? ? 1 3 ( ) , so that f f y y ? ? 1 ( ) b g and f f x x 1 ( ) . b g ? Thus
f is a bijection. Note : A diffeomorphism f of M onto itself is called a transformation
of M. A real-valued function on M ; i.e. f : M ? R is said to be a differentiable function of class C , if for every chart (U, ?) containing p of M, the function 3.4) f U R R n ?? ? ? ? ? ? 1 : ( ) is of class C . We shall often denote by F(M), the set of all differentiable functions on M and will sometimes denote by F(p), the set of functions on M which are differentiable at p of M. R n R M f U ? (U) f ?? ?1 .p ??( )p ? 1? ?
19 It is to be noted that such F(M) is i) an algebra over R ii) a ring over R iii) an associative algebra over R and iv) a module over R Where the defining relations are a) ( )( ) ( ) ( )
f g p f p g p ? ? ? b) ( )( ) ( ) ( ) fg p f p g p ? c) ( )( ) ( ), ? ? f
p f p ? ? ? f g F M , ( ), ? ?R, p M? . 1.4. Differentiable Curve : We are now in a position to define a curve on a manifold. A differentiable curve through p in M of class r C is a differentiable mapping ?:[,] a b R M ? ? , namely the restriction of a differentiable mapping of class r C of an open interval] c, d [containing [a, b]. such that 4.1) ?( )t p 0 ? , a t b ? ? 0 Also 4.2) ( ) ( ) ( ) ( ) (( )) x t u t u t i i i ? ? ? ? ? ? ? ? ? b g b g ? ? u t t t i n i ? ? ? 1 ( ), , ( ) ( ) ? b g We write it as 4.3) x t t i i ( ) ( ) ? ? The tangent vector to the curve ?( )t at p is a function R R n u i M R [ ] ? 0 t ? 0 ( ) p t ? ?
20 X F p R p : ( ) ? defined by 4.4) X f d dt f t p t t ? L N M O Q P ? ? (( )) ? 0 lim (( ) (( ) h f t h f t h t t ? ? ? L N M O Q P ? 0 0 ? ? where p t f F p ? ? ? ?( ), ( ) 0 It can be shown that it satisfies 4.5) X af bg a X f b X g p p p ( ) ( ) ( ) ? ? ? : Linearity 4.6) X fg g p X f f p X g p p p ( ) ( ) ( ) , ? ? f g F p , ( ) ? : Leibnitz Product Rule. Note : Each function X p : F (p) ? R, cannot be a tangent vector to some curve at p?M, unless it is a linear function and satisfies Leibnitz Product Rule. Exercises : 1. Let a curve ? on R n be given by ? i i i a b t ? ? , i n ?1 2, , ,? Find the tangent vector to the curve ? at the point ( ). a i 2. If C is a constant function on M and X is a tangent vector to some curve ? at p?M, then X p .C = 0 [Ans. i) (, , ,) b b b n 1 2 ? ii) use 4.5), 4.6) and the definition of constant function. Let us define 4.7) ( ) X Y f X f Y f p p p p ? ? ? 4.8) ( ) bX bX f p p ? , b?R If we denote the set of tangent vectors to M at p by T p (M), then from 4.7) and 4.8) it is easy to verify that T p (M) is a vector space over R. We are now going to determine the basis of such vector space. For each i = 1, ... , n, we define a mapping ? ?x F p R i : ( ) ?
21 by 4.9) ? ? ? ? x f f x t p i p i F H I K ? F H G I K J ( ) ( ) Note that ? ?x af bg i p F H I K ? ? ( ) ? ? ( ) ( ) ( ) af bg x t p i ? F H G I K J by 4.9) , a, b R, f, g?F (p) ? F H G I K J ? F H G I K J a f x t p b g x t p i i ? ? ? ? ( ) ( ) ( ) ( ) by a) of 1.3 ? F H G I K J ? F H G I K J a f x t p b g x t i i ? ? ? ? ( ) ( ) ( ) by a) of 1.3 ? F H I K ? F H I K a x f b x g i p i ? ? ? ? Thus such a mapping satisfies linearity property. It can be shown that ? ? ? ? ? ? x fg g p x f f p x g i p i p i p F H I K ? F H I K ? F H I K ( ) ( ) ( ) Let us define a differentiable curve ? : [a, b] R ? M by 4.10) ? ? i i t t t ( ) ( ) ? ? 0 , for fixed i ? i t( ) ,? 0 j i i n ? ? ? 1 2 1 1 , , , , , , ? ? then d dt f t f t t d t dt t t i i n i t t ? ? ? ?? ? ( ) ( ) ( ) ( ) b g b g L N M O Q P ? ? R S T U V W ? ? ? ? 0 0 1 by chain rule 0 ( ) ( ) i t f x t ? ? ? ? ? ? ? ? ?? ? for fixed i, by (4.3) {
22 ? ? ? f x t p i ( ) ( ) i p f x ? ? ? ? ? ? ? ? ? by (4.9) Thus we can claim that each i x ? ? ? ? ? ? ? ? , i n ?1 2, , ,? is a tangent vector to the curve ? defined above, at p t ? ?( ). 0 Again from the definition of the tangent vector, X f d dt f t p t t ? ? ?( ) | b g 0 ? ? R S T U V W ? ? ? ? ? ?? ? f t t d t dt i i i n t t ( ) ( ) ( ) b g 1 0 by chain rule ? F H G I K J ? ? ? dx t dt f t x t i t t i i n ( ) ( ) ( ) 0 0 1 ? ? ? b g by (4.3) ? F H G I K J F H G I K J ? ? ? dx t dt x t f i t t i n i p ( ) ( ) 0 1 ? ? We write it as 4.11) X p x p i i p i n ? F H I K ? ? ? ? ? ( ) 1 where 4.12) ? i i t t p dx t dt ( ) ( ) , ? F H G I K J ? 0 i n ?1, ,? Thus each ? i : M ? R, i n ?1, ,? is a differentiable function and every tangent vector,, say X p , to some curve, say ?( )t at p t ? ?( ) 0 can be expressed as a linear combination of the tangent vector ? ?x t i ( ) , i n ?1, ,? to the curve ? defined in (4.10)

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23 If possible, for a given linear combination of the form ? ? ? i i p x ( ) F H I K ? , where ? i ,s are functions on M, let us define a curve ? by ? ? ? ? : ( ) ( ) ( ) i i i t t p t ? ? 0 , 0 a t b ? ? then it can be shown that the tangent vector to this curve is ? ? ? i i p p x ( ) F H I K ? If we assume that ? ? ? i i p p x ( ) F H I K ? ? 0 then, ? ? ? i i p k i p x x ( ) F H I K ? ? 0 where x k : M ? R, 1,2, . K n ? ? or ? ? ? i k i p i p x x ( ) F H I K ? ? 0 ? ? k p( ) .? 0 for 1,2, . ? ? k n Thus the set ? ?x i n i p F H I K ? R S T U V W : , ,1 ? is linearly independent. Hence we state Theorem 1 : If (, ,)
x x n 1 ? is a local coordinate system in a neighbourhood U of p ? M,
then, the basis of the tangent space T p (M) is given by x x p n p 1 F H I K F H I K R S T U V W , ,? Let us define T(M) U p M T (M). p ? ? This T(M) is called the tangent space of M.
24 1.5. Vector Field : In classical notation, if to each point p of R 3 or in a domain U of R 3 , a vector : ( ) p p is specified, then, we say that a vector field is given on R 3 or in a domain U of R 3 . A vector field X on M is a correspondance that associates to each point p M, a vector X p T p (M). In fact, if f F(M), then Xf is defined to be a real-valued function on M, defined as follows 5.1) (Xf) (p) = X p f A vector field X is called differentiable if Xf is so for every f F (M). Using (4.11) of 1.4, a vector field X may be expressed as 5.2) X x i i where i 's are differentiable functions on M. Let ( )M denote the set of all differentiable vector fields on M. We define 5.3) ( ) X Y f Xf Yf ? ? ? ( ) ( ) bX f b Xf ?
It is easy to verify that ( )M is a vector space over R.
Also, for every f F(M), fX is defined to be a vector field on M, defined as 5.4) (fX) (p) = f (p)X p Let us define a mapping as [,] : F(M) F (M) as 5.5) [X, Y] f = X(Yf) − Y(Xf), X, Y ( )M Such a bracket is known as Lie bracket of X, Y. Exercises : 1. Show that for every X, Y, Z in (M), for every f, g in F(M), i) [
X, Y] (M) ii) [bX, Y] = [X, bY] = b[
X, Y], b R iii) [
X + Y, Z] = [X, Z] + [Y, Z] iv) [X, Y + Z] = [X, Y] + [X, Z] {
25
v) [X, X] = vi) [
X, Y] = − [Y, X] vii) X,[Y,Z] Y,[Z,X]
Z,[X,Y] ? ? ? ? : Jacobi Identity
viii) [fX, gY] = (fg) [
X, Y] + {f(Xg)}Y − {g(Yf)X} a) [X, fY] = f [X, Y] + (Xf)Y b) [
fX, Y] = f [X, Y] − (Yf)X 2.
In terms of a local co-ordinate system i) x x i i , L N M O Q P 0 ii) [
X, Y] = i j i i j i i j
x x F H G I K J , x j , where
X ? ? ? ? i i
x , Y ? ? ? ? j j x 3. Complete [X, Y] where i) X x 1 , Y x e x x 2 1 3 ii) X x x x 1 2 1 , Y x x 2 2 4.
Prove that i) (M) is
a F(M) module Hints : 1. viii) Note that { f(Yh)} (p) = f(p) (Yh) p by (5.4) of 1.5) = f(p) Y p h by (5.1) of 1.5) Again, {(fY)} (p) = (fY)(p) h by (5.1) = f(p) Y p h by (5.4) Thus {f(Yh)}(p) = {(fY)h}(p), p f(Yh) = (fY)h Use the above result, 5.5) of 1.5 & (4.6) of 1.4, the result follows after a few steps.
26 . 1.6. Integral Curve : In this article, we are going to give the geometrical interpretation of a vector field. Let Y be a vector field on M. The assignment of the vector Y p at each point p U M, is given by Y : p YY p TT p (M) A curve is an integral curve of Y if the range of is contained in U and for every a t b 0 in the domain [a, b] of , the tangent vector to at (t 0) = p coincides with Y p i.e. Y Y p ? ?( )t 0 Y f Y f p t ? ?( ) , 0 , f F(M) L N M O Q P d dt f t t t ( )( ) ? 0 by (4.4) of 1.4 Using 4.11) 1.4 one can write ? ? ? i i p i p x f ( ) F H I K ? L N M O Q P d dt f t t t ( )( ) ? 0 where ? i 's are functions on M. F H G I K J F H I K dx t dt x f i t t i p ( ) 0 As x i n i : , ,1 ? { } are linearly independent, we must have ? i i t t p dx dt ( ) ? F H G I K J ? 0 or ? ? i t t i t t t dx dt ( ) b g ? ? ? F H G I K J 0 0 or ? ? ? ? i n t t i t t t t t dx dt (( ), ( ), , ( ) 1 2 0 0 ? ? ? ? F H G I K J Using (4.3) of 1.4 we get
27 ? i n t t i t t x t x t x t dx dt (( ), ( ), , ( ) 1 2 0 0 ? ? ? ? F H G I K J Hence they are related by 6.1) dx dt x t x t i i n ? ? (( ), , ( ) 1 ? c h Exercises : 1. Find the integral curve of a zero vector. 2. Find
the integral curve of the following vector field i)
X x x x x ? ? 1 1 2 2 ? ? ? ? on R 2 ii) X e x x ? ? 1 1 ? ? on R iii) X x x x ? ? ? ? ? ? 1 1 2 2 ( )
on R 2

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Solution : 2.i) From (6.1) of 1.6, we see that dx dt x 1 1 ? , dx dt x 2 2 ? or dx x dt 1 1 ? , dx x dt 2 2 ? Integrating log x t 1 ? ?C , log x t D 2 ? ? say, where C, D are integrating constant. When t = 0, if x p 1 1 ? , x p 2 2 ? , then from x Ce t 1 ? and x De t 2 ? we find that p 1 = C, p 2 = D Thus ? : , p e p e t t 1 2 b g is the integral curve of X passing through the point 1 2 (,) p p 28 . 1.7 Differential of a mapping : Let f : M ? N be a a differentiable mapping of an n-dimensional manifold M to an m-dimensional manifold N. Let F(p) denote the set of all differentiable functions at p?M and F f p( ) b g denote the set of all differentiable functions at f p N ( ) . ? Such a map f, induces a map f F f p F p *: ( ) ( ) b g ? , usually called pull back map. and is defined by 7.1) f g g f *( ) , ? ? ? ? ( ) g F f p ? called the pull back of g by f, which satisfies 7.2) f ag bh a f g b f h * * * ( ) ( ) ( ) ? ? ? f gh f g f h * * * ( ) ( ) ( ) ? where ? ? , ( ) g h F f p ? and , ? a b R The map f, also induces a linear mapping f T M p * : ( ) ? ( ) ( ) f p T N such that 7.3) ? ? ? ? * * ( ) ( ) ( ) p p p f X g X g f X f g ? ? ? called the push forward of X by f. Such f * is also called derived linear map or Jacobian map or differential map of f on T p (M) f * ? push forward objects defined on objects defined on f * ? pull back N M f f *

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f M N p f p ? ( ) T p (M) T f(p) (N) { 29			

Let us write 7.4) f X f X p f p * * ( ) ( ) ( ) ? We can also define push forward of X by f, geometrically, in the following manner : Given a differential mapping f M N : , ? the differential of f at p M? is the
linear mapping f
T M p * : ( ) ? ( ) ( ) f p T N defined as follows : For each X p ? T p (M),
we choose a curve ?( )t in M such
that X p is the tangent vector to the curve ?( )
t at p
t ? ?( ). 0 Then f X p * ( ) is defined to be the tangent vector to the curve f t?( ) b g at f p f t ( ) ( ) ? ? 0 b g Exercises : 1. If f is a differentiable map from a manifold M into another manifold N and g is a differ- entiable map from N into another manifold L, then, show that i) ( ) * * * g f g f ? ? ? ii) ( ) * * * g f f g ? ? ? 2. If f is a transformation of M and g is a differentiable function on M, prove that i) f X Y f X Y * * [,] [,] ? ii) f f X g X f g * * *) ( ) b g ? iii) f gX g f f X * * ( ) ( )( ) ? ? ? 1 for all vector fields X, Y on M. Solution : 1. By definition,
f X p * ( ) is the tangent vector to the curve f t?( ) b g at f p f
t ( ) ( ) ? ? 0 b g where
X p is the tangent vector to the curve ?( )t at p
t ? ?( ). 0 Hence by (4.4) of 1.4
30 f g p * ( )X d i ? d dt g f t t t (( )? b g L N M O Q P ? 0 g F f p ? ( ) b g ? L N M O Q P ? d dt g f t t t ( ) ( ) ? ? b g 0 = X p ( ) g f? by 4.4) of 1.4 Hints 3. Given that f : M ? M is a transformation and hence for every p?M, f p q ( ) ,? say.. Thus, p f q ? ?1 ( ) consequently, from 7.3) of 1.7, we find that f X g f p p * (( ) d i { } ? X g f p p ( ) ( ), ? n s ? ?p M or f X g q X g f f q p p * ( ) ( ) ( ) ( ) d i { } n s ? ? ? 1 or f X g X g f f * ( ) ( ) b g b g ? ? ? 1 Using this relation, one can deduce the three results. We are now going to give a matrix representation of the linear mapping f * . Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where (, ,) x x n 1 ? is the local co-ordinate system in a neighbourhood of a point p of M and (,) y y m 1 ? is the local co-ordinate system in a neighbourhood of f p( ) of N, then f x f x y i p j i p j m j f p * ( ) ? ? ? ? ? ? ? F H I K ? F H G I K J ? ? 1 where f y f j j ? ? Proof : We write f x a y i p i j j f p j m * ( ) , ? ? ? ? F H I K ? F H G I K J ? ? 1 i n ?,...,
31 where a s i j, are unknown to be determined or f x y a y y i k i j j f p k j m * ( ) ? ? ? ? F H I K R S T U V W ? F H G I K J ? ? 1 where each y F f p k ?? (( )) k m ?1,..., using 7.3) of 1.7, we find ? ? ? x y f a i p k i j j k j m F H I K ? ? ? ( )? 1 or ? ?x f a i p k i k F H I K ? or ? ? f x a k i p i k F H G I K J ? by (4.9) of 1.4 Thus f x f x y i p j i p j f p j m * ( ) ? ? ? ? ? ? ? F H I K ? F H G I K J F H G I K J ? ? 1 Note : 1. The matrix of f * , denoted by (f *) is
given by ( ) *
f
f
x f x f x f x f x f x f x f
x
n

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n m m m n ? F H G G G G G I K J J J J ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 1 2 1 2 1 1 2 1 1 2 ? ? Note : 2. The kernel of f * is the set of X T M p p ? ( ) for which f X p * ( ) ? ? The image of f * is the set of ( ) ( ) ( ) f p f p Y T N ? for which, there exists X T M p p ? ( ) such that f X Y p f p * ( ) ( ) ? Now from a known theorem
32 dim (kernel f *) + dim (Range f *) = dim T p (M). We write it as 7.5) dim (kernel f *) + dim (Range f *) = dim T p (M) for each p ? M The dim (Range f *) is called the rank f * If rank f * = dim T p (M) we say i) f is an immersion if dim M ? dim N and f(M) is an immersed submanifold of N ii) f is an imbedding if f is one to one and an immersion and then f(M) is an imbedded submanifold of N iii) f is a submersion if dim M ? dim N. Exercises : 1. Show that f R R : ? 2 given by f(t) = (a cost, sint) is an immersion. 2. Find (f *) in the following cases i) f : R 2 ? R 2 given by f = ( ) ( ) ,
x x x x 1 2 2 2 1 2 2 3 ? b g ii) f : R 2 ? R 2 given by f = x e x x e x x x 1 2 1 2 2 2 ? ? ,
c h at (0, 0) where 1 2 (,) x x are the local co-ordinates on R 2 . 1.8 f-related vector Field : Let X and Y be fields on M and N respectively. Then, for p?M, let p p X T (M) ? and f (p) f(p) Y T (N) ? and such that 8.1) f X Y p f p * ( ) ( ) ? where f M N : ? is a differentiable mapping and f * is already defined in the previous article. In such a case, we say that the two vector fields X, Y are f-related.
33 For g F f p ? ( ) b g we see that f X g Y p f p g * ( ) ( ) n s ? Using 7.3) of 1.7 and (5.1) of 1.5 we find that X g f Yg f p p ( ) ( ) ( ) ? ? , p? Then 8.2) ( ) ( ) X g f Yg f ? ? If f is a transformation on M and f X X p f p * ( ) ( ) ? we say that, X is f-related to itself or X is invariant under f. We also write it as 8.3) f X X * ? Exercises : 1. Let X Y i i i , (,) ?1 2 be two f-related vector fields on M and N respectively.. Show that the vector fields [X 1 , X 2] and [Y 1 , Y 2] are also f-related. 2. Prove that two vector fields X, Y respectively on M and N are f-related if and only if f f X g X f g * * * ( ) ( ) b g ? where f : M ? N is a C map. 3. If f is a transformation on M, show that, for every X M ? ? ( ), there exists a unique f- related vector field to X. Solution : 1. From the definition of the Lie bracket, we see that [,]( ) X X
g f 1 2 ? ? ? ? ? ? X X g f X X g f 1 2 2 1 ( ) ( )
b g b g ? ? X
Y g f X Y g f 1 2 2 1 ( ) ( ) b g b g by (8.2) above ? ? Y Y g f Y Y g f 1 2 2 1 ( ) { ( )} l q by (8.2) above ? ? Y Y g Y Y g f 1 2 2 1 ( ) ( ) l q
34 [,]( ) [,] X X g f Y Y g f 1 2 1 2 ? ?
l q from the definition of the Lie Bracket. Hence from 8.2), one claims that [X 1 , X 2] and [Y 1 , Y 2] are f-related. . 1.9 One parameter group of transformations on a manifold : Definitioin Let a mapping ?: R M M ? ? is defined by ? ? :(,) ( ) t p p t ? which satisfy i) for each t R? , (,) ( ) t t p p ? ? ? is a transformation on M and 0 ( )p p ? ? ii) for all t, s, t + s ? R ? ? ? ? ? ? t s t s t s p p p ( ) ( )( ) ( ) b g ? ? ? ? Then the family ? t t R | ? l q of mappings is called a one-parameter group of transforma- tions on M. Exercise : 1. Let ? t t R | ? l q be a one-parameter group of mappings on M. Show that i) ? ? ? ? ? t t ( ) 1 ii) ? t t R | ? l q form an abelion group. Let us set 9.1) ?( ) ( ) t p t ? ? Then ?( )t is a differentiable curve on M such that ?( ) ( ) 0 0 ? ? ? p p by Def. (i) above Such a curve is called the orbit through p of M. The tangent vector, say X p to the curve ?( )t at p is therefore 9.2) X f d dt f t p t ? L N M O Q P ? ?( ) b g 0 ? ? ? lim ( ) ( ) t f p f p t t 0 ? b g , ? ?f F M( )
35 In this case, we say that ? t t R | ? l q induces the vector field X and X is called the generator of { }.? t The curve ?( )t defined by 9.1) is called the integral curve of X. Exercises : 2. Show that the mapping ?:
R R R ? ? 3 3 defined by ?(,) (, ,) t p p t p t p t ? ? ? ? ? 1 2 3 is a one-parameter group of transformations
on M and the generator is given by ? ? ? ? ? ? x x x 1 2 3 ? ? 3. Let M = R 2 and let ?: R M M ? ? be defined by ? t x y xe ye t t ,(,) , b g c h ? ? 2 3 Show that ? defines a one-parameter group of transformation on R 2 and find its generator.. Note : Since every 1-parameter group of transformations induces a vector field on M, the question now arises whether every vector field on M generates one parameter group of trans- formations. This question has been answered in the negative. Example : Let X e x x x ? ? ? 1 1 2 ? ? ? ? on M = R 2 . Then, dx dt e x 1 1 ? ? , dx dt 2 1? Thus e t A x? ? ? 1 , x t B 2 ? ? , where A, B are integrating constant. Let x p 1 1 ? , x p 2 2 ? for t = 0 Then, A e p ? ? 1 , B p? 2 .
36 Consequently the integral curve of X is ?( ) log , t t e t p p ? ? ? F H I K ? 1 1 2 which is not defined for all values of t in R. Thus, if ? ? ( ) ( ), t p t ? then, X does not generate one parameter group of transformations. Problem 7 leads us to the following definition : Let I ? be an open interval (,) ? ?? and U be a nbd of p of M. Let a mapping ? ? : ( ) I U U M t ? ? ? ? denoted by ? ? (,) ( ) t p p t ? be such that i) U is an open set of M ii) for each t I? ? ,? ? (,) ( ) t p
p t ? is a transformation of U onto an open set ? t U( ) of M
and 0 ( )p p ? ? iii) if t, s, t + s are in I ? and if ? s p( ) U ? ? ? t s t s p p ( ) ( ) b g ? ? Such a family ? t t I| ? ? l q of mappings is called a local one parameter group of transforma- tions, defined on I U ? ? . We are now going to establish the following theorem Theorem 1 : Let X be a
vector field on a manifold M. Then, X generates a local one- parameter group of

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transformations in a neighbourhood of a point of M. Proof : Let (, ,....)
x x x n 1 2 be a local coordinate system in a neighbourhood U of
p of M such that ?( ) (,...,) p ? 0 0 ?R n , where (U, ?) is the chart at p of M. Then x p u p i i ( ) ( )( ) , ? ? ?? 0 i n ?1, ..., Let X x x x i n i i ? ? ? ? ? (,...,) 1
37 be a given vector field on U, the neighbourhood of p ? M, where each ? i, s the components of X, are differentiable functioins on U of M. Then, for every X on M, we have a ? -related vector field on, n 1 (U) U CR ? ? with ? (p) =(,...,) 0 0 n 1 U CR . ? Let ? i, s be the components of the ? -related vector field on U 1 of R n . Then by the exist- ence theorem of ordinary differential equations, for each ? (p)? U 1 R n , there exists a ? 1 0? and a neighbourhood V 1 of ? (p), V 1 U 1 such that, for each q q q n ? (,...) 1 ?V 1 , q r ? ?( ), say, there exists n-tuple of C ? functions f t q f t q n 1 (,),... (,) defined on I ? 1 I ? 1 and mapping f i I: ? 1 ? V 1 U 1 , i n ?1,..., which satisfies the system of first order differential equations 1) df t dt t p i i ( ) , ( ) , ? ? ? b g i n ?1,..., with the initial condition 2) (0,) i i f q q? Let us write 3) ? t n q f t q f t q ( ) (,),..., (,) ? 1 b g We are going to show ? ? ? t s t s q q ? ?( ) ( ) . b g Note that if t, s, t + s are in I ? 1 and if ? s q( ) ? V 1 U 1 then each f t s q i (,), ? f t q i s , ( ) ? b g are defined on I ? 1 U 1 . Now let us set g t g t f t s q f t s q n n 1 1 ( ),..., ( ) (,),... (,) b g b g ? ? ? For simplicity, we write g t f t s q i i ( ) (,) b g b g ? ? Then each g t i ( ) is defined on I ? 1 U 1 and hence satisfies 1) with the initial condition 4) g o f s q i i ( ) (,) b g b g ?
38 Also, let us set h t h t f t q f t q n s n s 1 1 ( ),..., ( ) (, ( ),..., (, ( ) c h c h ? ? ? For simplicity, we write h t f t q i i s ( ) (, ( ) b g b g ? ? then each h t i ( ) is defined on I ? 1 U 1 and hence satisfies 1) with the initial condition h o f o q i i s ( ) , ( ) b g b g c h ? ? ? ? s i q( ) b g by 2) ? f s q i (,) b g by 3) ? g o i ( ) b g by 4) Hence from the uniqueness we must have g t h t i i ( ) ( ) b g b g ? Using 3) we must have ? ? ? t s t s q q ? ?( ) ( ) . b g Thus, we claim that, for every vector field defined in a neighbourhood U 1 of ? (p) of R n , there exists ? ? t t I| ? 1 n s as its local 1-parameter group of transformations defined on I ? 1 U 1 . Let us set V = ? ?1 (V 1) U and define ? ? : ( ) I V V M t ? ? ? ? as follows ? ? ? t r t q ( ) (,) ? ?1 b g Then i) ? ? (,) ( ) t r r t ? is a transformation of V onto ? t ( )V of M
39 ii) if t, s, t + s are in I ? and if ? s r( ) V, then ? ? ? ? ? ? t s s r t r ( ) (, ( ) b g b g c h ? ?1 ? ? ? ? ? 1 (,) t s q b g , after a few steps ? ? ? t s r( ) Thus for the given vector field X, defined in a neighbourhood U of p of M, there exists ? t t I| ? ? l q as its local 1-parameter group of transformations, defined on I ? V U of M. Note that if we define ? ? ? ? ( ) ( ) (,) , t r t q t ? ? ?1 b g q r ? ?( ) ? ? ? ? 1 ( ) ,t b g say,, then ? ? ?1 ( )t b g is the integral curve of X. This completes the proof. Theorem 2 : Let ? be a transformation of M. If a vector field X generates ? t as its local 1-parameter group of transformations, then, the vector field ? * X will generate ?? ? t ?1 as its local 1-parameter group of transformations. Proof : Left to the reader. Exercise : 4. Show that a vector field X on M is invariant under a transformation ? on M if and only if ? ? ? ? ? ?
t t ? where ? t is the local 1-parameter group of transformations induced by X. We
now give a geometrical interpretation of [X, Y], for every vector field X, Y on M. Theorme 3 : If X generates ? t as its local 1-parameter group of transformations, then, for every vector field Y on M. [,] lim ( ) * X Y t t Y Y q q t q ? ? ? 0 1 ? b g { } where q p t ? ? ( ) and ( ) ( ) ( ) * * ? ? ? t p t t Y Y p ? b g
40 To prove the theorem, we require some lemmas which are stated below : Lemma 1 : If ? (t, p) is a function on I ? M, where I ? is an open interval (,) ? ?? such that ? (0, p) = 0 , ? p?M then, there exists a function h (t, p) on I ? M such that t h (t, p) = ? (t, p) Moreover h (o, p) = ?? (o, p), Where ? ? ? ?d dt . Proof : It is sufficient to define h t p ts p d ts t (,) (,) ( ) ? ? z ? 0 1 Hence by the fundamental theorem of calculus h t p t ts p (,) (,) ? L N M O Q P 1 0 1 ? ? th t p t p (,) (,) ? ? Also from above h o p o p ds (,) (,) ? ? z ? 0 1 ? ? ? ? ? ? (,)[ ] (,) o p s o p 0 1 Lemma 2 : If f is a function on M and X is a vector field on M which induces a local 1-parameter group of transformations ? t then there exists a function g t defined on I ? V , V being the neighbourhood of p of M, where
g p g t p t ( ) (,) ? such that f p f p tg p
t t ? ( ) ( ) ( ) b g ? ?
41 Moreover, X f g o p g p p ? ? (,) ( ) 0 Symbolically, Xf g? 0 on M. Proof : Let us set ~ (,) ( ) ( ) , f t p f p f p t ? ? ? ? b g b g 0 ? p?M Then ~ (,) f t p is a function on I ? M such that ~ (,) ( ) ( ) , f o p f p f p ? ? ? ? ? 0 0 0 b g b g ? p?M Hence by Lemma 1, there exists a function, say, g(t, p) on I ? V , VV M being the neighbourhood of p of M, such
that tg t p f t p (,) ~ (,) ? ? g t p f p f p t t (,) ( ) ( ) ? ? ? ? b g b g 0 or, g o p t t f p f p X f t p (,)
lim ( ) ( ) ? ? ? ? 0 1 0 ? ? b g m r b g As, tg t p f p f p t (,) ( ) ( ) ? ? ? b g we find that f f tg t t ?? ? ? Proof of the main theorem : Let us write ? t p q ( ) ? ? p q q t t ? ? ? ? ? ? 1 ( ) ( )

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42 Now, ( ) ( ) ( ) ( ) ( ) ( ) * ? ? t t t Y f q Y f p Y f tg p b g m r l q l q ? ? ? ? by Lemma 2 or ( )( ) ( ) ( ) ( )( ) ( )( ) ( ) ( ) * Yf q Y q Yf q Yf p t Yg q t t t ? ? ? ? ? ? ? b g c h b g or, lim ( ) * t t Y Y f q t q ? ? F H I K 0 1 ? b g { } ? ? ? lim ( )( ) ( )( ) t Yf q Yf p t 0 ? ? ? lim ( ) ( ) t Yg q t t 0 ? b g = lim ( )( ) ( )( ) ( )( ) t t Yf q Yf p Yg q ? ? ? 0 1 0 l q = lim ( )( ) ( )( ) ( ), t t Yf q Yf p y Xf q ? ? ? 0 1 l q by Lemma 2 From the definition we find that, X f t t f q f q q t ? ? ? lim ( ) ( ) 0 1 ? b g m r or ? ? ? ? X f t t f p f q q lim ( ) ( ) 0 1 l q Taking f Yf? , we find from above after a few steps X Yf t t Yf q Yf p q ( ) lim ( )( ) ( )( ) ? ? ? 0 1 l q Thus we write, lim ( ) * t t Y Y f q t q ? ? F H I K 0 1 ? b g { } ? ? X Yf Y Xf q q ( ) ( ) ? [.] ( ), X Y f q l q after a few steps. [,] lim ( ) * X Y t t Y Y q q t q ? ? ? 0 1 ? b g { } Note : We abbreviate the above result as [,] lim ( ) * X Y t t Y Y t ? ? ? 0 1 ? b g m r Corollary : 1. Show that ? ? ? s s s t X Y t t Y Y b g b g b g o t * * * [,] lim ( ) ? ? ? ? 0 1

43 Proof : From the last theorem ? s X Y t t b g * [,] lim ? ? 0 1 ? ? ? s s t Y Y b g b g o t * * * ( ) , ? as ? s b g * is a linear mapping lim ( ) * * t t Y Y s s t ? ? 0 1 ? ? ? b g b g o t ? , from a known result Using the definition of local 1-parameter group of transformations, the result follows immedi- ately. Corollary 2 : Show that ( ) [.] ( ) * * ? ? s t t s X Y d Y dt ? ? F H G I K J ? b g Proof : Left to the reader Corollary 3 : Let X, Y generate ? t and ? s respectively, as its local 1-parameter group of transformations. Then ? ? ? ? t s s t ? ? ? if and only if [X, Y]. Proof : Let ? ? ? ? t s s t ? ? ? Then from Exercise 4, the vector field Y is invariant under ? t . Hence by 1.8 ( ) * ? t Y Y? Consequently from Theorem 3, [X, Y] = 0 Converse result follows from corollary 2.
A vector field X on a manifold M is said to be
complete if
it induces a one param- eter group of transformations on M. Theorem 4 : Every vector field on a compact manifold M is complete. Proof : Let X be a given vector field on M. Then by Theorem 1, X induces { }? t as its

44 local 1-parameter group of transformations in a neighbourhood U of p of M and t ? I ? R. If p runs over M, then for each p, we get a neighbourhood U(p) and I ? (p), where all such U(p) from an open coverings of M. Since M is compact, every open covering {U(p)} of M has a finite subcovering { ( ) : ,..., } U p i n i ?1 say. If we let ?? ? ? ? min ( ), ( ), ..., ( ) p p p n 1 2 l q then, there is a t such that for | |t ? ? ? t p( ):(,) ? ?? x M ? M is local 1-parameter group of transformations on M. We are left to prove that ? t p( ) is defined on R M. Case a) : t ? ? We write t k r ? ? ? ? 2 , | | , r ? ? 2 k being integer Then ? ? t k r ? ? ? 2 ? ? ? ? k r 2 ? ? 2 2 2 ..... r ? ? ? ? ? ? ? ?? ? ? k times Similarly for t ? ? ?, we can show that ? ? ? ? t r ? ? ? ? ? ? ? 2 2 .... Thus ? t is a local 1-parameter group of transformations on M. Combining all the cases, we claim that ? t is defined on R M. Hence X induces ? t as its 1-parameter group of transformations on a compact manifold M. Thus X is a complete vector field.

45 1.10 Cotangent Space : Note that ?(M) is a vector space over the field of real numbers. A mapping ? : ?(M) ? F(M) that satisfies ??(X+Y) = ??(X) + ??(Y) ??(bX) = b ??(X), b ? R and for all X, Y ? X(M), is a linear mapping over R. The linear mapping ? : ?(M) ? F(M) denoted by ? : X ? ??(X) is called a 1-form on M. Let D M M F M 1 b g b g b g n s ? ? ? ? ? ? , ,... : be the set of all 1-forms on M. Let us define 10.1) ? ? ? ? ? ? ? ? ? ? R S | T | b gb g b g b gb g b g X X X b X b X ( ) It can be shown that D 1 (M) is a vector space over R, called the dual of ? ??). For every p?M, ? X F M b g b g ? is a mapping ? X M R b g : ? defined by 10.2) ? ? X p X p p b gm rb g d i ?

46 so that ? p p T M R : b g ? Thus ? p ? dual of T M p b g . We write the dual of T M p b g by T M p * b g and is the cotangent space of T M p b g . Elements of T M p * b g are called the covectors at p of M or linear functionals on T M p b g . For every f ? F(M), we denote the total differential of f by df and is defined as 10.3) df X Xf p X f p p p p b g d i b gb g ? ? ?, We also write it as 10.4) (df) (X) = Xf Exercises : 1. Show that for every f ? F(M), df is a 1-form on M. 2. If x x x n 1 2 , ,...., d i are coordinate functions defined in a neighbourhood U of p ? M, show that each dx i n i , ,...., ?1 is a 1-form on U ? M. Solution : 2 Note that dx X Y X Y x i i c hb g b g ? ? ? , (10.4) above ? ? Xx Yx i i ? ? dx X dx Y i i c hb g c hb g , by (10.4) Similarly it can be shown that dx bX b dx X i i c hb g c hb g ? Thus each dx i n i , ,...., ?1 is a 1-form on R. From Exercise 2 above, it is evident that each ? ? * ( ) i p p dx T M ? , for i =1, ..., n. Wee now define

47 10.5) dx x i p i p j i c h ? ? ? F H I K ? Let ? p p T M ? * b g be such that 10.6) ? ? ? p j p j p x f F H G I K J ? d i where each f R j p d i ? If possible, let ? p p T M ? * b g be such that ? p p p n p n p f dx f dx ? ? ? 1 1 b g d i b g d i .... then ? ? ? p p p n p n p x f dx f dx 1 1 1 F H I K ? ? ? b g { } ( ) ( ) ( ) ? ? ?x f p p 1 1 F H I K ? ( ) by (10.5) Proceeding in this manner we will find that ? ? ? ? ? ? p p p p i x f x 1 1 F H I K ? ? F H I K b g by (10.6) As ? ?x i n i : ,..., ?1 { } are linearly independent, we must have ? ? p p ? . Thus any ? p ?T p * (M) can be expressed uniquely as 10.7) ? p i p i p f dx ? ? ( ) ( ) ? T p * (M) = span ( ) ,...,( ) dx dx p n n 1 n s Finally if ( ) ( ) , f dx i p i p i ? ? 0 then,

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48 ( )f dx x i p i p k p i c h ? ? F H I K ? ? 0 i.e. f k p b g ? 0. by (10.5) Similarly it can be shown that f f p n p 1 0 b g b g ? ? ? ..... Thus the set f f p n p 1 0 b g b g ? ? ? ... is linearly independent and we state Theorem 1 : If x x n 1, ,.... d i are local coordinate system in a neighbourhood U of p of M, then the linear functionals {( ) : ,...., } dx n i p ?1 on T p (M) form a basis of T (M). p * Note that ( )( ) dX X i ? Xx i by 10.4) = ? ? ? j j i x x ? by 5.2) of 1.5 10.8) ( )( ) dx X i i ? ? Thus, one can find ( )( ) ( ) df X Xf x f f x dx X i i i i ? ? ? ? ? ? ? ? ? ? from above Hence we write 10.9) df = ? ? f x dx i i ? For every ? ? D 1 (M), we define f ? to be a 1 form in M and write 10.10) ( )( ) ( ) f X f X ? ? ? b g Note : D 1 (M) is a F(M)-module

49 . 1.11 r-form, Exterior Product : An r-form is a skew-symmetric mapping ? ? ? : ( ) .... ( ) ( ) M M F M ? ? ? such that i) w is R-linear ii) if ? is a permutation of 1,2......r with (1, 2, ..., r)? ? ? ? ( ), ( ),... ( ) 1 2 r b g then 11.1) ? ? ? ? ? ? ? X X X r X X X r r 1 2 1 2 1 , , ( ) ( ) ( ) ......, ! (sgn) (, ...) d i ? ? where (sgn)? is + 1 or −1 according as ? is even or odd permutation . If ? is a r-form and ? is a s-form, then, the exterior product or wedge product of ? and ? denoted by ? ?? is a (r+s)-form. defined as 11.2 ( )(, ,...... ,) ,...... ? ?? ? X X X X X r r s 1 2 1 = 1 1 1 ( )! (sgn) (,....,) (,.......) ( ) ( ) ( ) ( ) r s X X X X r r r s ? ? ? ? ? ? ? ? ? ? ? ? where ? ranges over the permutation (1, 2,......r+s), X M i r s i ? ? ? ? ( ). , ,......., 1 2 For convenience, we write 11.3) f g fg ? ? , f g F M , ( ). ? It can be shown that, if ? is a r-form 11.4) ( )(,....,) (, ,....,)

f

X X f X

X X r r ? ? ? ? 1 1 2 ( )(,....,) (, ...,) ? ? ? ? f X X f X X r r 1 1 Again, if ? and ? are 1-forms, then 11.5) ( )(,) ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? ? ? ? X X X X X X

r 1 1 2 2 1 1 2

l q {

50 The exterior product obeys the following properties : 11.6) ? ? ? ? ? ? ? ? , ? ?? ? 0 f f f ? ? ? ? ? ? ? ? ? ? ? ? ( ) f g fg ? ? ? ? ? ? ? , ? ? ? ? ? ? ? ? ? 1 b g rs , ? : r - form ? : s - form ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? Exercises : 1. If ? is a 1-form and ? is a 2-form, show that ( )(, ,) ? ?? ?

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X X X 1 2 3 1 3 ? ? ? ? ? ? ( ) (,) ( ) (,) ( ) (,) X X X X X X X X X 1 2 3 2 3 1 3 1 2 ? ?		

l q 2. Compute i) ( ) ( ) 2 1 2 1 2 du du du du ? ? ? ii) ( ) ( ) 6 27 1 2 1 3 1 2 3 du du du du du du du ? ? ? ? ? ? Solution : 2 i) ( ) ( ) 2 1 2 1 2 du du du du ? ? ? ? ? ? ? ? ? 2 1 1 2 2 1 2 du du du du du du ( ) ( ) ? ? ? ? ? ? 2 1 2 2 1 du du du du as du du i i ? ? 0 = ? ? 3 1 2 du du by 11.6) Theorem 1 : In terms of a local coordinate system (, ,....,) x x x n 1 2 in a neighbourhood U of p of M, an r-form ? can be expressed uniquely as 11.7) ? ? ? ? ? ? ? ? f dx dx dx i i i i i i i i i r r r 1 2 1 1 2 2 ... ... where f i i i r 1 2 ... are differentiable functions on M. Proof : Let D r (M) denote the set of all differentiable r-forms on M. In terms of a local co- ordinate system (, ,....,) x x x n 1 2 in a neighbourhood U of p of M, the set dx dx i i i n i i r r 1 1 1 2 ? ? ? ? ? ? ? ... : ... l q form a basis of D r (M). Using 11.2) we find i) (...) dx dx i i r 1 ? ? (, ,....,) X X X r 1 2 ? 1 r! (sgn) ... ( ) ? ? ? dx X i 1 1 d i ? dx X i r r ?( ) d i i i i r 1 2 ? ? ? ?... {

51 where ? ranges over the permutation (1, 2, ..., r) and each X i ??(M). Let ii) X k = ? ? ? k j j j n m m m x ? ? 1 where ?'s are functions, called the components of X k . Using ii), we get from i) dx dx X X i i r r 1 1 ? ? ?..... (,....,) c h ? 1 r! (sgn) ... ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? F H I K F H I K dx x dx x i j jm i r r j jk m k 1 1 Using (10.5) of 1.10, we get from above iii) (...) dx dx i i r 1 ? ? (, ,....,) X X X r 1 2 ? 1 r! (sgn)? ? ? ? ? ? ? ( ) ( ) ... 1 1 i r i r i i i r 1 2 ? ? ? ?... Using ii) in (11.1) of 1.11, , we find ? (, ...,) X X X r 1 2 ? 1 r! (sgn)? ? ? ? ? ? ? ? ? ? ? ? ( ) ( ) ,...., 1 j j r j j m m s s x x ? ? F H I K As each ? is R-linear, we find from above ? 1 r! (sgn)? ? ? ? ? ? ? ? ? ? ? ? ( ) ,...., ( ) ... ,...., 1 j j j r j j m j s m m s s x x ? F H G I K J Changing the dummy indices j i j i m s r ? ? 1 ,...., we get ? 1 r! (sgn)? ? ? ? ? ? ? ? ? ? ? ? ( ) ,...., ( ) ... ,...., 1 1 1 1 1 i i i r i i i r r x x ? F H I K Using iii) we find from above ? ? ? ? ? ? ? (...) (, ...,) ,... ... ... dx dx X X X f i i i i i i i r i i i r r r 1 1 1 2 1 2 1 2 , where ? ? ? ? ? x x f i i r i i i 1 1 1 2 ,...., ... F H I K ? i i i r 1 2 ? ? ?...

52 Thus ?(, ...,) X X X r 1 2 ? ? ? ? ? ? ? ? f dx dx X X i i i i i i i i i i i r r r r r 1 2 1 1 1 2 1 ... ,... ... (...) (,...,), ? X X r 1 ,...., Hence we can write ? ? ? ? ? ? ? ? ? f dx dx dx i i i i i i i i i r r r 1 2 1 2 1 2 ... ... This completes the proof. Exercises : 3. Show that a set of 1-forms { , ,...., } ? ? ? ? 1 2 k is linearly dependent if and only if ? ? ? 1 2 0 ? ? ? ? ? ? k 4. Let { , ,...., } ? ? ? ? 1 2 k be k-independent 1-forms on M. If ? i be k 1-forms satisfying ? ? i i i ? ? ? 0 show that ? ? i ij j A ? ? with A A ij ji ? Solution : 3. Let the given set of 1-forms be linearly dependent. Hence any one of them, say, ? k ?1 can be expressed as a linear combination of the rest i.e. ? ? ? k b b ? ? ? ? ? ? 1 1 1 2 2 ? b b

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43% **MATCHING BLOCK 3/10** W

k k k k ? ? ? 2 2 ? ? , where each b R i ? ? ? ? ? ? 1 2 1 ? ? ? ? ? ? k k ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 2 1 1 2 2 2 2 ? (.....) b b b b k k k k k ? b 1 1 ? ? ? ? ? ? 2 ? ? ? ? ? ? ? ? 1 1 2 ? ? ? ? ? ? ? ? k k k k

b ? ? = 0 by 11.6) of this article. Converse follows easily. 4. As { ,..., } ? ? 1 k is a independent set of of 1-forms, we complete the basis of D 1 (M) by taking 1-forms ? ? k n ?1 ,..., . Thus the basis of D 1 (M) is given by { ,..., , ,..., }. ? ? ? ? 1 1 k k n ? Consequently any 1-from ? i , i k ?1,... can be expressed as
53 i) ? ? ? i im m ip p p k n m k A B ? ? ? ? ? ? ? , 1 1 i k ?1 2, ,... Given that ? ? i i i ? ? ? 0 i.e. ? ? ? ? ? ? 1 1 2 2 0 ? ? ? ? ? ? ? ? k k Using i) and 11.6) one gets after a few steps ( ) A A ij ji i j i j k ? ? ? ? ? ? ? ? ? ? ? ? ? B ji i j i k j k ? ? 0 As ? 's are given to be linearly independent, so we must have A A ij ji ? ? 0 and B ij ? 0 i.e. A A ij ji ? Consequently i) reduces to ? ? i ij j A ? ? with A A ij ji ? . 1.12. Exterior Differentiation : The exterior derivative, denoted by d on D is defined as follows : i) d (D r) D r+1 ii) for f ? D 0 , df is the total differential iii) if ? ?D r , ? ?D s then d d d r ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? ? ? 1 iv) d 2 = 0 From 11.7) of 1.11 we find that 12.1) 1 1 2 1 r r r i i i i i i i d df dx dx ? ? ? ? ? ? ? ? ? ? ? ?
54 Exercises : 1. Find the exterior differential of i) x y dy xy dx 2 2 ? ii) cos( ) xy dx dz 2 ? iii) x dy dz y dz dx z dx dy ? ? ? ? ? 2. Find the exterior differential of d d ? ? ? ? ? ? ? A form ? is said to be closed if 12.2) d? ? 0 If ? is a r–form and 12.3) d? ?? for some (r–1) form ? then, ? is said to be an exact form. Exercise : 3. Test whether ? is closed or not where i) ? ? ? ? F H I K xy dx x y dy 1 2 2 ii) ? ? ? e y dx e y dy x x cos sin Theorem 1 : If ? is a 1-form, then d X X ?(,) 1 2 ? 1 2 1 2 2 1 1 2 X X X X X X ? ? ? ( ) ( ) [, b g b g b g m r ? ? Proof : Without any loss of generality, one may take an 1-form as ? ? ? f dg f g , , F(M) ? d X X df dg X X ?(,) ( )(,) 1 2 1 2 ? ? Using 11.5) of 1.11, we find d
X X ?(,) 1 2 ? 1 2 1 2 2 1 ( ) ( ) ( ) ( ) ( ) ( ) df X dg X df X dg X ? l q
55 Using (10.4) of 1.10, we get d X
X ?(,) 1 2 ? ? 1 2 1 2 2 1 ( )( ) ( )( )
X f X g X f X g l q ? ? ? ? 1 2 1 2 1 2 2 1 2 1 X f X g f X X g X f X g f X X
g (( ) (( )
b g b g b g b g m r on using (4.6) of 1.4 Now ?( ) ( )( ) ( ) , X fdg X f dg X 1 1 1 ? ? b g as ( )( ) ( ) f
X f X ? ? ? b g ? f X g( ) 1 by (10.4) of 1.10 by ?( ) ( ) X f X g 2 2 ? Thus we get from above
d
X X X X X X f X X g X X g ? ? ? (,) ((( ) ( ) 1 2 1 2 2 1 1 2 2 1 1 2 ? ? ? ? ? b g b g l q ? ? ? 1 2 1 2 2 1 1 2 X X X X f X X
g ? ? (([,] b g b g b g m r ? d X X ?(,) 1 2 1 2 ? X X X X X
X 1 2 2 1 1 2 ? ? ? ( ) ( ) [,]
b g b g b g m r ? ? This completes the proof. Existence and Uniqueness of Exterior Differentiation : Without any loss of generality we may take an r-form as ? ? ? ? f dx dx i i i i i r r 1 2 1 ... ... , f i i r 1 ... ? F(M) Let us define an R-linear map d : D ? D as 12.4) d df i i i r ? ? 1 2 ... dx dx i i r 1 ? ?.... Clearly i) d (D r) D r+1 and ii) if ? is a 0-form, then d? is the total differential of ? . iii) Let ? ? D s and it is enough to consider
56 ? ? ? ? g dx dx j j j j s s 1 1 ... ... , g j j s 1 ... ? F(M) then d d f g dx dx dx dx i i i j j i i j j r s r s ( ) ... ... ? ?? ? ? ? ? ? ? ? 1 2 1 1 1 d i Using 12.1 we get d d f g dx dx dx i i j j l j j s s i s ? ?? ? ? ? ? ? ? b g ( ) .... 1 1 1 = ( ) .... g df f dg dx dx dx dx j j i i i i j j i j ji j s r r s r s 1 1 1 1 1 ? ? ? ? ? ? ? = g df dx dx dx j j i i i i j j s r r s 1 1 1 ... ... ? ? ? ? ? f dg dx dx dx dx i i j j i i j j r s r s 1 1 1 1 ... ... ? ? ? ? ? ? = df dx dx g dx dx f dx dx dg i i i i j j j j r i i i i j j r r i s s r r s 1 1 1 1 1 1 1 .... ( ) ...... ? ? ? ? ? ? ? ? ? ? ? dx dx j j i s = d d r ? ? ? ? ? ? ? ? ? ( )1 iv) Again using (10.9) of 1.10 in (12.4) we see that d f x dx dx dx i i i i i k k r k ? ? ? ? ? ? ? ? ? 1 ..... or d d d f x x dx dx dx i i i i i i i i s k s k r k s ( ) ..... ? ? ? ? ? ? ? ? ? ? ?? 2 1 = 0, If i i s k ? , then, dx dx i i s k ? =0 Thus existence of such d is established. It is easy to establish the uniqueness of d. Thus there exist a unique exterior differentiation on D. . 1.13 Pull-back Differential Form : Let M be an n-dimensional and N be an m-dimensional manifold and f M N : ?

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57 be a differentiable mapping. Let T p (M) be the tangent space at p of M where ( ) ( ) * f p T N is its dual. Let ( ) ( ) f p T N be the tangent space at f(p) of N where ( ) ( ) * f p T N is its dual. If (x 1 ,....x n) and (y 1 ,....y m) are the local corrdinate system at p of M and at f(p) of N respectively, then, it is known that { : 1,....., } { : 1,....., } i j i n and j m x y ? ? ? ? ? ? are respectively the basis of T p (M) and ( ) ( ) f p T N . Consequently {dx i : i=1,...n} and {dy j : j = 1, ..., m} are the basis of T p * (M) and ( ) ( ) * f p T N respectively.. Let ? be a 1-form on N. We define an 1-form on M, called the pull-back 1 form of ? on M, denoted by f * ? , as follows 13.1) ? ? * * ( ) ( ) * { ( )}( ) ( ) (( ), f p p p f p p p f X f X f X ? ? ? ? ? ? p of M. where f * , f * are already defined in 1.7 So, we write 13.2) f f f p p * ( ) * ( ) ( ) ? ? ? then, by 7.4) of 1.7, we get from 13.1, on using 13.2) 13.3) ( ) ( ) ( ) * ( ) * ( ), f X f X p of M p p f p f p ? ?? ? Therefore we may write, for a 1 form ? on N and a vector field X on M by 13.4) ( )( ) ( ) * * f X f X ? ?? Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where (, ,....) x x x n 1 2 is the local coordimate system in a neighbourhood of a point p of M and (....) y y m 1 is the local coordinate system in a neighbourhood of f(p) of N, then f dy f x dx i f p j i p i p i n * ( ) ( ) ( ) ( ) ? ? ? ? ? 1 where f y f j j ? . , i m ?1,.... Proof : Since * ( ) ( ) i f p f dy is a co-vector at P on M, it can be expressed as the linear combination of the basis co-vectors ( ) i p dx at P and we take f dy a dx j f p i j i p i n * ( ) ( ) ( ) ? ? ? 1

58 Where a i j 's are unknown s to be determined or { ( ) } ( ) * ( ) f dy x p a dx x p j f p k i j i i p k ? ? ? ? F H I K ? F H I K ? usinng 10.5 of 1.10 we find that ? ? * ( ) ( ) ? ? ? ? ? ? ? ? ? ? ? j j i i f p i k k k p f dy a a x for ( ) i i p k k p x dx x k ? ? ? ? ? ? ? ? ? ? ? ? ? i k By (13.1). one reduces to dy f x a j f p k p k j c h ( ) * ? ? F H I K R S | T | U V | W | ? using Theorem 1 of 1.7 we find dy f x y a j j i m f p s k p s f p k j c h ? ? F H G I K J F H G I K J ? ( ) ( ) ? ? ? ? Using (10.5) of (1.10) we find ? ? f x a j k p k j F H G I K J ? Thus we get f dy f x dx j f p j k p i p i n * ( ) ( ) ? F H G I K J ? ? ? ? c h 1 , j m ?1,..., ; f y f j j ? ? Note : 1. Using (10.9) of 1.10, one find from above theorem 13.5) f dy df j m j f p j p * ( ) ( ) ( ) , ,.... ? ?1 we can also write it as 13.6) f dy df d y f j f p j j * ( ) ( ) . ? ? c h

59 2. If ? is a 1-form, then, its pull-back 1-form f * ? is given by 13.7) f df j j j * ? ? ? ? , where ? j are the components of ? (Prove it.) Exercises : 1 If f M R : ? 3 be such that f u u u a (,) (cos , sin ,) ? ? ? ? ? where x u x u x a 1 2 3 ? ? ? cos , sin , ? ? ? then for a given 1-form ? , ? ? ? ? x dx dx x dx on R 1 1 2 2 3 3, compute f * ? . 2. If f M R : ? 3 be such that f u a u Sin a Sinu Sin a Cos , cos , , ? ? ? ? b g b g ? then for a given 1-form ? ? ? ? ? dx adx dx 1 2 3 on R 3 , determine f * ? . 3. Let ? be the 1-form in R o o 2 ? , l q by ? ? ? ? ? ? y x y dx x x y dy 2 2 2 2 . Let U be the set in the plane (,) r ? given by U r? ? ? ? 0 0 2 ; ? ? l q and let f : U ? R 2 be the map f r(,)? ? x r Cos ? ?, compute f * ? y r Sin ? ? Let us now suppose that ? be a r-form on N. In the same manner, as defined earlier, we define an r-form on M, called the pull-back r-form on M, denoted by f * ?, as follows : 13.8) f X X f p p r p * ( ) ( ) ,...,( ) ? d i e jd i 1 ? ? f p p r p f X f X ( ) * * ( ) ,..., ( ) , 1 d i ?p {

60 We also write it as 13.9) ( )(...,) (,...,) * * * f X X f X f X r r ? ? 1 1 ? Proposition : 1. Let f : M n ? N m be a map, ? and ? be r-forms on N and g be a 0-form on N. Then a) f f f * * * ( ) ? ? ? ? ? ? ? ? b) f g f g f * * * ( ) ( ) ? ? ? Proof : a) As ? and ? are r-forms on N, ( ) ? ?? is also so. Hence

f X X

X f p r * ( ) ( ) (, ...,) ? ?? d i 1 2 ? ?( ) (,...,) ( ) * * ? ? f p r

f X f X 1 ? ? ? ? f p r f p r f X f X f X f X ( ) * * ( ) * * (,...,) (,...,) 1 1 ? ? f X X f X X f

p r f p

r * ( ) * ( ) ,..., ,..., ? ?

d i e j b g d i e j b g 1 1 by 13.8) ? f f f f p f p f p * ( ) * ( ) * ( ) ( ) ( ) ( ) , ? ? ? ? ? ? ? ? f p( ) Hence f f f * * * ( ) ? ? ? ? ? ? ? ? b) Note that if ? is a r-form and g is a o-form, then g? is again a r-form. Using (13.8) one gets f g X X f p r * ( ) ( ) (,...,) ? d i 1 ? ( ) (,...,) ( ) * * g

f X

f X f X f p r ? 1 2 2 ? g fp f p ( ) ( ) ?

d i (, ,...,) * * *

f X f X f X r 1 2 ? ( )( ) (, ,...,) ( ) * * * g f p f X f X f X f p r ? ? d i 1 2 ? ( )( ) (,...,) ( ) * * g f p f X f X f

p r ? ? 1 ? f g p

f f X f X f

p r * * ( ) * * ( )( ) ((,...,) ?

d i 1

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61 or f g f g p f f p f p * ( ) * * ( ) ( ) ( )( ) ( ) ? ? d i d i ? or f g f g p f p p * * * ( ) ( ) ( ) ( ) , ? ? b g b g b g ? ?p Hence f g f g f * * * ( ) ( ) ( ). ? ? ? Exercises : 4. Show that f f f * * * ( ) ? ? ? ? ? ? ? ? 5. Prove that ( ) ( ) * * * f h h f ? ? ? ? Note : From Theorem 1 of 1.11, we see that, any r-form ? can be expressed as ? ? ? ? ? ? ? ? g dx dx i i i r i i i i i r r 1 2 1 1 2 ... ... where g i i i r 1 2 ... are differentiable functions on N. Then f f g dx dx i i i r i i i i i r r * * ... ... ? ? ? ? ? ? ? ? 1 2 1 1 2 e j ? ? ? ? f g f dx f dx i i i r i i r * ... * * ... 1 2 1 by the Proposition 1(b) and Exercise 4 above ? ? ? ? g f f dx f dx i i i i r r 1 1 ... * * ... ? d i Using 13.5) of 1.13 we see that 13.10) f g f df df i i i i i i i r r r * ... ... ? ? ? ? ? ? ? ? 1 1 2 1 ? e j Exercise : 7. Let M be a circle and ?M be R 2 so that f M M : ? ? be defined by x r x r 1 2 ? ? cos , sin ? ?

62 If ? ? ? a dx bdx 1 2 and ? ? ? 1 1 1 2 a dx b dx , find f * ( ) ? ?? Solution : In this case, ? ? ? ? 1 2 1 2 1 1 ? ? ? ? a b a b , , , df dr r d 1 ? ? cos sin ? ? ? df dr r d 2 ? ? sin cos ? ? ? ? f a dr r d b dr r d * (cos sin) (sin cos) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? (cos sin) (cos) a b dr br ar sin d ? ? ? ? ? and f * ? ? ? ? ? ? 1 1 a dr r d b dr r d (cos sin) (sin cos) ? ? ? ? ? ? ? ? ? F H I K ? ? F H I K 1 1 a b dr r b r a d cos sin cos sin ? ? ? ? ? Using Exercise 5, one finds that f * ( ) ? ?? ? ? f f * * ? ? ? ? ? ? ? a b Sin dr b r a r d cos cos sin ? ? ? ? ? b g b g m r ? ? F H I K ? ? F H I K R S T U V W 1 1 a b dr r b r a d cos sin cos sin ? ? ? ? ? ? ? ? F H I K ? ? (cos sin) cos sin a b r b r a dr d ? ? ? ? ? ? ? ? ? F H I K ? b r a r a b d dr cos sin cos sin ? ? ? ? ? b g 1 1 ? ? F H I K ? r a b b a dr d? where d dr dr d ? ? ? ? ? ? . Theorem 2 : For any form ?, d f f d ( ) ( ) * * ? ? ?

63 where the symbols have their usual meanings. Proof : We shall consider the following cases. i) ? is a o-form ii) ? is a r-form Case i) : In this case, let ? ? h, where h is a differentiable function Then f dh X * ( ) ( ) m r ?

dh f X( ) * ? ( ) * f X h by (10.4) of 1.10 = X h f( )? by (7.3) of 1.7 = d h f X ( )( ) ? by (10.4) of 1.10 = d f h X ( ) ( ) * m r by (10.4) of 1.10 or f dh d f h * * ( ) ( ) ? The result is true

in this case. Case ii) : In this case, we assume that the result is true for ( ) r ?1 form. Without any loss of generality, we may take an r-form ? as ? ? ? ? g

dx dx i i i i i r r 1 2 1 ... ... or f f g dx dx i i i i r r * * ... ... ? ? ? ? 1 1 d i ? ? ? f g dx dx i i i i r r * ... ... 1 1 d i ? ? ? ? ? ? f g dx dx f dx i i i i i r r r * ... * ... ( ) 1 1 1 d i or d f( ) * ? ? ? ? ? ? d f g dx dx f dx i i i i i r

r r * ... * ... ( ) 1 1 1 d i o t Using (12.1) of 1.12 we find that

d f( ) * ? ? ? ? ? ? ? d f g dx dx f dx i i i i i i r r r * ... * ... ( ) 1 1 1 d i o t +

64 ? ? ? ? ? ? ? ( ) ... ( ) * ... * 1 1 1 1 1 r i i i i i f g dx dx d f dx r

r r d i b g Note that dx i r is a 1-form and hence the theorem is true in this case. Thus d f dx f d dx i i r r * * ( ) ( ) b g b g ? ? 0 by (12.1) of 1.12

Hence d f( ) * ? ? ? ? ? ? d f g dx dx f dx i i i i i i r r r * ... * ... ( ) 1 1 1 d i o t ? ? ? ? ? f d g dx dx f dx i i i i i i r r r * ... * ... ( ) 1 1 1 d i o t , as the result is true for ( ) r ?1 form ? ? ? ? ? f dg dx dx i i i i r r * ... ... 1 1 1 d i o t ? f dx i r * ( ) by (12.1) of 1.12 ? ? ? ? ? ? ? f dg dx dx dx i i i i i r r r * ... ... 1 1 1 d i by known result Thus d f f d ( ) ( ) * * ? ? ? and hence the result is true for r-form also. Combining we claim that d f f d ( ) ( ) * * ? ? ? i.e. d and f commute each other.. REFERENCES 1. W.M.Boothby : An Introduction to Differentiable Manifolds and Riemannian Geometry. 2. Kobayashi & Nomizu : Foundations of Differentiable Geometry, Volume I 3. N. J. Hicks : Differentiable Manifold 4. Y. Matsushima : Differentiable Manifold

65 UNIT - 2 . 2.1 Lie group, Left translation, Right translation : Let G be a differentiable manifold. If G is a group and if the map (,) g g g 1 1 2 g 2 ? ? from G G to G and the map g g ? ?1 from G to G are both differentiable, then G is called a Lie group. Exmaple : Let GL(n, R) denote the set of all nonsingular n n matrices over real num- bers. GL(n, R) is a group under matrix multiplication. Define ? ( ) (, ,...., ; , ,...., ;...; , ,....,)

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A ? a a a a a a a a a

n n n n nn 11 12 1 21 22 2 1 2 then ?: (,) GL n R R n ? 2 is a mapping of class C . Hence GL(n, R) is a Lie group. Note : Lie groups are the fundamental building blocks for gauge theories. For every a ? G, a mapping L a : G ? G defined by 2.1) L x ax a ? , ? ?x G is called a Left translation

on G. Similarly, a mapping R a : G ? G defined by 2.2) R x xa

a ? , ? ?x G is called a right translation on G.

66 Note that L L x L bx abx a b a ? ? ( ) and L x abx a b ? ? L a L a = L ab R R x a b ? R bx xba a ( ) ? and R x xba ab ? ? R R R a b ba ? L R x L xb axb a b a ? ? ( ) and R L x R ax axb b a b ? ? ( ) ? L R R L a b b a ? Thus 2.3) L L L a b ab ? , R R R a b ba ? , L R R L a b b a ? Again L L x L ax bax abx L L x b a b a b ? ? ? ? ( ) , Thus 2.4) L L L L b a a b ? , unless G is commutative Taking b a? ?1 in 2.3) we find L L L a a aa ? ? ? 1 1 by 2.3) ? L e Thus 2.5) L L a a ? ? ? 1 1 ( ) It is evident that, for every a ? G, each L a and R a are diffeomorphism on G.. Exercise : 1 Show that the set of all left (right) translation on G form a group. 2. Let ? : G G 1 2 ? be a homeomorphism of a Lie group G 1 to another Lie group G 2 . Show that i) ? ? ? ? ? L L a a ? ( ) ii) ? ? ? ? ? L R b b ? ( ) , ? a b, in G.

67 3. Let ? be a 1-1 non-identity map from G to G. If ? ? ? ? L L g g ? is satisfied for all g ? G, then there is a h ? G such that ? ? R h . Solution : 2. From the definition of group homeomorphism of a Lie group G 1 to another Lie group G 2 , ? ? ? ( ) ( ) ( ), ab a b ? ? a b, in G 1 i) ( ) ( ) ( ) ( ) ( ) ? ? ? ? ? ? ? L x L x ax a x a a ? ? ? ? ? L x L x a a ? ? ? ? ( ) ( ) ( ) ( ) , ? ? x in G 1 ? ? ? ? ? ? L L a a ? ( ) Similarly ii) can be proved. 3. As G is a group, e ? G (identity). Further ? is a 1–1 map from G to G, so for e ? G, there is h in G such that ?( )e h? Note that ?( ) , e e? because, ? is not an identity map. Now for g ? G,, g ge ? ? ? ? ? ( ) ( ) g ge ? ?( ) L e g ? ( )( ) ? ? L e g ? ( )( ), L e g ?? as given ? Lg e?( ) b g = Lgh = gh = R h g ? ? ? ? R g h ,

68 . 2.2. Invariant Vector Field : We have already defined a vector field to be invariant under a transformation in 1.8. Note that, in a Lie group G, for every a, b in G, each L a , R b is a transformation on G. Thus we can define invariant vector field under L a , R b .

A vector field X on a Lie group G is called a left

invariant

vector field on G if 2.6) ( ) , * ( ) L X X a p L p a ? ? p ? G , where ( ) * La is the differential of

L a .

Thus from 1.7 ( ) * ( ) ( ) L X X a p L p L p a a d i ? We write it as 2.7) ( ) * L X X a ? Similarly for a right invariant vector field, write 2.8) ( ) * R X X a ? From 1.7) we know that ( ) ( ) * L X g X g L a p p a d i ? ? or ( ) ( ) * ( ) L X g X g L a p L p p a a d i ? ? If L p q a ( ) ? then p L q L q a q a a ? ? ? ? ? ? ( ) 1 1 1 Thus the above relation reduces to 2.9) ( ) ( ) * L X g X g L a q a q a b g ? ?1 ? Let g be

the set of all left invariant vector field on G. If X, Y, ? g, a, b ? R, then 2.10) ( ) ( ) *

L aX bY p ? ? ? a L X b L Y p p ( ) ( ) * * ? ? aX bY , ( ) * L p being linear explained in Unit 1. 2.11) ( ) [,] ( ) ,( ) * * * L X Y L X L Y p a p ? , see 1.7 = [X, Y]

69 Thus aX bY g ? ? and [,] . X Y g? Consequently g is a vector space over R and also a Lie- algebra. The Lie algebra formed by

the set of all

left invariant vector fields on G is called the Lie algebra of the Lie group G.

Note that

every left invariant vector field is a vector field i.e. g G ? ?( ) where ?( )G denotes the set of all vector field on G. The converse is not necessarily true. The converse will be true if a condition is satisfied by a vector field. The following theorem states such condition. Theorem 1 :

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A vector field X on a Lie group G is left invariant if			

and only if for every f F G ? ( ) 2.12) ( ) ( ) Xf L X f L a a ? ? ?

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Proof : Let X be a left invariant vector field on a Lie group G. Then for every f F G ? ( ) , we have from (2.6) ( ) * ( ) L X f X f			

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a p L p a n s ? or X f L Xf L p p a a ( ) ( ) ( ) ? ? by Q 1.7 or X f L p Xf L p a a ( ) ( ) ( )( ) ? ? l q ? , ? ?p G ? Xf L X f L a a ? ? ? ( ) Conversely let (2.12) be true. Reversing the steps one gets the desired result. Note : i) The behaviour of a Lie group is determined largely by its behaviour in the neighbourhood of the identity element e of G. The behaviour can be represented by an alge- braic structure on the tangent space of e, called the Lie algebra of the group. ii) Note that, two vector spaces U and V are said to be isomorphic, if a mapping f : U ? V is i) linear and ii) has an inverse f ?1 : V ? U Theorem 2 : As a vector space, the Lie subalgebra g of the
Lie group G is isomorphic to the tangent space T e (G) at the identity element e ? G..
70
Proof : Let us define a mapping ?: g T e ? (G) by i) ?( )X X e ? Note that, for every X, Y in g, X Y g ? ? and ?( ) ( ) X Y X Y e ? ? ? by i) ? ?X Y e e ? ? ? ? ( ) ( ) X Y Also for b R? , bX g? and ?( ) ( ) bX bX e ? by i) ? bX e ? bX by i) Thus ? is linear.. We choose X T G a a ? ( ) such that ii) ( ) , * L V X a e a ? , Where V T G e e ? ( ). Then ( ) * L X s s a ?1 ? ? ( ) ( ) * * L L s s a e 1 V from above ? ? L L s s a e ? 1 b g * V from 1.7 ? ? L a ss e 1 b g * V by (2.3) ? ( ) * L a e V ? X a , as chosen or ( ) * ( ) ( ) L X X s L s a L s a s s b g ? ? ? 1 1 by Q 1.7 or ( ) * L X X s ? ? X g? We define ? ? ? 1 :T (G) e g by
71 iii) ? ? ? 1 (V) e X Then ( ) ( ) ( ) ?? ? ? ? ? ? ? ? ? ? 1 1 V V X X e e e c h ii), where ( ) * L e is the identity differential on G.. or ( ) ?? ? ? 1 V V e e Further, ( ) ( ) ( ), ? ? ? ? ? ? ? ? ? ? 1 1 1 X X X e b g by i) ? ? ? 1 ( ) * L V e e b g by ii) ? ? ? 1 ( )V e ? X by iii) Thus an inverse mapping exists and we claim that g ? T (G) e Exercises : 1. If, X, Y are left invariant vector fields, show that [X, Y] is also so. 2. If c i j k n ij k (, , , ,...,) ?1 2 are structure constants on a
Lie group G with respect to the basis X X
X n 1 2 , ,..., l q of g, show that i) c c ij k ji k ? ? ii) c c c c c c ij k ks t js k ki t si k kj t ? ? ? 0 Solution : 1. From Q 1.7), we see that ( ) [,] * L X Y f a l q ?[,]( ) X Y f L a ? ? ? X Y f L Y X f L a a ((, ? ? b g b g from the definition of Lie Bracket ? ? X L Y f Y L X f a a ( ) ( ) * * b g m r b g m r by 1.7 ? ? X Yf Y Xf ( ) ( ) by (2.7) ?[,] X Y f from the definition of Lie Bracket
72 ? ( ) [,] [,], * L X Y X Y f a ? ? Using (2.7), we see that [X, Y] is a left invariant vector field. 2. Using problem 1 above, we see that every [,] X X g i j ? as X g i ? , i n ?1, ..., . Since X X X n 1 2 , ,..., l q is a basis of g, every [,] X X g i j ? can be expressed uniquely as, 1) [,] X X c X i j ij k k ? where c ij k ?R i) Note that if i j X X i j = ,[,] ? 0 So, let i j? . Then from a known result, [,] [,] X X X X i j j i ? ? Using 1) we find that c X c X ij k k ji k k ? ? As the set X X n 1 ,..., l q is a basis of g and hence linearly independent, we must have c c ij k ji k ? ? ii) Using Jacobi Identity, we find that [,], [,], [,],
X X X X X X X X X i j s j s i s i j ? ? ? ? Hence from 1) c X X c X X c X X
ij k k s js k k i si k k j [,] [,] [,] ? ? ? ? as [,] [,], bX Y b X Y b R ? ? Again applying 1) , we find that c c X c c X c c X ij k ks t t js k ki t t si k kj t t ? ? ? ? As X X n 1 ,..., l q is a basis and hence linearly independent, we must have c c c c c c ij k ks t js k ki t si k kj t ? ? ? ?
73 . 2.3 Invariant Differential Form : A differential form ?

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on a Lie group G is said to be left invariant if 2.13)			

L a
L p p a * ( ) , ? ? d i ? ? ?p G we write it as 2.14) L a * ? ?? and call L a * ?, the pull-back differential form of ?. Similarly, a differential form ?

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on a Lie group G is said to be right invariant if 2.15)			

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R a * ? ?? A differential form, which is both left and right invariant, is called a biinvariant differential form. Exercises : 1. If ? ? 1 2 , are left invariant differential forms, show that, each d? ? ? , 1 2 ? is also so. 2. Prove that a differential 1-form ? on a Lie group is left invariant if and only if for every left invariant vector field X on G, ?(X) is a constant function on G. 3. Let ? : G ? G be such that ?( ) , a a? ?1 ? ?a G. Show that a form ? is left invariant if and only if ? ? * is right invariant. 4. Prove that the set of all left invariant forms on G is an algebra over R. Such a set is denoted by A, say. 5. If g * denotes the dual space of g, then, prove that A ? g * where A is the set already defined in Exercise 4 above. Solution : 1. From Q 1.13, we see that L d d L a a * * ( )? ? 1 1 ? c h where L a * ? 1 is the pull-back 1 form of ? 1 Using on (2.14) on the right hand side of the above equation, we see that L d d a * ( )? ? 1 1 ?

74 Consequently, d? 1 is a left invariant differential form. It can be proved easily that ? ? 1 2 ? is a left invariant differential form. 2. Let us consider a differential 1-form ? . Then for every a G? , L a * ? will be defined as the pull-back differential 1-form. Consequently from the definition of pull-back. L X a L p p a * ( ) ( ) ? e j ? ? L p a p a L X ( ) * ( ) , d i ? ?p G Let us consider X to be left invariant. Then on using (2.6) on the right hand side of the above equation, we get 1) L X a L p p a * ( ) ( ) ? e j ? ? L p L p a a X ( ) ( ) e j Let us now consider ? to be left invariant 1-form. Then by (2.13), we get from 1) ? ? p p L p L p X X a a ( ) ( ) ( ) ? e j ? ? ap ap X( ) Taking p e? , we see that ? ? ? e e ae ae a a X X X ( ) ( ) ( ) ? ? Consequently, ?(X) is a constant function on G.. Conversely, if ?(X) is a constant function on G, then ( ) ( ) p p ap ap X X ? ? ? Hence 1) reduces to L X X a L p p p p a * ( ) ( ) ? ? e j ? or L a L p p a * ( ) ? ?? which is (2.13) Thus ? is a left invariant differential form. This completes the proof.

75 Theorem 1 : If g is a Lie subalgebra of a Lie group G and g * denotes the set of all left invariant form on G, then d X Y X Y ? ? (,) [,] ? ? 1 2 b g where ? ?g * , X Y g , ,? Note : Such an equatioin is called Maurer-Carter Equation. Proof : From theorem 1 of 1.12, we know that d X Y X Y Y Y X Y ? ? ? ? (,) ( ) ( ) [,] ? ? ? 1 2 b g b g b g m r for every vector field X, YY If X, Y are in g then by Exercise 2, ? ? ( ), ( ) X Y are constant functions on G. Hence by Exercise 2 of 1.4), X Y. ( ) , ? ? 0 Y X . ( ) ? ? 0 Thus the above equation reduces to d X Y X Y ? ? (,) [,] . ? 1 2 b g Exercise : 6. Show that d c c i jk i j k j k jk i j k k j ? ? ? ? ? ? ? ? ? ? ? ? ? 1 2 , , Solution : If X X X n 1 2 , ,..., l q is a basis of g and ? ? 1 ,..., n m r is the dual basis of g * , then 1) ? ? i j j i X( ) ? Hence from theorem 1 above d X X X X i j k i j k ? ? (,) [,] ? ? 1 2 d i ? ? R S | T | U V | W | ? 1 2 ? i jk m m c X from Exercise 2 of Q 2.2 1 1 ( ) 2 2 ? ? ? ? ? ? ? ? ? m i m i m m jk jk c X c ? ? 1 2 c jk i by i)

76 Again from 1.111 c X X c mn i m n j k mn i m n m n ( )(,) , , ? ?? ? ? ? 1 2 ? ? ? ? m j n k m k n j X X X X ( ) ( ) ( ) ( ) ? n s ? ? ? 1 2 c mn i m n j m k n k m j n , ? ? ? ? n s ? ? 1 2 c c jk i kj i o t ? ? 1 2 c c jk i jk i o t by i) of Exercise 1 of 2.2 ? ? 1 2 2 c jk i ? c jk i Thus d X X c X X i j k mn i m n m n j k ? ? ? (,) (,), , ? ? ? ? 1 2 ?x x j k , or d c mn i m n m n ? ? ? ? ? ? ? 1 2 , or d c i j k i j k j k ? ? ? ? ? ? ? ? 1 2 , Take i, j, k = 1, 2, 3 , then c c c c j k i j k j k i i i , ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 12 1 2 13 1 3 21 2 1 ? ? c i 23 3 2 ? ? ? ? ? ? ? c c i i 31 3 1 32 3 2 ? ? ? ? ? ? ? ? ? ? ? 2 2 2 12 1 2 13 1 3 23 2 3 c c c i i i ? ? ? ? ? ? as c c j k i h k i ? ? ? ? ? ? ? 2 c j k i j k j k ? ?

77 Thus, we write d c i j k i j k j k ? ? ? ? ? ? ? ? Hence d c i j k i j k k j ? ? ? ? ? ? ? ? . . 2.4 Automorphism : A mapping, denoted by ? a for every a ?G , ? a :G G ? defined by ? a x axa ( ) , ? ?1 ? ?x G is said to be an inner automorphism if i) ? ? ? a a a xy x y ( ) ( ) ( ) ? ii) ? a is injective iii) ? a is surjective such ? a is written as ada. Exercise : Show that if G is a Lie group, h ?G, then the map I G G h : ? defined by I h k hkh ( ) ? ?1 is an automorphism. An inner automorphism of a Lie group G is defined by 2.16) ( )( ) , ada x axa ? ?1 ? ?x G Now, ( ) ( ) ( ) ( )( ) L R x L R x L xa axa ada x a a a a a ? ? ? ? ? ? ? ? 1 1 1 1 ? L R ada a a ? ? 1 Using 2.3) we get

78 2.17) ada L R R L a a a a ? ? ? ? 1 1 Note that ada is a diffeomorphism. Theorem 1 : Every inner automorphism of a Lie group G induces an automorphism of the Lie algebra g of G. Proof : For every a ?G let us denote the inner automorphism on G by i) ( )( ) , ada x axa ? ?1 ? ?x G Now for every G, e ?G and from 1.7 such ada : G G ? induces a differential mapping (ada) * , ( ) : * ada ada T (G) T T (G) e (e) (G) e ? ? Such a mapping is a linear mapping and by Theorem 2 of 2.2, the Lie subalgebra g of a Lie group G is such that g T (G) e ? Thus to show every ada induces an automorphism of the Lie algebra g of G we are to show ii) (ada) * is a mapping from g to g iii) (ada) * is a homomorphism i.e. ( ) ( ) ( ) ( ) * * * ada X Y ada X ada Y ? ? ? ( ) ( ) ( ) * * ada bX b ada X ? ( ) [,] ( ) ( ) * * * ada X Y ada X ada Y ? ? , ? X Y, in g iv) (ada) * is injective v) (ada) * is surjective ii) Let Y ?G . Then on using 2.17) we get ( ) * * ada Y R L Y a a ? ?1 ? b g ? ? R L Y a a 1 b g b g * * as ( ) * * * f g f g ? ? ? ? ? R Y a 1 b g * Thus vi) ( ) * * ada R a ? ?1 b g Again, ( ) * * L R Y p a ?1 b g o t ? ? ( ) * * L R Y p a 1 b g o t , for every p ?G

79 ? ? L R Y p a ? 1 d i * ? ? R L Y a p 1 ? d i * by 2.3) ? ? R L Y a p 1 b g d i { } * * ? = R L Y a p ?1 b g d i * * = R Y a ?1 b g * as Y g? Consequently, from above, it follows that R Y a ?1 b g * ?g. Hence (ada) * is a mapping from g to g. iii) From 1.7) we know that such (ada) * is a linear mapping i.e. ( ) ( ) ( ) ( ) * * * ada X Y ada X ada Y ? ? ? ( ) ( ) ( ) , * * ada bX b ada X ? b R? Further, such (ada) * satisfies ( ) [,] ( ) , ( ) * * * ada X Y ada X ada Y ? Thus (ada) * is a homomorphism from g to g. iv) Clearly (ada) * is injective, on using vi) and the fact that R a ?1 is a translation on G.. v) For every a ?G, a ? ? 1 G and we set ( ) , * ada X Y ? ? 1 where X ?G we will show that Y ?G and ( ) . * ada Y X? Now, for s ?G, ( ) * L Y s ? ( ) ( ) ( ) ( ) * * * * L ada X L R La X s s a ? ? ? 1 1 ? by (2.17) ? ? ( ) ( ) ( ) * * * L R L X s a a 1 l q ?( ) ( ) * * L R X s a ?

80 =( ) ( ) ( ) * * * L R X R L X R X s a a s a ? ? ? ? ? ? ( ) * ada X 1 ? Y as defined. Thus Y g? Finally ( ) * ada Y ? ? L R Y a a ? 1 b g * by (2.17) ? ? ? L R ada X a a ? 1 1 b g * * ( ) as defined ? ? ? L R R L X a a a a ? ? 1 1 b g b g * * by (2.17) ? ? ? L R R L X a a a a ? ? 1 1 b g * by (1.7) ?( ) * L X e by (2.3) , where ( ) * L e is the identity differential = X Consequently, ( ) * ada is a surjective mapping. Combining ii) –– v), we thus claim ( ) : * ada g g ? is a Lie algebra automorphism. This completes the proof. Note : We also write ( ) * ada = AAda , for every a g? . and a ? Ada is called the Adjoint representation of G to g. . 2.5 One parameter subgroup of a Lie group Let a mapping a : R ? G denoted bya : t ? a(t)

81 be a differentiable curve on G. If for all s, t in R a t a s a t s ( ) ( ) ( ) ? ? then the family a t t R ( )| ? l q is called a one-parameter subgroup of G.. Exercises : 1. Let H = a t t R ( )| ? l q be a one-parameter subgroup of a Lie group G.. Show that H is a commutative subgroup of G. 2. If X is a left invariant vector field on G, prove that, it is complete We set 2.18) a t a e t t ( ) ( ) ? ? ? where ? t t: ?R l q

is one parameter group of transformations on G, generated by the left invariant vector field x.

Exercises : 3. Let ? t t| ?R l q be a one-parameter group of transformations on G, gener-- ated by X g? and ? t e a t ( ) ( ). ? If for every s g? , ? ? t s s t L L ? ? ? show that the set a t t R ( )| ? l q is a one-parameter subgroup of G and ? t a R t ? holds, for all t R? 4. Let the vector field X be generated by the one parameter group of transformations R t R a t | ? o t on G. Show that X is left invariant on G. Solution : As ? t t| ?R l q is a one-parameter group of transformations on G and a t R a t : ( ) ? ? ?G is a differentiable mapping, by definition a t a s L a s a t ( ) ( ) ( ) ( ) ? ? b g = L e a t s ( ) ( ) ? b g , as defined in the hypothesis ? L e a t s ( ) ( ) ?? d i ? ? s a t L e ? ( ) ( ) d i by the hypothesis

82 ? ? s a t e L ( ) ( ) e j ? ? s a t e ( ) b g ? ? s a t( ) b g ? ? s t q e( ) b g as defined ? ? ? s t e ? b g ( ) =? s t e ? ( ) is ? ( )t l q a one-parameter group of transformations on G ? ? ? t s e( ) , as s t t s ? ? ? in R ? ? a t s( ) Thus the set a t t R ( )| ? l q is a one-parameter subgroup of G.. Again ? ? ? ? t t t s t s s se L e L e ( ) ( ) ( ) ( )( ) ? ? ? b g ? ? ? L e L a s t s t ? ( ) ( ) b g by (2.18) ? sa t or ? t a s R s t ( ) ( ), ? ? ?s G ? ?? t a R t 4. From Exercise 3 above R a t t ? ? As it is given that R t R a t | ? o t generates the vector field X, from 1.9, we can say that X s is the tangent vector to the curve R a t and we write X f t t f R s f s s a t ? ? ? lim ( ) ( ) 0 1 e j { } ? ? ? ? ? lim (( )) ( )( ) t t f L R q s f L q s q a q t 0 1 1 1 d i { }

83 ? ? ? ? ? lim (( )) ( )( ) t t f L R q s f L q s q a q t 0 1 1 1 e j { } ? ? ? ? ? ? lim ( ) ( ) ( )( ) t t f L R q s f L q s q a q t 0 1 1 1 ? ? d i { } i) X f X f L s q s q ? ?1 ( ) ? from 1.9 We are left to prove that X g? . Note that, for q g? . L G G q : ? is a left translation on G and ( ) : ( ) ( ) ( ) * ( ) L T G T G T G q p L p qp q ? ? is its differential. Hence ( ) ( ) * L X f X f L q p q d i ? ? by 1.7, where f F G ? ( ) or ( ) ( ) * ( ) L X f X f L q L p p q q d i ? ? If L p s q ( ) ,? then p L s L s q q ? ? ? ? 1 1 ( ) ( ) by (2.5) ? p q s ? ?1 Consequently, the above equation reduces to ( ) ( ) * L X f X f L X f q s q s q s d i ? ? ?1 ? by i) ? ( ) , * L X X q s s d i ? ? ?s G ? ( ) , * L X q ? which shows that X is left invariant. Theorem 1 : If X,Y g, then [ ] lim Y,X Y Y t t Ada t ? 1 1 c h o t Proof : Every X g induces t t| R l q as its 1-parameter group of transformations on G. Hence by 1.9. [ ] [ ] lim Y,X X,Y Y Y * ? ? ? ? ? t t t ? 1 ? b g o t

84 Now from 2.4 A Y = Y * da ada t t 1 1 c h c h R L Y a a t t ? 1 e j * by 2.17) = R L Y a a t t e j e j{ } * * 1 R Y a t e j * , as Y g = t b g * Y by Exercise 3. Consequently, the above question reduces to, [ ] lim Y,X A Y Y t t da t ? 1 1 c h o t 2.6 Lie Transformation group (Action of a Lie group on a Manifold) A Lie group G is a Lie transformation group on a manifold M or G is said to act differentially on M if the following conditions are satisfied : i) Each a G induces a transformation on M, denoted by p pa, p M. ii) (a, p) : G M p M a is a differentiable map. iii) p ab pa b ( ) ( ) , a b p , , . G M We say that G acts on M on the right. Similarly, the action of G on the left can be defined. Exercise : 1. Let G =GL R 2 ( ) and M = R and :G M M be a differentiable mapping defined by a b p ap b 0 1 F H I K F H I K , , a 0, a b, R Show that is an action on M.

85 Solution : In this case, 1 0 0 1 F H I K G and i) 1 0 0 1 F H I K F H I K , p 1 0 p , = p ii) , , 0 1 0 1 a b a b p ? = F H I K F H I K a b ap b 0 1 , as defined a ap b b ( ) , as defined a ap a b b , F H I K F H I K aa a b b p 0 1 , as defined F H I K F H I K F H I K a b a b p 0 1 0 1 , Thus is an action on M. Definition : If G acts on M on the right such that 2.19) pa p, p M implies that a e then, G is said to act effectively on M. Note : There is no transformation, other than the identity one, which leaves every point fixed. If G acts on M on the right such that 2.20) pa p, p M , implies that a e for some p M then, G is said to act freelyeely on M. Note : In this case, it has isolated fixed points. Theorem 1 : If G acts on M, then the mapping : ( ) g M denoted by :A ( ) * A A

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86 is a Lie Algebra homomorphism Note : ( )A is called the fundamental vector field on M corresponding to A g . Proof : For every p G let p :G M be a mapping such that i) p a pa ( ) Such a mapping is called the fundamental map corresponding to p M? . We want to show that : ( ) g M is a Lie Algebra homomorphism i.e. we are to prove ii) ? ? ? ( ) ( ) ( ) X Y X Y ? ? ? iii) ? ? ( ) ( ), bX b X b R ? ? iv) ? ? ? [,] [,] X Y X Y ? It is evident from i) that v) p a a pa p ( ) ( ) R Let A g. Then from 2.5, A generates t t| R l q as its 1-parameter group of transfor-- mation on G, such that a t a e t t ( ) ( ) ? ? ? In this case, such a t( ) is the integral curve of A on G. The map * ( ) ( ) : ( ) ( ) ? ? ? ? p p e e p T G T M T (M) is the differential map of p and is a linear mapping by definition such that ( ) ( ). * p e p X T M Using the hypothesis of the theorem vi) p e e p p p d i l q l q * ( ) * A (A) (A) A

87 Note that for every A, B, in g, A + B is in g and hence (A + B) (A + B) l q d i p p e * p d i * (A + B) e e p p d i d i * * , A + B e e as p d i * is linear (A) (B) l q l q p p ? (A + B) = (A) + (B), p M. Also for b R bA g and hence ( ) * bA (bA) l q d i p p e p e p e p b d i d i * * { ( )} (A) A b A ? ? ? ( ) ( ) b b A A ? Thus is a linear mapping Now A e is the tangent vector to the curve a t a t ( ) at a e ( ) . 0 Consequently by 1.7, the vector field ? ? e ( ) * A T (M) T (M) ? ? ? ? p p e p is defined to be the tangent vector to the curve ? p t t a a pa p t ( ) ( ) ? ? R at p o p a e p ( ) ( ) . consequently, by vi), we see that A e * induce R a t p as its one-parameter group of transformations on M. Again [( ), ( )] [,] A B A B * * p p R S T U V W lim * * * t t p a p t 0 1 B R B e je j by Theorem 3 of 1.9 lim * * t t p a q t 0 1 d i e j { } * e B R B say, where vii) p q a t R ( ) viii) or q p p pa a a t t t R R e j 1 1 1 ( ) Thus R B R B a q a pa t t t e j e j * * * * 1 by vii) above R B a pa e t t e j e j * * 1 by vi) R B a pa e t t ? 1 e j * where R G M a pa t t ? 1 :

88 Hence for b G R R a pa a pa b t t t t b ? 1 1 e j e j ( ) ( ) =R a t t pa b ( ) 1 by i) = pa ba t t 1 by definition p t t a ba 1 c h by i) p t ada b 1 ( ) c h by 2.16) of 2.4 ? ? ? p t ada ? 1 d i ( )b ? R a pa p t t t ada ? ? 1 1 Consequently, R R B a q a pa e t t t B e j e j * * * ? ? ?? 1 reduces to R B a q p t e t B ada e j d i * * * ? ? ? ? 1 ? ? ? p t e ada d i c h e j * * 1 B ? ? ? p t e da d i c h e j * * A B 1 from the Note of 2.4 Thus we find ? ? ? ? ( ), ( ) lim ( ) * * * A B A B p p e p t e t t B da ? ? ? ? 0 1 1 d i c h e j { } ? ? ? ? ? p e t e t t da d i c h { } * * lim 0 1 1 B A B as ? p d i * is a linear mapping. ? ? p e d i * [ ]A,B by 1.9 ? ?[ ] A, B b g p by vi) ? ? ? ? [ ] ( ), ( ) A,B A B ? Thus the mapping ? ? : ( ) g ? M is a Lie Algebra homomorphism.

89 Theorem 2 : If G acts effectively on M, then the map ? ? : ( ) g ? M defined by ? ? : ( ) * A A A ? ? is an isomorphism. Proof : From Theorem 1, we know that such map ? ? : ( ) g ? M is a Lie Algebra homo- morphism. Hence we are left to prove that i) ? is injective and ii) ? is surjective. i) Let A, B ?g and ? ? ( ) ( ) A B ? Then ? ? ( ) , A B? ? as ? is a linear mapping. or ( ) * A B? ? ? i.e. ( ) * A B? is the null vector on M. Now A–B ?g and it will generate ? t e t( ) | ?R l q as its 1-parameter group of transformations on G such that ( ) A B? e is the tangent vector to the curve, say b t b e t t ( ) ( ) ? ? ? at b o e ( ) ? Consequently, the vector field ( ) ( ) * * A B A B ? ? ? ? p e d i is the tangent vector to the curve ? p t b b t pb R p t ( ) ( ) b g ? ? at ? ? p p b o e pe p ( ) ( ) . b g ? ? ? Thus ( ) ( ) * * A B A B ? ? ? ? p e d i generates R R b t p t ( )| ? o t as its 1-parameter group of trans- formations on M. But ( ) * A B? is the null vector on M. Hence the integral curve of ( ) * A B? will reduce to a single point of itself. Thus R b t p p ( ) ? or pb p t ? As G acts effectively on M, comparing this with 2.19) we get, b e t ? , ? ?p M. Again L q d i * ( ) A B A B ? ? ? as ( ) A B? ?g

90 ? L L q t t q ? ? ? ?? from 1.9 Thus ? ? ? t t t q q qe e ( ) ( ) ? ? b g d i L ? ? ? ( )( ) ( )( ) ( ) ? ? ? ? L L L q q t q t e e b ? ? ? qb q q t e Hence from 1.9 ( ) lim ( ) ( ) A B? ? ? ? q t f t t f q f q 0 1 ? b g m r reduces to ( ) lim ( ) ( ) . A B? ? ? ? ? q f t t f q f q 0 1 l q ? Thus A B= ? ? i.e. A = B. Hence ? ? ( ) ( ) A A ? implies that A = B. Consequently ? is injective. ii) As G acts effectively on M, ? is surjective. Thus the map is a Lie Algebra isomorphism and this completes the proof. Theorem 3 : If G acts freely on M, then, for every non-zero vector field A ?g, the vector field A * on M can never vanish. Proof : If possible, let A * be a null vector on M. Then, as done in the previous theorem, every A ? g will generate ? t e t( ) | ?R l q as its 1-parameter group of transformations on G and we will have ? t q q ( ) ? Consequently from the definition, as given in 1.9 A q t t f d dt f q ? L N M O Q P ? ? ( ) b g 0 ? ? ? lim ( ) ( ) t f q f q t t 0 ? b g = 0.

91 Hence A becomes a null vector, contradicting the hypothesis. Thus the vector field A * on M can never vanish. REFERENCE 1. P. M. Cohn : Lie groups 2. B. B. Sinha : An Introduction to Modern Different geometry 3. S. Helgason : Differential geometry, Lie groups and Symmetric spaces.

92 UNIT - 3 3.1 Linear Connection : The concept of linear (affine) connection was first defined by Levi-Civita for Riemannian manifolds, generalising the notion of parallelism for Eucliden Spaces. This definition is given in the sense of KOSZUL. A linear
connection
on a manifold M is a mapping ? ? ? : ( ) ( ) ( ) ? ? ? M M M denoted
by ? ? ? :( ) X,Y Y X
satisfying the following conditions : i) ? ? ? ? ? X X

X Y Z Y+ Z ( ) ii) Y Z (Y+Z) X X X ? ? ? ?? iii) X
f X Y
f Y ? ? ? iv) X X (f Y) (Xf)Y+f Y ? ? ? , ? ? ? X,Y,
Z (
M), F(
M) ? f The vector field ?
X Y is called the covariant derivative of Y in the direction of X
with respect to the connection If P is a tensor field of type (o, s) we define v) ? X P = XP, if s = o vi) ? X 1 2 n P Y Y Y b gb g , ,..., ? ? ? ? ? ? s 1 2 n 1 X i s i=1 X P Y , Y ,...,Y P Y ,..., Y ,...,Y ? ? ? ? Exercise 1 : Let M = R n and X, Y, ??(M) be such that Y= b i =1 ? ?x i i n ? where ? ? X i Y Xb c h ? ?x i Show that ? determines a linear connection on M.
93 Solution : Let X =a x i i ? ? , Z=c x i i ? ? with a c i i , ?F(M), i n = ,...,1 Then i) ? ? ? ? X Y Z X ( ) b c x i i i c h e j ? ? , as defined ? ? ? ? Xb Xc Xb Xc i i i i i i i x x x c h c h c h ? ? ? ? ? ? ? ? ? ? X X Y+ Z Similarly it can be shown that (Y+Z) Y Z X X X ? ? ? ? ? Again, ? ? ? f i i i i f b x f x X Y X Xb b gc h c h e j ? ? ? ? ? as ( ) ( ) f
f Y h Yh ? = Y X f ? and ? ? X Y X ( ) f f b x i i c h e j ? ? as ? ? ( ) ( ) X X f b f b x i i i c h ? ? as X(g) = (X)g+ (Xg) f f f ? ? ( ) ( ) X X f b x f b x i i i i ? ? ? ? ? ? ( )X Y+ Y X f
f Thus ? determines a linear connection on M. Let (, ,...,) x x x n 1 2 be a system of co-ordinates in a neighbourhood U of p of M. We define 3.1) ? ? ? ? ? x j i x = ? ?x k where ?F(M) Such are called the christoffel symbols or the connection co-efficients or the compo-
94 nents of the connection. Hence if X ? ? ? ? i i x , Y ? ? ? ? j j x where each ? ? i j , , ?F(M) i n ?1,..., we see that ? ? F H I K X Y = i ? ? ? ? ? ? x j j i x ? ? F H I K ? ? ? ? ? ? i x j j j x by iii) ? ? ? F H G I K J ? ?? ? ? ? ? ? ? i j i j j k x x x by iv) and 3.1) 3.2) ? ? F H G I K J X Y = ? ?? ? ? ? ? ? i k i i j k x x Exercise 2 : Let and ij be the connection co-efficients of the linear connection ? with respect to the local coordinate system (,...,) x x n 1 and (,...,) y y n 1 respectively. Show that in the intersection of the two coordinate neighbourhoods ? ? ? ? ? ? ? ? 2 x y y y x l i j k l t rs ? ? ? ? ? ? x y x y y x r i r j k t ? ? Solution : In the intersection of the two coordinates ? ? ? ? ? ? y x y x j l j l ? ? or ? ? ? ? ? ? ? ? ? ? ? ? y x y y x x y x x j s j j s l j l s ? ? ? ? ? Again, from 3.1) we see that ? ? ? ? ? ? ? ? ? ? ? ? y y x y x k y j y l j l i i ? ? ? ? ? F H G I K J from above
95 ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 2
x y y x x y x l i j l l j y l i by iv) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 2
x y y x x y x l i j l l j x y x l s i s from above ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 2
x y y x x y x y x l i i l l j s i x l s by iii) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 2 x y y x x y x y x
l i j l l j s i sl k k ? by 3.1) ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 2
x y y y x y x y x y x
l
i j k l k r i s i rs t t ? Changing s ? r l ? s k ? t ? 2
x y y y x y x y x y y x y
l i j k l k rs t r i s j k t k ? from above 2 ? ? ? ? ? ? ? ? ? ? ? ??? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? l k r s k t rs i j l i j t k k
x y x x y y y x y y x y y Since ? ?y
k n k : ? ??? R S T U V W 1 is a basis of the tangent space and hence linearly independent and the result follows immediately. 3.2 Torsion tensor field and curvature tensor field on a linear connection we define a mapping T : ? ? ? M M M b g b g b g ? ? by 3.2) T X,Y X,Y X Y Y X b g ? ? ? ? ? and another R : ? ? ? ? M M M b g b g b g ? ? ?
96 3.3)
R
X,Y Z Y Z X Z X, Y b g ? ? ? ? ? ? ? ? X Y Z
Then T is a tensor field of type (1,2)
and is called the torsion tensor field and R is a tensor field of type (1, 3), called the curvature tensor field of M. A linear connection is said to be symmetric if 3.4) T(X, Y) = 0 In such case 3.5)
X,Y X Y Y X ? ? ? ? Exercise : 1. Verify that i) T(X, Y) = −T(Y, X); ii) T X, Z T Y,
Z fX gY Z fT g ? ? ? , b g b g b g ; iii) T(fX, gY) = fg T(X, Y). 2. If ? ? ? ?
X Y X Y T X,Y b g , show that ? is a linear connection and T T ? ? 3. Show that i) T T
X,Y , Z T ,Z T Z, T X,Y , Z X Y Y X b g c h d i d i c h ? ? ? ? ? ii) R
X,X Y 0; R X,Y Z R Y,X Z b g b g b g ? ? ? ; R X,Y Z+ R Y,Z X R Z,X Y =0
b g b g b g ? iii) R T X,Y , Z R ,Z R Z, R X,Y , Z X Y Y X

b g c h d i d i c h ? ? ? ? ? iv)
R
X, fY Z R fY,Y Z R X,Y fZ f R X,Y
Z
b g b g b g b g ? ? ? Hence Show that R fX,gY hZ fgh R X,Y Z b g b g ? 4. Exercise 3 : Prove Ricci Identity a) for a 1-form w : ? ? ? ? ? ? ? F H I K ? ?
X Y Y X X,Y Z W R X,Y Z ? ? ? b g c h
97 b) for a 2-form W : ? ? ? ? ? ? ? ? ? ? X Y W Y X W X,Y W Z,P W R X,Y Z,P W Z,R X,Y
P
e j b g b g c h b g c h 5. If x x n 1 , ,??? d i is a local coordinate system and T x , x T x ,R x , y x R x i j ij k k i j k ijk h h ? ? ? ? ? ? ? ? ? ? ? ? ? ? F H G I K J ? F H G I K J ? Show that i) T and ij k ij k ji k ij k ? ? ? for a symmetric linear connection ii) R x x ijm k i jm k j im k jm t ti k im t jt k ? ? ? ? ? ? ? ? Solution : 1 i) From the definition
Y X
T(
Y,
X)
X Y [Y,X] ? ? ? ? ? Y X X Y [X,Y] ? ? ? ? ? ? ? X Y Y X [X,
Y] ? ? ? ? ? ? ? ?
T X Y (,)
Thus T
is skew-symmetric ii) fX gY Z T(
fX gY,Z) Z (fX gY) [fX gY,
Z] ? ? ? ? ? ? ? ? ? X Y Z Z f Z g Z (fX) (gY) [fX,Z] [gY,Z] ? ? ? ? ? ? ? ? ? ? ? g Y Z Zg Y [,] ( )
98 ? ? ? ? X Z Y Z f Z X [X,Z] g Z Y [Y,Z] ? ? ? ? ? ? ? ? ? ? ? ? fT X Z gT Y Z (,) (,)
Again,
using the definition, given in 3.1 and also from 1.5 we get Thus T is a bilinear mapping. 2. To prove that ? is a linear connection, we have to prove i), ii), iii), iv) of 3.1. Now
X
X (Y Z) (Y Z) T(X,Y Z) ? ? ? ? ? ? ? as defined X X Y Z T(X,Y) T(X,Z) ? ? ? ? ? ? X X Y Z, ? ? ? ?
as defined similarly, other results can be proved and hence ? is a linear connection. Now,,
X Y T(X,Y) Y X [X,Y], ? ? ? ? ? by definition X Y Y T(X,Y) X T(Y,X) [X,Y] , ? ? ? ?? ? ? as defined ? ? ? T X Y T X Y
T X
Y (,) (,) (,) by Ex 1 (i) above ? ?T X Y (,) ? T T ? ?
99 3. (iv) From the definition R(X,fY)Z X fY fY X [
X,fY] Z Z Z ? ? ? ? ? ? ? ? ? X Y Y X f[XY] (XfY) (f Z) f Z Z ? ? ? ? ? ? ? ? ? ? Y X Y Y X [X,Y] Y (Xf) Z f Z f Z f Z (Xf) Z ? ? ? ? ? ? ? ? ? ? ? ? ? ? X Y Y X [X,Y] f Z Z Z ? ? ? ? ? ? ? ? ? fR(X,Y)Z ?
by definition. 5. From the given condition i j i j j i i j x x T , , x x x x x x ? Using 3.1) we find k k ij ji k k 0 x x ? ? ? ? ? ? ? ? ? or, ? ? k k k ji ij ji k k T , x x ? ? ? ? ? ? ? ? as defined Since k :k 1, , n x ? ? ? ? ? ? ? ? ? ?? is a basis and hence linearly independent and thus i) k k k ij ij ji ? ? ? ?? If the linear connection is symmetric, then T = 0. consequently, the above equation reduces to k k ij ji ? ? ?
100 ii) From the definition, we see that i j j i i j i j m m m m ,
x
x x x x x R , x x x x x x ?
i j k k jm im k k x x x x ? as i j , 0 x x ? ? ? ? ? ? ? ? ? ? ?
k
k t k k jm jm ik im im i k t j k t x x x x x x ?
Changing the dummy indices t k,k t ? ? in the 2nd and 4th term we get k k t k k t k ijm jm jm it jm im jt k i k k j k k R x x x x x x x ? Since k :k 1, , n x ? ? ? ? ? ? ? ? ? ?? is a basis and hence linearly independent, we get from above k k k t k t k ijm jm im jm it im tj i j R x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? 3.2 Covariant Differential of
A Tensor Field of type (o, s) The covariant differential of a tensor field of type (0,

s)
is a tensor field of type (0, s + 1) and is defined as 3.6) ? ? x 1 1 2 X 1 X 1 2 S (P)(X ,X , , X) P (X ,
X ,; , X) ? ? ? ? ? ? ?? ??
Exercise : 1 Let i ? be the components of a vector field Y with respect to a local coordi- nate system 1 n ..... (x , ,x) i.e. i i Y x ? ? ? ? If i j ,? be the components of the convariant differential Y,? so that i i j i x Y , x ? ? ? ? ? ? ? then, show that i i i k j kj j , x ?? ? ? ? ? ? ? 2. Let ? be a 1 form and l x d l ? If we write i k,i k x x ? ? ? ? ? ? ? ? ? ? ? ? ? ?
101 h l x ik i k h d l x x x ?
102
103 UNIT - 4 ?
104 Theorem 1 : Every Riemannian manifold (M, g) admits a unique Riemannian Connection. Proof : To prove the existence of such a connection, let us define a mapping : (M) (M) (M) ? ? ?? ? ? denoted by X :(
X,Y) Y ? ? ? as follows 4.3) Clearly, X
X
X 2g(Y Z),W) 2g(Y,W),2g(Z,W) ? ? ? ? ? Xg(
Y Z,W) (Y Z)g(W,X) Wg(X,Y Z) g([X,Y Z],W) g(X,[W,Y Z]) ? ? ? ? ? ? ? ? ? ? g(Y Z,[W,X]) Xg(Z,W) Yg(W,X) Wg(X,Z) g([X,Y),W) g(X,[W,Y]) ? ? ? ? ? ? ?
g(Y,[W,X]) Xg(Z,W) Zg(W,X) Wg(X,Z) g([X,Z],W) ? ? ? ? ? g(X,[W,Z]) g(Z,[W,X]) ? ? 0? X X X 2g((Y Z) Y Z,W) 0, ? ? ? ? ? ? ? ? ? as g is linear Whence X X X (Y Z) Y Z ? ? ? ? ? ?
Similarly it can be shown that X Y X Y Z
Z Z, ? ? ? ? ? ? f X X Y f Y, ? ? ? X
X (fY) (Xf)Y f Y ? ? ? ?
Thus such a mapping determines a linear connection on M. Also, from (4.3) it can be shown that
X 2g(Y,Z)
Xg(Y,Z) Yg(Z,
X) Zg(X,Y) g([X,Y],Z) g(X,[Z,Y]) g(Y,[Z,X]) ? ? ? ? ? ? ?
105 X X 2Xg(Y,Z) 2g(Y,Z) 2g(Y, Z) 0 ? ? ? ? ? or, X
X X g(Y,Z) g(Y,Z) g(Y, Z) 0 ? ? ? ? ? ?
by v) of ?.3.1 or, X (g)(Y,Z) 0, ? ? X,Y,Z ?
Thus such a linear connection admits a metric connection. Further, it can be shown that X Y Y X [X,Y] 0 ? ? ? ? ? Hence such a metric connection admits a Riemannian connection To prove the uniqueness, let ? be another such connection. Then we must have
X
X
Xg(
Y,Z) g(Y,Z)
g(Y, Z) 0 ? ? ? ? ? and
X Y Y X [X,Y] 0 ? ? ? ? ? X X Xg(Y,Z) g(Y,Z) g(Y, Z) 0 ? ? ? ? ? and X Y Y X [X,Y] 0 ? ? ? ? ? Subtracting, X X
X X g(Y Y,Z) g(Y, Z Z) 0 X,Y,
Z ? ?? ? ? ?? ? ?
and X
X Y Y
Y Y X
X ? ? ? ? ? ? ? where form, we get X X Y Y 0 ? ?? ? X X
Y
Y ?? ? ?
Thus uniquences is established. This completes the proof
Exercise : 1 In terms of a local coordinate system 1 2 n .... {x ,x , ,x } in a neighbourhood U of p of a Riemannian Manifold (M, g) show that i) the components i jk ? defined in UNIT 3 is symmetric and ii) the Riemannian metric is covariantly constant. 2.
Let ? be a metric connection of a Riemannian manifold (M, g) and ? ? be another linear connecting given by X X Y Y T(X,Y) ? ? ? ? ? where T is the torsion tensor of M. Show that the following condition are equivalent i) g 0 ? ? ? and ii) g(T(X,Y),Z)

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g(Y,T(X,Y)) 0 ? ? 3. In terms of a local coordinate system 1 n .... {x , ,x } the components i jk ? of the Ri- emannian connection are given by
106
107
108 4.6)
g(
X,Y)Z,U) g(R(X,Y)U,Z) ? ? 4.7) g(R(X,Y)Z,U) g(R(Z,U)X,Y) ? ? Proof : Using 3.3), 3.5) one gets R(
X,Y)Z R(Y,Z)X R(Z,
X)Y [
X,[Y,Z]] [Y,[Z,X]] [Z,[X,Y]] 0 ? ? ? ? ? ? ?
by
Jacobi identity 4.5)
is Left to the reader To prove 4.6), one gets from 4.1) X (
g)(
Z,U) 0, X,
Z,U ? ? ? X X)Xg(
Z,U) g(Z,U) g(Z, U) ? ? ? ? ? ? or, Y Y X X (Xg(Z,U)) {g(Z,U) g(Z, U)} ? ? ? ? ? ?
or, X X Y(Xg(Z,U)) Yg(Z,
U)
Yg(Z, U) ? ? ? ? using)? on the right side we get
Y X
X Y Y Y
Y X
Y(Xg(Z,U) g(,Z,U) g(Z, U)
g(Z,
U) g(Z,
U) ? ? ? ? ? ? ? ? ? ? ? ? ?
Thus,
we find X(Yg(Z,U)) Y(Xg(Z,U)) [

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X,Y]g(Z,U) ? ? ? ? ? ? Z U X Y Y X X Y Y X [X,Y], [X,Y]

g Z Z U g
Z, U U ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? g(R(X,Y)Z,U) g(Z,R(X,Y)U) ? ?
Using the definition of [X, Y] f, on the left hand side, one finds g(
R(
X,Y)Z,U) g(Z,R(X,Y)U) 0 ? ? Again,
R(
X,Y)Z R(Y,Z)X R(Z,X)Y 0 ? ? ? g(R(X,Y)Z) g(R(Y,Z)X,U) g(R(Z,X)Y,
U) 0.....) ? ? ? ?
109 Similarly, we can write g(R(U,Z)X,Y) g(R(Z,X)U,Y) g(R(X,U)Z,Y) 0 ....) ? ? ? ? g(R(Y,X)U,Z) g(R(X,U)Y,Z) g(R(U,Y)X,Z) 0 ....) ? ? ? ?
g(R(Z,U)Y,X) g(R(U,Y)Z,X) g(R(Y,Z)U,X) 0 ....) ? ? ? ? Adding),),),),) ? ? ? ? ? and using 4.6) we get g(R(X,Y)Z,U) g(R(U,Z)X,Y) g(R(Y,X)U,
Z)
g(R(Z,U)Y,
X) 0 ? ? ? ?
Using Exercise 3(ii) 3.2 in the second and in the third term of the above equation. or,
g(

R(
X,Y)Z,U) g(R(Z,U)X,Y) g(R(X,Y)U,Z) g(R(Z,U)Y,X) 0 ? ? ? ? After a few steps one gets 2g(R(X,Y)Z,U) 2g(R(Z,U)X,Y) ? i.e.
g(R(X,Y)Z,U) g(R(Z,U)X,
Y) ?
Exercise 4. In terms of a local coordinate system 1 n .... {x , ,x } in a neighbourhood U of p of (M, g) show that i) m m m ijk jki kij R R R 0 ? ? ? ii) h h h ijk,m jmk,i mik,j R R R 0 ? ? ? iii) h h hm hk ijk jim R g R g ? ? iv) h h hm hj ijk kmi R g R g ? ? Solution : i) From ii) of Exercise 5 in 3.2 and also using the result m m jk kj ? ? ? the result follows immediately ii) Left to the reader ii) using ii) of Exercise 5 in 3.2, on finds h h h t h t h hm ijk jk ik jk ti ik tj i i R g x x ? h h h h hm hm hm hm jk jk ik ik i i j j g g g g x x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? t h t h hm hm jk ti ik tj g g ?? ? ? ? ?
110 Using Exercise 3 of 4.1 we get mj jk mk mh h h hm ijk jk i j k m i g g g g 1 R g 2 x x x x x ? mt ti mi mk ik hm mi h t ik jk j k i m j i t m g
g g g g g g 1 1 2 2 x x x x x x x
x ? mj tj mt t ik j t m g g g 1 2 x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? Similarly, one can write h hk ijm R g Thus, h h hm hk ijk ijm R g R g ? hj mj hm hi mi hm h h jk ik i m h j m h
g g g g g g 1 1 2 2 x x x x x x ?
hj jk kh hi ik hk h h jm im i k h j k h
g g g g g g 1 1 2 2 x x x x x x ?
h t h t h t h t th th th th jk im ik jm jm ik im jk g g g g ? ?? ? ? ? ? ? ? ? ? ? ? ? ? Thus, h h hm hk ijk ijm R g R g 0 ? ? or h h hm hk ijk ijm R g R g ? ? iv) From Exercise iii) above we write mh mi hi h h h h h hm hj ijk kmi jk jk jk i h m g g g 1 1 1 R g R g 2 2 2 x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? jh jk hk h h h mi mi mi k h j g g g 1 1 1 2 2 2 x x x ? ? ? ? ? ? ? ? ? ? ? ? ? hj jk mh hi mi hk h h jk mi i m h k j h
g g g g g g 1 1 2 2 x x x x x x ?
h t h t th th jk im mi jk 1 1 g g 0 2 2 ? ? ? ? ? ? ? ? ? ? h h hm hj ijk kmi R g R g ? ?
111
112 3.4.2 Riemann Curvature tensor field : The Riemann Curvature tensor field of 1st kind of M is a tensor field of degree (0, 4), denoted also by R R: (M) (M) (M) (M) F(M) ? ?? ?? ?? ? and defined by 4.10) R(
X,Y,Z,W) g(R(X,Y)Z,W),X,Y,Z,W ? in (M) ? Exercise : 1 Verify that i) R(X, Y, Z, W) = − R(Y, X, Z, W) ii) R(X, Y, Z, W) = − R(X, Y, W, Z) iii) R(X, Y,
Z, W) = − R(Z, W, X, Y) iv) R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0 v) U Z W (R)(X,Y,Z,W) (R)(X,Y,W,U) (R)(X,Y,U,Z) 0 ? ? ? ? ? ? ? 2.
If h ijk R and hm g are the components of the curvature tensor and the metric tensor with respect to a local coordinate system 1 2 n ...... x ,x , ,x then the components ijkm R of the Rieman Curvature tensor are given by h ijkm hm ijk R R g ? where ijkm i j k m R , , , x x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 3. A vector field z on (M, g) is called a gradient vector field if 4.11) g(Z,Y) d (Y) Yf, F(M) ? ? ? f f for every vector field Y and M. Show that for such Z X Y g(Z,Y) g(Z,X) ? ? ? for every vector field X on M. Solution : From 4.1) we see that X (g)(
Y,Z) 0 ? ? for all X, Y, Z in (M) ? or X
X Xg(Y,Z) g(Y,Z) g(Y, Z) ? ? ? ? Using 4.11), one finds X X
g(Z,Y)
X(Y) g(
Y,Z) ? ? ? ? f similarly
Y Y g(Z,X) Y(X) g(X,
Z) ? ? ? ? f X Y Y X g(Z,Y) g(Z,X)
X(Y) Y(X) g(X,Z) g(Y,Z) ? ? ? ? ? ? ? ? ? ? f f
113 or, X Y X Y g(Z,Y) g(Z,X) [X,Y] g(Y X,Z) ? ? ? ? ? ? ? ? f [X,Y] g[X,Y],
Z) ? ?
f by 4.2) [X,Y] [X,Y] ? ?
f
f
by 4.11) = 0
Thus X Y g(

Z,Y) g(Z,X) ? ? ? 3.4.3

Einstein Manifold : Let 1 2 n ..... {e ,e , e } be an orthonormal basis of p T (M) Then the Ricci tensor field, de- noted by S, is the covariant tensor field of degree 2 and is defined by ? ? ? ? n p p i P P i P P i 1 S(X ,Y) R (e ,X ,Y ,(e) ? ? ? We write it as 4.12) n i i i 1 S(X,Y) R(e ,X,Y,e) ? ? ? Such a tensor field S(X, Y) is also called the Ricci Curvature of M. If there is a constant ? such that 4.13) S(X,Y) g(X,Y) ? ? then M is called on Einstein Manifold. The function r on M, defined by ? ? ? ? ? ? n i i P P i 1 r(p) S e , e ? ? ? is called the scalar curvature of M. We write it as 4.14) n i i i 1 r S(e ,e) ? ? ? Exercise : 1. Show that the Ricci tensor field is symmetric. At any p M, ? we denoted by ? a plane section i.e., a two dimensional subspace of p T (M) . The sectional curvature of ? denoted by K(?) with orthonormal basis X, Y is defined as 4.15) K(?)=g(R(X, Y) Y, X) = R(X, Y, Y, X) If K(?) is constant for all plane section and for all points of p M,

114 Then (M, g) is called a manifold of constant curvature. For such a manifold 4.16)

R(

X,Y)Z k{g(Y,Z)X g(X,Z)Y} ? ?

where k( )? say Example : Euclidean space is of Constant Curvature

Exercise : 1, Show that a Riemannian manifold

of constant curvature is an Einstein Manifold. 2.

If M is a 3-dimensional Einstein Manifold, then, it is a manifold of constant curvature

Solution : Let 1 2 3 {X ,X ,X } be an orthonormal basis of p T (M) Then, the sectional curvature with orthonormal basis 1 2 X ,X denoted by 12 K( ) ? is given by 12 1 2 2 1 K( )

R(

X ,X ,X ,X) ? ? 2 1 1 2 R(X ,X ,X ,X) ? 21 K( ) ? ? Thus, ij ji K( ) K( ),i j ? ? ? ? Again from 4.12) 3 1 2 i 1 2 i i 1 S(

X ,X) R(

X ,X ,X ,X) ? ? ? 1 1 2 1 2 1 2 1 3 1 2 3 R(X ,X ,X ,X) R(X ,X ,X ,X)

R(X ,

X ,

X ,X) ? ? ? 21 31 0

K( ) K( ) ? ? ? ? ? 12 13 K( ) K( ) ? ? ? ? 2 2 21 23 S(X ,X) K( ) K( ) ? ? ? ? and 3 3 31 32 S(X ,X) K( )

K( ) ? ? ? ? As it is a 3-dimensional Einstein manifold, so from 4.13) 1 1 1 1

S(X ,X) g(X ,X) ? ? ? ? 1 2 1 2 S(X ,X) g(X ,X) 0 ? ? ?

115

116

using

the above result

in 4.8)

we get

X Y 1 Y Y {T(X,Y) (X)Y g(X,Y)p (Y)X g(Y,X)p} 2 ? ? ? ? ? ? ? ? ? ? ? Again using 4.17), one gets X Y Y Y (Y)X g(X,Y)

p ? ? ? ? ? ?

Exercise 1. If ?

and ? correspond to a semi-symmetric connection and the Levi-Civita connection respectively, then for any 1-form ? ? ? X X ( ) Y (X) (Y) (p)g(X,Y), ? ? ? ? ? ? ? ? ? ? where g(X,p) (X) ? ? ? 2. Let ? be the Levi-Civita Connection and ? be another linear connection such that X X Y Y (X)Y ? ? ? ?? where is a 1-form. Show that ? is a semi-symmetric connection for which X g 2 (X)g ? ? ? Hints : 1. Note that X X ( )Y X (Y) (Y) ? ? ? ? ? ? ? Use Theorem 1 in the second term on the right hand side, one gets the desired result. 2. Note

that

X Y T(

X,

Y) Y X [X,Y] ? ? ? ? ? X Y Y (X)Y X (Y)X [X,

Y] ? ? ?? ? ? ? ? ? T(X,Y) (Y)

X (X)

Y, ? ? ? ??

on using the hypothesis (Y)X (

X)Y, ? ? ? as T 0.? Again, X X X (g)(Y,Z) Xg(Y,Z) g(Y,Z) g(Y, ,Z) ? ? ? ? ? ? X X Xg(Y,Z) g(Y (X)Y,Z) g(Y,

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Z (X)Z) ? ? ? ?? ? ? ?? X (g)(Y,Z) 2 (X)g(Y,Z), ? ? ? ?
on using the hypothesis X g 2 (X)g, ? ? ? ? as g 0. ? ?
117
118 Or, 1 C(
X,Y)Z R(X,Y)Z {g(Y,Z)AX g(X,Z)AY S(Y,Z)X S(X,Z)Y} n 2 ? ? ? ? ? ? r {g(Y,Z)X g(X,Z)
Y} (
n 1)(n 2) ? ? ? ? Exercise : 1 If an n(n 3)? – dimensional Einstein Manifold is conformally flat than 2. If we write ijkl i j k l R R , , , x x x x ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ijkl i j k l C g C , , x x x x ? ij i j R S , x x ? ? ? ? ? ? ? ? ? ? ?
show that ? ? ijkl ijkl jk il ik jl jk il ik jl 1
C R g R g R R g R g n 2 ? ? ? ? ? ? ? ? ? jk il ik jl r g g g g (
n 1)(n 2) ? ? ? ? Hints : 1 Using 4.13) in 4.14, one gets r n ? ? Alsing above result, 4.13), one gets from 4.21) r Ax x n ? Using 4.20) in 4.22) and also the result deduced above, one gets the desired result after a few steps. 2. Using goldberg's result, one gets from the hypothesis ijkl i j k l C g C , , x x x x ? the desired result.
119 4.5 Conformally Symmetric Riemannian Manifold :
A Riemannian manifold (M, g) is said to be conformally symmetric if 4.23)
C 0 ? ? Where C is the Weyl Conformal Curvature tensor Theorem 1 : A conformally symmetric manifold is of constant scalar curvature if
Z W (S)(
Y,W) (S)(Y,Z) ? ? ? for all Y, Z, WW Proof : From 4.22) we see
that 1 C(
X,Y,Z,W) R(X,Y,Z,W) {g(Y,Z)g(AX,W) g(X,Z)g(AY,W) n 2 ? ? ? ? ? r S(Y,Z)g(X,W) S(X,Z)g(Y,W)} {g(Y,Z)g(X,W) g(X,Z)g(Y,
W)} (
n 1)(
n 2) ? ? ? ? ? ?
Taking co-variant derivative on both sides and using (4.23), we get U U U 1 (R)(
X,
Y,Z,W) {g(Y,Z)(S)g(X,W) g(X,Z)(S)g(Y,W) n 2 ? ? ? ? ? ? ? U U (S)(Y,Z)g(X,W) (S)(X,Z)g(Y,W)} ? ? ? ? U r {g(Y,Z)g(X,W) g(X,Z)g(Y,
W)} (n 1)(
n 2) ? ? ? ? ?
It is known from Exercise 1(v) of ? 4.2 that U
U W (R)(X,Y,Z,W) (R)(X,Y,W,U) (R)(X,Y,U,Z) 0 ? ? ? ? ? ?
Using the result deduced above, and also the hypothesis one gets
U
Z r{
g(Y,Z)g(X,W) g(X,Z)g(Y,W)} r{g(Y,W)g(X,U) g(X,W)g(Y,U)} ? ? ? ? ? W r{g(Y,U)g(X,Z) g(X,U)g(Y,
Z)} 0 ?? ? ?
Let i ...... {e :i 1, , n} ? be an orthonormal basis vectors.
120 Taking the sum for 1 i n ? ? for i X U e , ? ? we get on using the result ei i z rg(e ,z) r ? ? ? that w z z z w
w g(Y,Z) r g(Y,W) r ng(Y,W) r g(Y,W) r g(Y,Z) r ng(Y,Z) r 0 ? ? ? ? ? ? ? ? ? ? ? ? or w z g(Y,Z) r g(Y,W)
r 0 ? ? ? ? Finally taking the sum for 1 i n ? ? for i Y Z e , ? ? we get w r 0, n 1. ? ? ? Thus the manifold is of constant curvature. Definition : A linear transformation A is
symmetric or skew symmetric according as 4.24) g(AX,Y) g(X,AY) or g(AX,Y) g(X,AY) ? ? ? ? ? ? ? ?
Exercise : 1. Show that for a symmetric linear transformation A and a skew-symmetric linear transformation R, the new linear transformation T defined by, T A. R R. ? ? A is skew - symmetric. Theorem 2 : For a conformally flat n(n 3)? - dimensional Riemannian manifold, the curvature tensor R is of the form 1 r R(X,Y) (AX Y X AY) X Y n 2 (n 1)(n 2) ? ? ? ? ? ? ? ? ? where X Y? denotes the skew - symmetric endomarphism of the tangent space at every point defined by (
X Y)Z g(Y,Z)X g(X,Z)Y ? ? ?
121 Proof : Using the hypothesis, we find that (AX Y)Z (X AY) g(Y,Z)AX g(X,Z)AY S(Y,Z)X S(X,Z)Y ? ? ? ? ? ? ?
As the manifold is conformally flat,
we get on using the above result and the hypothesis, 1
r

R(X,Y)Z {(AX Y)Z (X AY)Z} {X Y)Z} n 2 (n 1)(n 2) ? ? ? ? ? ? ? ? ? i.e. 1 r R(X,Y) (AX Y X AY) X Y
n 2 (n 1)(n 2) ? ? ? ? ? ? ? ? ? Theorem 3 : If in a conformally flat manifold, for a symmetric linear transformation A, R(X, Y)A = A. R (X, Y) then 2 rA A X X 0 n 1 ? ? ? ? ? ? ? ? ? ? Proof : Note that R(X, Y) = – R(Y, X) As A is symmetric, so by Exercise 1 of this article A. R(X, Y) = R(X, Y). A is skew - symmetric. Thus R(Z, W)A is a skew symmetric linear transformation and from 4.24) we can write
g((R(Z, W)
A)X, X) = – g(X, (R (Z, W) A) X) or g(R(Z, W)A)X, X) = – g(X, R (
Z, W) AX) = – g(R(Z, W) AX, X), as g is symmetric. ? g(R(Z, W)AX, X) = 0 Using 4.7) one gets g(R (AX, X)Z, W) = 0 Whence R(AX, X)Z = 0 i.e., R(AX, X) = 0 Again (AX AX)Z 0 ? ? i.e., AX AX 0 ? ? for every Z. Using Theorem 2, one gets 2 1 r R(X,AX) (AX AX X A X) X AX n 2 (n 1)(n 2) ? ? ? ? ? ? ? ? ?
122 AS R(AX, X) = –R(X, AX) and R(AX, X) = 0, we get from above, 2 r X A X X AX 0 n 1 ? ? ? ? ? Note that X Y? is skew - symmetric and thus 2 r A X X AX X 0 n 1 ? ? ? ? ? 2 r A X X 0 n 1 ? ? ? ? ? ? ? ? ? ? ? Definition : A curve x(t),a t b ? ? ? ? is called a geodesic on M with a linear connection ? if 4.25) X X 0 ? ? Where X is the vector tangent to the integral curve ? at x(t). Note that the integral curves of a left invariant vector fields are geodesic. 4.7 Biinvariant Riemannian metric on a Lie group : A Riemannian metric g on a Lie group

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is said to be biinvariant if it is both left and right			

invariants. Exercise 1 : If g is a left invariant convariant tensor field of order 2 on G and X, Y are left invariant vector fields on G, show that g(X, Y) is a constant function. Theoxem 1 : If G is a Lie group admitting a biinvariant Riemannian metric g,
then 4.26)
g([
X,
Y], Z) = g(
X, [Y, Z]) 4.27) 1 R(X,Y)Z [[X,Y],Z] 4 ? ? 4.28) 1 g(R(X,Y)Z,
W) g([X,Y],[Z,
W]) 4 ? ? Proof : Since X, Y
are left invariant vector fields, X + Y is
also so and hence from 4.25) X Y X Y 0 ? ? ? ?
123 Using 4.25, we find from above i) X Y Y X 0 ? ? ? ?
since M admits a unique Riemannian connection, we must have
X
Y Y X [X,Y] 0 ? ? ? ? ? ii) or X 1 Y [X,Y] 2 ? ? from i) Now for a Riemannian Manifold Y (g)(X,Z) 0 ? ? or, Y Y Yg(X,Z) g(X,Y)
g(X, Z) 0 ? ? ? ? ?
Using Exercise 1
of this article and Exercise 2 of ? 1.4 we see that Y. g(X, Z) = 0 Thus from ii) we find that 1 1
g([
Y,X]Z) g(
X,[Y,Z]) 0 2 2 ? ? ? or, g([X,Y],Z) g(X,[Y,Z]) ? or,
g([X,Y],Z) g(X,[Y,Z]) ?
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Y X [X,Y] R(X,Y)Z Z Z ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 1 X,[Y,Z] Y,[X,Z] [
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Z 4 2 ? ? ? ? ? 1 [X,Y],Z 4 ? ?
Again ? ? ? ? 1 R(X,Y)Z,W) g [X,Y],Z , W 4 ? ? by 4.27) ? ? ? ? 1 g X,Y],Z , [
Z,
W] 4 ? ? by 4.26)
This completes the proof.
124
Theorem 2 : If G is a Lie group admitting a biinvariant Riemannian metric g and ? is a plane section in p T (M) where ? is determined by orthonormal left invariant vector fields X, Y at p on G, then the sectional curvature at p is zero if and only if [X, Y] = 0. Proof : From 4.15) K( ) g(
R(X,Y,)Y,X) ? ? 1 g([X,Y],[Y,X]) 4 ? ? by 4.28) 1 g([X,Y],[X,Y]) 4 ?
The result follows immediately as g is nonsingular. Theorem 3 : If G is a Lie group admitting a biinvariant Riemannian metric g, then for all left invariant vector fields, X, Y, Z, W, P. Proof : From Jacobi's identity [W, [
P, Z]] + [P, [Z, W]] + [Z, [W, P]] = 0 Taking P = [
X, Y], we get [W, [[X, Y], Z] + [[X, Y], [Z, W]] + [Z, [W, [X, Y]]] = 0 or [W, [[
X, Y], Z]] – [[X, Y], [W, Z]] = [[W, [X, Y]], Z] = [– [X, [Y, W]] – [Y, [W, X]], Z] by Jacobi Identity i) [W, [[X, Y], Z]] – [[X, Y], [W, Z]] = [[X, [W, Y]], Z] + [[W, X], Y],
Z] Again from the definition W W W W (R)(P,Z,X,Y) R(P,Z,X,Y) R(P,Z,X,Y) R(P, Z,X,Y) ? ? ? ? ? ? ? ? W W R(P,Z, X,Y) R(P,Z,X, Y) ? ? ? ? W W W 0 R(X,Y,Z, P) R(X,Y, Z,P) R(X,Y,Z,P) ? ? ? ? ? ? ? W P(X, Y,Z,
P) ? ?
125
REFERENCES 1. W. B. Boothby : An Introduction to differentiable Manifold and Riemannian Geometry. Using 4.28), one gets ? ? ? ? ? ? ? ? W 1 1 (R)(P,
Z,X,Y) g [X,Y], Z,[W,P g [W,Z],P ,[X,Y] 8 8 ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 g [W,X]Y [Z,P] g X,[W,Y],[Z,P 8 8 ? ? Using 4.26) successively we get ? ? ? ? ? ? ? ? ? 1 g [X,Y],Z ,W ,P g [X,Y],[W,Z] ,P 8 ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?? g [W,
X],Z ,P g [X,[W,Y] ,P] ? ? ? ? ? ? ? ? ? ? ? 1 1 g W, [X,Y ,Z ,P g [X,Y][W,Z] ,P 8 8 ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 1 g [X,[W,Y] ,Z],
P g [[W,X],Y],Z ,
P 8 8 ? ? = 0
by i) for all left invariant vector fields X, Z, Y, W, P. This completes the proof.
126 NOTES
127 NOTES
128 NOTES

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f M N p f p ? ( ) T p (M) T f(p) (N) { 29

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x,1,0,0],[1,x,1,1],[0,1,x,1],[0,1,1,x]]. ⌈| | | |[x1001x1101x1011x]| | | || [x 1 0 0 1 x 1 1 0 1 x 1 0 1 1 x].

W https://www.chegg.com/homework-help/questions-and-answers/determine-wheter-following-series-conve ...

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k3k3k3k3k3k]||| LHS = Ak+1=Ak.A =⌊||⌈3k−13k−13k−13k−13k−13k−13k−13k−13k−1]||⌉|⌊||⌈111111 111]||⌉ =⌊||⌈3

W https://www.toppr.com/ask/question/if-abeginbmatrix-1-1-1-1-1-1-1/

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A ? a a a a a a a a a

a0/1, a1a0 + 1/a1, a2(a1a0 + 1) + a0/a2a1 + 1,

W https://en.wikipedia.org/wiki/Continued_fraction

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A vector field X on a Lie group G is left invariant if

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Proof : Let X be a left invariant vector field on a Lie group G. Then for every f F G ? ( ) , we have from (2.6) ( ) * ( ) L X f X f

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on a Lie group G is said to be left invariant if 2.13)

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on a Lie group G is said to be right invariant if 2.15)

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X,Y]g(Z,U) ? ? ? ? ? ? Z U X Y Y X X Y Y X [X,Y], [X,Y]

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is said to be biinvariant if it is both left and right

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