NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS
POST GRADUATE

PG (MT) 06
GROUPS A & B

General Topology
•
Functional Analysis

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in a Subjects as introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amount to an involvement in invisible teaching. For whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Vice-Chancellor

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NETAJI SUBHAS OPEN UNIVERSITY

PG (MT)–06 : Groups A & B

General Topology and Functional Analysis

Group A

Group B

Functional Analysis

P.G.
(Mathematics)
SYLLABUS

General Topology VI A (Marks : 50)
PGMT-II

Topological spaces, Examples, Base for a Topology, Sub-subbase. Neighbourhood system of a point. Neighbourhood base, Limit point of a set, Closed sets. Closure of a set, Kuratowski closure operator. Interior and boundary of a set, Sub-space Topology, First and Second Countable spaces. Continuous function over a Topological space. Homeomorphism : Nets, Filters, Their convergence, Product space, Projection function, Open and closed function, Quotient spaces.

Separation axioms T_0, T_1, T_2, T_3, T_4 in Topological spaces, Product of T_2-spaces, Regular spaces, Normal spaces, Completely regular spaces. Tychonoff spaces, Urysohns Lemma in Normal spaces. Tietze extension Theorem. Embedding in cube, Embedding Lemma. Urysohn's metrization Lemma.

Open cover, sub-cover, Compactness, Countable open cover, Lindeloff space, Compact sets, Finite Intersection property. Tychonoff Theorem on product of compact spaces, Continuous image of a compact spaces, Locally compact spaces, One point compactification.

Connected spaces, Separated sets, Disconnection of a space, Union of connected sets, Closure of a connected set, Connected sets of reals, Continuous image of a connected space, Topological product of connected spaces, components, Totally disconnected spaces, Locally connected spaces.

Uniformity in a set, Base, Sub-base of a Uniformity, Uniform space, Uniform Topology, T_2-property of a Uniformity, Interior and closure of a set in terms of uniformity, Uniformly continuous function, Product Uniformity.

P.G.
(Mathematics)
SYLLABUS

General Topology VI A (Marks : 50)
PGMT-II

Topological spaces, Examples, Base for a Topology, Sub-base of [illegible] Neighbourhood base, Limit point of a set, [illegible] Closure of a set, Kuratowski closure operator, Interior and [illegible] Topology, First and Second Countable spaces, [illegible] Continuous function on a Topological Space, Homeomorphism, [illegible] Product Space, Projection function, Open and closed functions, [illegible] spaces.

Separation axioms: T_0, T_1, T_2 in Topological spaces, [illegible] Regular spaces, Normal spaces, Completely regular spaces, [illegible] Urysohn Lemma in Normal spaces, Tietze extension [illegible] Embedding theorem, Embedding Lemma, Urysohn's metrization theorem.

Compactness, [illegible] Countable compactness, [illegible] intersection property, Tychonoff Theorem, [illegible] Continuous image of a compact spaces, Locally compact spaces, One point compactification.

Connected spaces, Separated sets, Disconnection of a space, Union of connected sets, Closure of a connected set, Connectedness of [illegible] Continuous image of a connected space, Topological product of connected spaces, Components, Locally connected spaces, Locally connected [illegible]

Uniformity in a set, Base, Sub-base of a uniformity, Uniform space, Uniform topology, Property of a uniformity, Interior and closure of a set in terms of uniformities, Uniformly continuous function, Product uniformities.

Unit 1 ❑ Topological Spaces

(Topological spaces, Examples, Comparison of Topologies, Base for a Topology, its properties, Sub-base of a Topology, neighbourhood of a point, Neighbourhood system at a point, neighbourhood base, limit point if a set, closed sets, derived sets, closure of a set, Kuratowski closure operator, Interior, boundary of a set, Sub-space Topology, first and second Countable spaces).

§1. Let X be a non-empty set. Then the family ρ(X) of all subsets of X, including the empty set is called the **Power set of X.**

For example if X = {a, b, c}, then its power set ρ(X) = {ϕ, X, {a}, {b}, {c}, {a, b}, {b, c} {c, a}}, contains 8 members. In General, if X consists of n (distinct) members, then its Power set ρ(X) consists of 2^n member. This result is due to cantor.

Definition 1.1. A sub-family τ of ρ(X) is called a Topology on X if (T.1) ϕ, X ∈ τ ;

(T.2) Union of any number of members of τ is a member of τ ; and (T.3) Intersection of any two members of τ is a member of τ.

If τ is a Topology on X, the pair (X, τ) is called a Topological space.

Explanation : For any non-empty set X, the power ρ(X) satisfies all conditions (T.1) – (T.3) and forms a Topology on X ; This Topology is called the **Discrete Topology on X.** Similarly, the sub-family of ρ(X) consisting of ϕ and X only also forms a Topology on X called **Indiscrete Topology on X.** But any sub-family of ρ(X) does not form a Topology on X. For example, the sub-family comprising of empty set only does not form a Topology on X.

Example 1.1. Let X = {a, b, c} and τ = {ϕ, X, {a}}. Then τ is a Topology on X.

Definition 1.2. If (X, τ) is a Topological space, members of τ are called open sets in X.

Explanation : In a Topological space (X, τ) qualification for a subset to be on open set is exclusively its membership in τ. The more numerous is

τ in members the more open sets are there in X. Thus in discrete topology every subset, including singletons, is an open set ; but open sets are scarce in Indiscrete Topology in X.

§2. Comparison of Topologies :

If τ_1 and τ_2 are two topologies on X, then τ_1 is said to be weaker than τ_2 (or, τ_2 stronger than τ_1 or, τ_1 is courser than τ_2 or, τ_2 is finer than τ_1) **if every member of τ_1 is a member of τ_2.**

or, in symbol $\tau_1 \subset \tau_2$.

So Indiscrete Topology is the weakest and Discrete topology the strongest Topology on X, and **any other Topology τ on X** satisfies **Indiscrete Topology $\subseteq \tau \subset$ Discrete Topology.**

Example 2.1. Cofinite Topology on X.

Let X be an infinite set and τ consists of empty set ϕ and those subsets A of X such that X\A is a finite set. Then (T.1) axiom is satisfied. Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of members of τ, and $A = \bigcup_{\alpha \in \Delta} A_\alpha$; Then $X \backslash A = X \backslash \bigcup_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (X \backslash A_\alpha) \subset (X \backslash A_\alpha)$ for every α, and r h s is a finite set; So X\A is a finite set. Thus A $\in \tau$. Thus (T.2) condition is satisfied. Finally, Let $A_1, A_2 \in \tau$, then $X/(A_1 \cap A_2) = (X \backslash A_1) \cup (X \backslash A_2)$ = a Union of two finite sets = a finite set. Hence $A_1 \cap A_2$ is a member of τ. Hence τ is a Topology on X. This Topology is called Co-finite Topology on X.

Theorem 2.1. Intersection of any number of Topologies on X is a Topology on X.

Proof : Let $\{\tau_\alpha\}_{\alpha \in \Delta}$ be a collection of Topologies on X and put $\tau = \bigcap_{\alpha \in \Delta} \tau_\alpha$. Clearly (T.1) axion is O.K. for τ; and so is the case with (T.2). For (T.3) take U and V as two members of τ, and then they are members of each τ_α for $\alpha \in \Delta$. Since τ_α is a Topology on X, we have $U \cap V$ belongs to τ_α for every member α of Δ; and hence $U \cap V$ is a member of $\tau = \bigcap_{\alpha \in \Delta} \tau_\alpha$. So, τ is a Topology on X.

Remark : Theorem 2.1 is not true if word "intersection" is replaced by "Union". Following example supports this contention.

Example 2.2. : Take X = (a, b, c) and $\tau_1 = \{\phi, X, (a), (a, b)\}$ and $\tau_2 = \{\phi, X, (c), (b, c)\}$. Then τ_1 and τ_2 are topologies on X such that $\tau_1 \cup \tau_2 = \{\phi, X, (a), (c), (a, b), (b, c)\}$. As $(a) \cup (c) = (a, c)$ is not a member of $\tau_1 \cup \tau_2$ we find $\tau_1 \cup \tau_2$ is not a Topology on X i.e. **Union of Two topologies may not be a Topology.** To solve a problem in a Topological space (X, τ) sometimes it suffices to know and use a part of τ called a base for Topology τ that we presently define.

Definition 2.1. : A family $\wp$ of member of τ in a Topological space (X, τ) is called **a base of the Topology** τ if and only if **every member of** τ **is a Union of some members of** $\wp$.

Members of $\wp$ are called basic open sets.

For example, the family of all singletons is a base of the discrete Topology on X.

Theorem 2.2. : In a Topological space (X, τ) a sub-family $\wp$ of τ is a base of τ if and only if for any open set G in τ, and for any member $u \in G$, there is a member $B \in \wp$ to satisfy $u \in B \subset G$.

Proof : Condition is necessary : Suppose $\wp$ is a base for τ and G is a member of τ. So G is a Union of some members of $\wp$, say, G = $G = \bigcup_{\alpha \in \Delta} \{B_\alpha : B_\alpha \in \wp\}$. If $u \in G$, there is a member, say B_{α_0} for some $\alpha_0 \in \Delta$ such that $u \in B_{\alpha_0} \in \wp$; clearly $u \in B_{\alpha_0} \subset G$.

Condition is sufficient : Suppose condition as stipulated holds. Take a member G in τ. Then for every $p \in G$, we find a member, say, B_p from $\wp$ such that $p \in B_p \subset G$. So we can write $G \subset \bigcup_{p \in G} B_p$ and, of course, converse is true i.e. $\bigcup_{p \in G} B_p \subset G$. Combining them we have $G = \bigcup_{p \in G} B_p$ = a Union of members of $\wp$. Thus $\wp$ is a basis of τ.

Remark : A Given Topology on X may have Different bases. Also there is a caution. Not any family of subsets of X is a base for a Topology. For example, take X = (a, b, c) and then the family φ consisting of (a, b), (b, c), X and empty set ϕ **fails to form a base for a Topology on X.** Because if φ is a base for some topology on X, Unions of members of φ shall constitute a Topology on X, and this is not the case here. Intersection property is failing here $(\mathbf{a}, \mathbf{b}) \cap (\mathbf{b}, \mathbf{c}) \notin \varphi$).

We have following Theorem in this connection.

Theorem 2.3. A family $\wp$ of subsets of X forms a base for some topology on X if and only if

(i) $\phi \in \wp$ (ii) X is a union of members of $\wp$ and (iii) Given any two members B_1 and B_2 in $\wp$, and $x \in (B_1 \cap B_2)$, there is a member $B_3 \in \wp$ such that $x \in B_3 \subset (B_1 \cap B_2)$.

Proof : Necessary part follows from the Definition 2.1 and Theorem 2.2. For sufficient part suppose the $\wp$ satisfies the stated conditions, and let τ be the family of all posible unions (finite or infinite) of members of $\wp$. We check that τ is a Topology on X. For that purpose we atonce see that (T.1) is O.K.; and (T.2) is also clear in favour of τ. For (T.3) take two members C and D in τ; If $x \in (C \cap D)$, since C is a union of members of $\wp$, we find $U \in \wp$ such that

$$x \in U \subset C. \qquad \ldots\ldots (1)$$

Similarly find a member $V \in \wp$ such that

$$x \in U \subset D. \qquad \ldots\ldots (2)$$

From (1) and (2) and by hypothesis, we find a member $W \in \wp$ satisfying $x \in W\ (U \cap V) \subset (C \cap D)$. That means, we can write $C \cap D$ as a Union of members of $\wp$ showing that $(C \cap D) \in \tau$. Hence τ is a topology on X and $\wp$ is a base of τ. The proof is complete.

Example 2.3. The family of all open intervals along with ϕ forms a base for a Topology on the set R of reals known as usual or Euclidean Topology of R.

Solution : We assume ϕ as a member of this family. Take n = 1, 2, ... we see that the union $(-1, 1) \cup (-2, 2) \cup \ldots \cup (-n, n) \cup \ldots = \bigcup_{n=1}^{\infty}(-n,n)$ is equal to R; Further if (a, b) and (c, d) are two open intervals and $x \in (a, b) \cap (c, d)$, then open intervals intersect. Making n appropriately large, we make open interval $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ so small that $\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset (a, b)$ and $\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset (c, d)$ i.e. $\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset (a, b) \cap (c, d)$. Hence all conditions of Theorem 2.3 are fulfilled.

Theorem 2.4. A Topology τ_1 with a base $\wp_1$ is stronger than a Topology τ_2 with a base $\wp_2$ if and only if for $p \in X$ and for $V_2 \in \wp_2$ with $p \in V_2$, there is a member $V_1 \in \wp_1$ such that $p \in V_1 \subset V_2$.

Proof : The condition is sufficient : Let G be any member of τ_2 and $p \in G$. By base property we find a member $V_2 \in \wp_2$ with $p \in V_2 \subset G$. By stated condition we find a member $V_1 \in \wp_1$ such that

$$p \in V_1 \subset G.$$

So we can write $G = \cup\{V_1 \in \wp_2 : V_1 \subset G\}$. Hence $G \in \tau_1$. Thus $\tau_2 \subset \tau_1$ i.e. τ_1 is stronger than τ_2.

The condition is necessary : Let $\tau_2 \subset \tau_1$; So every member V_2 of $\wp_2$ being a member of τ_2 is a member of τ_1 whose base is $\wp_1$. So for $p \in V_2$, we find a member $V_1 \in \wp_1$ such that

$$p \in V_1 \subset V_2.$$

Example 2.4. All left-open (and right closed) **intervals** like **(a, b] (a, b $\in$ R and a < b)** along with ϕ form **a base** for a Topology called **the upper limit Topology of R which is stronger than usual topology of R.**

Solution : Here $\bigcup_{n=1}^{\infty}(-n,n]$ equals to R. If (a, b] and (c, d] are two such intervals and $u \in (a, b] \cap (c, d]$, then left-open intervals do intersect. Then taking n appropriately large, we make left-open interval $(u - \frac{1}{n}, u]$ so small that $(u - \frac{1}{n}, u] \subset (a, b] \cap (c, d]$; and therefore Theorem 2.3 applies. Further if (a, b) is an interval as a base member of usual Topology, and $p \in (a, b)$; we find a base member (a, c] of upper limit Topology such that $p \in (a, c] \subset (a, b)$. So Theorem 2.4 applies for desired conclusion.

Example 2.5. All right-open intervals like $[a, b)$ $a < b$, together with ϕ form a base for a Topology called the lower limit Topology of R which is stronger than usual Topology of R.

Solution : Similar to that of Example 2.4.

Definition 2.2. A family $S\wp$ of subsets of X is said to form a sub-base for a Topology τ of X, if and only if the family $\wp$ of all finite intersections of subsects in $S\wp$ forms a base for τ.

Members of $S\wp$ are called sub-basic open sets.

Example 2.5. Let $S\wp$ consists of all half rays like $(-\infty, a)$ and (a, ∞) as $a \in R$. Show that $S\wp$ forms a sub-base for a Topology of R(which Topology?).

Theorem 2.5. A Collection $S\wp$ of subsects of X is a sub-base for a Topology on X if and only if (a) $\phi \in S\wp$ and (b) X is the Union of some members of $S\wp$.

Proof : If $S\wp$ is a sub-base for a Topology, then ofcourse (a) and (b) hold. Conversely, let a family $S\wp$ of subsets of X obeys (a) and (b), and let $\wp$ denote the family of all finite intersections of members of $S\wp$. Then we have $\phi \in \wp$ and X is a Union of members of $\wp$. Further, if B_1, B_2 are any two members of $\wp$, let $B_1 = U_1 \cap U_2 \cap ... \cap U_{n_1}$ where $U_i \in S\wp$ and $B_2 = V_1 \cap V_2 \cap ... \cap V_{n_2}$ where $V_i \in S\wp$.

If $x \in (B_1 \cap B_2)$; Putting $B_3 = B_1 \cap B_2$, we find B_3 as a finite intersection of members from $S\wp$; and $B_3 \in \wp$, satisfying $x \in B_3 \subset (B_1 \cap B_2)$. Then Theorem 2.3 applies to complete the proof.

Remark : The Topology τ referred to in **Theorem 2.5 is the smallest Topology on X** containing members of $S\wp$, in the sense that τ **is weaker than every Topology on X containing members of** $S\wp$**.**

§3. Neighbourhood of a point, Neighbourhood system.

Let (X, τ) be a Topological space and $x \in X$.

Definition 3.1. A subset N_x of X is called a neighbourbood (or simply Nbd) of x if there is **an open set O_x in** τ such that

$$x \in O_x \subset N_x.$$

An open set O containing x can also be regarded as a nbd of x.

Explanation : A nbd. N_x of x is always non-empty because $x \in N_x$. Also whole set X is a nbd. of each if its points. If X is infinite, then X is the only nbd. of a given point x in X when τ is Indiscrete Topology, while there are many nbds. of x in X when τ is the discrete Topology. In the real number space R with usual topology a point x has neighbourhoods like open intervals containing x.

Theorem 3.1. A subset O of X is an open set if and only if O is a nbd. of each if its points.

Proof : Let O be an open set in (X, τ) i.e. $O \in \tau$, and $x \in O$. Put $N_x = O$ and we find $x \in O \subset N_x$, that confirms N_x as a nbd. of x.

Conversely, let a subset G enjoy the property as stated; and if $x \in G$, we find a nbd. G of x : So, there exists an open set, say, $O_x \in \tau$ such that $x \in O_x \subset G$. Then we write,

$$G = \bigcup_{x \in G} O_x$$

= a Union of some members of τ, and hence G is an open set.

Notation : If $x \in X$, let $\mathcal{N}_x$ denote the family of all nbds of x in (X, τ). $\mathcal{N}_x$ is also termed as **neighbourhood system at x.**

Properties of $\mathcal{N}_x$.

(a) If $N_x \in \mathcal{N}_x$, then $N_x \neq \phi$;

(b) If $N_x \in \mathcal{N}_x$ and $N_x \subset H$, then $H \in \mathcal{N}_x$ (H is a member of $\mathcal{N}_x$);

(c) Intersection of two members of $\mathcal{N}_x$ is a member of $\mathcal{N}_x$;

(d) If $N_x \in \mathcal{N}_x$, there is a member $N^* \in \mathcal{N}_x$ such that $N_x \in \mathcal{N}_u$ for every member $u \in N^*$.

Properties (a) – (c) are very much evident. We need not give proof. For (d), since N_x is a nbd. of x there is an open set, say, G satisfying

$$x \in G \subset N_x \qquad \ldots \qquad \ldots \qquad (1)$$

Since G is open, Theorem 3.1 says that G is a nbd. of each of its points i.e. $G \in \mathcal{N}_u$ for every member $u \in G$ and by (1) it follows that $N_x \in \mathcal{N}_u$ for $u \in G$ (here $G = N^*$).

Definition 3.2. A sub-family $\wp_x$ of $\mathcal{N}_x$ is said to be **a nbd. base of x** if for every nbd. N_x of x, there is a member $B_x \in \wp_x$ such that

$$B_x \subset N_x.$$

Explanation : A Given point of a Topological space (X, τ) may have more than one nbd. base. For example, in the space R of reals with usual topology a point x has a nbd. base consisting of all open intervals like $(x - \frac{1}{n}, x + \frac{1}{n})$, n = 1, 2, ...; also corresponding closed intervals constitute a nbd. base at x. In Euclidean 2-space R^2 with usual Topology, we find that every point $(x, y) \in R^2$ has a nbd. base consisting of all open oriented rectangles centred at (x, y). Also all open circular discs centred at (x, y) shall form a nbd. base at (x, y).

Topology from neighbourhood axioms :

Given $X \neq \phi$ if each point x in X is associated with a family of subsets under constraints of so called nbd. axioms, one can then derived a Topology on X. Following is a Theorem in this connection.

Theorem 3.2. Let each x in X be associated with a non-empty family $\mathcal{N}_x$ of subsets N_x of X satisfying

(a) $N_x \neq \phi$, and $x \in N_x$ for every member $N_x \in \mathcal{N}_x$

(b) if $N_x \in \mathcal{N}_x$ satisfies $N_x \subset W$, then $W \in \mathcal{N}_x$

(c) if $N_x^{(1)}, N_x^{(2)} \in \mathcal{N}_x$, then $(N_x^{(1)} \cap N_x^{(2)}) \in \mathcal{N}_x$

(d) if $N_x \in \mathcal{N}_x$, there is a member $N^* \in \mathcal{N}_x$ such that $N_x \in \mathcal{N}_u$ for every member $u \in N^*$.

For the proof which is lengthy, any standard book may be consulted.

§4. Limit point of a set. Closed sets.

Let A be a non-empty set in a Topological space (X, τ).

Definition 4.1. An element (point) p of X is called **a limit point of $A \neq \phi$ if every nbd N_p of p** meets A at **a point other than p.** Equivalently if $N_p \cap (A \backslash \{p\}) \neq \phi$.

If p is not a limit of A, then **p is said to be an Isolated print of A. In that case** we find **a nbd. N_p of p such** that $N_p \cap (A \backslash \{p\}) \neq \phi$ or equivalently **$N_p \cap A$ is either ϕ or $\{p\}$.**

Explanation : A limit point A may or may not be a point of A. It attracts every nbd. to intersect A at a point other than p. Naturally, the more are nbds. of p, the less is the chance of p to be a limit point of A and the less are nbds. of p, the more is the chance of p to be a limit point of A. Thus in (X, τ) with τ as discrete topology, a given non-empty set A possesses no limit point in X, because open sets are too numerous; If τ is the Indiscrete Topology, the subset A attracts every member of X as its limit point; Here only non-empty open set is X only.

Example 4.1. Obtain limit points (if any) of following sets of reals in the space R of reals with usual Topology.

(a) $A = (1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots)$, (b) The set Q of all rationals in R. (c) The set N of all antural numbers and (d) A finite subset of reals.

Definition 4.2. The set of all limit points of A in (X, τ) is called the derived set of A; it is denoted by A'.

Explanation : The set A' may be empty; for example, take a finite subset A of reals with usual topology. Here A has no limit points at all i.e. $A' = \phi$.

A' may be disjoint with A. A' may be a part of A; for example take A = the closed unit interval [0, 1] of reals with usual Topology. Here $A' = A$. The set A' may be strictly larger than A. For example take A as the set of all rationals in real number space R with usual topology. There $A' = R$ which is strictly larger than A.

Definition 4.3. A subset F in topological space (X, τ) is said to be **a closed set if $F' \subset F$ i.e. if every limit point of F is a point of F.**

For example, every closed interval, every finite subset of reals and the set N of all natural numbers are each a closed set in R with usual Topology.

Theorem 4.1. A subset F in (X, τ) is closed if and only if X/F (Complement of F in X) is an open set in X i.e. if and only if $(X\backslash F) \in \tau$.

Proof : The condition is necessary : Let F be a closed set in (X, τ). If F is empty, then its complement X is of course an open set $\in \tau$. Let F he non-empty and $u \in (X\backslash F)$. Then u is not a limit point of F and we find a nbd, and hence an open nbd (say) N_u of u such that $N_u \cap F = \phi$; this shows that $N_u \subset (X\backslash F)$, and $(X\backslash F)$ is rendered a nbd. of u; Thus $(X\backslash F)$ becomes a nbd. of each of its points, and so $(X\backslash F)$ is an open set in X.

The condition is sufficient : Suppose $X\backslash F$ is an open set in X and x is a limit point of F. It possible, let $x \notin F$. So $x \in (X\backslash F)$. Thus $(X\backslash F)$ is a nbd. of x such that

$$(X\backslash F) \cap F = \phi.$$

That contradicts the assumption that u is a limit point of F. Hence the proof is complete.

Notation : In (X, τ) denote **the family of all closed sets in X by $\mathscr{F}$**

Remark : By De-Mongan's Laws following statements are evident.

(1) $X, \phi \in \mathscr{F}$

(2) Intersection of any number of members of $\mathscr{F}$ is a member of $\mathscr{F}$

(3) Union of two members of $\mathscr{F}$ is a member of $\mathscr{F}$.

However, union of an infinite number of members of $\mathscr{F}$ may not be a member of $\mathscr{F}$. For example, in the real number space R with usual topology let us take the closed intervals.

$I_n = [\frac{1}{n}, 1]$ as $n = 1, 2, ...$ Then each I_n is a closed set of R such that

$\bigcup_{n=1}^{\infty} I_n = (0, 1]$ **which is not a closed set in R.**

§5. Closure of a set in (X, τ).

Given a subset A of X, its closure denoted by $\overline{A}$ is defined as

$\overline{A} = \cap$ **{F ⊂ X : F is a closed set in X ⊃ A}.**

The R.H.S. being intersection of a number of closed sets in (X, τ) is always a closed set in X. Thus $\overline{A}$ is always **a closed set containing A, and it is the smallest closed set to contain A.**

Explanation : Closure of A = $\overline{A}$ is a closed set no matter if A is closed or not. For example, if A = open unit interval (0, 1) of reals with usual topology we find its closure $\overline{A}$ = closed interval [0, 1] which is a closed set; but A = (0, 1) is not a closed set. Clearly if A is it self closed, then $\overline{A}$ = A and converse is also true.

Theorem 5.1. $\overline{A} = A \cup \overline{A}$.

Proof : Here $A \subset \overline{A}$; So $A' \subset \overline{A}' \subset \overline{A}$ because $\overline{A}$ is closed. Hence

$$(A \cup A') \subset \overline{A} \quad ... \quad (1)$$

Again if $x \in \overline{A}$, then every nbd. of x meets A. non-vaculously. For, if $x \notin \overline{A}$ then there is an open nbd. $N_2 = X \setminus \overline{A}$ of x which does not meet A.

Thus x is either in A or a limit point of A.

Accordingly, $x \in A \cup A'$.

Hence $$\overline{A} \subset (A \cup A') \quad ... \quad (2)$$

Combining (1) and (2) we get $\overline{A} = A \cup A'$.

Definition 5.1. (a) A subset G in (X, τ) is said to be **everywhere dense** or simply **dense** in X **if $\overline{G}$ = X.**

(b) Topological space (X, τ) is said to be **separable if there is a countable dense subset in X.**

For example, the set Q of all rationals in R is dense in R with respect to usual Topology of R, because of the fact that between any two reals there are many rationals.

Example 5.1. Let X be an uncountable set, and τ is a family of subsets of X consisting of ϕ, and complements of finite subsets of X. Show that (X, τ) is a Topological space where every infinite subset of X is dense.

Proof : By a routine exercise one checks that axioms of Topology T.1 – T.3 are satisfied and (X, τ) is a Topological space. Let A be an infinite set. If $x \in X$ and N_x be a nbd. of x ; We may assume N_x to be open. If $N_x \cap (A \setminus \{x\}) = \phi$, then we have $(A\setminus\{x\}) \subset (X\setminus N_x)$ or $A \subset (X\setminus N_x) \cup \{x\}$, r.h.s. being a finite set it implies that A is a finite set which is not so. Therefore, $N_x \cap (A \setminus \{x\}) \neq \phi$, that means x is a limit point of A; i.e. $\overline{A} = X$.

Kuratowski closure operator :

An operator that assigns to each subset A of X another subset of X denoted by A^c is called a Kuratouski closure operator if following four axioms known as Kuratowski closure axioms hold :

(K.1) $\phi^c = \phi$

(K.2) For any subset A of X, $A \subset A^c$

(K.3) For any two subsets A and B of X $(A \cup B)^c = A^c \cup B^c$

and (K.4) For any subset A, $(A^c)^c = A^c$.

Theorem 5.2. Let c be a Kuratowski closure operator on a non-empty set X, and Let $\mathscr{F}$ be the family of all subsets A of X for which $A^c = A$; and $\tau = \{G \subset X : (X\setminus G) \in \mathscr{F}\}$. Then τ is a Topology on X such that $A^c = \tau$-closure of A for every subset A of X.

Proof : ϕ and X belong to τ because their complements X and ϕ are respectively members of $\mathscr{F}$. Let $\{G_\alpha\}_{\alpha \in \Delta}$ be an arbitrary sub-family of τ and put $G = \bigcup_{\alpha\in\Delta} G_\alpha$. Then $X\setminus G = X\setminus \bigcup_{\alpha\in\Delta} G_\alpha = \bigcap_{\alpha\in\Delta}(X\setminus G_\alpha) \subset (X\setminus G_\alpha)$. Then $(X\setminus G)^c \subset (X\setminus G_\alpha)^c = (X\setminus G_\alpha)$ because $(X\setminus G_\alpha)$ is a member of $\mathscr{F}$. This is true for every index $\alpha \in \Delta$, and therefore $(X\setminus G)^c \subset \bigcap_{\alpha\in\Delta}(X\setminus G_\alpha) = X\setminus \bigcup_{\alpha\in\Delta} G_\alpha = X\setminus G$. That means $(X\setminus G)^c = X\setminus G$ and it is a member of $\mathscr{F}$ and hence $G \in \tau$. By a similar reasoning intersection of any two members of τ is member of τ. So τ is a topology on X. It remains to show that for every subset A of X A^c equals to $\overline{A}$ (= τ-closure of A).

Now $\overline{A}$ = Intersection of all τ-closed sets each containing A

= Intersection of members of $\mathscr{F}$ each containing A.

and, therefore, $\overline{A}$ is a member of $\mathscr{F}$ with $A \subset \overline{A}$.

Using Kuratowski closure axiom, $A^c \subset (\overline{A})^c = \overline{A}$ ... (1)

Again by Kuratowski axiom $(A^c)^c = A^c \supset A$.

That means, A^c is a member of $\mathscr{F}$ with $A^c \supset A$.

This gives, $\overline{A} \subset A^c$ ... (2)

Combining (1) and (2) we have $\overline{A} = A^c$ and the proof is complete.

Definition 5.2. (a) Given a non-empty set A of (X, τ) a point $x \in A$ is said to be an Interior point of A if there is an open set O in τ such that $x \in O \subset A$.

(b) Interior of A or simply Int A = $\{x \in A : x \text{ is an interior point of } A\}$.

Example 5.2. If A = Open interval $(0, 1) \cup \{2, 3, 4, ..., n, ...\}$

Find Int A (w.r.t. usual topology of reals).

Solution : Here every member of open unit interval (0, 1) is an interior point of A and none of its points like 2, 3, ... is an interior point; because member 2, 3, ... individually do not attract a whole open interval containing it, but contained in A. So Int A = (0, 1).

Example 5.3. In (X, τ) if G is an open set and A is any subset of X, then $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$ (bar denoting closure).

Solution : If $G \cap \overline{A} = \phi$, then $G \cap A = \phi$ because $A \subset \overline{A}$.

Conversely, let $G \cap A = \phi$ and $G \cap \overline{A} \neq \phi$. Take a member $u \in (G \cap \overline{A})$. Clearly $u \notin A$; So u is a limit point of A. Since $u \in G$ and G is open; G is a nbd. of u such that $G \cap A = \phi$ – that is not true. So conclusion remains valid as wanted.

Remark : $\text{Int } A \subset A \subset \text{closure } A$ where Int A is the largest open subset of A and closure A is the smallest closed (so called) superset of A to satisfy the inclusion.

Theorem 5.3. For any non-empty subset A of (X, τ).

(a) Int A is an open set

(b) A is open if and only if $A = \text{Int } A$.

(c) If $A \subset B$, then $\text{Int } A \subset \text{Int } B$.

(d) $\text{Int } A = X \setminus (\overline{X \setminus A})$.

Proof : The proofs for (a) – (c) are easy and left out. For (d) $\overline{X \setminus A}$ is a closed set containing $(X \setminus A)$; So complement $X \setminus (\overline{X \setminus A})$ is an open subset of A, and therefore we have

$$X \setminus (\overline{X \setminus A}) \subset \text{Int } (A) \quad \ldots \quad (1)$$

Again, $A \supset \text{Int } A$ gives $(X \setminus A) \subset X \setminus \text{Int } A$, which is a closed set containing $(X \setminus A)$; and therefore,

$$(\overline{X \setminus A}) \subset (X \setminus \text{Int } A)$$

So, $X \setminus (\overline{X \setminus A}) \supset \text{Int } A$ (Taking Complement) ... (2)

Combining (1) and (2) we produce

$\text{Int } A = X \setminus (\overline{X \setminus A})$.

Allied to closure and Interior of a set A in (X, τ), there is another operator called Boundary of A, denoted by Bdr (A) that we presently define.

Definition 5.3. Bdr (A) = closure (A)\Int (A).

($= \overline{A} - \mathring{A}$, $\mathring{A}$ denoting Int A).

For example, in Enclidean 2-space R^2 with usual topology i: A is taken an open circular disc given by

$$A = \{(x, y) ; x^2 + y^2 < r^2\}.$$

Then $\overline{A}$ = the closed circular disc = $\{(x, y) : x^2 + y^2 \leq r^2\}$. and $\mathring{A} = A$. and therefore

$\text{Bdr } A = \overline{A} \setminus \mathring{A}$

$= \overline{A} \setminus A$

$= \{(x, y) : x^2 + y^2 = r^2\}$, namely it represents the circle with radius r centred at (0, 0).

Definition 5.4. A subset G in (X, τ) is said to be nowhere dense in X if **Int (Closure G) = φ**

$$\textbf{(i.e. } (\overline{G})^{\circ} = \phi\textbf{).}$$

For Example, every finite subset of reals w.r.t. usual topology is a nowhere dense set. An infinite subset of reals may or may not be a nowhere dense set. **The set N of all natural numbers is, of course, nowhere dense.** And the set **Q of all rationals, an infinite set of reals is not nowhere dense set;** because **$\overline{Q}$ = whole space R of reals with usual topology.**

§6. Sub-space of (X, τ).

Let A be a subset of a topological space (X, τ). Let us put $\tau_A = \{\mathcal{U} \cap A ; u \in \tau\}$.

Then it is a routine exercise to check that the collection τ_A of subsets of A forms a Topology on A as per Definition 1.1. The pair (A, τ_A) is called a sub-space of (X, τ).

We have the following Theorem in this connection.

Theorem 6.1. Let (A, τ_A) be a sub-space of (X, τ). Then (a) if $\mathcal{B}$ is a base of τ, then $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$ is a base of τ_A.

(b) A subset H is a neighbourhood (nbd) of a point $x \in A$ if and only if $H = A \cap N_x$ where N_x is a nbd. of x in (X, τ).

(c) A subset C is closed in (A, τ_A) if and only if $C = A \cap G$, where G is a closed set in (X, τ).

Proof : (a) Take $x \in A$, since $\mathcal{B}$ is a base of τ, we find a member $B \in \mathcal{B}$ such that $x \in B$; Thus $x \in (B \cap A)$, where $(B \cap A) \in \mathcal{B}_A$. Suppose V_1, $V_2 \in \mathcal{B}_A$ and $x \in (V_1 \cap V_2)$; Then $V_1 = B_1 \cap A$, and $V_2 = B_2 \cap A$, where B_1, $B_2 \in \mathcal{B}$. Since $\mathcal{B}$ is a base of τ, and $x \in (B_1 \cap B_2)$, by property of base there is a member $B_3 \in \mathcal{B}$ to satisfy $x \in B_3 \subset (B_1 \cap B_2)$; Therefore $x \in V_3 \subset (V_1 \cap V_2)$ where $V_3 = (B_3 \cap A) \in \mathcal{B}_A$; Therefore $\mathcal{B}_A$ is a base of τ_A.

(b) Let H be a nbd. of x in (A, τ_A). Thus there is a member $V \in \tau_A$ to satisfy $x \in V \subset H$. But $V = U \cap A$ for some $U \in \tau$. Put $N_x = (U \cup H)$. So $x \in U \subset N_x$, wherefrom we find N_x is a nbd. of x in (X, τ) and also $H = A \cap N_x$; The converse goes by a similar argument.

(c) Let C be a closed set in (A, τ_A). Then $A \backslash C$ is open in (A, τ_A) i.e. $(A \backslash C) \in \tau_A$ and take $(A \backslash C) = A \cap O$ where $O \in \tau$. Then $G = (X \backslash O)$ is closed in (X, τ), such that $A \cap G = A \cap (X \backslash O) = A \backslash (A \cap O) = A \backslash (A \backslash C) = C$.

Conversely, let $C = A \cap G$ where G is a closed set in (X, τ). Then $(X \backslash G)$ is open in (X, τ), and therefore $A \cap (X \backslash G) \in \tau_A$. Now $A \cap (X \backslash G) = A \backslash (A \cap G) = A \backslash C$; showing that $(A \backslash C) \in \tau_A$ and therefore C is closed in (A, τ_A).

Example 6.1. Let G be a closed set in (X, τ), then a subset of G is closed in (X, τ) if and only if it is closed in (G, τ_G).

Solution : We know that a subset D of G given to be closed in (X, τ) is closed in (G, τ_G) if and only if $D = G \cap H$ where H is closed in (X, τ). R.H.S. set is intersection of two closed sets, and is closed in (X, τ) ; So LHS is a closed set in (X, τ).

Conversely, let a subset D of G is closed in (X, τ). Then $D = G \cap D$ is closed in (G, τ_G).

Example 6.2. Let A be an open set in (X, τ). Then a subset of A is open in (A, τ_A) if and only if it is open in (X, τ).

Solution : Given A as open set in (X, τ), let a subset B of A be open in (X, τ_A); So let $B = A \cap O$ where O is open in (X, τ). Now r.h.s. = Intersection of two open sets in (X, τ), and hence is an open set in (X, τ); Thus *l*.h.s = B is open in (X, τ).

Conversely, let B be a subset of A and B be an open set in (X, τ). As $B = A \cap B$, so B is open in (A, τ_A).

§7. First and Second Countable spaces :

Definition 7.1. A Topological space (X, d) is said to be a Second countable space if there is a countable base of the topology τ of X.

Example 7.1. The real number space R with usual topology τ is second countable.

Solution : Consider the family $\mathcal{B}_r$ of **all open intervals with end points as rational numbers.** Since the set of all rationals is countable, so is the family $\mathcal{B}_r$. As every open interval is a member of τ, we have

$$\mathcal{B}_r \subset \tau.$$

Let $x \in R$, and G is an open set $\in \tau$ with $x \in G$, we find an open interval, say (a, b) (a, b are reals) such that

$$x \in (a, b) \subset G$$

Since the set of **all** rational numbers is everywhere dense in R, we find two rationals u, v satisfying

$$a < u < x < v < b.$$

Clearly, (u, v) is an open interval with rational end points and is a member of $\mathcal{B}_r$ such that $x \in (u, v) \subset (a, b) \subset G$. Hence $\mathcal{B}_r$ is a countable base for τ, and (R, τ) is second countable.

Theorem 7.1. Every second countable Topological space is separable.

Proof : Let (X, τ) be a second countable space and let $\{B_1, B_2, ..., B_n, ...\}$ be a countable open base of τ.

Take $b_i \in B_i$, i = 1, 2, ... and put B = (b_1, b_2, ..., b_n, ...); Then B is a countable subset of X, and we show that B is dense in X i.e. $\overline{B} = X$. Take a member x $\in$ (X\B), and G is an open set containing x. By base property we find a member B_i such that

$$x \in B_i \subset G.$$

So $x \neq b_i$. Thus G intersects B at a point other than x. That is to say, x is a limit point of B. So a point of X is either a point of B or a limit point of B; So $\overline{B} = X$.

Remark : The Converse of Theorem 7.1. is false. Example 7.2 supports the statement.

Example 7.2. The real number space R with lower limit Topology generated by ϕ and all right-open intervals like [a, b), a < b is separable without being second countable.

Solution : Let Q be the set of all rationals. The Q is countable ; Because between any two reals there are many rationals, basic open sets like intervals [a, b), a < b includes members of Q, and hence Q is dense in R with lower limit Topology. But **this topology is not second countable.**

If possible, let $[a_i, b_i)$, i = 1, 2, ... be a countable base for this topology ($a_i < b_i$); Let u be a real $\neq a_i$ (i = 1, 2, ...) and v > u ; and then [u, v) is an open set of lower limit Topology such that none of $[a_i, b_i)$ satisfies $u \in [a_i, b_i) \subset [u, v)$. Because otherwise $u \leq a_i \leq p$ gives $a_i = u$ which is not the case. Hence conclusion as desired is valid in Example 7.2.

It is time to say when a Topological space is first countable.

Definition 7.2. A Topological space is said to be first countable if the nbd. system if each of its points has a countable base.

Explanation. Concerned Definitions tell us that **if a Topological space (X, τ) is second countable, then it is first countable.** But converse is not true. Because let X be uncountable, and τ is the discrete Topology. Then (X, τ) is first countable. For if $x \in X$, single nbd {x} constitutes nbd. base of the nbd. system $\mathcal{N}_x$ in (X, τ). But (X, τ) is not second countable. Because $\{x\}_{x \in X}$ = a family of open sets in τ possesses no countable sub-family whose union is X.

Example 7.3. The real number space **R with lower limit Topology (See Example 7.2) is a first countable space.**

Solution : Let $x \in R$, and put $\mathcal{B}_x = \{[x, r) : r \in Q$ which is the set of all rationals$\}$. Here $x < r$. Then $\mathcal{B}_x$ is a countable sub-family of the nbd. system at x in (R, τ_e), τ_e denoting the lower limit topology on R. $\mathcal{B}_x$ is a nbd. base at x ; because if N_x is any nbd. of x relative to τ_e, there is a right-open interval like [a, b) such that

$$x \in [a, b) \subset N_x.$$

Clearly $a \leq x < b$. Take a rational r such that $x < r < b$; Then $[x, r) \in \mathcal{B}_x$ such that $x \in [x, r) \subset N_x$. Our argument is over and Example 7.2 stands.

Exercise - A

Short answer type Questions

1. Given a non-empty set X any two Topologies are Comparable. **Either** prove it **or** give a counter example.
2. If $A = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...)$ obtain limit points, if any, of A if A is given (i) the discrete Topology (ii) the indiscrete Topology and (iii) the usual topology of reals.
3. In a Topological space if for any two open sets U and V we have $U \cap V = \phi$, show that $\overline{U} \cap V = U \cap \overline{V} = \phi$, bar denoting closure.
4. Show that if X is infinite and τ is the co-finite Topology of X, any two non-empty open sets have a non-empty intersection.
5. In a Topological space (X, τ) if $A \subset X$, show that Bdr (A) = ϕ if and only if A is clo-open.
6. Find two subsets A and B of reals with usual Topology such that Int (A) $\cup$ Int (B) $\neq$ Int $(A \cup B)$.

Exercise - B

1. Let X be an infinite set with $x_0 \in X$. If φ is the family consisting of all sets that do not contain x_0 and all sets $(X\backslash F)$ where F is a finite subset of X. Show that (X, φ) is a Topological space where every singleton other that $\{x_0\}$ is clo-open. Also show that $\{x_0\}$ is closed but not open.

2. Show that Interior operator in a Topological space (X, τ) is subject to following conditions :

   (i) $\text{Int}(X) = X$ (ii) $\text{Int}(A) \subset A$ (iii) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$

   (iv) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$ for any subsets A and B of X.

3. In a Topological space (X, τ) of $A \subset X$, show that

   (a) $\text{Int} A = A\backslash \text{Bdr}(A)$ (b) $\overline{A} = A \cup \text{Bdr}(A)$, bar denoting the closure and

   (c) $\text{Int} A \cap \text{Bdr} A = \phi$.

4. Let X be an infinite set, and for any subset A of X, let

   $A^c = A$ when A is a finite subset of X,

   $= X$ when A is an infinite subset of X.

   Verify that A^c satisfies Kuratowski closure axioms, and the resulting topology is the co-finite Topology on X.

5. In a Topology space (X, τ) show that (a) if G and H are open sets in X, then $\text{Int}(\overline{G \cap H}) = \text{Int}(\overline{G} \cup \overline{H})$, bar denoting the closure. (b) if G is open and H is dense, then $\overline{G \cap H} = \overline{G}$, bar denoting the closure.

6. If $X \neq \phi$, show that for any collection Topologies for X there is a unique largest Topology which is smaller than each member of the collection, and a unique smallest Topology which is larger than each member of the collection.

7. For any $A \subset X$, prove that (a) $(\overline{A})^c = \text{Int}(A^c)$ and (b) $(\text{Int} A)^c = \overline{A^c}$, where "c" indicates complementation.

Unit 2 ❑ Continuous Functions Over Topological Spaces

(Continuous function over Topological space, Homeomorphism, Their Characterisations, Continuity of Characteristic function, Nets, Filters, Their convergence, Mutual implications, Product spaces, Projection functions, Their properties; Open functions, Closed functions, Quotient spaces).

§1. Let (X, τ), (Y, U) denote Topological spaces.

Definition 1.1. A function $f : (X, \tau) \rightarrow (Y, U)$ is said to be continuous **if for every open set $u \in U$ in Y, $f^{-1}(u)$ is an open set $\in \tau$ in X.**

Definition 1.1(a). If $x = x_0 \in X$, then $f : (X, \tau) \rightarrow (Y, U)$ is said to be continuous at x_0 if corresponding to any nbd. W of $f(x_0) \in Y$, there is a nbd. V of x_0 in X such that $f(V) \subset W$.

If f is continuous at every point of X, then f is said to be continuous over X, or smiply $f : (X, \tau) \rightarrow (Y, U)$ is continuous.

Definition 1.2. A function $f : (X, \tau) \rightarrow (Y, U)$ is said to **be an open function if it sends open sets into open sets i.e. if $O \in \tau$, $f(O) \in U$.**

It is called **a closed function if it sends closed sets into closed sets i.e. if F is a closed set in X, then f(F) is a closed set in Y.**

Definition 1.3. A 1–1 and onto (bijective) function $f : (X, \tau) \rightarrow (Y, U)$ is said to be a Homeomorphism if f and f^{-1} are each continuous.

Explanation : Let (X, τ) be a discrete space, then any function $f : (X, \tau) \rightarrow (Y, U)$ becomes a continuous, because given any $u \in U$, $f^{-1}(u)$ is always an open set in discrete Topology on X. If τ_1 and τ_2 are two topologies on X such that τ_1 is stronger than τ_2. Then identity function $I : (X, \tau_1) \rightarrow (X, \tau_2)$ i.e. $I(x) = x$ for $x \in X$ is a continuous function. If f is a real-valued function of a real variable i.e., $f : R \rightarrow R$ then taking R with usual topology of reals and remembering that open intervals form a base if the Topology we get from Definition above that f is continuous at a point $x_0 \in R$ if corresponding to $\in > 0$ there is a +ve δ such that $f(x_0) - \in < f(x) < f(x_0) + \in$ i.e. $| f(x) - f(x_0) | < \in$ whenever $x_0 - \delta < x < x_0 + \delta$ i.e. $| x - x_0 | < \delta$. Thus definition of continuity of a function f as above is in agreement with $(\in - \delta)$. Definition of continuity of f in classical analysis.

Example 1.1 : Let R be taken with usual topology and $f : R \to R$ be taken as

$$f(x) = 0 \text{ if } x \leq 0$$

$$= x \text{ if } 0 x \leq 1$$

$$= 1 \text{ if } x \geq 1.$$

Then f is a closed function without being open.

Theorem 1.1. Let $f : (X, \tau) \to (Y, U)$. Then following statements are equivalent (That is to say, each implies the other).

(a) f is continuous.

(b) If F is a closed set in Y, then $f^{-1}(F)$ is closed in X.

(c) If $\mathcal{SB}$ be a sub-base for U in Y, and $G \in \mathcal{SB}$, then $f^{-1}(G) \in \tau$.

(d) for each nbd. W of f(x) in Y as $x \in X$, there is a nbd. V of x in X such that $f(V) \subset W$.

(e) $f(\overline{A}) \subset \overline{f(A)}$, for every subset A of X, bar denoting closure.

(f) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset B of X, bar denoting closure.

(g) f^{-1} (Int (C)) $\subset$ Int (f^{-1}(C)) for every subset C of Y.

Proof : (a) $\Rightarrow$ (b); so f^{-1} (an open set in Y) is an open set in X. For (b) Let F be a closed set in Y; then (Y\F) is open in Y, and by (a), f^{-1}(Y\F) is open in X i.e. X\f^{-1}(F) is open in X, and hence f^{-1}(F) is closed in X.

(b) $\Rightarrow$ (c); Let G be a sub-basic open set in Y, then (X\G) is a closed set in Y; by (b) f^{-1}(Y\G) = X\f^{-1}(G) is a closed set in X and therefore its complement f^{-1}(G) is an open set in X. So (c) holds.

(c) $\Rightarrow$ (d); Let $x \in X$ and Let W be a nbd. of f(x), and $U \in \mathcal{U}$ such that $f(x) \in U \subset W$. Without loss of generality take U as a basic open set in Y and therefore $U = W_1 \cap W_2 \cap \ldots \cap W_k$ as a finite intersection of sub-basic open sets $W_1, W_2, \ldots, W_k$ is $\mathcal{SB}$. Let $f(x) \in W_i \subset U \subset W$. Now $f^{-1}(W_i)$ is an open set in X by (c) with $x \in f^{-1}(W_i)$; Put $V = f^{-1}(W_i)$ which is a nbd. of x such that $f(V) = W_i \subset W$. So (d) holds.

(d) $\Rightarrow$ (e); Let A be any subset of X, and $v \in f(\overline{A})$. Take $u \in \overline{A}$ so that $f(u) = v$; If $u \in A$ we have $v = f(u) \in f(A) \subset \overline{f(A)}$ and we have finished. So assume $u \notin A$ but u is a limit point of A; Let W be a nbd. of f(u) in Y, by (d) there is a nbd. V of u such that $f(V) \subset W$. Now V cuts A, as u is a limit point of A, nonvacuously and so, $V \cap (A\backslash\{u\} \neq \phi$.

Now $f(V \cap (A\backslash\{u\})) \neq \phi$; or $f(V) \cap f(A\backslash\{u\}) \neq \phi$ and hence $W \cap (f(A)\backslash f(u)) \neq \phi$, since $f(V) \subset W$.

That means, f(u) is a limit point of f(A).

or, $v = f(u)$ is a limit point of f(A)

i.e. $v \in \overline{f(A)}$.

Therefore, we see $f(\overline{A}) \subset \overline{f(A)}$. So (e) holds.

(e) $\Rightarrow$ (f); Let $B \subset Y$, Put $A = f^{-1}(B) \subset X$; So, $f(A) = B$ by (e)

$f(\overline{A}) \subset \overline{f(A)}$ or, $f\left(\overline{f^{-1}(B)}\right) \subset \overline{B}$

i.e., $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$; So (f) holds.

(f) $\Rightarrow$ (g); Take $C \subset Y$.

Then we have $f^{-1}(\text{Int } C) = f^{-1}\{Y \setminus (\overline{Y \setminus C})\}$ (by relation between closure and Interior);

$$= X \setminus f^{-1}(\overline{Y \setminus C})$$

$$\subset X \setminus \overline{f^{-1}(Y \setminus C)} \quad \text{from (f)}$$

$$= X \setminus \overline{X \setminus f^{-1}(C)}$$

$$= \text{Int } f^{-1}(C).$$

Thus (g) holds.

(g) $\Rightarrow$ (a); Let G be an open set in Y; Then Int G = G, and by (g) $f^{-1}(G) = f^{-1}(\text{Int } G) \subset \text{Int } f^{-1}(G)$. That means, $f^{-1}(G)$ is an open set in X; and hence f is continuous. The proof is complete.

Theorem 1.2. For a bijective (1–1 and onto) continuous function f : $(X, \tau) \to (Y, U)$ following statements are equivalent;

(i) f is a homeomorphism

(ii) f is closed

(iii) f is open.

Proof : The proof is a routine verification based on the fact that for every subset A of X, $(f^{-1})^{-1}(A) = f(A)$; details are left out.

Example 1.2. In every Topological space (X, τ), the identity function I : $X \to X$ where $I(x) = x$, $x \in X$ is a homeomorphism.

Example 1.3. If the space R of reals is taken with usual Topology then for $a > 0$, the function $f : R \to R$ where $f(x) = ax$, $x \in R$ is a homeomorphism.

Definition 1.4. Topological spaces (X, τ) and (Y, U) are called homeomorphic if there is a homeomorphism $h : (X, \tau) \to (Y, U)$.

Explanation : Suppose (X, τ) and (Y, U) are homeomorphic spaces. A homeomorphism $h : (X, \tau) \to (Y, U)$ establishes 1–1 correspondence between elements of X and Y and between open sets of the two spaces. Thus a property of a Topological space X defined by means of open sets is transferred to the space Y that is homeomorphic to X. Examination of these properties, called **Toplogical properties** is essentially the **subject of 'TOPOLOGY'**.

Example 1.4. Let A be a non-empty proper subset of a Topological space (X, τ) and be R be the space of reals with usual Topology; then characteristic function $\chi_A : X \to R$ is continuous if and only if A is clo-open in X.

Solution : Let χ_A be the characteristic function of $A \subset X$, and so

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Suppose χ_A is continuous and take an open set $G \subset R$, then

$\chi_A^{-1}(G) = X$ if G contains 0 and 1, ... (i)

$= A$ if G contains 1 but not 0, ... (ii)

$= (X\backslash A)$ if G contains 0 but not 1, ... (iii)

$= \phi$ if G contains none of 0 and 1. ..... (iv)

Since $\chi_A^{-1}(G)$ is always open in X it follows from (i) – (iv) that A is clo-open in X. The converse is also true.

§2. Nets and Filters :

Definition 2.1. A binary relation denoted by $\geq$ is said to direct a non-empty set D if

(i) m, n and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$ ($\geq$ is transitive)

(ii) $m \in D$, then $m \geq m$ ($\geq$ is reflexive)

(iii) m and n are members of D, then there is a member p in D such that $p \geq m$ and $p \geq n$.

If $\geq$ directs D, then the pair $(D, \geq)$ is said to be a directed set.

For example the set $\mathcal{IN}$ of all natural numbers is a directed set with usual order of reals.

Explanation : A very common example of a directed set is the set N of all natural number with usual arithmetic ordering. So $(N, \geq)$ is a directed set. If (X, τ) is a Topological space and $\mathcal{N}_x$ is the family of all neighbourhoods of a point $x \in X$ is a directed set directed by set inclusion relation $\subset$ (being a subset of). Note that a directed set need not be a partially ordered set, since Definition 2.1 does not invite antisymmetry.

Definition 2.2. If $S : (D, \geq) \rightarrow X$ is a function where $(D, \geq)$ is a directed set, and X is a non-empty set, then S is called a net in X;

or, **equivalently, $S_n(= S(n)) \in X$ as $n \in D$.**

or, in symbol, a net $\{S_n; n \in D, \geq\}$ is in X.

A net $\{S_n : n \in D, \geq\}$ is said to be **eventually in X** if and only if there is a member $N \in D$ such that $S_n \in X$ if $n \geq N$.

A net $\{S_n : n \in D, \geq\}$ is said to be **Frequently in X if and only if for each $m \in D$, there is a member $n \in D$ such that $n \geq m$ and $S_n \in X$.**

Remark : A Sequence $\{x_n\}$ is a net with $I\mathcal{N}$ as a directed set.

Explanation : If a net $\{S_n; n \in D, \geq\}$ is Frequently in X, Let $E = \{n \in D : S_n \in X\}$; Then for each member $m \in D$, we find a member $p \in E$ with $p \geq n$. **Such a subset E of D is called cofinal. Cofinal subsets of a directed set are used in theory of subsets of a net.**

Definition 2.3. : A net $\{S_n, n \in D, \geq\}$ in a Topological space **(X, τ)** is said **to converge to an element $u \in X$, if and only if the net $\{S_n : n \in D, \geq\}$ is eventually in every neighbourhood of u in X.**

For example, if (X, τ) is a discrete space (τ is the discrete Topology), then a net $\{S_n, n \in D, \subset\}$ coverges to a point u iff $\{S_n, n \in D, \subset\}$ is eventual in $\{u\}$; That is to say, from some point on S_n's are all equal to u. On the other extreme case if (X, τ) is indiscrete, (only open sets are ϕ and X), then every net $\{S_n, n \in D, \subset\}$ converges to every point of X. Cousequently a given net may converge to several distinct points.

Theorem 2.1. In a Topological space **(X, τ)** (i) A point **u in X is a limit point of a subset A of X,** if and only if **there is a net in $A\backslash\{u\}$** such that the **net converges to u.**

(ii) **A point u belongs to closure of a non empty set A** i.e. **$u \in \overline{A}$** if and only if there is **a net in A converging to u.**

(iii) A subset A in X is **closed** if and only if **no net in A converges to a member of $(X\backslash A)$.**

Proof : (i) Let $u \in X$ be a limit point of a subset A if X. Then every nbd. N_u of u cuts A in a point other than u;

$$\text{i.e. } N_u \cap (A\backslash\{u\}) \neq \phi; \text{ Take } x_{N_u} \in N_u \cap (A\backslash\{u\}).$$

We know that the family $\mathcal{N}_u$ of **all** nbds. N_u of u forms a directed set with respect to set inclusion relation $\subset$.

Then $\{x_{N_u}, \mathcal{N}_u; \subset\}$ is a net in $A \setminus \{u\}$ such that if N_u and N'_u are nbds. of u with $N'_u \subset \mathcal{N}_u$, then by construction $x_{N'_u} \in N'_u \subset N_u$. Hence the net $\{x_{N_u}, \mathcal{N}_u; \subset\}$ is eventually in every nbd of u, implying that the net converges to u.

The converse is obviously straight forward.

(ii) $\overline{A}$ = closure of A = A $\cup$ derived set A′ (of A). For each member u $\in$ A′, there is a net (by preceding part) in A converging to u; For each member u $\in$ A, any net $\{S_n, n \in D, \geq\}$. Such that $S_n = u$ for all n $\in$ D becomes convergent net converging to u. Therefore, each point in $\overline{A}$ attracts a net in A that converges at that point.

(iii) This part is now clear because a set is closed if and only if $\overline{A} = A$.

Theorem 2.2. A function f : (X, τ) → (Y, U) is continuous at c ∈ X if and only if every net $\{S_n, n \in D, \geq\}$ in X that converges to c, the net $\{f(S_n)\} : n \in D, \geq\}$ converges to f(c) in Y.

In symbol, $f\left(\lim_n S_n\right) = \lim_n f(S_n)$.

Proof : The condition is necessary : Suppose f is continuous at x = c ∈ X. Let $\{S_n, n \in D, \geq\}$ be a net in X that converges to c. So it is **eventually in every nbd. of c in X.** Take $N_{f(c)}$ be a nbd. of f(c) in Y. By continunity if f at c, we have $f^{-1}(N_{f(c)})$ is a nbd. of c in X. So $\{S_n, n \in D, \geq\}$ is eventually in $f^{-1}(N_{f(c)})$. That is to say, $\{f(S_n), n \in D, \geq\}$ is eventually in $N_{f(c)}$, and the net $\{f(S_n), n \in D, \geq\}$ converges to f(c) in Y.

The condition is sufficient : Let the condition hold. If possible, let f be not continuous at c in X. We seek a contradiction.

Failure of continuity of f at c invites **a nbd. (say)** $N_{f(c)}$ **of f(c)** in Y such that $f^{-1}(N_{f(c)})$ is not a nbd. of c.

Thus for every member N_c of nbd. system $\mathcal{N}_c$ of c in X, we find a point x_{N_c} (say) $\in N_c$ such that $x_{N_c} \notin f^{-1}(N_{f(c)})$.

i.e. $f(x_{N_c}) \notin N_{f(c)}$ ... ... (*)

Put D = Collection of all such ordered pairs

$= \{(x_{N_c}, N_c)\}, N_c \in \mathcal{N}_c$.

Then consider the Directed set $D = \{(x_{N_c}, N_c), N_c \in \mathcal{N}_c, \subset\}$.

Define $S(x_{N_c}, N_c) = x_{N_c}$; Then the net $\{S, D, \subset\}$ has the property that it is eventually in **every nbd.** N_c of c; i.e. it converges to c in X, but the image net $\{f(S), D, \subset\}$ becomes such that $f(S(x_{N_c}, N_c)) = f(x_{N_c}) \notin N_{f(c)}$ (From *).

i.e. the net is not eventually in $N_{f(c)}$ = a nbd. of f(c) in Y. So the image not $\{f(S), D, \subset\}$ does not converge to f(c), although the net $\{S, D, \subset\}$ converges to c in X—a contradiction. The proof is complete.

Definition 2.4. A filter in a Topological space (X, τ) is a family $\mathcal{F}$ of subsets of X satisfying

(F.1) If $A \in \mathcal{F}$, and $A \subset B$, then $B \in \mathcal{F}$;

(F.2) If $A_1, A_2 \in \mathcal{F}$, then $(A_1 \cap A_2) \in \mathcal{F}$;

(F.3) $\phi \notin \mathcal{F}$.

For example, the nbd. system $\mathcal{N}_x$ if a point x in X is a Filter. Because above conditions (F.1 – F.3) are all O.K. in favour of the family $\mathcal{N}_x$. That is why $\mathcal{N}_x$ is often termed as nbd. Filter.

Definition 2.5. A Filter $\mathcal{F}$ is said to converge to a point $x \in X$, if and only if each nbd N_x of x is a member of $\mathcal{F}$ (that is, the nbd system $\mathcal{N}_x$ at x is a sub-family of $\mathcal{F}$).

Theorem 2.3. A function $f : (X, \tau) \to (Y, U)$ is continuous at $x = c \in X$ if and only if for every Filter $\mathcal{F}$ on X that converges to c, the image Filter $f(\mathcal{F})$ converges to f(c) in Y.

Proof : The condition is necessary : Let f he continuous at $x = c \in X$, and let $\mathscr{F}$ be a filter in X converging to c. Then the nbd. system $\mathscr{N}_c$ at $x = c$ is a sub-family of F i.e. $\mathscr{N}_c \subset \mathscr{F}$. We show that image Filter $f(\mathscr{F})$ in Y converges to f(c); That is to show the nbd. system $\mathscr{N}_{f(c)}$ at f(c) as a sub-family of $f(\mathscr{F})$ or

$$\mathscr{N}_{f(c)} \subset f(\mathscr{F}). \qquad \text{..... (1)}$$

Let $N_{f(c)}$ be a nbd. of f(c) in Y. i.e. $N_{f(c)} \in \mathscr{F}_{f(c)}$. By continuity of f at $x = c$, we find that $f^{-1}(N_{f(c)})$ is a nbd. of c in X and hence by assumption $f^{-1}(N_{f(c)}) \in \mathscr{F}$; That means $N_{f(c)} \in f(\mathscr{F})$ and (1) is verified.

The condition is sufficient; Suppose the condition holds. and f is not continuous at $x = c$. We derive a contradiction. We find a nbd $N_{f(c)}$ of f(c) in Y such that no nbd N_c of c in X satisfies $N_c \subsetneq f^{-1}(N_{f(c)})$.

i.e. no $f(N_c) \subset N_{f(c)}$, showing that $N_{f(c)} \notin f(\mathscr{N}_c)$.

Therefore image Filter $f(\mathscr{N}_c)$ does not converge to f(c) in Y, though nbd. filter $\mathscr{N}_c$ coverge to c in X—a contradiction. The proof is now complete.

Theorem 2.4. A point x is a limit point of a subset A of X, if and only if $A\setminus\{x\}$ is a member of some Filter converging to x.

Proof : Let x be a limit point of $A \subset X$. Then if $N_x \in \mathscr{N}_x$ we have $N_x \cap (A\setminus\{x\}) \neq \phi$.

If we put $\mathscr{G} = \{G \subset X : N_x \cap (A\setminus\{x\}) \subset G, N_x \in \mathscr{N}_x\}$, then it is not difficult ot cheek that $\mathscr{G}$ is a Filter generated by members $N_x \cap (A\setminus\{x\})$, $N_x \in \mathscr{N}_x$. Further $\mathscr{N}_x \subset \mathscr{G}$, and, hence, $\mathscr{G}$ is a filter converging to x in X. Of course, by construction, $A\setminus\{x\} \in \mathscr{G}$.

Conversely, Let $A\setminus\{x\}$ be a member of a filter (say) $\mathscr{H}$ that converges to x in X; That is, to say,

$$A\setminus\{x\} \in \mathscr{H}. \qquad \text{..... (1)}$$

Also the nbd, system $\mathscr{N}_x$ at $x \subset H$; That is to say, every nbd. N_x of x is a member of $\mathscr{H}$.

$$\text{or, } N_x \in \mathscr{H} \qquad \text{..... (2)}$$

Since $\mathscr{H}$ is Filter, we deduce from (1) and (2)

$$N_x \cap (A \setminus \{x\}) \neq \phi.$$

That means **x is a limit point of A.**

Nets and Filters Lead to essentially equivalent theories. Grounds for this suspicion rest in following Theorem.

Theorem 2.5. In a Topological space (X, τ). If $\{S_n, n \in D, \geq\}$ is a net converging to $u \in X$, they there is a Filter in X converging to u; and vice-versa.

Proof : Let the net $\{S_n, n \in D, \geq\}$ converge to $u \in X$.

For each $n \in D$, put $A_n = \{S_m, m \in D; m \geq n\}$. Since Intersection of any two such members contains another such member $\{A_n\}$ generate a Filter $\mathscr{F}$ in X. Now given any nbd. N_u of u, we know that the net is eventual in N_u; That is to say $A_n \subset N_u$ for some n. That means N_u is also a member of $\mathscr{F}$ i.e. the nbd. system $\mathscr{N}_u$ at u satisfies $\mathscr{N}_u \subset \mathscr{F}$; **Hence $\mathscr{F}$ converges to $u \in X$.**

Conversely, Let $\mathscr{F}$ be a Filter converging to $u \in X$. Put $D_{\mathscr{F}} = \{(x, F) : x \in F \text{ and } F \in \mathscr{F}\}$. Then $D_{\mathscr{F}}$ is a directed set with order $\geq$ by agreeing to $(Y, G) \geq (X, F)$ if and only if $G \subset F$. Consider the net $\{f(x, F) ; (x, F) \in D_{\mathscr{F}}, \geq\}$.

Where $f(x, F) = x$. Then it is a routine excercise to check that this net is eventual in every nbd N_u of u, and hence the net cônverges to $u \in X$.

§3. Product spaces.

There is a technique to construct a new Topological space out of a given number of Topological spaces. **That leads to the concept of product spaces.** Let (X_1, τ) and (X_2, V) be given Topological spaces, and let $X = X_1 \times X_2$. Consider a family $\mathcal{SB}$ of all subsets of $X_1 \times X_2 = X$ like $G_1 \times X_2$ and $X_1 \times G_2$ where $G_1 \in \tau$ and $G_2 \in V$. Then it is a routine exercise to see that $\mathcal{SB}$ is a sub-base for a Topology on the Cartesian Product X whose base B consisits of all members of the form $G_1 \times G_2$, and the Topology generated by the base $\mathcal{B}$ (or by sub-base $S\mathcal{B}$) on X is called the Product Topology on X. Then a subset W of $X_1 \times X_2 = X$ shall be an open set in the Product Topology if and only if to each member $(x_1, x_2) \in W$ there correspond open nbds U_1 of x_1 in X_1 and U_2 of x_2 in X_2 such that

$$U_1 \times U_2 \subset W.$$

The given spaces X_1 and X_2 are often called co-ordinate spaces and two functions p_{r_1} and p_{r_2} that carry a member (x_1, x_2) of $X_1 \times X_2$ into x_1 and into x_2 respectively

i.e. $p_{r_1} : X_1 \times X_2 \rightarrow X_1$ and $p_{r_2} : X_1 \times X_2 \rightarrow X_2$ where

$p_{r_1}(x_1, x_2) = x_1$ and $p_{r_1}(x_1, x_2) = x_2$ respectively are called projections into Co-ordinate spaces.

Projection functions p_{r_1} and p_{r_2} are here continuous functions. Because if U_1 is an open set in X_1 we see $p_{r_1}^{-1}(U_1) = U_1 \times Y$, and that is an open set in $X = X_1 \times X_2$.

Now suppose τ is a Topology for $X_1 \times X_2$ that make projection functions continuous. Then if U is open in X_1 and V is open in X_2, then $U \times V$ is open relative to τ, because $U \times V = p_{r_1}^{-1}(U) \cap p_{r_2}^{-1}(V)$, and this set is open relative to τ because p_{r_i}'s are continuous. Therefore τ is stronger than the product topology, and we conclude that the product topology is the weakest Topology on X to make projections continuous.

It is now a routine matter to extend the Definition of Product Topology for the cartesian product $X = X_1 \times X_2 \times ... \times X_n$ where each co-ordinate X_i is a given Topological space (X_i, τ_i). Thus a base for the Product Topology on X consists of all members of the from $U_1 \times U_2 \times ... \times U_n$ where each U_i in open in (X_i, τ_i), $i = 1, ..., n$. In particular, the real number space R with usual topology gives rise to the product space $R^n = R \times R \times ... \times R$ (n times) with Product topology; and R^n is more commonly known as the Euclidean n-space.

Here we see that each member of R^n is an ordered n-tuple of reals like $(x_1, x_2, \ldots x_n)$; $x_i \in R$. We look at it as a real function x on the set (1, 2, ..., n) where value at i as x_i $(= x(i))$.

Suppose we have an arbitrary family of Topological spaces (X_α, τ_α), $\alpha \in \Lambda$- an Index set. Then the cartesian product $X = X\{X_\alpha : \alpha \in \Lambda)$ may be looked as the set of all functions x on Λ such that $x_\alpha \in X_\alpha$ for each index $\alpha \in \Lambda$. In that case X_α is α-th Co-ordinate space and corresponding projection function $p_{r_\alpha} : X \to X_\alpha$ is given by p_{r_α} $(x \in X) = x_\alpha \in X_\alpha$.

Consider the family $\{p_{r_\alpha}^{-1}(U)$: where U is an open set is $(X_\alpha, \tau_\alpha)\}$. It is easy to check that this family of subsets of X is a sub-base for a Topology on X, and it is the smallest Topology on X to make each projection function continuous. This Topology on X is called the Product Topology and X is the Product space of given Topology spaces (X_α, τ_α). as $\alpha \in \Lambda$.

Theorem 3.1. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a family of Topological spaces and $X = X\{X_\alpha : \alpha \in \Lambda\}$ be the product space with Product Topology. Then each projection function $p_{r_\alpha} : X \to X_\alpha$ is an open function i.e. it sends open sets into open sets.

Proof : Let $p_{r_\alpha} : X \to X_\alpha$ be a projection function. Let G be an open set in X. If $x \in G$, we find a member of defining open base member, say, D of the product Topology such that $x \in D \subset G$. Then we have

$$p_{r_\alpha}(x) \in p_{r_\alpha}(D) \subset p_{r_\alpha}(G) \qquad \ldots\ldots (1)$$

We know that D looks like

$D = X\{G_\alpha : \alpha \in \Lambda\}$, where G_α is open in X_α and $G_\alpha \equiv X_\alpha$ for all α except a finite number of α's (say) $= \alpha_1, \alpha_2, \ldots, \alpha_n$.

Now, if α = one such α_k, we find

$$p_{r_{\alpha_k}}(D) = \{p_{r_{\alpha_k}}(y) : y \in D\} = \{y_{\alpha_k} : y_{\alpha_k} \in G_{\alpha_k}\} = G_{\alpha_k} \qquad \ldots\ldots (2)$$

And if α is none of $\alpha_1, \alpha_2, ..., \alpha_n$ we know

$p_{r_\alpha}(D) = X_\alpha$. Now (2) and (3) tell us $p_{r_\alpha}(D)$ is an open set and from (1) we conclude that $p_{r_\alpha}(G)$ is open.

Remark : A projection function may not send a closed set into a closed set. For example, in R^2, the closed set $\{(x, y) \in R^2 : xy = 1\}$ has a non-closed Projection on each Co-ordinate space.

Theorem 3.2. Let (Y, V) a topological space and $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a family of Topological space; function $f : Y \to X = X\{X_\alpha : \alpha \in \Lambda\}$ is continuous if and only if the composition function $p_{r_\alpha} f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$.

Proof : Let $f : Y \to X$ be a continuous function. Since X is the product space with product topology and we know that each projection function p_{r_α} on X is continuous, it follows that the composition $p_{r_\alpha} f$ is a continuous function.

Conversely, suppose the condition holds i.e. $p_{r_\alpha} f$ is continuous for each α, then for each open set U of X_α, we have $(p_{r_\alpha} f)^{-1}(U) = f^{-1}(p_{r_\alpha}^{-1}(U))$ and this is an open set by assumption; and product topology of X says that $p_{r_\alpha}^{-1}(U)$ is a member of sub-base for the product topology. This inverse image under f of a (any) member of defining sub-base member of the product Topology becomes an open set. That means, f is continuous.

§4. Quotient spaces.

Let (X, τ) be a Topological space, and Let $f : (X, \tau) \to Y$ be an onto function.

Let $\mathcal{U} = \{U \subset Y : f^{-1}(U) \in \tau\}$.

Then $\mathcal{U}$ forms a Topology on Y. Because

(i) $\phi, Y \in U$;

(ii) It $\{G_\alpha\}_{\alpha\in\Lambda}$ is a sub-family of $\mathcal{U}$, then $G = \bigcup_{\alpha\in\Lambda} G_\alpha \subset Y$ such that $f^{-1}(G) = f^{-1}(\bigcup_{\alpha\in\Lambda} G_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(G_\alpha) \in \tau$, because each $f^{-1}(G_\alpha) \in \tau$. So $G \in \mathcal{U}$:

(iii) Let $G, H \in \mathcal{U}$, and we have $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ which is a member of τ became $f^{-1}(G)$ and $f^{-1}(H)$ are so. Thus $G \cap H$ is a member of $\mathcal{U}$.

Thus $(Y, \mathcal{U})$ is a Topological space, called **quotient space.**

Theorem 4.1. The quotient Topology on Y is the largest topology such that $f : (X, \tau) \to Y$ is continuous.

Proof : Let V be a Topology on Y such that $f : (X, \tau) \to (Y, V)$ is continuous. Take $G \in V$, then by continuity of f we have $f^{-1}(G)$ is open in X i.e. $f^{-1}(G) \in \tau$, and by definition of quotient Topology G is a member of the quotient Topology on Y. Hence $V \subset$ Quotient Topology. The proof is complete.

Definition 4.1. A function f from one topological space into another is said to be a closed function if f sends each closed set into a closed set.

Theorem 4.2. If $f : (X, \tau) \to (Y, \mathcal{U})$ is continuous and onto such that f is either open or closed function, then $\mathcal{U}$ is the quotient Topology on Y.

Proof : Let $(X, \tau) \to (Y, \mathcal{U})$ be continuous and onto, and U be a subset of Y such that $f^{-1}(U)$ is open. Here $U = f(f^{-1}(U))$ is open (f is open) in $\mathcal{U}$. So every open set in quotient Topology is open in $\mathcal{U}$. If f is continuous and open, since quotient Topology is the largest Topology on Y for which f is continuous, $\mathcal{U}$ becomes the quotient Topology on Y.

If f is a closed function, we need to replace 'open' by 'closed' in argument above for desired conclusion.

Example 4.1. Let $X = \{(x, y) \in R^2 : x = 0 \text{ or } y = 0\}$. Show that projection function $P(x, y) = x$ for $(x, y) \in X$ is closed but not open.

Solution : Take any closed subset of X; it is mapped by P into singleton $\{0\}$ which is closed but not open.

Let (X, τ) be a given Topological space, and let R be an equivalence, relation on X. Then X is partitioned into disjoint equivalent classes; denote the set formed by these classes as X/R, called quotient set.

If $a \in X$, let D_a denote the 'equivalent' class containing a. Thus $D_a = D_b$ if and only if a and b enter the same 'equivalent' class i.e. if and only if $(a, b) \in R(a\ R\ b)$.

Now define $h : X \to (X/R)$ where $h(x) = D_x$, $x \in X$. Then h is onto. Also construct a family τ_R of subsets of (X/R) by the rule :-

$$\tau_R = \{K \subset (X\backslash R) : h^{-1}(K) \text{ is an open set in } X\}$$

That is to say, $\tau_R = \{K \subset (X\backslash R) : h^{-1}(K) \in \tau\}$.

Then we verify that τ_R is, indeed, a Topology on (X/R).

Because (i) ϕ, $(X\backslash R) \in \tau_R$.

(ii) If $\{K_\alpha\}_{\alpha\in\Delta}$ be a sub-family of τ_R, and $K = \bigcup_{\alpha\in\Delta} K_\alpha$.

Then $h^{-1}(K) = h^{-1}(\bigcup_{\alpha\in\Delta} K_\alpha) = \bigcup_{\alpha\in\Delta} h^{-1}(K_\alpha) \in \tau$ since each member $h^{-1}(K_\alpha) \in \tau$ which is a Topology on X. That shows $K \in \tau_R$.

(iii) Similarly we can show that intersection of two members of τ_R is a member of τ_R.

This topology τ_R on quotient set (X/R) is called the quotient Topology and Topological space $(X\backslash R)$ is said to be the quotient space.

Theorem 4.3. If π denote projection of (X, τ) onto the quotient space $(X\backslash R)$, then following statements are equivalent.

(a) π is an open function and (b) If G is open in X, R[G] is open.

Proof. For each subset A of X, we have $R[A] = \pi^{-1}(\pi[A])$.

(a) $\Rightarrow$ (b); Let π be open. If G is open in X, by continuity of projection function π. We have $\pi^{-1}(\pi[G])$ is open i.e. R[G] is open. Thus (b) holds.

(b) ⇒ (a); Let (b) hold. If $R[G] = \pi^{-1}(\pi[G])$ is open, then, by Definition of the quotient Topology, $\pi[G]$ is open, and so π is open.

Exercise - A

Short answer type questions

1. In $\mathbb{N}$ is the set of all natural numbers, construct a co-finite Topology τ_{IN} on $\mathbb{N}$. Is τ_{IN} a filter? Give reasons.

2. Show that a sequence $\{x_n\}$ is a net.

3. Give an example of a Projection function that is open, but not closed.

4. Let $P = (a = x_0 < x_1 < \ldots < x_n = b)$ be a Partition of a closed interval $[a, b]$. Let ξ denote the family of complements of all partitions of $[a, b]$. Show that ξ is a Directed set.

5. Let I and J be two non-degenerate intervals of reals with usual topology. Show that any homeomorphism $h : I \to J$ is monotonic.

6. If (X, τ) is a Topological space, and $f : X \to R$ of reals with usual topology, is continuous, show that the set $\{x \in X : -1 < f(x) < +1\}$ is an open set in X.

7. If $\varphi : (-1, 1) \to$ Reals with usual Topology is given by

$$\varphi(x) = \frac{x}{1 - |x|};\ \text{as } -1 < x < +1.$$

Examine if φ is a Homeomorphism.

Exercise - B

1. In a Topological space (X, τ) a subset G is open if and only if G is a member if every Filter that converges to a point of G.

2. Show that all limit points of a net in a Topological space form a closed set.

3. In a Topological space X let Φ_x denote the collection of all Filters each of which converges to $x \in X$; Show that $\cap\{\mathscr{F} : \mathscr{F} \in \Phi_x\}$ is equal to the nbd. system $\mathcal{N}_x$ at x.

4. Let $f : (X, \tau) \to (Y, \mathcal{V})$ be a continuous function. If $A \subset X$, show that restriction f_A is a continuous function on A. Is the converse true? Give reasons.

5. If (X, τ) and $(Y, \mathcal{V})$ are Topological spaces show that a bijective (1–1 and onto) function $f : X \to Y$ is a homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$ for every subset A of X, bar denoting the closure.

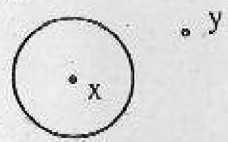
6. Show that the family of all subsets of a non-empty set X each of which contains a given element $x_0 \in X$ is a Filter on X. Examine its maximality.

Unit 3 ❑ Separation Axioms in Topological Spaces

(Separation axioms T_0, T_1, T_2, T_3, T_4 in Topological spaces, Their implications and characterisations, Product of T_2-space, Regular spaces, Normal spaces, Completely regular spaces, Tychonoff spaces, Urysohns Lemma in Normal space, Tietze extension Theorem, Embedding in cube, Embedding Lemma and Theorem, Metrization Urysohn's metrization Lemma).

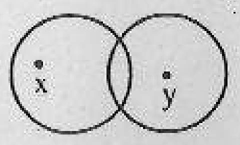
§1. If there are too many or too few open sets in a Topological space (X, τ), analysis thereupon may not be interesting and useful. For example, every function over (X, τ) with τ as discrete Topology becomes continuous. There arise several separation axioms in (X, τ) in terms of availability of open sets in X. These axioms are presented below in graded style : which one is weaker or stronger than the other. Let (X, τ) be a Topological space.

Definition 1.1. (X, τ) is called a **To-space** if given two distinct points in X, **there is an open set containing any one but not the other.**



For example, real number space R with usual topology is a T_0-space ; because given $x, y \in R$ with $x \neq y$, one can find an open interval containing x leaving y outside. Also there is a Topological space like (X, τ) where X = (a, b, c) (a, b and c are distinct) and $\tau = \{\phi, X, (a), (b, c)\}$, such that (X, τ) is not T_0; because distinct points b and c have no T_0-separation i.e. we do not find any open set in X to contain one without containing the other.

Definition 1.2. (X, τ) is called a **T_1-space** if given any two distinct elements in X, **there is an open set to contain each one of them without containing the other.**



Explanation : There are many T_1-spaces ; for example, space R of reals with usual Topology. Consider a Topological space (X, τ) where X = (a, b, c), a, b are c are distinct, and $\tau = \{\phi, X, (a), (a, b)\}$. Here a and b have no attracting open sets in τ as per T_1-stipulation. So (X, τ) is not T_1. Definitions above have been so framed that $T_1 \Rightarrow T_0$ i.e. every T_1-space is a T_0-space; opposite implication is however false.

Example 1.1. The space (X, τ) where $X = \{a, b\}$, $a \neq b$; and $\tau = \{\phi, X, (a)\}$ is T_0 without being T_1.

Solution : Here for distinct elements a and b we have an open set (a) containg a without containing b; and this pair does not have a T_1- separation; because only open set to cover b is (a, b) that cuts $\{a\}$.

Definition 1.3. (X, τ) is called a T_2-space or a Hausdorff space if given any two distinct members x and y in X, there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Explanation : Here $T_2 \Rightarrow T_1$; and if (X, τ) is a Topological space where $X = \{a, b\}$ $a \neq b$; and $\tau = \{\phi, x, (a), (b)\}$. Then (X, τ) is T_2. Example 1.2 shows there is a space (X, τ) that is T_1 without being T_2.

Example 1.2. Let X be an infinite set and let the collection $\mathcal{G}$ of subsets of X be as $\mathcal{G} = \{G \subset X : (X \backslash G)$ is a finite set (may be empty)$\}$. Then we verify that $\mathcal{G}$ is a Topology on X, **very often named as co-finite Topology on X. This Topological space $(X, \mathcal{G})$ is T_1 without being T_2.** Take two members $u, v \in X$ with $u \neq v$. Then $U = X \backslash \{v\}$ and $V = X \backslash \{u\}$ are two members of $\mathcal{G}$ containing u and v respectively such that v is outside U and u is not in V. **Hence $(X, \mathcal{G})$ is T_1. If possible, let any two distinct points x, y in X have T_2 separation.** So we find two open sets H and K in X such that $x \in H$, $y \in K$ and $H \cap K = \phi$. Clearly $(X \backslash H)$ and $(X \backslash K)$ are each finite subsets of X; and so.

	$(X \backslash H) \cup (X \backslash K)$	is a finite set in X,
i.e.	$X \backslash (H \cap K)$	is a finite set in X,
i.e.	X	is a finite set, because $H \cap K = \phi$.

This is a contradiction. Hence our assertion stands.

Theorem 1.1. If (X, τ) is T_0, then closures of distinct points in X are distinct.

Proof : Let (X, τ) be a T_0-space; and let $x, y \in X$ with $x \neq y$. We show that either $x \notin \{\bar{y}\}$ or $y \notin \{\bar{x}\}$, bar denoting the closure. By T_0-separation we obtain an open set U containing (say) x without containing y. That is to say,

$$x \in U \text{ and } y \notin U;$$

Thus a nbd. of x does not cut the singleton $\{y\}$; So x is not a dimit point of $\{y\}$. As $x \neq y$, we ha e $x \notin \overline{\{y\}}$. Of course $x \in \overline{\{x\}}$. That means $\overline{\{x\}} \neq \overline{\{y\}}$.

Theorem 1.2. (X,) **is T_1 if and only if every singleton is closed.**

Proof : Let (X, τ be T_1 and $x \in X$. If $y \in X$ and $y \neq x$, by T_1-separation we find an open set U such that

$$x \in U \text{ and } y \notin U.$$

Clearly then y is not a limit point of $\{x\}$ i.e. $y \notin \{x\}'$ (derived set of $\{x\}$). Of course $x \notin \{x\}'$. That means no member of X is a member of $\{x\}'$. Hence $\{x\}' = \phi$; So

$$\overline{\{x\}} = \{x\} \cup \{x\}' = \{x\}$$

i.e. $\overline{\{x\}}$ is closed.

Conversely, suppose every singleton in (X, τ) is closed. Take $x, y \in X$ with $x \neq y$. So singleton $\{x\}$ is closed, and hence $X \setminus \{x\}$ is an open set containing y (without containing x); and similarly $X \setminus \{y\}$ is an open set containing x (without containing y). Thus (X, τ) is T_1.

Theorem 1.3. (X, τ) **is T_2 if and only if every net in the space converges to atmost one point in X.**

Proof : The condition is necessary. Let (X, τ) be T_2 (Hausdroff). If x and y are two distinct points in X, they attract disjoint neighbourhoods U and V containing x and y respectively by T_2 separation. Since a net can not be eventually in each of two disjoint sets at the same time it follows that no net in X converges to both x and y simultaneously.

The condition is sufficient. Here assume that the condition holds. If possible let (X, τ) be not a T_2-space, and let us suppose a and b are two distinct members of X such that every nbd. of a intersects every nbd. of b. Now nbd. system $\mathcal{N}_a$ at a is a directed set and so is the nbd. system $\mathcal{N}_b$ at b. Let us order their product (Cartesian) $\mathcal{N}_a \times \mathcal{N}_b$ by agreeing that $(T, U) \geq (V, W)$ if and only if $T \subset V$ and $U \subset W$. For each member $(T, U) \in \mathcal{N}_a \times \mathcal{N}_b$ we have $T \cap U \neq \phi$. Take a point $x_{(T,U)} \in (T \cap U)$. Thus if $(V, W) \geq (T, U)$, then $x_{(V,W)} \in (V \cap W) \subset (T \cap U)$, and in consequence the net $\{x_{(T,U)}, \in (T, U)\, \mathcal{N}_a \times \mathcal{N}_b, \geq\}$ converges to both a and b simultaneously—**a contradiction.**

Theorem 1.4. A product of T_2-spaces is a T_2-space.

Proof : If x and y are two distinct points in product $X\{X_\alpha : \alpha \in \Delta\}$, then $x_\alpha \neq y_\alpha$ for some α in Δ. If each co-ordinate space is T_2, we find disjoint open nbds U and V of x_α and y_α respectively, and $P_{r_\alpha}^{-1}(U)$ and $P_{r_\alpha}^{-1}(V)$ become disjoint open nbds of x and y respectively in the product space $X\{X_\alpha : \alpha \in \Delta\}$ with product Topology.

Definition 1.4(a). (X, τ) is said to be a regular space if given any closed set F in X, and an outside point x in X ($\notin$ F), there are open sets U and V in X such that

$$x \in U \text{ and } F \subset V \text{ with } U \cap V = \phi.$$

(b) A regular space which is also T_1 is called a T_3-space.

Explanation : Take X = (a, b, c) and a Topology $\tau = \{\phi, X, (a), (b, c)\}$ in X. Here only closed sets are X, ϕ, (b, c) and (a). We verify that (X, τ) is a regular space, and this regular space is not T_1; because singleton (c) is not a closed set in (X, τ).

$$\text{Further } T_3 \Rightarrow T_2 \text{ (and hence } \Rightarrow T_1 \Rightarrow T_0).$$

As singletons are closed sets in T_1-space, we have $T_3 \Rightarrow T_2$.

Definition 1.5(a). (X, τ) is called a Normal space if given a pair of disjoint closed sets F and G in X, there are disjoint open sets U and V such that $F \subset U$ and $G \subset V$.

(b) A normal space which is also T_1 is said to be a T_4-space.

Explanation : Take X = (a, b, c, d, e, f) and τ = (ϕ, X, (e), (f), (e, f), (a, b, e), (c, d, f), (a, b, e, f), (c, d, e, f)}; Then (X, τ) is a normal space. There are only four pairs of disjoint non-empty closed sets ; they are ((a, b), (c, d)), ((a, b), (c, d, f)), ((a, b, e), (c, d)) and ((a, b, e), (c, d, f)). Each pair is separated by pair of disjoint open sets ((a, b, c), (c, d, f)).

This normal space (X, τ) is not regular; Because (a, b) is a closed set in X with e $\notin$(a, b); there is no disjoint pair of open sets in X to separate them.

Further $T_4 \Rightarrow T_3$. Because if F is a closed set, and x is a point $\notin$ F in a T_4-space (X, τ), we see that singleton $\{x\}$ is a closed set in X. So by normality of (X, τ) desired separation is immediate. So (X, τ) is a T_3-space.

Definition 1.6(a). (X, τ) is said to be **completely regular** if given any closed set F and a point $x \notin F$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(u) = 1$ for $u \in F$.

(b) **A completely regular space which is also T_1** is called **a Tychonoff space** often designated as $T_{3\frac{1}{2}}$-space.

Theorem 1.4. A space (X, τ) is **Normal if and only if** for every closed set F and open set H containing F there is an open set G such that

$$F \subset G \subset \overline{G} \subset H.$$

(Pushing a pair of open and closed sets $(G, \overline{G})$ in between a given pair of closed and open sets (F, H).

Proof : The condition is necessary : Let (X, τ) be a normal space and Let F and H be a pair of closed and open sets in X such that

$$F \subset H.$$

Then complement of $H = H^c$ is a closed set with $F \cap H^c = \phi$. By normality axiom we get a pair of disjoint open sets G and M satisfying

$$F \subset G \text{ and } H^c \subset M \text{ and } G \cap M = \phi.$$

Thus $G \subset M^c$; and $H^c \subset M$ gives $M^c \subset (H^c)^c = H$. As M^c is a closed set, we get

$$F \subset G \subset \overline{G} \subset M^c \subset H.$$

That is, $F \subset G \subset \overline{G} \subset H.$

The condition is sufficient. Suppose the condition holds. Let F_1 and F_2 be a pair of disjoint closed sets in X.

Then we have $F_1 \subset F_2{}^c$ which is open.

Hence by the condition assumed we find an open set G such that

$$F_1 \subset G \subset \overline{G} \subset F_2{}^c.$$

Now $\overline{G} \subset F_2{}^c$ gives $F_2 \subset \overline{G}^c$, and, of course, $G \subset \overline{G}$. So,

$$G \cap \overline{G}^c = \phi.$$

Thus $F_1 \subset G$ and $F_2 \subset \overline{G}^c$ where G and $\overline{G}^c$ is a pair of disjoint open sets in X. **Hence (X, τ) is normal.**

Theorem 1.5. (Urysohn's Lemma) : In a normal space X if A and B are disjoint closed sets in X, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

Proof : Let (X, τ) be normal, and A, B be a pair of disjoint closed sets i.e. $A \cap B = \phi$; So $A \subset B^c$ which is open, B^c denoting the complement of B. So Theorem 1.4. applies and we get an open set (say) $= G_{\frac{1}{2}}$ such that

$$A \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset B^c \qquad (1)$$

Now pair $(A, G_{\frac{1}{2}})$ is a pair of closed and open set with $A \subset G_{\frac{1}{2}}$, and, as before, calls for an open set, say $= G_{\frac{1}{4}}$ such that

$$A \subset G_{\frac{1}{4}} \subset \overline{G}_{\frac{1}{4}} \subset G_{\frac{1}{2}} \qquad (2a)$$

and by a similar reasoning there is an open set (say) $= G_{\frac{3}{4}}$ such that

$$\overline{G}_{\frac{1}{2}} \subset G_{\frac{3}{4}} \subset \overline{G}_{\frac{3}{4}} \subset B^c \qquad (2b)$$

So (2a) and (2b) give

$$A \subset G_{\frac{1}{4}} \subset \overline{G}_{\frac{1}{4}} \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset G_{\frac{3}{4}} \subset \overline{G}_{\frac{3}{4}} \subset B^c \qquad (2)$$

We continue this chain and for each dyadic rational of the form $\frac{m}{2^n}$ ($n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^{n-1}$) in [0, 1], we designate an open set G_l with property that for any two dyadic rationals l_1 and l_2 in [0, 1], $l_1 < l_2$ gives

$$A \subset G_{l_1} \subset \overline{G}_{l_1} \subset G_{l_2} \subset \overline{G}_{l_2} \subset B^c.$$

We know that collection of all such dyadic rationals in [0, 1] is dense in [0, 1].

Let us define $f : X \to [0, 1]$ by the rule :

$$f(x) = 0 \text{ if } x \in \text{every member } G_l$$
$$= \sup \{l : x \notin G_l\}.$$

So $0 \leq f(x) \leq 1$ for $x \in X$; Further from definition above we have $f(x) = 0$ if $x \in A$, and if $u \in B$, as $G_l \subset \overline{G}_l \subset B^c$, we see that $u \notin G_l$ for every

l. By dense property of the set of all l's we deduce from $f(u) = \sup \{l : u \notin G_l\} = 1$.

It remains to work that f is continuous. Since intervals like [0, a) and (b, 1] where 0 < a, b < 1 form sub-basic open sets in [0, 1] with usual Topology of reals. it suffices to show that f^{-1} [0, a) and f^{-1} [b, 1) are open sets in X. Now we can cheek

$$f^{-1}[0, a) = \{x \in X : 0 \le f(x) < a\}$$
$$= \{x \in X : x \in G_l \text{ form some } l < a\}$$
$$= \bigcup_{l<a} G_l \text{ which is an open set in X.}$$

Also $f^{-1}(b, 1) = \{x \in X : b < f(x) \le 1\}$

$$= \{x \in \overline{G}_l^c \text{ for some } l > b.$$
$$= \bigcup_{l>b} \overline{G}_l^c\text{, a union of open sets = an open set.}$$

The proof is now complete.

Corollary 1.1. Let X be a normal space, and let A and B be a pair of disjoint closed sets in X. Then there is a continuous function $g : X \to [a, b]$ such that $g(x) = a$ for $x \in A$ and $g(x) = b$ for $x \in B$.

Take $g = (b - a)f + a$ as f appearing in Urysohn's Lemma.

Example 1.3. In (X, τ) if for any pair of disjoint closed sets A and B there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$, then (X, τ) is Normal.

Solution : Under the given hypothesis, put $G = \left\{x \in X : f(x) < \frac{1}{2}\right\} = f^{-1}\left[0, \frac{1}{2}\right)$ and $H = \left\{x \in X : f(x) > \frac{1}{2}\right\} = f^{-1}\left(\frac{1}{2}, 1\right]$.

Since $\left[0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right]$ are open sets in [0, 1] with usual topology of reals, and since f is continuous it follows that G and H are a pair of disjoint open sets in X satisfying

$$A \subset G \text{ and } B \subset H.$$

So (X, τ) is normal.

Theorem 1.6. A product of Tychonoff spaces is a Tychonoff space.

Proof : Given a Topological space (X, τ), and a continuous function $f : X \to [0, 1]$ with usual topology of reals, let us make a convention :—

If $x \in X$, and U is a nbd. of x, we say that f is for a pair (x, U) if $f(x) = 0$ and $f(u) = 1$ if $u \in (X/U)$.

Then if $f_1, f_2, ..., f_n$ are functions for $(x, U_1), ... (x, U_n)$, (n = a +ve integer), and if $g(x) = \operatorname{Sup}_{1 \le i \le n}\{f_i(x)\}$, $x \in X$, we see atonce that g is for the pair $\left(x, \bigcap_{i=1}^{n} U_i\right)$.

Therefore, the space (X, τ) is completely regular if for each x and for each nbd. U of x (one may take U as a sub-basic member of τ), there is a function for the pair (x, U).

Suppose X = the product X $\{X_\alpha : \alpha \in \Delta\}$ of Tychonoff spaces X_α and take $x \in X$; Let U_α be a nbd. of x_α in X_α. If f is a function for (x_α, U_α), then $f_0 P_{r_\alpha}$ ($P_{r_\alpha} \equiv \alpha$th Projection function) is a function for $\left(x, P_{r_\alpha}^{-1}(U_\alpha)\right)$. Now family of sets like $P_{r_\alpha}^{-1}(U_\alpha)$ constitute a sub-base for the product Topology; and therefore the product space is completely regular. Since product of T_1-spaces is again a T_1-space, we have proved Theorem.

Theorem 1.7. (Tietze Extension Theorem) Let (X, τ) be a normal space, and F be a closed sub-space of X, and let $f : F \to [a, b]$ be a continuous function. Then f has an extension h over X with values in [a, b].

Proof : Invoking corollary 1.1, we may assume that $a = -1$ and $b = 1$. Put $f_0 = f$ and $A_0 = \left\{x \in F : f_0(x) \le -\frac{1}{3}\right\}$, $B_0 = \left\{x : f_0(x) \ge \frac{1}{3}\right\}$. So A_0 and B_0 are a pair of non-empty disjoint closed subsets in F, and hence are also closed in X, because F is closed. Now apply again corollary 1.1 to find a continuous function $g_0 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $g_0(A_0) = \left\{-\frac{1}{3}\right\}$ and $g_0(B_0) = \left\{\frac{1}{3}\right\}$.

Define $f_1 = f_0 - g_0$ on F; then $|f_1(x)| \le \frac{2}{3}$ for all $x \in F$.

If $A_1 = \left\{x : f_1(x) \le \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$, and

$$B_1 = \left\{ x : f_1(x) \geq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \right\}.$$

Then by reasoning as above, there is a continuous function $g_1 : X \to \left[\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right), \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right]$ such that $g_1(A_1) = \left\{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$ and $g_1(B_1) = \left\{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$. Next, define f_2 on F by $f_2 = f_1 - g_1 = f_0 - (g_0 + g_1)$, and see that $|f_2(x)| \leq \left(\frac{2}{3}\right)^2$. We continue this process to obtain a sequence $\{f_0, f_1, \ldots\}$ defined on F such that $|f_x(x)| \leq \left(\frac{2}{3}\right)^n$ and a sequence $\{g_0, g_1, \ldots\}$ defined on X such that $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ with property that on F we have $f_n = f_0 - (g_0 + g_1 + \ldots + g_{n-1})$. Define $S_n = g_0 + g_1 + \ldots + g_{n-1}$, we treat S_n as n-th partial sum of an infinite series of functions (real-valued) over X. We know that the space C(X, R) of all real-valued continuous functions with sup norm becomes complete ;

As $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ and since $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1$, We conclude that $\{S_n\}$ converges uniformly to a bounded continuous h such that $|h(x)| \leq 1$ over X. We may conclude our proof by the observation that $|f_n(x)| \leq \left(\frac{2}{3}\right)^n$ and $\{S_n\}$ converges uniformly over F to f_0 which is equal to f, and that h equals to f over F, and h is a continuous extension of f over whole space X which has the desired property.

Remark : Theorem 1.7 is not true if we remove assumption that F is closed. For example, take X to be the closed Unit interval [0, 1], and take $F \subset X$ as F = (0, 1]. Then look at $f : F \to [-1, 1]$ where $f(x) = \sin\frac{1}{x}$. Then f is continuous, but f does not admit a continuous extension over [0, 1].

§2. Embedding in Cube.

The Cartesian product of closed unit intervals Q = [0, 1] with product topology is called a cube.

So we look upon a cube as the set Q^A of all functions on A to Q. Suppose F is a family of functions such that each $f \in F$ acts on a Topological space X to a space Y_f (Y_f may be different for different $f \in F$). Put $X\{Y_f : f \in F\}$. There is a function $e : X \rightarrow$ the Product, often named as evaluation function where $e(x \in X)_f = f(x)$. It is seen that if members of F are continuous, then evaluation function e is continuous. F is said to distinguish points of X, if given $x, y \in X$ with $x \neq y$, there is a member $f \in F$ such that $f(x) \neq f(y)$. F is said to distinguish points and closed sets if for each closed set A in X and $x \in (X \backslash A)$, we find a member $f \in F$, such that $f(x) \notin \overline{f(A)}$.

Theorem 2.1. (Embedding Lemma). Suppose F is a family of continuous functions $f : X \rightarrow Y_f$ (X, Y_f are Topological spaces); then

(a) Evaluation function $e : X \rightarrow X\{Y_f : f \in F\}$ is continuous.

(b) e is an open function of X onto e[X] if F distinguishes points and closed sets.

(c) e is 1 – 1 if and only if F distinguishes points.

Proof : (a) we see that composition of evaluation function e with projection function p_{r_f} i.e. $p_{r_f}.e$ is equal to f; and here we know that $p_{r_f} \circ e$ is continuous, because $(p_{r_f} \circ e)(x) = f(x)$. Consequently e is continuous, by Theorem 3.2 (Unit II).

(b) It suffices to show that image under e of an open nbd. U of x contains $e[X] \cap$ a nbd. of e(x) in product. Take a member $f \in F$ such that $f(x) \notin$ closure of f(X/U). The set of all y in the product such that $y_f \notin \overline{f(X \setminus U)}$ is open, and its intersection with e[x] is a subset of e[U]. So, e is an open function of X onto e[X]. Now (c) is clear.

Theorem 2.2. (Embedding Theorem). (X, τ) is a **Tychonoff space if and only if** it is **homeomorphic** to a sub-space of **a cube.**

Proof : It is a routine verification that the closed unit interval with usual topology of reals is a completely regular space and it is also T_1. So it is a Tychonoff space. As product if Tychonoff spaces is a Tychonoff space. So a cube is a tychonoff space. Each sub-space of a cube is therefore a Tychonoff space. We now observe that if X is a Tychonoff space and F is is the collection of all continuous functions on X to closed unit interval, then by Embedding Lemma Evaluation function is a homeomorphism of X into the cube $Q^{\mathcal{F}}$.

§3. Metrization :

Metrization problem in Topology deals with obtaining necessary and sufficient conditions for a Topological space to be metrizable. A partial answer to the problem had been given by Urysohn as early as 1924 through a Theorem, better known as Urysohn's Lemma that we present below. We know that the sequence space l_2 of all real sequences $x = \{x_1, x_2, \dots x_n, \dots\}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$ is a metric space with a metric d given by $d(x, y) = \left(\sum_{n=1}^{\infty}(x_n - y_n)^2\right)^{1/2}$, where $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots) \in l_2$. This sequence space l_2 is a known as a real Hilbert space.

Theorem 3.1. (Urysohn's Metrization Lemma)

If (X, τ) is a secon countable normal T_1-space, then it is homeomorphic onto a subspace of l_2, and hence is metrizable.

Proof : Without loss of generality assume that X is infinite. Since (X, τ) is second countable, it has a countable open base, say $\mathcal{B} = (G_1, G_2, \dots, G_n, \dots)$, each G_i being different from φ and X. If $p \in G_k$, since X is Normal and T_1, we find a member G_i (say) of $\mathcal{B}$ such that

$$p \in G_i \subset \overline{G}_i \subset G_k . \quad \dots \quad (1)$$

Put $Q=\{(G_i,G_k):\overline{G}_i \subset G_k\}$, clearly Q is countable.

and let us write $Q = \{D_1, D_2, ...\}$.

For each ordered pair $D_n = (G_i, G_k)$, (say) with $\overline{G}_i \subset G_k$, we find the pair $(\overline{G}_i, G_k^c)$, ('c' meaning complement) as a pair of disjoint closed sets in normal space (X, τ), and hence it invites a continuous function $f_n : X \to [0, 1]$ such that

$$f_n(\overline{G}_i) = \{0\} \text{ and } f_n(G_k^c) = \{1\} \quad \ldots \quad (2)$$

Define a function f(x) on X by

$$f(x) = \left\{ f_1(x), \frac{1}{2} f_2(x), \ldots, \frac{1}{n} f_n(x), \ldots \right\}.$$

For each natural number n, we have $0 \le f_n(x) \le 1$, and hence $\left(\frac{f_n(x)}{n}\right)^2 \le \frac{1}{n^2}$, and we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Therefore $\sum_{n=1}^{\infty} \left(\frac{f_n(x)}{n}\right)^2 < \infty$ as $x \in X$. So f(x) is a member in l_2.

i.e. $f : X \to l_2$. Now **f has following properties** :-

(a) f is 1–1. Let $x, y \in X$ with $x \neq y$. By T_1-separation we find $G_k \in B$ such that $x \in G_k$ and $y \notin G_k$. Then from (1) we find $G_i \in B$ such that

$$x \in G_i \subset \overline{G}_i \subset G_k.$$

From (2) $f_n(x) = 0$, $x \in \overline{G}_i$ and $f_n(y) = 1$; That is to say, nth term of the sequence f(x) = 0, but nth term of the sequence f(y) shall be $\frac{1}{n}$. Thus $f(x) \neq f(y)$. Hence f is 1–1.

(b) f is continuous. Let $x_0 \in X$, and $\in > 0$ be given. Take $y \in X$ with $y \neq x_0$. Then

$$f(y) = \left\{ f_1(y), \frac{1}{2} f_2(y), \ldots, \frac{1}{n} f_n(y), \ldots \right\}$$

and $f(x_0) = \left\{ f_1(x_0), \frac{1}{2} f_2(x_0), \ldots, \frac{1}{n} f_n(x_0), \ldots \right\}$.

Since value of f_n lie in [0, 1], we have

$$\frac{|f_n(y) - f_n(x_0)|^2}{n^2} \leq \frac{1}{n^2} \qquad \ldots \quad (3)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we find an integer n_o (independent of y and x_0), such that

$$\sum_{n=n_0+1}^{\infty} \frac{1}{n^2} < \frac{1}{2} \in^2 \qquad \ldots \quad (4)$$

By continuity of f_n at x_0, for n =1, 2, ..., n_0 we take open sets V_n containing x_0 such that

$$\frac{|f_n(y) - f_n(x_0)|^2}{n^2} < \frac{\in^2}{2n_0} \text{ whenever } y \in V_n \qquad \ldots \quad (5)$$

Taking $V = \bigcap_{n=1}^{n_0} V_n$ we see V as an open set containing x_0 such that for

$$y \in V,\ d(f(y), f(x_0))^2 = \sum_{n=1}^{\infty} \frac{|f_n(y) - f_n(x_0)|^2}{n^2}$$

$$< \sum_{n=1}^{n_0} \frac{|f_n(y) - f_n(x_0)|^2}{n^2} + \frac{1}{2} \in^2 \quad \text{from (3) and (4)}$$

$$< \frac{\in^2}{2n_0} \cdot n_0 + \frac{\in^2}{2} \quad \text{by (5)}$$

$$= \in^2.$$

So, $d(f(y) - f(x_0)) < \in$ whenever $y \in V$.

Hence f is continuous of x_0.

(c) $f^{-1} : f(X) \to X$ is continuous.

Take $x_0 \in X$. There is $G_i \in B$ such that $x_0 \in G_i, \subset \overline{G}_i \subset G_k$ by (1). Let D_{n_0} = pair (G_i, G_k). Choose $\in$ such that $0 < \in < \frac{1}{2n_0}$.

Then $d(f(y), f(x_0)) < \in$ means

$$\sum_{n=1}^{\infty} \frac{|f_n(y) - f_n(x_0)|^2}{n^2} < \left(\frac{1}{2n_0}\right)^2$$

and hence $|f_{n_0}(y) - f_{n_0}(x_0)|^2 < \frac{1}{4}$.

Since $x_0 \in \overline{G}_i$, $f_{n_0}(x_0) = 0$, and hence $f_{n_0}(y) < \frac{1}{2}$.

Since $f_{n_0}(G_k^c) = \{1\}$, We see at once that $y \notin G_k^c$. Hence $y \in G_k$. Therefore $d(f(y), f(x_0)) < \in$ implies $y \in G_k$. So f^{-1} is continuous at $f(x_0) \in f(X)$. From properties (a), (b) and (c) it is shown that f is a homeomorphism of X onto a sub-space of l_2 i.e. X is homeomorphic to a sub-space of metric space l_2. The proof is now complete.

Exercise - A

Short answer type questions

1. Give an example with reason of a Topological space on which each real-valued continuous function is constant.
2. Show that each normal T_1-space is a Tychonoff space.
3. Show that every T_3-space is a T_2-space.
4. Show that a compact (see unit IV) subset of T_2-space is closed.
5. Show that a finite Topological space that is T_1 has discrete Topology.
6. In a Hausdorff space X if $x \in X$, show that

   $\bigcap\{\overline{N}_x : N_x \in \mathcal{N}_x\} = \{x\}$, bar denoting the closure, and $\mathcal{N}_x$ denoting the nbd. system at x.

Exercise - B

1. If X is an infinite set, Show that smallest T_1-Topology in X is its Co-finite Topology.
2. Let f, g : X → Y be continuous function where X is a Topological space and Y is Hausdorff, Show that the subset = $\{x \in X : f(x) = g(x)\}$ is a closed set in X.

3. Let X be a Topological sapce. Prove that following statements are equivalent :–

   (a) X is T_1.

   (b) Every singleton of X is closed.

   (c) Every finite subset of X is closed.

   (d) If $x \in X$, $\cap \{N_x : N_x \in \mathcal{N}_x\} = \{x\}$.

4. Show that every metric space is Normal.

5. Let X be a Hausdorff space and $f : X \to X$ be a continuous function. Show that the set $\{x \in X : f(x) \neq x\}$ is an open set in X.

6. Show that a Compact (see unit IV) T_2-space is metrizable if it is second countable.

7. Show that a homeomorphic image of a Hausdorff space is Hausdorff.

Unit 4 ❑ Compactness in Topological Spaces

(Open cover, Sub-cover, Compactness, Countable open cover, Lindelöff space, Lindelöff Theorem, Properties of compact sets, Finite intersection property, Its relation with compactness, Tychonoff Theorem on Product of Compact spaces, Continuous image of a compact space, Locally compact spaces, 1-point compactification).

§1. Heine-Borel Theorem is a wellknown phenomenon in real analysis. The essence of this lies in Compactness in a Topological space (X, τ).

Definition 1.1(a). A family $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Lambda}$ of **open sets** G_α in X is said to be **an open Cover for X if** $X = \bigcup_{\alpha \in \Lambda} G_\alpha$.

(b) A sub-family $\mathscr{S} \subset \mathscr{G}$ is said to be an **open sub-cover for X** if $\mathscr{S}$ is itself an open cover for X.

For example, the family θ of all **open intervals like (a, b),** a, bar reals with $a < b$ forms **an open cover** for the space R of reals with usual topology ; because each open interval (a, b) is an open set, and the sub-family θ_0 of θ consisting of all members like **(–n, n)**, n = 1, 2, ... also **forms an open cover for R,** and θ_0 is a sub-cover for R.

Definition 1.2. (X, τ) is said to be compact if each open cover for X has a finite open sub-cover for X.

Explanation : The real number space R with usual topology is not compact. Because open cover consisting of **all** open intervals (–n, n) does not have a finite open sub-cover for R. Ofcourse, **there are many compact spaces ;** for instance every finite Topological space is compact.

Example 1.1. Let X be an infinite set and τ be the Co-finite Topology for X ; Then (X, τ) is compact.

Solution : Let $\mathscr{G}$ be an open cover for X and fix a member G_{α_0} of $\mathscr{G}$. Then $(X \setminus G_{\alpha_0})$ is a finite set, say, $= (x_1, x_2, ..., x_m)$ in X. Since $\mathscr{G}$ is an open cover for X i.e. $X = \cup\{G_\alpha : \alpha \in \Lambda\}$; there are members $G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_m}$ of $\mathscr{G}$ such that $x_i \in G_{\alpha_i}$. i = 1, 2, ..., m. Then $G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_m}$ phus G_{α_0} all together form a finite sub-family of $\mathscr{G}$ such that $X = G_{\alpha_1} \cup G_{\alpha_2} \cup ... \cup G_{\alpha_m} \cup G_{\alpha_0}$. Hence (X, τ) is compact.

Definition 1.2. A subset E of (X, τ) is said to be compact if as a sub-space with relativised Topology E becomes a compact space.

Definition 1.3. (X, τ) is called a Lindelöff space if every open cover of X has a countable open sub-cover.

For example, it follows from Definitions that every compact space is a Lindelöff space.

Theorem 1.1. Every second countable space (X, τ) is Lindelöff.

Proof : Let (X, τ) be a second countable space with $(W_1, W_2, ..., W_n, ...)$ as a countable open base for τ. Now take $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for X. Corresponding to any $x \in X$, there exists some $\alpha \in \Delta$ such that $x \in G_\alpha$; we find a base member, W such that $x \in W \subset G_\alpha$. The corrèsponding base members {W} form a part of $\{W_1, W_2, ..., W_n, ...\}$ and is a countable sub-family of $(W_1, W_2, ...\}$, say, $\{W_{n_1}, W_{n_2}, ..., W_{n_k}, ...\}$; Now each $W_{n_i} \subset G_i$—some member of $\mathscr{G}$, i = 1, 2, ... Then $\{G_1, G_2, ...\}$ is a countable sub-family of $\mathscr{G}$ to act as a sub-cover for X; and (X, τ) is a Lindelöff space.

Corollary 1.1. The space R of all reals is a second countable space and is therefore a Lindelöff space, without being compact, with usual topology with basis consisting of intervals (a, b), a and b rationals. Thus a Lindelöff space need not be a compact space.

Remark : Theorem 1.1. is often named as Lindelöff Theorem.

Example 1.2. Continuous image of a Lindeloff space is a Lindelöff space.

Solution : Let $f : (X, \tau) \to (Y, V)$ be a continuous function where (X, τ) is Lindelöff, and (Y, V) is any Topological space. We show that f(X) is a Lindelöff sub-space of (Y, V).

Take $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for f(X); put $G_\alpha = f(X) \cap H_\alpha$ where H_α is open in (Y, V) for each $\alpha \in \Delta$. By continuity of f we have $f^{-1}(H_\alpha)$ is open in X for each $\alpha \in \Delta$.

Now $f(X) = \bigcup_{\alpha \in \Delta} G_\alpha \subset \bigcup_{\alpha \in \Delta} (H_\alpha)$;

So $X \subset f^{-1}\left(\bigcup_{\alpha \in \Delta} H_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(H_\alpha)$.

That is to say, $X = \bigcup_{\alpha \in \Delta} f^{-1}(H_\alpha)$; Hence the family $\{f^{-1}(H_\alpha)\}$ forms an open cover for X. As (X, τ) is Lindeloff, there is a countable open sub-cover for X, say,

$X = f^{-1}(H_1) \cup f^{-1}(H_2) \cup \ldots \cup f^{-1}(H_n) \cup \ldots$

So, $f(X) = \bigcup_{n=1}^{\infty} H_n$.

Thus $f(X) = f(X) \cap \left(\bigcup_{n=1}^{\infty} H_n\right) = \bigcup_{n=1}^{\infty} f(X) \cap H_n = \bigcup_{n=1}^{\infty} G_n$.

So, $\{G_n\}$ becomes a countable sub-family of $\{G_\alpha\}_{\alpha \in \Delta}$ to cover f(X). Hence f(X) is a Lindelöf sub-space of (Y, V).

Theorem 1.2. Every closed set in a compact space is compact.

Proof : Let (X, τ) be a compact space and E be a closed subset in X.

So its complement (X\E) is open in X. Suppose $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for E i.e. $E \subset \bigcup_{\alpha \in \Delta} G_\alpha$, each G_α being open in X. So the family $\{G_\alpha\}_{\alpha \in \Delta}$ and (X\E) form an open cover for X, by compactness of which we find a finite sub-family of this enlarged family as an open sub cover for X. Let $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ and possibly (X\E) form a finite open sub-cover for X, and hence from a finite open sub-cover for E. Clearly $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ is a finite open sub-cover for E, and E is compact.

Remark : Coverse of Theorem 1.2 is not true. For example, take X = a, b with topology $\tau = \{\phi, X, (a)\}$ Then the subset (a) is not closed in (X, τ). However it is compact in (X, τ).

Theorem 1.3. Every compact subset of a T_2-space is closed.

Proof : Let (X, τ) be a T_2-space and E be a compact subset of X. It suffices to show that $(X \backslash E)$ is open in X; without loss of generality take $(X \backslash E) \neq \phi$, and $x \in (X \backslash E)$. If $y \in E$, we have $x \neq y$, and by T_2-separation we find two disjoint open sets V_y and W_y such that $x \in V_y$ and $y \in W_y$ with $V_y \cap W_y = \phi$; So

$$V_y \subset (X \backslash W_y) \quad \ldots \quad (1)$$

Now $E \subset \bigcup_{y \in E} W_y$; showing that the family $\{W_y\}_{y \in E}$ is an open cover for E, by compactness of which we obtain a finite number of members, say, $W_{y_1}, W_{y_2}, \ldots, W_{y_n}$ out of this family such that

$$E \subset \bigcup_{k=1}^{n} W_{y_k} \quad \ldots \quad (2)$$

Look at the corresponding open sets $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ each containing x, and put

$$G = V_{y_1} \cap V_{y_2} \cap \ldots \cap V_{y_n} . \quad \ldots \quad (3)$$

Clearly G is an open set containing x such that (from (1) and (3))

$$G \subset (X \backslash W_{y_1}) \cap (X \backslash W_{y_2}) \cap \ldots \cap (X \backslash W_{y_n})$$

$$= X(W_{y_1} \cup W_{y_2} \cup \ldots \cup W_{y_n})$$

$$\subset (X \backslash E) \text{ by (2).}$$

Thus for each $x \in (X \backslash E)$, we find an open set G containing x such that G $\in (X \backslash E)$; Hence $(X \backslash E)$ is open in X.

Theorem 1.4. Show that a subset E of reals with usual Topology is compact if and only if E is closed and bounded.

Proof : Let E be a bounded and closed set of reals in respect of usual Topology of reals. Suppose [a, b] $(a < b)$ be a closed interval such that

$E \subset [a, b]$, since E is bounded. We appeal to Heine-Borel Theorem to say that [a, b] is compact ; and Now Theorem 1.2 applies to conclude that E is compact.

Conversely, suppose E is a compact set of reals with usual topology which is T_2; we apply Theorem 1.3 to see that E is closed. Finally, we see that $E \subset \bigcup_{n=1}^{\infty}(-n, n)$; So the family $\{(-n, n)\}_{n=1,2,\ldots}$ is an open cover for E, by compactness of which we find a finite number of members $(-n_1, n_1), \ldots, (-n_k, n_k)$ with $n_1 < n_2 < \ldots < n_k$ such that $E \subset \bigcup_{i=1}^{k}(-n_i, n_i) = (-n_k, n_k)$ and hence E is shown to be bounded.

§2. Definition 2.1. A family $\mathscr{F}$ of subsets $\{F_\alpha\}_{\alpha\in\Delta}$ in a topological space (X, τ) is said to have the finite intersection property (F.I.P) if every finite sub-family $\{F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}\}$ of $\mathscr{F}$ has non-empty intersection i.e.

$$\bigcap_{i=1}^{n} F_{\alpha_i} \neq \phi.$$

For example, every decreasing sequence $\{A_n\}$ of non-empty subsets A_n of X has F.I.P. ; because every finite sub-family of $\{A_n\}$ has the smallest member $(\neq \phi)$ as its intersection.

Theorem 2.1. A topological space (X, τ) is compact if and only if each family $F = \{F_\alpha\}_{\alpha\in\Delta}$ of closed sets with F.I.P has a non-empty intersection.

Proof : Let (X, τ) be compact and $F = \{F_\alpha\}_{\alpha\in\Delta}$ be a family of closed sets with F.I.P. If possible, Let $\bigcap_{\alpha\in\Delta} F_\alpha = \phi$. That means $X = \bigcup_{\alpha\in\Delta}(X|F_\alpha)$ where each $(X \backslash F_\alpha)$ is open ; and hence the family $\{(X\backslash F_\alpha) : F_\alpha \in F\}$ is an open cover for X. By compactness of (X, τ), there is a finite family, say, $(X\backslash F_1), (X\backslash F_2), \ldots, (X\backslash F_n)$ of this family to cover X.

So we have $X \subset \bigcup_{i=1}^{n}(X \backslash Fi)$, and taking the complement, we duduce that $\bigcap_{i=1}^{n} F_i^{i=1} = \phi$ – a contradiction that F has F.I.P.

Conversely, suppose the stated condition holds without (X, τ) being compact. We seek a contradiction. Let $\mathscr{G} = \{G_j\}_{j \in \Delta}$ be an open cover of X that has no finite sub-cover for X. Now $X = \bigcup_{j \in \Delta} G_j$; taking $F_j = (X \setminus G_j)$ we see that

$\phi = \bigcap_{j \in \Delta} (X \setminus G_j)$; hence the family $\mathscr{F}$ consisting of closed sets $F_j = (X \setminus G_j)$ as

$j \in \Lambda$ is such that $\phi = \cap\{F_j : F_j \in \mathscr{F}\}$—a contradiction; however by **assumption F does have F.I.P.** Therefore, there is a finite sub-family say $F_1, F_2, \ldots, F_n$ of F such that $\bigcap_{i=1}^{n} F_i = \phi$ i.e. $X = \bigcup_{i=1}^{n} (X | F_i) = \bigcup_{i=1}^{n} G_i$; that means $\mathscr{G}$ admits of a finite sub-cover for X, and (X, τ) is compact.

Theorem 2.2. (Tychonoff Theorem on Product) The Cartesian product of any number of compact spaces is compact w.r.t product Topology.

Proof : Let (X_α, τ_α) be compact spaces and : Let $X = X\{X_\alpha . \alpha \in \Delta\}$. We show that with product topology X is compact.

Put $\mathcal{SB} = \left\{ p_{r_\alpha}^{-1} (u_\alpha) : u_\alpha \in \tau_\alpha ; \alpha \in \Delta \right\}$ Then the family $\mathcal{SB}$ forms a sub-base for the product topology for X, p_{r_α} denoting the projection function : $X \to X_\alpha$ as $\alpha \in \Delta$ now X will be compact if each sub-family φ of $\mathcal{SB}$ such that no finite sub-collection of φ forms a covering of X, fails to form a covering of X.

For each $\alpha \in \Delta$, Let $\mathcal{B}_\alpha$ denote the family of all those open set $u_\alpha \in \tau_\alpha$ for which $p_{r_\alpha}^{-1}(u_\alpha) \in \varphi$. Then no finite sub-family of $\mathcal{B}_\alpha$ forms a covering for X_α ; and hence ashole family can not form a covering of X_α since (X_α, τ_α) is compact. So there is a point (say) $x_\alpha \in X_\alpha$ which is missing in any open set $u_\alpha \in \mathcal{B}_\alpha$. Then the point $x \in X$ with αth co-ordinate x_α, does not belong to any member of φ—meaning that φ does not form a cover for X.

Corollary : Each cube as the product of closed unit intervals is compact.

Definition 2.2. A subset H in (X, τ) is said to be **nowhere dense in X if Int** $(\overline{H}) = \phi$, bar denoting the closure.

For example every finite subset of reals with usual Topology is a nowhere dense set of reals.

Theorem 2.2(a). If **an infinite number of** Co-ordinate **spaces** are **not compact,** then **each compact set** in their **product** with product Topology **is no-where dense.**

Proof : Let (X_α, τ_α) be Topological spaces for each α of an infinite index set Δ be $X = X\{X_\alpha : \alpha \in \Delta\}$ with the product Topology.

Take G be a compact set in the product space X. Suppose u is an interior point of G. Take N as a nbd. of u with $N \subset G$; Without loss of generality take N as a member of the defining base. Thus N is of a form :-

$N = \bigcap\left\{p_{r_\alpha}^{-1}[V_\alpha] : \alpha \in F\right\}$, where **F is a finite subset of Δ,** and V_α is open in X_α. If $\beta \in (\Delta \backslash F)$, then $p_{r_\beta}(G) = X_\beta$, and X_β being continuous image of compact space shall be compact. Therefore all but a finite number of co-ordinate spaces are compact.

Theorem 2.3. Continuous image of a compact space is compact.

Proof : Let (X, τ) be a compact space and (Y, V) be a topological space, and $f : X \to Y$ be a continuous function. We show that f(X) is compact in Y. Take $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for f(X). So

$$f(X) \subset \cup\{G_\alpha ; G_\alpha \in \mathscr{G}\}.$$

Therefore $$X \subset f^{-1}\left(\bigcup_{\alpha \in \Delta} G_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(G_\alpha) \quad \ldots \quad (1)$$

By continuity of f each $f^{-1}(G_\alpha)$ is open in X, and (1) shows that the family $\{f^{-1}(G_\alpha)\}_{\alpha \in \Delta}$ is an open cover for (X, τ) by compactness of which it follows that there are a finite number of members, say, $f^{-1}(G_1), f^{-1}(G_2), \ldots, f^{-1}(G_n)$ such that

$$X \subset f^{-1}(G_1) \cup f^{-1}(G_2) \cup \ldots \cup f^{-1}(G_m) = f^{-1}\left(\bigcup_{i=1}^{n} G_i\right)$$

So $f(X) \subset \left(\bigcup_{i=1}^{n} G_i\right)$.

So given open cover G for f(X) has a finite sub-cover for f(X).

Thus f(X) is compact.

Corollary (1) If f is continuous function of a compact space onto a topological space Y, then Y is compact.

(2) A homeomorphic image of a compact space is compact.

(3) A real-valued continuous function on a compact space is bounded and attains its bounds.

Because, Let $f : (X, \tau) \to$ Reals be continuous where (X, τ) is compact. So f(X) is a compact set of reals with usual topology, and therefore f(X) is bounded and closed.

Theorem 2.4. **A 1-1 continuous function of a compact space onto a T_2-space** is a homeomorphism.

Proof : Let $f : (X, \tau) \to (Y, V)$ be a continuous function that is 1-1, where (X, τ) is compact and (Y, V) is T_2. It suffices to show that f is an open function or equivalently, we show that if F is a closed set in X, the image f(F) is closed in Y. We know that every closed set in compact space is compact ; So F is a compast set in X, and f sends F to a compact set i.e. f(F) is a compact set in Y-which is T_2. Hence f(F) is closed in Y.

Theorem 2.5. If A is a compact set of a T_2-space (X, τ) and $x \in (X \setminus A)$, there are open sets V and W such that $x \in V$ and $A \subset W$ and $(U \cap W) = \phi$.

Proof : Let A be a compact set in X which is T_2, and take $x \in (X \setminus A)$. If $y \in A$, then $x \neq y$, and by T_2-separation there are opon sets V_y and W_y such that $x \in V_y$, $y \in W_y$ and $V_y \cap W_y = \phi$; and, therefore, $x \notin \overline{W}_y$. Now the family $\{W_y : y \in A\}$ becomes an open cover for A which is compact. So there are a finite number of members (say) $W_{y_1}, W_{y_2}, ..., W_{y_n}$ such that

$A \subset W_{y_1} \cup W_{y_2} \cup ... \cup W_{y_n}$. Put $W = \bigcup_{i=1}^{n} W_{y_i}$. Then W is an open set such

that $A \subset W$ and $x \notin W_{y_i}$ $(i = 1, 2, ..., n)$ and hence $x \notin W \subset \overline{W}$. Take $V = X \setminus \overline{W}$. Then V and W are open sets to satisfy $x \in V$, $A \subset W$ and $V \cap W) = \phi$.

Corollary each compact set in a T_2-space is closed.

Theorem 2.6 Let A and B be two compact sets in a T_2-space X such that $A \cap B = \phi$. Then there are two open set V and W such that $A \subset V$ and $B \subset W$ with $V \cap W = \phi$.

(Consequently, a compact T_2-space is Normal).

Proof : For each $x \in A$, we have B is a compact in X with $x \notin B$ i.e. $(X \setminus B)$; Theorem 2.5 says there are open sets V_x and W_x satisfying $x \in V_x$ and $B \subset W_x$ with $V_x \cap W_x = \phi$. Also $B \cap \overline{V}_x = \phi$. Now family $\{V_x : x \in A\}$ of open sets is an open cover for A which is compact. So there are a finite number of members $V_1, V_2, \ldots, V_n$ of this family to cover A. Put $V = V_1 \cup V_2 \cup \ldots \cup V_n$. Then $A \subset V$ and $B \cap \overline{V}_i = \phi$ $(i = 1, 2, \ldots, n)$ gives $B \cap \overline{V} = \phi$. Let us take $W = (X \setminus \overline{V})$. Then V and W are open sets in X such that $A \subset V$ and $B \subset W$ with $V \cap W = \phi$.

(Consequential statement is clear).

§3. Locally Compact Spaces :

Definition 3.1. A topological space (X, τ) is said to be locally compact if each point in X has a compact neighbourhood (nbd).

Explanation : If (X, τ) is compact, then of course it is locally compact ; one may take X itself as a compact nbd. of each of its points. If τ is discrete and X is infinite then (X, τ) is Locally compact without being compact ; because for each $x \in X$, the singleton $\{x\}$ is a compact nbd. of x. However, family of all singletons is an open cover for X that has no finite sub-cover for X.

Example 3.1. The real number space R with usual topology is not compact. It is Locally Compact.

Solution : R being not bounded, it is not compact. However if $x \in R$, then a closed interval like $[x - \delta, x + \delta]$, $\delta > 0$ is a nbd. of x which is a closed and bounded set of reals and hence it is compact.

Theorem 3.1. Let (X, τ) be Locally compact T_2-space then the family of all closed compact nbds of a point in X is a nbd. base there.

Proof : Let (X, τ) be Locally compact adn a T_2-space. Take $x \in X$. Let C be a compact nbd. of x; suppose U is **any** nbd. if x.

$$\text{Put } W = \text{Int } (U \cap C).$$

Then $\overline{W}$ is a compact T_2-sub-space of X. So W contains a closed compact set V which is a nbd. of x in $\overline{W}$; but V is also a nbd. of x in W (relative to topology in W), and is therefore a nbd. of x in X.

Theorem 3.2. Every closed sub-space of a Locally compact space is locally compact.

Proof : Let E be a closed set of (X, τ) which is Locally compact. We show that (E, τ_E) (where τ_E is the relativised topology of τ on E) is Locally compact. Take $x \in E$. Since X is Locally compact, we find a compact nbd., say, G of x in X. Put $M = G \cap E$. So M is a nbd. of x in E. As M is a closed set in compact space G, we see that M is a compact subset of G i.e. M is a compact nbd. of x in (E, τ_E). Hence (E, τ_E) is Locally compact.

§4. One point compactification :

Let (X, τ) be a Locally compact T_2-space, and u is an element outside X ($u \notin X$). Put $X_u = X \cup \{u\}$. Define a Topology τ_u on X_u as :–

(i) All open sets in X as subsets of X_u are in τ_u.

(ii) All complements in X_u of compact sets in X are in τ_u and

(iii) $X_u \in \tau_u$.

We now check that above prescribed family is indeed a Topology in X_u. From (i) $\tau \subset \tau_u$, and by (iii) $X_u \in \tau_u$.

FOR UNION : It is O.K. in respect of members of τ (i).

Consider a family $\{(X_u \backslash A_\alpha) : A_\alpha \text{ is a compact set in X which is } T_2\}_{\alpha \in \Delta}$

Hence A_α is closed in X, and hence $\bigcap_{\alpha\in\Delta} A_\alpha$ is compact in X, and we have $\bigcup_{\alpha\in\Delta}(X_u/A_\alpha)=\left(X_u/\bigcap_{\alpha\in\Delta} A_\alpha\right)$, and by (ii) we have the union is a member of τ_u. Also if $G \in \tau$ and $(X_u\backslash H) \in \tau_u$ as in (ii) where H is a compact set in X, then

$$G \cup (X_u / H) = X_u \setminus ((X \setminus G) \cap H)$$

Since $(X\backslash G)$ is closed in X, and H is compact in X. Therefore $(X\backslash G) \cap H$ is compact in X, and from (ii) r.h.s. set is a member of τ_u.

FOR INTERSECTION : It is O.K. in respect of (i). Take two members $(X_u\backslash D)$ and $(X_u\backslash C)$ where C and D are compact sets in X as members from (ii). Then we have $(X_u\backslash D) \cap (X_u\backslash C) = (X_u\backslash(C \cup D))$. Where $C \cup D$ is also compact in X. Hence r.h.s member $\in \tau_u$ as in (ii). Also if, $A \in \tau_u$ as in (i) and $(X_u\backslash B)$, B is compact in X as in (ii), then B is also closed in X; because X is T_2. Therefore $(X_u\backslash B)\backslash(u)$ is open in X.

Now $A \cap (X_u\backslash B)$

$= A \cap \{(X_u\backslash B)\backslash(u)\}$, and hence is a member of τ as in (i).

Our verification is complete, and (X_u, τ_u) is a Topological space. It remains to show that τ **is actually equal to relativised Topology of τ_u on X.**

If $G \in \tau$, then $G \in \tau_u$ and $G = G \cap X$. Thus

$$\tau \subset \text{relativised Topology } \tau_u \text{ on X} \quad \text{..... (1)}$$

Let H be an open in relatised Topology τ_u in X. So we write $H = X \cap A$ for some member $A \in \tau_u$.

In case A is of type as in (i) we have H as a member of τ.

In case A is of type as in (ii), put $H = X \cap (X_u\backslash B)$ for some compact set B in X. Because X is T_2, we find B as a closed set in X. So $(X\backslash B)$ is a member of τ. Therefore from above

$H = X \cap (X_u\backslash B) = X \cap (X\backslash B)$, and so, $H \in \tau$. Therefore

$$\text{relativised topology } \tau_u \text{ in } X \subset \tau \quad \text{..... (2)}$$

Combining (1) and (2) proof is complete.

Theorem 4.1. (X_u, τ_u) is a compact T_2-space.

Proof : Take $x, y \in X_u$ with $x \neq y$. If x, y are members of X, we have finished. So let us suppose one of them equals u, say $u = y$. Now (X, τ) is Locally compact, so, there is a compact nbd. N_x of x in X. As X is T_2 we take N_x to be closed. Then

Int $(N_x) \cap (X_u \backslash N_x) = \phi$, where Int (N_x) and $X_u \backslash N_x$ are open sets in X_u containing x and u respectively. So (X_u, τ_u) is T_2.

To show that (X_u, τ_u) is compact, suppose $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover of X_u. Let $u \in G_{\alpha_0}$ for some $\alpha_0 \in \Delta$, and therefore G_{α_0} is such that $G_{\alpha_0} = (X_u \backslash D)$ where D is a compact set in X.

$$\text{i.e. } (X_u \backslash G_{\alpha_0}) = D \qquad \ldots\ldots (*)$$

Now $(X_u \backslash G_{\alpha_0}) \subset X = \bigcup_{\alpha \in \Delta} G_\alpha$, giving

$$(X_u \backslash G_{\alpha_0}) \subset \bigcup_{\alpha \in \Delta} (X \cap G_\alpha);$$

This shows that the family $\{(X \cap G_\alpha) : \alpha \in \Delta\}$ is an open cover for $(X_u \backslash G_{\alpha_0})$. (*) says that $(X_u \backslash G_{\alpha_0})$ is a compact set in X, and hence there is a finite sub-cover of this open cover for $(X_u \backslash G_{\alpha_0})$ say $(X \cap G_{\alpha_1}), (X \cap G_{\alpha_2}), \ldots, (X \cap G_{\alpha_n})$.

Thus $(X_u \backslash G_{\alpha_0}) \subset (X \cap G_{\alpha_1}) \cup (X \cap G_{\alpha_2}) \cup \ldots \cup (X \cap G_{\alpha_n})$

$$\text{i.e. } (X_u \backslash G_{\alpha_0}) \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$$

So $X_u = G_{\alpha_0} \cup (X_u \backslash G_{\alpha_0})$

$$\subset G_{\alpha_0} \cup G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}.$$

That means Given open cover admits of a finite sub-cover for X_u, and (X_u, τ_u) is compact.

Remark : Compact T_2-space (X_u, τ_u) arising out of a given Locally compact T_2-space (X, τ) in the manner described above is called **"One point compactification"** of X, and external point **u** is designated **as the point at infinity.**

Exercise - A

Short answer type questions

1. Show that a finite Topological space (X, τ) is compact.
2. Show that no open interval of reals with usual topology is compact.
3. Which one of followings subsets of reals with usual topology is compact?
   (a) $[1, 2] \cup [3, 4]$
   (b) The set N of all natural numbers. Give reasons.
4. In a Topological space (X, τ) any set E consisting of points of a convergent sequence together with its converging limit is compact. Prove it.
5. Show that union of a finite number of compact sets is a compact set in a Topological space. Is the union of an infinite number of compact sets a compact set? Give reasons.
6. In a compact T_2-space families of closed sets and compact sets are identical. Prove it by quoting relevant Theorems.
7. Show that an infinite discrete space is Locally compact without being compact.

Exercise - B

1. Show that every regular Lindelöff space is normal.
2. Show that a compact T_2-space is T_4.
3. Show that a sub-space of a compact space need not be compact, and verify that every closed sub-space of a compact space is compact.
4. Show that every real-valued continuous function on a compact space X there are points x and y in X such that $f(x) = \sup_X f$ and $f(y) = \inf_X f$.
5. Show that intersection of two compact sets in a Topological space X may not be compact. If A is a closed set and B is a compact set in X, show that $A \cap B$ is compact.
6. Show that a continuous image of a locally compact space need not be Locally compact.

Unit 5 ❑ Connectedness

(Connected spaces, Separated sets, Disconnection of a space, Union of connected sets, Closure of a connected set, Connected sets of reals, Continuous image of a connected space, Topological product of connected spaces, Components, Their properties, Totally disconnected spaces, Locally connected spaces).

§1. **A Connected Topological space (X, τ)**, roughly speaking, is such a **strong piece of objects** that it **does not allow its partition** into **two non-empty disjoint open (or closed) subsets.** To be more precise we present the following definitions.

Let (X, τ) denote a Topological space.

Definition 1.1. Two subsets A and B of X are said to be separated if $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$.

Explanation : Without loss of generality, assume $A \neq \phi$, $B \neq \phi$. Subsets A and B when separated are, of course disjoint. But there are more things to look at. Neither A nor B contains a limit point of other. In relative topology for $A \cup B$, both A and B are regarded as closed in $(A \cup B)$; or equivalently, A or B is taken as (each) open and hence each has status of a clo-open set in $(A \cup B)$. Take for example, open intervals (0, 1) and (1, 2) of reals with usual topology. They are disjoint subsets. But they are not separated as per Definition 1.1 above because number 1 belongs to closure of each.

Theorem 1.1. If A and B are subsets if (X, τ), and both A and B are closed (or both are open), then $(A \setminus B)$ and $(B \setminus A)$ are separated.

Proof : Let A and B be closed subsets in X. Then relative to $(A \cup B)$, A and B are closed, and therefore $(A \setminus B) = ((A \cup B) \setminus B))$ and $(B \setminus A)$ are open in $(A \setminus B) \cup (B \setminus A)$ and since they are complements relative to $((A \setminus B) \cup (B \setminus A))$, both become closed in $((A \setminus B) \cup (B \setminus A))$. Hence $(A \setminus B)$ and $(B \setminus A)$ are separated.

In case A and B are both open, proof is done by a similar dual arguments.

Definition 1.2(a). (X, τ) is sais to be **connected** if X **is not a union of two non-empty disjoint open sets in X.**

or equivalently, if X is **not a union of two non-empty disjoint closed sets in X.**

(b) A subject Y of X is said to be connected if Y as a sub-space of (X, τ) with relativised topology becomes connected.

or equivalently, Y is not a union of two non-empty disjoint open subsets (closed subsets) of Y in relativised topology on Y.

(c) (X, τ) is said to be disconnected if it is not connected i.e., if X admits of a decomposition like

$$X = G \cup H,$$

Where G and H are a pair of non-empty disjoint open (or closed) subsets of X. Such a decomposition of X is called a disconnected of X.

Explanation : Definitions say that (X, τ) is connected if and only if only clo-open sets are ϕ and X. A subset Y of (X, τ) is disconnected if Y has a partition like $Y = P \cup Q$ where $P \neq \phi$, $Q \neq \phi$, and P and Q are disjoint open subsets of Y. So we put $P = G \cap Y$ and $Q = H \cap Y$ where G and H are open sets in X. Therefore, $Y = (G \cap Y) \cup (H \cap Y)$ where r.h.s. members are non-empty and disjoint.

Conversely if $Y = (G \cap Y) \cup (H \cap Y)$ where G and H are open sets in (X, τ) whose intersections with Y are non-empty and disjoint, then Y is not connected.

Example 1.1. The subset Q of all rationals in real number space with usual topology is disconnected.

Solution : If $x, y \in Q$ with $x < y$, take an irrational number α such that $x < \alpha < y$.

Then we write $Q = \{(-\infty, \alpha) \cap Q\} \cup \{(\alpha, \infty) \cap Q\}$

Where r.h.s. members are each open sets relative to sub-space Q such that each is non-empty because $x \in (-\infty, \alpha) \cap Q$ and $y \in (\alpha, \infty) \cap Q$ and they are disjoint. Hence Q is disconnected.

Remark 1. : Above decomposition of Q is a disconnection for Q. By a different choice of α one gets another disconnection for Q. As the choices are many, there are many disconnections for Q.

Remark 2. : By a similar reasoning one sees that the set E of all irrationals in the real number spece R with usual topology is also disconnected, and there are many disconnections for E.

Remark 3. : We shall presently see that the real number space R with usual topology is a connected space. Thus we at once conclude that **Union of two disconnected sets may be a connected set.**

Theorem 1.2. If A and B are two non-empty separated sets in (X, τ), then $A \cup B$ is disconnected.

Proof : Let A and B be a pair of separated sets (non-empty) in (X, τ). Then we have $A \cap \overline{B} = \phi = \overline{A} \cap B$. Put $G = (X \setminus \overline{B})$ and $H = (X \setminus \overline{A})$.

Then G and H are disjoint open sets of X such that

$A \cup B = ((A \cup B) \cap G) \cup ((A \cup B) \cap H)$, which is a disconnection for $(A \cup B)$.

Theorem 1.3. A Subset Y of (X, τ) is disconnected if and only if Y is a union of two non-empty separated sets.

Proof : The condition is sufficient : This part follows from Theorem 1.2.

The condition is necessary : Let Y be a dis connected subset of (χ, τ). So Y admits of a partition like

$$Y = (Y \cap G) \cup (Y \cap H) \qquad (*)$$

Where G and H are open sets in X such that their intersections with Y are non-empty and disjoint. We check that members in r.h.s. of (*) are separated sets. It suffices only to verify that neither $(Y \cap G)$ nor $(Y \cap H)$ contains a limit point of the others. We use method of contradiction and let u be a limit point of $(Y \cap G)$ and $u \in (Y \cap H)$. Since H is an open set containing u, so

$$H \cap ((Y \cap G) \setminus \{u\}) \neq \phi \quad \text{. . . (1)}$$

But $(Y \cap G) \cap H = (Y \cap G) \cap (Y \cap H) = \phi$. by hypothesis . . . (2)

Now (1) and (2) are contradictory, and Theorem is proved.

Example 1.2. The Union $(0, 1) \cup (3, 4)$ is not a connected set of reals with usual topology.

Solution : Take a real number a such that $1 < a < 3$.

If $E = (0, 1) \cup (3, 4)$, we may write

$$E = ((-\infty, a) \cap E) \cup ((a, \infty) \cap E)$$

Where r.h.s. members are each non-empty open sets in E (with relativised topology) and they are disjoint. Hence E is disconnected.

Remark : We shall soon see that **every interval** of reals is **a connected set of reals** with usual topology; and Example 1.2 says that **Union of two connected sets may not be connected**. However, we have the following Theorem.

Theorem 1.4. In (X, τ) if $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of connected sets such that $\bigcap_{\alpha \in \Delta} A_\alpha \neq \phi$, then their Union $= \bigcup_{\alpha \in \Delta} A_\alpha$ is connected.

Proof : Let us assume the contrary and let $A = \bigcup_{\alpha \in \Delta} A_\alpha$ be disconnected. We seek a contradiction. Suppose A has a disconnection :

$$A = (A \cap G) \cup (A \cap H),$$

Where G and H are two open sets in X such that each of $(A \cap G)$ and $(A \cap H)$ is non-empty and $(A \cap G) \cup (A \cap H) = \phi$. Since $A = \bigcup_{\alpha \in \Delta} A_\alpha$, we have

$$A_\alpha \subset (G \cup H) \quad \text{. . . (*)}$$

Now A_α lies entirely either in G or in H; . . . (*) otherwise, $(A_\alpha \cap G) \neq \phi$, $(A_\alpha \cap H) \neq \phi$ and $A_\alpha = (A_\alpha \cap G) \cup (A_\alpha \cap H)$ gives rise to a disconnection of

A_α. Thus statement in (*) is valid for each member A_α. Further, if $A_\alpha \subset G$ and $A_\beta \subset H$ with $\alpha \neq \beta$, we see that $u \in \bigcap_{\alpha \in \Delta} A_\alpha \subset (A_\alpha \cap A_\beta) \subset (G \cap H)$. Ofcourse, $u \in A$. Hence $u \in A \cap (G \cap H) = (A \cap G) \cap (A \cap H)$— which is a contradiction. Therefore, members A_α enblock lie either in G or in H, and hence $\bigcup_{\alpha \in \Delta} A_\alpha = A$ lies either in G or in H—again a contradiction a desired. We have completed the proof.

Example 1.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{a, b, c\}, \{a, b\}\}$. Show that topological space (X, τ) is connected.

Solution : Here, the family of closed sets is $\{X, \phi, \{d\}, \{c, d\}, \{b, c, d\}\}$ and we check that only clo-open sets are ϕ and X. Therefore (X, τ) is connected.

Example 1.2. Let (X, τ) be a topological space where $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Examine if (X, τ) is connected.

Solution : Here $X = \{a\} \cup \{b, c\}$ which is a disconnection for X, and hence (X, τ) is disconnected.

Theorem 1.5. In a topological space (X, τ) let A be a connected set in X and B be a subset in X satisfying $A \subset B \subset \overline{A}$, then B is connected.

Proof : Assume the contrary. Let B have a disconnection like

$$B = (B \cap G) \cup (B \cap H) \qquad \ldots (1)$$

Where G and H are open sets in X such that $(B \cap G)$ and $(B \cap H)$ are non-empty and

$$(B \cap G) \cap (B \cap H) = \phi \qquad \ldots (2)$$

Now $A \subset B$; from (1) we see that $A \subset (G \cup H)$.

Now $A \cap G \neq \phi$ and $A \cap H \neq \phi$ give a disconnection of A as $A = (A \cap G) \cup (A \cap H)$ —making A disconnected. Hence A lies entirely either in G or in H. Suppose, $A \cap H = \phi$. Because H is open, we have $\overline{A} \cap H = \phi$. That means $B \cap H = \phi$ which is again a contradiction. We thus have reached the desired contradiction and the proof is complete.

Corollary 1.1. : Closure of a connected set is connected in a Topological space.

Theorem 1.6. A Topological space (X, τ) is connected if and only if given any two distinct points in X there is a connected sub-space of X containing both.

Proof : The condition is necessary : Let (X, τ) be connected. Given any two distinct points in X, the space X itself takes care of them as desired.

The condition is sufficient : Suppose the condition holds, but X is disconnected. We derive a contradiction.

Let $X = C \cup D$ be a disconnections of X, where C and D are a pair of non-empty disjoint open sets in X. Take $c \in C$ and $d \in D$; so $c \neq d$ in X. By hypothesis there is a connected subspace G of X containing c and d. Clearly $G \subset X = C \cup D$; Because G is connected, either $G \subset C$ or $G \subset D$. Let $G \subset C$; then $c, d \in G \subset C$ and so $d \in (C \cap D) = \phi$. This is absurd. Hence we have proved theorem.

§2. Consider the real number space R with usual topology. Here intervals are of various types like $(a, b) = \{x \in R : a < x < b\}$, $[a, b) = \{x \in R : a \leq x < b\}$. Similarly $(a, b]$, $(-\infty, a) = \{x \in R : x < a\}$; $(a, \infty) = \{x \in R; x > a\}$, and similarly $(-\infty, a]$, $[a, \infty)$ and $(-\infty, \infty) = \{x : x \in R\} = R$. An interval I of R may be characterised by the property :—

$$a, b \in I \text{ means the closed interval } [a, b] \subset I.$$

Theorem 2.1. A subset of R with usual topology is connected if and only if it is an interval.

Proof : The condition is necessary : Suppose E is a connected subset if R without being an interval. Then we find a pair of distinct members $a, b \in E$

such that $[a, b] \not\subset E$. Thus there is a member u such that $a < u < b$ and $u \notin E$.. Then write, $E = ((-\infty, u) \cap E) \cup ((u, \infty) \cap E)$ and that is a disconnection for E —a contradiction. Hence necessary part is proved.

The condition is sufficient : Let I be an interval of reals. If possible, let I have a disconnection

$$I = A \cup B,$$

where A and B are a pair of **non-empty disjoint closed sets in I.**

Take $x \in A$ and $z \in B$; since $A \cap B = \phi$ we have $x \neq z$, and without loss of generality, assume $x < z$. Because I is an interval we have the closed interval $[x, z] \subset I$. Thus

$$[x, z] \subset (A \cup B)$$

Put $y = \sup([x, z] \cap A)$. Then $x \leq y \leq z$; so $y \in I$. Since A is closed in I, we have

$$y \in A \qquad \ldots (1)$$

Therefore, $y \neq z$ and we have $y < z$.

By property of supremum, for large natural numbers n, all numbers $y + \frac{1}{n}$ belong to B; and since B is closed, passing on limit as $n \to \infty$, we have $\lim_{n \to \infty}\left(y + \frac{1}{n}\right) = y \in B$; this contradicts (1) because $A \cap B = \phi$. The proof is now complete.

Corollary : 1. The real number space R with usual topology is connected.

2. The only non-empty clo-open set in R is R itself.

§3. Theorem 3.1. Continuous image of a connected space is connected.

Proof : Let (χ, τ) be a connected space and $f : (X, \tau) \to (Y, \mu)$ be a continuous function where (Y, μ) is topological space. We show that f(x) is connected in Y. If possible, let f(X) be disconnected and let

$$f(X) = (G \cap f(X)) \cup (H \cap f(X))$$

Where G and H are open sets in Y such that $G \cap f(X)$ and $H \cap f(X)$ are a pair of non-empty disjoint open sets in f(X).

But continuity of f we know that $f^{-1}(G)$ and $f^{-1}(H)$ are open sets in X, and from (1) we find

$$X = f^{-1}(G) \cup f^{-1}(H),$$

Where members on r.h.s. are non-empty and disjoint. That means X is disconnected —a contradiction that X is connected. The proof is now complete.

Corollary : 3.1. Let (X, τ) and $(Y, \mathcal{V})$ be two Homeomorphic spaces. If X is connected, then Y is connected.

3.2. Any real valued continuous function over a closed interval [a, b] of reals possesses Intermediate-value property.

Because, f[a, b] = image of [a, b] under f is connected, and hence it is an interval of reals, by theorem 2, f(a), f(b) ε f [a, b]; and if f(a) < f(b), we have the closed inverval $[f(a), f(b)] \subset f[a, b]$.

Now if $f(a) < \mu < f(b)$, there is a member c between a and b such that $\mu = f(c)$.

Remark : In Corollary 3.2 domain of the continuous function is taken to be a closed interval. Statement is valid in respect of any interval.

Let us consider a topological space consisting of two mambers 0 and 1 with discrete topology. This discrete two pointic space {0, 1} is disconnected, because $\{0, 1\} = \{0\} \cup \{1\}$ is a disconnection. One can characterise a disconnected space with the help of this discrete space {0, 1}.

Theorem 3.2. A space (X, τ) is disconnected if and only if there is a continuous function $f : (X, \tau) \rightarrow \{0, 1\}$, which is onto.

Proof : The condition is necessary : Let (X, τ) be disconnected. X admits of a decomposition like $X = G \cup H$, where G and H are non-empty disjoint open sets in X.

Define $f : X \to \{0, 1\}$ by the rule :—

$$f(x) = 0 \text{ if } x \varepsilon G$$

and $\quad = 1 \text{ if } x \varepsilon H.$

Clearly f is an onto function. To check continuity of f we see that ϕ, $\{0, 1\}$, $\{0\}$ and $\{1\}$ are the only open sets in the discrete space $\{0, 1\}$ and that $f^{-1}(0)$ is always an open set in X, 0 being any open set in $\{0, 1\}$.

The condition is sufficient : Suppose the condition holds. Then X must be disconnected; because continuous image of a connected space is connected and here the discrete space $\{0, 1\}$ is disconnected.

Theorem 3.3. Topological product of two connected spaces is connected and conversely.

Proof : Let (X, τ) and (Y, v) be two topological spaces, and let $Z = X \times Y$ denote the topological product of X and Y with product topology. Suppose Z is connected. Then consider the projection functions $p_{r_1} : Z \to X$ and $p_{r_2} : Z \to Y$ that are each continuous; and $p_{r_1}(Z) = X$ and $p_{r_2}(Z) = Y$. Since continuous images of connected spaces ae connected it follows that each of X and Y is connected.

Conversely, assume that X and Y are connected spaces. Take a fixed memeber $y_0 \varepsilon Y$, and put $X_{y_0} = X_x\{y_0\}$. Then X and X_{y_0} are homeomorphic. Since X is connected it follows that X_{y_0} is connected. If $x \varepsilon X$, put $Y_x = \{x\} \times Y$. By a similar reasoning Y_x is connected, because Y is connected. As $(x_1 y_0) \varepsilon X_{y_0} \cap Y_x$, it follows that $X_{y_0} \cup Y_x$ is connected by Theorem 1.4. Finally, we write $X \times Y = \bigcup_{x \varepsilon X} (X_{y_0} \cup Y_x)$ and observe that $X_{y_0} \subset (X_{y_0} \cup Y_x)$ for every member $x \varepsilon X$, and because every individual member of r.h.s. is connected we finally see that $X \times Y$ is connected.

Remark : Theorem 3.3 remains valid for an arbitrary number of connected spaces.

Definition 3.1. A connected set C of (X, τ) is said to be a **component** in X if it is a **maximal connected sub set**; that is, C is not properly contained in any other connected set of X.

For example, in a **connected space (X, τ), X is itself a component.**

Example 3.1. Let (X, τ) be a topological space where $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. **Find all the components in X.**

Solution : Here X is not connected; because $X = \{a\} \cup \{b, c, d, e\}$ is a disconnection of X. However, there are connected sets in X. For example $\{a\}$ is a connected set; and we find $\{a\}$ and $\{b, c, d, e\}$ are the only components in X.

Theorem 3.4. Let (X, τ) be a Topological space. Then—

(a) Each point of X is contained in a component of X.

(b) The components of X determine a partition for X.

(c) Each connected set in X is contained in a component of X.

(d) A connected set in X that is both open and closed is a component of X.

(e) Each component of X is closed.

Proof : (a) Let $x \in X$. Put $C_x = \cup \{C_i : C_i$ is a connected set in X containing x$\}$. Because $x \in C_i$ for every i, it follows that C_x is a connected set. We now show that C_x is maximal. Let $C_x \subset D$ where D is a connected set containing x in X. By construction of C_x we have $D \subset C_x$; and therefore, $C_x = D$. Hence C_x is a component in X containing x.

(b) For each $x \in X$, construct C_x as in (a). Put $\zeta = \{C_x : x \in X\}$. We now verify that ζ becomes a partition of X. By construction, $X = \bigcup_{x \in X} C_x$. Now suppose $C_x \cap C_y \neq \phi$. Take $p \in C_x \cap C_y$, then we have $C_x \subset C_p$ and $C_y \subset C_p$. Now C_x and C_y are each connected set containing p, by maximality of C_p we have $C_p = C_x = C_y$. Therefore any two members of ζ are either disjoint or coincident. Hence (b) follows.

(c) Let A be connected set in X and take $x \varepsilon A$; by construction of C_x we find $A \subset C_x$.

(d) Let E be a connected set in X and let E be both open and closed. Now by (c) there is a component C of X such that $E \subset C$. Suppose E is a proper subset of C. Then we write $C = (C \cap E) \cup (C \cap E^c)$, 'c' denoting coplement of E. Because E is assumed to be clo-open, this decomposition is a disconnection of C— a contradiction, for a component C is connected. Hence result is E = C = a component.

(e) Let C be a component of X without being closed and $\overline{C}$ is strictly larger than C; Now $C \subset \overline{C}$ shows that $\overline{C}$ is connected because C is so— by corollary 1.1 that contradicts maximality of C which is a component. Hence C is closed.

Example 3.2. Give an example of a Topological space where components are not open.

Solution : Consider the sub-space Q of all rationals in real number space R with relativised topology in respect of usual topology on R. Here each component in Q is a singleton and this is not an open set in Q.

§4. Definition 4.1. : A topological space (X, τ) is said to be totally disconnected if for each pair of distinct points x and y in X, X has a disconnection $X = G \cup H$ with $x \varepsilon G$ and $y \varepsilon H$.

Explanation : A totally disconnected space (X, τ) is a T_2-space; because each pair of distinct points of X attracts a disjoint pair of open sets containing them individually. Ofcourse, a totally disconnected space is disconnected.

Example 4.1. The real number space R with upper limit Topology is totally disconnected.

Solution : We know that upper limit Topology for R is generated by left-open intervals like $(a, b]$, $a, b \in R$ with $a < b$. Let $x, y \in R$ with $x \neq y$, $x < y$; then we write

$$R = (-\infty, x] \cup (x, \infty),$$

where sets in r.h.s. are a pair of disjoint open sets in upper limit Topology containing x and y respectively. So R is totally disconnected.

Remark : With respect to usual topology R is a connected space.

Theorem 4.1. The components of a totally disconnected space are its singletons.

Proof : Let X be a totally disconnected space and let C be a component in X. We show that C does not have more than one point. Let $x, y \in C$ with $x \neq y$; as X is totally disconnected, X has a disconnection like $X = G \cup H$ where G, H are non-empty open disjoint sets with $x \in G$ and $y \in H$.

We write $C = C \cap X$

$$= C \cap (G \cup H) = (C \cap G) \cup (C \cap H),$$

showing that C is disconnected which is not the case. Hence theorem is proved.

Theorem 4.2. The product of totally disconnected spaces as totally disconnected in product topology.

Proof : Let $\{(X_i, \tau_i)\}_{i \in \Delta}$ be a family of totally disconnected spaces and let $X = X\{X_i : i \in \Delta\}$ be the product space with product topology. Take two distinct points $x = (x_i)$, and $y = (y_i)$ in X. Therefore, for some index $i = i_0 \in \Delta$, we have $x_{i_0} \neq y_{i_0}$ in co-ordinate space X_{i_0} which is assumed to be totally disconnected. There we find two disjoint open sets (say) G_{i_0} and H_{i_0} in X_{i_0} such that $x_{i_0} \in G_{i_0}$ and $y_{i_0} \in H_{i_0}$ and $X_{i_0} = G_{i_0} \cup H_{i_0}$. Take $G = X\{G_i$: where $G_i = X_i$ for all i except i_0 in $\Delta\}$ and $H = X\{H_i$: where $H_i = X_i$ for all i except

i_0 in $\Delta\}$. Then we see that $x \varepsilon G$, $y \varepsilon H$ with $G \cap H = \phi$, and G and H are rendered open sets of X in product topology such that

$$X = G \cup H$$

$\therefore$ That means X is totally disconnected.

§5. Locally connected spaces :

Definition 5.1. **(a)** A topological space (X, τ) is said to be locally connected at $x \varepsilon X$, **if every nbd. of x contains a open connected nbd. of x.**

or equivalently, **if open connected nbds of x form a base for the nbd. system at x.**

(b) (X, τ) is said to be locally connected if it is locally connected at each point of X.

Explanation : Unlike the relationship of compactness and local compactness of a space, local connectedness neither implies connectedness nor is implied by connectedness of the space. we have following examples in support of our contention.

Example 5.1. If $X = (0, 1) \cup (2, 3)$ is taken as a topological space with usual topology of reals, then X is locally connected without being connected.

Solution : Taking a real number α with $1 < \alpha < 2$, we write $X = ((-\infty, \alpha) \cap X) \cup ((\alpha, \infty) \cap X)$, and this is a disconnection for X. So X is not connected.

On the other hand if $u \varepsilon X$, say, $0 < u < 1$, and given any nbd. N_u of u in X, we can find on open interval like $(u - \delta, u + \delta)$, $\delta > 0$ such that $(u - \delta, u + \delta) \subset N_u$; as an open interval of reals is connected, if follows that N_u contains an open connected nbd. of u, and X is locally connected at u. If $1 < u < 2$, then also similar conclusion holds.

Example 5.2. Take $X = A \cup B$ as a sub-space of the Eudidean 2-space with usual topology where

$$A = \left\{(x, y) : 0 < x \leq 1; \text{ and } y = \sin\frac{1}{x}\right\},$$

and $\quad B = \{(0, y) : -1 \leq y \leq 1\}$.

Then X is connected without being locally connected.

Solution : Consider a function $f : (0, 1] \to R^2$ where $f(x) = \left(x, \sin\frac{1}{x}\right)$, $0 < x \leq 1$.

Then f is continuous, and hence the image $A = f(0, 1]$ is connected, because the interval (0, 1] is connected. Now we check that $X = \overline{A}$ and therefore **X is connected** because $\overline{A}$ is so.

However, **X is not locally connected at (0, 1) ε X**. Because open circular disc centred at (0, 1) with radius, say, $= \frac{1}{4}$ does not contain any connected open set containing the point (0, 1).

Theorem 5.1. (X, τ) is locally connected if and only if components of each open subspace of X are openin X.

Proof : The condition is necessary : Let (X, τ) be locally connected, and let Y be an open sub-space of X. Suppose C is a component of Y. Take $x \varepsilon C$. Since X is locally connected at x, there is an open cunnected set U in X such that $x \varepsilon U \subset Y$. Now $x \varepsilon C \cap U$ where U and C are connected; therefore $C \cup U$ is connected and $C \cup U \subset Y$. Since C is a component, by maximality of C, we have $(C \cup U) = C$ or $U \subset C$. That is $x \varepsilon U \subset C$; as x is an arbitrary member of C, we conclude that C is open.

The condition is sufficient : Suppose the condition holds. Let $x \varepsilon X$, and let N_x be an open nbd. of x in X. Take C as a component such that $x \varepsilon C \subset N_x$. By assumed condition C is open; this shows that there is an open connected nbd. C of x such that $C \subset N_x$. Thus X is locally connected at x; otherwords, X is locally connected.

Example 5.3. Continuous image of a locally connected space may not be locally connected.

Solution : Take X = {0, 1, 2, 3, . . .} with discrete topologies and $Y=\left\{0,1,\frac{1}{2},\frac{1}{3},\cdots\right\}$ as a sub-space of reals with usual topology. Consider a function

$\phi : X \to Y$ where $\phi(0) = 0$, and $\phi(n) = \frac{1}{n}$, n = 1, 2, . . .

The function ϕ is an 1–1 and onto continuous function, such that X is locally connected; but Y = f(x) is not locally connected, because induced topology on Y is not discrete, but singletons in Y are connected.

EXERCISE - A

Short answer type Questions

1. Show that a topological space (X, τ) with indiscrete topology τ is connected.
2. If X has more than two members, show that (X, τ) with discrete topology τ is disconnected.
3. Give an example to show that connectedness is not a hereditary property.
4. Examine if the real number space R with lower limit topology is connected.
5. Show that the set of all irrational numbers with topology of reals is a disconnected set.
6. Give an example of a topological spece with a non-discrete topology where each singleton is a component.

   (Hint : Take the sub-space Q of all rationals with usual topology of reals).

EXERCISE - B

1. If a Hausdorff space (X, τ) has an open base whose members are closed, show that X is totally disconnected.

2. Give an example to show that product space of an arbitrary number of locally connected spaces may not be locally connected.

3. Show that image of a locally connected space X under a continuous function which is an open function is locally connected.

4. Show that a compact locally connected space has a finite number of components.

5. Show that components of a totally disconnected space X are singletons of X.

6. Given a topological space (X, τ), a binary relation ρ in X is defined by $x \rho y$ $(x, y \in X)$ holds if and only if x and y belong to a connected set in X. Show that ρ is an equivalence relation on X and verify that ρ-equivalent classes are all the components of X.

Unit 6 ❑ Uniform Spaces

(Symmetric sets and composition of sets in XxX; uniformity $\mathscr{U}$ in X, base, sub-base for $\mathscr{U}$, uniform space (X, $\mathscr{U}$), uniform topology $\tau_{\mathscr{U}}$ for X, T_2-property of $\tau_{\mathscr{U}}$; Interior and closure of A in terms of member of $\mathscr{U}$, uniformly continuous function, Product Uniformity, Uniform continuty of a metric in (X, $\mathscr{U}$)).

§1. Let X be a non-empty set. So XxX $\neq \phi$. If A $\subset$ (XxX), then A^{-1} is defined as $A^{-1} = \{(y, x) : (x, y) \varepsilon A\}$. So that $(A^{-1})^{-1} = A$. If A and B are two subsets of XxX, then their composition $A_0B = \{(x, z) \varepsilon (XxX) : (x, y) \varepsilon B$ and $(y, z) \varepsilon A$ for some $y \varepsilon X\}$.

Now A_0B may not be the same as B_0A; however, composition is associative i.e. $A_0(B_0C) = (A_0B)_0C$ and also $(A_0B)^{-1} = B^{-1}{}_0A^{-1}$. The set of all pairs (x, x) as $x \varepsilon X$ is called the Diagonal, often denoted by Δ, of XxX. Also if U $\subset$ (XxX) and K $\subset$ X, the set $U[K] = \{y \varepsilon X : (x, y) \varepsilon U$ for some $x \varepsilon K\}$.

In particular, if K is a singleton, say = {x} in X, we have

$$U[x] = \{y \varepsilon X : (x, y) \varepsilon U\}$$

With these preliminaries we are ready to proceed further.

Definition 1.1. A subset U of XxX is said to be symmetric if $U = U^{-1}$.

Explanation : Let X = The set R of all reals, and $U = \{(x, y) \varepsilon RxR : |x-y|<1\}$; then $U = U^{-1}$, hence U is symmetric because if a pair (x, y) satisfies $|x-y|<1$, then $|y-x| = |-(x-y)| = |x-y| < 1$; So $(x, y) \varepsilon U$ if and only if $(x, y) \varepsilon U^{-1}$. However, if

$$H = \{(x,y) \varepsilon RxR : y - x < 1\}$$
$$= \left\{(x,y) \varepsilon R^2 : y < x+1\right\}$$

Then H is no symmetric; for $(0, 2) \varepsilon H$, but $(-2, 0) \notin H$, although $(-2, 0) \varepsilon H^{-1}$. Geometrically, H denotes the lower half-space of R^2 bounded above by the line y = x + 1; and

$$H^{-1} = \{(x,y) \varepsilon R^2 : (y,x) \varepsilon H\}$$
$$= \{(x,y) \varepsilon R^2 : x < y+1\}$$
$$= \{(x,y) \varepsilon R^2 : y > x-1\}$$

= upper half - space bounded below by the line $y = x - 1$.

And further

$$H \cap H^{-1} = \{(x,y) \varepsilon R^2 : x-1 < y < x+1\}$$
$$= \{(x,y) \varepsilon R^2 : |x-y| < 1\} = U$$

= Open strip in R^2 bounded by the lines $y = x - 1$ and $y = x + 1$.

Theorem 1.1. If U and V are two subsets of XxX such that V is symmetric; then $V \circ U \circ V = U\{V[x] \times V[y]\} : (x,y) \varepsilon U$.

Proof : Here, $V \circ U \circ V$ = **the set of all pairs (u, v) such that (u, x) ε V, (x, y) ε U and (y, v) ε V for some x and some y.** Since V is symmetric; this is the collection of all (u, v) such that u ε V[x] and v ε V[y] for some (x, y) ε U. But u ε V[x] and v ε V[y] if and only if (u, v) ε V[x] × V[y], and hence $V \circ U \circ V$

$$= \{(u,v) : (u,v) \varepsilon V[x] \times V[y] \text{ for some } (x,y) \varepsilon U\}$$
$$= U\{V[x] \times V[y] : (x,y) \varepsilon U\}$$

Definition 1.2. A Uniformity $\mathscr{U}$ on X is a non-empty family of subsets of XxX satisfying the following conditions known as axioms of Uniformity :

(u.1) Each member of $\mathscr{U}$ contains the diagonal Δ;

(u.2) If u ε $\mathscr{U}$, then u^{-1} ε $\mathscr{U}$;

(u.3) If u ε $\mathscr{U}$, there is a member V ε $\mathscr{U}$ such that $V_o V \subset U$;

(u.4) If u and v are two members of $\mathscr{U}$, then $(U \cap V)$ ε $\mathscr{U}$, and

(u.5) If u ε $\mathscr{U}$, and $U \subset V \subset XxX$, then v ε $\mathscr{U}$.

If $\mathscr{U}$ is a Uniformity on X, then the pair (X, $\mathscr{U}$) is called a Uniform space.

Definition 1.3. (a) A sub-family $\mathscr{B}$ of a Uniformity $\mathscr{U}$ on X is called **a base** for $\mathscr{U}$ if and only if each member of $\mathscr{U}$ contains a member of $\mathscr{B}$.

(b) A sub-family $S\mathscr{B}$ of $\mathscr{U}$ is said to be a sub-base for $\mathscr{U}$ if and only if the family of all finite intersections of members of $S\mathscr{B}$ forms a base for $\mathscr{U}$.

Example 1.1. Every metric space is a Uniform space.

Solution : Let (X, d) be a metric space. For each +ve r let

$$V_{d,r} = \{(x, y) \varepsilon (XxX) : d(x, y) < r\}.$$

Then we note that $V_{d,r}^{-1} = V_{d,r}$; $V_{d,r} \cap V_{d,t} = V_{d,u}$ where $u = \min[r, t]$; and $V_{d,r} \circ V_{d,r} \subset V_{d,2r}$.

Now it is a routine exercise to verify that the family of all sets of form $V_{d,r}$ forms a base for a uniformity for X; and X becomes a Uniform space.

Analogous to base and sub-base for a Topology in X, we have the following Theorem that is easy to prove and the proof is left out as such.

Theorem 1.2. A family $\mathscr{B}$ of subsets of XxX is a base for a Uniformity for X if and only if

(a) Each member of $\mathscr{B}$ contains Δ;

(b) If $B \varepsilon \mathscr{B}$, then B^{-1} contains a member of $\mathscr{B}$;

(c) If $B \varepsilon \mathscr{B}$, there is a member C in $\mathscr{B}$ such that $C_o C \subset B$; and

(d) If B_1, $B_2 \varepsilon \mathscr{B}$, then there is a member $B_3 \varepsilon \mathscr{B}$ such that $B_3 \subset (B_1 \cap B_2)$.

Proof : Let (X, $\mathscr{U}$) be a Uniform space.

Define $\tau = \{G \subset X$: for each $x \varepsilon G$, there is a member $U \varepsilon \mathscr{U}$ such that $U[x] \subset G\}$. We now verify that τ is, indeed, a Topology in X. Definition says that union of members of τ is a member of τ. Let G, H be two members of τ; and let $x \varepsilon (G \cap H)$. So, there are members U and V in $\mathscr{U}$ such that $U[x] \subset G$ and $V[x] \subset H$. Therefore, $(U \cap V)[x] \subset (G \cap H)$; since $(U \cap V)$ is a member of $\mathscr{U}$, it follows that $G \cap H$ is a member of τ, and (X, τ) is a Topological space. This topology τ is rather abbreviated as $\tau_{\mathscr{U}}$ since it is being induced by the uniformity $\mathscr{U}$, and very often named as a Uniform Topology on X.

Theorem 1.3. Let (X, $\mathcal{U}$) be a Uniform space with uniform Topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$. If $A \subset X$, then

$\tau_{\mathcal{U}}$-IntA = {x ε A : there is a member U ε $\mathcal{U}$ such that $U[x] \subset A$}. (or, **simply, we write Int A** instead $\tau_{\mathcal{U}}$–Int A).

Proof : Let $B = \{x \in A : U[x] \subset A$ for some member $U \in \mathcal{U}\}$. We show that $B \in \tau_{\mathcal{U}}$. Take $x \in B$, then we find a member $U \in \mathcal{U}$ such that $U[x] \subset A$; Also we find a member $V \in \mathcal{U}$ such that $V_0 V \subset U$. If $u \in V[x]$ and $y \in V[u]$, we have $(u, y) \in V$ and $(x, u) \in V$; Therefore, $(x, y) \in V_0 V$ i.e. $y \in V_0 V[x]$.

$$\text{or, } V[u] \subset V \circ V[x] \subset U[x] \subset A.$$

Therefore $u \in B$. In otherword, $V[x] \subset B$. Hence B is open. Further, B contains every open subset of A, and consequently it is the Largest open subset of A i.e. B = Int A.

Remark 1. If $U \in \mathcal{U}$, $U[x]$ is a nbd. of x in $(X, \tau_{\mathcal{U}})$.

2. The family of all set $U[x]$ as U comes from $\mathcal{U}$ is a base for the nbd. system at x.

Theorem 1.4. Let (X, $\mathcal{U}$) be a Uniform space with a Uniformity $\mathcal{U}$ and $\tau_{\mathcal{U}}$ be the Uniform Topology on X induced by $\mathcal{U}$. If $A \subset X$, then $\tau_{\mathcal{U}}$-closure of A (or simply Closure of A)

$$= \cap \{U[A] : U \in \mathcal{U}\}.$$

Proof : A point $x \in \overline{A}$ (Closure of A w.r.t. $\tau_{\mathcal{U}}$) if and only if for each $U \in \mathcal{U}$, $U[x]$ cuts A non-vacuously. Now $U[x]$ intersects A non-vacuously if and only if $x \in U^{-1}[A]$.

Because, $U[x] \cap A \neq \phi$ iff $p \in U[x] \cap A$;

i.e. iff $p \in U[x]$ and $p \in A$;

i.e. iff $(x, p) \in U$ and $p \in A$;

i.e. iff $(p, x) \in U^{-1}$ and $p \in A$;

i.e. iff $x \in U^{-1}[A]$.

Since each member of $\mathcal{U}$ contains a symmetric member $\in \mathcal{U}$.

We conclude, $x \in \bar{A}$ iff $x \in U[A]$ for each $U \in \mathscr{U}$.

The proof is now complete.

Given a uniformity $\mathscr{U}$ in X, the uniform Topology $\tau_{\mathscr{U}}$ may be used to construct the product Topology for $X \times X$. Members of $\mathscr{U}$ have an intimate relation with this product Topology. We shall presently see that the family of all open symmatric members of $\mathscr{U}$ is a base for $\mathscr{U}$. To that end, we need the following Lemma :

Lemma 1.1. If $U \in \mathscr{U}$ there is a symmetric member $V \in \mathscr{U}$ such that

(i) $V \circ V \circ V \subset U$

and (ii) $V \circ V \circ V = \cup\{V[x] \times V[y] : (x, y) \in V\}$.

Proof : (i) By axion of Uniformity we find a member $H \in \mathscr{U}$ such that

$$H \circ H \subset U \qquad \text{..... (1)}$$

Take $W = H \cap H^{-1}$; so W is a symmetric member of $\mathscr{U}$ such that

$$W \circ W \subset H \circ H \subset U \text{ from (1).}$$

Taking W in place of U, we obtain a symmetric member $V \in \mathscr{U}$ such that $V \circ V \subset W$; therefore $V \circ V \circ V \circ V \subset W \circ W$; and $V \circ V \circ V \subset V \circ V \circ V \circ V \subset W \circ W \subset U$ from above.

Theorem 1.1. now applies; we take $U = V$ in Theorem 1.1 and find atonce that $V \circ V \circ V = \cup\{V[x] \times V[y] : (x, y) \in V\}$.

This is (ii), and the proof of the Lemma is complete.

Theorem 1.5. If $U \in \mathscr{U}$, then Int $(U) \in \mathscr{U}$; and the family of all open symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$.

Proof : If $E \subset X \times X$, we have Int $E = \{(x, y) : U[x] \times V[y] \subset E$ for some $U, V \in \mathscr{U}\}$. By axion of University $(U \cap V) \in \mathscr{U}$; so

Int $E = \{(x, y) : V[x] \times V[y] \subset E$ for some $V \in \mathscr{U}\}$. Now Lemma 1.1 says that there is a symmetric member $V \in \mathscr{U}$ such that $V \circ V \circ V \subset U$, and

$V \circ V \circ V = \cup\{V[x] \times V[y] : (x, y) \in V\}$. Hence every point of $V \in$ Int U i.e. $V \subset$ Int U, and by Uniformity axiom Int $U \in \mathscr{U}$, and in consequence the family of all open symmetric members of $\mathscr{U}$ is a base for $\mathscr{U}$.

By Theorem 1.4 for $x \in X$ we have **$\tau_{\mathscr{U}}$-closure $\{x\} = \cap \{U[x] : U \in \mathscr{U}\}$. So $(X, \tau_{\mathscr{U}})$ is T_2 (Hansdorff) if and only if $\cap \{U : U \in \mathscr{U}\}$ is equal to Diagonal Δ.** In that case $(X, \tau_{\mathscr{U}})$ is also said to be separated.

§2. Let $(X, \tau_{\mathscr{U}})$ and $(Y, \tau_{\mathscr{V}})$ be two uniform spaces with uniform topologies $\tau_{\mathscr{U}}$ and $(X, \tau_{\mathscr{V}})$ respectively induced by given uniformities $\mathscr{U}$ and $\mathscr{V}$, and let $f : (X, \tau_{\mathscr{U}}) \to (Y, \tau_{\mathscr{V}})$ be a function.

Definition 2.1. $f : X \to Y$ is said to be uniformly continuous if and only if for each member $V \in \mathscr{V}$, the set $\{(x, y) \in X \times X : (f(x), f(y)) \in V\}$ is a member of $\mathscr{U}$.

The above statement may be re-phrased like :-

If $S\mathscr{B}(\mathscr{V})$ is a subbase for $\mathscr{V}$, then f is said to be uniformly continuous iff $f_2^{-1}(V) \in \mathscr{U}$ for each member $V \in S\mathscr{B}(\mathscr{V})$, where $f_2(x, y) = (f(x), f(y))$.

Explanation : Let X = R, and for each +ve r let $V_r = \{(x, y) \in R \times R : | x - y | < r\}$; Then the family $\{V_r\}_{r>0}$ of subsets of $R \times R$ forms a base for a Uniformity $\mathscr{U}$ known as usual uniformity for reals. So, the induced uniform topology $\tau_{\mathscr{U}}$ shall consist of members like $I_r = V_r[x] = \{ y \in R : (x, y) \in V_r\} = \{y \in R : | x - y | < r\} = \{y \in R : x - r < y < x + r\}$ = an open interval $(x - r, x + r)$ as $x \in R$; These members act as basic open set in $\tau_{\mathscr{U}}$ — confirming that $\tau_{\mathscr{U}}$ is the usual topological of reals. So, as per Definition 2.1 above, a real-valued function f of a real variable is uniformly continuous if given a $\in > 0$, there is a +ve δ such that $| f(x) - f(y) | < \in$ whenever $| x - y | < \delta$. This is in agreement with usual and familiar notion of uniform continuity of f. We also know that uniform continuity of f implies its continuity. The same is also true in a general uniform space.

Theorem 2.1. Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be two uniform spaces with induced uniform topology $\tau_{\mathscr{U}}$ and $\tau_{\mathscr{V}}$ respectively, then every uniformly continuous function : $X \to Y$ is continuous relative to uniform Topology.

Proof : Let $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous function ; Take H to be a nbd. of f(x) in $(Y, \tau_{\mathscr{V}})$, $x \in X$. So we find a member V $\in \mathscr{V}$ such that

$V[f(x)] \subset H$,

and $f^{-1}\{V[f(x)]\}$

$= \{y \in X : f(y) \in V[f(x)]\}$

$= \{y \in X : (f(x), f(y) \in V\}$

$= f_2^{-1}(V) [x]$ where $f_2(x, y) = (f(x), f(y))$

$=$ a nbd. of x in $(X, \tau_{\mathscr{U}})$.

That means $f^{-1}(H)$ becomes a nbd. of x, and f is rendered continuous at $x \in X$. Since x is any arbitrary member of X, we have proved Theorem.

§3. Suppose for each member α in an Index set Δ, $(X_\alpha, \mathscr{U}_\alpha)$ is a uniform space ; then **the product Uniformity for $X\{X_\alpha : \alpha \in \Delta\}$ is the smallest Uniformity such that projection functions from the Product $X\{X_\alpha : \alpha \in \Delta\}$ into each co-ordinate space $(X_\alpha, \mathscr{U}_\alpha)$ are uniformly continuous.**

The family of sets of the form $\{(x, y) : (X_\alpha, Y_\alpha) \in U\}$, for $\alpha \in \Delta$ and $U \in \mathscr{U}_\alpha$ forms a sub-base for the product Uniformity. If $x \in X\{X_\alpha : \alpha \in \Delta\}$, then sub-basic members of the nbd. system at x (with respect to the Product Uniform Topology), shall be obtained from the sub-basic members for the Product uniformity. Thus the family of all sets like $\{y : (X_\alpha, Y_\alpha) \in U\}$ becomes a sub-base for the Nbd. system at x. Clearly, a base for the Nbd. system at x with respect to induced topology from the product Uniformity is the family of all finite intersections of sets of the form $\{Y : Y_\alpha \in U[X_\alpha]\}$ for $\alpha \in \Delta$ and $U \in \mathscr{U}_\alpha$. But this family is also a base for the Nbd. system at x with respect to the product Topology; and therefore the product topology is the topology of the product Uniformity.

Theorem 3.1. A function f on a Uniform space to a product of Uniform spaces is uniformly continuous if and only if composition of f with each projection function into a Co-ordinate space is uniformly continuous.

Proof : If f is a uniformly continuous function with values in the product $X\{X_\alpha : \alpha \in \Delta\}$, then each projection function p_{r_α} is uniformly continuous, and we know that composition $p_{r_\alpha} \circ f$ is again uniformly continuous.

Conversely, if each $p_{r_\alpha} \circ f$ is uniformly continuous for each $\alpha \in \Delta$, and if U is a member of $\mathcal{U}_\alpha$ in X_α, then $\{(u, v) : (p_{r_\alpha} \circ f(u), p_{r_\alpha} \circ f(v)) \in U\}$ is a member of the uniformity $\mathcal{V}$ of domain f. Now we write this set as f_2^{-1} $[\{(x, y) : (x_\alpha, y_\alpha) \in U\}]$. So, inverse under f_2 of each member of a sub-base for the product Uniformity is a member of $\mathcal{V}$, and therefore f becomes uniformly continuous.

Theorem 3.2. Let $(X, \mathcal{U})$ be a Uniform space and let d be a metric for X. $d : X \times X \to R$ is uniformly continuous if and only if $\{(x, y) : d(x, y) < r\}$ is a member of $\mathcal{U}$ for each $r > 0$.

Proof : For each $r > 0$, take $V_{d,r} = \{(x, y) : d(x, y) < r\}$. It suffices to show that $V_{d,r} \in \mathcal{U}$ if and only if d is uniformly continuous. Let $U \in \mathcal{U}$, then sets $\{(x, y), (u, v)) : (x, u) \in U\}$ and $\{((x, y), (u, v)) : (y, v) \in U\}$ belong to the product uniformity, and we find that the family of sets of form $\{((x, y), (u, v)) : (x, u) \in U$ and $(y, v) \in U\}$ is a base for the product Uniformity. If d is uniformly continuous, then for each $r > 0$, there is $U \in \mathcal{U}$ such that if (x, u) and (y, v) belong to U, then $| d(x, y) - d(u, v) | < r$. Say, in particular $(u, v) = (y, y)$, then it follows that if $(x, y) \in U$, then $d(x, y) < r$. Hence $U \subset V_{d,r}$ and therefore $V_{d,r} \in \mathcal{U}$.

For converse part, if both (x, u) and (y, v) belong to $V_{d,r}$ then $| d(x, y) - d(u, v) | < 2r$ because $d(x, y) \le d(x, u) + d(u, v) + d(y, v)$ and $d(u, v) \le d(x, u) + d(x, y) + d(y, v)$. It follows that if $V_{d,r} \in \mathcal{U}$ for each +ve r, then d is uniformly continuous.

Theorem 3.2. Opens the gate to develop relation between Uniformities and metries (or pseudometrics). The reader may see the Literature as in Kelley's book in Chapter of Uniform spaces.

EXERCISE - A

Short answer type questions

1. Construct a Uniformity $\mathcal{U}$ for the space R of reals to induce the usual topology for R.

2. Show that a metric space is a uniform space.

3. Over a non-empty set X obtain (a) the largest uniformity and (b) the smallest uniformity for X.

EXERCISE - B

1. Show that the family of closed symmetric members of a Uniformity $\mathcal{U}$ is a base for $\mathcal{U}$.

2. Describe the product Uniformity in the product $X\{X_\alpha : \alpha \in \Delta\}$ where each $(X_\alpha, \mathcal{U}_\alpha)$ as $\alpha \in \Delta$ is a uniform space.

3. Prove that a continuous function of a compact Uniform space into a Uniform space is uniformly continuous.

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1. General Topology	:	Kelley
2. Topology	:	Dugundji
3. Topological Structures	:	Thron
4. General Topology	:	Adhikari, Chatterjee, Ganguly
5. Modern Analysis	:	G. F. Simons
6. General Topology	:	K. K. Jha

Author

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Functional Analysis

UNIT 1

(***Contents*** : Metric spaces, metric Topology, convergent and Cauchy sequences, completeness, metric space of all real sequences, complete metric spaces l_p, $C[a,b]$; Metric sub-spaces, separable metric space, continuous functions, Homeomorphism, Isometry, Compact metric spaces, Sequential compactness, Arzela-Ascoli Theorem)

§ 1.1 METRIC SPACES :

Let X be a non-empty set; so the Cartesian product $X_x X$ of all ordered pairs (x, y) of elements $x, y \in X$ is also non-empty.

Definition 1.1.1. A function $d : X_x X \to R$ (reals) is called a metric or a distance function over X if it satisfies following conditions, known as metric or distance axioms :

(M.1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$. (Property of non-negativity),

(M.2) $d(x, y) = d(y, x)$ for all $x, y \in X$. (Property of symmetry).

(M.3) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y and $z \in X$. (Property of triangle inequality).

If d is a metric on X, then the pair (X, d) is called a metric space. In a metric space (X, d) if $x_0 \in X$ and r is *a* +ve real, we have

Definition 1.1.2. The subset $\{x \in X : d(x_0, x) < r\}$ of X denoted by $B_r(x_0)$ is called an open ball in X, centred at x_0 with radius $= r$.

For example, if $d(x, y) = |x - y|$ for any two reals $x, y \in R$, then (R, d) is a metric space and for $x_0 \in R$ and r any +ve r, open ball $B_r(x_0) = \{x \in R : |x - x_0| < r\}$

$= \{x \in R : x_0 - r < x < x_0 + r\}$

= an open interval $(x_0 - r, x_0 + r)$ with mid point x_0 and length $= 2r$.

Similarly, in the metric space $¢$ of all complex numbers with usual metric we find an open ball $B_r(z_0)$ looks like an open circular disc with centre at $z_0 \in ¢$ having radius $= r$.

Definition 1.1.3. The subset $\{x \in X : d(x_0, x) \leq r\}$ of a metric space (X, d) is called a closed ball centred at x_0 with radius $= r$.

The subset $\{x \in X : d(x_0, x) = r\}$ of X is called a sphere centred at x_0 with radius = r. It is also called boundary (Bdr) of open (closed) ball centred at x_0 having radius = r.

The open balls in a metric space (X, d) form a base for a Topology, called metric Topology τ_d (induced by the metric d) on X. So every metric space (X, d) is a topological space with metric topology τ_d. This metric topology τ_d is Hausdorff (T_2).

Definition 1.1.4. A sequence $\{x_n\}$ in (X, d) is said to be a convergent sequence if there is a member $u \in X$ such that, $\lim_{n\to\infty} d(u, x_n) = 0$.

Or, equivalently, given any +ve ε, there is an index N such that $d(u, x_n) < \varepsilon$, when $n \geq N$.

If $\{x_n\}$ is a convergent sequence in (X, d) with $u \in X$ and $\lim_{n\to\infty} d(u, x_n) = 0$, we write $\lim_n x_n = u \in X$, and u is a unique member of X, because metric space is Hausdorff.

Definition 1.1.5. A sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, d) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Or, equivalently, given any +ve ε, there is an index N satisfying $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

It is an easy exercise to see that in a metric space every convergent sequence is cauchy, but converse is false.

Definition 1.1.6. A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) is convergent in X.

For example, real number space R with usual metric $d(x,y) = |x - y|;\ x, y \in R$ is a complete metric space. This is what is known as Cauchy's General Principle of convergence; and essentially by same reason the Euclidean n-space R^n consisting of all n tuples of reals like $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n),\ x_i \in R$ is also a complete metric space with usual/Euclidean metric d where $d^2(\underset{\sim}{x}, \underset{\sim}{y})$

$$= \sum_{i=1}^{n} |x_i - y_i|^2;\ \underset{\sim}{x} = (x_1, x_2, \ldots, x_n),\ \underset{\sim}{y} = (y_1, y_2, \ldots, y_n) \in R^n$$

Example 1.1.1. The collection S of all sequences of reals is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$, where $\underset{\sim}{x} = (\xi_1, \xi_2, \ldots\ldots),$

$\underset{\sim}{y} = (\eta_1, \eta_2, ...) \in S$. The r.h.s. series is convergent because each term is domoninated by a corresponding term of a convergent geometric series. Here is a routine exercise to see that metric axioms are all satisfied. For completeness part we remark on passing that if $a_{n,m} \geq 0$, then $a_{n,m} \to 0$ if and only if $\dfrac{a_{n,m}}{1+a_{n,m}} \to 0$ as $n, m \to \infty$.

Take $\{\underset{\sim}{x}_n\}$ as a Cauchy sequence of elements in S

where $\underset{\sim}{x}_n = (\xi_1^{(n)}, \xi_2^{(n)}, ... \xi_i^{(n)}, ...)$.

Corresponding to a + ve ε we find an index N such that

$$\rho(\underset{\sim}{x}_n, \underset{\sim}{x}_m) < \varepsilon \text{ for all } n, m \geq N$$

or, $$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(m)}|}{1+|\xi_i^{(n)} - \xi_i^{(m)}|} < \varepsilon \text{ for all } n, m \geq N \quad \text{.................................... (1.1.7)}$$

As individual term in series above is ≥ 0, we appeal to the remark made earlier to say that $|\xi_i^{(n)} - \xi_i^{(m)}| \to 0$ as $n, m \to \infty$. And hence for each co-ordinate i by Cauchy's General Principle of Convergence, $\{\xi_i^{(n)}\}$ is convergent.

Put $\lim_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, i = 1, 2, ...

Taking $\underset{\sim}{x}_0 = \left(\xi_1^{(0)}, \xi_2^{(0)}, ...\right)$ we find $\underset{\sim}{x} \in S$ and passing on limit as $m \to \infty$ in (1.1.7) we have

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(0)}|}{1+|\xi_i^{(n)} - \xi_i^{(0)}|} \leq \varepsilon \text{ for } n \geq N.$$

That means, $\lim_{n\to\infty} \rho(\underset{\sim}{x}_n, \underset{\sim}{x}_0) = 0$

or $\lim_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0 \in S$

So the sequence space S becomes a complete metric space.

Remark : The convergence of sequence of elements in S as shown above is known as co-ordinatewise convergence; that is to say, $\lim_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0$ in S,

where $\underset{\sim}{x}_n = \left\{\xi_i^{(n)}\right\}$ and $\underset{\sim}{x}_0 = \left\{\xi_i^{(0)}\right\}$, if and only if $\lim_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, for $i = 1, 2, 3, ...$; **The convergence is not necessarily uniform.**

Example 1.1.2. The sequence space l_p $(1 < p < \infty)$ consisting of all sequences $\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n, ...)$ of reals with $\sum_{i=1}^{\infty} |\xi_i|^p < +\infty$ is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p \right)^{1/p}$, for $\underset{\sim}{x} = (\xi_1, \xi_2, ...), \underset{\sim}{y} = (\eta_1, \eta_2, ...) \in l_p$.

Example 1.1.3. The function space $C[a,b]$ consisting of all real valued continuous functions over the closed interval $[a, b]$ is a complete metric space with sup metric $\rho(f, g) = \sup_{a \le t \le b} |f(t) - g(t)|$, as $f, g \in C[a,b]$

The last two examples appear in Book PGMT 2A. They are referred to there.

§ 1.2 SUB-SPACES :

Let Y be a non-empty subset of a metric space (X, d). There is a natural metric, namely the restriction d_Y of d to $Y \times y$.

Definition 1.2.1. The metric space (Y, dy) is called a sub-space of (X, d).

Theorem 1.2.1. A subset A in Y is open in (Y, dy) if and only if there is a subset A_1 in X that is open in (X, d) such that $A = Y \cap A_1$.

Proof : Let $x \in X$ and $y \in Y$ and r be a +ve number n and let $B_x(x, r)$ and $B_Y(y, r)$ denote open balls centred at x and at y respectively with radius $= r$ in (X, d) and in (Y, dy).

Then we have $B_Y(y,r) = Y \cap B_x(y,r)$ for all $y \in Y$, and $r > 0$ ....(1.2.1)

Take A as an open set in (Y, d_Y), then we know that A is a Union of some open balls of (Y, d_Y); say of $\{B_Y(y,r)\}$ as $y \in A$ and $r > 0$.

Thus $A = UB_Y(y,r)$

$= U\{Y \cap B_X(y,r)\}$ by (1.2.1)

$= Y \cap \{UB_X(y,r)\}$

$= Y \cap A_1$ (say)

where A_1 is a union of open balls in (X, d) and A_1 is an open set in (X, d).

Conversely, let $A = Y \cap A_1$, where A_1 is an open set in (X, d). For $y \in A$, there is an open ball $B_X(y,r) \subset A_1$, and hence $B_Y(y,r) = Y \cap B_X(y,r) \subset (Y \cap A_1) = A$. So every member of A attracts an open ball in (Y, d_Y) *i.e.* A is an open set in (Y, d_Y). The proof is complete.

Corollary : A is closed in (Y, d_Y) if and only if there is a subset A_1 of X that is closed in (X, d) such that $A = Y \cap A_1$. (If $A = Y \cap A_1$, we have $Y \setminus A = Y \cap (X \setminus A_1)$, and now proceed).

Definition 1.2.2. A metric space (X, d) is said to be separable if and only if there is a countable subset D of X such that D is dense in (X, d) (or equivalently, $\overline{D}$ (closure of D) = X).

For example, real number space R with usual metric is separable, because the set Q of all rationals if R is dense in R, where we know that Q is countable.

Theorem 1.2.2. A sub-space of a separable metric space is separable.

Proof : Let (Y, d_Y) be a sub-space of (X, d) which is a separable metric space. Let $A = \{x_1, x_2, ..., x_n, ...\}$ be a countable set in X such that $\overline{A} = X$.

If $y \in Y$, then for each +ve integer m the open ball $B\left(y, \frac{1}{m}\right)$ meets A at some point, say = x_n.

Thus $x_n \in \left\{A \cap B\left(y, \frac{1}{m}\right)\right\}$.

So, Open ball $B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi$

Put $\Delta = \left\{(n, m) : B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi\right\}$. Thus $\Delta \neq \phi$. For each $(n, m) \in \Delta$, take a member $y_{n,m} \in \left\{B\left(x_n, \frac{1}{m}\right) \cap Y\right\}$, and put $B = \{y_{n,m} : (n, m) \in \Delta\}$. Therefore B is a countable subset of Y because Δ is so. We now verify that B is dense in (Y, d_Y). Take $y \in Y$ and $r > 0$; choose +ve integer m so that $\frac{1}{m} \leq \frac{1}{2} r$. As said above there is an integer n such that $x_n \in B\left(y, \frac{1}{m}\right)$. Then $(n, m) \in \Delta$, and we have

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \leq r.$$

That means, $y_{n,m} \in B(y, r)$. Therefore $y \in \overline{B}$ in (Y, d_Y), or, B is dense in (Y, d_Y).

§ 1.3 CONTINUOUS FUNCTIONS :

Let (X, d) and (Y, ρ) be two metric spaces.

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be continuous at a point $c\in X$, if and only if given a +ve ε, there is a +ve δ (depending on ε and c) such that $e\left(f(x),f(c)\right)<\varepsilon$ whenever $d(x,c)>\delta$.

or equivalently, $f\left(B(c,\delta)\right)\subset B\left(f(c),\in\right)$.

f is said to be a continuous function if f remains continuous each point of X.

Further details on continuous functions over metric spaces may be seen in Book PGMT 2A.

Homeomorphism, Isometry :

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be a homeomorphism if f is 1–1, onto (bijective) and both f and f^{-1} are continuous functions.

If there is a homeomorphism between (X, d) and (y, ρ), then two metric spaces (X, d) and (y, ρ) are called homeomorphic.

Explanation : If f is a homeomorphism of X onto Y, then f^{-1} is so between Y and X. Also it is a routine matter to see that composition of two homeomorphisms is again a homeomorphism; thus in the family of all metric spaces the binary relation 'of being homeomorphic' is an equivalence relation.

Example 1.3.1. Consider the metric space R of reals with usual metric and a function $T:R\to R$ given by $T(x)=x+a$, where a is a fixed real number, and $x\in R$. Then this translation function (equals to Identity function when a = 0) is a homeomorphism; here $T^{-1}:R\to R$ is given by $T^{-1}(x)=x-a; x\in R$. Similarly one shows that for any non-zero real λ, multiplication function $M_\lambda:R\to R$ given by $M_\lambda(x)=\lambda x, x\in R$ is a homeomorphism, where $M_\lambda^{-1}=M_{\lambda^{-1}}$.

We know that family of all open sets in (X, d) forms a Topology, called metric topology τ_d on X induced by d. Any property in a metric space (X, d) that can be formulated entirely in terms of members of τ_d (open sets) is known as **a Topological property.**

Consequently, homeomorphic metric spaces have the same topological properties like convergence of sequences in the space and continuity of functions over the space. Following example shows **completness is not a topological property in a metric space.**

Example 1.3.1. Take $X=\{1, 2, 3, ...\}$ and $Y=\{1, \frac{1}{2}, \frac{1}{3}, ...\}$. Regarded as a subspace of the space R of reals with usual metric we find that spaces X and Y are discrete metric spaces (every subset being both open and closed); thus the function $h: X \to Y$ where $h(n)=n^{-1}$ is a homeomorphism of X onto Y. Since X is a closed subset of R which is a complete metric space, the space X is complete. On the other hand Y is not complete.

Definition 1.3.2. A function $f: Y \to Y$ that is onto (surjective) is said to be an Isometry if $e(f(x), f(y))=d(x, y)$ for all $x, y \in X$.

Explanation : Identity function on X is an Isometry of X onto itself. Also a transformation of rotation like $x'=x\cos\theta+y\sin\theta$, $y'=-x\sin\theta+y\cos\theta$ is an Isometry of Eudidean 2-space R^2 onto itself with usual metric. Also an Isometry is a homeomorphism. Thus two metric spaces that are isometric are indistinguishable in respect of their metric properties.

Example 1.3.2. In metric space (X, d) take $x_0 \in X$.

For $x \in X$, Let $f_x : x \to R$ (space of reals with usual metric) be given as

$$f_x(y)=d(y, x)-d(y, x_0) \text{ for } y \in X.$$

Then show that $x \to f_x$ is an isometry of X into $C(X)$ where $C(X)$ is metric space of all real valued continuous functions over X with sup metric

$$\|f-g\|=\sup_{y\in X}|f(y)-g(y)|<\infty.$$

As distance function d is continuous, it follows that f_x is continuous for all $x \in X$.

Solution : Take $u, v \in X$; so we have

$$\left.\begin{aligned} &f_u(y)=d(y, u)-d(y, x_0) \\ \text{and } &f_v(y)=d(y, v)-d(y, x_0) \quad \text{for all } y \in X \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(1.3.1)$$

So, $|f_u(y)-f_v(y)|=|d(y, u)-d(y, v)|\le d(u, v)$ which is independent of $y \in X$,

taking the $\sup\limits_{y\in X}$ over L.H.S. we obtain

$$\sup_{y\in X} | f_u(y) - f_v(y) | \le d(u,v)$$

or $\| f_u - f_v \| \le d(u,v)$ ....1.3.2)

Putting $y = u$ in (1.3.1) we have,

$$f_u(u) = -d(y, x_v) \text{ and } f_v(u) = d(u,v) - d(y, x_0)$$

So, $| f_u(u) - f_v(u) | = d(u,v)$

Now $\sup_{y\in X} | f_u(y) - f_v(y) | \ge | f_u(u) - f_v(u) | = d(u,v)$ .... (1.3.3)

from (1.3.2) and (1.3.3) we obtain

$$\| f_u - f_v \| = d(u,v).$$

Thus $x \to f_x$ invites an Isometry of X into $C(X)$.

§ 1.4 COMPACT METRIC SPACES :

Some important properties of reals as we encounter in real analysis had motivated more important concepts in a metric space like completeness and compactness. Cauchy's General Principle of Convergence is the driving force behind completeness in a metric space. Essence of Heine-Borel Theorem could be found in concept of compactness in a metric space.

In consequence, it had been an inevitable task with urgency to identify compact subsets in a metric space. Russian Mathematicians like Alexandrov and Urysohn had been responsible to put forward notion of compactness via 'open cover' in the space; on the other hand close to Bolzano-Weirstrass property is classical analysis concept of sequential compactness owed to Frechet in a metric space. And now we know for certain that these two routes are equivalent in describing compactness in a metric space. For details in this context see the book PGMT 2A.

It has been possible to discover that a subset in Euclidean n-space R^n with usual metric is compact if and only if the subset is a bounded and closed set in R^n.

Given a metric space X it is often hard to decide which subsets of X are compact, and which are not. Our present task is the job of identifying compact subsets of a very important and useful function space of some continuous functions that we presently discuss below. The concerned target theorem in this connection is Ascoli-Arzela Theorem.

Definition 1.4.1. Let (X, d) denote a metric space.

(a) A family $Q = \{A_1\}_{i \in \Lambda}$ of open sets A_i in (X, d) is said to be an open cover for X if every element of X belongs to at least one member A_i of the family Q. That is to say, $X \subset \underset{i \in \Delta}{U} A_i$.

(b) A sub-family of an open cover for X which by itself is an open cover for X is called sub-cover for X.

(c) (X, d) is said to be a compact metric space if every open cover for X has **a finite sub-cover for X.**

Explanation : By a finite sub-cover we mean that the sub-cover consists of a finite number of members only. Consider a family $\{(-n, n)\}_{n \in N}$ (N = set of all natural numbers). Its members are open intervals, and hence open sets in the metric space R of reals with usual metric. It is an open cover for R; because $R = \underset{n=1}{U}(-n, n)$. Clearly, this open cover possesses no finite sub-cover for R. That is why, R is not compact.

Definition 1.4.2. A subset G of (X, d) is said to be compact if as a sub-space of (X, d) it is compact under definition 1.4.1.

For example, although R is not compact with usual metric any finite subset of R becomes compact.

Definition 1.4.3. (X, d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence in X.

It is a bit lengthy exercise to conclude that a metric space is compact if and only if it is a sequentially compact. See book PGMT 2A.

The function space $C[a, b]$ of all real-vaheed continuous functions over a closed interval $[a, b]$.

We know that $C[a, b]$ is a complete metric space with respect to sub metric $\rho(f, g) = \sup_{a \le t \le b} |f(t) - g(t)|$, $f, g \in C[a, b]$. But $C[a, b]$ is not compact with respect to sub metric, because $C[a,b]$ is not bounded; for all constant functions like $f_n(t) = n$ for $a \le t \le b$ satisfy $\rho(f_n, 0) = n \to \infty$ as $n \to \infty$. However there are compact sets in $C[a, b]$. In searching then we need some Definitions.

Definition 1.4.1. (a) A subset M of $C[a, b]$ is said to be **uniformly bounded** if there is a +ve constant K such that $|x(t)| \le K$ for all t in $a \le t \le b$ and for all members $x \in M$.

(b) Subset M is said to be **equi-continuous** if given any +ve ε, there is a +ve δ (depending on ε only) such that $|x(t_1) - x(t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta (t_1, t_2 \in [a,b])$ for all members $x \in M$.

Example 1.4.1. Show that the subset $\{f_n\} \subset C[0,1]$ is equibounded where $f_n(t) = 1 + \frac{t}{n}; 0 \le t \le 1$.

Solution : Here $|f_n(t)| = |1 + \frac{t}{n}| \le 1 + |\frac{t}{n}| \le 1 + \frac{1}{n} \le 2$ for all n and for all t in $0 \le t \le 1$. So the conclusion stands.

Theorem 1.4.1 (Arzela-Ascoli Theorem) : A subset M of $C[a,b]$ is compact if and only if M is uniformly bounded and equi continuous.

Proof : The condition is necessary : Let M be a compact subset of $C[a,b]$ (*w.r.t.* sup metric). Then M is bounded, because a compact set in a metric space is bounded and closed. Thus we find a closed ball say $\overline{B}_r(x_0)$ centred at $x_0 \in C[a,b]$ with radius = r, such that

$$M \subset \overline{B}_r(x_0)$$

Thus $\sup\limits_{a \le t \le b} |x(t) - x_0(t)| \le r$

Now $x(t) = x(t) - x_0(t) + x_0(t)$ and

$$\sup_{a \le t \le b} |x(t)| \le \sup_{a \le t \le b} |x(t) - x_0(t)| + \sup_{a \le t \le b} |x_0(t)| \le r + k, \text{ say,}$$

where $k = \sup\limits_{a \le t \le b} |x_0(t)|$.

That means $|x(t)| \le (r + R) = K$ (say) for all t in $a \le t \le b$ and for all $x \in M$. Hence M is uniformly bounded.

For equi-continuity take a +ve ε.

Since M is compact, we find an $\frac{\varepsilon}{3} - net = \left(x_1(t), x_2(3), \ldots, x_n(t)\right)$ for M.

Since every real-valued continuous function over a closed interval is uniformly continuous. So here each of the members $x_1, x_2, \ldots x_k$ of $C[a,b]$ is uniformly continuous in $[a,b]$.

So, for each $x_i(t)$ we find a +ve δ_i such that

$$|x_i(t_1)-x_i(t_2)|<\frac{\varepsilon}{3} \quad \text{whenever } |t_1-t_2|<\delta_i,\ t_1,t_2\in[a,b].$$

Now take a +ve $\delta = \min_{1\le i\le k}\{d_i\}$. Then we have

$$|x_i(t_1)-x_i(t_2)|<\frac{\varepsilon}{3} \text{ whenever } |t_1-t_2|<\delta, t_1,t_2\in[a,b] \text{ for all } i=1,2,\ldots\ldots, k.$$

Now for every member $x\in M$, we find a member, say, x_i from $\frac{\varepsilon}{3}$–net, such that

$$\rho(x,x_i)<\frac{\varepsilon}{3} \quad (\rho = \text{sup-metric of } C[a,b]).$$

If $t_1,t_2\in[a,b]$ and $|t_1-t_2|<\delta$ we have

$$|x(t_1)-x(t_2)|\le x(t_1)-x_i(t_1)|+|x_i(t_1)-x_i(t_2)|+|x_i(t_2)-x(t_2)|$$

$$\le \sup_{a\le t\le b}|x(t)-x_i(t)|+|(x_i(t_1)-x_i(t_2)|+\sup_{a\le t\le b}|x_i(t)-x(t)|$$

$$<\rho(x,x_i)+\frac{\varepsilon}{3}+\rho(x,x_i)<\varepsilon.$$

This inequality holds for all $t_1,t_2\in[a,b]$, with $|t_1-t_2|<\delta$ and for all members $x\in M$. So M is equi-continuous.

The condition is sufficient : Suppose M is uniformly bounded and equi-continuous ; we show that M is compact. Because $C[a,b]$ is complete and so is M; It suffices to show that every sequence in M has a Cauchy subsequence. Let $D=(t_2, t_3, t_4, \ldots..)$ be a countable dense set of reals in $[a,b]$.

Suppose $S_1=(f_{11}, f_{12}, f_{13}, \ldots..)$ be any sequence of elements in M. By uniform boundedness property of M. We find a +ve K such that

$|f(t)|\le K$ for all t in $a\le t\le b$ and for all $f\in M$. .... (1.4.6)

Let us examine real sequence

$$\{f_{11}(t_2), f_{12}(t_2), f_{13}(t_2), \ldots, f_{1m}(t_2), \ldots\}$$

From (1.4.6) it is clear that this is a bounded sequence of reals and has a convergent subsequence.

Let $S_2=(f_{21}, f_{22}, f_{23}, \ldots\}$ be a sub-sequence of S_1 above such that $\{f_{21}(t_2), f_{22}(t_2), f_{23}(t_2), \ldots\ldots\}$ converges.

Now examine real sequence $\{f_{21}(t_3), f_{22}(t_3), f_{23}(t_3), ...\}$, and by similar reasoning as above, we have

$S_3 = \{f_{31}, f_{32}, f_{33}, ...\}$ as a subsequence of S_2 such that

$\{f_{31}(t_3), f_{32}(t_3), f_{33}(t_3), ...\}$ is convergent.

We continue this chain to construct $S_1, S_2, S_3, ...$ of sequences of functions like :

$$S_1 = \{f_{11}, f_{12}, f_{13}, ...\}$$

$$S_2 = \{f_{21}, f_{22}, f_{23}, ...\}$$

$$S_3 = \{f_{31}, f_{32}, f_{33}, ...\}$$

....

where S_m constitutes a subsequence of $S_{m-1} (m = 2, 3, ...)$ with the property that $\{f_{n1}(t_n), f_{n2}(t_n)\ f_{n3}(t_n), ...\}$ is a convergent sequence of reals.

Now put $f_n = f_{nn}$ $(n = 2, 3, 4, ...)$ then $\{f_1, f_2, f_3, ...\}$ is the diagonal subsequence of S_1. From mode of construction

$x_n \in D$ and $\{f_1(t_n), f_2(t_n), ..., f_i(t_n), ...\}$ is a convergent real sequence.

If $i > k$, consider $|f_{ii}(t_n) - f_{kk}(t_n)|$ for $i > k > n$ and knowing that both $f_{ii}(t_n), f_{kk}(t_n)$ are members of convergent real sequence

$$\{f_{n1}(t_n), f_{n2}(t_n), (f_{n3}(t_n), ...\}$$

We have $|f_i(t_n) - f_k(t_n)| \to 0$ as $i, k \to \infty$. Thus $\{f_1(t_n), f_2(t_n), f_3(t_n), ...\}$ is a Cauchy sequence of reals.

Finally, take any +ve ε. Since M is equi-continous and $S \subset S_1 \subset M$, we find a +ve δ such that $|f_n(t) - f_n(t') < \frac{\varepsilon}{3}$ whenever $|t - t'| < \delta$, $t, t' \in [a, b]$ for all members $f_n \in S$.

Now consider the family $\{t_n - \delta, t_n + \delta)\}$ of open intervals with mid point $t_n \in D$.

It is routine verification with dense property of D in $[a, b]$ that this family of open intervals becomes an open cover for $[a, b]$. By compactness of $[a, b]$ we obtain a finite sub-over, say

$$[a, b] = \underset{t_n \in D}{U} (t_n - \delta, t_n + \delta) \text{ and } 2 \le n \le n_0$$

Again $\{f_1(t_n), f_2(t_n), ...\}$ is Cauchy; thus a +ve integer K_0 is there such that

$$|f_i(t_n) - f_k(t_n)| < \frac{\varepsilon}{3} \text{ for all } 2 \le n \le n_0$$

If t is any position of $[a, b]$, we find n with $2 \le n \le n_0$ so that $t_n - \delta < t < t_n + \delta$ and for $i, K \ge K_0$ we have

$$|f_i(t) - f_k(t)| \le |f_i(t) - f_i(t_n)| + |f_i(t_n) - f_k(t_n)|$$

$$+ |f_k(t_n) - f_k(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

That means $\sup\limits_{a \le t \le b} |f_i(t) - f_k(t)| \le \varepsilon$ for $i, k \ge k_0$

or, $\rho(f_i, f_k) \le \in$ for $i, k \ge k_0$

or, $S = \{f_1, f_2, ...\}$ is a Cauchy subsequence of S_1.

The proof is now complete.

EXERCISE A

Short-answer type questions :

1. Show that compactness is not a heriditary property in a metric space.
2. Give an example to show that a closed bounded set in a metric space may not be compact.
3. Show that $f(x) = x + a$ or $= -x + a$ where a is a fixed real is an Isometry on the space R of reals with usual metric.
4. Prove that any bounded sequence of reals has a convergent subsequence.
5. In a metric space (X, d) if $\lim\limits_{n \to \infty} x_n = x \in X$, show that $\{x_n\} \cup \{x\}$ is copact.

EXERCISE B

Broad questions

1. Show that the closed ball $\tilde{B} = \left\{x : \sup\limits_{0 \le t \le 1} |x(t) \le 1\right\}$ of $C[0, 1]$ with supmetric is not compact.

2. Prove that only Isometries of the space R of reals with usual metric are $f(x) = x + a$ and $f(x) = -x + a$ where a is a real number.
3. Give an example of a Homeomorphism that is not an Isometry.
4. Let f be a real-valued function on a compact metric space (X, d), show that f assumes its maximum and minimum on X.
5. Verify that closed Unit ball in sequence space l_2 is bounded without being totally bounded.
6. Let X denote the metric space of all real polynomials $p(t)$ in $0 \leq t \leq 1$; show that X is not a complete metric space with respect to sup metric.

UNIT 2

(*Contents* : Linear spaces, Dimension of a linear space, Normed linear space (NLS), Banach space, $C[a,b]$ as a Banach space, Quotient space of a NLS, Convex sets, their algebra, Bounded linear operator; its continuity, Unbounded linear operator, Norm ||T|| of a bounded linear operator T; Formulae for ||T||.)

§ 2.1 LINEAR SPACES

Definition 2.1.1. Let R (q) denote the field of reals (complex numbers) that are also called scalars. A linear space (Vector space) V is a collection of objects called vectors satisfying following conditions :

I. V is additively an Abelian (commutative) Group, the identity element of which is called the Zero vector denoted by 0.

II. For every pair (α v), α being a scalar and $v \in V$, there is a vector, denoted by $\alpha.v$ (**not** $v\alpha$), called a scalar multiple of v such that

(a) $1.v = v$ for all $v \in V$.

(b) $\alpha.(u+v) = \alpha.u + \alpha.v$ for all scalars α and for all vectors $u, v \in V$.

(c) $(\alpha+\beta).v = \alpha.v + \beta.v$ for all scalars α and β and for all vectors $v \in V$.

(d) $\alpha.(\beta.v) = (\alpha.\beta).v$ for all scalars α and β and for all $v \in V$.

Example 2.1.1. Let R^n be the collection of all n tuples of reals like $x = (x_1, x_2, ...x_n)$; x_i being reals. Then R^n becomes a linear space with real scalar field where addition of vectors and scalar multiplication of vectors are defined as

$$x + y = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \text{ and}$$

$\alpha.x = \alpha(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ...\alpha x_n)$; $x, y \in R^n$ and α any real scalar.

Here R^n is also called Euclidean n-space with the zero vector $\underline{0} = (0,0,...,0)$ (all co-ordinates are zero), and it is a real Linear space.

Example 2.1.2. Let $C[a,b]$ denote the collection of all real valued continuous functions over a closed interval $[a,b]$. Then $C[a,b]$ is a real linear space (associated scalar field being that of reals) where vector sum and scalar multiplication are defined as under :

$$(f+g)(t) = f(t) + g(t); \quad a \le t \le b, \text{ and } f, g \in C[a,b]$$

and $(\alpha f)(t) = \alpha f(t)$; $a \le t \le b$ and α any real scalar.

As we know that sum of two continuous functions is a continuous function and so is a scalar multiple of a continuous function, we see that $f+g$ and αf are members of $C[a,b]$ where $f,g \in C[a,b]$ and α is any scalar. Here the zero vector equals to the zero function $\left(0(t)=0; a \le t \le b\right)$ over the closed interval $[a,b]$.

There are many other linear spaces like the sequence spaces $l_p (1 < p < \infty)$, polynomial space $\rho[a,b]$, function space $L_2[a,b]$, that we encounter in our discussion to follow.

Definition 2.1.2. (a) If A and B are subsets of a linear space V then $A+B=\{a+b : a \in A \text{ and } b \in B\}$.

(b) For any scalar λ,

$$\lambda A = \{\lambda a : a \in A\}$$

The subset $A-B=A+(-1)B$; and taking λ = zero scalar we find $0A=\{\underline{0}\}$. Further we see that $A+B=B+A$, because vector addition is commutative, However $A-B \neq B-A$. Taking A and B as singleton and $A=\{(1,0)\}$, $B=\{(0,0\}$ in Euclidean 2-space R^2, we find $A-B=\{(1,0)\}$ and $B-A=\{(-1,0)\}$.

Further for any scalar α we have $\alpha A=\{\alpha a : a \in A\}$.

Here is a caution. In general, $A+A \neq 2A$.

Because take $A=\{(1,0),(0,1)\}$; Then we have

$2A=\{(2,0),(0,2)\}$ which is not equal to $A+A$

where $A+A=\{(2,0),(0,2),(1,1)\}$.

Given a fixed member $a \in V$, the subset $a+B=\{a+b : b \in B\}$ is called a translate of B.

§ 2.2. Let X denote a linear space over reals/complex scalars. Given $x_1, x_2, \ldots, x_n$ in X, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ as scalars, the vector $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ of X is called a linear combination of $x_1, x_2, \ldots, x_n$.

A subset E of X is said to span (generate) X if and only if every member of X is a linear combination of some elements of E.

Elements $x_1, x_2, \ldots, x_n$ of E are said to be linearly dependent if and only if there are corresponding number of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \underline{0}$$

A finite number of elements $x_1, x_2, ..., x_k$ of X are said to be linearly independent if they are not linearly dependent. This amounts to say that if

$$\sum_{i=1}^{k} \alpha_i x_i = \underline{0} \text{ implies } \alpha_1 = \alpha_2 = ... = \alpha_k = 0.$$

An arbitrary system of elements of X is called linearly independent if every finite subset of the given system becomes linearly independent.

Observe that if a set of vectors in X contains a linearly dependent subset, whole set becomes linearly dependent. Also note that a linearly independent set of vectors does not contain the zero vector.

Definition 2.2.1. A non-empty sub-set L of a linear space X is called a sub-space of X if $x + y$ is in L whenever x and y are both in L, and also αx is in L, whenever x is in L and α is any scalar.

Example 2.2.1. Let S be any non-empty subset of X. Let L = the set of all linear combinations of elements of S. Then L is sub-space of X, called the sub-space spanned (generated) by S.

The subset = $\{\underline{0}\}$ is a sub-space, called the Null-space.

Theorem 2.2.1. Let $x_1, x_2, ..., x_n$ be a set of vectors of X with $x_1 \neq \underline{0}$. This set is linearly dependent if and only if some one of vectors $x_2, ..., x_n$, say x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Proof : Suppose the given set of vectors is linearly dependent. There is a smallest k with $2 \leq k \leq n$ such that $x_1, x_2, ..., x_k$ is linearly dependent; and we have $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = 0$ with not all α's are zero scalars. Necessarily, we have $\alpha_k \neq 0$; otherwise $x_1, x_2, ..., x_{k-1}$ would form a linearly dependent set.

In consequence $x_k = -\dfrac{\alpha_1}{\alpha_k} x_1 - \dfrac{\alpha_2}{\alpha_k} x_2 - .... - \dfrac{\alpha_{k-1}}{\alpha_k} x_{k-1}$.

That means x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Conversely, if one assumes that some x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$; then we have

$$x_k = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_{k-1} x_{k-1}$$

That means $x_1, x_2, ..., x_k$ are linearly dependent, and in turn we have the set $(x_1, x_2, ..., x_k)$ as linearly dependent.

Definition 2.2.2. In a linear space X suppose there is a +ve integer n such that X contains **a set of n vectors** that are **linearly independent,** while **every set of $n + 1$** vectors in X is **linearly dependent,** then X is called finite dimensional and n is called **dimension of** X {Dim(X)}.

The Null-space is finite dimensional of **dimension 0.**

If X is **not** finite dimensional it is called **infinite dimensional.**

Definition 2.2.3. A finite set B in linear space X is called a basis of X if B is linearly independent, and f the sub-space spanned (generated) by B is all of X.

Explanation : If $x_1, x_2, ..., x_n$ is a basis for X, every member $x \in X$ can be expressed as $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ where scalar coefficients α_i's are uniquely determined; so x does not have a **different** linear combination of basis members.

Suppose $\text{Dim}(X) = n$ $(n \geq 1)$. Then X has a basis consisting of n members; For, X certainly contains vectors $x_1, x_2, ..., x_n$ that form a linearly independent set. Now for any member $x \in X$, the set of vectors $x_1, x_2, ..., x_n$ plus x u of $n + 1$ vectors must be linearly dependent. Now Theorem 2.2.1 applies to conclude that x is in the sub-space generated by $x_1, x_2, ..., x_n$. Hence $x_1, x_2, ..., x_n$ form a basis of X.

§ 2.3 NORMED LINEAR SPACES :

Definition 2.3.1. A linear space X is called a Normed Linear Space (*NLS*) if there is a non-negative real valued function denoted by $\| \;\|$, called a norm on X whose value at $x \in X$ denoted by $\| x \|$ satisfies following conditions (N.1) – (N.3), called norm axioms :–

(N.1) $\quad \| x \| \geq 0$, and $\| x \| = 0$ if and only if $x = \underline{0}$.

(N.2) $\quad |\alpha x \| = |\alpha| \| x \|$ for any scalar α and for any $x \in X$.

(N.3) $\quad \| x + y \| \leq \| x \| + \| y \|$ for any two members x and y in X.

If $\| \;\|$ is a norm on X, the ordered pair $(X, \| \;\|)$ is designated as a *NLS*. If norm changes, *NLS* also changes.

In a *NLS* $(X, \| \;\|)$ one can define a metric ρ by the rule : $\rho(x, y) = \| x - y \|$ for all $x, y \in X$. It is an easy task to check that ρ satisfies all metric axioms; and (X, ρ) becomes **a metric space** with the metric topology called **Norm Topology** because of its induction from norm $\|\ \|$. We write $\lim_{n\to\infty} x_n = x$ in X off $\lim_{n\to\infty} \| x_n - x \| = 0$; this convergence in *NLS X* is known as convergence in Norm. Similarly, we define a Cauchy sequence in *NLS X*.

A subset B in a *NLS X* is said to be bounded if there is a +ve K such that $\| x \| \le K$ for all $x \in B$

Let $x_0 \in X$, and take a +ve number r. Then in NLS X, the set $\{x \in X : \| x - x_0 \| < r\}$ is called an open ball denoted by $B_r(x_0)$ centred at x_0 having radius $= r$. Similarly, we have a closed ball $\overline{B}_r(x_0) = \{x \in x : \| x - x_0 \| \le r\}$; and in agreement with usual open sphere we encounter in Co-ordinate Geometry we have a sphere $S_r(x_0) = \{x \in X : \| x - x_0 \| = r$ centred at x_0 with radius $= r$.

Definition 2.3.2. A *NLS* $(X, \|\ \|)$ is said to be a Banach space if it is a complete metric space with metric induced from the norm function $\|\ \|$ on X.

Example 2.3.1. The space $C[a,b]$ of all real-valued continuous functions over closed interval $[a,b]$ is a Banach space with supnorm $\| f \| = \sup_{a\le t\le b} | f(t) |$; $f \in C[a,b]$.

Solution : It is routine exercise to see that $C[a,b]$ is a real linear space in respect of usual addition and scalar multiplication of continuous functions.

Now put $\| f \| = \sup_{a\le t\le b} | f(t) |$ for $f \in C[a,b]$ wherein we recall that $| f |$ is also continuous function over closed interval $[a,b]$ with a finite sup value $= \| f \| \ge 0$. Also $\| f \| = 0$ if and only f equals to the zero function. So (N.1) axiom is satisfied; For (N.2) take α any scalar (real), then we have for $f \in C[a,b]$,

$$\|\alpha f \| = \sup_{a\le t\le b} \| (\alpha f)(t) | = \sup_{a\le t\le b} \| \alpha\, f(t) | = | \alpha | \sup_{a\le t\le b} | f(t) | = | \alpha | \| f \|.$$

Also, if $f, g \in C[a,b]$ we have $\| f + g \| = \sup_{a\le t\le b} | (f+g)(t) |$

$$= \sup_{a\le t\le b} | f(t) + g(t) | \le \sup_{a\le t\le b} | f(t) | + \sup_{a\le t\le b} | g(t) | = \| f \| + \| g \|.$$

Thus $C[a,b]$ is a NLS; Now take $\{f_n\}$ as a Cauchy sequence in $C[a,b]$; So $\| f_n - f_m \| \to 0$ as, $n, m \to \infty$. Give a $\varepsilon > 0$, we find an index N satisfying

$$\| f_n - f_m \| < \varepsilon \text{ whenever } n, m \ge N.$$

That is, $\sup_{a \le t \le b} | f_n(t) - f_m(t) | < \varepsilon$ ....

Thus for $a \le t \le b$, we have $| f_n(t) - f_m(t) | \le \sup_{a \le t \le b} | f_n(t) - f_m(t) | < \varepsilon$ whenever $n, m \ge N$. Above inequality shows that the sequence $\{f_n\}$ of continuous functions over the closed interval $[a,b]$ converges uniformly to a function say f over $[a,b]$ and also f becomes a continuous function over $[a,b]$. So $f \in C[a,b]$. Taking $m \to \infty$ in (2.3.1) we find

$$| f_n(t) - f(t) | \le \varepsilon \text{ whenever } n \ge N \text{ and for all } t \text{ in } a \le t \le b.$$

This gives $\sup_{a \le t \le b} | f_n(t) - f(t) | \le \varepsilon$ whenever $n \ge N$

or, $\| f_n - f \| \le \varepsilon$ for $n \ge N$

That means, $\lim_{n \to \infty} f_n = f \in C[a,b]$. Thus $C[a,b]$ is a Banach space.

Theorem 2.3.1. Let X be a *NLS* with norm $\| \ \|$. Then

(a) $\big| \| x \| - \| y \| \big| \le \| x - y \|$ for any two members $x, y \in X$.

(b) $\| \ \| : X \to$ Reals is a continuous function.

Proof : (a) We write $\| x \| = \| x - y + y \| \le \| x - y \| + \| y \|$

or, $\| x \| - \| y \| \le \| x - y \|$ .... (2.3.1)

Interchanging x and y we have

$\| y \| - \| x \| = \| y - x \| = \| x - y \|$ .... (2.3.2)

From (2.3.1) and (2.3.2) we write

$$\pm (\| x \| - \| y \|) \le \| x - y \|$$

or, $\big| \| x \| - \| y \| \big| \le \| x - y \|$

(b) Let $\{x_n)$ be a sequence of elements in X converge to x_0.

So $\| x_n - x_0 \| \to 0$ as $x \to \infty$. By (a) we have

$$\big| \| x_n \| - \| x_0 \| \big| \le \| x_n - x_0 \| \to 0 \text{ as } n \to \infty.$$

That means, $\lim_{n \to \infty} \| x_n \| = \| x_0 \|$. Hence norm function $\| \ \|$ is continuous at x_0; As x_0 may be taken as any point in X, (b) follows.

Remark : If $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} y_n = y_0$ in *NLS X*, then

(a) $\lim_{n\to\infty}(x_n \pm y_n) = x_0 \neq y_0$

(b) $\lim_{n\to\infty}(\lambda x_n) = \lambda x_0$ for any scalar λ.

Definition 2.3.3. Two norms $\| \|_1$ and $\| \|_2$ in a linear space X are said to be equivalent if there two +ve constants a and b such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \text{ for all } z \in X.$$

Example 2.3.2. Consider $NLS = R^2$ (Euclidean 2-space) with two norms $\| \|_1$ and $\| \|_2$ defined by $\|x, y\|_1 = \sqrt{x^2 + y^2}$ and $\|x, y\|_2 = \max(|x|, |y|)$ for $(x, y) \in R^2$. Show that two norms are equivalent.

Solution : We have for $(x, y) \in R^2$, $|x|^2 \le |x|^2 + |y|^2$ and $|y|^2 \le |x|^2 + |y|^2$

Thus $\|(x, y)\|_2 = \max(|x|, |y|) \le \sqrt{|x|^2 + |y|^2} = \sqrt{x^2 + y^2} = \|(x, y)\|_1$

or, $\|(x, y)\|_2 \le \|(x, y)\|_1$ (2.3.1)

Again $\|(x, y)_1^2 = x^2 + y^2 = |x|^2 + |y|^2 \le 2\{\max(|x|, |y|)\}^2 = 2\|(x, y)\|_2$

or, $\|(x, y)_1 \le \sqrt{2}\|(x, y)\|_2$ (2.3.2)

Combining (2.3.1) and (2.3.2) we produce

$$\|(x, y)\|_2 \le \|(x, y)\|_1 \le \sqrt{2}\|(x, y)\|_2$$

Therefore two norms as given are equivalent in $NLS = R^2$.

Explanation : If two norms $\| \|_1$ and $\| \|_2$ are equivalent in a NLS X, then identify function : $(X, \| \|_1) \to (X, \| \|_2)$ is a homeomorphism. (In fact, it is a linear homeomorphism).

§ 2.4 QUOTIENT SPACE :

Let $(X, \| \|)$ be a NLS and F be a linear sub-space of X.

If $x \in X$, let $x + F = \{x + y : y \in F\}$.

These subsets $x + F$ as $x \in X$ are cosets of F in X.

Put $X/F = \{x + F : x \in X\}$.

One observes that $F = \underline{0} + F$, $x_1 + F = x_2 + F$ if and only if $x_1 - x_2 \in F$, and as a result, for each pair $x_1, x_2 \in X$, either $(x_1 + F) \cap (x_2 + F) = \Phi$

or, $x_1 + F = x_2 + F$

Further, if $x_1, x_2, y_1, y_2 \in X$, and $(x_1 - x_2) \in F$, $(y_1 - y_2) \in F$, then

$(x_1 + y_1) - (x_2 + y_2) \in F$, and for any scalar α $(\alpha x_1 - \alpha x_2) \in F$ because F is Linear sub-space.

We define two operations in $X \backslash F$ by the following rule :-

(i) $(X \backslash F \times (X \backslash F) \to (X|F)$

where $(x+F, y+F) \to (x+F) + (y+F) = (x+y) + F$

and (ii) $R(\text{¢}) \times (X \backslash F) \to (X|F)$

where $(\alpha, x+F) \to \alpha(x+F) = \alpha x + F$

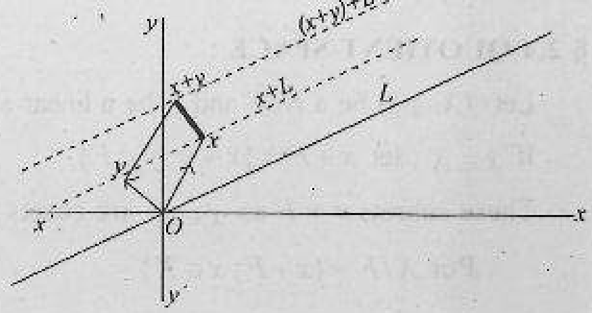
for all $x, y \in X$ and α any scalar.

It is now a routine exercise to verify that $(X \backslash F)$ is a linear space in respect of above 'addition' and 'scalar multiplication'. Note that zero vector of this Linear space $(X|F)$ equals to F.

Definition 2.4.1. The linear space $X \backslash L$ where L is a linear subspace of *NLS* X is called the quotient space (or quotient space of X modulo L).

Example 2.4.1. Geometrically describe the quotient space R^2/L where R^2 = the Euclidean 2-space and L is the sub-space represented by a line through origin $(0,0) \in R^2$.

Solution : Given a sub-space L as represented by a line through $(0,0) \in R^2$, $\underset{\sim}{x}$ is any position of R^2, then $\underset{\sim}{x} + L$ geometrically represents a straight line through $\underset{\sim}{x}$ parallel to the line represented by L; that is say that $\underset{\sim}{x} + L$ is a translate of L through



x. Further if y is any other position of R^2, then by Law of parallelogram we obtain the position $x + y$ and here $(x+L)+(y+L)=(x+y)+L$ is re-presented by the straight line through $x + y$ and it is parallel to L; that is—it is the translate of L through $(x + y)$ in R^2.

Example 2.4.2. Obtain the quotient space $C[0,1]/L$ where $C[0,1]$ is the linear space of all real valued continuous functions over the closed interval $[0,1]$ and L consists of those members $f \in C[0,1]$ with $f(1) = 0$, *i.e.* vanishing at $t = 1$.

Solution : If $f, g \in L$, then $f(1) = g(1) = 0$; Now $(f+g)(1) = f(1) + g(1) = 0$; So $f + g \in L$ (note that sum of two continuous functions over $[0,1]$ is again a continuous functions over $[0,1]$), and for any scalar α we have $\alpha f \in L$ when $f \in L$. Therefore L is a sub-space of $C[0,1]$.

Let us look at members of $C[0,1]\setminus L$. Take $f \in C[0,1]$ where $f(1) = a$ (say). Then for any other member $g \in C[0,1]$ sharing the value a at $t = 1$, *i.e.* $g(1) = a$; we note that $(g - f) \in C[0,1]$ such that $(g-f)(1) \in g(1) - f(1) = a - a = 0$; showing that $(g - f) \in L$ *i.e.* $g \in f + L$. So these members g plus f all belong to $f + L$.

Now if $h \in C[0,1]$ with $h \notin (f + L)$ (2.4.1)

So, $h - f \notin L$

i.e. h and f differ at $t = 1$.

i.e. $h(1) \neq f(1) = a$

We similarly construct a member $(h + L)$ of $C\setminus L$, where

$$(h+L)\cap(f+L)=\phi \tag{2.4.2}$$

or else, we find a member ϕ in both implying

$\varphi - h \in L$ and $\varphi - f \in L$

therefore $\varphi(1) - h(1) = 0$ and $\varphi(1) - f(1) = 0$

i.e. $\varphi(1) = h(1)$ and $\varphi(1) = f(1)$

i.e. $h(1) = f(1)$

that means $h \in (f + L)$, which is not the case by (2.4.1).

Theorem 2.4.1. Let L be a closed linear sub-space of *NLS X*, and let $\|x+L\| = Inf\{\|x+y\|: y \in L\}$, for all $x \in X$, then above is a norm function on the quotient space $(X \setminus L)$. Further if X is Banach space, so will be $(X \setminus L)$.

Proof : For any member $x + L$ of $X \setminus L$, from definition we have

$$\|x+L\| \geq 0 \text{ for any } x \in X.$$

Now assume that $\|x+L\| = 0$ for some $x \in X$.

$$i.e.\ Inf\{\|x+y\|: y \in L\} = 0$$

As $y \in L$ if and only if $-y \in L$, we have

$$Inf\{\|x-y\|: y \in L\} = 0.$$

Since L is closed, $x \in L$ (distance of x from L is zero);

That means $x+F = F =$ the zero vector of the quotient space X/L.

For verification (N.2) take α any non-zero scalar. Then

$$\|\alpha(x+L)\| = \|\alpha x + L\|$$

$$= Inf\{\|\alpha x + y\|: y \in L\}$$

$$= Inf\{\|\alpha(x+\tfrac{y}{\alpha})\|: y \in L\}$$

$$= |\alpha|\, Inf\{\|x+(\tfrac{1}{\alpha})y\|: y \in L\}$$

$$= |\alpha|\|x+L\| \text{ , because } L \text{ is a linear sub-space of } X.$$

For triangle inequality (N.3) take $x, y \in L$

Then $\|(x+t)+(y+L)\| = \|(x+y)+L\|$ (L is a linear sub-space).

$$= Inf\{\|x+y+u\|: u \in L\}$$

$$= Inf\{\|x+y+\tfrac{u}{2}+\tfrac{u}{2}\|: u \in L\}$$

$$\leq Inf\{\|x+\tfrac{u}{2}\| + \|y+\tfrac{u}{2}\|: u \in L\}$$

$$\leq Inf\{\|x+\tfrac{u}{2}\|: u \in L\} + Inf\ \|y+\tfrac{u}{2}\|: u \in L\}$$

$$= Inf\{\|x+h\|: u \in L\} + Inf\ \|y+K\|: K \in L\};\ L \text{ is a sub-space.}$$

$$= \|x+L\| + \|y+L\|$$

Thus quotient space $X\backslash L$ is a *NLS.*

Now suppose X is a Banach space. We show that the quotient space X/L is so. Let $\{x_n+L\}$ be a Cauchy sequence in (X/L). So corresponding to each +ve integer k we find an index N_k such that

$$\| x_n - x_m + L \| < \frac{1}{2^k}, \text{ whenever } m,n \geq N_k \tag{2.4.1}$$

We define by Induction a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\| x_{n_k} - x_{n_{k+1}} + L \| < \frac{1}{2^k}$$

Take $n_1 = N_1$, and suppose $n_2, n_3, \ldots, n_k$ have been so defined that $n_1 < n_2 < \ldots < n_k$ and $N_j \leq n_j$ $(j = 1,2,\ldots,k)$.

Let $n_{k+1} = \max\{N_{k+1}, n_k + 1\}$. This enables one to obtain an increasing sequence $\{n_k\}$ and (*) follows from (2.4.1)

Put $y_k = x_{n_k}$. Then by induction we define a sequence $\{z_k\}$ in L such that $z_k \in (y_k + L)$ and $\| z_k - z_{k+1} \| < \frac{1}{2^{k+1}}$, $k = 1,2, \ldots\ldots$

Choose $z_1 \in (y_1 + L)$, suppose $z_2, \ldots, z_k$ have been so chosen to satisfy above condition. Then $y_k + L = z_k + L$ and by (2.4.1) we have $\| z_k - y_{k+1} + F \| < \frac{1}{2^k}$. By definition of norm in $(X\backslash L)$

we find $z_{k+1} \in (y_{k+1} + L)$ such that $\| z_k + z_{k+1} \| \leq z_k - y_{k+1} + L \| + \frac{1}{2^k}$.

Then $\| z_k + z_{k+1} \| < \frac{1}{2^{k+1}}$ as wanted.

That means $\sum_{k=1}^{\infty} \| z_k - z_{k+1} \|$ is convergent, and hence $\sum_{k=1}^{\infty} (z_k - z_{k+1})$ is convergent.

But $\sum_{k=1}^{\infty} (z_k - z_{k+1}) = (z_1 - z_2) + (z_2 - z_3) + \ldots + (z_m - z_{m+1}) = z_1 - z_{m+1}$.

So, $\{z_m\}$ is convergent; Put $\lim_{k\to\infty} z_k = z$; since $z_k \in (y_k + L)$

we have $\|(z+L)-(y_k+L)\| = \|z-y_k+L\| \le \|z-z_k\|$.

That means $\lim_{k\to\infty}\{y_k+L\} = z+L$. Thus given Cauchy sequence $\{x_n+L\}$ has a convergent subsequence $\{x_{n_k}+F\}$.

Hence $\{x_n+L\}$ is convergent in $(X\backslash L)$. This proves that $(X\backslash L)$ is a Banach space.

§ 2.5 CONVEX SETS IN *NLS* :

Let $(X, \|\ \|)$ be a *NLS*, and C be a non-empty subset of X.

Definition 2.5.1. C is said to be a convex set if for any real scalar α in $0 \le \alpha \le 1$, and any two members $x_1, x_2 \in C$ we have $\alpha x_1 + (1-\alpha)x_2$ is a member of C.

Or, equivalently, for any two reals α, β with $0 \le \alpha, \beta \le 1$ $\alpha + \beta = 1$, $(\alpha x_1 + \beta x_2) \in C$.

Or, equivalently, the segment consisting of members $tx_1 + (1-t)x_2$ $(0 \le t \le 1)$ is a part of C.

For example, in an Euclidean space like R^n, cubes, ball, sub-spaces are all examples of convex sets in R^n.

Theorem 2.5.1. Intersection of any number of convex sets in a *NLS* is a convex set, but their union may not be so,

Proof : Suppose $\{C_\alpha\}_{\alpha\in\Lambda}$ be a family of convex set in *NLS* $(X, \|\ \|)$ and put $C = \bigcap_{\alpha\in\Delta} C_\alpha$; Let $C \ne \phi$ and let $x, y \in C$ take $0 \le \alpha \le 1$. Now $x, y \in \bigcap_{x\in\Lambda} C_\alpha$, so for every α, x and y are members of C_α which is convex, thus, $(\alpha x + (1-\alpha)y \in C_\alpha$. Therefore $\alpha x + (1-\alpha)y$ is a member of every C_α and hence is a member of $\bigcap_{x\in\Delta} C_\alpha = C$. Thus C is shown to be a convex set in X.

Union of two convex sets may not be a convex set. Every triangular region in Euclidean plane is a convex set but the figure ⧖ as a union of two such convex sets fails to be a convex set.

Theorem 2.5.2. A subset C in a *NLS* is convex if and only if $sC + tC = (s+t)C$ for all +ve scalars s and t.

Proof : For all scalars s and t we have

$$(s+t)C \subset sC+tC \quad \text{.......... (2.5.1)}$$

If C is convex and s, t are +ve scalars we have

$$\frac{s}{s+t}C+\frac{t}{s+t}C \subset C$$

$$\text{Or} \quad sC+tC \subset (s+t)C \quad \text{.......... (2.5.2)}$$

Combining (2.5.1) and (2.5.2) we have

$$sC+tC=(s+t)C$$

Conversely, suppose $(s+t)C=sC+tC$ holds for all +ve scalars; If $0 \le \alpha \le 1$, take $s=\alpha$ and $t=1-\alpha$ and then we find $\alpha C+(1-\alpha)C \subset C$. So C is convex.

Theorem 2.5.3. A ball (open or closed) of a *NLS* is a convex set.

Proof : $\bar{B}(x_0,r)$ be a closed ball in a *NLS* $(X,\|\ \|)$.

Let $x,y \in \bar{B}(x_0,r)$; So $\|x-x_0\| \le r$ and $\|y-x_0\| \le r$. If $0 \le t \le 1$, and $u=tx+(1-t)y$, we have

$$\|u-x_0\|=\|tx+(1-t)y-(tx_0+(1-t)x_0)\|=\|t(x-x_0)+(1-t)(y-y_0)\|$$

$$\le t\|x-x_0\|+(1-t)\|y-y_0\| \le tr+(1-t)r=r.$$

That shows $u \in \bar{B}(x_0,r)$. So, $\bar{B}(x_0,r)$ is shown to be convex. The proof for an open ball shall be similar.

Example 2.5.1. If $(X, \|\ \|)$ is a Banach space and L is a closed sub-space of X, show that L is a Banach space.

Solution : If L is a closed sub-space of X, then L becomes a closed set of a complete metric space X, the metric being induced from the norm $\|\ \|$. And we know that every closed sub-space of a complete metric space is a complete metric sub-space and hence here L is a Banach space. (as a sub-space of X).

§ 2.6 BOUNDED LINEAR OPERATORS OVER A *NLS* $(X, \|\ \|)$:

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. (Here, same notation $\|\ \|$ has been used for norm function; it is to be noted that norm functions in X and Y are, in general, different).

Definition 2.6.1. A function (or mapping or transformation) (function, map, mapping, transformation are synonyms of the same mathematical object) $T: X \to Y$ is called a linear operator if (1) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for any two members x_1 and x_2 in X, and

(2) $T(\alpha x_1) = \alpha T(x_1)$ for any scalar α and for any member $x_1 \in X$.

Explanation : For a linear operator $T: X \to Y$ condition (1) in Definition 2.6.1 is termed as linearity condition which says Image of the sum is equal to sum of the images. Condition (2) is known as that homogeneity. For example, if $X = Y = R =$ the space of reals with usual norm (Euclidean norm) and $T: R \to R$ is given by $T(x) = \alpha x$ where $x \in R$ and α is a fixed real (zero or non-zero), we verify that T is a linear operator; and we shall presently see that any linear operator : $R \to R$ shall be of the form $T(x) = \alpha x$ for some fixed scalar α for all $x \in R$.

Definition 2.6.2. The operator $T: X \to Y$ defined by $T(x) = \underline{0}$ in Y. For all X, is called **the zero operator, denoted by 0.**

Remark : (a) The zero operator : $X \to Y$ is a Linear operator.

(b) The identity operator, $I : X \to X$ where $I(x) = x$ for all $x \in X$ is a linear operator.

Theorem 2.6.1. Let $T : X \to Y$ be a linear operator. If T is continuous at one point of X, then T is continuous at every other point of X.

Proof : Suppose T is continuous at $x_0 \in X$; so given $\varepsilon > 0$, there is a +ve δ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|(x) - (x_0)\| < \delta$. Suppose $x_1 (\neq x_0)$ be another point of X. Then if $\|x - x_1\| < \delta$, we write $\|x - x_1\| = \|x_0 - (x - x_1 + x_0)\|$.

Thus $\|(x - x_1 + x_0)\| < \delta$ shall give by virtue of continuity of T at x_0,

$$\|T(x - x_1 + x_0) - T(x_0)\| < \varepsilon$$

or, $\|T(x) - T(x_1) + T(x_0) - T(x_0)\| < \varepsilon$ because T is linear.

or, $\|T(x) - T(x_1)\| < \varepsilon$. Therefore T is continuous at $x = x_1$.

Corollary : A linear operator over a NLS X is continuous **either everywhere or nowhere in X.**

Definition 2.6.3. A linear operator $T: X \to Y$ is called bounded if there is a +ve constant M such that

$$\| T(x) \| \leq M \| x \| \text{ for all } x \in X.$$

or equivalently $\frac{\| T(x) \|}{\| x \|} \leq M$ for all non-zero numbers $x \in X$.

Theorem 2.6.2. Let $T: X \to Y$ be a linear operator. Then T is continuous if and only if T is bounded.

Proof : Let $T: X \to Y$ be a continuous linear operator; if possible let T be not bounded. So for every +ve integer n we find a member $x_n \in X$ such that

$$\| T(x_n) \| > n \| x_n \| \quad \text{...} \quad (2.6.1)$$

Now x_n is non-zero vector in X, put $u_n = \frac{x_n}{n \| x_n \|}$,

clearly $\| x_n \| = \frac{1}{n} \cdot \frac{1}{\| x_n \|} = \frac{1}{n} \to 0$ as $n \to \infty$. So we see $\lim_{n \to \infty} u_n = \underline{0}$ in X; By continuity of T we have $\lim_{n \to \infty} T(u_n) = T(\underline{0}) = \underline{0}$ in Y. ($T(\underline{0}) = \underline{0}$, because T is linear);

Therefore we have $\| T(u_n) \| \to 0$ as $n \to \infty$ (*)

On the other hand, $\| T(u_n) \| = \left\| T\left(\frac{x_n}{n \| x_n \|} \right) \right\|$,

$$= \left\| \frac{1}{n \| x_n \|} T(x_n) \right\|, \text{ because } T \text{ is linear}$$

$$= \frac{1}{n} \cdot \frac{1}{\| x_n \|} \| T(x_n) \| > 1 \text{ by (2.6.1)}$$

Now $\| T(u_n) \| > 1$ and (*) are contradictory.

So, we have shown that $T: X \to Y$ is bounded.

Conversely, suppose linear operator $T: X \to Y$ is bounded. Then we find a +ve scalar such that

$$\| T(x) \| \leq M \| x \|;$$

So given $\in > 0$, there is a +ve $\delta = \frac{\in}{2M}$ (here), such that

$$\|T(x)\| < \varepsilon \quad \text{whenever } \|x\| < \delta$$

i.e. $\|T(x) - T(\underline{0})\| < \varepsilon$ whenever $\|x - \underline{0}\| < \delta$ because $T(\underline{0}) = \underline{0}$ in *Y*. That means, *T* is continuous at $x = \underline{0}$ in *X*, and therefore Theorem 2.6.1 says that *T* is continuous at every non-zero position of *X*. The proof is now complete.

Examples of bounded and unbounded linear operators.

Example 2.6.1. Consider a transformation *T* of rotation in Euclidean 2-space R^2 given by $T(x, y) \to (x', y')$ where

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \qquad (*)$$

Now it is easy to verify that $T : R^2 \to R^2$, under (*) is a linear operator in respect which rotation takes place around origin (0,0) with axes of co-ordinates being rotated through angle θ to give new axes of co-ordinates.

In *NLS* R^2 with usual norm $\|(x, y)\| = x^2 + y^2$, we see that

$$\|T(x, y)\|^2 = \|(x', y')\| = x'^2 + y'^2 = (x\cos\theta + y\sin\theta) + (-x\sin\theta + y\cos\theta)^2$$

$$= x^2 + y^2 = \|(x, y)\|^2.$$

Thus $\|T(x, y)\| = \|(x, y)\|$; and this is true for all points (*x, y*) in R^2 , and we conclude that *T* is a bounded linear operator.

Example 2.6.2. Consider the Banach space $C[0,1,]$ of all real-valued continuous functions over the closed interval $[0,1]$ with respect to sup norm

$$\|f\| = \sup_{a \le t \le 1} |f(t)|;\ f \in C[0,1]$$

Let *K*(*s,t*) be a real-valued continuous function over the square

$$\{0 \le s \le t;\ 0 \le t \le 1\}.$$

Now define $T : C[0,1] \to C[0,1]$ by the rule : let $T(f) = F$

where $F(s) = \int_0^1 k(s,t) f(t)\,dt$; as $f \in C[0,1]$.

It is a routine exercise to check that F is continuous over $[0,1]$ and T is a linear operator.

Now, $\|T(f)\| = \|F\| = \sup\limits_{0\le s\le 1} |F(s)| = \sup\limits_{0\le s\le 1} |\int_0^1 k(s,t) f(t) dt|$

$$\le \sup_{0\le s\le 1} \int_0^1 |k(s,t)||f(t)| dt \le M \int_0^1 |f(t)| dt \text{ where } M = \sup_{0\le s\le 1,\, 0\le t\le 1} |k(s,t)|;$$

$$\le M . \sup_{0\le t\le 1} |f(t)| \int_0^1 dt = M . \|f\|. \text{ This is true for every member } f \in C[0,1].$$

Therefore, T is shown to be bounded.

Example 2.6.3. Let $C^{(1)}[0,1,]$ denote the class of real-valued continuous functions that are continuously differentiable over $[0,1]$. Then $C^{(1)}[0,1]$ is a sub-space of $C[0,1]$ which is Banach space with sup norm. Consider the Differential operator $D: C^{(1)}[0,1] \to C[0,1]$ when $D(f) = \varphi$; $f \in C^{(1)}[0,1]$ and $\frac{d}{dt} f(t) = \varphi(t)$ in $0 \le t \le 1$. We can easily verify that D is a linear operator; presently we see that D is not bounded.

Let us take $f_n \in C^{(1)}[0,1]$ where $f_n(t) = \sin n\pi t$ in $0 \le t \le 1$. Then we have $Df_n = \varphi_n$ where $\varphi_n(t) = \frac{d}{dt}(\sin n\pi t) = n\pi \cos n\pi t$ in $0 \le t \le 1$.

Therefore, $\|f_n\| = \sup\limits_{0\le t\le 1} |\sin n\pi t| = 1$ and

$$\|D(f_n)\| = \|\varphi_n\| = \sup_{0\le t\le 1} |n\pi \cos n\pi t| = n\pi$$

Here $\frac{\|D(f_n)\|}{\|f_n\|} = \frac{n\pi}{1} \to \infty$ as $n \to \infty$

That means D can not be bounded.

Definition 2.6.4. Let $T : X \to Y$ be a bounded (or equivalently, continuous) linear operator. Then the norm of T, denoted by $\|T\|$ is defined as

$$\|T\| = Inf\{M > 0 : \|T(x)\| \le M\|x\| \text{ for all } x \in X\}$$

(A set of +ve reals has always *Inf.* value).

Theorem 2.6.3. Let $T : X \to Y$ be a bounded linear operator. Then

(a) $\|T(x)\| \le \|T\|\|x\|$ for all $x \in X\}$

(b) $\|T\| = \sup\limits_{\|x\|\le 1}\{\|T(x)\|\}$

(c) $\|T\| = \sup\limits_{\|x\|=1}\{\|T(x)\|\}$

(d) $\|T\| = \sup\limits_{x\ne \underline{0}}\left\{\dfrac{\|T(x)\|}{\|x\|}\right\}$

Proof : (a) From definition of operator norm we see that for any +ve ε we have $\|T(x)\| \le (\|T\| + \varepsilon)\|x\|$ for all $x \in X$.

Taking $\varepsilon \to 0_+$ we have $\|T(x)\| \le \|T\|\|x\|$

(b) If $\|x\| \le 1, x \in X$, we have $\|T(x)\| \le \|T\|\|x\| \le \|T\|$

Therefore $\sup\limits_{\|x\|\le 1}\|T(x)\| \le \|T\|$, (1)

From Definition of operator norm $\|T\|$, given any +ve ε, we find $x_\varepsilon \in X$ such that $\|T(x_\varepsilon)\| > (\|T\| - \varepsilon)\|x\|$.

Take $u_\varepsilon = \dfrac{x_\varepsilon}{\|x_\varepsilon\|}$ we see $\|u_\varepsilon\| = 1$ such that

$$\|T(u_\varepsilon)\| = \frac{1}{\|x_\varepsilon\|}\|T(x_\varepsilon)\| > \frac{1}{\|x_\varepsilon\|}.(\|T\| - \varepsilon)\|x_\varepsilon\| = \|T\| - \varepsilon$$

As $\|u_\varepsilon\| = 1$, this gives $\sup\limits_{\|x\|\le 1}\|T(x)\| \ge \|T(u_\varepsilon)\| > \|T\| - \varepsilon$. As $\varepsilon > 0$ is arbitrary we produce $\sup\limits_{\|x\|\le 1}\|T(x)\| \ge \|T\|$ (2)

From (1) and (2) we have (b), namely, $\sup\limits_{\|x\|\le 1}\|T(x)\| = \|T\|$

(c) the proof shall be like that of (b).

(d) we have $\|T(x)\| \le \|T\|\|x\|$ for all $x \in X$.

So, $\frac{\|T(x)\|}{\|x\|} \le \|T\|$ for $x \in X$ with $x \ne 0$.

Since r.h.s does not depend on non-zero $x \in X$, we have

$$\sup_{x \ne \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \le \|T\| \qquad (3)$$

Again given a +ve ε $(0 < \varepsilon < \|T\|)$ we find a member $x_\varepsilon \in X$ such that

$\|T(x_\varepsilon)\| > (\|T\| - \varepsilon)\|x_\varepsilon\|$; clearly $x_\varepsilon \ne \underset{\sim}{0}$.

Thus $\frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Therefore $\sup_{x \ne \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \ge \frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Now taking $\varepsilon \to 0_+$ we find

$$\sup_{x \ne \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \ge \|T\| \qquad (4)$$

Combining (3) and (4) we have $\sup_{x \ne \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} = \|T\|$.

EXERCISE A

Short answer type questions :

1. In a linear space X if $x \in X$ show that $-(-x) = x$.
2. If a finite set of vectors in a linear space contains the zero vector show that it is a linearly dependent set.
3. In Euclidean 2-space R^2 describe geometricaly open ball centred at (0,0) with radius = 1 in respect of (a) $\|x\|_1 = \sqrt{x_1^{\ 2} + x_2^{\ 2}}$ (b) $\|x\|_2 = |x_1| + |x_2|$ and (c) $\|x\|_3 = \max\{|x_1|, |x_2|\}$ where $x = (x_1, x_2) \in R^2$.
4. Obtain a condition such that function suit and $\sin \lambda t$ are linearly independent in the space $C[0,2\pi]$.
5. Construct a basis of Euclidean 3-space R^3 containing (1,0,0) and (1,1,0).

EXERCISE B

Broad asnwer type questions

1. If $C[a,b]$ is the linear space of all real-valued continuous functions over the closed interval $[a,b]$, show that $C[a,b]$ is a Normed Linear space with respect to $\| f \| = \int_a^b | f | dt$, $f \in C[a,b]$. Examine if $C[a,b]$ is a Banach space with this norm.

2. In a *NLS X*. verify that for a fixed member $a \in X$, the function $f : X \to X$ given by $f(x) = x + a$; $x \in X$ is a homeomorphism. Hence deduce that translate of an open set in X is an open set.

3. Examine if the sub-space $\rho[0,1]$ of all real polynomials over the closed interval $[0,1]$ is a closed sub-space of the Banach space $C[0,1]$ with sup norm.

4. Prove that in a *NLS* the closure of the open unit ball is the closed unit ball.

5. Let $(X, \| \|)$ and $(Y, \| \|)$ be two *NLS* over the same scalars and $T : X \to Y$ be a linear operator that sends a convergent sequence in X to a bounded sequence in Y. Prove that T is a bounded linear operator.

6. Let $T : C[0,1] \to$ itself, where $C[0,1]$ is the Banach space of all real-valued continuous functions over the closed unit interval with sup norm such that $T(x) = y$ where

$$y(t) = \int_0^t x(u)du; \quad x \in C[0,1] \text{ and } 0 \le t \le 1$$

Find the range of T, and obtain $T^{-1} : (\text{range } T) \to C[0,1]$.

Examine if T^{-1} is linear and bounded.

UNIT 3

(*Contents* : Every Finite Dimensional *NLS* is a Banach space, Equivalent norms, Riesz Lemma, Finite Dimensionality of *NLS* by compact unit ball, Linear operators over finite Dimensional *NLS* and matrix representation; Isomorphism, Boundedness of linear operators over finit Dimensional *NLS*, space $Bd\mathscr{L}(X,Y)$ of bounded linear operators, and its completeness).

§ 3.1 FINITE DIMENSIONAL *NLS*

Theorem 3.1.1. Every finite dimensional *NLS* is a Banach space. To prove this Theorem we need a Lemma.

Lemma 3.1.1. Let $(x_1, x_2, ..., x_n)$ be a set of linearly independent vectors in a *NLS* $(X, \| \|)$; then there is a +ve β such that

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq \beta(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$$ for every set of scalars $\alpha_1, \alpha_2 ..., \alpha_n$.

Proof : Put $S = \sum_{i=1}^{n} |\alpha_i|$. Without loss of generality we take $S > 0$.

Then above inequality is changed into

$$\| \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n \| \geq \beta, \text{ where } \beta_i = \frac{d_i}{S} \qquad (*)$$

and $\sum_{i=1}^{n} |\beta_i| = 1$.

If suffices to establish (*) for any set of scalars $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{i=1}^{n} |\beta_i| = 1$. We apply method of contradiction. Suppose there is a sequence $\{y_m\}$ with

$$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n; \text{ and } \sum_{i=1}^{n} |\beta_i^{(m)}| = 1 \text{ for } m = 1, 2, ...$$

such that $\| y_m \| \to 0$ as $m \to \infty$

Now $|\beta_i^{(m)}| \leq \sum_{i=1}^{n} |\beta_i^{(m)}| = 1$

Hence for a fixed i the sequence $\{\beta_i^{(m)}\} = \{\beta_i^{(1)}, \beta_i^{(2)}, ...\}$ is bounded. So Bolzano-Weirstrass Theorem says that $\{\beta_i^{(m)}\}$ has a sub-sequence that converges to (say) β_i.

Let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ shall give a sub-sequence, say $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). We continue this process. At nth stage we produce a subsequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2}, ...\}$ of $\{y_m\}$ whose term

$$y_{n,m} = \sum_{i=1}^{n} \delta_i^{(m)} x_i, \quad \sum_{i=1}^{n} |\delta_i^{(m)}| = 1$$

such that $\lim_{m\to\infty} \delta_i^{(m)} = \beta_i$, Hence we see

$\lim_{m\to\infty} y_{n,m} = \sum_{i=1}^{n} \beta_i x_i = y$ (say) when $\sum_{i=1}^{n} |\beta_i| = 1$. That means all β_i's are not zero. Since $x_1, x_2, ..., x_n$ are linearly independent it follows that $y \neq \underset{\sim}{0}$.

Now $\lim_{m\to\infty} y_{n,m} = y$ gives

$$\lim_{m\to\infty} \| y_{n,m} \| = \| y \|.$$

Since $\{y_{n,m}\}$ is a sub-sequence of $\{y_m\}$ and $\| y_m \| \to 0$ as $m \to \infty$, So $\| y_{n,m} \| \to 0$ as $m \to \infty$ and so $\| y \| = 0$ giving $y = \underset{\sim}{0}$, a contradiction. Therefore Lemma is proved.

Proof of Theorem 3.1.1. Suppose $\{y_m\}$ be a Cauchy sequence in a finite dimensional *NLS* (X, $\|\ \|$). Let Dim(X) = n, and (e_1, e_2, ..., e_n) forms a basis in X. So each y_m has a unique representation.

$$y_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + ... + \alpha_n^{(m)} e_n$$

Give a +ve ε, as $\{y_m\}$ is Cauchy, we find an index N such that

$$\| y_m - y_r \| < \varepsilon \text{ for } m, r \geq N.$$

Now $\varepsilon > \| y_m - y_r \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(r)}) e_i \|$

$$\geq \beta \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| \text{ by Lemma 3.1.1}$$

whenever $m, r > N$. Therefore

$$|\alpha_i^{(m)} - \alpha_i^{(r)}| \leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(r)}| < \frac{\varepsilon}{\beta} \text{ for } m, r > N$$

Therefore, each of the n sequences

$\{\alpha_i^{(m)}\}$ $(i = 1, 2, ..., n)$ becomes a Cauchy sequence of scalars (reals/complex), and by Cauchy's General Principle of convergence becomes a convergent sequence with, say,

$$\lim_{m\to\infty} \alpha_i^{(m)} = \alpha_i^{(0)} \text{ (say)}, i = 1, 2, ..., n.$$

Put $y = \alpha_1^{(0)} e_1 + \alpha_2^{(0)} e_2 + ... + \alpha_n^{(0)} e_n$; so $y \in X$.

Further, $\lim_{m\to\infty} \alpha_i^{(m)} = \alpha_i^{(0)}$ for $i = 1, 2, ..., n$ gives,

$$\| y_m - y \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(0)}) e_i \| \leq \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(0)}| \| e_i \| \to 0 \text{ as } m \to \infty.$$

i.e. $\lim_{m\to\infty} y_m = y \in X$. So given Cauchy sequence $\{y_m\}$ in X is convergent in X; and $(X, \| \ \|)$ is Banach space.

Theorem 3.2.1. Any two norms in a finite dimensional *NLS* X are equivalent.

Proof : Let Dim $(X) = n$ and $(e_1, e_2, ..., e_n)$ form a basis for X. If $x \in X$, we write $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ uniquely.

Applying Lemma 3.1.1 we find a +ve β such that

$$\| x \|_1 \geq \beta(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$$

If $\mu = \max_{1 \leq i \leq n} \| e_i \|_2$; Then we have

$$\| x \|_2 \leq \sum_{i=1}^{n} |\alpha_i| \| e_i \| \leq \mu \sum_{i=1}^{n} |\alpha_i| \leq \frac{\mu}{\beta} \| x \|_1$$

or, $\beta_{\overline{\mu}} \| x \|_2 \leq \| x \|_1$, the other half of desired inequality comes by interchanging norms $\| \ \|_1$ and $\| \ \|_2$. The proof is now complete.

Theorem 3.1.3. A *NLS* $(X, \|\ \|)$ is finite Dimensional if and only if the closed unit ball (centred at $\underset{\sim}{0}$) is compact.

To prove this theorem we need support of another result popularly known as Riesz Lemma.

Lemma 3.1.2 (Riesz Lemma). Let L $(\neq X)$ be a closed sub-space of a *NLS* $(X, \|\ \|)$. Given a +ve $\in$ $(0 < \in < 1)$ there is a member $y \in \left(\frac{X}{L}\right)$ with $\|y\| = 1$ such that $\|y - x\| > 1 - \varepsilon$ for all $x \in L$.

Proof : Take $y_0 \in \left(\frac{X}{L}\right)$ and put $d = dist(y_0, L)$

$$= \underset{x \in L}{Inf} \|y_0 - x\|.$$

Since L is closed and y_0 is outside L, we have $d > 0$. Given a +ve ε, choose $\eta > 0$ such that

$$\frac{\eta}{d + \eta} < \varepsilon$$

So we find a member $x_0 \in L$ such that

$$d \leq \|y_0 - x_0\| < d + \eta$$

Take $y = \dfrac{y_0 - x_0}{\|y_0 - x_0\|}(y_0 \neq x_0)$; then $\|y\| = 1$, and we have

$y_0 = x_0 + \|y_0 - x_0\| y$. Since y_0 is outside L, we find y also outside L *i.e.* $y \in \left(\frac{X}{L}\right)$.

If $x \in L$, we have $\|y - x\| = \left\| \dfrac{y_0 - x_0}{\|y_0 - x_0\|} - x \right\|$

$$= \frac{1}{\|y_0 - x_0\|} \big\| y_0 - x_0 - x \|y_0 - x_0\| \big\| = \frac{1}{\|y_0 - x_0\|} \|y_0 - x'\| \text{ (say)}$$

where $x' = x_0 + \|y_0 - x_0\| x$; clearly $x' \in L$ because $x_0, x \in L$.

Therefore, $\|y - x\| > \dfrac{1}{d + \eta} \|y_0 - x'\| \geq \dfrac{d}{d + \eta} = 1 - \dfrac{\eta}{d + \eta} = 1 - \varepsilon$.

The proof is now complete.

Proof of Theorem 3.1.3. First suppose that closed unit ball $\widehat{B}_1(\underset{\sim}{0})=\{x \in X : \|x\| \leq 1\}$ in a *NLS* $(X, \|\ \|)$ is compact and hence is sequentially compact. We show that $\mathrm{Dim}(X)<\infty$.

Suppose no. take $x_1 \in X$ with $\|x_1\|=1$ and L_1 as the sub-space spanned by $x_1(\neq \underset{\sim}{0})$. Then L_1 is a closed sub-space of X without being equal to X. So we apply Riesz Lemma (Lemma 3.1.2) when we take $\in=\frac{1}{2}$. Then we find $x_2 \in (X \setminus L_1)$ with $\|x_2\|=1$ and $\|x_1-x_2\|>\frac{1}{2}$.

Take L_2 as the sub-space spanned by x_1 and x_2. By the argument same as above we find L_2 as a proper closed sub-space of X and attracts Riesz Lemma. Thus there is $x_3 \in (X \setminus L_2)$ with $\|x_3\|=1$ and $\|x_3-x_1\|>\frac{1}{2}$, $\|x_3-x_2\|>\frac{1}{2}$.

We continue this process to obtain a sequence $\{x_n\}$ with $\|x\|=1$ *i.e.* $x_n \in \tilde{B}_1(\underset{\sim}{0})$ such that $\|x_n-x_m\|>\frac{1}{2}$ for $n \neq m$. That means $\{x_n\}$ does not admit if any convergent subsequence : a contradiction that $\widehat{B}_i(\underset{\sim}{0})$ is sequentially compact. Hence we have shown that $\mathrm{Dim}(X)<\infty$.

Conversely let $(X, \|\ \|)$ be finite dimensional. Then it is a well known property that a subset in X is norm-compact if and only if that subset is bounded and closed. Here the closed unit ball $\widehat{B}_i(\underset{\sim}{0})$ is bounded; and hence it must be compact. The proof is now complete.

§ 3.2 LINEAR OPERATORS OVER FINITE DIMENSIONAL SPACES :

Let R^n denote the Euclidean n-space. Then an $m \times n$ real matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ defines a Linear operator $T : R^n \to R^m$ where $T(\underset{\sim}{x})=\underset{\sim}{y}$;

$\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n)$ and $\underset{\sim}{y} = (\eta_1, \eta_2, ..., \eta_n)$ such that

$$\sum_{j=1}^{n} a_{ij}\xi_j = \eta_i \quad i = 1, 2, ..., m.$$

Verification is an easy exercise and is left out.

Conversely, given a linear operator $T : R^n \to R^m$. We show that it is represented by an $(m \times n)$ real matrix. Let us take $(e_1, e_2, ..., e_n)$ as a basis in R^n where $e_i = \Big(\underbrace{0\;1\;0}_{i\text{th place}}\Big)$, $i = 1, 2, ..., n$. And let $f_1 = \underbrace{(1, 0, 0, ...0)}_{m \text{ places}}$, $f_2 = (0, 1, 0, 0, ...0)$, $f_n = (0, 0, ..., 1)$ form the analogous basis in R^m.

Let $T(e_j) = \underset{\sim}{a}_j \in R^m$

$$= \alpha_{1j}f_1 + \alpha_{2j}f_2 + ... + \alpha_{mj}f_m \quad \text{(say)} \quad (j = 1, 2, ..., n)$$

In general, if $\underset{\sim}{x} = (\xi_1, \xi_2, ... \xi_n) \in R^m$ and if $T(\underset{\sim}{x}) = \underset{\sim}{y} \in R^m$

we have $\eta_1 f_1 + \eta_2 f_2 + ... + \eta_m f_m = \underset{\sim}{y}$ and

$$\underset{\sim}{y} = T(\underset{\sim}{x}) = T\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j T(e_j) = \sum_{j=1}^{n} \xi_j \underset{\sim}{a}_j$$

$$= \sum_{j=1}^{n} \xi_j \left(\sum_{i=1}^{m} \alpha_{ij} f_i\right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \xi_j\right) f_i$$

Or, $\displaystyle\sum_{i=1}^{m} \eta_i f_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \xi_j\right) f_i$ gives $\displaystyle\eta_i = \sum_{j=1}^{n} \alpha_{ij} \xi_j$; $i = 1, 2, ..., m$.

Therefore, T is represented by the matrix $\left((\alpha_{ji}\right)_{m \times n}$.

Remark : Given a linear operator $T : R^n \to R^m$, there is an $(m \times n)$ matrix to represent T. Entries (reals) in this matrix depend upon the choice of basis in underlying spaces. If basis changes co-efficients entering representative matrix change; However order of the matrix does not change.

Example 3.2.1. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval $[0,1]$ with degree ≤ 3. Let $D:\rho_3[0,1]\to\rho_2[0,1]$ be the differential operator. Show that D is a linear operator and obtain a representative matrix for D.

Solution : Here $\rho_3[0,1]$ (and similarly $\rho_2[0,1]$) is a real linear space with $\text{Dim}\,\rho_3[0,1]=4$ ($\text{Dim}(\rho_2[0,1]=3$). Let us take (p_0, p_1, p_2, p_3) as a basis for $\rho_3[0,1]$ where $p_0(t)=1,\ p_1(t)=t,\ p_2(t)=t^2$ and $p_3(t)=t^3$ in $0\leq t\leq 1$.

Then we have $D(p_0)=0,\quad D(p_1)=1,\ D(p_2)=2t$ and $D(p_3)=3t^2$; and we write

$$\begin{aligned} 0 &= 0p_0+0p_1+0p_2 \\ 1 &= 1p_0+0p_1+0p_2 \\ 2t &= 0p_0+2p_1+0p_2 \end{aligned}$$

and $\qquad 3t^2 = 0p_0+0p_1+3p_2$

And therefore representative matrix $((a_{ij}))_{3\times4}$ for D is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times4}$$

Remark : Representative matrix for linear operator changes if basis is changed.

Example 3.2.2. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval $[0,1]$ with degree ≤ 3.

Let $T:\rho_3[0,1]\to\rho_3[0,1]$ be a linear operator given by

$T(a_0+a_1x+a_2x^2+a_3x^3)=a_0+a_1(x+1)+a_2(x+1)^2+a_3(x+1)^3$ for every member $a_0+a_1x+a_2x^2+a_3x^3\in\rho_3[0,1]$; obtain representative matrix for T relative to basis (i) $(1,x,x^2,x^3)$ and (ii) $(1,1+x,1+x^2,1+x^3)$ of $\rho_3[0,1]$

Solution : Here $\text{Dim}\,\rho_3[0,1]=4$; So required matrix for linear operator T is of order 4×4; where $T:\rho_3[0,1]\to\rho_3[0,1]$.

Now (i) $(1, x, x^2, x^3)$ forms a basis for $\rho_3[0,1]$.

Now we have,

$T(1) = 1$, $T(x) = (x + 1)$, $T(x^2) = (x + 1)^2$ and $T(x)^3 = (x + 1)^3$. So we write with respect to basis above

$$T(1) = 1 \quad = 1.1 + 0.x + 0.x^2 + 0.x^3$$
$$T(x) = 1 + x \quad = 1.1 + 1.x + 0.x^2 + 0.x^3$$
$$T(x^2) = (x + 1)^2 \quad = 1.1 + 2.x + 1.x^2 + 0.x^3$$
$$T(x^3) = (x + 1)^3 \quad = 1.1 + 3.x + 3.x^2 + 1.x^3$$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Here basis is $(1, 1+x, 1+x^2, 1+x^3)$ of $\rho_3[0,1]$

We have $T(1) = 1$, $T(1+x) = 1 + (1 + x)$, $T(1+x^2) = 1 + (1+x)^2$ and $T(1+x^3) = 1 + (1+x)^3$

Therefore relative to basis $(1, 1+x, 1+x^2, 1+x^3)$ we write

$$T(1) \quad = 1 = 1.1 + 0.(1+x) + 0.(1+x^2) + 0.(1+x^3)$$
$$T(1+x) = 2+x = 1.1 + 1.(1+x) + 0.(1+x^2) + 0.(1+x^3)$$
$$T(1+x^2) = 1+1+2x+x^2 = -1.1 + 2.(1+x) + 1.(1+x^2) + 0.(1+x^3)$$
$$T(1+x^3) = 1+1+3x+3x^2+x^3 = -5.1 + 3.(1+x) + 3.(1+x^2) + 1.(1+x^3)$$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note : Basis taken and treated above should be termed as ordered basis. In ordered basis order of arrangement of vectors is basis in important. For example, in

Euclidean 3-space R^3 we know (e_1, e_2, e_3) is a basis in R^3, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Then each of (e_1, e_2, e_3), (e_2, e_1, e_3) and (e_1, e_3, e_2) is an ordered basis and they are different ordered basis for R^3.

§ 3.2(A) ISOMORPHIC LINEAR SPACES :

Definition 3.2.1. Two linear spaces X and Y over the same scalars are said to be isomorphic (or, linearly isomorphic) if there is a linear operator $T : X \to Y$ that is 1–1 (injective) and onto (surjective). The operator T is called an Isomorphism.

Theorem 3.2.1. Linear isomorphism between linear spaces over same scalars on the class Γ, of all such spaces is an equivalence relation.

Proof : If $X \in \Gamma$, the identity operator $I : X \to X$ is an isomorphism. So the binary relation of being isomorphic is reflexive; let $X, Y \in \Gamma$ such that X is isomorphic to Y with $\varphi : X \to Y$ as an isomorphism; Then $\varphi^{-1} : Y \to X$ is also an isomorphism. Thus Y is isomorphic to X. Hence relation of isomorphism is symmetric. Finally, if $f : X \to Y$ and $g : Y \to Z$ are isomorphism, then $(g . f) : X \to Z$ is also an isomorphism. Therefore, the relation of isomorphism is transitive. Thus it is an equivalence relation.

Theorem 3.2.2. Every real linear space X with $\dim(X) = n$ is isomorphic to the Euclidean n-space R^n.

Proof : Let $(u_1, u_2, \ldots, u_n)$ form a basis in X. So if $u \in X$ we write

$$u = \xi_1 u_1 + \xi_2 u_2 + \ldots + \xi_n u_n \text{ uniquely.}$$

Define an operator $T : X \to R^n$ by the rule :

$$T(u) = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n \text{ where } u = \xi_1 u_1 + \xi_2 u_2 + \ldots + \xi_n u_n \in X$$

Then it is easily verified that T is a linear operator. Furhter, if $u = \sum_{i=1}^{n} \xi_i u_i$ and $v = \sum_{i=1}^{n} \eta_i u_i$ with $u \neq v$ are members of X, then we have

$$(\xi_1, \xi_2, \ldots, \xi_n) \neq (\eta_1, \eta_2, \ldots, \eta_n) \text{ or } T(u) \neq T(u);$$

thus T is 1–1. Finally, for $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$

We have $\sum_{i=1}^{n} \alpha_i u_i \in X$ such that $T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

So T is onto. Therefore X is isomorphic to R^n.

Notation : If two linear space X and Y are isomorphic we use the symbol $X \cong Y$.

Corollary : Any two real linear spaces of same finite dimension are isomorphic

Because if X and Y are finite dimensional real linear spaces with Dim(X) = Dim(Y), we apply Theorem 3.2.2. to say $X \cong R^n$; and hence $X \cong Y$.

Theorem 3.2.3. Every linear operator over a finite dimensional *NLS* is bounded (hence continuous).

Proof : Let (X, || ||) and (Y, || ||) be two *NLS* over same scalars and Dim(X) $<\infty$, say, being equal to n, and let (e_1, e_2, ..., e_n) be a basis for X. Then each member $x \in X$ has a unique representative as $x = \xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$ where ξ_i's are scalars. Let us define a norm $\|x\|'$ by the formula :

$$\|x\|' = \sum_{i=1}^{n} |\xi_i|.$$

It is an easy task to check that $\|x\|'$ is indeed a norm in X. Since X is finite dimensional, we know that any two norms in X are equivalent.

Therefore there is a +ve M satisfying

$$\|x\|' \le M\|x\| \text{ for all } x \in X$$

$$i.e. \quad \sum_{i=1}^{n} |\xi_i| \le M\|x\| \qquad \text{.......... (*)}$$

If $T : X \to Y$ is a linear operator and $x = \sum_{i=1}^{n} \xi_i e_i \in X$, we have

$$\|T(x)\| = \left\|T\left(\sum_{i=1}^{n} \xi_i e_i\right)\right\| = \left\|\sum_{i=1}^{n} \xi_i T(e_i)\right\|$$

$$\le \sum_{i=1}^{n} |\xi_i| \, \|T(e_i)\|$$

$$\le \max(\|T(e_i)\|, \|T(e_2)\|, ..., \|T(e_n)\|).M\|x\|$$

(from (*)) $= L\|x\|$, (say).

This being true for all $x \in X$, we conclude that T is bounded.

§ 3.3 SPACE OF ALL BOUNDED LINEAR OPERATORS Bd$\mathscr{L}(X,Y)$

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. Then zero operator $O : X \to Y$ where $O(x) = \underset{\sim}{0} \in Y$ *as* $x \in X$ is a bounded linear operator. Therefore $Bd\mathscr{L}(X,Y) \neq \phi$. It is a routine exercise to check that $Bd\mathscr{L}(X,Y)$ becomes a linear space with respect to addition and scalar multiplication as given by

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for all $x \in X$; and $T_1, T_2 \in Bd\mathscr{L}(X,Y)$ and

$(\lambda T_1)(x) = \lambda T_1(x)$ for all $x \in X$ and for all scalars λ and $T_1 \in Bd\mathscr{L}(X,Y)$

Theorem 3.3.1. *$Bd\mathscr{L}(X,Y)$ is a Normed Linear space, and it is a Banach space when Y is so.*

Proof : Let us take the norm in linear space $Bd\mathscr{L}(X,Y)$ as operator norm $\|T\|$ as $T \in Bd\mathscr{L}(X,Y)$. We verify that all norm axioms are satisfied here.

For (N.1) it is obvious that $\|T\| \geq 0$ always for any member $T \in Bd\mathscr{L}(X,Y)$; zero operator O has the norm $\|O\| = 0$.

Suppose $\|T\| = 0$ *i.e.* $\sup\limits_{\|x\| \leq 1} \|T(x)\| = 0$. So if $\|x\| \leq 1$, we have

$$\|Tx\| \leq \sup_{\|x\| \leq 1} \|T(x)\| = 0 \text{ gives } \|T(x)\| = 0 \quad \ldots\ldots (1)$$

If $\|x\| > 1$, then put $y = \dfrac{x}{\|x\|}$; Thus $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = 1$; so as got above

$$\|T(y)\| = 0 \text{ or } 0 = \|T(y)\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| = \frac{1}{\|x\|}\|T(x)\| \text{ giving}$$

$$\|T(x)\| = 0 \quad \ldots\ldots (2)$$

So (1) and (2) say that $T(x) = \underset{\sim}{0}$ for all $x \in X$ *i.e.* T equals to the zero operator.

For (N.2) take λ to be any scalar.

$$\text{Then } \|\lambda T\| = \sup_{\|x\| \leq 1} \|(\lambda T)(x)\|$$

$$= \sup_{\|x\| \leq 1} \|\lambda T(x)\| = \sup_{\|x\| \leq 1} \{|\lambda| \|T(x)\|\}$$

$$= |\lambda| \sup_{\|x\| \leq 1} \|T(x)\| = |\lambda| \|T\|.$$

So (N.2.) is satisfied.

For triangle inequality, if T_1, T_2 are members of $Bd\mathscr{L}(X,Y)$ we have for

$$x \in X\,,\ \|(T_1+T_2)(x)\| = \|T_1(x)+T_2(x)\| \le \|T_1(x)\| + \|T_2(x)\|$$

$\le \|T_1\|\|(x)\| + \|T_2\|\|x\| = (\|T_1\| + \|T_2\|)\|x\|$; this is true for all $x \in X$,

Therefore $\|T_1+T_2\| \le \|T_1\| + \|T_2\|$, and that is the triangle inequality.

Therefore $Bd\mathscr{L}(X,Y)$ is a Normed Linear space (*NLS*) with respect to operator norm.

Now suppose that Y is a Banach space. We show that $Bd\mathscr{L}(X,Y)$ is so. Take $\{T_n\}$ as a Cauchy sequence in $Bd\mathscr{L}(X,Y)$ *i.e.* $\|T_n - T_m\| \to 0$, as $n, m \to \infty$

If $x \in X$, we have $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$\le \|T_n - T_m\|\|x\| \to 0$ as $n, m \to \infty$. That means, $\{T_n(x)\}$ is a Cauchy sequence in $(Y, \|\ \|)$ which is complete.

Let $\lim\limits_{n\to\infty} T_n(x) = y \in Y$

Let us define $T : X \to Y$ by the rule :

$$T(x) = \lim_{n\to\infty} T_n(x) \text{ as } x \in X.$$

Now it is easy to see that T is a linear operator.

Further, $\left|\|T_n\| - \|T_m\|\right| \le \|T_n - T_n\| \to 0$ as $n, m \to \infty$.

That means $\{\|T_n\|\}$ is a sequence of non-negative reals and this is Cauchy sequence and therefore is bounded. So we find a +ve K satisfying

$$\|T_n\| \le K \text{ for all } n.$$

So, $\|T(x)\| = \|\lim\limits_{n\to\infty} T_n(x)\| = \lim\limits_{n\to\infty} \|T_n(x)\|$

$$\le \lim_{n\to\infty} \|T_n\|\|x\| \le K\|x\| \text{ by above inequality.}$$

This being true for all $x \in X$, we find $T : X \to Y$ as a bounded linear operator *i.e.* $T \in Bd\mathscr{L}(X,Y)$.

Finally, from Cauchyness of $\{T_n\}$, given a +ve ε, we have

$$\| T_{n+p} - T_n \| < \varepsilon \text{ for } n \ge n_0 \text{ and } p = 1,2,......$$

Take $\| x \| \le 1$ in X, So $\| T_{n+p}(x) - T_n(x) \| = \| (T_{n+p} - T_n)(x) \|$

$$\le \| T_{n+p} - T_n \| \| x \| \le \| T_{n+p} - T_n \| . < \varepsilon \text{ for } n \ge n_0$$

Let us pass on limit as $p \to \infty$, then we have

$$\| T(x) - T_n(x) \| \le \varepsilon \text{ whenever } n \ge n_0$$

This is the case whenever $\| x \| \le 1$; taking sup we have

$$\sup_{\|x\| \le 1} \| T(x) - T_n(x) \| \le \varepsilon \text{ whenever } n \ge n_0$$

Now $\| T - T_n \| = \sup_{\|x\| \le 1} \| (T - T_n)(x) \|$

$$= \sup_{\|x\| \le 1} \| (T(x) - T_n(x) \|$$

$$\le \varepsilon \text{ whenever } n \ge n_0$$

So we obtain $\lim_{n \to \infty} T_n = T \in Bd\mathscr{L}(X, Y)$ in operator norm.

The proof is now complete.

Example 3.3.1. Show $Bd\mathscr{L}(R^n, R^n)$ is finite dimensional with dimension n^2.

Solution : By matrix representation theorem we know that every member $T \in Bd\mathscr{L}(R^n, R^n)$ has a representative matrix of order $n \times n$ (*i.e.* a square matrix of size n). With respect to a fixed basis in R^n, we also see that $Bd\mathscr{L}(R^n, R^n)$ and the linear space $m_{n \times n}$ is finite dimensional with $\text{Dim}(m_{n \times n}) = n^2$.

Therefore $\text{Dim}(Bd\mathscr{L}(R^n, R^n)) = n^2$

Example 3.3.2. A *NLS* $(X, \| \|)$ is a Banach space if and only if $\{x \in X : \| x \| = 1\}$ is complete.

Solution : Suppose $(X, \| \|)$ is a Banach space; then the given set $\{x \in X : \| x \| = 1\}$ is a closed subset of X, and hence is complete.

Conversely, suppose $S = \{x \in X : \| x \| = 1\}$ is complete. Now let $\{x_n\}$ be a Cauchy sequence in X, so $\| x_n - x_m \| \to 0$ as $n, m \to \infty$

Therefore $\big| \| x_n \| - \| x_m \| \big| \le \| x_n - x_m \| \to 0$ as $n, m \to \infty$. Thus scalar sequence $\{\| x_n \|\}$ is Cauchy, and by Cauchy General Principle of convergence $\{\| x_n \|\}$ is convergent; put $\lim_{n\to\infty} \| x_n \| = \alpha$. If $\alpha = 0$ we see $\{x_n\}$ to be convergent in X and we have finished. Or else $\alpha > 0$. Without loss of generality we assume that $\alpha = 1$. Let us put $y_n = \frac{x_n}{\| x_n \|}$ making $\| y_n \| = 1$ *i.e.* $y_n \in S$. If possible, let $\{y_n\}$ be not Cauchy. Then there is a +ve ε_0 (say) and there are indices $n_k (\ge k), m_k (\ge k)$ such that

$$\| y_{n_k} - y_{m_k} \| \ge \varepsilon_0, \quad k = 1, 2, \ldots\ldots$$

$$\text{or, } \varepsilon_0 \le \left\| \frac{x_{n_k}}{\| x_{n_k} \|} - \frac{x_{m_k}}{\| x_{m_k} \|} \right\| \le \left\| \frac{x_{n_k}}{\| x_{n_k} \|} - x_{n_k} \right\| + \| x_{n_k} - x_{m_k} \| + \left\| x_{m_k} - \frac{x_{m_k}}{\| x_{m_k} \|} \right\|$$

$$= \| x_{n_k} \| \left| 1 - \frac{1}{\| x_k \|} \right| + \| x_{m_k} \| \left| 1 - \frac{1}{\| x_{m_k} \|} \right| \to 0 \quad \text{as} \quad k \to \infty;$$ arriving at contradiction that ε_0 is +ve. Therefore we conclude that $\{y_n\}$ is Cauchy in S by completeness of which let $\lim_{n\to\infty} y_n = y_0 \in S$. That is $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \| x_n \| y_0 = y_0$.

Hence $\{x_n\}$ is convergent in X and X is shown as a Banach space.

EXERCISE A

Short answer type questions

1. Let X be the linear space spanned by f and g where $f(x) = \sin x$ and $g(x) = \cos x$. For any real θ, let $f_1(x) = \sin(x+\theta)$ and $g_1(x) = \cos(x+\theta)$. Show that f_1 and g_1 are members of X, and they are linearly independent.

2. Let A and B be two subsets of a *NLS* X and let $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Show that if A or B is open then $A + B$ is open.

3. Let $m_{2\times2}$ be the linear space of all real 2×2 matrics and $E = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

If $T : m_{2\times2} \to m_{2\times2}$ is taken as $T(A) = EA$ for $A \in m_{2\times2}$, show that T is a linear operator.

4. If C is a convex subset of a *NLS* X and $x_0 \in X$, and α is a non-zero scalar, show that $x_0 + C$ and αC are convex sets.
5. Show that $T : C[a,b] \to R$ (real space with usual norm) defined by the rule :

$$T(f) = \int_a^b tf(t)dt; \quad f \in C[a,b].$$

Show that T is a bounded linear operator.

EXERCISE B

1. Let A and B be two subsets of a *NLS* X, and let $A+B = \{a+b : a \in A \text{ and } b \in B\}$. If A and B are compact, show that $A + B$ is compact.
2. Let M be a closed linear sub-space of a *NLS* $(X, \|\ \|)$, and X/M be the quotient space, and $T : X \to X/M$ where $T(x) = x + M$ for $x \in X$.

   Show that T is a bounded linear operator with $\|T\| \le 1$.
3. Show that the space of all real polynomials of degree $\le n$ is the closed interval $[a,b]$ is isomorphic to the Euclidean $(n+1)$-space R^{n+1}.
4. Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be *NLS* over same scalars and $F, T : X \to Y$ be bounded linear operators such that F and T agree over a dense set in X, show that $F \equiv T$.
5. If X is a finite Dimensional *NLS*, and Y is a proper sub-space of X, then show that there is a member $x \in X$ with $\|x\| = 1$. satisfying dist$(x, Y) = 1$.

UNIT 4

(*Contents* : Bounded Linear functionals, sub-linear functionals, Hahn-Banach Theorem; Its applications, Conjugate spaces of a *NLS*, Canonical mapping, Embedding of a *NLS* into its second conjugate space under a linear isometry, reflexive Banach space; Open mapping theorem, Closed Graph Theorem.).

§ 4.1 LINEAR FUNCTIONALS :

Let $(X, \| \|)$ be a *NLS* over reals/complex numbers.

Definition 4.1.1. A Scalar-valued Linear operator f over X is called a Linear functional.

For example if X = Banach space $C[0,1]$ with sup norm, then $f : X \to$ Reals (with usual norm) is a linear functional when $f(x) = \int_0^1 x(t)dt; \quad x \in C[0,1]$.

Explanation : Linear functionals are special kind of Linear operators, and thus enjoy all the properties of Linear operators like sending dependent set of the domain into a similar such elements in range.

Let us consider the collection of all continuous (bounded) linear functionals over X *i.e.* we have the space $Bd\mathscr{L}(X,R)$ whenever X is a real *NLS*. We have seen that the space $Bd\mathscr{L}(X,R)$ is always a *NLS* with operator norm $\|f\|$; f being a member of $Bd\mathscr{L}(X,R)$. We have also seen that the *NLS* $Bd\mathscr{L}(X,R)$ is a Banach space because R is so.

Definition 4.1.2. The space $Bd\mathscr{L}(X,R)$ denoted by X^* is called first conjugate space (Dual space) of X.

Thus first conjugate space or simply conjugate space X^* of any *NLS* $(X, \| \|)$ is always a Banach space irrespective of X being complete or not.

By a similar construction one can produce $Bd\mathscr{L}(X^*,R)$ = the space of all bounded linear functionals over X^*; this Banach space $X^{**} = (X^*)^*$ is called second conjugate (Dual) space of X; and so on.

Most of theory of conjugate spaces rests on one single theorem, known as famous Hahn-Banach Theorem that asserts that any continuous linear functional on a linear subspace of X can be extended to a continuous linear functional over X by keeping the norm-value of the functional unchanged. The proof of Hahn-Banach Theorem is lengthy but necessarily indispensable item in Functional Analysis.

Before we take up Hahn-Banach Theorem in setting of a *NLS* we proceed as under :

Definition 4.1.2. Let X be a real linear space. Then $p : X \to$ Reals satisfying (i) $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$ and (ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$, $x \in X$ is called a sub-linear functional.

Note : Condition (i) above is known as condition of sub-additivity and condition (ii) above is called positive homogeneity.

It is not difficult to see that norm function in a *NLS* X is a sub-linear functional over X.

Theorem 4.1.1. (Hahn-Banach Theorem in a linear space)

Let M be a subspace of a real linear space X, and p is a sub-linear functional over X and f is a linear functional on M such that $f(x) \le p(x)$ for all $x \in M$.

Then there is a linear functional F over X which is an extension of f (over M) such that

$$F(x) \le p(x) \text{ for all } x \in X.$$

The proof of this Theorem rests upon following Lemma.

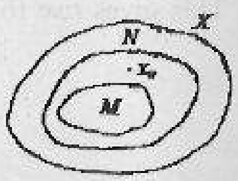
Lemma 4.1.1. Suppose M is a subspace ($\ne X$) of a real linear space X and $x_0 \in (X \setminus M)$. Let N be the subspace spanned by M and $\{x_0\}$ *i.e.* $N = [M \cup \{x_0\}]$; suppose $f : M \to R$ is a Linear functional such that

$f(x) \le p(x)$ for all $x \in M$, where $p : X \to R$ is a sub-linear functional (over X).

Then f can be extended to a linear functional F defined on N such that

$$F(x) \le p(x) \quad \text{for } x \in N.$$

Proof : Since $f(x) \le p(x)$ over M, we have for $y_1, y_2 \in M$.



$$f(y_1 - y_2) = f(y_1) - f(y_2) \le p(y_1 - y_2) = p(y_1 + x_0 - y_2 - x_0)$$

$$\le p(y_1 + x_0) + p(-y_2 - x_0)$$

or, $\quad -p(-y_2 - x_0) - f(y_2) \le p(y_1 + x_0) - f(y_1)$ .... (1)

(separation of terms involving y_1 and y_2)

Now fix y_1 and allow y_2 to change over M. From (1) we see that the set of reals $\{-p(-y_2 - x_0) - f(y_2)\}$ possesses sup.

Put $a = \sup\limits_{y_2 \in M} \{-p(-y_2 - x_0) - f(y_2)\}$; and in a similar argument, put

$b = \operatorname*{Inf}_{y_1 \in M}\{p(y_2 + x_0) - f(y_1)\}$. The relation (1) says, $a \le b$.

Take a real c_0 between a and b *i.e.* $a \le c_0 \le b$

Therefore as $y \in M$ we have

$$-p(-y - x_0) - f(y) \le c_0 \le p(y + x_0) - f(y) \quad \text{.............. (2)}$$

Since $x_0 \notin M$, we write $x_0 \in N$ as $x = y + \alpha x_0$, and this representation is unique.

Consider $F : N \to R$ defined by the rule :

$F(y + \alpha x_0) = f(y) + \alpha c_0$, as $(y + \alpha x_0) \in N$ ($y \in M$ & α a scalar). It is easy to check that F is a linear functional over N such that $F(y) = f(y)$ as $y \in M \subset N$.

In other words F is an extension of f from M to N. We verify further that $F(x) \le p(x)$ for all $x \in N$. To achieve this we are to consider following two cases : When $x \in N$, we have $x = y + \alpha x_0$, where α is a scalar.

Case I . When $\alpha > 0$; we consider R.H.S. of inequality (2) with y replaced by $\frac{y}{\alpha}$; this gives $c_0 \le p\left(\frac{y}{\alpha} + x_0\right) - f\left(\frac{y}{\alpha}\right)$.

Multiplying throughout by α and using the fact that p is sub-linear we have.

$$f(y) + \alpha c_0 \le p(y + \alpha x_0)$$

$$\text{or,} \quad F(x) \le p(x)$$

Case II. When $\alpha < 0$, we use L.H.S. of inequality (2) with y replaced by $\frac{y}{\alpha}$. This gives rise to

$$-p\left(-\frac{y}{\alpha} - x_0\right) - f\left(\frac{y}{\alpha}\right) \le c_0$$

$$\text{or,} \quad -p\left(\frac{y}{\alpha} - x_0\right) \le c_0 + f\left(\frac{y}{\alpha}\right)$$

Multiplying throughout by α and reversing the sign we have,

$$(-\alpha)p\left(-\frac{y}{\alpha} - x_0\right) \ge \alpha c_0 + f(y)$$

Since $-\alpha > 0$, we have $p(y+\alpha x_0) \geq \alpha c_0 + f(y)$

or, $p(x) \geq F(x)$

or, $F(x) \leq p(x)$

When $\alpha = 0$, we readily see $F(y) = f(y)$. The proof of Lemma is now complete.

Proof of Theorem 4.1.1. To prove the theorem we invite partial ordering in a set and use Zorn's Lemma which says that in a partially ordered set if every chain has an upper bound, then there is a maximal member in the set.

Here let Γ denote the collection of all linear functionals $\{\hat{f}\}$ such that each $\hat{f}$ is an extension of f such that $\hat{f}(x) \leq p(x)$ over domain of $\hat{f} \equiv D_{\hat{f}}$.

Lemma 4.1.1 tells us that Γ is non-empty. Let us partially order Γ as for $\hat{f}_1, \hat{f}_2 \in \Gamma$ we say, $\hat{f}_1 < \hat{f}_2$

if $\hat{f}_2$ is an extension of $\hat{f}_1$ with $D_{\hat{f}_2} \supset D_{\hat{f}_1}$, and $\hat{f}_2 = \hat{f}_1$ over $D_{\hat{f}_1}$.

We may verify that α is a partial order relation in Γ where we show that every chain (totally ordered subset) in Γ has an upper bound in Γ. To that goal, let $\tau = \{\hat{f}_\alpha\}$ be a totally ordered subset of Γ. We find some member $\hat{f} \in \Gamma$ to act as an upper bound for τ.

Construct $\hat{f}$ whose domain $= \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$. If $x \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ there is a member α such that $x \in D_{\hat{f}_\alpha}$ and let $\hat{f}(x) = f_{\hat{\alpha}}(x)$

By routine work we verify that $\underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ is a sub-space of X; taking $x, y \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ we find two indices α_1 and α_2 such that $x \in D_{\hat{f}_{\alpha_1}}$ and $y \in D_{\hat{f}_{\alpha_2}}$.

Since τ is totally ordered either $D_{\hat{f}_{\alpha_1}} \subset D_{\hat{f}_{\alpha_2}}$ or $D_{\hat{f}_{\alpha_2}} \subset D_{\hat{f}_{\alpha_1}}$, and in either of the cases we have

$(x+y) \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ and similarly $\alpha x \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ and $\underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ is a sub-space of X. Finally we show $\hat{f}$ is well-defined.

Suppose $x \in D_{\hat{f}_\alpha}$ and $x \in D_{\hat{f}_\beta}$; by definition

$\hat{f}(x) = f_{\hat{\alpha}}(x)$ and $\hat{f}(x) = f_{\hat{\beta}}(x)$

By total ordering of τ either $\hat{f}_\alpha$ is an extension of $\hat{f}_\beta$ or vice-versa.

So $\hat{f}_\alpha(x) = \hat{f}_\beta(x)$. Thus we have

$\hat{f}(x) \le p(x)$ for $x \in D_{\hat{f}}$ and for any member $\hat{f}_\alpha$ of τ, we have $\hat{f}_\alpha \alpha \hat{f}$. So $\hat{f} \in \Gamma$ is an upper bound of τ. So we apply Zorn's Lemma to obtain a maximal member (say) F in Γ. And F is the desired extension of f as a linear functional with $F(x) \le p(x)$ for all $x \in X$; that domain of F equals to X follows maximality of F; Otherwise by argument as above one can have an extension of F to some other functional—a contradiction of maximality of F. The proof of theorem is now complete.

Remark : Theorem 4.1.1 is also true for complex spaces, for which one has to furnish proof.

Theorem 4.1.2. (Hahn-Banach Theorem in a *NLS*).

Suppose f is a bounded linear functional on a sub-space M of *NLS* X. There is a bounded linear functional F which is an extension of f from M to X having the same norm as that of f.

Proof : If $x \in M$ we have $|f(x)| \le \|f\| \|x\|$.

Define $p: X \to R$ by the rule :

$$p(x) = \|f\| \|x\| \text{ for } x \in X..$$

Then we verify that p is a sub-linear functional over X.

Such that $f(x) \le p(x)$ for $x \in M$.

Now apply Theorem 4.1.1 (Hahn-Banach Theorem in real space) to get a linear functional F which is an extension of f from M to X such that

$$|F(x)| \le p(x) \quad \text{for all } x \in X.$$

$$i.e. \ |F(x)| \le \|f\| \|x\| \quad \text{for all } x \in X.$$

This is true for all $x \in X$; So we conclude that F is a bounded linear functional over X with $\|F\| \le \|f\|$ .... (1)

Further, over M we have $f(x) = F(x)$

So $|f(x)| = |F(x)| \le \|F\| \|x\|$ for all $x \in M$. This gives

$$\|f\| \le \|F\| \qquad \text{............... (2)}$$

Now (1) and (2) together say $\|f\| = \|F\|$

§ 4.2 SOME CONSEQUENCES OF HAHN-BANACH THEOREM :

Application I. Given a real *NLS* $(X, \| \|)$ and a non-zero member $x_0 \in X$. There is a bounded linear functional F over X such that $F(x_0) = \| x_0 \|$ with $\| F \| = 1$.

Proof : Consider the sub-space M of X spanned by x_0;

Then $M = [x_0] = \{\alpha x_0 : \alpha$ any real scalar$\}$

Define $f : M \rightarrow$ Reals by the rule :

$$f(\alpha x_0) = \alpha \| x_0 \|; \text{ as } (\alpha x_0) \in M.$$

Then f is a linear functional over M and $| f(x) | = | \alpha | \| x_0 \| = \| \alpha x_0 \|$ for all $x = \alpha x_0 \in M$ and hence we have $\| f \| \leq 1$. *i.e.* f is a bounded linear functional.

Further if $u = \alpha x_0$ is a member of M with $\| u \| = 1$ we have

$$| f(u) | = | \alpha | \| x_0 \| = \| \alpha x_0 \| = \| u \| = 1$$

$$\therefore \| f \| \geq | f(u) | = 1 \text{ giving } \| f \| = 1.$$

Now an application of Hahn-Banach Theorem gives a bounded linear functional F over X satisfying

$$F(x) = f(x) \quad x \in M$$

and $\quad \| F \| = \| f \| = 1$

This gives $F(x_0) = f(x_0) = \| x_0 \|$ and $\| F \| = 1$.

Corollary : For a non-null *NLS* $(X, \| \|)$, its conjugate space X^* is non-null. (Hints : because F appearing in corollary is non-zero member of X^*).

Application II. For every $x \in X$, $\| x \| = \sup\limits_{f(\neq 0) \in X^*} \dfrac{| f(x) |}{\| f \|}$.

Proof : From Application I we find a non-zero bounded linear functional $f_0 \in X^*$ such that $f_0(x) = \| x \|$ and $\| f_0 \| = 1$.

Therefore, $\sup\limits_{f \neq 0 \in X^*} \dfrac{| f(x) |}{\| f \|} \geq \dfrac{| f_0(x) |}{\| f_0 \|} = \| x \|$

i.e. $\sup\limits_{f(\neq 0) \in X^*} \dfrac{| f(x) |}{\| f \|} \geq \| x \|$ .... (1)

On the other hand, if f is any non-zero member of X^*, we have

$$|f(x)| \le \|f\| \|x\|$$

or $\dfrac{|f(x)|}{\|f\|} \le \|x\|$, r.h.s. being independent of f

we have, $$\sup_{f(\neq 0)\in X^*} \frac{|f(x)|}{\|f\|} \le \|x\| \qquad \text{.............. (2)}$$

From (1) and (2) one has $\|x\| = \sup\limits_{f(\neq 0)\in X^*} \dfrac{|f(x)|}{\|f\|}$.

Corollary : If $f(x) = 0$ for every non-zero bounded linear functional $f \in X^*$, then $x = \underset{\sim}{0}$ in X.

Application III. Let M be a closed subspace of X and $M \neq X_0$, if $u \in (X \setminus M)$ and $d = dist(u, M) = \inf\limits_{m\in M} \|u - m\|$,

Then $d > 0$, and there is a bounded linear functional $f \in X^*$ such that

(i) $f(x) = 0$ for $x \in M$

(ii) $f(u) = 1$

and (iii) $\|f\| = \dfrac{1}{d}$.

Proof : Here M is a closed sub-space ($\neq X$); so $d > 0$.

Take N = Linear subspace spanned by M and u

i.e. $N = [M \cup \{u\}]$; So every member of N is of the form $m + tu$ where t is a real scalar, and $m \in M$.

Define $g : N \to R$ by the rule :

$$g(m + tu) = t \text{ as } (m + tu) \in N.$$

It is easy to check that g is a linear functional over N such that g vanishes over M *i.e.* $g(m) = 0$ for $m \in M$, and $g(u) = 1$ (taking $t = 1$).

Now $|g(m + tu)| = |t| = \dfrac{|t| \cdot \|m + tu\|}{\|m + tu\|} = \dfrac{\|m + tu\|}{\|\frac{m}{t} + u\|}$

$$= \frac{\| m+tu \|}{\| u - (\frac{m}{t}) \|} \leq \frac{\| m+tu \|}{d} = \frac{1}{d} \| m+tu \|.$$

because $d = \inf_{v \in m} \| u - v \| \leq \| u - (-\frac{m}{t}) \|$.

This is true for all member $(m+tu) \in N$; and hence g is a bounded linear functional over N with $\| g \| \leq \frac{1}{d}$.

So, $\| g \| \leq \frac{1}{d}$ .... (1)

Again from $d = \inf_{v \in m} \| u - m \|$; we find a sequence $\{m_n\}$ in M

such that $\| u - m_n \| \to d$ as $n \to \infty$

i.e. $\lim_{n \to 0} \| u - m_n \| = d$ .... (2)

Now $| g(m_n - u) | \leq \| g \| \| m_n - u \|$

or, $| g(m_n) - g(u) | \leq \| g \| \| m_n - u \|$

or, $| 0 - 1 | \leq \| g \| \| m_n - u \|$; ($g$ vanishing over M and $g(u) = 1$).

or, $1 \leq \| g \| \| m_n - u \|$

Now passing on limit as $n \to \infty$ we produce

$$1 \leq \| g \| d$$

giving, $\| g \| \geq \frac{1}{d}$ .... (3)

Combining (1) and (3) we have $\| g \| = \frac{1}{d}$

Finally, Hahn-Banach Theorem says that g has an extension f from N to the whole space X as a bounded linear functional with $\| f \| = \| g \|$; As f and g agree over $M \subset N$, we have the result as wanted.

Application IV. Let M be a sub-space of *NLS* $(X, \| \|)$ and $M \neq X$; if $u \in (X \setminus M)$ such that $dist(u, M) > 0$, say $= d$.

Then there is a bounded linear functional $F \in X^*$ satisfying

(i) $F(x) = 0$ over M (for $x \in M$)

(ii) $F(u) = d$

and (iii) $\| F \| = 1$.

Proof : Let N = Linear sub-space spanned by M plus u, *i.e.* $N = [M \cup \{u\}]$

Now define $f : N \to$ Reals by rule :

$f(m + tu) = td$ (d as above), where $m + tu$ is a representative member of N($m \in M$, t a scalar).

Clearly f is a linear functional over N, such that for $t = 0$, f vanishes over M and $f(u) = d$ ($t = 1$).

Also for $t \neq 0$, $\| m + tu \| = \| -t\left(-\frac{m}{t} - u\right) \|$ (here $\frac{-m}{t} \in M$)

$$= |t| \; \| -\frac{m}{t} - u \| \geq |t| d.$$

So, $| f(m + tu) = |t| d \leq \| m + tu \|$; this inequality stands even for $t = 0$.

That means, f is a bounded linear functional over N with $\| f \| \leq 1$.

For $\in > 0$, we find by Infimum property, a member $m \in M$ such that $\| m - u \| < d + \in$.

Put $v = \frac{m - u}{\| m - u \|}$, making $\| v \| = 1$ and $v \in N$ (because, v is the form $m' + t'u$).

So, $| f(v) | = \frac{d}{\| m - u \|} > \frac{d}{d + \in} = \frac{d}{d + \in} \| v \| \; (\because \| v \| = 1)$

That means, $\| f \| \geq \frac{d}{d + \in}$. Now this is true for every +ve $\in$, and taking $\in \to 0_+$, we find $\| f \| \geq 1$.

i.e. $\| f \| \geq 1$ .... (2)

Combining (1) and (2) we find $\| f \| = 1$. Now we apply Hahn-Banach Theorem to find an extension F of f from N to the whole space X as a bounded linear functional over X with $\| F \| = \| f \|$; since F agrees with f over M, we have the result as desired.

§ 4.3 CONJUGATE SPACES X^*, X^{**}, ... OF A *NLS* $(X, \|\ \|)$:

Let $(X, \|\ \|)$ be a *NLS*; then X^*, $X^{**} = (X^*)^*$, .. are first, second, ...conjugate space of X.

Theorem 4.3.1. If X^* is separable, then so is X.

Proof : Suppose D is a countable dense subset of X^*. Let D_1 be the subset of D which is dense in the surface $\{f \in X^* : \| f \| = 1\}$ of the closed unit ball of X^*; let us write $D_1 = \{f_1, f_2, ..., f_n, ...\}$ with $\| f_n \| = 1$ for all n. From $\| f_n \| = 1$, we find a member say x_n with $\| x_n \| = 1$ such that

$$| f_n(x_n) | > \frac{1}{2}.$$

Consider the linear sub-space L of X spanned by $\{x_1, x_2, ..., x_n\}$

i.e. $L = [x_1, x_2, ..., x_n .....]$ and Put $M = \bar{L}$ (closure of L). The M is also a linear sub-space of X.

Suppose, $M \neq X$ .... (1)

Take $x_0 \in (X \setminus M)$, then $d = dist(x_0, M) > 0$ because M is closed.

By application of Hahn-Banach Theorem we obtain a bounded linear functional $F \in X^*$ with $\| F \| = 1$ such that F vanishes ($F = 0$) over M and $F(x_0) \neq 0$.

Clearly F is a member of the set $\{f \in X^* : \| f \| = 1\}$ and $F(x_n) = 0$ for all n.

Now $f_n(x_n) = f_n(x_n) - F(x_n) + F(x_n)$ gives

$$| f_n(x_n) | \leq | f_n(x_n) - F(x_n) | + | F(x_n) |$$
$$= | (f_n - F)(x_n) |$$

Thus $\frac{1}{2} < | f_n(x_n) | \leq \| f_n - F \| \| x_n \|$

or, $\frac{1}{2} < \| f_n - F \|$ for all n; This contradicts that $\{f_1, f_2, ..., f_n, ..\}$ is dense in the set $\{f \in X^* : \| f \| = 1\}$.

So, $M = X$.

That is $\bar{L} = X$;

Now L contains that subset formed by finite linear combinations of $x_1, x_2,.., x_n,..$ with rational coefficients; and that subset becomes countable dense in X. The proof is now complete.

Remark : Converse of Theorem 4.3.1 is not true. The *NLS* l_1 consisting of all those real sequences $\underline{x} = (x_1, x_2 ... x_n, ...)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ with norm $\| \underline{x} \| = \sum_{i=1}^{\infty} |x_i|$ is separable but its conjugate space l_∞ consisting of all bounded sequences of reals is not separable.

Example 4.3.1. Let $(X, \| \, \|)$ be a *NLS* over reals, and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Show that there is a bounded linear functional f over X such that $f(x_1) \neq f(x_2)$.

Solution : Here $x_1, x_2 \in X$ with $x_1 \neq x_2$ *i.e.* $x_1 - x_2 \neq \underline{O}$ in X. So an application of Hahn-Banach Theorem there is a bounded linear functional $f \in X^* (f \neq 0)$ such that

$$f(x_1 - x_2) \neq 0$$

or, $f(x_1) - f(x_2) \neq 0$

or, $f(x_1) \neq f(x_2)$.

Given a *NLS* $(X, \| \, \|)$ we show that there is a natural embedding of X in its second conjugate space X^{**} through a mapping, called the Canonical mapping that we presently define using X^*.

Theorem 4.3.2. Given $x \in X$, let $\hat{x}(x^*) = x^*(x)$ for all $x^* \in X^*$. Then $\hat{x}$ is a bounded linear functional over X^*, and the mapping $x \to \hat{x}$ is a Linear Isometry of X into X^{**}.

Proof : Let $x \in X$, $x_1^*, x_2^* \in X^*$; then we have

$$\hat{x}(x_1^* + x_2^*) = (x_1^* + x_2^*)(x) = x_1^*(x) + x_2^*(x) = \hat{x}(x_1^*) + \hat{x}(x_2^*).$$

Also if λ is any scalar we have $\hat{x}(\lambda x_1^*) = (\lambda x_1^*)(x) = \lambda x_1^*(x) = \lambda \hat{x}(x_1^*)$.

Therefore $\hat{x}$ is a linear functional over X^*.

Now we show that $\| x \| = \sup_{\|x^*\| \leq 1} \{ |x^*(x)| \}$.

By Hahn-Banach Theorem we find a member $x^* \in X^*$ with $\|x^*\| = 1$ and $\|x^*(x)\| = \|x\|$.

Therefore $\|x\| \le \sup_{\|x^*\|\le 1} \{|x^*(x)|\}$ .... (1)

Again $\|x^*(x)\| \le \|x^*\| \|x\| \le \|x\|$ when $\|x^*\| \le 1$

Therefore $\|x\| \ge |x^*(x)|$ when $\|x^*\| \le 1$

Thus $\|x\| \ge \sup_{\|x^*\|\le 1} |x^*(x)|$. .... (2)

From (1) and (2) we have

$$\|x\| = \sup\{|x^*(x)| : x^* \in X^* \text{ with } \|x^*\| \le 1\}.$$

which is $= \sup\{|\hat{x}(x^*)| : x^* \in X^* \text{ with } \|x^*\| \le 1\}$

$= \|\hat{x}\|$.

It shows that $\hat{x}$ is a bounded linear functional over X^* with $\|\hat{x}\| = \|x\|$.

Finally, let $x_1, x_2 \in X$ and $x^* \in X^*$, then

$$\begin{aligned}\widehat{(x_1 + x_2)}(x^*) &= x^*(x_1 + x_2) \\ &= x^*(x_1) + x^*(x_2) \\ &= \hat{x}_1(x^*) + \hat{x}_2(x^*).\end{aligned}$$

Similarly for any scalar α we have $\widehat{(\alpha x_1)}(x^*) = x^*(\alpha x_1)$

$$= \alpha x^*(x_1)$$

$$= \alpha \hat{x}_1(x^*)$$

Therefore the mapping $x \to \hat{x}$ is linear; and since $\|\hat{x}\| = \|x\|$, this mapping is Isometry.

That is, $x \to \hat{x}$ is a Linear Isometry of X onto the linear sub-space $\{\hat{x} : x \in X^*\}$ of X^{**}.

Definition 4.3.1. Given a *NLS* $(X, \|\ \|)$, Linear Isometry $x \to \hat{x}$ is called the Canonical mapping of X into its second conjugate space X^{**}.

Definition 4.3.2. A *NLS* $(X, \| \|)$ is called reflexive if and only if the Canonical mapping $x \to \hat{x}$ maps X onto X**.

Thus a necessary condition for X to be reflexive is that X is a Banach space. However there are Banach spaces without being reflexive.

§ 4.4 OPEN MAPPING THEOREM AND CLOSED GRAPH THEOREM :

Like a big and important theorem of Hahn-Banach we have another big theorem known as open mapping theorem in Functional analysis. There one is concerned with open mappings that send open sets into open sets. Open mapping theorem states conditions under which a bounded linear operator shall be an open mapping.

Definition 4.4.1. Let X and Y be two metric spaces. Then a mapping $f : X \to Y$ is called an open mapping if G is an open set in X, its image under $f = f(G)$ is an open set in Y.

Theorem 4.4.1. Let $(X, \| \|)$ and $(Y, \| \|)$ be two Banach spaces; and $T : X \to Y$ be a bounded linear operator which is onto (surjective). Then T is an open mapping.

The proof of the above theorem shall rest on following Lemma that we prove first.

Lemma 4.4.1 Let $T : X \to Y$ be a bounded linear operator which is onto and let $B_0 = B_1(\underset{\sim}{0})$ be the open unit ball in X, then $T(B_0)$ contains an open ball centred at $\underset{\sim}{0}$ in Y.

Proof : We may complete the proof in three stages as under :

(a) $\overline{T(B_0)}$ (closure of $T(B_0)$) contains an open ball B^*.

(b) If B_n = open ball $B_{\frac{1}{2^n}}(\underset{\sim}{0})$ in X, then $\overline{T(B_n)}$ shall contain an open ball V_n centred at $\underset{\sim}{0}$ in Y.

and (c) $T(B_0)$ contains an open ball cenred at $\underset{\sim}{0}$ in Y.

(a) Consider open ball $B_1 = B_{\frac{1}{2}}(\underset{\sim}{0}) \subset X$. If $x \in X$, we find large real k so that $x \in kB_1$. Therefore we write

$X = \bigcup_{k=1}^{\infty} kB_1$; Since T is onto and linear, we have

$Y = T(X) = T\left(\bigcup_{k=1}^{\infty} kB_1\right) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}$, taking closure did not add more points to the Union = Y. As Y is a Banach space, we invite Baire Category

Theorem to conclude that one component say $\overline{kT(B_1)}$ contains an open ball. That means $\overline{T(B_1)}$ contains an open ball, say, $B^* = B(y_0, \varepsilon)$. So we write

$$B^* - y_0 = B(\underset{\sim}{o}, \varepsilon) \subset \overline{T(B_1)} - y_0$$

(b) We show that $B^* - y_0 \subset \overline{T(B_0)}$, where B_0 stands as appearing in theorem. This is accomplished by showing :

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

Take $y \in \overline{T(B_1)} - y_0$; then $(y + y_0) \in \overline{T(B_1)}$ and remembering that $y_0 \in \overline{T(B_1)}$ we find

$$u_n = T(w_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} u_n = y + y_0$$

$$v_n = T(z_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} v_n = y_0 .$$

Since $w_n, z_n \in B_1$ and B_1 is of radius $= \frac{1}{2}$ we have

$$\| w_n - z_n \| \le \| w_n \| + \| z_n \| < \frac{1}{2} + \frac{1}{1} = 1; \text{ So that } (w_n - z_n) \in B_0 .$$

From $T(w_n - z_n) - T(w_n) - T(z_n) = u_n - v_n \to y$ as $n \to \infty$.

Therefore, $y \in \overline{T(B_0)}$. Since $y \in \left(\overline{T(B_1)} - y_0\right)$ is an arbitrary we have shown

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

From $B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_1)} - y_0$ above we have

$$B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_0)} \qquad \text{.................} \quad (1)$$

Take $B_n = B(\underset{\sim}{0}, 2^{-n}) \subset X$. Since T is linear, we have $\overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$;
From (1) one obtains

$$V_n = B(\underset{\sim}{0}, \frac{\varepsilon}{2^n}) \subset \overline{T(B_n)} \qquad \text{.......................} \quad (2)$$

(c) Finally, we show that $V_1 = B(\underset{\sim}{0}, \frac{1}{2}\varepsilon) \subset T(B_0)$.

Take $y \in V_1$. From (2), for $n = 1$, we have $V_1 \subset \overline{T(B_1)}$.

Hence $y \in \overline{T(B_1)}$ and we find $v \in \overline{T(B_1)}$ such that $\| y - v \| < \frac{\varepsilon}{4}$

Now $v \in \overline{T(B_1)}$ implies $v \in T(x_1)$ for some $x_1 \in B_1$.

Therefore $\| y - T(x_1) \| < \frac{\varepsilon}{4}$

Using this and (2) above with $n = 2$ we see that $(y - T(x_1)) \in V_2 \subset \overline{T(B_2)}$.

As before we find $x_2 \in B_2$ such that $\| y - T(x_1) \| - T(x_2) \| < \frac{\varepsilon}{8}$

Hence $(y - T(x_1) - T(x_2) \in V_3 \subset \overline{T(B_3)}$, and so on. In nth step we take $x_n \in B_n$ such that

$$\left\| y - \sum_{k=1}^{n} T(x_k) \right\| < \frac{\varepsilon}{2^{n+1}}, \quad n = 1, 2, \ldots \qquad \text{.............. (3)}$$

Put $z_n = x_1 + x_2 + \ldots + x_n$; Since $x_k \in B_k$, we have $\| x_k \| < \frac{1}{2^k}$ that means $n > m$,

$$\| z_n - z_m \| \le \sum_{k=m+1}^{n} \| x_k \| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \text{ which } \to 0 \text{ as } m \to \infty.$$

So $\{z_n\}$ is Cauchy; let $\lim_{n\to\infty} z_n = x$ (X is a Banach space).

Also $x \in B_0$ since B_0 has radius = 1, and

$$\sum_{k=1}^{\infty} \| x_k \| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

As T is continuous, we have $\lim_{n\to\infty} T(z_n) = T(x)$ and (3) shows that $T(x) = y$.

So $y \in T(B_0)$.

Proof of Theroem 4.4.1. If A is an open set in X, we show that $T(A)$ is open in Y, by showing that every $y \in T(x) \in T(A)$ attracts an open ball centred at $y = T(x)$ within $T(A)$.

Take $y = T(x) \in T(A)$. As A is open there is an open ball centred at $x \subset A$. Hence $A - x$ contains an open ball centred at $\underset{\sim}{0} \in X$. Let radius of that open ball $= r$. Put $k = \frac{1}{r}$ or $r = \frac{1}{k}$. Then $k(A - x)$ contains the open unit ball $B(\underset{\sim}{0}.1)$. Now Lemma 4.4.1 says that $T\left(k(A-x)\right) = k[T(A)-T(x)]$ contains an open ball centred at $\underset{\sim}{0}$, and so does $T(A) - T(x)$. Hence $T(A)$ contains an open ball centred at $y = T(x)$. As y is an arbitrary member of $T(A)$, we have shown that $T(A)$ is open.

Corollary : Under open mapping theorem if T is bijective, T^{-1} is bounded.

Example 4.4.1. Let $T : R^2 \to R$ be defined by $T(x,y) = x$ for $(x,y) \in R^2$. Show that T is an open mapping. Examine if $T : R^2 \to R^2$ where $T(x, y) = (x, 0)$, $(x, y) \in R^2$ is an open mapping.

Solution : Here $T : R^2 \to R$ given by $T(x, y) = x$ is a projection mapping and we know that it is a bounded linear operator such that T is onto. So we apply open mapping theorem to conclude that T is an open mapping (In fact, T sends open circular disc of R^2 onto an open interval).

If $T : R^2 \to R^2$ is given by $T(x, y) = (x, 0)$; there Image of an open circular disc under T is not like that. So T is not an open mapping.

We know that all linear operators are bounded. For instance, differential operator is an unbounded linear operator. Closed Linear operators that we introduce presently behave satisfactorily in this respect. Another important theorem, known as closed Graph Theorem states sufficient conditions under which a closed linear operator on a Banach space is bounded.

Let $(X \,||\ ||)$ and $(Y, ||\ ||)$ be *NLS* with same scalars.

Definition 4.4.2. A linear operator $T : X \to Y$ is called a closed linear operator if its graph $G(T) = \{(x, y) \in (X \times Y) : y = T(x), x \in X\}$ is a closed set in *NLS* $X \times Y$ with norm $||(x, y)|| = ||x|| + ||y||$, $(x, y) \in (X \times Y)$.

Theorem 4.4.2. Let X and Y be Banach spaces, and $T : X \to Y$ be a closed linear operator. Then T is a bounded linear operator.

Proof : First we verify that $X \times Y$ with norm $||(x, y)|| = ||x|| + ||y||$ as $(x, y) \in (X \times Y)$ is also a Banach space.

Let $\{z_n = (x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$.

Then $||z_n - z_m|| = ||x_n - x_m|| + ||y_n - y_m||$

Thus $\| x_n - x_m \| \le \| z_n - z_m \| \to 0$ as $n, m \to \infty$ shows that $\{x_n\}$ is Cauchy in X, and since X is complete,

$$\text{let } \lim_{n\to\infty} x_n = x \in X, \quad \text{and similarly let } \lim_{n\to\infty} y_n = y \in Y.$$

These together imply that $\lim_{n\to\infty} z_n = z = (x, y) \in (X \times Y)$. Thus we see that $X \times Y$ is a Banach space. Graph $G(T)$ being a closed set in $X \times Y$, it follows that $G(T)$ is complete (infact, $G(T)$ is a Banach space as a subspace of $X \times Y$)

Consider a mapping $p : G(T) \to X$ given by $p(x, T(x)) = x \in X$. Then p is linear operator over $G(T)$. p is also bounded, because

$$\| p(x, T(x)) \| = \| x \| \le \| x \| + \| T(x) \| = (x, T(x)) \|$$

Further, p is bijective, with p^{-1} given by

$p^{-1} : X \to G(T)$ mapping $x \to (x, T(x))$ as $x \in X$. By applying open mapping theorem we find p^{-1} to be bounded. Hence there is a +ve K such that

$$\| (x, T(x)) \| \le K \| x \| \quad \text{for} \quad x \in X.$$

Therefore $\| T(x) \| \le \| T(x) \| + \| x \| = \| (x, T(x)) \| \le K \| x \|$.

That means T is bounded. The proof is complete.

Example 4.4.2. If X and Y are Banach spaces over same scalars, and $T : X \to Y$ is a linear operator. Show that Graph $G(T)$ is a subspace of $X \times Y$.

Solution : Let $(x_1, T(x_1))$ and $(x_2, T(x_2))$ be two members of $G(T)$ as $x_1, x_2 \in X$, where $G(T) = \{(x, T(x)) : x \in X\} \subset (X \times Y)$.

$$\begin{aligned} \text{Then} \quad (x_1, T(x_1)) + (x_2, T(x_2)) &= (x_1 + x_2, T(x_1) + T(x_2)) \\ &= (x_1 + x_2, T(x_1 + x_2)) \quad (T \text{ is linear}) \\ &\in G(T). \end{aligned}$$

If λ is any scalar $\lambda(x_1, T(x_1)) = (\lambda x_1, \lambda T(x_1)) = (\lambda x_1, T(\lambda x_1)) \in G(T)$.

Therefore $G(T)$ is a sub-space of $(X \times Y)$.

EXERCISE A

Short answer type questions

1. Show that a norm in a linear space X is a sub-linear functional over X.
2. Show that a sub-linear functional p in a linear space X satisfies (a) $p(\underline{0}) = 0$ and (b) $p(-x) \geq -p(x)$ for $x \in X$.
3. Show that non-null *NLS* X has a non-null conjugate space X^*.
4. If $f(x) = f(y)$ for every bounded linear functional on a *NLS* X, show that $x = y$ in X.
5. If X and Y are Banach spaces show that the Null space $N(T)$ of a closed linear operator $T : X \to Y$ is a closed sub-space of X.
6. If two non-zero linear functionals f_1 and f_2 over a linear space have the same Null space, then show that f_1 and f_2 are proportional.

EXERCISE B

1. Let X be a *NLS*, and $x_0 \in X$ such that $|f(x_0)| \leq c$ for all $f \in X^*$ with $\|f\| = 1$, show that $\|x_0\| \leq c$.
2. If X is a *NLS* which is reflexive, show that X^* is reflexive.
3. If X and Y are Banach spaces over the same scalars, and $T : X \to Y$ is a closed linear operator, then show that (a) if C is compact in X, $T(C)$ is closed in Y, and (b) if K is compact in Y, $T^{-1}(K)$ is closed in X.
4. Let f be a non-zero linear functional in a linear space X, and x_0 is a fixed element in $(X \setminus N(f))$, ($N(f)$ = Null space of $f = \{x \in X : f(x) = 0\}$), then any member x in X has a unique representation $x = \alpha x_0 + y$ where $y \in N(f)$. Prove it.
5. Show that $T : C[a,b] \to R$ defined by $T(f) = \int_a^b f dt$, $f \in C[a,b]$ is a bounded linear functional over $C[a,b]$ and find $\|T\|$.
6. Show that f defined over $C[-1,1]$ by the rule :

$$f(x) = \int_{-1}^{0} x dt - \int_{0}^{1} x dt, \quad x \in C[-1,1]$$

is a bounded linear functional over $C[-1, 1]$ and find $\|f\|$.

UNIT 5

(***Contents*** : Inner product spaces, Cauchy-Schwarz inequality, I.P. spaces as *NLS*, continuity of I.P. function, Law of parallelogram, orthogonal (orthonormal) system of vectors, Projection Theorem in Hilbert space H; Reisz Theorem for a bounded linear functional over H, Bessel's inequality, Grahm-Smidst orthogonalisation process, complete orthonormal system in H.)

§ 5.1 INNER PRODUCT SPACE

In a Normed Linear space principle operations involved are addition of vectors and scalar multiplication of vectors by scalars as in elementary vector algebra. Norm in such a space generalizes elementary idea of length of a vector. What is still more missing in an *NLS* is an analogue if well known dot product $a.b = a_1b_1 + a_2b_2 + a_3b_3$, and resulting formulas among other things like (i) length $|a| = \sqrt{a.a}$ and (ii) relation of orthogonality $a.b = 0$. These are important tools in numerous applications.

History of Inner product spaces is older than that of *NLS*. Theory had been initiated by Hilbert through his work on integral equations. An inner product space is a Linear space with an inner-product structure that we presently define.

Suppose X denotes a complex Linear space.

Definition 5.1.1. X is said to be an Inner Product space or simply I.P. space if there is a scalar-valued function known Inner product function, denoted by, $<, >$ over $X_x X$ satisfying

(I.P. 1) $<x + y, z> = <x, z> + <y, z>$ for all $x, y, z \in X$,

(I.P. 2) $<\alpha x, y> = \alpha<x, y>$ for all scalars α and for all $x, y \in X$,

(I.P. 3) $<y, x> = \overline{<x,y>}$ for all $x, y \in X$, bar denoting complex conjugate.

(I.P. 4) $<x, x> \geq 0$ for all $x \in X$ and it is $= 0$ if and only if $x = \underline{0}$ in X.

From I.P. axioms above one can immediately derive the following :

(a) $<x,\alpha y> = \overline{\alpha} <x,y>$ for all scalars α and $x, y \in X$.

(b) $<\lambda x + \mu y, z> = \lambda <x,z> + \mu <y,z>$ for all $x, y, z \in X$ and for all scalars λ, μ.

(c) $<x,\alpha y + \beta z> = \overline{<\alpha x + \beta z, x>} = \overline{\alpha <y,x>} + \overline{\beta <z,x>}$

$= \overline{\alpha}\,\overline{<y,x>} + \overline{\beta}\,\overline{<z,y>} = \overline{\alpha} <x,y> + \overline{\beta} <x,z>$, because conjugate of a complex scalar is itself.

Example 5.1.1. Unitary space $\mathbb{C}^n = \underbrace{\mathbb{C}\times\mathbb{C}\times\ldots\times\mathbb{C}}_{n\ copies}$ whose $\mathbb{C}$ is the space of all complex number is an I.P. space with I.P. $< >$ given by

$<\underset{\sim}{z}, \underset{\sim}{w}) = z_1\overline{w}_1 + z_2\overline{w}_2 + \ldots + z_n\overline{w}_n$ where $\underset{\sim}{z} = (z_1, z_2, \ldots z_n)$ and $\underset{\sim}{w} = (w_1, w_2, \ldots w_n) \in \mathbb{C}^n$.

Solution : Here $\overline{<\underset{\sim}{z}, \underset{\sim}{w}>} = \overline{z_1\overline{w}_1 + z_2\overline{w}_2 + \ldots + z_n\overline{w}_n}$

$= \overline{z_1\overline{w}_1} + \overline{z_2\overline{w}_2} + \ldots + \overline{z_n\overline{w}_n} = \overline{z}_1 w_1 + \overline{z}_2 w_2 + \ldots + \overline{z}_n w_n$

$= <\underset{\sim}{w}, \underset{\sim}{z}>$; and this (I.P. 3); rest of axioms are routine check-ups.

In an I.P. space $(X, < >)$ of $x \in X$, let us define $\|x\|^2 = <x,x>$ which is always a non-negative quantity and is equal to 0 if and only if $x = \underset{\sim}{0}$ in X.

Theorem 5.1.1. Every I.P. space is an *NLS*. To prove this Theorem we need help from following Lemma that is an independent proposition as well.

Lemma 5.1.1 (Cauchy-Schwarz inequality/C-S inequality)

In an I.P. space $(X, < >)$ if $x, y \in X$,

$$|<x,y> \leq \|x\|\|y\|.$$

Proof : Without loss of generality take $y \neq \underset{\sim}{0}$ in X. (taking $y = \underset{\sim}{0}$ L.H.S. = R.H.S. For any scalar λ we have

$$\|x + \lambda y\|^2 \geq 0$$

or, $<x + \lambda y, x + \lambda y> = 0$

or, $<x,y> + \lambda\overline{\lambda} <y,y> + \overline{\lambda} <x,y> + \overline{\lambda} <y,x> \geq 0$

or, $\|x\|^2 + |\lambda|^2 \|y\|^2 + \overline{\lambda} <x,y> + \lambda\overline{<x,y>} \geq 0$

Let us now choose $\lambda = -\dfrac{<x,y>}{<y,y>}$

$$= -\frac{<x,y>}{\|y\|^2}.$$

Then L.H.S. of above inequality

$$= \|x\|^2 + \frac{|x,y|^2}{\|y\|^2} - \frac{|<x,y>|^2}{\|y\|^2} - \frac{|<x,y>|^2}{\|y\|^2} = \|x\|^2 - \frac{|<x,y>|^2}{\|y\|^2}$$

Therefore above inequality assumes the form

$$\|x\|^2 - \frac{|<x,y>|^2}{\|y\|^2} \geq 0$$

or $$|<x,y> \leq \|x\|\|y\|.$$

Proof of Theorem 5.1.1. Norm axioms (N.1) and (N.2) follow from (I.P. 4); and the fact $\|\alpha x\|^2 = <\alpha x, \alpha x> = \alpha\bar{\alpha} <x,x> = |\alpha|^2 \|x\|^2$.

This gives $\|\alpha x\|^2 = |\alpha| \|x\|$

For triangle inequality (N.3), let $x, y \in X$, then we have

$$\|x+y\|^2 = <x+y, x+y> = \|x\|^2 + <x,y> + <y,x> + \|y\|^2.$$

Thus $\|x+y\|^2 \leq \|x\|^2 + |<x,y> + |<y,x>| + \|y\|^2$

$$= \|x\|^2 + 2|<x,y>| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$ by Lemma 5.1.1.

$$= (\|x\| + \|y\|)^2.$$

Therefore $\|x+y\| \leq \|x\| + \|y\|$.

The proof is now complete.

Remark : Equality sign in C-S inequality holds if and only if $y = 0$ or $0 = \|x + \lambda y\|^2$ *i.e.* $x = -\lambda y$ or $x + \lambda y = 0$ showing that x and y to be linearly dependent.

Theorem 5.1.2. In an I.P. space $(X, <>)$, show that I.P. function is a continuous function.

Proof : Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_n z_n = x$ and $\lim_{x\to\infty} y_n = y$ in norm. That is to say, $\lim_{n\to\infty} \|x_n - x\| = 0 = \lim_{n\to\infty} \|y_n - y\|$.

Now $|<x_n, y_n> - <x,y>| = |<x_n, y_n> - <x_n, y> + <x_n, y> - <x,y>|$

$$= |<x_n, y_n - y> + <x_n - x, y>|$$

$$\leq |< x_n, y_n - y >| + |< x_n - x, y >|$$
$$\leq \| x_n \| \| y_n - y \| + \| y \| \| x_n - x \|;$$

Since $\lim_{x\to\infty} x_n = x$ in norm, $\{x_n\}$ is norm bounded; So there is an M (+ve) such that $\| x_n \| \leq M$ for all n.

Therefore above inequality assumes the form

$\leq M \| y_n - y \| + \| y \| \| x_n - x \| \to 0$ as $n \to \infty$. This shows that $\lim_{n\to\infty} < x_n, y_n > = < x, y >$ and I.P. function is continuous at (x, y).

Definition 5.1.2. An I.P. space X is said to be a Hilbert space if X is a complete *NLS* with norm $\| \ \|$ as induced from I.P. function.

Thus every Hilbert space is a Banach space. But opposite is not true.

Very often a Hilbert space is denoted by H and an I.P. space is termed as a pre-Hilbert space.

Theorem 5.1.3. If x and y are two members in a Hilbert space H, then

$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$. (Law of parallelogram).

Proof : Here $\| x + y \|^2 + \| x - y \|^2 = < x + y, x + y > + < x - y, x - y >$

$$= \| x \|^2 + < x, y > + < y, x > + \| y \|^2 + \| x \|^2 - < x, y > - < y, x > + \| y \|^2$$

$$= 2 \| x \|^2 + 2 \| y \|^2.$$

Remark : In school Geometry it is known that sum of squares raised on sides of a parallelogram is equal to the sum of squares raised on its diagonals. This is exactly what is in Theorem 5.1.3 above. Hence the name is Law of parallelogram.

Example 5.1.2. The sequence space l_2 of all real sequences $\underset{\sim}{x} = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ with $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ is a real Hilbert space.

Solution : We know that l_2 is a real linear space where let us define an I.P. function $< \underset{\sim}{x}, \underset{\sim}{y} > = \sum_{i=1}^{\infty} \xi_i \eta_i$, the r.h.s. series is convergent because

$|\xi_i \eta_i| \leq \frac{1}{2}(|\xi_i|^2 + |\eta_i|^2$ $[(\underset{\sim}{x} = (\xi_1, \xi_2, \ldots), \ \underset{\sim}{y} = (\eta_1, \eta_2, \ldots) \in l_2]$. $i = 1, 2, \ldots\ldots$

By routine exercise we check that all I.P. axioms are O.K. in l_2, and l_2 is an I.P.

space with real scalars. Further, with respect to the induced norm $\| \underset{\sim}{x} \|^2 = \sum_{i=1}^{\infty} |\xi_i|^2$ it is also known that l_2 becomes a complete *NLS*. Hence l_2 is a Hilbert space.

Example 5.1.3. The sequence space $l_p (1 < p < \infty)$ consisting of all real sequences $\underset{\sim}{x} = (\xi_1, \xi_2, ...)$ with $\left(\sum_{i=1}^{\infty} |\xi_i| p \right)^{1/p} < \infty$ is a Banach space without being a Hilbert space with I.P. function to induce Banach-space norm.

Solution : We have already seen that sequence space $l_p (1 < p > \infty)$ is a Banach space with norm $\| x \| = \left(\sum_{i=1}^{\infty} |\xi_i| p \right)^{1/p}$, as $\underset{\sim}{x} = (\xi_1, \xi_2, ...) \in l_p$. We now show that this norm does not come from an I.P. function on l_p. This is verified by showing that this norm does not satisfy Law of Parallelogram. Take $\underset{\sim}{x} = \left(1, 1, \frac{0}{\text{block}}\right)$, $\underset{\sim}{y} = \left(1, -1, \frac{0}{\text{block}}\right)$ from l_p. Then we find $\| \underset{\sim}{x} \| = \| \underset{\sim}{y} \| = 2^{1/p}$ and $\| \underset{\sim}{x} + \underset{\sim}{y} \| = 2$ $= \| \underset{\sim}{x} - \underset{\sim}{y} \|$. Therefore, if $p \neq 2$ parallelogram law fails.

§ 5.2 ORTHOGONAL ELEMENTS IN HILBERT SPACE

Let H denote a Hilbert space.

Definition 5.2.1. (a) Two members x and y in a Hilbert space H are called orthogonal if $< x, y > = 0$;

We write in this case $x \perp y$.

(b) Given a non-empty subset L of H, an element $x \in$ H is said to be orthogonal to L, denoted by $x \perp y$ if $< x, l > = 0$ for every member $l \in L$.

Theorem 5.2.1. (Pythogorian Law) If $x, y \in H$ and $x \perp y$, then

(i) $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

(ii) $\| x - y \|^2 = \| x \|^2 + \| y \|^2$

Proof : (i) $\| x + y \|^2 = < x + y, x + y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2$

$= \| x \|^2 + < x, y > + \overline{< x, y >} + \| y \|^2 = \| x \|^2 + \| y \|^2$ since $<x, y> = 0$.

(ii) the proof is similar to above.

Theorem 5.2.2. Every closed convex subset of a Hilbert space H has a unique member of smallest norm.

Proof : Let C be a closed convex subset of H, and let $d = \text{Inf}\{\|x\| : x \in C\}$.

Let $\{x_n\}$ be a sequence in C such that $\lim\limits_{n\to\infty} \|x_n\| = d$. for $x_n, x_m \in C$ we have $\frac{1}{2}(x_n + x_m) \in C$, because C is convex.

So, $\left\|\frac{x_n + x_m}{2}\right\| \geq d$ or, $\|x_n + x_m\| \geq 2d$ .... (1)

By Law of Parallelogram we have

$$\|x_n - x_n\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$
$$\leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

Since $\lim\limits_{n\to\infty} \|x_n\| = d$ and similarly $\|x_m\| \to d$ as .... (2)

$m \to \infty$; taking limit $n, m \to \infty$ in (2) we get

$\lim\limits_{n,m\to\infty} \|x_n - x_m\| = 0$; showing that $\{x_n\}$ is Cauchy in C.

As C is closed, Let $\lim\limits_{n\to\infty} x_n = x \in C$. Thus $\|x\| = \lim\limits_{n\to\infty} \|x_n\| = d$.

Hence $x \in C$ has a smallest norm. For uniqueness of x, let $x' \in C$ so that $\|x'\| = d$. By convexity of C we have $\frac{x + x'}{2} \in C$ and also $\|\frac{x+x'}{2}\| \geq d$. Again by Law of Parallelogram we have

$$\|\frac{x+x'}{2}\|^2 = \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \frac{\|x - x'\|^2}{2}$$
$$< \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x'\|^2 \quad \text{if } x \neq x'$$

$= d^2$; giving $\|\frac{x+x'}{2}\| < d$—a contradiction of $\|\frac{x+x'}{2}\| \geq d$ as arrived at early. The proof is now complete.

Theorem 5.2.3 (Projection Theorem). Let L be a closed subspace of H and $L \neq H$; Then every member $x \in H$ has a unique representation $x = y + z$ where $y \in L$ and $x \perp L$.

Proof : If x is a member of $L \subset H$; we write $x = x + \underset{\sim}{0}$ when $\underset{\sim}{0} \in \perp L$.

Let us take $x \in (H \setminus L)$, and put

$$d = \operatorname*{Inf}_{a \in L} \| x - a \|^2 = dist(x, L)$$; Because L is closed we have $d > 0$,

and there is a sequence $\{a_n\}$ of member a_n in L such that

$$\lim_{n \to \infty} d_n = \| x - a_n \|^2 = d. \qquad \text{............... (1)}$$

Take any non-zero member a in L. As L is a sub-space of H, we have for any scalar ε, $(a_n + \varepsilon a) \in L$ and therefore

$$\| x - (a_n + \varepsilon a) \|^2 \geq d$$

or, $$< x - a_n - \varepsilon d,\ x - a_n - \varepsilon a > \geq d$$

or, $$\| x - a_n \|^2 - \bar{\varepsilon} < x - a_n, a > - \varepsilon < a, x - a_n > + |\varepsilon|^2 \| a \|^2 \geq d.$$

Now take $\varepsilon = \dfrac{< x - a_n, a >}{\| a \|^2}$; with such a choice of ε, we have

$$\| x - a_n \|^2 - \frac{|< x - a_n, a >|^2}{\| a \|^2} \geq d$$

or, $$|< x - a_n, a >|^2 \leq \| a \|^2 (d_n - d)$$

or, $$|< x - a_n, a >| \leq \| a \| \sqrt{d_n - d} \qquad \text{............... (*)}$$

Inequality holds for $a = \underset{\sim}{0}$ in L; So for any $a \in L$ we have

$$|< a_n - a_m, a >| \leq |< a_n - x, a >| + |< x - a_m, a >|$$

i.e. $$|< a_n - a_m, a >| \leq \| a \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right) \qquad \text{from (*)}$$

Putting $a = a_n - a_m$, we have

$$|< a_n - a_m, a_n - a_m >| \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$$

i.e. $$\| a_n - a_m \|^2 \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$$

or, $\quad \| a_n - a_m \| \le \left(\sqrt{d_n - d} + \sqrt{d_m - d}\right)$, where r.h.s. $\to 0$ as $n, m \to \infty$ by (1).

That means $\{a_n\}$ is Cauchy in L.

Since L is closed, let $\lim\limits_{n\to\infty} a_n = y \in L$.

Now in $|< x - a_n, a >| \le \| a \| \sqrt{d_n - d}$, let us pass on the $\lim\limits_{n\to\infty} a_n = y$ and get $|< x - y, a >| = 0$

i.e. $< x - y, 0 > = 0$; This is true for any member a in L; Therefore $(x - y) \perp L$. Let us put $z = x - y$.

Then we have $x = y + z$ where $y \in L$ and $z \perp L$.

For uniqueness of this representation, let $x = y + z = y' + z'$ where $y' \in L$ and $z' \perp L$. Thus y, y' come from L and $z, z' \perp L$. Clearly, $y - y' = z' - z$, and

$\| y - y' \|^2 = < y - y', y - y' > = < y - y', z' - z > = 0$ where $\| z' - z \| \perp L$. Therefore $y = y'$ and hence $z = z'$. The proof is now complete.

Remark : In representation Theorem 5.2.3. where $x = y + z$, y is called projection of x on L. It is obvious that collection M of all elements, orthogonal to L forms a sub-space. M is also closed because of continuity of I.P. function. That is why z is called projection of x on M which is called orthogonal complement of L. Further, Hilbert space H is then sum of two orthogonal sub-spaces L and M. Here we see orthogonal sum is a special case of the Direct sum. Thus projection Theorem 5.2.3 gives a decomposition of any member in Hilbert space H into its projections onto two complementary orthogonal sub-spaces.

§ 5.3. It is important to know that the general form of a bounded Linear functional acting on a given space. Such formulae in respect of some *NLS* are known; their derivations could be much complicated. Situation is, however, surprisingly simple for a Hilbert space H.

Theorem 5.3.1 (Riesz Theorem on representation of functional over H).

Let f be a bounded linear functional over a Hilbert space H. Then $f(x) = < x, y >$ for all $x \in H$ and for some $z \in H$ uniquely determined by f such that $\| z \| = \| f \|$.

Proof : If f is the zero functional over H. We take $z = \underset{\sim}{0}$ in H to do the job. Suppose that f is a non-zero bounded linear functional over H. Consider the null-space $N(f)$ of f where

$N(f)=\{x \in H : f(x)=0\}$. Clearly $N(f)$ is a closed linear sub-space of H without being equal to H.

Take a non-zero $z_0 \in \perp N(f)$

Let $x \in$ H. Put $v = f(x)z_0 - f(z_0)x$

So that $f(v) = f(f(x)z_0) - f(f(z_0)x)$

$$= f(x)f(z_0) - f(z_0)f(x) \quad ; (f \text{ is linear})$$

$$= 0$$

That means $v \in N(f)$; by choice z_0 is orthogonal to v

So $0 = <v, z_0> = <f(x)z_0 - f(z_0)x, z_0>$

$$= f(x) <z_0, z_0> - f(z_0) <x, z_0>$$

$$= \| z_0 \|^2 f(x) - f(z_0) <x, z_0>$$

Giving $f(x) = \dfrac{f(z_0)}{\| z_0 \|} <x, z_0>$

$$= \left\langle x, \overline{\frac{f(z_0)}{\| z_0 \|}} z_0 \right\rangle$$

$$= <x, y> \text{ (say), where } z = \overline{\frac{f(z_0)}{\| z_0 \|^2}} z_0 . \qquad \text{.............. (1)}$$

This is the representative formula for $f(x)$ as wanted.

For uniqueness of z, let $f(x) = <x, z_1> = <x, x_2>$ for all $x \in$ H.

Then we have $<x, z_1> = <x, z_2>$ or, $<x, z_1 - z_2> = 0$

put $x = z_1 - z_2$; So $<z_1 - z_2, z_1 - z_2> = 0$ or, $\| z_1 - z_2 \|^2 = 0$ or, $z_1 = z_2$.

Finally, We have $| f(x) | = |<x, z>| \leq \| x \| \| z \|$

This gives $\| f \| \leq \| z \|$ .... (1)

Again taking $z = x$ in (1) we have $<z, z> = f(z)$

$$\text{or, } \| z \|^2 \leq \| f \| \| z \|$$

$$\text{or, } \| z \| \leq \| f \| \qquad \text{.............. (2)}$$

Combining (1) and (2) we have $\| f \| \le \| z \|$.

Converse of Theorem 5.3.1. is true. This is what Example 5.3.1 has to say.

Example 5.3.1. Let z be a fixed member in a Hilbert space H. Show that

$f(x) = < x, z >$ for all $x \in$ H is a bounded linear functional over H with $\| f \| = \| z \|$.

Solution : Here $f : H \to$Scalar such that for $x_1, x_2 \in H$.

Then $f(x_1 + x_2) = < x_1 + x_2, z > = < x_1, z > + < x_2, z > = f(x_1) + f(x_2)$.

And for any scalar α $f(\alpha x_1) = < \alpha x_1, z > = \alpha < x_1, z > = \alpha f(x_1)$.

Thus f is Linear. Further $| f(x) | = |< x, z >| \le \| x \| \| z \|$ (by C-S inequality)

This is true for all $x \in$ H. Therefore f is a bounded linear functional such that

$$\| f \| \le \| z \| \qquad \text{.............. (1)}$$

Taking $x = z$ in $f(x) = <x, y>$ we have

$$\| z \|^2 = < z, z > = f(z) \le \| f \| \| z \|$$

$$\text{or, } \| z \| \le \| f \| \qquad \text{.............. (2)}$$

(1) plus (2) gives $\| f \| \le \| z \|$.

Corollary to Theorem 5.3.1. Every Hilbert space H is reflexive.

Because by Theorem 5.3.1. together example put up above says that every bounded linear functional over H. *i.e.* every member of H^* arises out of a member of H and conversely. This correspondence gives rise to an isomorphism between H and H^*; and we say that H is self-dual and this in turn implies that here Canonical mapping between H and H^{**} is a surjection. Hence H is reflexive.

§ 5.4 ORTHONORMAL SYSTEM IN HILBERT SPACE *H*.

Definition 5.4.1. (a) A non-empty subset $\{e_i\}$ of Hilbert space H is said to be an orthonormal system if

(i) $i \ne j$, $e_i + e_j$ *i.e.* any two distinct members of $\{e_i\}$ are orthogonal.

and (ii) $\| e_i \| = 1$ for every i *i.e.* any vector of the system is non-zero unit vector in H.

(b) If an orthonormal system of H is countable, we can enumerate its elements in a sequence say it as an orthonormal sequence.

For example in Euclidean n-space R^n which is a real Hilbert space the fundamental unit vectors $e_1 = (1,0,0,...,0)$, $e_2 = (0,1,0,0,...,0)$ ...... $e_n = (1,0,...,0,1)$ form an orthonormal system of vectors in R^n.

Example 5.4.1. Let $L_2[0,2\pi]$ be the real Hilbert space of all square integrable functions f over $[0,2\pi]$ with I.P. function

$$< f,g > = \int_0^{2\pi} fg dt; \qquad f,g \in L_2 \ [0,2\pi].$$

$$\therefore \| f \| = \sqrt{\sum_0^{2\pi} f^2 dt}.$$

Then $e_0(t) = \frac{1}{\sqrt{2\pi}}, e_n(t) = \frac{\cos nt}{\sqrt{\pi}}$ $\{n = 1, 2, ...)$ and $0 \le t \le 2\pi$.

form an orthonormal sequence in $L_2[0,2\pi]$; because

$$\int_0^{2\pi} \cos mt \ \cos nt \ dt = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n = 1,2... \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

Theorem 5.4.1. An orthonormal system in H is linearly independent.

Proof : Let $\{e_i\}$ be an orthonormal system in H, and let for a finite subset, say, $e_1, e_2, ...e_n$ of the system we have

$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = \underset{\sim}{0}$ where α_i's are scalars. Then for $1 \le j \le n$ we have

$$0 = < \underset{\sim}{0}, e_j > = \left\langle \sum_{i=1}^{n} \alpha_i e_i, e_j \right\rangle = \sum_{i=1}^{n} \alpha_i, < e_i, e_j >$$

$= \alpha_j < e_j, e_j > = \alpha_j$; (other terms being zero because of mutual orthogonality). So $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. That means any finite sub-system of the given system is linearly independent. Hence proof is done.

Definition 5.4.2. Let $\{e_i\}$ be an orthonormal system in H and $x \in H$; Then scalars $c_i = < x, e_i >$ are called Fouries co-efficients of x w.r.t the system.

Theorem 5.4.2. Suppose $\{e_1, e_2, e_3, ..., e_n, ...\}$ be an orthonormal sequence in H;

then for $x \in$ H,

$$\sum_{i=1}^{\infty} |< x, e_i >|^2 \leq \| x \|^2$$

(This inequality is very often termed as Bessel's inequality).

Proof : Let n be a +ve integer. If c_i are Fouries coefficients of x w.r.t. $\{e_i\}$, we have

$$0 \leq \| x - \sum_{i=1}^{n} c_i e_i \|^2 = \left\langle x - \sum_{i=1}^{n} c_i e_i, x - \sum_{i=1}^{n} c_i e_i \right\rangle$$

$$= \| x \|^2 - \left\langle x, \sum_{i=1}^{n} c_i e_i \right\rangle - \left\langle \sum_{i=1}^{n} c_i e_i, x \right\rangle + \left\langle \sum_{i=1}^{n} c_i e_i, \sum_{k=1}^{n} c_k e_k \right\rangle$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c_i} < x, e_i > - \sum_{i=1}^{n} c_i < e_i, x > + \sum_{i=1}^{n} c_i < e_i, \sum_{k=1}^{n} c_k e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c_i} c_i - \sum_{i=1}^{n} c_i \overline{c_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} c_i \overline{c_k} < e_i, e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} | c_i |^2 - \sum_{i=1}^{n} | c_i |^2 + \sum_{i=1}^{n} | c_i |^2 = \| x \|^2 - \sum_{i=1}^{n} | c_i |^2$$

Therefore, $\sum_{i=1}^{n} | c_i |^2 \leq \| x \|^2$ or, $\sum_{i=1}^{n} |< x, e_i >|^2 \leq \| x \|^2$.

This is true for any +ve integer n, and thus $\sum_{i=1}^{\infty} |< x, c_i >|^2$ is convergent and

$\sum_{i=1}^{n} |< x, c_i >|^2 \leq \| x \|^2$.

Theorem 5.4.3. In a separable Hilbert space H every orthonormal system is countable.

Proof : Let $E = \{e_i\}$ be an orthonormal system in H which is separable. If $e_i \neq e_j$ we have $< e_i, e_j > = 0$ and $\| e_i \| = 1 = \| e_j \|$.

Therefore $<e_i - e_j, e_i - e_j> = \| e_i \|^2 - <e_i, e_j> - <e_j, e_i> + \| e_j \|^2 = 0+1+1=2$

So, $\| e_i - e_j \|^2 = 2$

or, $\| e_i - e_j \| = \sqrt{2}$.

By separability of H, we find a countable set $\{y_1, y_2, ..., y_n, ...\}$ which is dense in E. So we find two members, say, y_i and y_i such that

$$\| y_i - e_i \| < \frac{\sqrt{2}}{3} \text{ and } \| y_j - e_j \| < \frac{\sqrt{2}}{3}.$$

So $\sqrt{2} = \| e_i - e_j \| = \| e_i - y_i + y_i - y_j + y_j - e_j \|$

$$\leq \| e_i - y_i \| + \| y_i - y_j \| + \| y_j - e_j \|$$

$$< \frac{2\sqrt{2}}{3} + \| y_i - y_j \|.$$

Showing $\| y_i - y_j \| > \frac{\sqrt{2}}{3}$. clearly $i \neq j$; This establishes an H correspondence between members of E with members of a subset of a countable set. Therefore E is countable.

Gram-Schmidt Orthogolisation Process : Subject is that in a Hilbert space H one can transform a linearly independent set of elements in H into an orthonormal system in H by a technique known by above name.

Let x_1, x_2,... be an independent system of vectors in H (So none is zero vector)

Put $e_1 = \frac{x_1}{\| x_1 \|}$ and let $y_2 = x_2 - c_{21}e_1$ where $c_{21} = <x_2, e_1>$.

Next we put $e_2 = \frac{y_2}{\| y_2 \|}$; By verification we see $<e_1, e_1> = 1$, $<e_2, e_2> = 1$, and $<e_1, e_2> = 0$.

Now let $y_3 = x_3 - (c_{31}e_1 + c_{32}e_2)$ where we choose $c_{31} = <x_3, e_1>$, $c_{32} = <x_3, e_2>$.

Next we put $e_3 = \frac{y_3}{\| y_3 \|}$, and as before we have

$$<e_3, e_3> = 1, \; <e_3, e_2> = 0 = <e_3, e_1>.$$

We continue this process, if $e_1, e_2, ..., e_{k-1}$ have been constructed, let us take

$$y_k = x_k - \sum_{i=1}^{k-1} c_{ki} e_i$$

where $c_{ki} = < x_k, e_i >$; so that y_k is orthogonal to $e_1, e_2, ..., e_{k-1}$; Define $e_k = \dfrac{y_k}{\| y_k \|}$. Inductively, we construct e_n as a linear combination of $x_1, x_2, ...$ and x_n. This way we are led to orthonormal system ($e_1, e_2, ... e_n, ...$) from ($x_1, x_2, ... x_n, ...$).

Definition 5.4.3. In a Hilbert space H an orthonormal system E is called a complete orthonormal system if there is no orthonormal system in H to contain E as a proper subset.

For example, in Euclidean n-space R^n (a real Hilbert space) the set of all fundamental unit vectors $\{e_1, e_2, ..., e_n\}$ where $e_j = (\underline{0} \ \underset{j\text{th place}}{\underline{1}} \ \underline{0})$, $j = 1, 2, ..., n$ is a complete orthonormal system in R^n.

Theorem 5.4.4. In a Hilbert space H let $\{e_1, e_2, ..., e_n ...\}$ be an orthonormal sequence in H. Then following statements are equivalent (one implies other).

(a) $\{e_i\}$ is complete.

(b) $< x, e_i > = 0$ for all i implies $x = \underset{\sim}{0}$ in H.

(c) $x = \sum_{i=1}^{\infty} < x, e_i > e_i$ for each $x \in H$.

(d) $\sum_{i=1}^{\infty} |< x, e_i >|^2 = \| x \|^2$ for every $x \in H$.

Proof : (a) $\Rightarrow$ (b); Let (a) be true. Suppose (b) is false. Then we find a non-zero x in H such that $< x, e_i > = 0$ for $i = 1, 2, ...$

Put $e = \dfrac{x}{\| x \|}$. So that $\| e \| = 1$, and $< e, e_i > = 0$ for all i,

Therefore $\{e_1, e_2, ..., ...e_n, ...\} \cup \{e\}$ becomes an orthonormal system containing given system properly—a contradiction that $\{e_1, e_2, ..., e_n\}$ is complete. Hence (b) is established.

<u>(b) $\Rightarrow$ (c)</u> Let $S_n = \sum_{i=1}^{n} < x, e_i > e_i$;

Then $\sum_{i=1}^{\infty} < x, e_i > e_i = \lim_{n\to\infty} S_n = S$ (say)

If $1 \le j \le n$, $< x, e_j > - < S_n, e_j >$

$$= < x, e_j > - < \sum_{i=1}^{n} < x, e_i > e_i, e_j >$$

$$= < x, e_j > - < x, e_j > = 0$$

Thus $< S_n, e_j > = < x, e_j >$

Now $< x - \sum_{i=1}^{n} < x, e_i > e_i, e_j > = < x - S, e_j > = < x, e_j > - < S, e_j >$

$$= < x, e_j > - < \lim_{n\to\infty} S_n, e_j > = < x, e_j > - \lim_{n\to\infty} < S_n, e_j > = < x, e_j > - < x, e_j > = 0$$

That means $e_j \perp \left(x - \sum_{i=1}^{\infty} < x, e_i > e_i \right)$; therefore from (b) we have

$$x - \sum_{i=1}^{n} < x, e_i > e_i = \underset{\sim}{0} \quad i.e. \quad x = \sum_{i=1}^{\infty} < x, e_i > e_i .$$

<u>(c) ⇒ (d)</u>. We have $\| x \|^2 = < x, x > = < \sum_{i=1}^{\infty} < x, e_i > e_i . \sum_{j=1}^{\infty} < x, e_j > e_j >$

$$= < \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > e_i, < \lim_{n\to\infty} \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} < \sum_{i=1}^{n} < x, e_i > e_i, \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > \overline{< x, e_i >} = \lim_{n\to\infty} \sum_{i=1}^{n} |< x, e_i >|^2$$

$$= \sum_{i=1}^{\infty} |< x, e_i >|^2 .$$

(d) $\Rightarrow$ (a). Let (*d*) hold and if possible let $\{e_i\}$ be not complete. Then we find an orthonormal system strictly larger than $\{e_1, e_2, \ldots, e_n, \ldots\}$; say larger system looks as $\{e, e_1, e_2, \ldots, e_n, \ldots\}$ where, of course, $\| e \| = 1$ and $<e, e_i> = 0$ for $i = 1, 2, \ldots$ Now (d) applies (taking $x = e$), and we have

$$\| e \|^2 = \sum_{i=1}^{\infty} |< e, e_i >|^2 = 0$$ — a contradiction. So we have proved (a).

Example 5.4.2. Let $\{x_n\}$ be a sequence in Hilbert space H and $x \in H$ such that $\lim_{n \to \infty} \| x_n \| = \| x \|$, and $\lim_{n \to \infty} < x_n, x >\| = < x, x >$. Show that $\lim_{n \to \infty} x_n = x$.

Solution : Given $\lim_{n \to \infty} \| x_n \| = \| x \|$ and $\lim_{n \to \infty} < x_n, x > = < x, x > = \| x \|^2$.

Now $\| x_n - x \|^2 = < x_n - x, x_n - x > = \| x_n \|^2 - < x_n, x > - < x, x_n > + \| x \|^2$

$$= \| x_n \|^2 - < x_n, x > - \overline{< x_n, x >} + \| x \|^2$$

$$\to \| x \|^2 - \| x \|^2 - \| x \|^2 + \| x \|^2 = 0 \text{ as } n \to \infty.$$

Therefore $\lim_{n \to \infty} x_n = x$

Example 5.4.3. In a real Hilbert space H if $\| x \| = \| y \|$, show that $<x + y, x - y> = 0$. Interpret the result Geometrically if H = Euclidean 2-space R^2.

Solution : Let H be a real Hilbert space and $x, y \in H$ that such $\| x \| = \| y \|$.

Now $< x + y, x - y > = < x, x > - < x, y > + < y, x > - < y, y >$

$= \| x \|^2 - < x, y > + < x, y > - \| y \|^2$ (because it is a real Hilbert space, $< x, y > = \overline{< x, y >}$)

$= 0$

That means $(x + y) \perp (x - y)$.

It Euclidean 2-space R^2, fig is an equilateral parallelogram *i.e.* a Rhombus with adjacent sides represented by x and y with $\| x \| = \| y \|$; and we know that in a Rhombus Diagonals cut at right angles.

EXERCISE A

Short answer type questions

1. If in an I.P. space $< x, u > = < x, v >$ for all x in the space, show that $u = v$.

2. Show that Banach space $c[a,b]$ with sup norm is not a Hilbert space with an I.P. to induce the sup norm.
3. If f is a bounded linear functional over Euclidean 3-space R^3, show that f can be represented by a dot product

$$f(x) = x.z = \xi_1\rho_1 + \xi_2\rho_2 + \xi_3\rho_3 .$$

4. Show that in a Hilbert space H convergence of $\sum_{j=1}^{\infty} \| x_j \|$ implies convergence of $\sum_{j=1}^{\infty} x_j$
5. If ¢ denotes the Unitary space of all complex numbers. If $z_1, z_2 \in$ ¢, show that $< z_1, z_2 > = z_1, \overline{z}_2$ defines an I.P. function on ¢.

EXERCISE B

1. If x and y are two non-zero elements in a Hilbert space H, show that $\| x+y \| \leq \| x \| + \| y \|$ where equality holds if and only if $y = \alpha x$ for a suitable scalar α.
2. Let c be a convex set in a Hilbert space H, and $d = \text{Inf}\{\| x \| : x \in c\}$. If $\{x_n\}$ is a sequence in c such that $\lim_{n\to\infty} \| x_n \| = d$, show that $\{x_n\}$ is a Cauchy sequence.
3. If $\{e_n\}$ is any orthonormal sequence in a Hilbert space H and $x, y \in H$, show that

$$\left| \sum_{n=1}^{\infty} < x, e_n > < y, e_n > \right| \leq \| x \| \| y \|$$

4. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal set in a Hilbert space H where n is fixed. If $x \in H$ be a fixed member, show that for scalars $\alpha_1, \alpha_2, ..., \alpha_n$ $\| x - \sum_{i=1}^{n} \alpha_i e_i \|$ is minimum when $\alpha_i = < x, e_i >, i = 1, ..., n$.
5. Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space H. For $x \in H$, define $y = \sum_{k=1}^{\infty} < x, e_k > e_k$; show that $(x - y) \perp e_k$ $(k = 1, 2, ...)$.
6. Show that for the sequence space l_2 (a real Hilbert space) its conjugate space l_2^* is isomorphic to l_2.

UNIT 6

(***Contents*** : Adjoint of bounded linear operator in a Hilbert space H, Algebra of adjoint operators, proudct of adjoints, self-adjoint operators in H, their algebra, Norm of self-adjoint operator, space of self-adjoint operators, every bounded linear operator in H as a sum of self-adjoint operators, eigen value and eigen vectors of self-adjoint operator.)

§ 6.1 Let H be a complex Hilbert space and let $Bd\alpha(H,H)$ denote the space of all bounded linear operators $T: H \to H$. Take one such $T: H \to H$ as a bounded linear operator. Let $y \in$ H.

Define $f_y : H \to$ scalars by the rule :

$$f_y(x) = <T(x), y> \text{ as } x \in H \qquad \text{........... (1)}$$

Notice that if $x_1, x_2 \in$ H, we have

$$f_y(x_1 + x_2) = <T(x_1 + x_2), y> = <T(x_1) + T(x_2), y> \text{ because } T \text{ is linear}$$

$$= <T(x_1), y> + <T(x_2), y> \text{ using property inner product}$$

$$= f_y(x_1) + f_y(x_2);$$

Similarly $f_y(\alpha x_1) = \alpha f_y(x_1)$ for any scalar α.

That means, f_y is a linear functional over H.

Plus $| f_y(x) | = |<T(x), y>| \leq \| T(x) \| \| y \|$ by C-S inequality,

$$\leq \| T \| \| x \| \| y \| = (\| T \| \| y \|) \| x \| \text{ for all } x \in H.$$

Therefore, f_y is a bounded linear functional over H, and as we had seen earlier, Riesz representation Theorem says, there is a unique member, say $y^* \in H$ such that

$$f_y(x) = <x, y^*> \qquad \text{........... (2)}$$

where we remember that y^* is determined by f_y. From the text as put up above one sees that given $y \in H$, there is a unique member $y^* \in H$ (via f_y).

Let us define $T^* : H \to H$ by formula :

$$T^*(y) = y^* \text{ as described above} \qquad \text{........... (3)}$$

This operator T^* is called adjoint operator to T in H and as explained above they are connected by relation

$< T(x), y > = < x, T^*(y) >$ from (1), (2) and (3) above **for all** $x, y \in H$.

Explanation : T^* is well defined over H. Because, suppose that for all $x, y \in H$, we have simultaneously

$$< T(x), y > = < x, T^*(y) >$$

and $\quad < T(x), y > = < x, T_1^*(y) >$ for another $T_1 : H \to H$.

Therefore we see $< x, T^*(y) > = < x, T_1^*(y) >$ for all $x, y \in H$.

meaning thereby $T^*(y) = T_1^*(y)$ for $y \in H$. *i.e.* $T^* \equiv T_1^*$

Theorem 6.1.1. $T^* : H \to H$ is a bounded linear operator $(T^* \in Bd\alpha(H, H))$.

Proof : Let $x, y, z \in H$. Then $< x, T^*(y+z) > = < T(x), y+z >$

$$= < T(x), y > + < T(x), z > = < x, T^*(y) > + < x, T^*(z) >$$

$$= < x, T^*(y) + T^*(z) >.$$

Therefore, $T^*(y+z) = T^*(y) + T^*(z)$ .... (1)

Again for a scalar λ, $< x, T^*(\lambda y) > = < T(x), \lambda y >$

$$= \bar{\lambda} < T(x), y > = \bar{\lambda} < x, T^*(y) > = < x, \lambda T^*(y) >.$$

Therefore, $T^*(\lambda y) = \lambda T^*(y)$ .... (2)

(1) and (2) together say that T^* is a linear operator.

Again, for $y \in H$ we have

$$\| T^*(y) \|^2 = < T^*(y), T^*(y) > = < T T^*(y), y >$$

$$\leq \| T T^*(y) \| \| y \| \leq \| T \| \| T^*(y) \| \| y \|$$

That means, $\| T^*(y) \| \leq \| T \| \| y \|$, and therefore T^* is a bound linear operator over H with $\| T^* \| \leq \| T \|$.

Corollary 1. $T^{**} \equiv T$.

Now T^* is a bounded linear operator; and from the relation $< T(x), y > = < x, T^*(y) >$ let us put T^* in place of T to get for all $x, y \in H$,

$$< T^*(x), y > = < x, T^{**}(y) >$$

Interchange x and y to get

$$<T^*(y), x> = <y, T^{**}(x)>$$

Taking conjugates, $<T^{**}(x), y> = <x, T^*(y)> = <T(x), y>$ ....(*)

Now (*) remains true for all $y \in H$, therefore we deduce that

$TT^*(x) = T(x)$ and this being true for all $x \in H$ we finally obtain $T^{**} = T$.

Corollary 2. $\|T^*\| = \|T\|$.

We do already thave $\|T^*\| \leq \|T\|$; let us apply this in favour of T^* to get

$$\|T^{**}\| \leq \|T^*\|$$

or, $\|T\| \leq \|T^*\|$

Therefore, $\|T\| = \|T^*\|$.

§ 6.2 ALGEBRA OF ADJOINT OPERATORS IN HILBERT SPACE *H*.

Let A and B be two bounded linear operators $: H \to H$ *i.e.* $A, B \in Bd\alpha(H, H)$. Then $A + B$ and αA (α any scalar) are also members of $Bd\alpha(H, H)$.

Theorem 6.2.1. (a) $(A+B)^* = A^* + B^*$ and (b) $(\alpha A)^* = \bar{\alpha} A^*$, where A^* denotes adjoint of A.

Proof : (a) For all x, $y \in H$ we have $<A(x), y> = <x, A^*(y)>$ and $<B(x), y> = <x, B^*(y)>$.

Now $<x, (A+B)^* y> = <(a+B)(x), y>$

$$= <A(x) + B(x), y>$$

$$= <A(x), y> + <B(x), y>$$

$$= <x, A^*(y)> + <x, B^*(y)>$$

$$= <x, A^*(y) + B^*(y)>$$

$$= <x, (A^* + B^*)(y)>$$

This shows that $(A+B)^* = A^* + B^*$

(b) $<x,(\alpha A)^*(y)> = <(\alpha A)(x), y>$

$$= <\alpha A(x), y>$$
$$= \alpha < A(x), y>$$
$$= \alpha < x, A^*(y)>$$
$$= < x, \bar{\alpha} A^*(y)>$$
$$= < x, (\bar{\alpha} A^*)(y)>$$

This being true for all $x, y \in H$, we have $(\alpha A)^* = \bar{\alpha} A^*$.

For A and B belonging to $Bd\alpha(H,H)$, let us define $(AB) : H \to H$ by following rule of composition;

$(AB)(x) = A(B(x))$ for $x \in H$. In this way $(BA) : H \to H$ is also defined. It is a routine verification that $(AB) : H \to H$ is a linear operator such that for $x \in H$,

$$\|(AB)(x)\| = \|A(B(x))\| \le \|A\| \|B(x)\| \le \|A\| \|B\| \|x\|.$$

This is true for all $x \in H$; Therefore (AB) is also a bounded linear operator over H i.e. $(AB) \in Bd\alpha(H,H)$.

Theorem 6.2.2. $(AB)^* = B^* A^*$.

Proof : For $x, y \in H$, we have $< A(x), y > = < x, A^*(y) >$

and $< B(x), y > = < x, B^*(y) >$

Now $< (AB)(x), y > = < x, (AB)^*(y) >$ which is the same as,

$$< x, (AB)^*(y) > = < (AB)(x), y >$$
$$= < A(B(x)), y >$$
$$= < B(x), A^* y >$$
$$= < x, B^*(A^*(y)) >$$

$= < x, (B^* A^*)(y) >$; Therefore we have $(AB)^* = B^* A^*$.

Theorem 6.2.3. For any $A \in Bd\alpha(H,H)$, $\|A^* A\| = \|A\|^2 = \|AA^*\|$.

Proof : We always have $\|A^* A\| \le \|A^*\| \|A\| = \|A\| \|A\| = \|A\|^2$ because A^* is also a member of $Bd\alpha(H,H)$

i.e. $\quad \| A^* A \| \leq \| A \|^2$ .... (1)

Again $\| A \|^2 = \sup_{\|x\| \leq 1} \{ \| A(x) \|^2 \}$

$$= \sup_{\|x\| \leq 1} \{ | < A(x), A(x) > | \}$$

$$= \sup_{\|x\| \leq 1} \{ | < A^*(A(x)), x > | \}$$

$$= \sup_{\|x\| \leq 1} \{ | < (A^* A)(x), x > | \}$$

$$\leq \sup_{\|x\| \leq 1} \{ \| (A^* A)(x) \| \| x \| \} \text{ form } C\text{-}S \text{ inequality,}$$

$$\leq \| A^* A \| .$$

That is, $\| A \|^2 \leq \| A^* A \|$. ....(2)

From (1) and (2) we have $\| A^* A \| = \| A \|^2$. Now applying this equality to A^* one obtains $\| AA^* \| = \| (A^*)^* A^* \| = \| A^* \|^2 = \| A \|^2$. The proof is now complete.

Corollary : If $A \in Bd\alpha(H, H)$ is such that $AA^* = A^* A$ (*i.e.* A and A^* commute), then $\| A^2 \| = \| A \|^2$.

§ 6.3 SELF-ADJOINT OPERATORS OVER HILBERT SPACE *H*.

Definition 6.3.1. A member $T \in Bd\alpha(H, H)$ *i.e.* T being a bounded linear operator over H is called self-adjoint if $T^* = T$.

Theorem 6.3.1. (a) If T_1 and T_2 are self-adjoint operators over H, then $T_1 + T_2$ is so.

(b) If T_1 is self-adjoint and α any real scalar, then αT_1 is self-adjoint.

(c) For any member $T \in Bd\alpha(H, H)$, T^*T, TT^* and $T + T^*$ are self-adjoint.

(d) If T_1 and T_2 are self-adjoint, then $T_1 T_2$ is self-adjoint if and only if $T_1 T_2 = T_2 T_1$ (T_1 and T_2 commute).

Proof : (a) $(T_1 + T_2)^* = T_1^* + T_2^* = T_1 + T_2$

(b) $(\alpha T_1)^* = \bar{\alpha} T_1^* = \bar{\alpha} T_1 = \alpha T_1$ because α is a real scalar.

(c) $(T^*T)^* = T^* TT^{**} = T^*T$, $(TT^*)^* = T^{**}T^* = TT^*$;
and $(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$.

and finally (d) $(T_1T_2)^* = T_2{}^*T_1{}^* = T_2T_1$; Therefore $(T_1T_2)^* = T_1T_2$ if and only if $T_1T_2 = T_2T_1$.

Theorem 6.3.2. The class of all self-adjoint operators forms a closed real sub-space of $Bd\alpha(H,H)$, and hence it is a Banach space.

Proof : If 0 and I denote the zero operator and identity operator, we have 0 and I are members of $Bd\alpha(H,H)$. Further $0^* = 0$ and $I^* = I$; Now if A and B are self-adjoint operators with α and β two real scalars, we have

$$(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^* = \alpha A^* + \beta B^*$$
$$= \alpha A + \beta B$$

Showing thereby that $\alpha A + \beta B$ is also self-adjoint.

Further if $\{A_n\}$ is a sequence of self-adjoint operators over H such that $\lim_{n\to\infty} A_n = A$ in operator norm, *i.e.* $\| A_n - A \| \to 0$ as $n \to \infty$. Then we know that A is a bounded linear operator on Hilbert space. So that

$$\| A - A^* \| \le \| A - A_n \| + \| A_n - A_n{}^* \| + \| A_n{}^* - A^* \|$$
$$= \| A - A_n \| + \| (A_n - A)^* \|$$
$$= \| A - A_n \| + \| A_n - A \| \to 0 \text{ as } n \to \infty.$$

Hence $A = A^*$ and A is self-adjoint.

Theorem 6.3.3. Let A be a bounded linear operator : $H \to H$ such that for all $x, y \in H$, $< A(x), y > = 0$, then A equals to the zero operator and conversely.

Proof : For the zero operator we always have $< 0(x), y > = < \underset{\sim}{0}, y > = 0$. Conversely let for all $x, y \in H$, $<A(x), y> = 0$. Let us fix $x \in H$ and consider $< A(x), y > = 0$ for all $y \in H$. That means $A(x) = \underset{\sim}{0}$ in H; Now let x be free and we see $A(x) = \underset{\sim}{0}$ for $x \in H$; showing $A = 0$.

Corollary : If A is a bounded linear operator : $H \to H$ satisfies $< A(x), x > = 0$ for all $x \in H$, then A is the zero operator.

If $x, y \in H$ and α, β are any two scalars we have

$$0 = <(\alpha x + \beta y), \ \alpha x + \beta y>$$

$$= <\alpha A(x) + \beta A(y), \ \alpha x + \beta y> \quad (A \text{ is Linear})$$

$$= \alpha\bar{\alpha} < A(x), x> + \alpha\bar{\beta} < A(x), y> + \beta\bar{\alpha} < A(y), x> + \beta\bar{\beta} < A(y), y>$$

$= \alpha\bar{\beta} < A(x), y> + \beta\bar{\alpha} < A(y), x>$ other terms are zero by given condition.

Let us take $\alpha = 1$ and $\beta = 1$, then we have

$$< A(x), y> + < A(y), x> = 0 \quad \text{..........(1)}$$

Again take $\alpha = i$ and $\beta = 1$, then above gives

$$i < A(x), y> - i < A(y), x> = 0$$

or, $$< A(x), y> - < A(y), x> = 0 \quad (2)$$

Adding (1) and (2) we deduce $< A(x), y> = 0$, and now apply Theorem 6.3.3 for desired conclusion.

Theorem 6.3.4. Let $T \in Bd\alpha(H, H)$ ($T : H \to H$ is a bounded linear operator). Then T is self-adjoint if and only if $< A(x), x>$ is a real scalar for all $x \in H$ (Hilbert space).

Proof : Suppose T is a self-adjoint operator over H, and let $x \in H$; we have

$$\overline{< T(x), x>} = < x, T(x)> = < x, T^*(x)> = < T(x), x>$$

Therefore scalar $< T(x), x>$ is a real scalar.

Conversely, let $< T(x), x>$ is real for all $x \in H$.

Then $< T(x), x> = \overline{< T(x), x>} = < \overline{x, T^*(x)}> = < T^*(x), x>$

Thus $< T(x), x> - < T^*(x), x> = 0$

or, $< T(x) - T^*(x), x> = 0$

or, $< (T - T^*)(x), x> = 0$

This being true for all x in H, we conclude that

$T - T^*$ = zero operator

or, $T = T^*$

i.e. T is a self-adjoint operator.

Theorem 6.3.5. If H is a Hilbert space and $T \in Bd\alpha(H,H)$, such that T is self-adjoint, Then $\| T \| = \sup_{\|x\|=1} |< T(x), x >|$

Proof : If T is self-adjoint, it is ofcourse a bounded linear operator over H. Then for any x with $\| x \| = 1$ in H.

$$|< T(x), x >| \leq \| T(x) \| \| x \| \qquad \text{by C-S inequality,}$$

$$\leq \| T \| \| x \| . \| x \| = \| T \| .$$

Therefore, $\sup_{\|x\|\leq 1} |< T(x), x >| \leq \| T \|$ ....(1)

Let K= $\sup_{\|x\|\leq 1} |< T(x), x >|$.

Now we show that $\| T \| \leq K$

If $T(u) = 0$ for all u with $\| u \| = 1$ in H, then we see that $T = 0$ (zero operator), and in that case we have finished.

Otherwise for any z with $\| z \| = 1$ such that $T(z) \neq \underset{\sim}{0}$, put $v = \sqrt{\| T(z) \|}.z$ and $w = \frac{1}{\sqrt{\| T(z) \|}} T(z)$. Then $\| v \|^2 = \| w \|^2 = \| T(z) \|$. Let us now put $y_1 = v + w$ and $y_2 = v - w$. Then on straight calculation and using the fact that T is self-adjoint, we have

$$< T(y_1), y_1 > - < T(y_2), y_2 > = 2(< T(v), w > + < T(w), v >$$

$$= 2(< T(z), T(z) > + < T^2(z), z >) = 4 \| T(z) \|^2 \qquad \text{............(2)}$$

Now for every $y \neq \underset{\sim}{0}$, and $x = \frac{y}{\| y \|}$, we have

$$y = \| y \| x \text{ and } < T(y), y > = \| y \|^2 |< T(x), x >|$$

$$\leq \| y \|^2 \leq \sup_{\|u\|=1} |< T(u), u >| = K \| y \|^2 .$$

Now $|< T(y_1), y_1 > - < T(y_2), y_2 >| \leq |< T(y_1), y_1 >| + |< T(y_2), y_2 >|$

$$\leq K(\| y_1 \|^2 + \| y_2 \|^2)$$

$$= 2K(\| v \|^2 + \| w \|^2)$$

$$= 4K \| T(z) \|$$

From here and (2) we get $4\| T(z) \|^2 \leq 4K \| T(z) \|$

Hence $\| T(z) \| \leq K$

So taking supremum over all z with norm | one obtains $\| T \| \leq K$

together with $K \leq \| T \|$ from (1) we finally get $\| T \| = K$.

Theorem 6.3.6. Let $T \in Bd\alpha(H, H)$, H being Hilbert space show that following statements are equivalent.

(a) $T^*T = I$ (Identity operator)

(b) $< T(x), T(y) > = < x, y >$ for all $x, y \in H$

(c) $\| T(x) \| = \| x \|$ for all $x \in H$

Proof : (a) $\Rightarrow$ (b). Let (a) hold. Then for all $x, y \in H$, we have

$$< T^*T(x), y > = < I(x), y > = < x, y >$$

or, $< T(x), T(y) > = < x, y >$, (b) follows.

(b) $\Rightarrow$ (c) ; suppose (b) is true. Taking $y = x$ in (b).

We have $< T(x), T(x) > = < x, x >$

or, $\| T(x) \|^2 = \| x \|^2$

or, $\| T(x) \| = \| x \|$

$(c) \Rightarrow (a)$; Then $\| T(x) \| = \| x \|$ gives $\| T(x) \|^2 = \| x \|^2$

or, $< T(x), T(x) > = < x, x >$

or, $< T^*(T(x)), x > = < x, x >$

or, $< T^*T(x), x > - < x, x > = 0$

or, $((T^*T - I)(x), x) = 0$; Here we apply corollary of Theorem 6.3.3 to conclude that $T^*T - T = 0$ or, $T^*T = I$.

§ 6.4 EIGEN VALUES AND EIGEN VECTORS OF OPERATOR ON HILBERT SPACE *H*.

Let T be a bounded Linear operator : $H \to H$ *i.e.* $T \in Bd\alpha(H, H)$.

Definition 6.4.1. A non-zero vector $x \in H$ is said to be an eigen vector corresponding to a scalar λ called an eigen value of T if

$T(x) = \lambda x$

or, $T(x) - \lambda I(x) = 0$ (I denoting Identity operator on H)

or, $(T - \lambda I)(x) = 0$

Theorem 6.4.1. Let $T : H \to H$ be a self-adjoint operator. Then (1) all eigen values of T (if they exist) are real, and (2) Eigen vectors corresponding to different eigen values of T are orthognonal.

Proof : (a) Let λ be an eigen value of T and x a corresponding eigen vector. Then $x \neq \underset{\sim}{0}$ and $T(x) = \lambda x$.

Since T is self-adjoint, we have

$$\lambda < x, x > = < \lambda x, x > = < T(x), x > = < x, T(x) > = < x, \lambda x >$$

$= \bar{\lambda} < x, x >$ where $< x, x > = \| x \|^2$ is +ve as $x \neq \underset{\sim}{0}$, and this gives $\lambda = \bar{\lambda}$ (since $\| x \| > 0$) and therefore λ is real.

(b) Let λ and μ be two different eigen values of T, and let x and y be eigen vectors (non-zero) corresponding to eigen values λ and μ respectively.

Then we have $T(x) = \lambda x$ and $T(y) = \mu y$. Since T is self-adjoint and eigen values are real, we have

$$\lambda < x, y > = < \lambda x, y > = < T(x), y > = < x, T(y) >$$

$$= < x, \mu y > = \mu < x, y >, \quad \mu \text{ being real.}$$

Since $\lambda \neq \mu$ we conclude that $< x, y > = 0$ or, $x \perp y$ holds.

Theorem 6.4.2. If $T \in Bd\alpha(H, H)$ such that $T^*T = TT^*$, then if x is an eigen vector of T with eigen value λ, then x is also an eigen value of T^* with eigen value $\bar{\lambda}$, and conversely.

Proof : Consider the operator $T - \lambda I$ in H. Then

$$(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda} I - \lambda T^* + | x |^2,$$

and similarly $(T - \lambda I) + (T - \lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + | \lambda |^2 I$

Given $T^*T = T^*T$. Therefore

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I) \text{ puting } T-\lambda I = S$$

We have $SS^* = S^*S$,

Thus for $x \in H$, $SS^*(x) = S^*S(x)$

or, $\quad < SS^*(x), x > = < S^*S(x), x >$

or, $\quad < S^*(x), S^*x > = < S(x), S(x) >$

or, $\quad \| S^*(x) \|^2 = \| S(x) \|^2$

or, $\quad \| (T^* - \bar{\lambda} I)(x) \|^2 = \| (T - \lambda I)(x) \|^2$

or, $\quad \| (T - \lambda I)(x) \|^2 = \| (T^* - \bar{\lambda} I)(x) \|^2$

or, $\quad \| T - \lambda x \| = \| T^* - \bar{\lambda} x \|$.

This shows that $T(x) = \lambda x$ if and only if $T^*(x) = \bar{\lambda} x$.

Example 6.4.1. Let $L_2[0,1]$ be the real Hilbert space of all square integrable functions over the closed interval $[0,1]$ with I.P. function $< x, y > = \int_0^1 x(t)\, y(t) dt$ as $x, y \in L_2[0,1]$.

Show that $T : L_2[0,1] \to L_2[0,1]$ defined by $T(x) = y \in L_2[0,1]$ where $y(t) = t\, x(t)$ in $0 \le t \le 1$ is a bounded linear operator which is self-adjoint having no eigen values.

Solution : Here T is a linear operator because if $x, y \in L_2[0,1]$ and if $T(x+y) = z$ where $z(t) = t(x+y)(t)$, in $0 \le t \le 1$, we have

$$T(x+y)(t) = z(t) = t\big(x(t) + y(t)\big) = tx(t) + ty(t)$$

$$= T(x)(t) + T(y)(t) \qquad \text{in } 0 \le t \le 1.$$

$\therefore$ $T(x+y) = T(x) + T(y)$ and similarly for any real scalar α, $T(\alpha x) = \alpha\, T(x)$.

Further, $T(x)(t) = tx(t)$ in $0 \le t \le 1$.

$$\therefore \quad \| T(x) \|^2 = \int_0^1 t^2 x^2(t)dt \le \sup_{0\le t\le}\{t^2\}\int_0^1 x^2(t)dt$$

$$= 1.\| x \|^2;$$

Thus $\| T(x) \| \le \| x \|$; that shows that T is a bounded linear operator in $L_2[0,1]$.

***T* is self-adjoint.** Let $x, y \in L_2[0,1]$, then we have

$$< x, T(y) > = \int_0^1 x(t)ty(t)dt = \int_0^1 tx(t)y(t)dt$$

and $$< y, T(x) > = \int_0^1 y(t)t\, x(t)dt = \int_0^1 tx(t)y(t)dt$$

Therefore $< x, T(y) > = < y, T(x) >$; That shows T as self-adjoint.

If λ is an eigen value of T, and a non-zero $x \in L_2[0,1]$ is an eigen vector of T corresponding to the eigen value λ, we have

$$T(x) = \lambda x$$

or, $tx(t) = \lambda x(t)$ in $0 \le t \le 1$

or, $(t - \lambda)x(t) = 0$ in $0 \le t \le 1$

Since x is non-zero, we have $t = \lambda$ in $0 \le t \le 1$, which is not the case. Thus no such λ is there, *i.e.* T possesses no eigen value.

Theorem 6.4.3. Every bounded linear operator T on a Hilbert space H is equal to a sum $A + iB$ where A and B are self-adjoint operator in H.

Proof : Let us define A and B as follows :

$$A = \frac{1}{2}(T + T^*), \text{ and } B = \frac{1}{2i}(T - T^*).$$

Then $A^* = \frac{1}{2}(T^* + T) = A$ and $B^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = B$; So each of A and B is a self-adjoint operator on H such that $A + iB = T$.

Remark : Representation of T as $T = A + iB$ is unique. Because, Let $T = C + iD$ where C and D are self-adjoint operator on H; then $T^* = (C + iD)^* = C - iD$ and hence $T + T^* = 2C$ and $T - T^* = 2iD$; Thus $C = A$ and $D = B$.

EXERCISE A

Short answer type questions

1. Find the eigen values and eigen vectors of $\begin{pmatrix} a, & b \\ -b, & a \end{pmatrix}$ $b \neq 0$ and a, b are reals.
2. Examine if zero operator and Identity operator in a Hilbert space H are self-adjoint.
3. If T is a self-adjoint operator in a Hilbert space H, show that for every natural number n, T^n is self-adjoint.
4. If T is a self-adjoint operator in a Hilbert space H, and S is any bounded Linear operator in H, show that S^*TS is self-adjoint.
5. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not possess any eigen vector.

EXERCISE B

1. Given a square matrix $A = ((a_{ji}))_{n \times n}$ having eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, show that kA has eigen values $k\lambda_1, k\lambda_2, ..., k\lambda_n$; and A^2 has eigen values $\lambda_1^2, \lambda_2^2, ...\lambda_n^2$.
2. Let $T : l_2 \to l_2$ be defined by $T(\xi_1, \xi_2, ...) = (0, 0, \xi_1, \xi_2, ...)$ as $(\xi_1, \xi_2, ...) \in l_2$; Examine if T is a bounded linear operator in l_2 and if T is self-adjoint in l_2.
3. Show that in a Hilbert space H, $T_1{}^*T_1 = T_2{}^*T_2$ if and only if $\| T_1(x) \| = \| T_2(x) \|$ for all $x \in H$.
4. In H if T is self-adjoint show that $T(x) = \underset{\sim}{0}$ in H if and only if $T\,T(x) = \underset{\sim}{0}$.
5. Let $T : H \to H$ and $W : H \to H$ be bounded Linear operators and $S = W^*TW$. Show that if T is self-adjoint and +ve, so will be S.

NOTES

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

—*রবীন্দ্রনাথ ঠাকুর*

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে ; সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

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Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

—*Subhas Chandra Bose*

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