মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— *রবীন্দ্রনাথ ঠাকুর*

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধুলিসাৎ করতে পারি।

— *সুভাষচন্দ্র বসু*

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

— *Subhas Chandra Bose*

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CBCS • UG • HMT • MATHEMATICS • CC-MT-01

NETAJI SUBHAS OPEN UNIVERSITY

Choice Based Credit System (CBCS)

SELF LEARNING MATERIAL

HMT
MATHEMATICS

CC-MT-01

Under Graduate Degree Programme

PREFACE

In a bid to standardise higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific generic elective, ability* and *skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry acquired credits. I am happy to note that the University has been accredited by NAAC with grade 'A'.

UGC (Open and Distance Learning Programmes and Online Learning Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the *six* semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University

Under Graduate Degree Programme

Choice Based Credit System (CBCS)

Subject : Honours in Mathematics (HMT)

ALGEBRA

Course Code : CC - MT - 01

প্রথম মুদ্রণ ঃ নভেম্বর, 2021

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Netaji Subhas Open University

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UG: Mathematics (HMT)

ALGEBRA

[CC - MT - 01]

Unit-1 ❐ Complex Numbers

Structure

1.1 Objectives

In this unit, we would be able to know the definition of a complex number and its conjugate. We shall know the polar representation of a complex number and De Moivre's Theorem. We would be able to find the n^{th} roots of unity. Here we shall learn the addition, subtraction, multiplication, division and different types of mathematical operations of complex numbers. We would also be able to find the modulus and amplitude of a complex number with its geometrical representation.

1.1.1 Definition

An expression of the form $a + ib$, where a and b are both real, is called a complex number. It is usually denoted by $z = a + ib$.

A complex number z is also defined as an ordered pair of real numbers a and b and we write $z = (a, b)$ subject to the following conditions:

(i) If $z_1 = (a, b)$ and $z_2 = (c, d)$ be two complex numbers, then $(a, b) = (c, d)$, if and only if $a = c$ and $b = d$.

(ii) An operation 'addition' denoted by '+' is defined as
$(a, b) + (c, d) = (a + c, b + d)$.

(iii) An operation 'multiplication' denoted by '.' or '×' is defined as
$(a, b). (c, d) = (ac - bd, ad + bc)$.

(iv) If k is a real number, then $k (a, b) = (ka, kb)$.

1.1.2 Remarks

(i) In general, $(a, b) \neq (b, a)$.

(ii) For the complex number $z = (a, b)$, a is called the real part of z, which is written as $Re (z)$ and b is called the imaginary part of z, which is written as $Im(z)$.

(iii) The complex number $(a, 0)$ represents the real number a. So any real number can be regarded as a complex number with imaginary part equal to 0.

(iv) The complex number $(0, b)$ represents purely imaginary number ib.

(v) If $a = 0$ and $b = 0$, then the complex number $(0, 0)$ is 0.

(vi) If we write the complex number $(0, 1)$ by i, the imaginary unit, then the rule of multiplication gives

$$i^2 = i.i = (0, 1). (0, 1) = (-1, 0) = -1.$$

(vii) We see that $(a, 0) + (0, b) = (a, b)$ i.e. $a + ib = (a, b)$.

(viii) If $z = (a, b)$ be a complex number, then the negative of z is the complex number (x, y) such that $(a, b) + (x, y) = (0, 0)$. It is denoted by $-z$. Then,
$-z = (-a, -b)$.

(ix) If $(a, b) = 0$, then $a = 0, b = 0$.

(x) Inverse of z is denoted by $\frac{1}{z}$.

1.1.3 Theorem

If $z_1 = a + ib$, $z_2 = c + id$ and $z_3 = e + if$ be complex numbers, then

(i) $z_1 + z_2 = z_2 + z_1$ (commutative law of addition)

(ii) $z_1 . z_2 = z_2 . z_1$ (commutative law of multiplication)

(iii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative law of addition)

(iv) $z_1 . (z_2 . z_3) = (z_1 . z_2) . z_3$ (associative law of multiplication)

(v) $z_1 . (z_2 + z_3) = z_1 . z_2 + z_1 . z_3$ (distributive law)

Proof : The proofs of (i), (ii) and (iii) are trivial.

Proof for (iv) : We have

$$z_1.(z_2.z_3) = (a + ib)\{c + id).(e + if)\} = (a, b)(ce - df, cf + de)$$
$$= (ace - adf - bcf - bde, acf + ade + bce - bdf)$$

and
$$(z_1.z_2).z_3 = \{(a, b).(c, d)\}(e, f)\} = (ac - bd, bc + ad).(e, f)$$
$$= (ace - bde - bcf - adf, bce + ade + acf - bdf).$$

Hence proved.

Proof for (v) : We have

$$z_1.(z_2 + z_3) = (a, b).\{(c, d) + (e, f)\}$$
$$= (a, b)(c + e, d + f)$$
$$= (ac + ae - bd - bf, bc + be + ad + af)$$

and
$$z_1.z_2 + z_1.z_3 = (a, b).(c, d) + (a, b).(e, f)$$
$$= (ac - bd, bc + ad) + (ae - bf, be + af)$$
$$= (ac + ae - bd - bf, bc + be + ad + af).$$

Hence proved.

1.1.4 Conjugate of a complex number

If two complex numbers are such that their real parts are equal and their imaginary parts are equal in magnitude but opposite in signs, then these are said to be conjugate complex numbers. Thus if $z = a + ib$ be a complex number then $a - ib$ is the conjugate of z and is denoted by $\bar{z}$ and vice versa.

1.1.5 Properties of conjugate complex numbers

If z_1 and z_2 be any two complex numbers, then

(i) $\bar{\bar{z}}_1 = z_1$

(ii) $z_1 + \bar{z}_1 = 2Re(z_1)$

(iii) $z_1 - \bar{z}_1 = 2i\, Im(z_1)$

(iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(v) $\overline{z_1.z_2} = \bar{z}_1.\bar{z}_2$

(vi) $z_1.\bar{z}_1$ = a positive real number unless $z_1 = 0$

(vii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

Verifications of (i)-(iv) are trivial.

If $z_1 = a + ib$, $z_2 = c + id$ and $\overline{z_1} = a - ib$, $\overline{z_2} = c - id$, then

$$\overline{z_1.z_2} = \overline{(a+ib).(c+id)}$$

$$= \overline{(ac-bd)+i(bc+ad)}$$

$$= (ac - bd) - i\,(bc + ad)$$

and $\quad \overline{z_1}.\overline{z_2} = (a - ib).\ (c - id)$

$$= (ac - bd) - i\,(bc + ad)$$ which establishes (v).

Now, $\quad z_1.\overline{z_1} = (a + ib).\ (a - ib)$

$$= (a^2 + b^2) + i\,(ab - ab) = a^2 + b^2$$

which is a positive real number, which proves (vi).

Lastly, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \overline{\left(\dfrac{z_1.\overline{z_2}}{z_2.\overline{z_2}}\right)} = \overline{\left(\dfrac{(a+ib).(c-id)}{(c+id).(c-id)}\right)}$

$$= \overline{\left(\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right)}$$

$$= \overline{\left(\frac{(ac+bd)}{c^2+d^2}+i\frac{(bc-ad)}{c^2+d^2}\right)}$$

$$= \frac{(ac+bd)}{c^2+d^2}-i\frac{(bc-ad)}{c^2+d^2}$$

and $\quad \dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{a-ib}{c-id} = \dfrac{(a-ib).(c+id)}{(c-id).(c+id)}$

$$= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$= \frac{(ac+bd)}{c^2+d^2}-i\frac{(bc-ad)}{c^2+d^2}$$

which proves (vii).

1.2 Introduction

You have earned gradually the knowledge of the system of integers viz. all positive integers, all negative integers and zero.

From the concept of solving (i.e. to find the value of the unknown) a linear equation $ax - b = 0$, ($a \neq 0$ and b are integers). The solution is $x = \frac{b}{a}$. Thus we have extended our knowledge to positive and negative fractions. While solving a quadratic equation of the type $x^2 = 2$, we realize that it has no solution of the type discussed above.

Then for the sake of requirement, we have extended our knowledge and by way of definition we have divided the numbers into two categories–rational and irrational numbers. With this idea, whenever I ask you to solve the equation (a) $2x = 0$, (b) $3x - 5 = 0$, (c) $4x + 1 = 0$, (d) $4x^2 = 25$ or $x^2 = 3$, you will at once answer these questions (a) $x = 0$, (b) $x = \frac{5}{3}$, (c) $x = -\frac{1}{4}$, (d) $x = \pm\frac{5}{2}$ and $x = \pm\sqrt{3}$.

Now, if you come across to solve an equation of type $x^2 = -1$ or $x^2 = -4$, you would not be able to answer it because you know that square of any number of your knowledge is never negative. At this stage your answer will be 'it is not solvable'. To make this type of equations solvable, we are compelled to extend our knowledge of the number system. Thus we call the previous numbers as set of real numbers and the solution of the type of the last equations as imaginary numbers.

To make the above equations solvable, we introduce the symbol i which is assumed to be $\sqrt{-1}$. This i is called the imaginary unit. Thus $i^2 = -1$. Therefore $x^2 = i^2$ or $x = \pm i$ and $x^2 = 4i^2$ or $x = \pm 2i$. Thus solutions of the above equations are obtained.

1.3 Polar or Geometrical representation of a complex number

Argand Diagram : Let us imagine a plane called Argand Plane on which two mutually perpendicular straight lines XOX' and YOY' are taken. We shall call XOX' as real axis and YOY' as imaginary axis. On the Argand plane, let us take a point P to represent the complex number $z = a + ib$.

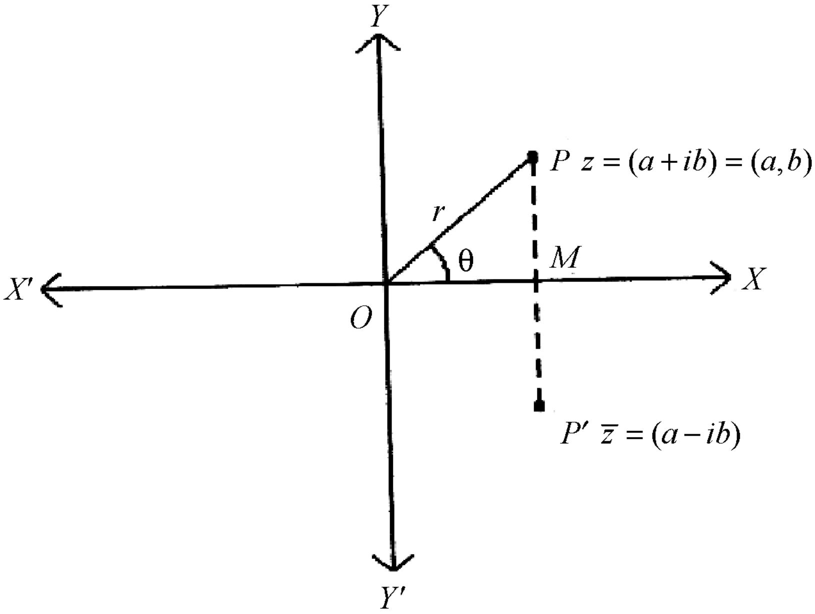


Figure 1 : Argand plane

If $b = 0$, then z is real and the point lies on the real axis and if $a = 0$, then z is purely imaginary and the point lies on the imaginary axis.

If PM is drawn perpendicular on OX, $OM = a$ and $MP = b$ so that (a, b) is the cartesian co-ordinates of P. Let P' be the image of P on the real axis. Obviously the cartesian co-ordinates of P' are $(a, -b)$ which represents the complex number $\bar{z} = a - ib$, the conjugate of $z = a + ib$. OP is joined. Let $OP = r$ and $\angle MOP = \theta$, then (r, θ) is the polar co-ordinates of P whose cartesian co-ordinates are (a, b).

We have $a = r \cos \theta$ ... (1)

and $b = r \sin \theta$ ... (2)

So $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \dfrac{b}{a}$.

Therefore $z = a + ib = r (\cos\theta + i \sin\theta)$.

This is called polar form or trigonometric form of representation of the complex number z.

This is also called De Moivre's form of z.

Here $r = OP$ is called the modulus of z and is denoted by $|z|$ or $mod\,(z)$ which is positive. Therefore $|z| = r = +\sqrt{a^2 + b^2}$.

The angle θ satisfying the equations (1) and (2) simultaneously is called argument or amplitude of z. It is denoted by arg z or amp z. The argument of z is obtained from the equations

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Since $2n\pi+\theta$, for all integers n, satisfies these equations, may also be argument of z. For unique representation of a complex number in modulus-amplitude form, we have considered $-\pi < \theta \le \pi$ which we call the principal value of the argument.

1.3.1 Remember

(i) $1 = \cos 0 + i \sin 0$

(ii) $-1 = \cos \pi + i \sin \pi$

1.3.2 Properties of Moduli and Amplitudes of complex number

Let z_1 and z_2 be any two complex numbers.

(i) $|z_1.z_2| = |z_1|.|z_2|$ and $amp\ (z_1.\ z_2) = amp\ (z_1) + amp\ (z_2)$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$ and $amp\left(\frac{z_1}{z_2}\right) = amp(z_1) - amp(z_2)$

(iii) $|z_1+z_2| \le |z_1|+|z_2|$

(iv) $|z_1-z_2| \ge \left||z_1|-|z_2|\right|$

Let $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then $|z_1| = r_1$, $|z_2| = r_2$, $amp\ (z_1) = \theta_1$ and $amp\ (z_2) = \theta_2$.

(i) $z_1 . z_2 = r_1(\cos\theta_1 + i\sin\theta_1) . r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$.

Therefore $|z_1.z_2| = r_1r_2 = |z_1|.|z_2|$ and $amp\ (z_1.z_2) = \theta_1+\theta_2 = amp\ (z_1) + amp\ (z_2)$.

(ii) $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}.\frac{(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}.$$

Therefore $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and $amp\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = amp\ (z_1) - amp\ (z_2)$.

(iii) $|z_1 + z_2| = |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)|$

$= |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$

$= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2}$

$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$

$\leq \sqrt{r_1^2 + r_2^2 + 2r_1r_2}$, since $\cos(\theta_1 - \theta_2) \leq 1$

i.e., $\leq \sqrt{(r_1 + r_2)^2} = r_1 + r_2 = |z_1| + |z_2|$.

Hence $|z_1 + z_2| \leq |z_1| + |z_2|$. This is known as Triangle Inequality.

(iv) $|z_1 - z_2| = |(r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)|$

$= \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$

$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$

$\geq \sqrt{r_1^2 + r_2^2 + 2r_1r_2(-1)}$, since $-\cos(\theta_1 - \theta_2) \geq -1$

i.e., $\geq \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|$.

Therefore $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$.

Exercise : Prove that (i) $|z| = |\overline{z}|$ **(ii)** $z\overline{z} = |z|^2$.

Solution : Left for the students.

1.4 Cube roots of unity

Let us solve the equation $x^3 - 1 = 0$ ... (1)

We have $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

Therefore, equation (1) gives either $x - 1 = 0$ i.e. $x = 1$ which is a real root of equation (1) or $(x^2 + x + 1) = 0$.

Therefore, $x = \dfrac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1} = \dfrac{-1 \pm \sqrt{3}i}{2}$

which are two imaginary roots of equation (1).

Now, $\left(\dfrac{-1+\sqrt{3}i}{2}\right)^2 = \dfrac{+1-3-2\sqrt{3}i}{4} = \dfrac{-2-2\sqrt{3}i}{4} = \dfrac{-1-\sqrt{3}i}{2}$.

Again, $\left(\dfrac{-1-\sqrt{3}i}{2}\right)^2 = \dfrac{+1-3+2\sqrt{3}i}{4} = \dfrac{-2+2\sqrt{3}i}{4} = \dfrac{-1+\sqrt{3}i}{2}$

i.e., if ω be one of the imaginary cube roots of 1, then ω^2 is the other.

Therefore cube roots of unity are 1, ω and ω^2.

We see that

(i) $\omega^2 + \omega + 1 = \dfrac{-1+\sqrt{3}i}{2} + \dfrac{-1-\sqrt{3}i}{2} + 1 = 0$.

(ii) $\omega^3 = \omega^2.\omega = \dfrac{-1+\sqrt{3}i}{2} . \dfrac{-1-\sqrt{3}i}{2} = \dfrac{1+3}{4} = 1$.

1.5 Worked out Examples (I)

Example 1 : Express the following complex numbers in polar form :

(i) i (ii) $-i$ (iii) $1+i$ (iv) $1-i$ (v) $-1-i$.

Solution :

(i) Let $i = r(\cos\theta + i \sin\theta)$. Therefore $r \cos\theta = 0$ and $r \sin\theta = 1$.

Therefore $r = \sqrt{0^2 + 1^2} = 1$ and since the point (0,1) lies on the y-axis, $\theta = \dfrac{\pi}{2}$.

Therefore $i = 1.\left\{\cos\dfrac{\pi}{2} + i \sin\dfrac{\pi}{2}\right\}$.

(ii) Let $-i = r(\cos\theta + i \sin\theta)$. Therefore $r \cos\theta = 0$ and $r \sin\theta = -1$.

Therefore $r = \sqrt{0^2+(-1)^2} = 1$ and since the point (0, –1) lies on the negative y-axis, $\theta = -\frac{\pi}{2}$.

Therefore $-i = 1.\left\{\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right\}$.

(iii) Let $1+i = r(\cos\theta + i\sin\theta)$. Therefore $r\cos\theta = 1$ and $r\sin\theta = 1$.

Therefore $r = \sqrt{1^2+1^2} = \sqrt{2}$ and $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$.

As the point (1, 1) lies in the first quadrant, the unique value of θ is $\frac{\pi}{4}$.

Therefore $1+i = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

(iv) Let $1-i = r(\cos\theta + i\sin\theta)$. Therefore $r\cos\theta = 1$ and $r\sin\theta = -1$.

Therefore $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$ and $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$.

As the point (1, –1) lies in the fourth quadrant, the unique value of θ is $-\frac{\pi}{4}$.

Therefore $1-i = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right\}$.

(v) Let $-1-i = r(\cos\theta + i\sin\theta)$.

Therefore $r\cos\theta = -1$ and $r\sin\theta = -1$.

Therefore $r = \sqrt{(-1^2+(-1)^2)} = \sqrt{2}$ and $\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$.

As the point (–1, –1) lies in the third quadrant and remembering

$-\pi < \arg(-1-i) < -\frac{\pi}{2}$, $\theta = -\left(\pi-\frac{\pi}{4}\right) = -\frac{3\pi}{4}$.

Therefore $-1-i = \sqrt{2}\left\{\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right\}$.

Example 2 : Express $\frac{2+3i}{3-5i}$ in the form $A + i\,B$ where A and B are real numbers.

Solution : $\frac{2+3i}{3-5i} = \frac{(2+3i)}{(3-5i)}\cdot\frac{(3+5i)}{(3+5i)} = \frac{6-15+i(9+10)}{3^2+5^2}$

$$= \frac{-9+19i}{34} = \frac{-9}{34} + i\frac{19}{34}$$

which is of the form $A + i\,B$ where $A = \frac{-9}{34}$ and $B = \frac{19}{34}$.

Example 3 : Find the value of i^n where n is zero or any positive integer.

Solution : We have $(i)^0 = 1$, $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = (i)^2.\, i = -i$, $(i)^4 = \{(i)^2\}^2 = 1$.

Hence, if $n = 4m$, $i^n = (i)^{4m} = \{(i)^4\}^m = (1)^m = 1$.

If $n = 4m + 1$, $i^n = (i)^{4m+1} = \{(i)^4\}^m.\,(i)^1 = (1)^m.\, i = i$.

If $n = 4m + 2$, $i^n = (i)^{4m+2} = \{(i)^4\}^m.\,(i)^2 = (1)^m.(-1) = -1$.

If $n = 4m + 3$, $i^n = (i)^{4m+3} = \{(i)^4\}^m.\,(i)^3 = (1)^m.(-i) = -i$.

Example 4 : Find $|z|$ and *amp z*, where $z = -1 + \sqrt{3}i$.

Solution : Let $-1 + \sqrt{3}i = r(\cos\theta + i\sin\theta)$, r is $|z|$ and θ is *amp z*.

Therefore $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$. $\therefore r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$.

Therefore $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$. Since z lies in the second quadrant,

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Example 5 : If ω be one of the imaginary cube roots of unity, then show that $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = 9$.

Solution : We know that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

Therefore $\omega^4 = \omega^3 . \omega = \omega$ and $\omega^8 = \omega^6 . \omega^2 = \omega^2$.

Therefore $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$$= \{(1-\omega)(1-\omega^2)\}^2 = \{1-(\omega+\omega^2)+\omega^3\}^2 = (1+1+1)^2 = 9.$$

Example 6 : If $z = x + iy$ be a complex number and $\dfrac{z+1}{z-i}$ be purely imaginary, then show that z lies on the circle whose centre is at $\frac{1}{2}(-1+i)$ and radius is $\frac{1}{\sqrt{2}}$.

Solution : $\dfrac{z+1}{z-i} = \dfrac{(x+1)+iy}{x+i(y-1)} = \dfrac{(x+1)+iy}{x+i(y-1)} \cdot \dfrac{x-i(y-1)}{x-i(y-1)}$

$$= \frac{x(x+1)+y(y-1)}{x^2+(y-1)^2} + i\frac{xy-(x+1)(y-1)}{x^2+(y-1)^2}.$$

This will be purely imaginary if $x(x+1) + y(y-1) = 0$

or, $x^2 + y^2 + x - y = 0$ or, $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$

which represents a circle with centre at $\left(-\frac{1}{2}, \frac{1}{2}\right)$, i.e., at $-\frac{1}{2}+i\frac{1}{2}$ and radius $= \frac{1}{\sqrt{2}}$.

1.6 Model Questions (I)

1. (a) Give the polar representation of the following complex numbers :

(i) $1+i\sqrt{3}$ (ii) $\sqrt{3}-i$ (iii) $\dfrac{-1+i\sqrt{3}}{1+i}$.

(b) Express the following in the form $A + iB$ where A and B are real numbers:

(i) $(1-i)(6-8i)$ (ii) $i(1+i)(2+i)$ (iii) $\dfrac{1-i}{i}+\dfrac{i}{1-i}$

(iv) $\dfrac{(1-i)^2}{2+i}$ (v) $\dfrac{1}{1-\cos\theta+2i\sin\theta}$.

2. Simplify :

(i) i^{644} (ii) i^{565} (iii) i^{814}

(iv) i^{451} (v) $\frac{1+2i}{2-i}$ (vi) $\frac{(1+i)^3}{1-i^3}$

(vii) $\frac{2}{1-i}+\frac{3}{1+2i}-\frac{2}{1+i}$ (viii) $\frac{i^5+i^9+i^{14}}{i^6+i^{11}+i^{13}}$.

3. Find the modulus of the following :

(i) i (ii) $-i$ (iii) $1+i$

(iv) $4-3i$ (v) $\frac{(1-i)^2}{1+i}$ (vi) $\frac{2+3i}{2-3i}-\frac{2-3i}{2+3i}$

(vii) $(1+i)(1-i)$.

4. Find the principal amplitude of each of the following complex numbers :

(i) $2i$ (ii) $-3i$ (iii) $1-\sqrt{3}i$

(iv) $\frac{-i}{1-i}$.

5. Find the value of $arg\left(\frac{z_1}{z_2}\right)$ where $z_1=2i$ and $z_2=-1-i$.

6. Prove that $Im\,(z)=0$ where $z=\left(\frac{\sqrt{3}+i}{2}\right)^5+\left(\frac{\sqrt{3}-i}{2}\right)^5$.

7. If ω be an imaginary cube root of unity, then show that

(i) $(1+\omega-\omega^2)^7=-128\omega^2$

(ii) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=32$

(iii) $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

(iv) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)=1$

(v) $\frac{x\omega^2+y\omega+z}{x\omega+y+z\omega^2}=\omega$

(vi) $x^3+y^3+z^3-3xyz=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$.

8. For any two complex numbers z_1 and z_2, show that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}.$$

9. If z_1 and z_2 be conjugate complex numbers and z_3 and z_4 be also conjugate complex numbers, then show that $amp\left(\frac{z_1}{z_4}\right) = amp\left(\frac{z_3}{z_2}\right)$.

10. If $\omega^{2n} + \omega^n + 1 = 0$, examine whether n is divisible by 3.

11. If the ratio $\frac{z-i}{z+i}$ be purely imaginary, then show that the point z lies on the circle whose centre is at the origin and whose radius is 1.

12. Prove that the complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$, lie on the x-axis.

13. Objective questions : Verify:-

(i) $z + \bar{z} = 0$, if and only if $Re\ (z) = 0$.

(ii) $z.\bar{z} = 0$, if and only if $z = 0$.

(iii) $amp\ (bi)$, $(b > 0)$ is $\frac{\pi}{2}$.

(iv) $amp\ a$, $(a < 0)$ is π.

(v) $arg\ z + arg\ \bar{z} = 0$, if $z = 0$.

(vi) One square root of $3 + 4i$ is $2 + i$.

(vii) The smallest integer for which $\left(\frac{1-i}{1+i}\right)^n = 1$, is 4.

(viii) $\frac{1+i}{1+\frac{1}{i}}$ is purely imaginary and its amplitude is $\frac{\pi}{2}$.

1.6.1 Answers

1. (a) (i) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (ii) $2\ \{\cos(-\frac{\pi}{6})+i\ \sin(-\frac{\pi}{6})\}$

(iii) $\sqrt{2}(\cos\frac{5\pi}{12}+i\ \sin\frac{5\pi}{12})$.

(b) (i) $A = -2$, $B = -14$ (ii) $A = -3$, $B = 1$ (iii) $A = -\frac{3}{2}$, $B = -\frac{1}{2}$

(iv) $A = -\frac{2}{5}$, $B = -\frac{4}{5}$ (v) $A = \frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}, B = \frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}$.

2. (i) 1 (ii) i (iii) -1 (iv) $-i$ (v) i (vi) $2i$ (vii) $\frac{1}{5}(3+4i)$.

3. (i) 1 (ii) 1 (iii) $\sqrt{2}$ (iv) 5 (v) $\sqrt{2}$ (vi) $\frac{24}{13}$ (vii) 2.

4. (i) $\frac{\pi}{2}$ (ii) $-\frac{\pi}{2}$ (iii) $-\frac{\pi}{3}$ (iv) $-\frac{\pi}{4}$. 5. $-\frac{3\pi}{4}$. 10. No.

1.7 De Moivre's Theorem

If n is an integer, positive or negative, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

If n is a fraction, positive or negative, then $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$. [$\theta$ is a real number].

Proof

Case 1: Let n be a positive integer. We shall use principle of mathematical induction to prove the proposition.

For $n = 1$,

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos(1.\theta) + i\sin(1.\theta)$.

Therefore the theorem is true for $n = 1$.

Let us assume that the theorem is true for a particular value of n, say k, $(k \geq 1)$ i.e., we assume

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ ... (i)

Multiplying both sides of (i) by $(\cos\theta + i \sin\theta)$, we get

$$(\cos\theta + i \sin\theta)^{k+1} = (\cos k\theta + i \sin k\theta)(\cos\theta + i \sin\theta)$$
$$= (\cos k\theta \cos\theta - \sin k\theta \sin\theta) + i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)$$
$$= \cos(k+1)\theta + i \sin(k+1)\theta.$$

This shows that it is true for $n = k+1$.

Thus whenever the theorem is true for $n = k$, it is true for $n = k+1$. But the theorem has been proved to be true for $n = 1$, so it must be true for $k = 1+1=2$ and as it is true for $n = 2$, it must be true for $n = 2+1=3$ and so on. Hence the theorem is true for all positive integers n.

Case II : Let n be a negative integer. When n is a negative integer, we take $n = -m$, where m is a positive integer.

Then $(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{-m} = \dfrac{1}{(\cos\theta + i\sin\theta)^m}$

by Case 1, m being a positive integer

$$= \frac{1}{\cos m\theta + i \sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta)^2 + (\sin m\theta)^2} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta, \qquad \text{since } -m = n.$$

Hence the theorem is true for all negative integers n.

Case III : Let n be a fraction, positive or negative. Let $n = \dfrac{p}{q}$, where p is any integer positive or negative and q is an integer > 1.

Now we have $\left(\cos\dfrac{p}{q}\theta + i\sin\dfrac{p}{q}\theta\right)^q = \left(\cos q\dfrac{p}{q}\theta + i\sin q\dfrac{p}{q}\theta\right)$

(by De Moivre's Theorem for positive integral index)

$= \cos p\theta + i \sin p\theta = (\cos\theta + i \sin\theta)^p$ ... (2)

Now, extracting the q-th root of both sides of equation (2), we have $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the q-th roots of $(\cos\theta + i\sin\theta)^p$

i.e., $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

i.e., $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$, since $n = \frac{p}{q}$.

Thus the proof of De Moivre's theorem is complete.

Note 1 : When $n = 0$, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^0 = 1$

and $\cos(n\theta) + i\sin(n\theta) = \cos(0.\theta) + i\sin(0.\theta) = \cos 0 + i\sin 0 = 1$.

Therefore De Moivre's theorem is also true for $n = 0$.

Corollary : $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$;

for, $(\cos\theta - i\sin\theta)^n = \{\cos(-\theta) + i\sin(-\theta)\}^n = \{(\cos\theta + i\sin\theta)^{-1}\}^n$

$$= (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta.$$

Note 2 : De Moivre's theorem holds for all real values of n. When n is irrational, the number of values of $(\cos\theta + i\sin\theta)^n$ is infinite.

1.7.1. Application of De Moivre's Theorem

We like to state a very important theorem without proof.

1.7.2 Theorem :

Let $\frac{p}{q}$ be a rational number. p, q are integers prime to each other where $q > 1$.

Then $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ has exactly q distinct values which are given by

$$\cos\frac{p(2k\pi+\theta)}{q} + i\sin\frac{p(2k\pi+\theta)}{q}, \text{for } k = 0, 1, 2, \ldots\ldots, (q-1).$$

1.7.3 Extraction of any assigned root of a complex number

We are to find out all the n-th roots of a complex number $z = a + ib$. We express z in its De Moivre's form or polar form. We have $z = a + ib = r(\cos\theta + i\sin\theta)$, when $r\cos\theta = a$, $r\sin\theta = b$, then $r = \sqrt{a^2+b^2}$.

$z = r(\cos\theta + i\sin\theta) = r\{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}$,

where k is an integer. Then we see that $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left\{\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right\}$,

where $k = 0, 1, 2, \ldots\ldots, n-1$.

1.8 n-th roots of unity

Let us consider the equation $x^n - 1 = 0$.

Now $x^n = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$, where k is zero or any integer.

Therefore $x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{n}}$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{ for } k = 0, 1, 2, \ldots, (n-1).$$

If n be even, the real roots of $x^n = 1$ are ± 1 and the imaginary roots are given by

$$\cos\frac{2k\pi}{n} \pm i\sin\frac{2k\pi}{n}, \text{ where } k = 1, 2, \ldots, (\frac{1}{2}n-1).$$

If n be odd, the real root of $x^n - 1 = 0$ is 1 only and the imaginary roots are given by

$$\cos\frac{2k\pi}{n} \pm i\sin\frac{2k\pi}{n}, \text{ where } k = 1, 2, \ldots, \frac{1}{2}(n-1).$$

Let us now consider the equation $x^n + 1 = 0$.

If $x^n + 1 = 0$, then $x^n = -1 = \cos\pi + i\sin\pi = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$, where k is zero or any integer.

Therefore $x = \cos\dfrac{(2k+1)\pi}{n} + i\sin\dfrac{(2k+1)\pi}{n}, k = 0, 1, 2, \ldots, (n-1)$.

1.9 Expansions of cos $n\theta$ and sin $n\theta$, where n is a positive integer and θ is real

From De Moivre's theorem, we get

$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$

$$= \cos^n\theta + \binom{n}{1}\cos^{n-1}\theta(i\sin\theta) + \binom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2 + \ldots\ldots + (i\sin\theta)^n$$

$$= \left(\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\right)$$

$$+ i\left(\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots\right).$$

Equating real and imaginary parts, θ being real, we get

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots$$

and $$\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots$$

If n is even, the last term of the expansion of cos $n\theta$ is $(-1)^{\frac{n}{2}}\sin^n\theta$

and that of sin $n\theta$ is $(-1)^{\frac{n-2}{2}}.\binom{n}{n-1}\cos\theta\sin^{n-1}\theta$.

If n is odd, the last term of the expansion of cos $n\theta$ is $(-1)^{\frac{n-1}{2}}\binom{n}{n-1}\cos\theta\sin^{n-1}\theta$

and that of sin $n\theta$ is $(-1)^{\frac{n-1}{2}}\sin^n\theta$.

1.9.1 Expansions of $\cos^n\theta$ and $\sin^n\theta$ when n is a positive integer and θ is real

Let $z = \cos\theta + i\sin\theta$. Therefore $\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$.

Also $z^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{z^n} = \cos n\theta - i\sin n\theta$.

So $z+\frac{1}{z} = 2\cos\theta$, $z^n + \frac{1}{z^n} = 2\cos n\theta$

and $z-\frac{1}{z} = 2i\sin\theta$, $z^n - \frac{1}{z^n} = 2i\sin n\theta$.

Now $2^n\cos^n\theta = \left(z+\frac{1}{z}\right)^n$

$$= z^n + \binom{n}{1}z^{n-1}.\frac{1}{z^1} + \binom{n}{2}z^{n-2}.\frac{1}{z^2} + ...+\binom{n}{n-1}z.\frac{1}{z^{n-1}} + \frac{1}{z^n}$$

$$= \left(z^n+\frac{1}{z^n}\right)+\binom{n}{1}\left(z^{n-2}+\frac{1}{z^{n-2}}\right)+\binom{n}{2}\left(z^{n-4}+\frac{1}{z^{n-4}}\right)+......$$

$$= 2\cos n\theta+\binom{n}{1}.2\cos(n-2)\theta+\binom{n}{2}.2\cos(n-4)\theta+......$$

Therefore

$$\cos^n\theta = \frac{1}{2^{n-1}}\{\cos n\theta+\binom{n}{1}\cos(n-2)\theta+\binom{n}{2}\cos(n-4)\theta+......\}.$$

Since n is a positive integer, the number of terms on the R.H.S is finite.

Similarly, from the equality, $(2i\sin\theta)^n = \left(z-\frac{1}{z}\right)^n$, we can find the value of $\sin^n\theta$.

1.9.2 Expansion of tan $n\theta$, when n is a positive integer and θ is real, can be obtained from the relation $\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$.

1.9.3 Expansion of cos α and sin α in ascending powers of α, when n is a positive integer and α is measured in radians

We have $\cos n\theta = \cos^n\theta-\binom{n}{2}\cos^{n-2}\theta\sin^2\theta+\binom{n}{4}\cos^{n-4}\theta\sin^4\theta-......$

$$= \cos^n\theta-\frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta-......$$

Similarly $\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta+......$

Putting $n\theta = \alpha$, *i*.e. $n = \frac{\alpha}{\theta}$, ($\theta$ is also measured in radians),

we have $\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)}{2!}\cos^{n-2}\theta\sin^2\theta$

$$+\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{4!}\cos^{n-4}\theta\sin^4\theta - \ldots\ldots$$

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2$$

$$+\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \ldots\ldots$$

and $$\sin\alpha = \alpha\cos^{n-1}\theta.\frac{\sin\theta}{\theta} - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \ldots\ldots$$

Keeping α fixed and making $n \to +\infty$, we see that $\theta \to 0$ and $\frac{\sin\theta}{\theta} \to 1$ and therefore, we have $\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \ldots\ldots$

and $$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \ldots\ldots$$

1.10 Worked out Examples (II)

Example 1 : Find all the values of $(-1+i)^{\frac{1}{5}}$.

Solution : Let $-1 + i = r(\cos\theta + i\sin\theta)$. Therefore $r\cos\theta = -1$, $r\sin\theta = 1$.

Hence $r = \sqrt{(-1)^2+1^2} = \sqrt{2}$ and $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$.

These equations are simultaneously satisfied with $\theta = \frac{3\pi}{4}$.

Therefore $-1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{2}\left\{\cos\left(2k\pi+\frac{3\pi}{4}\right)+i\sin\left(2k\pi+\frac{3\pi}{4}\right)\right\}$,

where k is zero or any integer.

Therefore $(-1+i)^{\frac{1}{5}}=2^{\frac{1}{10}}\left\{\cos\frac{1}{5}\left(2k\pi+\frac{3\pi}{4}\right)+i\sin\frac{1}{5}\left(2k\pi+\frac{3\pi}{4}\right)\right\}$ for $k=0,1,2,3,4$.

Example 2. Solve (i) $x^6-1=0$ (ii) $x^7+1=0$.

Solution : (i) Here $x^6=1=\cos 2k\pi+i\sin 2k\pi$, where k is zero or an integer.

Therefore $x=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{6}}=\cos\frac{2k\pi}{6}+i\sin\frac{2k\pi}{6}$, where $k=0,1,2,...,5$.

So $x=\cos\frac{2k\pi}{6}\pm i\sin\frac{2k\pi}{6}=\cos\frac{k\pi}{3}\pm i\sin\frac{k\pi}{3}$, where $k=1,2$., [imaginary roots]

and $x=\pm 1$, [real roots].

(ii) When $x^7+1=0$, $x^7=-1=\cos(2k+1)\pi+i\sin(2k+1)\pi$ where k is zero or an integer.

Therefore $x=\cos\frac{(2k+1)}{7}\pi+i\sin\frac{(2k+1)}{7}\pi$, for $k=0,1,2,3,4,5,6$.

Example 3. Apply De Moivre's theorem to express (i) $\cos 3\theta$ in powers of $\cos\theta$ and (ii) $\sin 5\theta$ in powers of $\sin\theta$.

Solution : (i) We have

$$\begin{aligned}\cos 3\theta+i\sin 3\theta &= (\cos\theta+i\sin\theta)^3\\ &= \cos^3\theta+3\cos^2\theta\,.\,i\sin\theta+3\cos\theta\,.\,i^2\sin^2\theta+i^3\sin^3\theta\\ &= \cos^3\theta-3\cos\theta\sin^2\theta+i(3\cos^2\theta\sin\theta-\sin^3\theta).\end{aligned}$$

Equating real parts of both sides,

$$\begin{aligned}\cos 3\theta &= \cos^3\theta-3\cos\theta\sin^2\theta=\cos^3\theta-3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta-3\cos\theta.\end{aligned}$$

(ii) We have

$\cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin \theta)^5$

$= \cos^5\theta + 5\cos^4\theta \,.\, i \sin \theta + 10 \cos^3\theta \,.\, i^2 \sin^2 \theta$

$+ 10 \cos^2\theta \,.\, i^3 \sin^3 \theta + 5 \cos\theta \,.\, i^4 \sin^4 \theta + i^5 \sin^5\theta$

$= \cos^5 \theta - 10 \cos^3 \theta \sin^2\theta + 5 \cos\theta \sin^4 \theta$

$+ i (5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta)$.

Equating imaginary parts of both sides,

$\sin 5\theta = 5 \cos^4\theta \sin\theta - 10\cos^2 \theta \sin^3\theta + \sin^5\theta$

$= 5 (1 - \sin^2\theta)^2 \sin \theta - 10 (1 - \sin^2\theta) \sin^3\theta + \sin^5\theta$

$= 5 (1 - 2 \sin^2\theta + \sin^4\theta) \sin\theta - 10 (1-\sin^2\theta) \sin^3\theta + \sin^5\theta$

$= 5 \sin \theta - 20 \sin^3\theta + 16 \sin^5\theta$.

Example 4. Use De Moivre's theorem to express $\sin^4 \theta \cos^4\theta$ in terms of sines and cosines of multiples of θ.

Solution : Let $z = \cos \theta + i \sin\theta$, $\frac{1}{z} = z^{-1} = (\cos \theta + i \sin\theta)^{-1} = \cos\theta - i \sin\theta$.

Therefore $z + \frac{1}{z} = 2\cos\theta, \qquad z^n + \frac{1}{z^n} = 2\cos n\theta.$

$z - \frac{1}{z} = 2i\sin\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$

Therefore $(2i \sin\theta)^4 . (2\cos\theta)^4 = \left(z - \frac{1}{z}\right)^4 . \left(z + \frac{1}{z}\right)^4 = \left(z^2 - \frac{1}{z^2}\right)^4$

or, $2^8 \sin^4\theta \cos^4\theta = z^8 + 4 (z^2)^3 . \left(-\frac{1}{z^2}\right) + 6(z^2)^2\left(-\frac{1}{z^2}\right)^2$

$+ 4z^2 . \left(-\frac{1}{z^2}\right)^3 + \left(-\frac{1}{z^2}\right)^4$

$= \left(z^8 + \frac{1}{z^8}\right) - 4\left(z^4 + \frac{1}{z^4}\right) + 6$

$= 2 \cos 8\theta - 4.2 \cos 4\theta + 6$, [since $z^n + \frac{1}{z^n} = 2 \cos n\theta$]

Therefore $\sin^4\theta \cos^4\theta = \frac{1}{2^7} (\cos 8\theta - 4 \cos 4\theta + 3)$.

Example 5 : Find the value of $\lim_{x\to 0}\frac{x-\sin x}{x^3}$, x is in radians.

Solution : $\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{x-(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......)}{x^3}$

$$= \lim_{x\to 0}\frac{1}{x^3}\left(\frac{x^3}{3!}-\frac{x^5}{5!}+......\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{3!}-\frac{x^2}{5!}+......\right)=\frac{1}{3!}=\frac{1}{6}.$$

1.11 Summary and Keywords

Summary :

I. Polar representation of a complex number.

If $z = x + iy$, (x, y are real numbers) be a complex number, then $z = r(\cos\theta + i\sin\theta)$ is its polar representation, where $x = r\cos\theta$, $y = r\sin\theta$.

Its modulus = $|z| = r = \sqrt{x^2+y^2}$ and its amplitude = amp (z) = θ, satisfying

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

Conjugate of $z = \bar{z} = x - iy = r(\cos\theta - i\sin\theta)$.

II. De Moivre's Theorem.

If n be an integer, positive or negative, then $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$.

If n be a fraction, positive or negative, then $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

If n is of the form $\frac{p}{q}$, then $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ has exactly q distinct values which are given by $\cos\frac{p(2k\pi+\theta)}{q} + i\sin\frac{p(2k\pi+\theta)}{q}$, for $k = 0, 1, 2,......, (q-1)$.

III. n-th roots of unity.

n-th roots of unity are given by the following n values of $\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ for $k = 0, 1, 2, \ldots, (n-1)$.

Keywords : Complex numbers, modulus, amplitude, distinct roots.

1.12 Model Questions (II)

1. If n be a positive integer, prove that

   $(\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) \ldots (\cos\theta_n + i \sin\theta_n)$

   $= \cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i \sin(\theta_1 + \theta_2 + \ldots\ldots \theta_n)$.

   Hence deduce De Moivre's Theorem for positive integral indices.

2. Find θ for which $(\cos\theta + i \sin\theta)\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = i$.

3. Express $\frac{(\cos 2\theta + i \sin 2\theta)^4}{(\cos\theta + i \sin\theta)^6}$ in $A+i\,B$ form where A and B are real.

4. Find modulus and amplitude of $\frac{(\sin\theta + i \cos\theta)^4}{(\cos\theta - i \sin\theta)^6}$.

5. Show that $(\sin 2\theta + i \cos 2\theta)^n = \cos n\left(\frac{\pi}{2} - 2\theta\right) + i \sin n\left(\frac{\pi}{2} - 2\theta\right)$, n being an integer.

6. If n be a positive integer, then prove that

   $$\left(\frac{1 + \sin\varphi + i \cos\varphi}{1 + \sin\varphi - i \cos\varphi}\right)^n = \cos\left(\frac{n\pi}{2} - n\varphi\right) + i \sin\left(\frac{n\pi}{2} - n\varphi\right) = (\sin\varphi + i \cos\varphi)^n.$$

7. Solve : (i) $x^9 - 1 = 0$ (ii) $x^8 + 1 = 0$ (iii) $x^8 + x^5 - x^3 - 1 = 0$.

8. If $\alpha = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$ and r and p be prime to n, then prove that

   $1 + \alpha^p + \alpha^{2p} + \ldots\ldots + \alpha^{(n-1)p} = 0$.

9. If $a + \frac{1}{a} = 2\cos\alpha$ and $b + \frac{1}{b} = 2\cos\beta$, then show that $\cos(\alpha - \beta)$ is one of the values of $\frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$.

10. Using De Moivre's theorem, express $\sin 7\theta$ in terms of $\sin\theta$.

11. Using De Moivre's theorem, find the value of

(a) $\sin^5\theta$ (b) $\cos^7\theta$ (c) $2^5\sin^4\theta\cos^2\theta$.

12. If α and β are the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$, find the equation whose roots are α^n and β^n, where n is a positive integer.

13. If $x = \cos\theta + i\sin\theta$ and $1 + \sqrt{1-a^2} = na$, then prove that

$$1 + a\cos\theta = \frac{a}{2n}(1 + nx)\left(1 + \frac{n}{x}\right).$$

14. If n be a positive integer, prove that $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$.

15. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then prove that

(a) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(c) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$.

16. Find θ, when $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$.

1.12.1 Answers

2. $\frac{\pi}{3}$. 3. $\cos 2\theta + i\sin 2\theta$. 4. $1, 2\theta$.

7. (i) $\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}, k = 0$ to 8;

(ii) $x = \cos\frac{(2k+1)}{8}\pi + i\sin\frac{(2k+1)}{8}\pi$, for k=0 to 7

(iii) $\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}, k = 0, 1, 2, 3, 4$ and

$$\cos\frac{(2r+1)\pi}{3} + i\sin\frac{(2r+1)\pi}{3}, r = 0, 1, 2.$$

10. $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

11. (a) $\frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

(b) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$.

(c) $\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

12. $x^2 - 2x\cos n\theta + 1 = 0$.

16. 3° approximately.

Unit-2 ❐ Functions of Complex Numbers (Exponential and Trigonometric)

Structure

2.1 Objectives

After learning this unit, we shall be able to do the following :

- Any complex number can be expressed as an exponential function of the form $r.\ e^{i\theta}$.
- Circular function of a complex number can be written as an exponential function and vice-versa.
- Logarithm of a complex number can be expressed as a complex number of the form $A+iB$.

2.2 Introduction

In this unit, we shall learn the definitions of exponential functions, circular functions, hyperbolic functions, logarithmic functions of complex variables. The definitions of these functions which are known to us for any real variable will follow when the imaginary part of the complex number is taken to be zero.

As for example, let us consider a complex number $z = x + iy$, where x and y are any real number. Then the exponential function

$e^z = e^{x+iy} = e^x . e^{iy} = e^x (\cos y + i \sin y)$.

Now, if we take imaginary part equal to zero, *i.e.*, $y = 0$, we get $e^z = e^x . (\cos 0 + i \sin 0) = e^x$ which is the exponential function of a real variable. Similarly for other functions. We know that, for any real number x, the infinite series $1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ is an absolutely convergent series whose sum is denoted by e^x, *i.e.*, $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$

Similarly, for all real numbers x, the trigonometric series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...... + (-1)^n \frac{x^{2n}}{(2n)!} ......$$

and $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} ......$$

2.3 Definition

The series $1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!} + ...$ can be shown to be absolutely convergent for all complex numbers z and is denoted by *exp* (z) or $E(z)$. For uniformity, whether z is real or complex, we write this series as e^z.

Then, if z be a complex number, $e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!} + ...$

Similarly, for any complex number z, we define $\sin z$ and $\cos z$ as the sum function of the absolutely convergent power series given by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...... \text{ and } \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ......$$

Other trigonometric functions are defined as

$$\tan z = \frac{\sin z}{\cos z}, \ \cot z = \frac{\cos z}{\sin z}, \ \operatorname{cosec} z = \frac{1}{\sin z} \text{ and } \sec z = \frac{1}{\cos z}.$$

Corollary 1 : For $z = 1$, $e = 1+\frac{1^1}{1!}+\frac{2^2}{2!}+......$

Corollary 2 : For $z = 0$, $e^0 = 1$.

Like real index, e^x obeys the laws of indices. If z_1 and z_2 be two complex numbers, then $e^{z_1} = 1+z_1+\frac{z_1^2}{2!}+......$ and $e^{z_2} = 1+z_2+\frac{z_2^2}{2!}+......$

Now $e^{z_1} e^{z_2} = \left(1+z_1+\frac{z_1^2}{2!}+......\right)\left(1+z_2+\frac{z_2^2}{2!}+......\right)$

$$= 1+\frac{z_1+z_2}{1!}+\left(\frac{z_1^2}{2!}+z_1z_2+\frac{z_2^2}{2!}\right)+......$$

$$= 1+(z_1+z_2)+\frac{(z_1+z_2)^2}{2!}+......$$

Since the infinite series representing e^{z_1} and e^{z_2} are known to be absolutely convergent, the product $e^{z_1}.e^{z_2}$ is also an absolutely convergent series.

Hence $e^{z_1}.e^{z_2} = e^{z_1+z_2}$.

Corollary 3 : $e^{z_1}.e^{z_2}.......e^{z_n} = e^{z_1+z_2+..+z_n}$.

Corollary 4 : $(e^z)^n = e^{nz}$.

Corollary 5 : $(e^{mz})^n = (e^{nz})^m$.

Corollary 6 : $e^z.\ e^{-z} = 1$.

Corollary 7 : $e^{z_1} \div e^{z_2} = e^{z_1-z_2}$.

We have $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+......$

Let $z = ix$, where x is a real number.

Then we have $e^{ix} = 1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+......$

$$= \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)$$

Therefore $e^{ix} = \cos x + i \sin x$.

Similarly $e^{-ix} = \cos x - i \sin x$.

From these results, we have $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$.

These are known as Euler's exponential values. These values can be extended to complex variables also, *i.e,* if z be a complex variable, we can define

$$e^{iz} = \cos z + i \sin z,$$

$$e^{-iz} = \cos z - i \sin z \text{ and hence } \cos z = \frac{e^{iz}+e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

2.3.1 A few properties of sine and cosine

(i) We have $\cos(-x) = \frac{1}{2}\left(e^{i(-x)}+e^{-i(-x)}\right) = \frac{1}{2}\left(e^{-ix}+e^{ix}\right) = \cos x.$

(ii) We have $\sin(-x) = \frac{1}{2i}\left(e^{i(-x)}-e^{-i(-x)}\right) = \frac{1}{2i}\left(e^{-ix}-e^{ix}\right) = -\sin x.$

(iii) $\sin^2 x + \cos^2 x = \left\{\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\right\}^2 + \left\{\frac{1}{2}\left(e^{ix}+e^{-ix}\right)\right\}^2$

$$= \frac{1}{4}\left\{\left(e^{ix}+e^{-ix}\right)^2 - \left(e^{ix}-e^{-ix}\right)^2\right\}$$

$$= \frac{1}{4}.4e^{ix}.e^{-ix} = 1.$$

(iv) (a) $\sec^2 x = 1 + \tan^2 x$. (b) $\operatorname{cosec}^2 x = 1 + \cot^2 x$.

(v) $\sin x \cos y \pm \cos x \sin y = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right).\frac{1}{2}\left(e^{iy}+e^{-iy}\right)$

$$\pm\frac{1}{2}\left(e^{ix}+e^{-ix}\right).\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)$$

$$= \frac{1}{4i}\left(e^{ix}.2e^{\pm iy} - e^{-ix}.2e^{\mp iy}\right)$$

$$= \frac{1}{2i}\left\{e^{i(x\pm y)} - e^{-i(x\pm y)}\right\} = \sin(x \pm y).$$

Similarly we have

(vi) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

(vii) Putting $x = y$ in $\sin(x + y) = \sin x \cos y + \cos x \sin y$, we get $\sin 2x = 2 \sin x \cos x$.

Similarly

(viii) $\cos 2x = \cos^2 x - \sin^2 x$.

2.3.2 Periodic function

Definition : A function $f(x)$ of a complex variable x is said to be a periodic function with period k, if $f(x+k) = f(x)$.

Corollary : If m be any integer and $f(x)$ be a periodic function of x of period k, then $f(x + mk) = f(x)$.

2.3.3 Theorem

If x be a complex number, then $\sin x$, $\cos x$, $\tan x$, e^x, e^{ix} are periodic functions of x with period $2n\pi$, $2n\pi$, $n\pi$, $2n\pi i$, $2n\pi$ respectively, where n is any integer.

Proof :

We know that $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$ for all integers n.

We have

$\sin(2n\pi + x) = \sin 2n\pi \cos x + \cos 2n\pi \sin x = 0.\cos x + 1.\sin x = \sin x$,

$\cos(2n\pi + x) = \cos 2n\pi \cos x - \sin 2n\pi \sin x = 1.\cos x - 0.\sin x = \cos x$,

$\tan(n\pi + x) = \tan x$,

$$e^{2n\pi i + x} = e^{2n\pi i}.e^x = (\cos 2n\pi + i \sin 2n\pi)e^x = e^x,$$

$$e^{i(2n\pi + x)} = \cos(2n\pi + x) + i \sin(2n\pi + x) = \cos x + i \sin x = e^{ix}.$$

2.4 Logarithm of a complex number

Definition : If there is a complex number $N\ (\neq 0)$, corresponding to a complex number z such that $e^z = N$, then z is defined to be the logarithm of N.

Since we know $e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1$, so for all integers n,

$$N = e^z = e^z.1 = e^z.e^{2n\pi i} = e^{z + 2n\pi i}.$$

Therefore, according to the definition, $(x + 2n\pi i)$ is also logarithm of N, *i.e.*, logarithm of a complex number is a multiple-valued function, and we write this as Log N.

Therefore Log $N = z + 2n\pi i = \log N + 2n\pi i$.

$\log N$ is called the principal value of logarithm, when $n = 0$.

If we take the complex number in modulus-amplitude form, we have

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = \text{mod}(z)$ and $\theta = \text{amp}(z)$.

Then Log $z = \log r + i(\theta + 2n\pi) = \log(\text{mod}(z)) + i(\text{amp}(z) + 2n\pi)$.

Principal value is given by $\log z = \log(\text{mod}(z)) + i\,\text{amp}(z)$, taking $n = 0$.

Properties of logarithms :

For any two non-zero complex numbers x and y,

(i) Log (xy) = Log x + Log y.

Let $x = e^{z_1}, y = e^{z_2}$. Therefore $xy = e^{z_1}.e^{z_2} = e^{z_1+z_2}$.

Since Log $x = z_1 + 2n\pi i$, Log $y = z_2 + 2n'\pi i$ and Log $(xy) = z_1 + z_2 + 2n''\pi i$, where n, n', n'' are integers.

Therefore Log x + Log y $= z_1 + z_2 + 2(n + n')\pi i = z_1 + z_2 + 2n''\pi i$ = Log (xy), taking $n'' = n + n'$ as integer.

(ii) $\text{Log}\left(\frac{x}{y}\right) = \text{Log}\,x - \text{Log}\,y$. Here $\frac{x}{y} = \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$, etc.

Similarly, we have

(iii) Log $(x^y) = y$ Log $x + 2n\pi i$, where n is any integer.

(iv) For $x > 0$, Log $(-x) = \log x + (2n+1)\pi i$ and $\log(-x) = \log x + \pi i$.

(v) For each value of m, Log z is some value of Log z^m but not conversely, m being a rational number.

2.4.1 Definition of a^z, where $a\ (\neq 0)$ and z are complex numbers

We define a^z as $a^z = e^{z\text{Log}\,a} = e^{z(\log a + 2n\pi i)}$, n being any integer.

The principal value of a^z is $e^{z\log a}$.

2.5 Hyperbolic functions

For any complex number z, we define hyperbolic functions as under :

Hyperbolic sine : $\sinh z = \dfrac{e^z - e^{-z}}{2}$

Hyperbolic cosine : $\cosh z = \dfrac{e^z + e^{-z}}{2}$

Similarly $\tanh z = \dfrac{\sinh z}{\cosh z}$, $\coth z = \dfrac{1}{\tanh z}$,

$\operatorname{cosech} z = \dfrac{1}{\sinh z}$, $\operatorname{sech} z = \dfrac{1}{\cosh z}$.

Some important formulae :

Let x, y and z be complex numbers.

(i) $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x + \tanh^2 x = 1$, $\coth^2 x - \operatorname{cosech}^2 x = 1$.

(ii) $\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$, $\tanh(-x) = -\tanh x$.

(iii) $\sinh(0) = 0$, $\cosh(0) = 1$, $\tanh(0) = 0$.

(iv) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

(v) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

(vi) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

(vii) $\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$.

(viii) $\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$

$\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$

$\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$

$\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$.

(ix) $2 \sinh x \cosh y = \sinh(x + y) + \sinh(x - y)$

$2 \cosh x \sinh y = \sinh(x + y) - \sinh(x - y)$

$2 \cosh x \cosh y = \cosh(x + y) + \cosh(x - y)$

$2 \sinh x \sinh y = \cosh(x + y) - \cosh(x - y)$.

(x) $\sinh(2n\pi i + x) = \sinh x$, $\cosh(2n\pi i + x) = \cosh x$ and $\tanh(n\pi i + x) = \tanh x$, where n is any integer.

These formulae can directly be proved from definitions.

2.6 Inverse Circular Functions

For real values of x, y, a and b, if $\cos(x + iy) = a + ib$, then we define $(x + iy)$ as inverse cosine of $(a + ib)$

For all integers n, $\cos(x + iy) = \cos(2n\pi \pm (x + iy)) = a + ib$.

Therefore inverse of cosine function of a complex number is a many valued function and we shall write it as $\mathrm{Cos}^{-1}(a + ib)$, *i.e.*,

$$\mathrm{Cos}^{-1}(a + ib) = 2n\pi + \cos^{-1}(a + ib).$$

The principal value (p.v.) is obtained by putting $n = 0$. The real part of the p.v. lies in $(0, \pi)$.

Similarly, $\mathrm{Sin}^{-1}(a + ib) = n\pi + (-1)^n \sin^{-1}(a + ib)$.

The principal value is obtained by putting $n = 0$. The real part of the p.v. lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\mathrm{Tan}^{-1}(a + ib) = n\pi + \tan^{-1}(a + ib)$.

Putting $n = 0$, the principal value is obtained. The real part of the p.v. lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Also, $\mathrm{Sec}^{-1}(a + ib) = 2n\pi + \sec^{-1}(a + ib)$

$\mathrm{Cosec}^{-1}(a + ib) = n\pi + (-1)^n \mathrm{cosec}^{-1}(a + ib)$

$\mathrm{Cot}^{-1}(a + ib) = n\pi + \cot^{-1}(a + ib)$.

2.6.1 Some relations between hyperbolic and circular functions

Let z be a complex number ; then

(i) $\cosh z = \cos(iz)$, $\quad i \sinh z = \sin(iz)$, $\quad i \tanh z = \tan(iz)$.

Proof : $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh z$

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i.\frac{e^{z} - e^{-z}}{2} = i \sinh z$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = i \tanh z.$$

(ii) $\cosh(iz) = \cos z$, $\quad \sinh(iz) = i \sin z$, $\quad \tanh(iz) = i \tan z$.

Proof : $\cosh(iz) = \cos(i.\, iz) = \cos(-z) = \cos z$

$$\sinh(iz) = \frac{1}{i}\sin(i.iz) = \frac{1}{i}\sin(-z) = -\frac{1}{i}\sin(z) = i\sin(z)$$

$$\tanh(iz) = \frac{\sinh(iz)}{\cosh(iz)} = i \tan z.$$

(iii) We define the inverse of Trigonometric and Hyperbolic functions.

If $\sinh z = w$, then $z = \sinh^{-1} w$, etc.

(iv) $\sinh^{-1} x = -\, i \sin^{-1}(ix)$

$\cosh^{-1} x = -i \cos^{-1} x$

$\tanh^{-1} x = -\, i \tan^{-1}(ix)$.

2.6.2 Gregory's series

We have $i \tan\theta = \dfrac{i \sin\theta}{\cos\theta}$

or, $\dfrac{1 + i\tan\theta}{1 - i\tan\theta} = \dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = \dfrac{e^{i\theta}}{e^{-i\theta}} = e^{2i\theta}$, by compo. and divi.

Considering the principal value of logarithm of both sides, we get

$$2i\theta = \log(1 + i\tan\theta) - \log(1 - i\tan\theta)$$

$$= \left(i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta - ......\right) - \left(-i\tan\theta - \frac{1}{2}i^2\tan^2\theta - \frac{1}{3}i^3\tan^3\theta - ......\right)$$

$$= 2i\left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - ......\right).$$

Therefore $\theta = \left(\tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - ......\text{to infinity}\right)$, which is expansion of θ in powers of tan θ.

This is known as Gregory's series.

It is convergent when $|\tan\theta| \le 1$, *i.e.*, when $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Putting tan $\theta = x$, *i.e.*, $\tan^{-1} x = \theta$, we get

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - ......$ when $|x| \le 1$.

2.7 Worked out Examples

Example 1 : If x be a complex number, then from the definition of sin x and cos x, show that

(i) $\sin 2x = 2\sin x \cos x$

(ii) $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$.

Solution : (i) We have R.H.S. $= 2\sin x\cos x = 2.\ \frac{e^{ix} - e^{-ix}}{2i}.\frac{e^{ix} + e^{-ix}}{2}$

$$= \frac{e^{2ix} - e^{-2ix}}{2i} = \sin 2x = \text{L.H.S.}$$

(ii) R.H.S $= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \left(\frac{e^{i\frac{x}{2}} + e^{-i\frac{x}{2}}}{2}\right)^2 - \left(\frac{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}{2i}\right)^2$

$$= \frac{e^{ix} + 2e^0 + e^{-ix}}{4} - \frac{e^{ix} - 2e^0 + e^{-ix}}{-4}$$

$$= \frac{e^{ix} + 2 + e^{-ix} + e^{ix} - 2 + e^{-ix}}{4} = \frac{2e^{ix} + 2e^{-ix}}{4}$$

$$= \frac{e^{ix} + e^{-ix}}{2} = \cos x = \text{L.H.S.}$$

(iii) R.H.S. $= 3\sin x - 4\sin^3 x = 3.\ \frac{e^{ix} - e^{-ix}}{2i} - 4.\frac{(e^{ix} - e^{-ix})^3}{(2i)^3}$

$$= \frac{3(e^{ix} - e^{-ix})}{2i} + \frac{e^{3ix} - 3e^{2ix}.e^{-ix} + 3e^{ix}.e^{-2ix} - e^{-3ix}}{2i}$$

$$= \frac{3e^{ix} - 3e^{-ix} + e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{2i} = \frac{e^{i.3x} - e^{-i.3.x}}{2i} = \sin 3x = \text{L.H.S.}$$

Example 2 : Show that $\sin\left\{i\log\frac{a-ib}{a+ib}\right\} = \frac{2ab}{a^2+b^2}$.

Solution : Let $a = r\cos\theta$ and $b = r\sin\theta$.

Therefore $\tan\theta = \frac{b}{a}$.

Now

$$\sin\left\{i\log\frac{a-ib}{a+ib}\right\} = \sin\left\{i\log\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right\} = \sin\left\{i\log\frac{e^{-i\theta}}{e^{i\theta}}\right\}$$

$$= \sin\left\{i\log e^{-2i\theta}\right\}$$

$$= \sin\{i(-2i\theta)\} = \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2ab}{a^2+b^2}.$$

Example 3 : Prove that Log $(-1) = (2n + 1)\ \pi i$.

Solution : Log $(-1) = \log(-1) + 2n\pi i$, for all integers n

$= \log(\cos\pi + i\sin\pi) + 2n\pi i$

$= \log(e^{i\pi}) + 2n\pi i = i\pi + 2n\pi i$

$= (2n + 1)\ \pi i.$

Example 4 : Find the value of log i.

Solution : We know $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$.

Therefore mod $(i) = 1$ and amp $(i) = \frac{\pi}{2}$.

Therefore $\log(i) = \log(\text{mod } i) + i(\text{amp } i) = \log 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$.

Example 5 : Prove that Log $i^2 \neq 2$ Log i, but $\log i^2 = 2\log i$.

Solution : Log i^2 = Log $(-1) = \log(\text{mod}(-1)) + i\ \text{amp}(-1) + 2n\pi i$

$= \log 1 + i\pi + 2n\pi i = (2n + 1)\ \pi i$, for any integer n.

2 Log $i = 2\ [\log|i| + i\ \text{amp}\ (i) + 2k\pi i]$, for any integer k

$= 2\ [\log|i| + i\frac{\pi}{2} + 2k\pi i] = (4k + 1)\ \pi i$.

Obviously, $2n + 1$ and $4k + 1$ are not always equal. Therefore Log $i^2 \neq 2$Log i.

Again, $\log i^2 = \log(-1) = i\pi$ and $2\log i = 2\left(i\frac{\pi}{2}\right) = i\pi$. Hence proved.

Example 6 : Find a complex number z, for which $e^z = i$.

Solution : $e^z = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}}.\ e^{2n\pi i}$, for any integer n.

So, $e^z = e^{i\frac{\pi}{2} + 2n\pi i} = e^{(4n+1)\frac{\pi}{2}i}$.

Hence $z = (4n+1)\frac{\pi}{2}i$, , for $n = 0$ or any integer.

Example 7 : Prove that $\operatorname{Sin}^{-1}\sqrt{-1} = n\pi + (-1)^n\, i \log(\sqrt{2}+1)$, n being zero or any integer.

Solution : Let $\operatorname{Sin}^{-1}\sqrt{-1} = \theta$. Therefore $\sin\theta = \sqrt{-1} = i$

and $\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-(-1)} = \pm\sqrt{2}$.

Now $e^{i\theta} = \cos\theta + i\sin\theta = \pm\sqrt{2} + i.i = \pm\sqrt{2} - 1$.

If $e^{i\theta} = \sqrt{2}-1$, then $i\theta = \operatorname{Log}(\sqrt{2}-1) = 2k\pi i + \log(\sqrt{2}-1)$,

where k is zero or any integer.

So, $\theta = 2k\pi + \frac{1}{i}\log\left(\sqrt{2}-1\right) = 2k\pi - i\log\left(\sqrt{2}-1\right)$

$$= 2k\pi - i\log\frac{1}{\sqrt{2}+1} = 2k\pi + i\log\left(\sqrt{2}+1\right) \qquad \ldots (1)$$

If $e^{i\theta} = -\sqrt{2}-1$, then $i\theta = \operatorname{Log}(-\sqrt{2}-1) = 2m\pi i + \pi i + \log\left(\sqrt{2}+1\right)$

$$= (2m+1)\ \pi i + \log\left(\sqrt{2}+1\right),$$

where m is zero or any integer

or $\theta = (2m+1)\pi + \frac{1}{i}\log\left(\sqrt{2}+1\right) = (2m+1)\pi - i\log\left(\sqrt{2}+1\right) \qquad \ldots(2)$

Considering (1) and (2), we have

$\theta = n\pi + (-1)^n \log\left(\sqrt{2}+1\right)$, where n is zero or any integer.

Example 8 : Prove that $\cos\left(\log i^i\right) = 0$.

Solution : $i^i = e^{i\operatorname{Log} i} = e^{i[\log 1 + i\frac{\pi}{2} + 2n\pi i]}$, since $|i| = 1$ and $\operatorname{amp} i = \frac{\pi}{2}$

$$= e^{-[\frac{\pi}{2} + 2n\pi]}.$$

Therefore $\log i^i = \log e^{-[\frac{\pi}{2}+2n\pi]} + i.0,$ taking principal value of log i^i

$$= -\left(\frac{\pi}{2}+2n\pi\right).$$

Therefore cos $\left(\log i^i\right) = \cos[-\left(\frac{\pi}{2}+2n\pi\right)] = \cos\left(2n\pi+\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$.

Example 9 : Express $(4 + 3i)^{1+i}$ in $A + iB$ form.

Solution : We have $(4 + 3i)^{1+i} = e^{(1+i)\,\mathrm{Log}\,(4+3i)}$

$= e^{(1+i)[\log 5 + i\tan^{-1}\frac{3}{4}+2k\pi i]}$ $\left[\text{Here}\,|4+3i| = \sqrt{4^2+3^2} = 5 \text{ and amp } (4+3i) = \tan^{-1}\frac{3}{4}\right]$

$= e^{\left\{\log 5-(2k\pi+\tan^{-1}\frac{3}{4})\right\}+i\left\{\log 5+2k\pi+\tan^{-1}\frac{3}{4}\right\}}$

$= e^{P+iQ}$, where $P = \log 5 - \left(2k\pi + \tan^{-1}\frac{3}{4}\right)$ and $Q = \log 5 + 2k\pi + \tan^{-1}\frac{3}{4}$

$= e^P.e^{iQ} = e^P(\cos Q + i\sin Q) = e^P\cos Q + ie^P\sin Q$

$= A + iB$, where $A = e^P \cos Q$ and $B = e^P \sin Q$.

Example 10 : Deduce from Gregory's series :

$$\frac{\pi}{2} = \sqrt{3}\left\{1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\ldots\ldots\right\}$$

Solution : We know Gregory's series is given by

$\tan^{-1}x = x - \frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots, |x| \le 1$ and $-\frac{\pi}{4} \le \tan^{-1}x \le \frac{\pi}{4}$.

Taking $x = \frac{1}{\sqrt{3}}$, we get $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{1}{3.3\sqrt{3}} + \frac{1}{5.3^2\sqrt{3}} - \frac{1}{7.3^3\sqrt{3}} + \ldots\ldots$

or, $\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left[1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^2}+\ldots\ldots\right]$

or, $\frac{\pi}{2} = \sqrt{3}\left[1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\ldots\ldots\right]$

2.8 Summary and Keywords

Summary :

I. Functions of a complex number.

If z be a complex number, then $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... + \frac{z^n}{n!} + ...$

If $z = i\theta$, then $e^{i\theta} = \cos\theta + i\sin\theta$.

Euler's exponential values of $\cos x$ and $\sin x$ are given by $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and

$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.

II. Logarithm of a complex number.

Logarithm of a complex number N $(\neq 0)$ is a many valued function. We write $\text{Log}\, N = \log N + 2n\pi i$, where $\log N$ is the principal value of logarithm for $n = 0$.

III. Definition of a^z, where a $(\neq 0)$ and z are complex numbers.

We define $a^z = e^{z\, \text{Log}\, a} = e^{z\,(\log a + 2n\pi i)}$, n being any integer.

IV. Hyperbolic functions.

If z be a complex number, then $\sinh z = \frac{1}{2}(e^z - e^{-z}), \cosh z = \frac{1}{2}(e^z + e^{-z})$.

$\tanh z = \frac{\sinh z}{\cosh z}, \coth z = \frac{1}{\tanh z}, \text{cosech}\, z = \frac{1}{\sinh z}, \text{sech}\, z = \frac{1}{\cosh z}$.

V. Relations among hyperbolic and circular functions.

(i) $\cosh z = \cos(iz)$ (ii) $\sinh z = -i\sin(iz)$ (iii) $\tanh z = -i\tan(iz)$.

KEYWORDS : Exponential, Logarithm, Inverse, principal value.

2.9 Model Questions

1. Show that $a^z = 1 + z\log a + \frac{(z\log a)^2}{2!} + \frac{(z\log a)^3}{3!} + ...,$

2. Show that (i) log $(-x) = \log x + \pi i$. (ii) $\text{Log}(-i) = \frac{1}{2}(4n-1)\pi i$.

3. Using the definitions of sine and cosine of complex numbers, show that

   (i) $\sin 2x = 2 \sin x \cos x$.

   (ii) $\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.

   (iii) $\sin x + \sin y = 2 \sin\frac{1}{2}(x+y).\cos\frac{1}{2}(x-y)$.

   (iv) $\cos (x - y) = \cos x \cos y + \sin x \sin y$.

4. Prove that $\tan \left\{i \log\frac{a-ib}{a+ib}\right\} = \frac{2ab}{a^2-b^2}$.

5. Prove that $i \log \frac{x-i}{x+i} = \pi - 2\tan^{-1} x$.

6. Prove that $\log i + \log (-1 + i) \neq \log \{i (-1+ i)\}$.

7. If $\tan \log (x + iy) = a + ib$, where $a^2 + b^2 = 1$,

   then show that $\tan \log (x^2+y^2) = \frac{2a}{1-a^2-b^2}$.

8. If $\tan (u + iv) = x + iy$, (u, v, x, y are real), then prove that $x^2 + y^2 + 2x \cot 2u = 1$

   and $x^2+y^2+2y\left(\frac{e^{-2v}+e^{2v}}{e^{-2v}-e^{2v}}\right)+1=0$.

9. Find the general value and the principal value of each of the following:

   (i) Log (–2). (ii) Log (–1). (iii) Log (1+i). (iv) Log $\sqrt{i}$. (v) Log (3+4i).

10. Find the value of

    (i) i^i. (ii) $(-i)^i$. (iii) $(i)^{-i}$. (iv) $(-i)^{-i}$. (v) $\sin (\log i^i)$.

11. Express the following in $A + iB$ form, where x and y are real:

    (i) $\sin (x + iy)$. (ii) $\text{cosec} (x + iy)$. (iii) $\tan(x + iy)$.

    (iv) x^i. (v) e^{x+iy}. (vi) $\text{Log} \sin (x + iy)$. (vii) $\cosh (x+iy)$.

    (viii) $\text{Cos}^{-1} i$. (ix) $\tan^{-1} (\alpha \pm \beta)$. (x) π^i.

12. Show that the ratio of the principal values of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2) + i\cos(\log 2)$.

13. Show that the solution of the equation $\cos x = 2$ is given by

$$x = 2n\pi \pm i \log(2 + \sqrt{3}),\ n \text{ being an integer.}$$

14. Deduce from Gregory's series $\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots\ldots$

15. Prove that $\text{Tan}^{-1}(1+i) = \frac{1}{2}\left\{(2n+1)\pi + \tan^{-1}(-2)\right\} + \frac{i}{4}\log 5$.

16. Show that $\text{Sin}^{-1}(ix) = n\pi + (-1)^n i \log\left(x + \sqrt{1+x^2}\right)$.

17. If $x = \log\tan\left(\frac{\pi}{4} + \frac{1}{2}y\right)$, then prove that $y = -i\ \text{Log}\tan\left(\frac{\pi}{4} + \frac{1}{2}ix\right)$, x and y being real.

18. Solve : $e^z = 1 + i\sqrt{3}$.

19. Show that the equation $\tan\left\{i\log\frac{x-iy}{x+iy}\right\} = 2$ represents the rectangular hyperbola $x^2 - y^2 = xy$.

20. If z be a complex number, prove that the equation $e^z = 0$ has no solution and the general solution of the equation $\sinh z = 2$ is given by

$$z = n\pi i + (-1)^n \log(\sqrt{5} + 2).$$

2.9.1 Answers

9. (i) $\log 2 + (2n+1)\pi i,\ \log 2 + \pi i.$ (ii) $(2n+1)\pi i,\ \pi i$.

(iii) $\frac{1}{2}\log 2 + \left(2n + \frac{1}{4}\right)\pi i,\ \frac{1}{2}\log 2 + \frac{\pi}{4}i$. (iv) $(8n+1)\frac{\pi}{4}i,\ \frac{\pi}{4}i$.

(v) $\frac{1}{2}\log 5 + (2n\pi + \tan^{-1}\frac{4}{3})i,\ \frac{1}{2}\log 5 + i\tan^{-1}\frac{4}{3}$.

10. (i) $e^{-(4n+1)\frac{\pi}{2}}$. (ii) $e^{(2n+\frac{1}{2})\pi}$. (iii) $e^{(2n+\frac{1}{2})\pi}$. (iv) $e^{(4n-1)\frac{\pi}{2}}$. (v) -1.

11. (i) $\frac{1}{2}(e^y + e^{-y})\sin x + i.\frac{1}{2}(e^y - e^{-y})\cos x$.

(ii) $\frac{2\sin x\cosh y}{1-\cos 2x\cosh 2y} - i\frac{2\cos x\sinh y}{1-\cos 2x\cosh 2y}$.

(iii) $\frac{2\sin 2x}{\cosh 2x+\cosh 2y} + i\frac{\sinh 2y}{\cosh 2x+\cosh 2y}$.

(iv) $e^{-2n\pi}\{\cos(\log x) + i\sin(\log x)\}$ (v) $e^x\cos y + ie^x\sin y$.

(vi) $\frac{1}{2}\log\left\{\frac{1}{2}(\cosh 2y - \cos 2x) + i\{2n\pi + \tan^{-1}(\cot x.\tanh y\}\right\}$.

(vii) $\cosh x \cos y + i \sinh x \sin y$. (viii) $\left(2n\pi \pm \frac{\pi}{2}\right) \mp i\log(\sqrt{2}+1)$.

(ix) $n\pi + \tan^{-1}(\alpha \pm \beta) + i.\ 0$. (x) $e^{2n\pi}\{\cos(\log\pi) + i\sin(\log\pi)\}$.

14. Hint. Put $\theta = \frac{\pi}{4}$ in Gregory's series and get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

$$= (1-\frac{1}{3}) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots\ldots$$

18. $z = \log 2 + \left(2n + \frac{1}{3}\right)\pi i$.

Unit-3 ❒ Theory of Equations

Structure

3.1 Objectives

This unit gives us

- the short cut process of division of a polynomial by a binomial or a trinomial,
- general properties of equations
- statement of Fundamental Theorem of classical algebra,
- statement of Descartes' Rule of signs.

3.2 Introduction

Before going to discuss the general properties of equations, we are to know about the algebraic expressions. Here under we like to give the definition of a polynomial, the idea of division algorithm, remainder theorem. This will help a lot in solving algebraic equations.

3.3 Polynomials

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$, often denoted by $f(x)$, where n is a positive integer or zero and the coefficients a_0, a_1, a_2......,a_n

($a_0 \neq 0$), are in general any number (real or complex), is called a polynomial in x of degree n. a_0x^n is called the leading term. The algebraic symbol x is called a variable or an unkown. Henceforth, in all our discussions, we shall use, a_i (i = 0, 1, 2,..., n – 1, n) as real or rational numbers.

A polynomial will be called a complete polynomial if all the lower degree terms beginning from the highest are present, otherwise it is an incomplete polynomial. Thus $3x^4 + 4x^3 - 8x^2 + x - 9$ is a complete polynomial where as $4x^5 - 3x^3 + 5x + 10$ is an incomplete polynomial.

A polynomial $f(x)$ of degree one, two, three or four is called linear, quadratic, cubic or quartic (or biquadratic) respectively.

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$

and $g(x) = b_0x^m + b_1x^{m-1} + b_2x^{m-2} + ... + b_{m-1}x + b_m$

be two polynomials of degree n and m respectively, then

(i) $f(x)$ and $g(x)$ will be identically equal if $n = m$ and $a_i = b_i$ for i = 0, 1, 2,...

(ii) $f(x) + g(x)$ is the sum of two polynomials and this is a polynomial of degree n or m according as $n > m$ or $n \leq m$,

(iii) $f(x).\ g(x)$ is the product of two polynomials and this is a polynomial of degree $(n + m)$.

(iv) If $F(x) = f(x).\ g(x)$, then $f(x)$ and $g(x)$ are called factors of $F(x)$.

3.3.1 Division Algorithm

Let $f(x)$ and $g(x)$ be two polynomials such that the degree of $f(x)$ is greater than or equal to that of $g(x)$. Then to divide $f(x)$ by $g(x)$ is nothing but to find two unique polynomials $Q(x)$ and $R(x)$ such that $f(x) = g(x) \times Q(x) + R(x)$, where degree of $R(x)$ is less than the degree of $g(x)$. This is known as division algorithm. The polynomials $Q(x)$ and $R(x)$ are respectively called the quotient and remainder of this division.

If $R(x) = 0$, then $f(x)$ is said to be divisible by $g(x)$ or $g(x)$ is said to be a factor of $f(x)$.

3.3.2 Synthetic Division

This is a short-cut process to divide a polynomial $f(x)$ by a binomial of the form $(x - h)$. Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n, a_0 \neq 0$ be divided by $(x - h)$ to have the quotient $Q(x) = b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + ... + b_{n-2}x + b_{n-1}$ and the remainder

R which is obviously a constant. Then, by division algorithm, we get the identity

$$f(x) = (x - h)\, Q(x) + R$$

or, $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$

$= (x-h)(b_0x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \ldots\ldots + b_{n-2}\,x + b_{n-1}) + R$

$= b_0x^n + (b_1 - hb_o)x^{n-1} + (b_2 - hb_1)x^{n-2} + \ldots\ldots + (b_{n-1} - hb_{n-2})x + R - hb_{n-1}.$

Equating the coefficients of like powers of x on both sides, we get

$$a_0 = b_0,\ a_1 = b_1 - hb_0,\ a_2 = b_2 - hb_1, \ldots\ldots,\ a_{n-1} = b_{n-1} - hb_{n-2},\ a_n = R - hb_{n-1}$$

i.e., $b_0 = a_0, b_1 = a_1 + hb_0, b_2 = a_2 + hb_1, \ldots, b_{n-1} = a_{n-1} + hb_{n-2,} R = a_n + hb_{n-1}.$

The coefficients of the quotient polynomial and the remainder can casily be calculated according to the following scheme :

h	a_0	a_1	a_2	a_3	……	a_{n-1}	a_n
	$\downarrow$	b_0h	b_1h	b_2h	……	$b_{n-2}h$	$b_{n-1}h$
	$a_0(=b_0)$	b_1	b_2	b_3	……	b_{n-1}	R

3.3.3 Remainder Theorem

If a polynomial $f(x)$ be divided by a binomial $(x-h)$, then the remainder is $f(h)$.

Proof : When $f(x)$ is divided by $(x - h)$, let $Q(x)$ be the quotient and R (independent of x) be the remainder, then $f(x) = (x - h)\, Q(x) + R$.

Putting $x = h$, $f(h) = 0.\ Q(h) + R = R$. Therefore $R = f(h)$.

If $f(h) = 0$, *i.e.*, $R = 0$, then $f(x) = (x - h)\, Q(x)$ *i.e.*, $(x-h)$ is a factor of $f(x)$.

We say $f(x)$ is divisible by $(x - h)$.

Remark : From above, we see that $f(x)$ is divisible by $(x - a)$, if and only if $f(a) = 0$. The value (or values) of x for which the polynomial $f(x)$ vanishes is (are) called zeros of the polynomial. When $f(a) = 0$, we say $x = a$ is a root of the equation $f(x) = 0$.

3.4 Worked out Examples (I)

Example 1: Use synthetic method of division to find the quotient and the remainder, when $x^3 + 5x^2 + 1$ is divided by $x + 3$.

Solution : Here $f(x) = 1.x^3 + 5.x^2 + 0.x + 1$ and $h = -3$.

–3	1	5	0	1
	↓	–3	–6	18
	1	2	–6	19

Quotient = $x^2 + 2x - 6$ and Remainder = 19.

Example 2 : Find $Q(x)$ and R, when $4x^3 + 2x^2 - 8x - 5$ is divided by $2x - 1$.

Solution : Let $Q(x)$ be the quotient and R be the remainder.

Hence $f(x) \equiv 4x^3 + 2x^2 - 8x - 5 = (2x - 1)\, Q(x) + R = \left(x - \frac{1}{2}\right).\, 2Q(x) + R.$

So we first divide $f(x)$ by $x - \frac{1}{2}$ as under :

$\frac{1}{2}$	4	2	–8	–5
	↓	2	2	–3
	4	4	–6	–8

Therefore we get $2Q(x) = 4x^2 + 4x - 6$ or $Q(x) = 2x^2 + 2x - 3$ and $R = -8$.

Example 3: Express $2x^5 + 5x^3 - 4x$ as a polynomial in $x - 1$. Also find $f(x + 1)$.

Solution : We divide $2x^5 + 5x^3 - 4x$ by $x - 1$ using synthetic division in succession.

1	2	0	5	0	–4	0
		2	2	7	7	3
	2	2	7	7	3	3
		2	4	11	18	
	2	4	11	18	21	
		2	6	17		
	2	6	17	35		
		2	8			
	2	8	25			
		2				
	2	10				

Therefore $f(x) = 2x^5 + 5x^3 - 4x$

$= 2(x-1)^5 + 10(x-1)^4 + 25(x-1)^3 + 35(x-1)^2 + 21(x-1) + 3.$

Also, writing $x + 1$ for x, we get

$f(x+1) = 2x^5 + 10x^4 + 25x^3 + 35x^2 + 21x + 3.$

Example 4 : Find the remainder, when $3x^4 - 4x^3 + 2x^2 - 9x + 1$ is divided by $2x + 1$.

Solution : Let $f(x) = 3x^4 - 4x^3 + 2x^2 - 9x + 1$. When $f(x)$ is divided by $2x + 1$, the remainder will be $f\left(-\frac{1}{2}\right)$. Now,

$$f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^4 - 4\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) + 1$$

$$= \frac{3}{16} + \frac{1}{2} + \frac{1}{2} + \frac{9}{2} + 1 = \frac{107}{16}.$$

Example 5 : Find the value of m, if $4x^3 - 3x^2 + 2x + m$ is divisible by $x + 2$.

Solution : Let $f(x) = 4x^3 - 3x^2 + 2x + m$. If $f(x)$ be divided by $x + 2$, then the remainder is $f(-2) = 4.(-2)^3 - 3(-2)^2 + 2(-2) + m$

$= -32 - 12 - 4 + m = m - 48.$

But $f(x)$ is divisible by $x + 2$, so the remainder must be zero.

Therefore $f(-2) = 0$ or $m - 48 = 0$ or $m = 48$.

3.5 Model Questions (I)

1. If $f(x) = 2x^5 - 5x^4 + 4x^2 - 8$, then find (i) $f(0)$ (ii) $f(1)$ (iii) $f(-2)$.
2. Find the remainder when
   (i) $x^3 + 5x^2 + 1$ is divided by $x + 3$.
   (ii) $x^4 + 5x^2 - 3x + 2$ is divided by $x + 2$.
   (iii) $x^3 + 2x^2 + 3x - 4$ is divided by $2x - 1$.
3. Show that $2x^4 - 7x^3 + 4x + 15$ is exactly divisible by $x - 3$.

4. Show that $x + 3$ is a factor of $x^3 + 5x^2 - 18$.

5. If $x^4 + 2x^3 - 13x^2 + ax + 24$ is divisible by $x + 4$, find a.

6. Find quotient and remainder, using synthetic division, when

   (i) $2x^4 - x^2 - 5x + 6$ is divided by $x - 2$.

   (ii) $x^3 + 5x^2 + 1$ is divided by $x + 3$.

   (iii) $2x^4 - 6x^3 - 9x + 21$ is divided by $2x - 3$.

7. Show that that $2x^4 - 7x^3 + 8ax^2 - 3bx + 17$ is divisible by $x - 2$, when $32a - 6b - 7 = 0$.

8. Express

   (i) $4x^3 - 3x^2 + 6x - 5$ as a polynomial in $(x + 1)$.

   (ii) $2x^4 - 4x^2 + 1$ as a polynomial in $(x - 3)$.

9. If $f(x) = 3x^4 - x^3 + 5x^2 - 4x - 9$, find $f(x + 2)$.

10. (a) Find quotient and remainder, when $5x^4 - 9x^3 + 6x^2 + 16x - 13$ is divided by $x^2 - 3x + 2$.

    (b) Use synthetic division to find remainder, when $x^4 - 1$ is divided by $x + 2$.

11. Find the polynomials whose zeros are (i) 1, 2, 3 (ii) 1, –1, 2, – 2.

12. Find the condition that $x^3 + 3px + q$ may have a factor of the form $(x - a)^2$.

3.5.1 Answers

(1) (i) –8 (ii) –7 (iii) –136. 2. (i) 19 (ii) 44 (iii) $-\frac{39}{8}$. 5. $a = -14$.

6. (i) $Q = 2x^3 + 4x^2 + 7x + 9$, $R = 24$. (ii) $Q = x^2 + 2x - 6$, $R = 19$.

   (iii) $Q = x^3 - \frac{3}{2}x^2 - \frac{9}{4}x - \frac{63}{8}$, $R = -\frac{21}{8}$.

8. (i) $4(x+1)^3 - 15(x+1)^2 + 24(x+1) - 18$.

   (ii) $2(x-3)^4 + 24(x-3)^3 + 104(x-3)^2 + 192(x-3) + 127$.

9. $3x^4 + 23x^3 + 71x^2 + 100x + 43$.

10. (a) $Q = 5x^2 + 6x + 14$, $R = 46x - 41$. (b) $R = 15$.

11. (i) $x^3 - 6x^2 + 11x - 6$. (ii) $x^4 - 5x^2 + 4$.

12. $4p^3 + q^2 = 0$.

3.6 General Properties of Equations

3.6.1 Equation and Identity

Let $f(x)$ be a polynomial of degree $n > 1$. If $f(x) = 0$ is satisfied for certain specific value or values of x, then $f(x)$ is called an algebraic equation of degree n.

The values of x for which $f(x) = 0$ is satisfied are called the roots of the equation $f(x) = 0$.

On the other hand, if $f(x) = 0$ is satisfied for any value of x, then it is called an identity.

Example : Show that (i) $x^2 - 3x + 2 = 0$ is an equation

(ii) $5(x - 2) + (x - 8) - 6(x - 3) = 0$ is an identity.

3.6.2 Fundamental Theorem of Classical Algebra

Statement : Every algebraic equation has a root, real or imaginary.

3.6.3 Consequences of Fundamental Theorem.

Theorem 1 : Every algebraic equation of degree n has n and exactly n roots.

Proof : Let $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$, $a_0 \neq 0$ be an equation of degree n.

By fundamental theorem, $f(x) = 0$ has a root, say α_1. Therefore $f(\alpha_1) = 0$.

Therefore, by remainder theorem, $(x - \alpha_1)$ is a factor of $f(x)$.

Therefore, if $f(x)$ be divided by $x - \alpha_1$, the quotient $Q_1(x)$ will be a polynomial of degree $(n - 1)$.

Therefore we write $f(x) \equiv (x - \alpha_1). Q_1(x) = 0$.

Now, the equation $Q_1(x) = 0$ must have a root, say α_2. Then $Q_1(x) \equiv (x - \alpha_2) Q_2(x) = 0$ and we write $f(x) \equiv (x - \alpha_1)(x - \alpha_2). Q_2(x) = 0$, where $Q_2(x)$ is a polynomial of degree $(n - 2)$.

Proceeding in this way and noticing that after dividing $f(x)$ by each linear factor, the degree of the quotient polynomial is diminished by one. We will have n linear factors of the form $(x - \alpha_1)$, $(x - \alpha_2)$,....., $(x - \alpha_n)$ and then $f(x)$ can be put in the form

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n).\ Q_n(x),$$

where the degree of $Q_n(x)$ is $n - n = 0$. Therefore $Q_n(x)$ must be a constant. Since the above equation is an identity, equating coefficient of x^n on both sides, we get $Q_n(x) = a_0$.

So the equation can be put in the form $a_0 (x - \alpha_1)(x - \alpha_2)\ldots\ldots(x - \alpha_n) = 0$.

Since this is true for any one of the n values of x, viz. $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$, the equation $f(x) = 0$ has n roots.

Now, we shall show that the equation $f(x) = 0$ cannot have more than n roots.

If possible, let $f(x) = 0$ have a root α which is distinct from each of the above roots α_i $(i = 1, 2, \ldots\ldots, n)$. Hence $x = \alpha$ will satisfy the equation $f(x) = 0$.

Therefore $\quad f(\alpha) = 0$

i.e., $\ a_0 (\alpha - \alpha_1)(\alpha - \alpha_2)\ldots\ldots(\alpha - \alpha_n) = 0$.

But, by assumption, $a_0 \neq 0$ and α being distinct from $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $\alpha - \alpha_1 \neq 0$,

$\alpha - \alpha_2 \neq 0, \ldots\ldots, \alpha - \alpha_n \neq 0$ and hence $f(\alpha)$ cannot be zero. So that we arrive at a contradiction, *i.e.*, α cannot be a root of $f(x) = 0$.

Therefore we conclude that an algebraic equation of degree n has exactly n roots.

Cor. Two or more roots of the equation $f(x) = 0$ may be equal. If r number of roots be equal, we say that $f(x) = 0$ has a multiple root of multiplicity r.

Note . If $f(x)$ is of degree n and $f(x) = 0$ is satisfied by more than n values of x, then $f(x) = 0$ is an identity.

Theorem 2 : In an algebraic equation with rational coefficients, irrational roots occur in conjugate pairs.

Proof : Let $f(x) = 0$ be an algebraic equation with rational coefficients of degree n (≥ 2). Let $\alpha + \sqrt{\beta}$, where α is rational and $\sqrt{\beta} \neq 0$ is irrational be a root of $f(x) = 0$. We divide the polynomial by $\{x - (\alpha + \sqrt{\beta})\}\{x - (\alpha - \sqrt{\beta})\}$. The quotient $Q(x)$ will be a polynomial of degree $(n - 2)$ and the remainder will be at most linear.

So we write $f(x)=\{x-(\alpha+\sqrt{\beta})\}\{x-(\alpha-\sqrt{\beta})\}Q(x)+ax+b$, ...(1)

where a and b of the remainder $ax + b$ are rational.

We have assumed that $(\alpha+\sqrt{\beta})$ is a root of $f(x) = 0$, therefore $f(\alpha+\sqrt{\beta})=0$.

From equation (1), we have

$$a(\alpha+\sqrt{\beta})+b=0$$

or, $(a\alpha+b)+a\sqrt{\beta}=0$.

Equating rational and irrational parts to zero, we have

$$a\alpha + b = 0 \qquad ...(2)$$

and $$a\sqrt{\beta}=0. \qquad ...(3)$$

But $\sqrt{\beta}$ cannot be 0. Therefore, from equation (3), we get $a = 0$ and from equation (2), we get $b = 0$. Hence equation (1) becomes

$$f(x)=\{x-(\alpha+\sqrt{\beta})\}\{x-(\alpha-\sqrt{\beta})\}Q(x) \qquad ...(4)$$

From equation (4), it is evident that when $x=\alpha-\sqrt{\beta}, f(\alpha-\sqrt{\beta})=0$. So $\alpha-\sqrt{\beta}$ is another root of $f(x) = 0$, *i.e.*, if $\alpha+\sqrt{\beta}$ is a root of $f(x) = 0$, then $\alpha-\sqrt{\beta}$ is also a root of it.

Theorem 3 : An algebraic equation with real coefficients, imaginary roots occur in conjugate pairs.

Proof : Let $f(x) = 0$ be an algebraic equation with real coefficients of degree $n\ (\geq 2)$. Let $\alpha + i\beta$, where α and $\beta \neq 0$ are real, be a root of $f(x) = 0$. We divide the polynomial $f(x)$ by $\{x - (\alpha + i\beta)\}\ \{x - (\alpha - i\beta)\}$. The quotient $Q(x)$ will be a polynomial of degree $(n-2)$ and the remainder will be at most linear.

So we write $f(x) = \{x - (\alpha + i\beta)\}\ \{x - (\alpha - i\beta)\}\ Q(x) + ax + b$, ...(1)

where a and b of the remainder $ax + b$ are real.

We have assumed that $(\alpha + i\beta)$ is a root of $f(x) = 0$. Therefore $f(\alpha + i\beta) = 0$.

From equation (1), we have

$$a(\alpha + i\beta) + b = 0$$

or. $(a\alpha + b) + ia\beta = 0$.

Equating real and imaginary parts to zero, we have

$$a\alpha + b = 0 \qquad \dots (2)$$

and $$a\beta = 0 \qquad \dots (3)$$

But $\beta \neq 0$. Therefore, from equation (3), we get $a = 0$ and from equation (6), we get $b = 0$. Hence equation (1) becomes

$$f(x) = \{x - (\alpha + i\beta)\}\{x - (\alpha - i\beta)\}Q(x). \qquad \dots (4)$$

From equation (4), it is evident that when $x = \alpha - i\beta$, $f(\alpha - i\beta) = 0$. So, $\alpha - i\beta$ is another root of $f(x) = 0$, *i.e.*, if $\alpha + i\beta$ is a root of $f(x) = 0$, then $\alpha - i\beta$ is also a root of it.

Example 1 : If one root of the equation $x^4 - 3x^3 - 5x^2 + 9x - 2 = 0$ be $2 - \sqrt{3}$, find the other roots. [Ans. $2 + \sqrt{3}$, 1, –2]

Example 2 : If one root of the equation $x^4 - 2x^3 + 6x^2 + 22x + 13 = 0$ be $2 + 3i$, find the other roots. [Ans. $2 - 3i$, –1, –1]

3.6.4 Some Important Properties of Algebraic Equations

Property 1 : Let $f(x)$ be a polynomial with real coefficients. If for two numbers α and β $(\alpha < \beta)$, $f(\alpha)$ and $f(\beta)$ are of opposite signs, then there exists at least one real number γ, $\alpha < \gamma < \beta$, such that $f(\gamma) = 0$.

Since a polynomial $f(x)$ is a continuous function of x, therefore it assumes all values between $f(\alpha)$ and $f(\beta)$ as x changes its value from α to β. Since $f(\alpha)$ and $f(\beta)$ are of opposite signs, for at least one value of γ, $\alpha < \gamma < \beta$, $f(\gamma) = 0$, *i.e.*, γ is a root of $f(x) = 0$.

Property 2 : If $f(\alpha)$ and $f(\beta)$ are of opposite signs, then the equation $f(x) = 0$ has an odd number of real roots between α and β.

Property 3 : If $f(\alpha)$ and $f(\beta)$ are of the same sign, then equation $f(x) = 0$ has no real root or an even number of real roots between α and β.

Property 4 : An equation of an odd degree must have at least one real root, opposite in sign to that of the last term (*i.e.*, constant term), the leading term being positive.

Property 5 : An equation of an even degree, whose last term (*i.e.*, the constant term) is negative, has at least two real roots of opposite signs.

3.6.5 Multiple Roots

Let α_i $(i = 1, 2,......, n)$ be n roots of the equation $f(x) = 0$ of degree n, of which the first r roots $(r < n)$ are equal to α_1. Then we can write

$$f(x) = (x - \alpha_1)^r \phi(x),$$

where $\phi(\alpha_1) \neq 0$ and we say that α_1 is a root of $f(x) = 0$ of multiplicity r.

Theorem : If α be a root of the equation $f(x) = 0$ of multiplicity r, then α is a root of the equation $f'(x) = 0$ of multiplicity $r - 1$ where $f'(x)$ is the first derived function of $f(x)$.

3.6.6. Rolle's Theorem

Between any two consecutive roots of the equation $f(x) = 0$, there is at least one real root or an odd number of real roots of the equation $f'(x) = 0$.

3.7 Descartes' Rule of Signs

This gives us a rule for the determination of the maximum number of real roots of any polynomial equation with real coefficients.

When two consecutive coefficients of a polynomial $f(x)$ (complete or incomplete) have same signs, we say these two coefficients present a continuation of signs, but if they have opposite signs, they present a variation or change of signs.

Let us consider a polynomial

$$5x^{15} + 2x^{14} + 3x^{12} + x^{10} - 8x^6 - 6x^5 + 7x^4 - 3x^3 + 4x^2 - 10x - 12.$$

The signs of the coefficients are

$$+ + + + - - + - + - -.$$

We see that the number of continuations is 5 and the number of variations is 5.

Statement : A polynomial equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.

Note. If the number of positive roots of $f(x) = 0$ be less than the number of changes of sign in $f(x)$, it will be less than by an even number. Similarly, if the number of negative roots of $f(x) = 0$ be less than the number of changes of sign in $f(-x)$, it will be less than by an even number.

As for example, if the number of changes of sign in $f(x)$ be 1, then $f(x) = 0$ has exactly one positive root. Again, if the number of changes of sign in $f(x)$ be 3, then $f(x) = 0$ has 3 or 1 positive real root and so on. Similar argument may be applied for negative real roots.

Verification : At first, we shall show that if a polynomial $f(x)$ with real coefficients be multiplied by $(x - \alpha)$ where α is a positive real number, then the number of variations of signs of the product $(x - \alpha).\, f(x)$ will be greater than the number of variations of signs in $f(x)$ by an odd number.

Let the signs of the terms of the polynomial be + + + − − + − − − + − −.

The signs of the product $(x - \alpha).\, f(x)$ are shown below :

+	+	+	−	−	+	−	−	−	+	−	−	
+	−											
+	+	+	−	−	+	−	−	−	+	−	−	
	−	−	−	+	+	−	+	+	+	−	+	+
+	±	±	−	∓	+	−	∓	∓	+	−	∓	+

The signs in the product reflect that

(i) each continuation of sign in $f(x)$ is replaced by an ambiguity (*i.e.* ± or ∓)

The sign of ambiguity indicates that the sign may be + or − or the corresponding term is zero.

(ii) The signs before and after an ambiguity or set of ambiguities are unlike.

(iii) One change of sign is introduced at the end of the product.

From (i) and (ii), it is clear that the product has at least as many changes of sign as $f(x)$, even in the most unfavourable case in which all the ambiguities are continuous. From (iii), we see that the product has one more change of sign than $f(x)$. We can also see that no changes of signs are lost for any term which may be missing from $f(x)$.

Hence the product has at least one more change of sign than $f(x)$.

Now let $f(x) = Q(x).(x - \alpha)(x - \beta)$...... where α, β,...... are the positive roots of $f(x) = 0$. We know that if $Q(x)$ be multiplied by $x - \alpha, x - \beta$,...... in succession, at least one change of sign will be introduced by each multiplication. Hence $f(x)$ has at least as many changes of sign as $f(x) = 0$ has positive roots.

Corollary : Let n be the degree of $f(x)$ and let

p = the number of changes of sign in $f(x)$,

q = the number of changes of sign in $f(-x)$,

r = the number of positive roots of $f(x) = 0$,

s = the number of negative roots of $f(x) = 0$.

Then (i) if $p + q < n$, the equation $f(x) = 0$ has at least $n - (p + q)$ imaginary roots;

(ii) if all the roots of $f(x) = 0$ are real, then $r = p$ and $s = q$.

3.7.1 A Few Important Results from Descartes' Rule of Signs.

From Descartes' Rule of Signs, we have

(i) If the signs of the terms of an equation be all positive, then it cannot have a positive root.

(ii) If the signs of the terms of a complete equation be alternately positive or negative, then it cannot have a negative root.

(iii) If an equation, with all positive signs of its coefficients, involves only even powers of x, it cannot have any real root.

(iv) If an equation, with all positive signs of its coefficients, involves only odd powers of x, then it has the root zero and no other real root.

3.8 Worked out Examples (II)

Example 1: Find the equation whose roots are 1, –2, 3, – 4.

Solution : The required equation is $(x - 1)\{x - (-2)\}(x - 3)\{x - (-4)\} = 0$

or, $(x-1)(x+2)(x-3)(x+4) = 0$, or, $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$.

Example 2: Solve the equation $x^4 - 2x^3 + 6x^2 + 22x + 13 = 0$, if 2 + 3$i$ is a root.

Solution: One root of the equation being 2 + 3i, another root of it must be 2 – 3i (as all the coefficients of the equation are real).

Now $(x - 2 - 3i)(x - 2 + 3i) = (x - 2)^2 + 9$, *i.e.*, $x^2 - 4x + 13$ will be a factor of L.H.S. of the equation. Dividing it by $x^2 - 4x + 13$, we get the quotient $x^2 + 2x + 1$. So two other roots of the equation will be given by $x^2 + 2x + 1 = 0$ giving $x = -1, -1$. Hence the roots are –1, –1, $2 \pm 3i$.

Example 3: Find an equation of degree four with rational coefficients, if one root be $\sqrt{3} + i\sqrt{2}$.

Solution: Since surd roots as well as the imaginary roots occur in conjugate pairs, the roots of the equation will be $\sqrt{3} + i\sqrt{2},\ \sqrt{3} - i\sqrt{2},\ -\sqrt{3} + i\sqrt{2},\ -\sqrt{3} - i\sqrt{2}$.

So the required equation is

$$(x - \sqrt{3} - i\sqrt{2})(x - \sqrt{3} + i\sqrt{2})(x + \sqrt{3} - i\sqrt{2})(x + \sqrt{3} + i\sqrt{2}) = 0$$

or, $x^4 - 2x^2 + 25 = 0$.

Example 4: Apply Descartes' Rule of Signs regarding the number of real and imaginary roots of the equation $3x^5 - 4x^2 + 8 = 0$.

Solution: We have $f(x) = 3x^5 - 4x^2 + 8$ and therefore $f(-x) = -3x^5 - 4x^2 + 8$.

Number of changes of sign in $f(x)$ is 2 and number of changes of sign in $f(-x)$ is 1.

So the number of positive real roots ≤ 2 and the number of negative real root is exactly 1.

Since the degree of $f(x)$ is 5, it has 5 roots in all.

Therefore possible cases regarding number of roots of $f(x) = 0$ are as under :

Number of positive real roots	Number of negative real roots	Number of imaginary roots
2	1	2
0	1	4

Conclusion : The equation $f(x) = 0$ has at least two imaginary roots.

Example 5: Solve the equation $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0$, which has equal roots.

Solution: We have $f(x) = x^4 - 11x^3 + 44x^2 - 76x + 48$

$$\therefore \quad f'(x) = 4x^3 - 33x^2 + 88x - 76.$$

We find the highest common factor of $f(x)$ and $f'(x)$ which is $x - 2$. Therefore $(x - 2)^2$ is a factor of $f(x)$. Now $f(x) = (x-2)^2(x^2 - 7x + 12) = (x-2)^2(x-3)(x-4)$.

So the roots are 2, 2, 3 and 4.

3.9 Summary and Keywords

Summary :

I. Synthetic division : A short cut process of division of a polynomial by a binomial.

II. Remainder Theorem : If $f(x)$ be divided by $x - a$, then the remainder will be $f(a)$. If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

III. Fundamental Theorem of Algebra : Every algebraic equation has a root, real or imaginary.

IV. Rolle's Theorem : If $x - a$ be the highest common factor of $f(x)$ and $f'(x)$, where $f'(x)$ is the first derivative of $f(x)$, then $(x - a)^2$ is a factor of $f(x)$ and therefore $x = a$ is a root of $f(x) = 0$ of multiplicity 2.

V. Descartes' Rule of Signs : An equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.

Keywords : Polynomial, Fundamental, Real, Imaginary, Remainder, Multiplicity, Continuation, Variation.

3.10 Model Questions (II)

1. Find the equation whose roots are 1, 2, 3 and 4.
2. Solve the equation $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.

3. Find the equation of fourth degree two of whose roots are $1 + i$ and $1+\sqrt{2}$.
4. Find the equation of fourth degree with rational coefficients one root of which is $\sqrt{2}+\sqrt{3}i$.
5. Apply Descartes' Rule of Signs to determine the nature of roots of the following equations :

    (i) $x^4+16x^2+7x-10=0$

    (ii) $x^n-1=0$

    (iii) $x^{10}-1=0$.
6. Apply Descartes' Rule of Signs to show that the equation $x^7-2x^4+3x^3-1=0$ has at least four imaginary roots.
7. Find the exact number of real roots of the equation $x^6-x^3+2x^2-3x-1=0$.
8. (a) If α,β,γ be the roots of the equation $x^3+x+1=0$, then prove that $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)=1$.

    (b) If the roots of the equation $x^{n+1}-1=0$ be $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n$, then show that $(1-\alpha_1)(1-\alpha_2)\ldots..(1-\alpha_n)=n+1$.
9. If $a>b>c>0$ are all real, then show that the roots of the equation $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{1}{x}$ are all real.
10. Show that the equation $x^3-2x-5=0$ has no negative real root.
11. Find the multiple root, if any, of the equation $x^3-3x^2-9x+27=0$.
12. Show that the equation $x^3-3x+2=0$ has a multiple root.
13. Use Descartes' Rule of Signs to show that the equation $x^7-3x^4+2x^3-1=0$ has at least four imaginary roots.
14. (a) If p, q, r are positive, show that the equation $f(x) = x^4 + px^2 + qx - r = 0$ has one positive, one negative and two imaginary roots.

    (b) Show that the equation $x^5-4x^4+5=0$ has at least two imaginary roots.

15. Solve the equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$, which has equal roots.

16. Show that the roots of the equation $\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + ... + \frac{L^2}{x-l} = x - m$, connot have any imaginary root where a, b, c,......, l, and m are real and distinct.

3.10.1 Answers :

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.
2. 1, 3, 5, 7.
3. $x^4 - 4x^3 + 5x^2 - 2x - 2 = 0$.
4. $x^4 + 2x^2 + 25 = 0$.
5. (i) One positive real root, one negative real root and two imaginary roots.
   (ii) One positive real root, $n - 1$ imaginary roots.
   (iii) One positive real root, one negative real root and 8 imaginary roots.
7. 2.
11. 3 of multiplicity 2.
12. $x = 1$ of multiplicity 2.
15. 1, 1, 1 and 3.

Unit-4 ❒ Relations connecting the Roots and Coefficients of an Equation

Structure

4.1 Objectives

On learning this unit, we will be able to do the following :

- We may use the relations between roots and coefficients in diffierent area of algebra.
- We would be able to find the values of the expressions of symmetrical forms of roots.
- By transformation, we can increase or decrease the roots of an equation by a constant.
- With the help of the transformation, we would be able to form an equation whose roots are the roots of a given equation with sign changed and many others.

4.2 Introduction

In Unit-3, we have learnt about the roots and coefficients of a polynomial equation. In the present unit, we shall discuss interesting relations between the roots and coefficients of an equation.With the help of these relations, we shall be able to find the values of several symmetric functions of the roots.

Next we shall learn different types of transformations which will help us to form equations having specified roots.

4.3 Relations connecting the Roots and Coefficients of an Equation

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be n roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0.$$

We have the identity

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n \equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)......(x-\alpha_n)$$

or, $$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_n}{a_0} = x^n - \left(\sum \alpha_1\right)x^{n-1} + \left(\sum \alpha_1\alpha_2\right)x^{n-2}$$

$$- ......+ (-1)^n \alpha_1\alpha_2......\alpha_n.$$

Equating coefficients of like powers of x from both sides, we get

$$\sum \alpha_1 = -\frac{a_1}{a_0},\ \sum \alpha_1\alpha_2 = \frac{a_2}{a_0},\ ......,\ \alpha_1\alpha_2...\alpha_n = (-1)^n\frac{a_n}{a_0}.$$

Corollary 1 : If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, then $\alpha+\beta+\gamma = -\frac{b}{a}$, $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$.

Corollary 2 : If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$ax^4 + bx^3 + cx^2 + dx + e = 0$, then $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$,

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a},$$

$$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta = -\frac{d}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}.$$

4.3.1 Symmetric functions of the roots

An expression involving all the roots of a polynomial equation is said to be symmetric if it remains unaltered when any two roots are interchanged.

For example, $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$ (also written as $\sum\alpha\beta\gamma$) is a symmetric function of the roots $\alpha, \beta, \gamma, \delta$ of a biquadratic equation.

4.4 Worked out Examples (I)

Example 1 : Solve the equation $x^3 - 3x^2 - 6x + 8 = 0$, if the roots are in A.P.

Solution : Let the roots be $\alpha - \beta$, α, $\alpha + \beta$.

Therefore sum of the roots = $\alpha - \beta + \alpha + \alpha + \beta = 3$

or, $3\alpha = 3$, giving $\alpha = 1$.

Sum of the roots taken two at a time is

$$(\alpha - \beta)\alpha + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = -6$$

Since $\alpha = 1$, $(1-\beta) + (1+\beta) + (1-\beta)(1+\beta) = -6$

or, $1-\beta+1+\beta+1-\beta^2 = -6$

or, $\beta^2 = 9$

Therefore $\beta = 3$ or -3.

Hence the roots are -2, 1, 4.

Example 2 : If one root of the equation $x^3 + px^2 + qx + r = 0$ be equal to the sum of other two roots, then prove that $p^3 + 8r = 4pq$.

Solution : Let the roots of the equation be α, β, γ.

Therefore, by the given condition, $\alpha = \beta + \gamma$ ...(1)

Again, $\alpha + \beta + \gamma = -p$, ...(2)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ...(3)

and $\alpha\beta\gamma = -r$. ...(4)

From the relations (1) and (2), $\alpha = -\frac{p}{2}$

From the relation (3), $\beta\gamma + \alpha(\beta + \gamma) = q$

or, $\beta\gamma + \alpha^2 = q$

Therefore $\beta\gamma = q - \alpha^2 = q - \frac{p^2}{4}$

From the relation (4), we have $\alpha\beta\gamma = -r$

or, $-\frac{p}{2}.\left(q - \frac{p^2}{4}\right) = -r$

Therefore $p^3 + 8r = 4pq$.

Example 3 : Solve the equation $x^4 + 3x^3 - 4x^2 - 9x + 9 = 0$, given that the product of two of the roots is equal to the product of the other two.

Solution : Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation.

Therefore $\alpha+\beta+\gamma+\delta=-3$ ...(1)

Again, $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=-4,$ ...(2)

$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=9$ ...(3)

and $\alpha\beta\gamma\delta=9.$ ...(4)

Also, by the given condition, $\alpha\beta=\gamma\delta.$ ...(5)

From the relation (3), we have

$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=9$$

As $\alpha\beta=\gamma\delta,$ $\alpha\beta(\alpha+\beta+\gamma+\delta)=9$

or, $\alpha\beta.(-3)=9$

Therefore $\alpha\beta=-3=\gamma\delta.$

From the relation (1), we have $\gamma+\delta=-3-(\alpha+\beta).$

Therefore, from the relation (2), we have

$$\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=-4$$

or, $-3-3+(\alpha+\beta)\{-3-(\alpha+\beta)\}=-4$

or, $(\alpha+\beta)^2+3(\alpha+\beta)+2=0$

Therefore $\alpha+\beta=-1 \text{ or } -2.$

Taking $\alpha+\beta=-2$ and $\alpha\beta=-3,$ we have

$$\alpha+\frac{-3}{\alpha}=-2$$

or, $\alpha^2+2\alpha-3=0$

or, $(\alpha+3)(\alpha-1)=0.$

Therefore $\alpha = 1, -3.$

When $\alpha=1,$ $\beta=-3$ and when $\alpha=-3,$ $\beta=1.$

Taking $\alpha+\beta=-1$ and $\alpha\beta=-3,$ we have

$$\alpha-\frac{3}{\alpha}=-1$$

or, $\alpha^2+\alpha-3=0$

or, $\alpha = \dfrac{-1 \pm \sqrt{1+12}}{2} = \dfrac{-1 \pm \sqrt{13}}{2}$

and corresponding $\beta = \dfrac{-1 \mp \sqrt{13}}{2}$.

Since $\gamma + \delta = -1,\ -2$ according as $\alpha + \beta = -2,\ -1$. Therefore the values of γ and δ will also be same as above. Hence the roots are $1, -3, \dfrac{-1 \pm \sqrt{13}}{2}$.

Example 4 : If α, β, γ are the roots of the equation $px^3 + qx^2 + 1 = 0$, find the value of $\sum \frac{1}{\alpha}$.

Solution : Since α, β, γ are the roots of the equation $px^3 + qx^2 + 1 = 0$, then $\sum \alpha = -\frac{q}{p}$, $\sum \alpha\beta = 0$ and $\alpha\beta\gamma = -\frac{1}{p}$.

Therefore $\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\sum \alpha\beta}{\alpha\beta\gamma} = \frac{0}{\alpha\beta\gamma} = 0$.

Example 5 : If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then find the value of $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$.

Solution : We have $\sum \alpha = 0, \sum \alpha\beta = p$ and $\alpha\beta\gamma = -q$. Then

$$(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma) = \left(\sum \alpha - 2\alpha\right)\left(\sum \alpha - 2\beta\right)\left(\sum \alpha - 2\gamma\right)$$

$$= (-2\alpha)(-2\beta)(-2\gamma) = -8\alpha\beta\gamma = 8q.$$

Example 6 : Find the sum of the squares of the reciprocals of the roots of the equation $x^3 - 2x + 1 = 0$.

Solution : Let α, β, γ be the roots of the equation $x^3 - 2x + 1 = 0$.

Therefore $\sum \alpha = 0, \sum \alpha\beta = -2$ and $\alpha\beta\gamma = -1$.

We are required to find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

Now, $$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2-2\left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)$$

$$=\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right)^2-2\left(\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}\right)$$

$$=\left(\frac{-2}{-1}\right)^2-2\left(\frac{0}{-1}\right)=4.$$

4.5 Model Questions (I)

1. Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$, if the sum of two roots is zero.
2. Solve the equation $x^3 - 6x^2 + 3x + 10 = 0$, if the roots are in A.P.
3. Solve the equation $2x^3 - x^2 - 22x - 24 = 0$, two of its roots being in the ratio 3 : 4.
4. If the roots of the equation $x^3 + ax^2 + bx + c = 0$ are in G.P., then show that $b^3 = a^3c$.
5. Solve the equation $2x^3 - 21x^2 + 42x - 16 = 0$, if the roots are in G.P.
6. Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$, whose roots are of the form $a, -a, b, -b, c$.
7. If the roots of the equation $x^3 + a_1x^2 + a_2x + a_3 = 0$ are in A.P. find the relation connecting a_1, a_2, a_3.
8. If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of $\sum\frac{1}{\alpha}$.
9. If α, β, γ are the roots of the equation $x^3 + 3x^2 + 3x - 2 = 0$, then find the value of

   (i) $\sum\alpha^2$. (ii) $\sum\alpha^3$. (iii) $\sum\alpha^2\beta$. (iv) $\sum\frac{1}{\alpha}$. (v) $\sum\frac{1}{\beta+\gamma}$.
10. If a, b, c be the roots of $x^3 + qx + r = 0$, then show that $a^3 + b^3 + c^3 = -3r$.
11. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, then find the value of $(\beta+\gamma-\alpha)^{-1}+(\gamma+\alpha-\beta)^{-1}+(\alpha+\beta-\gamma)^{-1}$.

12. If α, β, γ are the roots of the equation $x^3 - 2x^2 + 3x - 4 = 0$, then find the value of $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$.

13. If α, β, γ, δ are the roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$, then find the value of (i) $\sum\frac{\alpha}{\beta}$. (ii) $\sum\alpha^2\beta^2$.

14. If $\alpha,\beta,\gamma,\delta$ are the roots of the equation $x^4 - 4x + 3 = 0$, then show that the value of $\sum\alpha^4$ is (–12).

4.5.1 Answers

1. –2, 2, 3. 2. –1, 2, 5. 3. $4, -\frac{3}{2}, -2$. 5. $\frac{1}{2}, 2, 8$.

6. $\pm 1, \pm 2, 5$. 7. $9a_1a_2 - 2a_1^3 = 27a_3$. 8. $\frac{q}{r}$.

9. (i) 3. (ii) 6. (iii) –15. (iv) $\frac{3}{2}$. (v) $-\frac{12}{11}$. 11. $\frac{q}{2r}$. 12. $-\frac{3}{2}$.

13. (i) $\frac{pr-4s}{s}$. (ii) $q^2 - 2pr + 2s$.

4.6 Transformation of Equations

Sometimes we require to form a new equation so that the roots of this equation is related to the roots of a given equation in a definite way. The new equation so formed is called transformed equation and the method of forming this equation is known as transformation of equation.

4.6.1. How to do it : Let an equation $f(x) = 0$ be given. We are to form a new equation $g(y) = 0$ whose roots are related with the roots of the given equation by a certain relation $\phi(x,y) = 0$. Eliminating x between $f(x) = 0$ and $\phi(x,y) = 0$, we get the transformed equation $g(y) = 0$. The equation $\phi(x,y) = 0$ is known as equation of transformation.

Let us now discuss a few transformations of equation.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be n roots of the equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0 \quad \ldots(1)$$

I. Formation of equation whose roots are the roots of equation (1) with sign changed

We are to form an equation whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$. Here $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the values of x for $f(x) = 0$ and $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ are the values of y for the new equation $g(y) = 0$.

So the equation of transformation is $y = -x$ or $x = -y$. Eliminating x from equation (1), we get $f(-y) = 0$, which is the transformed equation.

II. Multiplication of the roots by a constant *m*

We are to form an equation whose roots are $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$. Here $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the values of x for $f(x) = 0$ and $m\alpha_1, m\alpha_2, \ldots, m\alpha_n$ are the values of y for the new equation $g(y) = 0$.

So the equation of transformation is $y = mx$ or $x = \frac{y}{m}$.

Eliminating x from equation (1), we get

$$a_o\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \ldots\ldots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0$$

or, $$a_0y^n + a_1my^{n-1} + a_2m^2y^{n-2} + \ldots\ldots + a_{n-1}m^{n-1}y + a_nm^n = 0,$$

which is the transformed equation.

Note. To transform a given equation to another equation whose roots are those of the original equation divided by a certain constant *m*, we just multiply the roots of the original equation by $\frac{1}{m}$. Here, equation of transformation will be $y = \frac{x}{m}$ or $x = my$ and the transformed equation is

$$a_0m^ny^n + a_1m^{n-1}y^{n-1} + a_2m^{n-2}y^{n-2} + \ldots\ldots + a_{n-1}my + a_n = 0.$$

III. Formation of equation whose roots are reciprocals of the roots of equation (1)

We shall form an equation whose roots will be $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \ldots\ldots, \frac{1}{\alpha_n}$.

Therefore equation of transformation is $y = \frac{1}{x}$ or $x = \frac{1}{y}$.

Eliminating x from equation (1), we get

$$a_0\frac{1}{y^n}+a_1\frac{1}{y^{n-1}}+a_2\frac{1}{y^{n-2}}+......+a_{n-1}\left(\frac{1}{y}\right)+a_n = 0$$

or, $$a_n y^n + a_{n-1}y^{n-1}+a_{n-2}y^{n-2}+......+a_2y^2+a_1y+a_0 = 0.$$

This is the equation having roots $\frac{1}{\alpha_1},\frac{1}{\alpha_2},\frac{1}{\alpha_3},......,\frac{1}{\alpha_n}$.

IV. Formation of equation whose roots are the roots of equation (1), each diminished by *h*

We shall form an equation whose roots will be $\alpha_1 - h$, $\alpha_2 - h$,......, $\alpha_n - h$ ($h > 0$).
Therefore equation of transformation is $y = x - h$ or $x = y + h$.
Eliminating x from equation (1), we get

$$a_0(y+h)^n + a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+......+a_{n-1}(y+h)+a_n = 0.$$

V. Formation of equation whose roots are the roots of equation (1), each increased by *h*

To form this equation, as before, the equation of transformation is $y = x + h$ or $x = y - h$.
Eliminating x from equation (1), we get

$$a_0(y-h)^n + a_1(y-h)^{n-1}+a_2(y-h)^{n-2}+......+a_{n-1}(y-h)+a_n = 0.$$

VI. To find an equation whose roots are the squares of the roots of equation (1)

We are to form an equation whose roots are α_1^2, α_2^2,......,α_n^2. The equation of transformation is $y = x^2$.

To get the transformed equation, we take the even power of x on one side and odd powers of x on another side of equation (1). Squaring both sides, we shall get an equation containing all even powers of x. The transformed equation will be obtained by putting y for x^2.

VII. To form an equation whose roots are the cubes of the roots of equation (1)

We are to form an equation whose roots are α_1^3, α_2^3,......, α_n^3. As before, the equation of transformation is $y = x^3$.

The transformed equation will be obtained by putting x for $y^{\frac{1}{3}}$ in equation (1) and then simplifying after suitable transposing and cubing both sides.

4.6.2 Removal of a term of an equation

Let the roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

be diminished by h. Then the transformed equation is

$$a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + ... + a_{n-1}(y+h) + a_n = 0.$$

On simplifying, we get

$$a_0y^n + (na_0h + a_1)y^{n-1} + \left(\frac{1}{2}n(n-1)a_0h^2 + (n-1)a_1h + a_2\right)y^{n-2} + ... + a_n = 0.$$

The second term will be removed, if

$$na_0h + a_1 = 0, \ i.e., h = -\frac{a_1}{na_0}, \ a_0, a_1 > 0.$$

Similarly, the third term will be removed if h be chosen such that

$$\frac{1}{2}n(n-1)a_0h^2 + (n-1)a_1h + a_2 = 0.$$

4.6.3. Equation of the squared differences of the roots of a cubic equation

Let α, β, γ be the roots of the cubic equation

$$x^3 + qx + r = 0. \qquad ...(1)$$

We are to form an equation whose roots are $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ and $(\alpha-\beta)^2$.

Since α, β, γ are the roots of the cubic (1), we have $\alpha + \beta + \gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = q$ and $\alpha\beta\gamma = -r$.

Let $y = (\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma$

$$= (-\alpha)^2 - \frac{4\alpha\beta\gamma}{\alpha} = \alpha^2 + 4\frac{r}{\alpha}$$

or, $$\alpha^3 - \alpha y + 4r = 0. \qquad ...(2)$$

Since α is a root of equation (1),

therefore $$\alpha^3 + q\alpha + r = 0. \qquad ...(3)$$

Subtracting (2) from (3), we get

$$(q+y)\alpha - 3r = 0.$$

Therefore $$\alpha = \frac{3r}{y+q}.$$

Therefore equation of transformation is $x = \dfrac{3r}{y+q}$.

Putting this value of x in equation (1),

$$\left(\frac{3r}{y+q}\right)^3 + q.\frac{3r}{y+q} + r = 0$$

or, $$(y+q)^3 + 3q(y+q)^2 + 27r^2 = 0$$

or, $$y^3 + 6qy^2 + 9q^2y + (4q^3 + 27r^2) = 0.$$

4.7 Worked out Examples (II)

Example 1 : Obtain the equation whose roots are twice the roots of the equation $x^3 + 3x^2 + 4x + 5 = 0$.

Solution : Let α, β, γ be the roots of the given equation.

We are to form an equation whose roots are 2α, 2β, 2γ.

Therefore equation of transformation is $y = 2x$ or $x = \frac{1}{2}y$.

Putting this value of x in the given equation, the required equation is

$$\left(\frac{y}{2}\right)^3 + 3.\left(\frac{y}{2}\right)^2 + 4.\left(\frac{y}{2}\right) + 5 = 0$$

or, $$y^3 + 6y^2 + 16y + 40 = 0.$$

Example 2 : Find the equation whose roots are the reciprocals of the roots of the equation $2x^3 + 3x^2 - 8x + 5 = 0$.

Solution : Let the roots of the given equation be α, β, γ.

We are to form an equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Therefore equation of transformation is $y = \frac{1}{x}$ or $x = \frac{1}{y}$.

So, the required transformed equation is $2.\left(\frac{1}{y}\right)^3 + 3.\left(\frac{1}{y}\right)^2 - 8.\left(\frac{1}{y}\right) + 5 = 0$

or, $$5y^3 - 8y^2 + 3y + 2 = 0.$$

Example 3 : Diminish the roots of the equation $3x^3 - 5x^2 + 3x - 2 = 0$ by 3.

Solution : Equation of transformation is $y = x - 3$.
Using method of synthetic division

3	3	–5	3	–2
		9	12	45
	3	4	15	43
		9	39	
	3	13	54	
		9		
	3	22		

Therefore the transformed equation is $3y^3 + 22y^2 + 54y + 43 = 0$.

Alternate method : Putting $x = y + 3$ in the given equation, we get the required equation as

$$3(y+3)^3 - 5(y+3)^2 + 3(y+3) - 2 = 0.$$

On simplification, $$3y^3 + 22y^2 + 54y + 43 = 0.$$

Example 4 : Increase the roots of the equation $x^3 - 5x^2 + 6x + 41 = 0$ by 2.
Solution : Equation of transformation is $y = x + 2$.
Using method of synthetic division

–2	1	–5	6	41
		–2	14	–40
	1	–7	20	1
		–2	18	
	1	–9	38	
		–2		
	1	–11		

Therefore the transformed equation is $y^3 - 11y^2 + 38y + 1 = 0$.

Example 5 : Find the equation whose roots are the squares of the roots of the equation $x^4 + 2x^3 - 3x^2 + x + 1 = 0$.

Solution : If $\alpha, \beta, \gamma, \delta$ be the roots of the given equation, then we are to form an equation whose roots are α^2, β^2, γ^2 and δ^2.

Therefore equation of transformation is $y = x^2$.

The given equation can be written as, by transposing even and odd powers of x,

$$x^4 - 3x^2 + 1 = -(2x^3 + x).$$

Squaring both sides, $(x^4 - 3x^2 + 1)^2 = (2x^3 + x)^2 = 4x^6 + 4x^4 + x^2$.

Putting $x^2 = y$, we get the required equation as

$$(y^2 - 3y + 1)^2 = 4y^3 + 4y^2 + y$$

or, $y^4 + 9y^2 + 1 - 6y^3 - 6y + 2y^2 = 4y^3 + 4y^2 + y$

or, $y^4 - 10y^3 + 7y^2 - 7y + 1 = 0$.

Example 6 : Find the equation whose roots are the cubes of the roots of the equation $x^3 - 3x - 2 = 0$.

Solution : Here the equation of transformation is $y = x^3$.

The given equation can be put as $x^3 - 2 = 3x$.

Cubing both sides, $(x^3 - 2)^3 = (3x)^3 = 27x^3$.

Putting $x^3 = y$, we get the required equation as

$$(y - 2)^3 = 27y$$

or, $y^3 - 3y^2.2 + 3.y.2^2 - 2^3 = 27y$

or, $y^3 - 6y^2 - 15y - 8 = 0$.

Example 7 : Remove the second term of the equation $2x^3 + 6x^2 - x - 3 = 0$.

Solution : Let the second term be removed if the roots are increased by h. Then equation of transformation is $y = x + h$. Therefore $x = y - h$.

Therefore the transformed equation is $2(y-h)^3 + 6(y-h)^2 - (y-h) - 3 = 0$

or, $2(y^3 - 3y^2h + 3yh^2 - h^3) + 6(y^2 - 2yh + h^2) - y + h - 3 = 0$

or, $2y^3 - 6(h-1)y^2 + (6h^2 - 12h - 1)y - (2h^3 - 6h^2 - h + 3) = 0$.

The second term will be removed if $h - 1 = 0$, *i.e.*, if $h = 1$.

Therefore the transformed equation is $2y^3 - 7y + 2 = 0$.

Example 8 : Form the equation of squared differences of $x^3+6x^2+9x+4=0$.

Solution : First we remove the second term of the given equation.

Let the second term be removed if the roots are diminished by h.

Then $h=-\frac{6}{1.3}=-2$.

Therefore the equation of transformation is $y = x + 2$.

–2	1	6	9	4
		–2	–8	–2
	1	4	1	2
		–2	–4	
	1	2	–3	
		–2		
	1	0		

Therefore the transformed equation is $y^3-3y+2=0$. ...(1)

Since the difference of any two roots remains unaltered by this transformation, the equation of squared differences of the given equation will be the same as that of the equation $y^3-3y+2=0$. ...(2)

Let α, β, γ be the roots of this equation.

Then $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\gamma\alpha=-3$, $\alpha\beta\gamma=-2$.

Now we shall form an equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2 \text{ and } (\alpha-\beta)^2$.

Let $$z=(\beta-\gamma)^2=(\beta+\gamma)^2-4\beta\gamma$$

$$=(-\alpha)^2-4\frac{\alpha\beta\gamma}{\alpha}=\alpha^2+\frac{8}{\alpha}$$

or, $$\alpha^3-\alpha z+8=0 \quad ...(3)$$

Since α is a root of equation (2),

therefore $$\alpha^3-3\alpha+2=0 \quad ...(4)$$

Subtracting (4) from (3), we get

$$(3-z)\alpha+6=0$$

or, $$\alpha = \frac{6}{z-3}.$$

Putting the value of α in equation (4), we get the required equation as

$$\left(\frac{6}{z-3}\right)^3 - 3\left(\frac{6}{z-3}\right) + 2 = 0$$

or, $$2(z-3)^3 - 18(z-3)^2 + 216 = 0$$

or, $$z^3 - 18z^2 + 81z = 0.$$

Example 9 : If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}, \frac{\gamma}{\alpha+\beta}$.

Solution : Since α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, therefore $\alpha+\beta+\gamma = -p$, $\alpha\beta+\beta\gamma+\gamma\alpha = q$, $\alpha\beta\gamma = -r$.

Let y be a root of the required equation. Then

$$\frac{\alpha}{\beta+\gamma} = \frac{\alpha}{\alpha+\beta+\gamma-\alpha} = \frac{\alpha}{-p-\alpha}$$

(expressing in terms of a root of the given equation)

Therefore the equation of transformation is

$$y = \frac{x}{-p-x}$$

or, $$-py - xy = x$$

or, $$x = -\frac{py}{y+1}.$$

Putting this value of x in the given equation, we get the required equation as

$$\left(-\frac{py}{y+1}\right)^3 + p\left(-\frac{py}{y+1}\right)^2 + q\left(-\frac{py}{y+1}\right) + r = 0$$

or, $$r(y+1)^3 - pqy(y+1)^2 + p^3y^2(y+1) - p^3y^3 = 0$$

or, $$r(y^3 + 3y^2 + 3y + 1) - pqy(y^2 + 2y + 1) + p^3y^3 + p^3y^2 - p^3y^3 = 0$$

or, $$(r - pq)y^3 + (3r - 2pq + p^3)y^2 + (3r - pq)y + r = 0.$$

4.8 Summary and Keywords

Summary :

I. Relations between roots and coefficients of an equation.

II. Transformation of Equations : Solving of an equation is sometimes simplified by transforming it into another equation whose roots have some assigned relation to those of the proposed equation. Such transformations are specially useful in the solution of cubic and biquadratic equations.

Keywords : Equation, Roots, Coefficients, Relation, Transformations.

4.9 Model Questions (II)

1. Find the equation whose roots are

   (i) twice the roots of $x^3+3x+1=0$.

   (ii) $\frac{1}{3}$rd of the roots of $x^3-2x^2+x-5=0$.

2. Find the equation whose roots are roots of the following equations with their signs changed :

   (i) $5x^4-3x^3+9x-18=0$

   (ii) $3x^5-9x^4+2x^3+5x^2-6x+1=0$.

3. Remove the fractional coefficients of the equation $x^4-\frac{5}{4}x^3-\frac{8}{27}x+\frac{7}{81}=0$.

   [Hint. Multiply the roots by 3.]

4. Increase the roots of the equation $4x^4+32x^3+83x^2+76x+21=0$ by 2.

5. Diminish the roots of the equation $x^4-3x^3+2x^2+7x-5=0$ by 2.

6. Remove the second term of the equation $x^3+6x^2+9x+4=0$.

7. Remove the third term of the equation $x^4-4x^3-18x^2-3x+2=0$.

8. Form the equation whose roots are the reciprocals of the roots of the equation

$$x^5-4x^4-5x^3+8x^2-8x+15=0.$$

9. Find the equation whose roots are the squares of the roots of $x^3-ax^2+bx-1=0$.

10. Find the equation whose roots are the cubes of the roots of $x^3-2x^2-3x+1=0$.

11. If α, β, γ are the roots of $x^3+3x+1=0$, find the equation whose roots are

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha},\ \frac{\beta}{\gamma}+\frac{\gamma}{\beta} \text{ and } \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}.$$

12. If α, β, γ are the roots of $x^3+qx+r=0$, find the equation whose roots are $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$ and $\frac{1-\gamma}{1+\gamma}$. Hence find the values of $\sum\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\alpha}{1+\alpha}\cdot\frac{1-\beta}{1+\beta}\cdot\frac{1-\gamma}{1+\gamma}$.

13. If α, β, γ are the roots of $2x^3+3x^2-x-1=0$, form the equation whose roots are $\frac{\alpha}{\beta+\gamma}, \frac{\beta}{\gamma+\alpha}$ and $\frac{\gamma}{\alpha+\beta}$.

14. If α, β, γ are the roots of $x^3+qx+r=0$, find the equation whose roots are $\frac{1}{\beta+\gamma-\alpha}, \frac{1}{\gamma+\alpha-\beta}$ and $\frac{1}{\alpha+\beta-\gamma}$ and hence find the value of $\sum\frac{1}{\beta+\gamma-\alpha}$.

15. If α, β, γ are the roots of $x^3+qx+r=0$, form the equation whose roots are

(i) $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}$ and $\frac{\gamma\alpha}{\beta}$.

(ii) $\frac{\beta^2+\gamma^2}{\alpha^2}, \frac{\gamma^2+\alpha^2}{\beta^2}$ and $\frac{\alpha^2+\beta^2}{\gamma^2}$.

(iii) $\beta\gamma+\frac{1}{\alpha}, \gamma\alpha+\frac{1}{\beta}, \alpha\beta+\frac{1}{\gamma}$.

16. If α, β, γ are the roots of $x^3-7x+6=0$, find the equation whose roots are $(\alpha-\beta)^2, (\beta-\gamma)^2$ and $(\gamma-\alpha)^2$.

17. If α, β, γ are the roots of $x^3+px^2+qx+r=0\ (r\neq 0)$, form the equation whose roots are (i) $\alpha^3, \beta^3, \gamma^3$. (ii) $\alpha-\frac{1}{\beta\gamma}, \beta-\frac{1}{\gamma\alpha}, \gamma-\frac{1}{\alpha\beta}$.

18. If α, β, γ are the roots of $x^3-9x+9=0$, then show that $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=\pm 27$.

19. If α, β, γ are the roots of $x^3+2x^2+3x+1=0$, find the equation whose roots are $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$.

20. If α, β, γ are the roots of $x^3-3x^2+8x-5=0$, form the equation whose roots are $2\alpha+3$, $2\beta+3$, $2\gamma+3$.

    [Hint. First multiply the roots by 2 and then increase the roots by 3.]

21. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4-3x^3+2x^2-7x+5=0$, find the equation whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$, $\delta+1$.

22. If α, β, γ are the roots of $2x^3+3x^2-x-1=0$, find the equation whose roots are $2\alpha-3$, $2\beta-3$, $2\gamma-3$.

23. Transform the equation $x^3-6x^2+5x+12=0$ into an equation lacking the second term.

24. Transform the equation $72x^3-54x^2+45x-7=0$ into an equation whose coefficient of the leading term is unity and other coefficients are the least possible integers.

4.9.1 Answers

1. (i) $y^3+12y+8=0$. (ii) $27y^3-18y^2+3y-5=0$.

2. (i) $5y^4+3y^3-9y-18=0$. (ii) $3y^5+9y^4+2y^3-5y^2-6y-1=0$.

3. $y^4-5y^3-8y+7=0$.

4. $4y^4-13y^2+9=0$.

5. $y^4+5y^3+8y^2+11y+9=0$.

6. $y^3-3y+2=0$.

7. $y^4+8y^3-111y-196=0$ or $y^4-8y^3+17y-8=0$.

8. $15x^5-8x^4+8x^3-5x^2-4x+1=0$.

9. $y^3+(2b-a^2)y^2+(b^2-2a)y-1=0$.

10. $y^3-23y^2-42y+1=0$.

11. $y^3+3y^2+30y+54=0$.

12. $(r-q-1)y^3+(3-q+3r)y^2+(q+3r-3)y+q+r+1=0;$

$\dfrac{3-q+3r}{1+q-r}, \dfrac{q+r+1}{1+q-r}.$

13. $7y^3-18y^2+6y+4=0$.

14. $8ry^3-4qy^2-1=0, \dfrac{q}{2r}$.

15. (i) $ry^3+q^2y^2-2qry+r^2=0$.

(ii) $r^2y^3+(3r^2+2q^3)y^2+(3r^2-4q^3)y+r^2+2q^3=0$.

(iii) $ry^3+q(1-r)y^2+(1-r)^3=0$.

16. $(y-7)^3-21(y-7)^2+972=0$

17. (i) $\{q(r+y)-p^2y\}^2=\{pqy-(r+y)^2\}\{p(r+y)-q^2\}$.

(ii) $r^3y^3+pr(1+r)y^2+q(1+r)^2y+(1+r)^3=0$.

19. $y^3-6y^2+11y-5=0$.

20. $y^3-15y^2+95y-217=0$.

21. $y^4-7y^3+17y^2-10y-6=0$.

22. $x^3+12x^2+43x+44=0$.

23. $x^3-7x+6=0$.

24. $y^3-9y^2+90y-168=0$.

Unit-5 ❒ Cubic and Biquadratic Equations

Structure

5.1 Objectives

Here we shall be able to do the following :

- We shall be able to find the roots of a cubic equation by Cardan's Method.
- We shall be able to find the roots of a biquadratic equation by Euler's, Descartes' and Ferrari's Method.

5.2 Introduction

In this unit, we shall discuss the methods of solving cubic and biquadratic equations. Solving of these equations are extremely useful in research work and in different applications, specially in physics, architecture, etc.

5.3 Cubic Equations

In this section, we like to find the solution of a general cubic equation with binomial coefficients $ax^3+3bx^2+3cx+d=0,\ a\neq 0$. In fact, Tartaglia found out a method of solving the standard form of this equation, which goes by the name of his student Cardan.

5.3.1. Standard form of a cubic

Let us consider the cubic equation

$$ax^3+3bx^2+3cx+d=0,\ a\neq 0. \qquad ...(1)$$

First, we reduce this equation to its standard form in which the second term is lacking.

To remove the second term, let us increase the roots of equation (1) by h.

Therefore equation of transformation is $y=x+h$ or $x=y-h$.

Therefore equation (1) becomes

$$a(y-h)^3+3b(y-h)^2+3c(y-h)+d=0$$

or, $$a(y^3-3hy^2+3h^2y-h^3)+3b(y^2-2hy+h^2)+3c(y-h)+d=0$$

or, $$ay^3+3(-ah+b)y^2+3(ah^2-2bh+c)y+(-ah^3+3bh^2-3ch+d)=0.$$

The second term will be removed if $-ah+b=0$, *i.e.*, if $h=\frac{b}{a}$.

Therefore the transformed equation becomes

$$ay^3+3\left(a.\frac{b^2}{a^2}-\frac{2b^2}{a}+c\right)y+\left(-a.\frac{b^3}{a^3}+3b.\frac{b^2}{a^2}-\frac{3cb}{a}+d\right)=0$$

or, $$ay^3+3\left(-\frac{b^2}{a}+c\right)y+\left(\frac{2b^3}{a^2}-\frac{3bc}{a}+d\right)=0$$

or, $$a^3y^3+3(ac-b^2)ay+(a^2d-3abc+2b^3)=0 \qquad ...(2)$$

By putting $ay=z$ in equation (2), we further transform the equation to the form

$$z^3+3Hz+G=0, \qquad ...(3)$$

where $H=ac-b^2$ and $G=a^2d-3abc+2b^3$.

Equation (3) is called the standard form or reduced cubic of Cardan.

If z be a root of equation (3), then $y=\frac{z}{a}$ is a root of equation (2)

and hence $x=y-h=\frac{z}{a}-\frac{b}{a}=\frac{1}{a}(z-b)$ is a root of the equation (1).

Therefore the relation between the roots of the general cubic (1) and those of the standard cubic (3) is $ax+b=z$.

5.3.2 Cardan's Method of Solution of Standard Cubic

To solve the standard cubic $z^3+3Hz+G=0$ ...(1)
according to Cardan, we assume $z=m+n$.

Therefore $z^3=m^3+n^3+3mn(m+n)=m^3+n^3+3mnz$

or, $z^3-3mnz-m^3-n^3=0$ ...(2)

Comparing (1) and (2), we get $m^3+n^3=-G$ and $mn=-H$ or $m^3n^3=-H^3$.
So m^3 and n^3 are the roots of the quadratic equation $t^2+Gt-H^3=0$.
Solving this equation, we get two roots

$$m^3=\frac{1}{2}(-G+\sqrt{G^2+4H^3}) \text{ and } n^3=\frac{1}{2}(-G-\sqrt{G^2+4H^3}).$$

From these, we get three cube roots as m, $m\omega$, $m\omega^2$ and n, $n\omega$, $n\omega^2$, where ω and ω^2 are imaginary cube roots of unity.

Keeping in mind that $mn=-H$, we get only three pairs of values of m and n viz. (m, n), $(m\omega, n\omega^2)$ and $(m\omega^2, n\omega)$ are to be taken. So the solution of the equation (1) is given by

$$m+n,\ m\omega+n\omega^2 \text{ and } m\omega^2+n\omega. \quad ...(3)$$

5.3.3 Nature of the roots of a cubic

The general cubic equation is $ax^3+3bx^2+3cx+d=0$. ...(1)

By the transformation, $z=ax+b$, we get the standard form viz.

$$z^3+3Hz+G=0, \quad ...(2)$$

where $H=ac-b^2$ and $G=a^2d-3abc+2b^3$.

Nature of the roots is not changed by substitution. So the nature of the roots of equation (1) remains the same as the nature of the roots of equation (2). Again the roots of equation (2) are obtained by using the quadratic equation

$$t^2+Gt-H^3=0. \quad ...(3)$$

The quantity G^2+4H^3 is the discriminant of equation (3). Hence the nature of the roots depends upon the value of G^2+4H^3.

We have found that the roots of equation (2) are

$$m+n,\ m\omega+n\omega^2 \text{ and } m\omega^2+n\omega. \quad ...(4)$$

The following three cases arise :

(i) If $G^2+4H^3>0$, the roots of equation (3) are real and distinct. Therefore, from equation (4), we conclude that, equation (1) has one real root and two imaginary roots.

(ii) If $G^2+4H^3=0$, the roots of equation (3) are equal, *i.e.*, $m = n$ and therefore, from equation (4), we see that, the roots of equation (1) are $2m$, $-m$, $-m$ (since $1 + \omega + \omega^2 = 0$). So all the roots are real and two of them are equal.
If $G = H = 0$, then all the roots are real and equal.

(iii) If $G^2 + 4H^3 < 0$, the roots of equation (3) are imaginary. m and n cannot be found by any arithmetical process. This is called irreducible case of Cardan. In this case, De Moivre's theorem may be applied to solve the equation.

5.4 Worked out Examples (I)

Example 1 : Reduce the equation $x^3+6x^2-12x+32=0$ to its standard form.

Solution : Comparing this equation with $ax^3+3bx^2+3cx+d=0$, we get

$$a = 1,\ b = 2,\ c = -4 \text{ and } d = 32.$$

Under the transformation $z = ax + b = x + 2$, the given equation will be reduced to the standard form $z^3 + 3Hz + G = 0$ where $H = ac - b^2 = -4 - 4 = -8$ and

$$G = a^2d - 3abc + 2b^3 = 32 + 24 + 16 = 72.$$

Therefore the required standard form is $z^3 - 24z + 72 = 0$.

Example 2 : Solve $x^3 - 30x + 133 = 0$ by Cardan's method.

Solution : Let $x = m + n$.

Therefore $x^3 = m^3 + n^3 + 3mn(m+n) = m^3 + n^3 + 3mnx$

or, $$x^3 - 3mnx - (m^3 + n^3) = 0.$$

Comparing this equation with the given one, we get $-3mn = -30$ and $-(m^3 + n^3) = 133$

or, $m^3 + n^3 = -133$ and $mn = 10$, *i.e.*, $m^3n^3 = 1000$.

Therefore m^3 and n^3 are the roots of the quadratic

$$t^2 + 133t + 1000 = 0$$

or, $$t = \frac{-133 \pm \sqrt{(133)^2 - 4000}}{2} = \frac{-133 \pm 117}{2} = -125,\ -8.$$

Therefore $m^3 = -125$ and $n^3 = -8$.

Therefore $m = -5, -5\omega, -5\omega^2$ and $n = -2, -2\omega, -2\omega^2$, where ω and ω^2 are imaginary cube roots of unity.

Since $mn = 10$, we take $m = -5$, $n = -2$; $m = -5\omega$, $n = -2\omega^2$; $m = -5\omega^2$, $n = -2\omega$

Therefore, $x = m + n = -5 - 2,\ -5\omega - 2\omega^2,\ -5\omega^2 - 2\omega$

or, $x = -7,\ \frac{1}{2}(7 + 3\sqrt{3}\,i),\ \frac{1}{2}(7 - 3\sqrt{3}\,i),$

where $\omega = \frac{1}{2}(-1 + \sqrt{3}i)$ and $\omega^2 = \frac{1}{2}(-1 - \sqrt{3}i)$.

Example 3 : Solve $x^3 + 9x^2 + 15x - 25 = 0$.

Solution : To solve the equation, we are to reduce this equation to its standard form.

Let us put $y = x + h$ or $x = y - h$.

Therefore the equation becomes $(y-h)^3 + 9(y-h)^2 + 15(y-h) - 25 = 0$

Coefficient of y^2 is taken to be zero.

Therefore $-3h + 9 = 0$ or, $h = 3$.

Therefore $y = x + 3$.

–3	1	9	15	–25
		–3	–18	9
	1	6	–3	–16
		–3	–9	
	1	3	–12	
		–3		
	1	0		

Using synthetic division, we get the transformed equation

$$y^3 - 12y - 16 = 0. \qquad ...(1)$$

Let $y = m + n$

or, $y^3 = m^3 + n^3 + 3mn(m+n) = m^3 + n^3 + 3mny$

or, $y^3 - 3mny - (m^3 + n^3) = 0. \qquad ...(2)$

Comparing equations (1) and (2), we get $mn = 4$ and $m^3 + n^3 = 16$.

Therefore m^3 and n^3 are the roots of $t^2 - 16t + 64 = 0$

or, $$t = \frac{16 \pm \sqrt{256 - 4 \times 1 \times 64}}{2 \times 1} = 8,\ 8.$$

Therefore $m^3 = 8$ and $n^3 = 8$.

Therefore $m = 2, 2\omega, 2\omega^2$ and $n = 2, 2\omega, 2\omega^2$.

Since $mn = 4$, we have $y = 2+2,\ 2\omega+2\omega^2,\ 2\omega^2+2\omega$, *i.e.*, $y = 4, -2, -2$.

Hence the required roots of the given equation (using $x = y - 3$) are $4-3, -2-3$ and $-2-3$, *i.e.*, $1, -5, -5$.

Example 4 : Solve the equation $x^3 - 3x + 1 = 0$ by Cardan's method.

Solution : The given equation $x^3 - 3x + 1 = 0$ ...(1)

is already in standard form.

Comparing this with $z^3 + 3Hz + G = 0$, we get $H = -1$ and $G = 1$.

Now $G^2 + 4H^3 = 1 - 4 = -3 < 0$.

Therefore this is irreducible case of Cardan.

Let $x = m + n$.

Therefore $x^3 - 3mnx - (m^3 + n^3) = 0$. ...(2)

Comparing equations (1) and (2), $mn = 1$ and $m^3 + n^3 = -1$.

Therefore m^3 and n^3 are roots of $t^2 + t + 1 = 0$.

Therefore $t = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}i}{2}$.

Therefore $m^3 = -\dfrac{1}{2} + \dfrac{\sqrt{3}i}{2}$ and $n^3 = -\dfrac{1}{2} - \dfrac{\sqrt{3}i}{2}$.

Let $m^3 = r(\cos\theta + i\sin\theta)$ and $n^3 = r(\cos\theta - i\sin\theta)$.

Therefore $r\cos\theta = -\dfrac{1}{2}$ and $r\sin\theta = \dfrac{\sqrt{3}}{2}$.

Therefore $r = \sqrt{\dfrac{1}{4} + \dfrac{3}{4}} = 1$.

$\therefore \cos\theta = -\dfrac{1}{2}$ and $\sin\theta = \dfrac{\sqrt{3}}{2}$.

Therefore $\theta = \dfrac{2\pi}{3}$.

Therefore $m^3 = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$ and $n^3 = \cos\dfrac{2\pi}{3} - i\sin\dfrac{2\pi}{3}$.

Hence $m = \cos\dfrac{1}{3}\left(\dfrac{2\pi}{3} + 2k\pi\right) + i\sin\dfrac{1}{3}\left(\dfrac{2\pi}{3} + 2k\pi\right)$, for $k = 0, 1, 2$

and $n=\cos\frac{1}{3}\left(\frac{2\pi}{3}+2k\pi\right)-i\sin\frac{1}{3}\left(\frac{2\pi}{3}+2k\pi\right)$ for $k=0, 1, 2$.

Hence $x=m+n=2\cos\frac{1}{3}\left(\frac{2\pi}{3}+2k\pi\right)$ for $k=0, 1, 2$.

Therefore required roots are $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{3}$ and $2\cos\frac{14\pi}{3}$.

5.5 Model Questions (I)

Reduce the equations to the standard form :

1. (i) $x^3-3x^2+12x+16=0$. (ii) $x^3+3x^2+10x+6=0$.

2. Solve the following equations by Cardan's Method :

(i) $x^3-18x-35=0$. (ii) $x^3-9x+28=0$.

(iii) $x^3-15x-126=0$. (iv) $8x^3-36x+27=0$.

(v) $x^3-12x+65=0$. (vi) $x^3+6x^2-12x+32=0$.

(vii) $x^3-3x^2+12x+16=0$. (viii) $x^3-6x-4=0$.

(ix) $x^3-7x+6=0$.

5.5.1. Answers

1. (i) $y^3+9y+26=0$. (ii) $y^3+7y-2=0$.

2. (i) $5, -\frac{5}{2}\pm\frac{1}{2}\sqrt{3}i$. (ii) $-4, 2\pm\sqrt{3}i$.

(iii) $6, -3\pm2\sqrt{3}i$. (iv) $\frac{3}{2}, \frac{3}{4}(-1\pm\sqrt{5})$.

(v) $-5, 1-3\omega, 1+3\omega^2$. (vi) $-8, -2\omega, -2\omega^2$.

(vii) $-1, -4\omega, -4\omega^2$.

(viii) $2\sqrt{2}\cos\frac{\pi}{12}, 2\sqrt{2}\cos\frac{3\pi}{4}, 2\sqrt{2}\cos\frac{17\pi}{12}$. (ix) 1,2, –3.

5.6 Biquadratic Equations

Let us consider the general biquadratic equations with binomial coefficients

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0, \ (a \neq 0) \quad \ldots (1)$$

To reduce this equation to its standard form, we decrease the roots of equation (1) by h by putting $y = x - h$.

Therefore the transformed equation is

$$a(y+h)^4 + 4b(y+h)^3 + 6c(y+h)^2 + 4d(y+h) + e = 0.$$

The second term will be removed if coefficients of $y^3 = 0$, *i.e.*, if $4ah + 4b = 0$

i.e., if $h = -\frac{b}{a}$.

$-\frac{b}{a}$	a	$4b$	$6c$	$4d$	e
		$-b$	$-\frac{3b^2}{a}$	$\frac{3b^3}{a^2}-\frac{6bc}{a}$	$-\frac{3b^4}{a^3}+\frac{6b^2c}{a^2}-\frac{4bd}{a}$
	a	$3b$	$6c-\frac{3b^2}{a}$	$4d+\frac{3b^3}{a^2}-\frac{6bc}{a}$	$e-\frac{3b^4}{a^3}+\frac{6b^2c}{a^2}-\frac{4bd}{a}$
		$-b$	$-\frac{2b^2}{a}$	$\frac{5b^3}{a^2}-\frac{6bc}{a}$	
	a	$2b$	$6c-\frac{5b^2}{a}$	$4d+\frac{8b^3}{a^2}-\frac{12bc}{a}$	
		$-b$	$-\frac{b^2}{a}$		
	a	b	$6c-\frac{6b^2}{a}$		
		$-b$			
	a	0			

The transformed equation is

$$ay^4+\frac{6}{a}(ac-b^2)y^2+\frac{4}{a^2}(a^2d-3abc+2b^3)y+\frac{1}{a^3}(a^3e-4a^2bd+6ab^2c-3b^4)=0.$$

Putting $H=ac-b^2, G=a^2d-3abc+2b^3$ and $I=ae-4bd+3c^2$, we have

$$a^3e-4a^2bd+6ab^2c-3b^4=a^2(ae-4bd+3c^2)-3(a^2c^2-2ab^2c+b^4)$$
$$=a^2I-3(ac-b^2)^2=a^2I-3H^2.$$

The transformed equation becomes $ay^4+\frac{6}{a}Hy^2+\frac{4}{a^2}Gy+\frac{1}{a^3}(a^2I-3H^2)=0$

or, $a^4y^4+6Ha^2y^2+4Gay+(a^2I-3H^2)=0.$

Multiplying the roots of this equation by a and writing $z=ay$, we get

$$z^4+6Hz^2+4Gz+(a^2I-3H^2)=0 \qquad ...(2)$$

This is the standard form of the biquadratic.

If z is a root of the equation (2), then the corresponding root of equation (1) is $x=\frac{1}{a}(z-b)$.

5.6.1 Euler's Method of Solution of a Biquadratic

Let us solve the biquadratic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0,\ a\neq 0 \qquad ...(1)$$

Reducing it to the standard form

$$z^4+6Hz^2+4Gz+(a^2I-3H^2)=0, \qquad ...(2)$$

where $H=ac-b^2, G=a^2d-3abc+2b^3$ and $I=ae-4bd+3c^2$.

To solve equation (2), let $z=\sqrt{p}+\sqrt{q}+\sqrt{r}$. ...(3)

Squaring both sides, we get $z^2=p+q+r+2(\sqrt{pq}+\sqrt{qr}+\sqrt{rp})$

or, $z^2-(p+q+r)=2(\sqrt{pq}+\sqrt{qr}+\sqrt{rp})$.

Squaring both sides, we get

$$z^4-2(p+q+r)z^2+(p+q+r)^2=4(pq+qr+rp)+8\sqrt{pqr}(\sqrt{p}+\sqrt{q}+\sqrt{r})$$

or, $z^4-2(p+q+r)z^2-8\sqrt{pqr}.z+(p+q+r)^2-4(pq+qr+rp)=0.$ ...(4)

Comparing equations (2) and (4), we get

$$p+q+r=-3H,$$

$$\sqrt{pqr}=-\frac{1}{2}G$$

and $(p+q+r)^2-4(pq+qr+rp)=a^2I-3H^2$.

From the last relation, we have

$$4(pq+qr+rp)=(-3H)^2-(a^2I-3H^2)=12H^2-a^2I.$$

So we get $\quad p+q+r=-3H,$

$$pq+qr+rp=3H^2-\frac{1}{4}a^2I$$

and $\quad pqr=\frac{1}{4}G^2$.

Therefore p, q, r are the roots of the equation

$$t^3+3Ht^2+(3H^2-\frac{1}{4}a^2I)t-\frac{1}{4}G^2=0. \quad ...(5)$$

Equation (5) is known as Euler's cubic. Equation (5) can be written as

$$(t+H)^3-\frac{a^2}{4}I(t+H)-\frac{1}{4}(G^2+4H^3-a^2IH)=0$$

or, $(t+H)^3-\frac{a^2}{4}I(t+H)-\frac{1}{4}(-a^3J)=0,$

where $J = ace + 2bcd - ad^2 - b^2e - c^3$

or, $4(t+H)^3 - a^2 I(t+H) + a^3 J = 0$.

Now putting $t + H = a^2\theta$, we get

$$4a^3\theta^3 - Ia\theta + J = 0. \quad ...(6)$$

This is known as the reduced cubic of the biquadratic.

If $\theta_1, \theta_2, \theta_3$, be the roots of the equation (6), then the corresponding roots of (5) will be $p=a^2\theta_1-H$, $q=a^2\theta_2-H$, $r=a^2\theta_3-H$ and the values of z will be obtained from equation (3) as

$$z=\pm\sqrt{a^2\theta_1-H}\pm\sqrt{a^2\theta_2-H}\pm\sqrt{a^2\theta_3-H}.$$

Since the total number of values of z, *i.e.*, roots of equation (2) will be only four, we shall select the signs of $\sqrt{p}$, $\sqrt{q}$ and $\sqrt{r}$ in such a way that the conditions $\sqrt{pqr} = -\frac{1}{2}G$ is satisfied. After finding out the four values of z, four roots of the equation (1) will be given by $x = \frac{1}{a}(z-b)$.

5.6.2 Descartes' Method of Solution of a Biquadratic

Let the equation be $ax^4+4bx^3+6cx^2+4dx+e=0,\ a\neq 0$ ...(1)

and the standard form be $z^4+6Hz^2+4Gz+(a^2I-3H^2)=0$ ...(2)

To solve this equation, the L.H.S. of the equation is expressed as product of two quadratic expressions in certain manner.

Let this equation be represented by

$$(z^2+mz+n)(z^2-mz+k)=0 \qquad ...(3)$$

or, $$z^4+(n+k-m^2)z^2+m(k-n)z+nk=0. \qquad ...(4)$$

Comparing equations (2) and (4), we get

$$n+k-m^2=6H, \qquad ...(5)$$

$$m(k-n)=4G \qquad ...(6)$$

and $$nk=a^2I-3H^2. \qquad ...(7)$$

From (5) and (6), we get

$2k=6H+m^2+\frac{4G}{m}$ and $2n=6H+m^2-\frac{4G}{m}$.

Putting these values in (7), we get

$$\left(6H+m^2+\frac{4G}{m}\right)\left(6H+m^2-\frac{4G}{m}\right)=4(a^2I-3H^2)$$

or, $$(6H+m^2)^2-\frac{16G^2}{m^2}=4(a^2I-3H^2)$$

or, $$36H^2+12Hm^2+m^4-\frac{16G^2}{m^2}-4(a^2I-3H^2)=0$$

or, $$m^6+12Hm^4-4(a^2I-12H^2)m^2-16G^2=0.$$

This is a cubic equation in m^2 and solving this equation, we can find the values of m^2 from which k and n can be found out. We then get two quadratic equations

$$z^2+mz+n=0 \text{ and } z^2-mz+k=0.$$

The roots of these quadratic equations are the roots of the equation (1).

5.6.3. Ferrari's Method of Solution of a Biquadratic

Let us consider the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,\ a \neq 0 \qquad ...(1)$$

To solve this equation, the L.H.S. of (1) is expressed as difference of two squares.

Let us multiply L.H.S. of (1) by a. We then get

$$a^2x^4 + 4abx^3 + 6acx^2 + 4adx + ae = 0$$

or, $$(ax^2 + 2bx + \lambda)^2 - (2mx + n)^2 = 0 \text{ (say)} \qquad ...(2)$$

or, $a^2x^4 + 4abx^3 + (4b^2 + 2a\lambda - 4m^2)x^2 + (4b\lambda - 4mn)x + (\lambda^2 - n^2) = 0$.

Equating coefficients of like terms, we get

$$4b^2 + 2a\lambda - 4m^2 = 6ac \text{ or, } m^2 = \frac{1}{2}(2b^2 + a\lambda - 3ac) \qquad ...(3)$$

$$4b\lambda - 4mn = 4ad \text{ or, } mn = b\lambda - ad \qquad ...(4)$$

and $$\lambda^2 - n^2 = ae \text{ or, } n^2 = \lambda^2 - ae. \qquad ...(5)$$

Eliminating m and n from (3), (4) and (5), we get

$$\frac{1}{2}(\lambda^2 - ae)(a\lambda + 2b^2 - 3ac) = (b\lambda - ad)^2$$

or, $\lambda^3 - 3c\lambda^2 + (4bd - ae)\lambda + (3ace - 2b^2e - 2ad^2) = 0$.

This is a cubic equation in λ.

Corresponding to a real root λ, we find the values of m^2, mn and n^2 from (3), (4) and (5) respectively. From these, we get m and n. Hence equation (2) is known and we have

$$\{ax^2 + 2(b + m)x + \lambda + n\}\{ax^2 + 2(b - m)x + \lambda - n\} = 0.$$

From $ax^2 + 2(b + m)x + \lambda + n = 0$, we get two roots and

from $ax^2 + 2(b - m)x + \lambda - n = 0$, we get two roots.

These four roots are the roots of equation (1).

5.7 Worked out Examples (II)

Example 1 : Solve the equation $x^4-2x^2+8x-3=0$ by Euler's method.

Solution : This is a biquadratic equation in its standard form.

Let $\quad x=\sqrt{p}+\sqrt{q}+\sqrt{r}$

or, $\quad x^2=p+q+r+2(\sqrt{pq}+\sqrt{qr}+\sqrt{rp})$

or, $\quad \{x^2-(p+q+r)\}^2=4(\sqrt{pq}+\sqrt{qr}+\sqrt{rp})^2$

or, $x^4-2(p+q+r)x^2+(p+q+r)^2-4\{pq+qr+rp+2\sqrt{pqr}(\sqrt{p}+\sqrt{q}+\sqrt{r})\}=0$

or, $\quad x^4-2(p+q+r)x^2-8\sqrt{pqr}.x+(p+q+r)^2-4(pq+qr+rp)=0$.

Comparing this equation with the given equation, we have

$$p+q+r=1,$$

$$\sqrt{pqr}=-1$$

and $\quad (p+q+r)^2-4(pq+qr+rp)=-3$

or, $\quad 1-4(pq+qr+rp)=-3$

or, $\quad pq+qr+rp=1$.

Since $p+q+r=1, pq+qr+rp=1$ and $pqr=1$, therefore p,q,r are the roots of the equation $\quad t^3-t^2+t-1=0$

or, $\quad t^2(t-1)+1(t-1)=0$

or, $\quad (t^2+1)(t-1)=0$.

Therefore $\quad t=1, \pm i$.

Let p = 1, q = i and r = – i.

Therefore $\quad \sqrt{p}=\pm 1$.

Since $\quad q=i=\frac{2i}{2}=\frac{1-1+2i}{2}=\frac{1^2+i^2+2i}{2}=\left(\frac{1+i}{\sqrt{2}}\right)^2$,

$$\sqrt{q}=\pm\frac{1+i}{\sqrt{2}}$$

and $$\sqrt{r} = \pm\frac{1-i}{\sqrt{2}}$$

Since $\sqrt{pqr} = -1$, we take

(a) $\sqrt{p} = -1, \quad \sqrt{q} = \frac{1}{\sqrt{2}}(1+i), \quad \sqrt{r} = \frac{1}{\sqrt{2}}(1-i)$

(b) $\sqrt{p} = -1, \quad \sqrt{q} = -\frac{1}{\sqrt{2}}(1+i), \quad \sqrt{r} = -\frac{1}{\sqrt{2}}(1-i)$

(c) $\sqrt{p} = 1, \quad \sqrt{q} = \frac{1}{\sqrt{2}}(1+i), \quad \sqrt{r} = -\frac{1}{\sqrt{2}}(1-i)$

(d) $\sqrt{p} = 1, \quad \sqrt{q} = -\frac{1}{\sqrt{2}}(1+i), \quad \sqrt{r} = \frac{1}{\sqrt{2}}(1-i)$.

Hence the required roots of the given equation are $-1 \pm \sqrt{2}$ and $1 \pm \sqrt{2}i$.

Example 2 : Solve $x^4 + 2x^2 + 56x - 187 = 0$ by Descartes' method.

Solution : Let the equation be expressed as product of two quadratic expressions

$$(x^2 + mx + n)(x^2 - mx + k) = 0 \qquad ...(1)$$

or, $$x^4 + (n + k - m^2)x^2 + m(k - n)x + kn = 0.$$

Comparing this equation with the given equation, we get

$$n + k - m^2 = 2,$$
$$m(k - n) = 56$$

and $$nk = -187.$$

Therefore $$k + n = m^2 + 2$$

and $$k - n = \frac{56}{m}.$$

Solving, $$2k = m^2 + 2 + \frac{56}{m}$$

and $$2n = m^2 + 2 - \frac{56}{m}.$$

Therefore $$\left(m^2 + 2 + \frac{56}{m}\right)\left(m^2 + 2 - \frac{56}{m}\right) = 4kn = -748$$

or, $$(m^2 + 2)^2 - \frac{3136}{m^2} = -748$$

or, $$m^4 + 4m^2 + 752 - \frac{3136}{m^2} = 0$$

or, $$m^6 + 4m^4 + 752m^2 - 3136 = 0.$$

By trial, we get $m^2 = 4$ or $m = 2$.

Therefore $k = \frac{1}{2}(4+2+28) = 17$ and $n = \frac{1}{2}(4+2-28) = -11$.

Therefore equation (1) becomes

$$(x^2 + 2x - 11)(x^2 - 2x + 17) = 0.$$

Therefore

either $(x^2 + 2x - 11) = 0$ which gives $x = \frac{-2 \pm \sqrt{4+44}}{2} = -1 \pm 2\sqrt{3}$

or, $(x^2 - 2x + 17) = 0$ which gives $x = \frac{2 \pm \sqrt{4-68}}{2} = 1 \pm 4i$.

Example 3 : Solve $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.

Solution : Let

$$x^4 - 2x^3 - 5x^2 + 10x - 3 \equiv (x^2 - x + \lambda)^2 - (2mx + n)^2 = 0 \quad ...(1)$$

or, $$x^4 - 2x^3 + (1 + 2\lambda - 4m^2)x^2 + (-2\lambda - 2mn)x + \lambda^2 - n^2 = 0.$$

Comparing coefficients of like terms, we get

$$1 + 2\lambda - 4m^2 = -5 \quad \text{or, } m^2 = \frac{1}{2}(\lambda + 3)$$

$$-2\lambda - 4mn = 10 \quad \text{or, } mn = -\frac{1}{2}(\lambda + 5)$$

and $$\lambda^2 - n^2 = -3 \quad \text{or, } n^2 = \lambda^2 + 3.$$

Eliminating m and n, we get

$$\frac{1}{2}(\lambda^2 + 3)(\lambda + 3) = \frac{1}{4}(\lambda + 5)^2$$

or, $$2(\lambda^3 + 3\lambda^2 + 3\lambda + 9) = \lambda^2 + 10\lambda + 25$$

or, $$2\lambda^3 + 5\lambda^2 - 4\lambda - 7 = 0.$$

By trial, $\lambda = -1$ is a root of this equation. Therefore we get

$$m^2 = \frac{1}{2}(-1 + 3) = 1,$$

$$mn = -\frac{1}{2}(-1 + 5) = -2$$

and $$n^2 = (-1)^2 + 3 = 4.$$

Therefore $m=1,\ n=-2$.

So equation (1) becomes

$$(x^2-x-1)^2-(2x-2)^2=0$$

or, $\quad (x^2+x-3)(x^2-3x+1)=0.$

Therefore $\quad x^2+x-3=0$ gives $x=\frac{1}{2}\left(-1\pm\sqrt{13}\right)$

and $\quad x^2-3x+1=0$ gives $x=\frac{1}{2}\left(3\pm\sqrt{5}\right)$.

5.8 Summary and Keywords

Summary :

I. General form of a Cubic equation

$ax^3+3bx^2+3cx+d=0,\ a\neq 0$ ...(1)

Its Standard form :

$z^3+3Hz+G=0,$ ...(2)

where $H = ac - b^2$ and $G = a^2d - 3abc + 2b^3$.

II. Nature of the roots of the cubic equation

The nature of the roots of (1) is the same as that of (2).

(i) If $G^2 + 4H^3 > 0$, the equation (1) has one real root and two imaginary roots.

(ii) If $G^2 + 4H^3 < 0$, we get irreducible case of Cardan. (1) has two imaginary roots.

(iii) If $G = H = 0$, all the roots of (1) are real and equal.

(iv) If $G^2 + 4H^3 = 0$, all the roots of (1) are real, two of them are equal.

III. General form of Biquadratic equation

$ax^4+4bx^3+6cx^2+4dx+e=0,\ a\neq 0.$

Its Standard form is $z^4+6Hz^2+4Gz+(a^2I-3H^2)=0,$ ...(3)

where $H = ac - b^2$, $G = a^2d - 3abc + 2b^3$ and $I = ae - 4bd + 3c^2$.

IV. Methods of solving the equation (3) by (a) Euler (b) Descartes' and (c) Ferrari are discussed in details.

Keywords :

Equation, Roots, Cubic, Biquadratic, Nature, Irreducible.

5.9 Model Questions (II)

1. Solve by Euler's method :

   (i) $x^4-2x^2+8x-3=0$. (ii) $x^4+4x^3-6x^2+20x+8=0$.

   (iii) $x^4-3x^2-6x-2=0$. (iv) $x^4-6x^3+16x^2-24x+16=0$.

2. Solve by Descartes' method :

   (i) $x^4-3x^2+1=0$.

   (ii) $x^4-4x^3+12x^2-16x+15=0$.

   (iii) $x^4-4x^3-20x^2+64x-20=0$.

   (iv) $x^4+3x^2+2x+12=0$.

3. Solve by Ferrari's method :

   (i) $x^4+12x-5=0$.

   (ii) $x^4-18x^2+32x-15=0$.

   (iii) $x^4+3x^3+x^2-2=0$.

   (iv) $x^4-2x^3-5x^2+10x-3=0$.

4. Solve $x^4-6x^2-16x-15=0$ by expressing it in the form $(x^2+\lambda)^2-(mx+n)^2=0$.

5.9.1 Answers

1. (i) $-1\pm\sqrt{2},\ 1\pm\sqrt{2}i$. (ii) $1\pm\sqrt{3}i,\ -3\pm\sqrt{7}$.

   (iii) $-1\pm i, 1\pm\sqrt{2}$. (iv) $2,\ 2, 1\pm\sqrt{3}i$.

2. (i) $\frac{1}{2}\left(-1\pm\sqrt{5}\right),\ \frac{1}{2}\left(1\pm\sqrt{5}\right)$. (ii) $1\pm\sqrt{2}i, 1\pm 2i$.

   (iii) $-1\pm\sqrt{11},\ 3\pm\sqrt{7}$. (iv) $1\pm\sqrt{3}i,\ -1\pm\sqrt{2}i$.

3. (i) $1\pm 2i,\ -1\pm\sqrt{2}$. (ii) 1, 1, 3, –5.

   (iii) $-1\pm\sqrt{3},\ \frac{1}{2}\left(-1\pm\sqrt{3}i\right)$. (iv) $\frac{1}{2}\left(-1\pm\sqrt{13}\right),\ \frac{1}{2}\left(3\pm\sqrt{5}\right)$.

4. $1\pm\sqrt{6},\ -1\pm\sqrt{2}i$.

Unit-6 ❒ Inequalities

Structure

6.2 Objectives

In mathematics, use of inequalities is very important. Here we shall learn the relations among Arithmetic Mean, Geometric Mean and Harmonic Mean. We shall learn proof of Cauchy-Schwartz's inequality. We shall be acquainted with some other useful inequalities. These inequalities are widely used in solving many problems.

6.1 Introduction

In this chapter, we shall deal only with the real numbers and n is assumed to be a positive integer. We recall some known inequalities which are very useful in mathematics. We use '>', '<' as sign of inequalities.

6.3 Some definitions and important properties

(a) For the real numbers a and b, we say a is less than b if $a - b$ is negative. Using sign of inequality, we write $a < b$. On the other hand, a is greater than b if $a - b$ is positive and we write $a > b$. If a is equal to b, we write $a = b$. If a is less than or equal to b, we write $a \leq b$. Again, if a is greater than or equal to b, we write $a \geq b$.

(b) For any two real numbers a and b, we always have either $a < b$ or $a = b$ or $a > b$.

(c) Square of any real number must be non-negative, *i.e.*, if a be a real number then $a^2 \geq 0$.

(d) If $a > b$ and $b > c$ then $a > c$.

(e) If $a > b$, then $a + k > b + k$ and $a - k > b - k$, where k is any real number.

(f) If $a > b$, then $ka > kb$, if $k > 0$ and $ka < kb$, if $k < 0$. Also $\frac{a}{k} > \frac{b}{k}$ and $\frac{a}{k} < \frac{b}{k}$, according as $k > 0$ or $k < 0$.

(g) Let $a \neq 0$ and $b \neq 0$. If $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

(h) If $a_1 > b_1$, $a_2 > b_2$,......, $a_n > b_n$, then $a_1 + a_2 + ... + a_n > b_1 + b_2 + ... + b_n$ and $a_1 . a_2 ..... a_n \geq b_1 . b_2 ..... b_n$ (assuming $a_i \geq 0$ and $b_i \geq 0$, $i = 1, 2, ..., n$).

(i) If $x > 0$ and $a > b > 0$, then $a^x > b^x$.

(j) If $a > 1$ and $x > y > 0$, then $a^x > a^y$.

(k) If $0 < a < 1$ and $x > y > 0$, then $a^x < a^y$.

6.4 Worked out Examples (I)

Example 1 : If $a > 0$ and $b > 0$, then $\frac{a+b}{2} > \sqrt{ab}$.

Solution : We have $\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$

or, $a + b - 2\sqrt{ab} > 0$

or, $a + b > 2\sqrt{ab}$

Therefore $\frac{a+b}{2} > \sqrt{ab}$.

Example 2 : If $a, b, c,$ be any three real numbers, then prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution : we have $a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{1}{2}\{(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)\}$$

$$= \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

$$\geq 0.$$

Hence $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Example 3 : Show that $(n!)^2 > n^n$, for $n > 2$.

Solution : We have

$$(n!)^2 = (n!).\ (n!)$$

$$= 1.2.3 \ldots r \ldots n \times n.(n-1).(n-2) \ldots (n-r+1) \ldots .3.2.1 \qquad \ldots(1)$$

Now $\qquad r(n-r+1) > n,$

if $\qquad r^2 - r(n+1) + n < 0$

i.e., if $\qquad (r-1)(r-n) < 0$

i.e., if $\qquad 1 < r < n$.

Therefore, from equation (1), we get

$$(n!)^2 > n.n....n. \ldots .n \quad (n \text{ terms}), \text{ for } n > 2$$

i.e., $\qquad (n!)^2 > n^n$, for $n > 2$.

Example 4 : If a, b, c are positive and not all equal, then prove that

$$(a+b+c)(bc+ca+ab) > 9abc.$$

Solution : We have $(a+b+c)(bc+ca+ab) - 9abc$

$$= abc + a^2(c+b) + abc + b^2(a+c) + abc + c^2(a+b) - 9abc$$

$$= a^2(c+b) + b^2(a+c) + c^2(a+b) - 6abc$$

$$= a(b^2 + c^2 - 2bc) + b(c^2 + a^2 - 2ca) + c(a^2 + b^2 - 2ab)$$

$$= a(b-c)^2 + b(c-a)^2 + c(a-b)^2 > 0.$$

Hence $(a+b+c)(bc+ca+ab) > 9abc$.

6.5 Model Questions (I)

1. Prove that, for all $x > 0$, $x + \frac{1}{x} \geq 2$ and for $x < 0$, $x + \frac{1}{x} \leq -2$.

2. If a, b, c be any three real numbers, show that

$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b} \geq a+b+c.$$

3. (a) If a, b, x, y be all positive, then show that $\frac{(a+b)xy}{ay+bx} \leq \frac{ax+by}{a+b}$.

(b) If a, b, c, d are all > 1, then prove that $8(abcd+1) > (a+1)(b+1)(c+1)(d+1)$.

[Hint : Prove first $2(ab+1) > (a+1)(b+1)$, etc.]

4. Prove that $1!.3!.5!....... (2n-1)! > (n!)^n$.

5. If a, b, c are any three real numbers, then prove that

$$\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2}+\frac{a+b}{a^2+b^2} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

6. Show that the minimum value of $\frac{(4+x)(7+x)}{3+x}$ is 9.

[Hint : Put $3 + x = y$.]

6.6 Inequalities among the Means

If $a_1, a_2, ..., a_n$ be n positive real numbers, then

$$A = \frac{1}{n}(a_1+a_2+......+a_n),$$

$$G = \sqrt[n]{a_1.a_2......a_n},$$

$$H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+......+\frac{1}{a_n}}$$

are respectively called the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, ..., a_n$.

6.6.1 Relation among Arithmetic Mean, Geometric Mean and Harmonic Mean

The relation among A, G and H is $A \geq G \geq H$, the sign of equality holds only when $a_1 = a_2 = ... = a_n$.

Proof : If a_1 and a_2 be positive, then

$$\left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \geq 0$$

or, $a_1 + a_2 - 2\sqrt{a_1 a_2} \geq 0$

or, $$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}\ .$$

The sign of equality holds only when $a_1 = a_2$.
Similarly, if a_3 and a_4 be positive, then

$$\frac{a_3 + a_4}{2} \geq \sqrt{a_3 a_4}\ .$$

The sign of equality holds only when $a_3 = a_4$.

Hence $$a_1 a_2 . a_3 a_4 \leq \left(\frac{a_1 + a_2}{2}\right)^2 . \left(\frac{a_3 + a_4}{2}\right)^2$$

or, $$a_1 a_2 a_3 a_4 \leq \left(\frac{a_1 + a_2}{2} . \frac{a_3 + a_4}{2}\right)^2$$

i.e., $$\leq \left(\frac{\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2}}{2}\right)^4 = \left(\frac{a_1 + a_2 + a_3 + a_4}{4}\right)^4 .$$

The sign of equality holds only when $a_1 = a_2 = a_3 = a_4$.
Proceeding in this way, for $2^3 = 8$ positive numbers a_1, a_2, ..., a_8, we get

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 \leq \left(\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8}{8}\right)^8 .$$

The sign of equality holds only when $a_1 = a_2 = ... = a_8$.
Then, if n be a power of 2, *i.e.*, if $n = 2^m$, m being a positive integer, we get

$$a_1 a_2 ..... a_n \leq \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^n \qquad ...(1)$$

The sign of equality holds only when $a_1 = a_2 = ... = a_n$.
If n is not a power of 2, let $n + p$ be power of 2, where p is a positive integer. Now let us consider n positive numbers a_1, a_2,......, a_n and p positive integers each equal to A, where

$$A = \frac{a_1 + a_2 + ... + a_n}{n}. \qquad ...(2)$$

From equation (1), we have

$$a_1a_2.....a_n.A^p \le \left(\frac{a_1+a_2+......+a_n+pA}{n+p}\right)^{n+p}$$

i.e., $$a_1a_2.....a_n.A^p \le \left(\frac{nA+pA}{n+p}\right)^{n+p},$$

since from equation (2) we have $a_1+a_2+......+a_n = nA$.

Therefore $$a_1a_2.....a_n.A^p \le \left(\frac{(n+p)A}{n+p}\right)^{n+p} = A^{n+p}$$

i.e., $$a_1a_2.....a_n \le A^n = \left(\frac{a_1+a_2+......+a_n}{n}\right)^n.$$

The sign of equality holds only when $a_1 = a_2 = ... = a_n$.

Therefore $$\frac{a_1+a_2+......+a_n}{n} \ge \sqrt[n]{a_1a_2.....a_n}$$

or, $$A \ge G. \qquad ...(3)$$

The sign of equality holds only when $a_1 = a_2 = ... = a_n$.

Let us now consider n positive numbers $\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n}$.

Using the inequality $A \ge G$ for these numbers, we have

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+......+\frac{1}{a_n}}{n} \ge \left(\frac{1}{a_1}.\frac{1}{a_2}......\frac{1}{a_n}\right)^{\frac{1}{n}}.$$

The sign of equality holds only when $\frac{1}{a_1} = \frac{1}{a_2} = ... = \frac{1}{a_n}$, *i.e.,* $a_1 = a_2 = ... = a_n$

or, $$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+......+\frac{1}{a_n}} \le (a_1a_2.....a_n)^{\frac{1}{n}}$$

or, $$H \le G$$

or, $$G \ge H. \qquad ...(4)$$

Combining (3) and (4), we get $A \ge G \ge H$, the sign of equality holds only when all the numbers are equal.

6.6.2 Theorem of Weighted Means

If $a_1, a_2, \ldots, a_n$ and $x_1, x_2, \ldots, x_n$ be two sets of n positive numbers, those in the second set being rational, then

$$\left(\frac{a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n}{n}\right)^{x_1+x_2+\ldots+x_n} \geq a_1^{x_1}a_2^{x_2}\ldots\ldots a_n^{x_n}.$$

The sign of equality occurs only when $a_1 = a_2 = \ldots\ldots = a_n$.

Proof : As $x_1, x_2, \ldots\ldots x_n$ are positive rational numbers, there exist positive numbers

$y_1, y_2, \ldots\ldots, y_n$ and g such that $x_1 = \frac{y_1}{g},\ x_2 = \frac{y_2}{g},\ \ldots\ldots, x_n = \frac{y_n}{g}$.

Now we consider

y_1 numbers each equal to a_1,

y_2 numbers each equal to a_2,

....

y_n numbers each equal to a_n.

Their A.M. ≥ their G.M.

So $$\frac{a_1y_1 + a_2y_2 + \ldots\ldots + a_ny_n}{y_1 + y_2 + \ldots\ldots + y_n} \geq (a_1^{y_1}a_2^{y_2}\ldots\ldots a_n^{y_n})^{\frac{1}{y_1+y_2+\ldots+y_n}}.$$

The sign of equality occurs only when $a_1 = a_2 = \ldots\ldots = a_n$.

Putting $y_1 = x_1g,\ y_2 = x_2g,\ \ldots,\ y_n = x_ng$ on both sides, we get

$$\frac{a_1x_1g + a_2x_2g + \ldots\ldots + a_nx_ng}{x_1g + x_2g + \ldots\ldots + x_ng} \geq (a_1^{x_1g}a_2^{x_2g}\ldots\ldots a_n^{x_ng})^{\frac{1}{x_1g+x_2g+\ldots\ldots+x_ng}}$$

or, $$\frac{a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n}{x_1 + x_2 + \ldots\ldots + x_n} \geq (a_1^{x_1}a_2^{x_2}\ldots\ldots a_n^{x_n})^{\frac{1}{x_1+x_2+\ldots\ldots+x_n}}.$$

The sign of equality occurs only when $a_1 = a_2 = \ldots\ldots = a_n$.

6.6.3 Extreme Values of Sum and Product.

Theorem : If $a_1, a_2, \ldots\ldots, a_n$ be n positive variables and c is a constant, then

(i) if $a_1 + a_2 + \ldots\ldots + a_n = c$, the value of $a_1a_2\ldots\ldots a_n$ is greatest when

$a_1 = a_2 = \ldots\ldots = a_n = \frac{c}{n}$, so that the greatest value of $a_1\ a_2\ \ldots\ldots\ a_n$ is $\left(\frac{c}{n}\right)^n$.

(ii) if $a_1 \, a_2 \, ... \, a_n = c$, the value of $a_1 + a_2 + ... + a_n$ is least when $a_1 = a_2 = ... = a_n$, so that the least value of $a_1 + a_2 + ... + a_n$ is $nc^{\frac{1}{n}}$.

Proof : (i) Let $a_1, a_2, ..., a_n$ be n positive numbers such that $a_1 + a_2 + ..., + a_n = c$. Using the relation G.M ≤ A.M., we get

$$(a_1 a_2 ..... a_n)^{\frac{1}{n}} \le \frac{a_1 + a_2 + ... + a_n}{n} = \frac{c}{n}.$$

Therefore $$a_1 a_2 ..... a_n \le \left(\frac{c}{n}\right)^n.$$

The sign of equality occurs only when $a_1 = a_2 = ... = a_n$.

Therefore the maximum value of $a_1 a_2 ..... a_n = \left(\frac{c}{n}\right)^n$ and it occurs when all the numbers are equal.

Proof : (ii) Let $a_1, a_2, ... , a_n$ be n positive numbers such that $a_1 \, a_2 \, ... \, a_n = c$. Using the relation A.M. ≥ G.M., we get

$$\frac{a_1 + a_2 + ... + a_n}{n} \ge (a_1 a_2 ..... a_n)^{\frac{1}{n}} = c^{\frac{1}{n}}$$

i.e., $$a_1 + a_2 + ... + a_n \ge nc^{\frac{1}{n}}.$$

The sign of equality occurs only when $a_1 = a_2 = ... = a_n$.

Therefore the minimum value of $a_1 + a_2 + ... + a_n = nc^{\frac{1}{n}}$ and it occurs when all the numbers are equal.

Example : Find the minimum value of $4x + 3y$ for positive values of x and y, subject to the condition $2x^3y^2 = 3$.

Solution : We have $x^3y^2 = \frac{3}{2}$. If λ, μ are any constants, we have

$$(\lambda x)(\lambda x)(\lambda x)(\mu y)(\mu y) = \lambda^3\mu^2x^3y^2 = \frac{3}{2}\lambda^3\mu^2 = \text{ a constant}.$$

Therefore $\lambda x+\lambda x+\lambda x+\mu y+\mu y=3\lambda x+2\mu y$ is minimum, when

$$\lambda x=\mu y=\left(\frac{3}{2}\lambda^3\mu^2\right)^{\frac{1}{5}}.$$

Hence the minimum value of $3\lambda x+2\mu y$ is $5.\left(\frac{3}{2}\lambda^3\mu^2\right)^{\frac{1}{5}}$.

Putting $3\lambda=4$ and $2\mu=3$, *i.e.*, $\lambda=\frac{4}{3}$ and $\mu=\frac{3}{2}$, we have the minimum value of $4x+3y$ as $5.\left\{\frac{3}{2}\left(\frac{4}{3}\right)^3\left(\frac{3}{2}\right)^2\right\}^{\frac{1}{5}}=5\sqrt[5]{8}.$

6.7 Cauchy-Schwartz's Inequality

We shall now state and prove Cauchy-Schwartz Inequality.

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sets of n real numbers, then

$$(a_1b_1+a_2b_2+\ldots\ldots+a_nb_n)^2\le(a_1^2+a_2^2+\ldots\ldots+a_n^2)(b_1^2+b_2^2+\ldots\ldots+b_n^2),$$

the sign of equality occurs when $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots\ldots=\frac{a_n}{b_n}$.

Proof : Let $A=a_1^2+a_2^2+\ldots\ldots+a_n^2$,

$B=a_1b_1+a_2b_2+\ldots\ldots+a_nb_n$

and $C=b_1^2+b_2^2+\ldots\ldots+b_n^2$.

For all real values of x, we have $(a_1x+b_1)^2+(a_2x+b_2)^2+\ldots\ldots+(a_nx+b_n)^2\ge 0$.

The sign of equality occurs if and only if $a_1x+b_1=a_2x+b_2=\ldots\ldots=a_nx+b_n=0$

i.e., when $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots\ldots=\frac{a_n}{b_n}=-\frac{1}{x}$. Otherwise we get

$$(a_1^2+a_2^2+\ldots\ldots+a_n^2)x^2+2(a_1b_1+a_2b_2+\ldots\ldots+a_nb_n)x+(b_1^2+b_2^2+\ldots\ldots+b_n^2)>0$$

or, $\quad Ax^2+2Bx+C>0$

or, $$x^2+2\frac{B}{A}x+\frac{C}{A}>0$$

or, $$\left(x+\frac{B}{A}\right)^2+\frac{AC-B^2}{A^2}>0.$$

This is true if and only if $AC - B^2 > 0$, *i.e.*, when $B^2 < AC$. Hence we prove that

$$(a_1b_1+a_2b_2+......+a_nb_n)^2<(a_1{}^2+a_2{}^2+......+a_n{}^2)(b_1{}^2+b_2{}^2+......+b_n{}^2).$$

6.8 Other useful inequalities*

6.8.1 Weirstrass Inequalities

If a_1, a_2, ..., a_n and b_1, b_2, ..., b_n are positive numbers less than 1 whose sum is denoted by S_n, then

(i) $$1-S_n<(1-a_1)(1-a_2)......(1-a_n)<\frac{1}{1+S_n}$$

and (ii) $$1+S_n<(1+a_1)(1+a_2)......(1+a_n)<\frac{1}{1-S_n},$$

where in the last inequality, it is supposed that $S_n < 1$.

6.8.2 Tchebychef's Inequality

If a_1, a_2, ..., a_n and b_1, b_2, ..., b_n be two sets of real numbers such that $a_1 \le a_2 \le ...... \le a_n$ and $b_1 \le b_2 \le ..... \le b_n$, then

$$(a_1+a_2+......+a_n)(b_1+b_2+......+b_n)\le n(a_1b_1+a_2b_2+......+a_nb_n).$$

Example : Prove that $\sqrt{1}+\sqrt{2}+\sqrt{3}+......+\sqrt{n}<n\sqrt{\frac{n+1}{2}}$.

6.8.3. Jensen's Inequality

If a_1, a_2, ..., a_n be a set of n positive numbers and r, s be two positive rational numbers such that $r < s$, then $(a_1{}^r+a_2{}^r+......+a_n{}^r)^{\frac{1}{r}}>(a_1{}^s+a_2{}^s+......+a_n{}^s)^{\frac{1}{s}}$.

Example 1: Prove that $(a^3+b^3+c^3)^4>(a^4+b^4+c^4)^3$.

Example 2 : Show that if in a $\triangle ABC$, $b^3 + c^3 = a^3$, then A is an acute angle.

* Reference for Advanced students

6.8.4 Minkowski's Inequality

If $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two sets of positve numbers and m be any positive rational number not equal to 0 or 1, then

$$(a_1^m + a_2^m + ... + a_n^m)^{\frac{1}{m}} + (b_1^m + b_2^m + ... + b_n^m)^{\frac{1}{m}}$$

$$> \text{ or } < \{(a_1 + b_1)^m + (a_2 + b_2)^m + ... + (a_n + b_n)^m\}^{\frac{1}{m}},$$

according as m does not or does lie between 0 and 1.

Example : Show that $(a+3)^3 + (b+4)^3 + (c+5)^3 < 343$ where a,b,c are all positive numbers and $a^3 + b^3 + c^3 = 1$.

6.8.5 Holder's Inequality

If $a_1, a_2, ..., a_n; b_1, b_2, ...b_n; ......; k_1, k_2, ..., k_n$ be m sets of positive numbers and $\alpha, \beta, ..., \lambda$ be m rational numbers such that $\alpha + \beta + ... + \lambda = 1$, then

$$a_1^\alpha b_1^\beta ......k_1^\lambda + a_2^\alpha b_2^\beta ......k_2^\lambda + ... + a_n^\alpha b_n^\beta ....k_n^\lambda$$

$$< (a_1 + a_2 + ... + a_n)^\alpha (a_1 + b_2 + ... + b_n)^\beta ......(k_1 + k_2 + ... + k_n)^\lambda.$$

Example : If a, b, c and d be positive numbers, then show that

$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq (1 + abcd)^4$.

6.8.6 If $a_1, a_2, ..., a_n$ are n positive numbers not all equal to one another, then

$$n\sum a_r^{x+y} > \text{ or } < \sum a_r^x . \sum a_r^y$$

according as x and y have the same or opposite signs.

6.8.7 If a is any positive number except 1 and x, y are positive rational numbers, then

$$\frac{a^x - 1}{x} > \frac{a^y - 1}{y}, \text{ if } x > y.$$

6.8.8 If a and b are positive and unequal and x is any rational number except 1, then

$$x.a^{x-1}(a-b) > a^x - b^x > xb^{x-1}(a-b)$$

unless $0 < x < 1$, in which case,

$$x.a^{x-1}(a-b) < a^x - b^x < x.b^{x-1}(a-b).$$

6.8.9 If $b = 1$ in the second parts of the inequalities stated above, we get

$$a^x - 1 > x(a-1) \text{ or } a^x - 1 < x(a-1)$$

according as x does not or does lie between 0 and 1.

6.8.10 m-th Power Theorem

If a_1, a_2, ..., a_n be n positive numbers which are not all equal to one another and m is any rational except 0 or 1, then

$$\frac{a_1^m + a_2^m + ...+ a_n^m}{n} > \text{ or } < \left(\frac{a_1 + a_2 + ...+ a_n}{n}\right)^m$$

according as m does not or does lie between 0 and 1.

Example 1. If a and b be two positive numbers and $a + b = 4$, then prove that

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq 12\frac{1}{2}.$$

Example 2. Prove that $a^5 + b^5 + c^5 > abc(ab+bc+ca)$.

6.9 Worked out Examples (II)

Example 1: Prove that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) > 9.$

Solution : Applying A.M. > G.M. for the numbers a, b, c we have

$$\frac{a+b+c}{3} > \sqrt[3]{abc} \qquad ... (1)$$

and for the numbers $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, we have

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} > \sqrt[3]{\frac{1}{abc}} \qquad ...(2)$$

Multiplying (1) and (2), the result follows.

Note : The result (2) can be obtained applying $AM > HM$ of the numbers a,b,c.

Example 2 : Prove that $a^a b^b > \left(\frac{a+b}{2}\right)^{a+b} > a^b b^a$.

Solution : Consider a quantities each equal to $\frac{1}{a}$ and b quatities each equal to $\frac{1}{b}$. Then using the inequality A.M. > G.M., we get

$$\frac{\left(\frac{1}{a}+\frac{1}{a}+\text{...... to } a \text{ times}\right)+\left(\frac{1}{b}+\frac{1}{b}+\text{......to } b \text{ times}\right)}{a+b}$$

$$>\left[\left(\frac{1}{a}.\frac{1}{a}\text{......to } a \text{ factors}\right)\left(\frac{1}{b}.\frac{1}{b}\text{.......to } b \text{ factors}\right)\right]^{\frac{1}{a+b}}$$

or, $$\frac{a.\frac{1}{a}+b.\frac{1}{b}}{a+b}>\left(\frac{1}{a^a}.\frac{1}{b^b}\right)^{\frac{1}{a+b}}$$

or, $$\frac{2}{a+b}>\left(\frac{1}{a^a b^b}\right)^{\frac{1}{a+b}}$$

or, $$\left(\frac{2}{a+b}\right)^{a+b}>\frac{1}{a^a b^b}$$

or, $$a^a b^b>\left(\frac{a+b}{2}\right)^{a+b} \qquad ...(1)$$

Next consider b quantities each equal to a and a quantities each equal to b. Then applying the inequality A.M. > G.M., we get

$$\frac{1}{a+b}[(a+a+\text{...... to } b \text{ times})+(b+b+\text{......to } a \text{ times})]$$

$$>[(a.a\text{.....to } b \text{ factors}).(b.b\text{...... to } a \text{ factors})]^{\frac{1}{a+b}}$$

or, $$\frac{a.b+b.a}{a+b}>(a^b.b^a)^{\frac{1}{a+b}}$$

or, $$\frac{2ab}{a+b}>\left(a^b.b^a\right)^{\frac{1}{a+b}}.$$

But $\frac{a+b}{2}$ is the A.M. of a and b. Also $\frac{2}{\frac{1}{a}+\frac{1}{b}}=\frac{2ab}{a+b}$ is the *H.M.* of a and b.

Since A.M. > H.M., we get $\frac{a+b}{2}>\frac{2ab}{a+b}$.

Therefore, we have $\frac{a+b}{2} > (a^b b^a)^{\frac{1}{a+b}}$

or, $\left(\frac{a+b}{2}\right)^{a+b} > a^b b^a$ ...(2)

Therefore, from (1) and (2), we get $a^a b^b > \left(\frac{a+b}{2}\right)^{a+b} > a^b b^a$.

Example 3 : If the sum of the sides of a triangle is given, prove that the area is greatest when the triangle is equilateral.

Solution : Let a, b and c be the sides of a triangle.

Therefore $a+b+c = 2s$ (given).

The area of a triangle

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

or, $$\triangle^2 = s(s-a)(s-b)(s-c).$$

Now $(s-a)+(s-b)+(s-c) = 3s-(a+b+c) = 3s-2s = s =$ a constant. Hence the value of $(s-a)(s-b)(s-c)$ is greatest when all the factors are equal, *i.e.*, when $s-a = s-b = s-c$, *i.e.*, when $a = b = c$, *i.e.*, the triangle is equilateral.

Example 4. Find the greatest value of $(a+x)^3(a-x)^4$ for any real value of x numerically less than a which is a constant.

Solution : The given expression is greatest when

$$\left(\frac{a+x}{3}\right)^3 . \left(\frac{a-x}{4}\right)^4 \text{ is so.}$$

Sum of the factors = $3.\frac{a+x}{3} + 4.\frac{a-x}{4} = 2a$, which is constant.

Hence the expression will be maximum when all the factors are equal, *i.e.*, when $\frac{a+x}{3} = \frac{a-x}{4}$ or $x = -\frac{a}{7}$ and the required greatest value is $\frac{6^3.8^4}{7^7}.a^7$.

6.10 Summary and Keywords

Summary

I. Basic Inequalities

(i) If $x > 0$ and $a > b > 0$, then $a^x > b^x$

(ii) If $a > 1$ and $x > y > 0$, $a^x > b^x$

(iii) If $0 < a < 1$ and $x > y > 0$, then $a^x < b^x$.

II. If a_1, a_2, ..., a_n be n positive real numbers, then

$$A = \frac{1}{n}.(a_1 + a_2 + ...+ a_n),\ G = \sqrt[n]{a_1 a_2 .....a_n}$$

and $H = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ...+ \frac{1}{a_n}}$ are defined as the arithmetic mean, geometric mean and harmonic mean respectively of $a_1, a_2, ..., a_n$.

The relations among A, G and H are $A \geq G \geq H$, the sign of equality occurs only when $a_1 = a_2 = ... = a_n$.

III. m-th power theorem

If a_1, a_2,......, a_n be n positive real numbers which are not all equal and m be any rational number except 0 and 1,

then $\dfrac{a_1^m + a_2^m + ...+ a_n^m}{n} >$ or $< \left(\dfrac{a_1 + a_2 + ...+ a_n}{n}\right)^m$ according as m does not or does lie between 0 and 1.

IV. Cauchy-Schwartz Inequality

If a_1, a_2, ..., a_n and b_1, b_2, ..., b_n be two sets of n real numbers, then $(a_1b_1 + a_2b_2 + ...+ a_nb_n)^2 \leq (a_1^2 + a_2^2 + ...+ a_n^2)(b_1^2 + b_2^2 + ...+ b_n^2)$, the sign of equality holds only when $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = ... = \dfrac{a_n}{b_n}$.

V. A few other Inequalities

Keywords : Inequality, Mean, Maximum, Minimum

6.11 Model Questions (II)

1. Prove that (i) $a^2b + b^2c + c^2a \geq 3abc$.

(ii) $(b+c)(c+a)(a+b) > 8abc$.

(iii) $(ab + xy)(ax + by) > 4abxy$.

2. Prove that $(n+1)^n > 2^n n!$.
3. If a, b, c are three positive real numbers,

   then show that $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 6$ and $\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b} \geq \frac{9}{a+b+c}$.
4. Prove that $(n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$.
5. If a, b, c be positive, then show that

   $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \geq 0$

   and hence deduce that $a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)$.
6. If a, b, c be in H.P., then show that $a^2 + c^2 > 2b^2$.
7. If a, b, c be positive and $a + b + c = 1$, then show that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$

   and $(1-a)(1-b)(1-c) < \frac{8}{27}$.
8. If a_1, a_2, ..., a_n be n positive numbers and $s = a_1 + a_2 + ... + a_n$, then show that

   $$\frac{s}{s-a_1}+\frac{s}{s-a_2}+......+\frac{s}{s-a_n} \geq \frac{n^2}{n-1}.$$
9. (a) Prove that the greatest value of $x^2 y^3$ is $\frac{3}{16}$ where x and y are connected by the relation $3x+4y=5$.

   (b) Prove that the minimum value of $3x + 4y$ for positive values of x, y subject to the condition $x^2y^3 = 6$ is 10.
10. If $a_1, a_2, ..., a_n$ are n positive numbers, then show that

    $$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+......+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} \geq n.$$
11. If a, b, c, d are positive, then show that

    $$\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c} \geq \frac{16}{a+b+c+d}.$$

12. Prove that the greatest value of $xyz\,(d - ax - by - cz)$ is $\frac{d^4}{4^4 abc}$, provided that all the factors are positive and a,b,c,d are given positive numbers.

13. If x, y, z be positive and $x + y + z = 1$, then show that

$$8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}.$$

14. (a) Find the greatest value of x^2y^3 subject to the condition $3x + 2y = 1$.

(b) Find the least value of $3x + 2y$ for positive values of x and y subject to the condition $x^3y^4 = 16$.

(c) Find the greatest value of xyz where x, y, z are positive real numbers subject to the condition $xy + yz + zx = 48$.

15. (a) If $a_1, a_2, \ldots\ldots, a_n$ be n positive rational numbers whose sum is s, then show that

$$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\left(\frac{s}{a_3}-1\right)^{a_3}\cdots\cdots\left(\frac{s}{a_n}-1\right)^{a_n} \le (n-1)^s.$$

[Hint : Consider the set of positive numbers $\left(\frac{s}{a_i}-1\right)$ with weight a_i for $i = 1, 2, \ldots\ldots, n$ and apply weighted means theorem.]

(b) Show that $8(1^3 + 2^3 + \ldots\ldots + n^3) > n(n+1)^3$.

16. If $a_1, a_2, \ldots\ldots a_n$, $b_1, b_2, \ldots\ldots, b_n$ and $c_1, c_2, \ldots\ldots, c_n$ be all positive, then show that

$$(a_1b_1c_1 + a_2b_2c_2 + \ldots\ldots + a_nb_nc_n)^2$$
$$< (a_1^2 + a_2^2 + \ldots\ldots + a_n^2)(b_1^2 + b_2^2 + \ldots\ldots + b_n^2)(c_1^2 + c_2^2 + \ldots\ldots + c_n^2).$$

6.11.1 Answers

14. (a) $\frac{3}{6250}$. (b) 7. (c) 64.

Unit-7 ❐ Matrices

Structure

7.1 Objectives

In this unit, we shall know the definition of a Matrix, Determinant of a square matrix, its different algebraic operations, singular and non-singular matrices etc.

We shall also be able

- to find adjoint of a matrix
- to find inverse of a matrix
- to perform elementary transformations
- to reduce a matrix to echelon form
- to reduce a matrix to its normal form
- to find rank of a matrix

7.2 Introduction

Matrices play a very important role in modern mathematics especially in many branches of science, technology, economics, etc. We are to deal with sets of numbers representing quantities. Matrices have various applications in every sphere of life. Since matrix is in general a two-dimensional array of numbers, we use double subscript to represent any one of its elements or entries. By convention, the first subscript refer to row and the second to column. Thus a_{ij} refers to the element in the i^{th} row, j^{th} column. No relation exists between the number of rows and the number of columns. Any matrix which has the same number of rows and columns is called a square matrix. Any square matrix with n rows and n columns is called an n^{th} order matrix.

7.3 Basic Definitions

7.3.1 Definition of Matrix

A set of $m\times n$ numbers, real or complex, arranged in a rectangular array of m rows and n columns is called a rectangular matrix or a matrix of order $m\times n$ (read as m by n matrix).

The general form of an $m\times n$ matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

The element in the i-th row and the j-th column of it is denoted by a_{ij} and this matrix is shortly written as $(a_{ij})_{m\times n}$ or $[a_{ij}]_{m\times n}, i=1,2,\ldots\ldots,m$ and $j = 1,2, \ldots\ldots, n$. A matrix is denoted by A, B, C,......etc. It is to be noted that matrix is nothing but an operator which is the management of numbers. It should be noted at the very outset that a matrix has no numerical value.

7.3.2 Square matrix, Determinant

If $m = n$, then the matrix is called a square matrix of order n (or of order m), *i.e.*,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is a square matrix of order 3.

Definition of a Determinant.

Corresponding to every square array of a square matrix A, we get a determinant. This is called the determinant of a square matrix A. This will be denoted by $|A|$ or det A. Every determinant has a numerical value.

7.3.3 Row matrix

If $m = 1$, it is called a row matrix of order n (it is also called an n-dimensional row vector).

Thus $A = [a_{11}\ a_{12}\ ......\ a_{1n}]$ is a row matrix.

7.3.4 Column matrix

If $n = 1$, it is called a column matrix of order m (it is also called an m-dimensional column vector).

$$\text{Thus } A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \text{ is a column matrix.}$$

7.3.5 Equality of matrices

If the matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

7.3.6 Zero matrix

If all the elements of a matrix be zero, it is called a null matrix or a zero matrix. A null matrix of order $m\times n$ is denoted by $O_{m\times n}$.

$$\text{Thus } O_{3\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the elements of a matrix are not zero, then it is called a non-zero matrix.

If $A = (a_{ij})_{m\times n}$ is a matrix, then $(-a_{ij})_{m\times n}$ is called the negative of the matrix A and is denoted by $(-A)$.

7.3.7 Diagonal matrix

Let $A = (a_{ij})_{n \times n}$ be a square matrix. If $a_{ij} \neq 0$ for all $i = j$ and $a_{ij} = 0$ for $i \neq j$, then A is a diagonal matrix.

Thus $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a diagonal matrix.

7.3.8 Scalar matrix

A scalar matrix is a diagonal matrix with all its diagonal elements equal $(\neq 1)$.

Thus $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ is a scalar matrix.

7.3.9 Identity matrix or unit matrix

A diagonal matrix with all its diagonal elements 1 is called a unit matrix.

Thus $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a unit matrix.

7.3.10 Triangular matrix

If all the elements of a matrix below the diagonal are 0, *i.e.*, $a_{ij} = 0$ for $i > j$, it is called an upper triangular matrix. If all the elements of a matrix above the diagonal are 0, *i.e.*, $a_{ij} = 0$ for $i < j$, it is called a lower triangular matrix.

Thus $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ is an upper triangular matrix and $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$ is a lower triangular matrix.

7.3.11 Singular and Non-singular matrix

A square matrix A is said to be singular, if det A = 0.

A square matrix A is said to be non-singular, if det $A \neq 0$.

Thus $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is a singular matrix as det A = 0 and $B = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ is a non-singular matrix as det $B \neq 0$.

7.4 Matrix Algebra

We have so far defined different types of matrices. In this section, we shall discuss the conditions under which matrices are conformable for addition, subtraction, multiplication, etc.

7.4.1 Addition of matrices

If the sizes of two matrices are same, then they are said to be conformable for addition. If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$, their sum, $A + B$ is defined as $A + B = (a_{ij} + b_{ij})_{m \times n}$.

7.4.2 Subtraction of matrices

If the sizes of two matrices are the same, then the matrices are said to be conformable for subtraction. If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$, we define $A - B$ as $A + (-B) = (a_{ij} - b_{ij})_{m \times n}$.

Example 1: Let $A = \begin{bmatrix} 5 & 2 & 3 & 4 \\ 6 & 7 & 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 6 & -1 \end{bmatrix}$.

Then $A + B = \begin{bmatrix} 5+1 & 2+1 & 3+2 & 4+3 \\ 6+2 & 7+4 & 9+6 & 0+(-1) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 5 & 7 \\ 8 & 11 & 15 & -1 \end{bmatrix}$

and $A - B = \begin{bmatrix} 5-1 & 2-1 & 3-2 & 4-3 \\ 6-2 & 7-4 & 9-6 & 0-(-1) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 4 & 3 & 3 & 1 \end{bmatrix}$.

7.4.3 Multiplication of a matrix by a scalar

Let $A=\left(a_{ij}\right)_{m\times n}$ be a matrix and c be a scalar, their product is defined as the matrix $c.A=\left(c.a_{ij}\right)_{m\times n}$.

Example : If $A=\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 9\end{bmatrix}$ and c = 3, then $cA=3.\begin{bmatrix}2 & 3 & 5\\ 6 & 7 & 9\end{bmatrix}=\begin{bmatrix}6 & 9 & 15\\ 18 & 21 & 27\end{bmatrix}$.

7.4.4 Multiplication of two matrices

Two matrices A and B are conformable for the product AB if the number of columns of A is equal to the number of rows of B, *i.e.*, if $A=\left(a_{is}\right)_{m\times p}$ and $B=\left(b_{sj}\right)_{p\times n}$, where i = 1, 2, ..., m; s = 1, 2, ..., n, j = 1, 2, ..., n, then the product $AB=(c_{ij})_{m\times n}$, where $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+......+a_{ip}b_{pj}$.

Example : If $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}_{2\times 3}$ and $B=\begin{pmatrix}2 & 3 & 4\\ 3 & 4 & 5\\ 4 & 5 & 6\end{pmatrix}_{3\times 3}$,

then $AB=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}_{2\times 3}.\begin{pmatrix}2 & 3 & 4\\ 3 & 4 & 5\\ 4 & 5 & 6\end{pmatrix}_{3\times 3}$

$$=\begin{pmatrix}1.2+2.3+3.4 & 1.3+2.4+3.5 & 1.4+2.5+3.6\\ 4.2+5.3+6.4 & 4.3+5.4+6.5 & 4.4+5.5+6.6\end{pmatrix}_{2\times 3}$$

$$=\begin{pmatrix}20 & 26 & 32\\ 47 & 62 & 77\end{pmatrix}_{2\times 3}.$$

Since $B=\begin{pmatrix}2 & 3 & 4\\ 3 & 4 & 5\\ 4 & 5 & 6\end{pmatrix}_{3\times 3}$ and $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}_{2\times 3}$, the product BA is not defined.

7.4.5 Idempotent matrix

If A be a matrix such that $A^2 = A$, then A is said to be idempotent. A matrix A is said to be nilpotent of index p, if p be the least positive integer for which $A^p = O$ (a null matrix).

7.4.6 Laws of Algebraic operations on matrices

The laws are as follows :

(i) If A and B are matrices of same order, then $A + B = B + A$, *i.e.*, matrix addition is commutative.

(ii) If A, B and C are matrices of same order then $(A + B) + C = A + (B + C)$ *i.e.*, matrix addition is associative.

(iii) If A and B are matrices of same order and c and d be scalars, then

(a) $c.(A \pm B) = cA \pm cB$

(b) $(c \pm d)A = cA \pm dA$.

(iv) If A and B are any two matrices, then

(a) AB or BA may not be defined

(b) if AB be defined, BA may not be defined and vice versa.

(c) Both AB and BA are defined if and only if the number of rows of A = the number of columns of B and the number of columns of A = the number of rows of B.

(d) In general, matrix multiplication is not commutative.

(e) If A and B are conformable for the product AB, B and C are conformable for the product BC, then $A(BC) = (AB)C$, *i.e.*, matrix multiplication is associative.

(f) If A, B and C be matrices of such orders that products AB and AC and the sum $B + C$ exist, then $A(B + C) = AB + AC$, *i.e.*, matrix multiplication is distributive over matrix addition.

(g) We also have $(A + B)C = AC + BC$ and $A(B - C) = AB - AC$, if the algebraic operations are defined.

(h) For the matrices A, B and O (null matrix) of suitable orders, $AB = O$ does not imply either $A = O$ or $B = O$.

For example, let us take $A=\begin{pmatrix}0 & 5\\0 & 6\end{pmatrix}$ and $B=\begin{pmatrix}2 & 3\\0 & 0\end{pmatrix}$. Both A and B are non-null. But $AB=\begin{pmatrix}0 & 5\\0 & 6\end{pmatrix}\begin{pmatrix}2 & 3\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}=O$, a null matrix.

(i) For matrices A, B and C of suitable orders, $AB = AC$ and $A \neq O$, does not imply $B = C$.

7.5 Transpose of a Matrix

If A be a matrix, then the matrix obtained from A by changing the rows into columns and the columns into rows is called the transpose of A. It will be denoted by A^T. So if $A=(a_{ij})_{m\times n}$, then $A^T=(a_{ji})_{n\times m}$.

Theorem : If A and B are matrices of suitable orders and c be a scalar, then

(i) $(A^T)^T = A$.

(ii) $(A \pm B)^T = A^T \pm B^T$.

(iii) $(cA)^T = cA^T$.

(iv) $(AB)^T = B^T A^T$.

Proof of (iv) : Let order of A be $m\times n$; then order of B is $n \times p$, so that the product AB is defined and order of AB is $m\times p$. Therefore order of $(AB)^T$ is $p\times m$.

Again order of B^T is $p \times n$ and that of A^T is $n \times m$.

So order of $(AB)^T$ = order of $B^T A^T = p\times m$.

So it is sufficient to prove that for any i (=1, 2, ..., p) and any j (=1, 2, ..., m), the (i, j)-th element of $(AB)^T$ = the (i, j)th element of B^TA^T.

Now the (i, j)-th element of $(AB)^T$

= the (j, i)-th element of AB

= the sum of the products of the elements of the j-th row of A with the corresponding elements of the i-th column of B

= the sum of products of the elements of the j-th column of A^T with the corresponding elements of the i-th row of B^T

= the sum of products of the elements of the i-th row of B^T with the corresponding elements of the j-th column of A^T

= the (i, j)-th element of B^TA^T.

This completes the proof.

Corollary : For three matrices A, B, C of suitable orders, we have $(ABC)^T=C^TB^TA^T$

7.6 Symmetric and Skew-symmetric Matrices

Here we shall define symmetric and skew-symmetric matrices with suitable examples and their properties will be discussed.

7.6.1 Definition

A square matrix $A=(a_{ij})_{n\times n}$ is called symmetric, if $A^T=A$, *i.e.*, if $a_{ji}=a_{ij}$ for all i and j.

Thus $A=\begin{pmatrix}2 & 5 & 3\\ 5 & 8 & 9\\ 3 & 9 & 4\end{pmatrix}$ is a symmetric matrix.

A square matrix $A=\left(a_{ij}\right)_{n\times n}$ is called skew-symmetric, if $A^T=-A$, *i.e.*, if $a_{ji}=-a_{ij}$ for all i and j. It is to be noted that for a skew-symmetric matrix $A=(a_{ij})_{n\times n}$, we have by definition, $a_{ij}=-a_{ji}$ for all $i, j = 1, 2, ..., n$.

Therefore, in particular, for all $i = 1, 2, ..., n$, $a_{ii}=-a_{ii}$ or, $2a_{ii}=0$.

Thus all the diagonal elements of a skew-symmetric matrix are zero.

Therefore $\begin{pmatrix}0 & 2 & 5\\ -2 & 0 & 7\\ -5 & -7 & 0\end{pmatrix}$ is a skew-symmetric matrix.

7.6.2 Theorems on Symmetric and Skew-symmetric matrices

Theorem 1 : Every diagonal matrix is symmetric.

Theorem 2 : For any matrix A, AA^T and A^TA are symmetric.

For, $(AA^T)^T=(A^T)^TA^T=AA^T$

and $(A^TA)^T=A^T(A^T)^T=A^TA$.

Theorem 3 : For a square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

For, $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

and $(A-A^T)^T = A^T - (A^T)^T = A^T - A = -(A-A^T)$.

Theorem 4 :

(i) If A and B are symmetric, then AB is symmetric if and only if $AB = BA$.

(ii) If A and B are skew-symmetric, then AB is skew-symmetric if and only if $AB = -BA$.

Theorem 5 :

Every square matrix can be expressed as a sum of a symmetric and a skew-symmetric matrix uniquely.

Proof : Let A be a square matrix.

Let $B = \frac{1}{2}(A+A^T)$ and $C = \frac{1}{2}(A-A^T)$; then we have $A = B + C$.

Now $B^T = \left[\frac{1}{2}(A+A^T)\right]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T + A) = B$. Therefore B is symmetric

and $C^T = \left[\frac{1}{2}(A-A^T)\right]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A-A^T) = -C$.

Therefore C is skew-symmetric.

7.7 Worked out Examples (I)

Example 1. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 5 \end{bmatrix}$ is non-singular.

Solution :

$$\text{We have det } A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 3 & 8 \end{vmatrix}, R_2' = R_2 - R_1, R_3' = R_3 + R_1$$

is non-singular.

Example 2. Find, where possible, $A+B$, $A-B$, AB and BA, with reasons when

$$A=\begin{bmatrix}4 & 2 & -1\\ 3 & -7 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}2 & 3\\ -3 & 0\\ -1 & 5\end{bmatrix}.$$

Solution: The order of A is 2×3 and that of B is 3×2. Since the orders of A and B are not same, they are not conformable for the sum $A + B$ as well as for the difference $A - B$.

As the number of columns of A = the number of rows of B = 3, they are conformable for the product AB whose order will be 2×2. Again, as the number of columns of B = the number of rows of A = 2, they are conformable for the product BA whose order will be 3×3.

$$AB=\begin{bmatrix}4 & 2 & -1\\ 3 & -7 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ -3 & 0\\ -1 & 5\end{bmatrix}=\begin{bmatrix}8-6+1 & 12+0-5\\ 6+21-1 & 9+0+5\end{bmatrix}=\begin{bmatrix}3 & 7\\ 26 & 14\end{bmatrix} \text{ and}$$

$$BA=\begin{bmatrix}2 & 3\\ -3 & 0\\ -1 & 5\end{bmatrix}\begin{bmatrix}4 & 2 & -1\\ 3 & -7 & 1\end{bmatrix}=\begin{bmatrix}8+9 & 4-21 & -2+3\\ -12+0 & -6+0 & 3+0\\ -4+15 & -2-35 & 1+5\end{bmatrix}=\begin{bmatrix}17 & -17 & 1\\ -12 & -6 & 3\\ 11 & -37 & 6\end{bmatrix}.$$

Example 3. Find a 3×1 non-zero real matrix B such that $AB = O$, where

$$A=\begin{bmatrix}1 & 3 & 2\\ 2 & 1 & 0\\ 3 & 4 & 2\end{bmatrix}.$$

Solution :

Let $B=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$. Given $AB=O$. $\therefore \begin{bmatrix}1 & 3 & 2\\ 2 & 1 & 0\\ 3 & 4 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ or, $\begin{bmatrix}x+3y+2z\\ 2x+y\\ 3x+4y+2z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

From this matrix equation, we get $x+3y+2z=0$, $2x+y=0$ and $3x+4y+2z=0$.

Solving, one non-trivial solution is $x=2, y=-4, z=5$.

$$\therefore B = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \text{ or, } B = \begin{bmatrix} 2c \\ -4c \\ 5c \end{bmatrix},$$ where c is a non-zero real number.

7.8 Model Questions (I)

1. (a) Find the values of x, y, z and t for which the matrices

$$\begin{pmatrix} x-y & y-t \\ z+t & x+z \end{pmatrix} \text{ and } \begin{pmatrix} y-z & x-z \\ 2+t & 3+y \end{pmatrix} \text{ are equal.}$$

(b) Find the matrix A, so that

$$A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -2 \\ 2 & 4 & 6 \end{bmatrix}.$$

2. Find two matrices A and B so that

$$2A-3B = \begin{pmatrix} -6 & -1 & 3 \\ -1 & 3 & 6 \\ 3 & -6 & -1 \end{pmatrix} \text{ and } A+B = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}.$$

3. Express $\begin{pmatrix} 3 & 2 & -6 \\ 0 & -1 & 4 \\ 5 & -2 & 0 \end{pmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

4. If $A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, then show that $(A+B)^2 \neq A^2 + 2AB + B^2$.

5. Evaluate $(A+I_3)(A-I_3)$, if $A+I_3 = \begin{pmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{pmatrix}$ and I_3 represents the 3×3 identity matrix.

6. Let A and B be matrices such that $AB = O$, where O is the null matrix. Does it imply that $A = O$ or $B = O$. Give an example in your support.

7. If $A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -3 & 0 \\ -1 & 5 \end{bmatrix}$, evaluating AB and BA, show that

$AB \neq BA$.

8. If A be a skew-symmetric matrix, then show that the matrix A^2 is symmetric.

9. If $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, show that $(A-2I)(A-3I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, I is the 2×2 unit matrix.

10. Show that the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A + 3I = O$, where O is the null matrix and I is the unit matrix of order 2×2.

11. If $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be two matrices, then prove that

$(A+B)(A-B) \neq A^2 - B^2$.

12. If $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then verify $A^3 = I$.

13. (a) If the matrix $A = \begin{pmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{pmatrix}$ satisfies $AA^T = I_3$, then show that

$\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$.

(b) If $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$, then show that $AA^T = I_3$.

14. Verify $(AB)^T = B^T A^T$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

15. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ is not singular.

16. Determine the matrices A and B when $A + B = 2B^T$ and $3A + 2B = I_3$ where I_3 represents the 3×3 identity matrix.

17. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of order 3.

18. Find a 3×1 non-zero real matrix B such that $AB = O$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & -5 \end{bmatrix}$.

19. If A and B are symmetric matrices, then prove that $AB - BA$ is a skew-symmetric matrix.

20. Show that the matrix $A^T BA$ is symmetric or skew-symmetric according as B is symmetric or skew-symmetric.

7.8.1 Answers

1. (a) $x = 4, y = 3, z = 2, t = 1$.

   (b) $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$.

2. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

3. $\begin{pmatrix} 3 & 1 & -\frac{1}{2} \\ 1 & -1 & 1 \\ -\frac{1}{2} & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & -\frac{11}{2} \\ -1 & 0 & 3 \\ \frac{11}{2} & -3 & 0 \end{pmatrix}$.

5. $\begin{pmatrix} -12 & -12 & 9 \\ -6 & -13 & -4 \\ 3 & -6 & -18 \end{pmatrix}$.

16. $A = B = \frac{1}{5} I_3$.

18. $B = \begin{bmatrix} 2c \\ c \\ 0 \end{bmatrix}$, where c is a non-zero real number.

7.9 Adjoint or Adjugate of a Matrix

We suppose that the students are well-equipped with Determinants. Before going to the definition and properties of adjoint of a matrix, we require the following results on determinants.

7.9.1 Some Important Results.

If A and B be square matrices of order n, then

(a) $|A| = |A^T|$

(b) $|AB| = |A|\ |B|$

(c) $|cA| = c^n |A|$, where c is a scalar.

The cofactor of the element a_{ij} of a determinant of order n is $(-1)^{i+j} \times$ determinant of order $n - 1$, obtained by omitting the row and the column containing a_{ij}. It is denoted by A_{ij} for $i, j = 1, 2, \ldots\ldots, n$.

7.9.2 Definition

Let $A = (a_{ij})_{n \times n}$ be a square matrix and let A_{ij} be the cofactor of a_{ij} in det A for $i, j = 1,2, \ldots\ldots, n$. If B be the square matrix $(A_{ij})_{n \times n}$, then the transpose of B, *i.e.*, B^T is called the adjoint or adjugate of A and is denoted by Adj A or adj A.

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then Adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

7.9.3 Properties of Adjoint:

Let $A=(a_{ij})_{n\times n}$ be a square matrix. Then

(a) $A.(\text{Adj}\,A)=(\text{Adj}\,A).\,A=|A|\,I_n$

(b) $\text{Adj}(A^T)=(\text{Adj}\,A)^T$

(c) $\text{Adj}(cA)=c^{n-1}(\text{Adj}\,A)$

(d) $|\text{Adj}\,A|=|A|^{n-1},\ |A|\neq 0$.

Proof of (a) :

We have $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\text{Adj}\,A=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \ldots & \ldots & \ldots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$.

The element in the i-th row and the j-th column of A. Adj A is

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\ldots\ldots+a_{in}A_{jn}=\begin{cases} 0, \text{ if } i\neq j \\ |A|, \text{ if } i=j \end{cases}$$

Therefore $A.\,\text{Adj}\,A=\begin{bmatrix} |A| & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |A| \end{bmatrix}=|A|\,I_n.$

Similarly it can be proved that (Adj A). $A=|A|.\,I_n$.

Proof of (b) :

Let $A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$, then $A^T=\begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}$

and $\text{Adj}\, A^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$. Again $\text{Adj}\, A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$.

Therefore $[\text{Adj } A]^T = \text{Adj } A^T$.

Proof of (c) : $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

$$\therefore \text{Adj}(cA) = \begin{bmatrix} c^{n-1}A_{11} & c^{n-1}A_{21} & \cdots & c^{n-1}A_{n1} \\ c^{n-1}A_{12} & c^{n-1}A_{22} & \cdots & c^{n-1}A_{n2} \\ \vdots & \cdots & \cdots & \cdots \\ c^{n-1}A_{1n} & c^{n-1}A_{2n} & \cdots & c^{n-1}A_{nn} \end{bmatrix} = c^{n-1}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= c^{n-1}(\text{Adj } A).$$

Proof of (d) : We have $A.(\text{Adj } A) = |A|\, I_n$

or, $|A.(\text{Adj } A)| = \big||A|I_n\big| = |A|^n |I_n| = |A|^n.1 = |A|^n$

or, $|A||\text{Adj } A| = |A|^n$

or, $|\text{Adj } A| = |A|^{n-1}$, since $|A| \neq 0$.

7.9.4 Reciprocal Matrix

Let $A = \left[a_{ij}\right]_{n\times n}$ be a non-singular square matrix. The matrix $\dfrac{\text{Adj}\, A}{(\det A)}$ is called the reciprocal matrix of A.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then reciprocal of $A = \frac{1}{(\det A)} \cdot \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

7.10 Inverse of a Matrix

Corresponding to any non-zero real number x, there exists a unique real number $\frac{1}{x}$ or x^{-1} such that $x.\frac{1}{x}$ or $x.x^{-1}$ is 1, then we say $\frac{1}{x}$ or x^{-1} is the reciprocal or inverse of x. With this idea, we can divide a real number y by another real number x as $\frac{y}{x}$ or $y.\, x^{-1}$. We like to investigate whether the operation of division of two matrices is possible. For this purpose, we require the concept of inverse of a matrix.

7.10.1 Definition

Let A be a square matrix of order n. If there exists another square matrix B of the same order such that $AB = BA = I$ where I is the unit matrix of order n, then B is called the inverse of A and is denoted by A^{-1} and A is said to be invertible.

7.10.2 Properties of Inverse Matrix

(a) The inverse of a matrix, if exists, is unique.

Proof : If possible, let B and C be two inverses of A. Then $AB = BA = I$ and $AC = CA = I$. Since matrix multiplication is associative,

therefore $C.(AB) = (CA)B$

or, $C.I = I.B \quad \Rightarrow C = B$.

(b) The inverse of a square matrix A exists, if and only if A is non-singular.

Proof: Let the inverse of a square matrix A exist and $A^{-1} = B$. Then, by definition, $AB = BA = I$.

Therefore $|AB| = |A||B| = |I| = 1$.

Hence $|A| \neq 0$, *i.e.*, is non-singular.

Next, let A be non-singular, *i.e.*, $|A| \neq 0$.

Since we know $A.(\text{adj}A) = (\text{adj}A).A = |A|.I$,

therefore $A.\dfrac{\text{adj } A}{|A|} = \dfrac{\text{adj } A}{|A|}.A = I$.

So, inverse of A exists and $A^{-1} = \dfrac{\text{adj } A}{|A|}$.

This completes the proof.

Note : The method of finding out the inverse of a non-singular matrix A is to find $\dfrac{\text{adj} A}{|A|}$, which is nothing but the reciprocal matrix of A.

(c) If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : We have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= AIA^{-1} = AA^{-1} = I$$

and $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB = B^{-1}B = I.$$

So, by definition, $B^{-1}A^{-1}$ is the inverse of AB, *i.e.*, $(AB)^{-1} = B^{-1}A^{-1}$.

(d) If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

Proof : A is an $n \times n$ square matrix. Since A is invertible, therefore A^{-1} exists.

Again $\quad (A^{-1})A = A^{-1}A = I$

and $\quad A(A^{-1}) = AA^{-1} = I$.

Therefore A is the inverse of A^{-1} by definition and so $(A^{-1})^{-1} = A$.

(e) The operations 'inverse' and 'transpose' of a matrix are commutative, *i.e.*,

$$(A^{-1})^T = (A^T)^{-1}.$$

Proof : We have

$$(A^T)(A^{-1})^T = (A^{-1}A)^T, \text{ since } (AB)^T = B^T A^T$$

$$= I^T = I.$$

Again $\quad (A^{-1})^T(A^T) = (AA^{-1})^T$

$= I^T = I$.

This shows that the inverse of A^T is $(A^{-1})^T$, *i.e.*, $(A^T)^{-1} = (A^{-1})^T$.

7.11 Orthogonal Matrix

We have defined transpose and inverse of square matrices. We have proved that the operations of transposition and inversion are commutative. Now we consider the square matrices of real numbers for which transpose and inverse coincide.

7.11.1 Definition

A square matrix A is said to be orthogonal, if $AA^T = A^T A = I$.

Thus $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an orthogonal matrix.

7.11.2 Properties of Orthogonal Matrices

(a) Orthogonal matrices are non-singular.

Let A be an orthogonal matrix.

Therefore $\quad A^T A = I \quad$ or, $|A^T A| = |I| = 1$

or, $\quad |A^T||A| = 1 \quad$ or, $|A|^2 = 1, \qquad$ since $|A^T| = |A|$.

Therefore $\quad |A| \neq 0$.

(b) The value of the determinant of an orthogonal matrix is +1 or –1.

From above, we see that $|A|^2 = 1$.

Therefore $|A| = \pm 1$.

(c) Unit matrix is orthogonal.

For, $I^T I = I.I = I$.

(d) The product of two orthogonal matrices is orthogonal.

Let A and B be two orthogonal matrices.

Therefore $A^T A = I$ and $B^T B = I$.

Now $(AB)^T AB = (B^T A^T)(AB) = B^T (A^T A)B = B^T IB = B^T B = I$.

$\therefore AB$ is orthogonal.

(e) The transpose of an orthogonal matrix is orthogonal.

Let A be an orthogonal matrix. Therefore $A^T A = I$.

We shall show that A^T is orthogonal, *i.e.*, to prove $(A^T)^T A^T = I$.

We have $\quad (A^T)^T A^T = (AA^T)^T = I^T = I$.

(f) From above, it is evident that if A is an orthogonal matrix, then $A^T = A^{-1}$.

(g) The inverse of an orthogonal matrix is orthogonal.

Let A be an orthogonal matrix. Therefore $A^T A = I$. We are to establish

$$(A^{-1})^T A^{-1} = I.$$

We have $\quad (A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I^{-1} = I$.

7.11.3 Laws of Indices of Matrices

If s is a positive integer, we define

$$A^s = AAA \text{ ...... to } s \text{ factors}$$

$$A^{-s} = (A^{-1})^s = A^{-1}A^{-1}A^{-1} \text{ ...... to } s \text{ factors}$$

$$A^0 = I.$$

Therefore $\quad (A^s)^{-1} = (AAA \text{ ...... to } s \text{ factors})^{-1}$

$$= A^{-1}A^{-1}A^{-1} \text{ ...... to } s \text{ factors} = (A^{-1})^s.$$

Similarly we have $\quad A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$, etc.

7.12 Rank of a Matrix : Elementary Transformations

To find rank of a matrix, we are to know the minor of a matrix.

A minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns.

7.12.1 Definition

A number r is said to be the rank of a non-zero $m \times n$ matrix A, if

(i) there exists at least one r-th order non-singular minor of A and

(ii) every minor of order $\geq r+1$ is singular.

Example 1: Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$. Here we see that $|A| \neq 0$. So the rank of A is 2.

Example 2: Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$. Here we see that the only second order minor $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ is singular and a first order minor (1) is non-singular as $|1| = 1 \neq 0$. Therefore rank of A is 1.

Example 3: Let $A = \begin{pmatrix} 2 & 3 & 4 & 0 \\ 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This is a 3×4 matrix. So the maximum order of a square sub-matrix of A is 3, which is min (3,4). As all the third order minors of A are singular, its rank is not 3. Then we consider for second order sub-matrix. There exists a square sub-matrix $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$ which is non-singular. So rank of A is 2.

Note : (i) For an n-th order non-singular square matrix A, rank of $A = n$.

(ii) Rank of n-th order unit matrix is n.

(iii) Rank of a null matrix is zero.

(iv) Rank of the matrix A = Rank of the matrix A^T.

(v) Rank of a matrix every element of which is non-zero real number 'a' is 1.

7.12.2 Determination of Rank by Sweep-out Process

As per definition, the process of finding the rank of a matrix discussed above is very easy. But when the size of a matrix is large and the rank is also large, the determination of rank becomes laborious and time consuming.

However, determination of rank is possible easily by transforming the matrix to echelon matrix which is known as sweep-out process. Here we are going to discuss the process step-by-step.

7.12.3 Elementary Transformations

By elementary transformation of matrix, we shall understand three transformations to the rows and three transformations to the columns of a matrix.

(i) Interchange of any two rows or two columns : The interchange of i-th row with j-th row will be denoted by $R_i \leftrightarrow R_j$. The interchange of i-th column with j-th column will be denoted by $C_i \leftrightarrow C_j$. For example,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{pmatrix} c_1 & b_1 & a_1 \\ c_3 & b_3 & a_3 \\ c_2 & b_2 & a_2 \end{pmatrix}$$

(ii) Multiplication of the i-th row (or j-th column) by $c\ (\neq 0)$. This will be denoted by $R_i \to cR_i$ (or $C_j \to cC_j$).

For example, $\begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} \xrightarrow{R_1 \to 2R_1} \begin{pmatrix} 4 & 2 \\ 5 & 4 \end{pmatrix} \xrightarrow{C_2 \to 2C_2} \begin{pmatrix} 4 & 4 \\ 5 & 8 \end{pmatrix}$

(iii) Addition of c times j-th row to i-th row, denoted by $R_i \to R_i + cR_j$. Addition of c times j-th column to i-th column, denoted by $C_i \to C_i + cC_j$.

For example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{pmatrix} 1 & 2 \\ 5 & 8 \end{pmatrix} \xrightarrow{C_1 \to C_1 + 2C_2} \begin{pmatrix} 5 & 2 \\ 21 & 8 \end{pmatrix}$

Note: When the elementary operations are applied to rows, they are called row operations and when applied to column, they are known as column operations. These operations transform a matrix from one form to another and are known as elementary transformations.

7.12.4 An important Theorem

The rank of a matrix remains unaltered under the elementary transformations.

7.12.5 Elementary matrices

A square matrix obtained from a unit matrix by any one of the elementary row or column operations is called an elementary matrix.

Notations : (i) E_{ij} is the matrix obtained from unit matrix by $R_i \leftrightarrow R_j$

(ii) $E_i\,(k)$ is the matrix obtained from unit matrix by $R_i \to k.R_i,\ k \neq 0$

(iii) $E_{ij}\,(k)$ is the matrix obtained from unit matrix by $R_i \to R_i + k.R_j$

(iv) $E'_{ij}(k)$ is the transpose of $E_{ij}(\text{k})$.

Example :

(i) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to 3R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \to C_2 + 2C_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem : Every non-singular matrix is a product of elementary matrices.

Example : Express $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as the product of elementary matrices.

Solution : Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ by $R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \text{by } R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{by } R_2 \to -\tfrac{1}{2}R_2$$

$$= I_2.$$

Hence we have $I_2 = E_{12}(1)E_{21}(-3)AE_2(-\frac{1}{2})$.

$$\therefore A = \left[E_{21}(-3)\right]^{-1}\left[E_{12}(1)\right]^{-1} I_2 \left[E_2(-\tfrac{1}{2})\right]^{-1}$$

$$= E_{21}(3)E_{12}(-1)E_2(-2).$$

Thus A is expressed as the product of elementary matrices.

7.12.6 Row Equivalent or Column Equivalent Matrices

If a matrix B is obtained from a matrix A by applying a finite number of elementary row operations, then A and B are said to be row equivalent.

On the other hand, if a matrix B is obtained from a matrix A by applying a finite number of elementary column operations, then A and B are said to be column equivalent.

7.13 Echelon form of a Matrix

Here we shall define Echelon form of a matrix with suitable examples and discuss some theorems on it.

7.13.1 Definition

A matrix is said to be in echelon form

(i) if the number of zeros preceding the first non-zero element of a row increases as we pass from row to row downwards and

(ii) all zero-rows will follow all non-zero rows.

Example : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, etc are echelon form of matrices.

7.12.2 Theorem 1: A matrix can be made row equivalent (or column equivalent) to an echelon matrix by elementary operations.

Example : Let $A=\begin{pmatrix} 0 & 0 & 2 & 3 & 0 \\ 3 & 0 & 2 & 1 & 0 \\ -3 & 3 & 2 & 2 & 0 \end{pmatrix}$.

A can be made row equivalent to an echelon matrix in the following way :

$$A=\begin{pmatrix} 0 & 0 & 2 & 3 & 0 \\ 3 & 0 & 2 & 1 & 0 \\ -3 & 3 & 2 & 2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ -3 & 3 & 2 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 3 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 3 & 4 & 3 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 3 & 0 & 2 & 1 & 0 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix} = B \text{ (an echelon matrix)}.$$

7.13.3 Theorem 2 : Two row equivalent matrices have same rank.

7.13.4 Theorem 3 : The number of non-zero rows of an echelon matrix is its rank.

In the last example, we see that B has three non-zero rows. So rank of B is 3 and since A is equivalent to B, rank of A is also 3. Thus rank of A is easily found out.

7.13.5 Row-reduced echelon matrix

A row-reduced echelon matrix will have 1 as the first non-zero element in a non-zero row and each column containing that 1 will have all other elements zero. First few rows in this form will be non-zero and the remaining, if there be any, are all zero rows.

An example of a row-reduced echelon matrix is $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

In a matrix A of this echelon form, if there be r non-zero rows, the rank of the matrix A is r as every square sub-matrix of order $(r + 1)$ contains a zero row. So the rank of the above matrix is 3.

7.13.6 Normal form

If a non-zero $m\times n$ matrix, by elementary transformation, be reduced to an equivalent matrix in which each of the first r elements along the principal diagonal is unity and every other element is zero, then this new matrix is said to be in normal form of the given matrix of order $m\times n$, and the rank of this matrix will be r.

By a series of elementary row and column operations, any matrix can be converted to the normal form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, if its rank be r.

Thus the easy way of finding the rank of a matrix is to reduce it to its normal form.

7.14 Worked out Examples (II)

Example 1: Find the inverse of $A = \begin{bmatrix} 5 & 3 \\ -2 & 2 \end{bmatrix}$.

Solution : $|A| = \begin{vmatrix} 5 & 3 \\ -2 & 2 \end{vmatrix} = 10 + 6 = 16 \neq 0$

and $\operatorname{Adj} A = \begin{bmatrix} 2 & 2 \\ -3 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -3 \\ 2 & 5 \end{bmatrix}$.

So $A^{-1} = \dfrac{1}{|A|}\operatorname{Adj} A = \dfrac{1}{16}\begin{bmatrix} 2 & -3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{-3}{16} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$.

Example 2 : Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & 6 & -2 \end{bmatrix}$.

Solution : det $A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & 6 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -5 & 2 \\ 1 & 5 & -3 \end{vmatrix} C_2' = C_2 - C_1, C_3' = C_3 - C_1$

$$= 15 - 10 = 5 \neq 0$$

and $\text{Adj}\, A = \begin{bmatrix} \begin{vmatrix} -3 & 4 \\ 6 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 1 & 6 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 6 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^T$

$$= \begin{bmatrix} -18 & 8 & 15 \\ 8 & -3 & -5 \\ 7 & -2 & -5 \end{bmatrix}^T = \begin{bmatrix} -18 & 8 & 7 \\ 8 & -3 & -2 \\ 15 & -5 & -5 \end{bmatrix}.$$

So $\quad A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{5}\begin{bmatrix} -18 & 8 & 7 \\ 8 & -3 & -2 \\ 15 & -5 & -5 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{18}{5} & \frac{8}{5} & \frac{7}{5} \\ \frac{8}{5} & -\frac{3}{5} & -\frac{2}{5} \\ 3 & -1 & -1 \end{bmatrix}.$$

Example 3 : If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2 - 4A - 5I = O$. Hence find A^{-1}.

Solution : We have $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}.$

Therefore $A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4.\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \text{ (null matrix)}.$$

Here $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \neq 0.$ So A^{-1} exists.

Since $A^2 - 4A - 5I = O$,

therefore $AAA^{-1} - 4AA^{-1} - 5IA^{-1} = O$

or, $AI - 4I - 5A^{-1} = O$

or, $5A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

Therefore $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$

Example 4 : Find A^{-1}, by elementary row operation, where $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$.

Solution : We have $A = IA$

or, $$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

or, $$\begin{bmatrix} 1 & 3 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A \quad \text{by } R_1 \to (-1)R_1$$

or, $$\begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \quad \text{by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 + R_1$$

or, $$\begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \quad \text{by } R_2 \to -\tfrac{1}{2} R_2$$

or, $$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & -3 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{7}{2} & -\frac{11}{2} & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A \quad \text{by } R_1 \to R_1 - 3R_2,\ R_3 \to R_3 + 11R_2,\ R_4 \to R_4 - 4R_2$$

or, $$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{7}{6} & \frac{11}{6} & -\frac{1}{3} & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} A \quad \text{by } R_3 \to -\tfrac{1}{3}R_3$$

or, $$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & 0 \\ \frac{7}{6} & \frac{11}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 1 \end{bmatrix} A \quad \text{by } R_2 \to R_2 + R_3,\ R_4 \to R_4 - R_3$$

or, $$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & 0 \\ \frac{7}{6} & \frac{11}{6} & -\frac{1}{3} & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A \quad \text{by } R_4 \to 6R_4$$

or, $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A \quad \text{by } R_1 \to R_1 + \tfrac{1}{2}R_4,\ R_2 \to R_2 - \tfrac{1}{3}R_4,\ R_3 \to R_3 + \tfrac{1}{6}R_4$$

Hence $$A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}.$$

Example 5: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$.

Solution : The greatest order of square sub-matrix of A is 3.

Now $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix}$ $\quad R_3' = R_3 - R_2$

$= 0,$ since R_1 and R_3 are identical.

So the rank of the matrix is not 3.

Next consider second order of square sub-matrices of A and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is such a sub-matrix.

The value of $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$. Hence the rank of A is 2.

Example 6 : Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by sweep-out operations.

Solution : $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$ by $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$\sim\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -3 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ by } R_3 \to R_3 - R_2,\; R_4 \to R_4 - R_2$$

Since A is an echelon matrix having two non-zero rows, hence the rank of the matrix is 2.

Example 7: Reduce the matrix $A=\begin{bmatrix}1 & 1 & -1\\ 2 & -3 & 4\\ 3 & -2 & 3\end{bmatrix}$ to the normal form and hence show that its rank is 2.

Solution : $A=\begin{bmatrix}1 & 1 & -1\\ 2 & -3 & 4\\ 3 & -2 & 3\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0\\ 2 & -5 & 6\\ 3 & -5 & 6\end{bmatrix}$ by $C_2 \to C_2 - C_1,\; C_3 \to C_3 + C_1$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 1\\ 3 & 1 & 1\end{bmatrix} \text{ by } C_2 \to -\tfrac{1}{5}C_2,\; C_3 \to \tfrac{1}{6}C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} \text{ by } R_2 \to R_2 - 2R_1,\; R_3 \to R_3 - 3R_1$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix} \text{ by } C_3 \to C_3 - C_2$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

$$\sim\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}, \text{which is in normal form.}$$

$\therefore$ rank of A is 2.

7.15 Summary and Keywords

Summary :

I. Different forms of Matrices.

Square Matrix, Row Matrix, Column matrix, Zero Matrix, Diagonal Matrix, Identity Matrix, Triangular Matrix.

Singular and Non-singular matrices

II. Adjoint of a Matrix, Inverse of a Matrix, Reciprocal Matrix, Orthogonal Matrix.

III. Rank of a Matrix.

A number r is said to be the rank of an $m\times n$ matrix A, if

(i) there exists at least one r-th order non-singular minor of A and

(ii) every minor of order $\geq r+1$ is singular.

IV. Echelon form of Matrices

A matrix is said to be in echelon form

(i) if the number of zeros preceding the first non-zero element of a row increases as we pass from row to row downwards and

(ii) all zero rows will follow all non-zero rows.

Keywords : Matrix, Square, Determinant, Adjoint, Inverse, Reciprocal, Orthogonal, Rank, Echelon.

7.16 Model Questions (II)

1. Find the inverse of a unit matrix of order 3.
2. Find the rank of a unit matrix of order 3.

3. Find the inverse of

(i) $A=\begin{bmatrix}3 & 2 & -1\\ 1 & 1 & -1\\ 5 & 1 & -1\end{bmatrix}$ (ii) $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 3 & 1 & -5\end{bmatrix}$ (iii) $A=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$.

4. If $A=\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$, then show that $A^2-5A-7I_2=O$, where I_2 is the 2×2 unit matrix and O is the 2×2 null matrix. Hence find A^{-1}.

5. If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, then find A^2 and show that $A^2=A^{-1}$.

6. Find A, if $\operatorname{Adj} A=\begin{bmatrix}-2 & 3 & 1\\ 6 & -8 & -2\\ -4 & 7 & 1\end{bmatrix}$ and $\det A = 2$.

7. If $A=\begin{bmatrix}1 & 0 & -1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, then show that $(A^{-1})^2+A=I$.

8. If $A=\begin{bmatrix}0 & 1 & -1\\ 2 & 0 & 0\\ 1 & -1 & 0\end{bmatrix}$ and I is the 3×3 unit matrix, then find $(A^2+I)(A+I)^{-1}$.

9. Find the rank of $A=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 4 & 6 & 8 & 10\end{pmatrix}$.

10. Find the rank of $A=\begin{pmatrix}2 & 0 & 10\\ -4 & 2 & 0\\ 8 & 0 & 2\end{pmatrix}$.

11. Find the rank of $A = \begin{pmatrix} 2 & 3 & -1 & 4 \\ 4 & 6 & -2 & 8 \\ -6 & -9 & 3 & -12 \end{pmatrix}$.

12. Find the rank of the following matrices by sweep-out process :

(i) $\begin{pmatrix} 2 & 4 & 6 & 0 \\ 0 & 8 & 10 & 2 \\ 2 & 4 & 6 & 0 \\ 0 & 8 & 10 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 2 & 2 \\ 2 & -2 & -2 \\ 6 & 2 & 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 9 & 6 & -6 \\ 0 & 9 & -9 \end{pmatrix}$.

13. Find the value of x for which the rank of the matrix $\begin{pmatrix} 4 & 8 & 4 \\ 4 & 2 & 4 \\ 2 & 0 & x \end{pmatrix}$ is 3.

14. Find the rank of the matrix $\begin{pmatrix} a & -1 & -1 \\ -1 & a & -1 \\ -1 & -1 & a \\ 1 & 1 & 1 \end{pmatrix}$, when (i) $a \neq -1$ and (ii) $a = -1$.

15. Find A, where $A^{-1} = \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{pmatrix}$.

16. Reduce the matrix A to row-reduced echelon form and hence find its rank,

where $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$.

17. Reduce the matrix $A=\begin{pmatrix}2 & 3 & -1 & -1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{pmatrix}$ to the normal form and find its rank.

18. Reduce the matrix A to row-reduced echelon form and show that $r(A) = 2$,

where $A=\begin{pmatrix}1 & 0 & -5 & 6\\ 3 & -2 & 1 & 2\\ 5 & -2 & -9 & 14\\ 4 & -2 & -4 & 8\end{pmatrix}$.

7.16.1 Answers

1. I_3. 2. 3. 3. (i) $A^{-1}=\frac{1}{2}\begin{pmatrix}-1 & 3 & 1\\ 6 & -8 & 2\\ -4 & 7 & 1\end{pmatrix}$. (ii) $A^{-1}=-\frac{1}{4}\begin{pmatrix}-13 & 6 & 1\\ 14 & -8 & -2\\ -5 & 2 & 1\end{pmatrix}$.

(iii) $A^{-1}=\frac{1}{6}\begin{pmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{pmatrix}$. 4. $A^{-1}=\frac{1}{7}\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}$. 5. $A^2=\begin{pmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{pmatrix}$.

6. $\begin{pmatrix}3 & 2 & 1\\ 1 & 1 & 1\\ 5 & 1 & -1\end{pmatrix}$. 8. $\begin{pmatrix}0 & 1 & 0\\ 0 & 1 & -2\\ -2 & 1 & -2\end{pmatrix}$. 9. 1. 10. 3 . 11. 1.

12. (i) 2. (ii) 2. (iii) 2. (iv) 2. 13. $x \neq 2$. 14. (i) 3. (ii) 1.

15. $A=\begin{pmatrix}-1 & -3 & 3 & -1\\ 1 & 1 & -1 & 0\\ 2 & -5 & 2 & -3\\ -1 & 0 & 1 & 1\end{pmatrix}$. 16. Rank = 2. 17. Rank = 3.

Unit-8 ❒ System of Linear Equations

Structure

8.1 Objectives

Here we shall learn

- to solve non-homogeneous n linear equations with n unknowns by matrix method for $n \geq 3$
- the condition for existence of unique solution of the system
- when the system has no solution
- when the system has infinitely many solutions.

8.2 Introduction

Equations involving three or more variables can be solved easily by matrix method. In this unit, we shall deal with the problems of solving a system of non-homogeneous linear equations using matrices.

We shall consider the system consisting of n linear equations with n unknowns (variables) for $n \geq 3$. We shall discuss the consistency of the system of equations.

We shall also discuss to have the solution of the Homogeneous system of equations.

8.3 Linear Equations with three unknowns

In mathematics, a system of linear equations is a collection of two or more linear equations involving the same set of variables.

Let us start with three linear equations involving three unknowns x, y, z as under :

$$\left.\begin{array}{l} a_{11}x+a_{12}y+a_{13}z=b_1 \\ a_{21}x+a_{22}y+a_{23}z=b_2 \\ a_{31}x+a_{32}y+a_{33}z=b_3 \end{array}\right\}$$

where a_{ij} $(i, j = 1,2,3)$ and b_1, b_2, b_3 are constants.

We take $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which is known as the coefficient matrix,

unknown matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then the given system of equations can be equivalently written in matrix form as $AX = B$.

If A be non-singular, A^{-1} exists and we have $A^{-1}(AX) = A^{-1}B$

or, $(A^{-1}A)X = A^{-1}B$, *i.e.*, $X=A^{-1}B$ (since $A^{-1}A=I$), which gives the solution of the given system of equations.

For example,

$$\begin{aligned} 3x+2y-z&=1, \\ 2x-2y+4z&=-2, \\ -2x+y-2z&=0 \end{aligned}$$

is a system of three equations in three unknowns. x, y and z.

Here $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -2 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

The given equations are written as a single matrix equation $AX = B$.

Now $\det A = 3(4-4) - 2(-4+8) - 1(2-4) = -8 + 2 = -6 \neq 0$. Therefore A^{-1} exists and

$$A^{-1} = \frac{1}{\det A}.(\text{Adj}A) = -\frac{1}{6}.\begin{bmatrix} 0 & 3 & 6 \\ -4 & -8 & -14 \\ -2 & -7 & -10 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = -\frac{1}{6}.\begin{bmatrix} 0 & 3 & 6 \\ -4 & -8 & -14 \\ -2 & -7 & -10 \end{bmatrix}.\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{6}.\begin{bmatrix} -6 \\ 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

Therefore we get $x = 1$, $y = -2$, $z = -2$.

A solution to a linear system is an assignment of values of the unknowns such that all the equations are simultaneously satisfied. So solution of the above system is $x = 1$, $y = -2$ and $z = -2$.

8.4 General Form

A general system of m linear equations with n unknowns can be written as

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m,$$

where $x_1, x_2, \ldots\ldots x_n$ are unknowns, $a_{11}, a_{12}, \ldots\ldots, a_{mn}$ are the coefficients of the system and $b_1, b_2, \ldots\ldots b_m$ are the constant terms.

8.4.1 Vector Equations

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots\ldots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The collection of all possible linear combinations of the vectors on the left hand side is called their span and the equations have solution just when the right hand vector is within that span. If every vector within that span has exactly one expression as a linear combination of the given left hand vectors, then any solution is unique. In any event, the span has a basis of linearly independent vectors that do guarantee exactly one expression and the number of vectors in that basis (its dimension) cannot be larger than m or n, but it can be smaller. This is important because if we have m independent vectors a solution is guaranteed regardless of the right hand side and otherwise not guaranteed.

8.4.2. Matrix Equation to General Form

The vector equation is equivalent to a matrix equation of the form

$$AX = B,$$

where A is an $m \times n$ matrix, X is a column vector with n entries and B is a column vector with m entries. Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \ldots\ldots & \ldots\ldots & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The number of vectors in a basis for the span is now expressed as the rank of the matrix. The solution set for the equations $x - y = -1$ and $3x + y = 9$ is a single point (2,3). Geometrically, these two equations represent a pair of intersecting lines, the point of intersection being (2, 3).

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots\ldots, x_n$ such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

A linear system may behave in any one of three possible ways :

(i) The system has infinitely may solutions

(ii) The system has a single unique solution

(iii) The system has no solution.

Note : In general,

1. a system with fewer equations than unknowns has infinitely many solutions, but it may have no solution. Such system is known as under determined system.

2. a system with the same number of equations and unknowns has a single unique solution.

3. a system with more equations than unknowns has no solution. Such a system is also known as over determined system.

8.4.3. Independence

The equations of a linear system are independent if one of the equations can be derived algebraically from the others. When the equations are independent, each equation contains new information about the variables and removing any one the equation increases the size of the solution set. For linear equations, logically independence is the same as linear independence. For example, $3x + 2y = 6$ and $6x + 4y = 12$ are not independent, they are the same equation when scaled by a factor of two and they will produce identical graphs. This is an example of equivalence in a system of linear equations.

8.4.4 Consistency

A linear system is inconsistent if it has no solution and otherwise it is said to be consistent. Let us consider the equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2, \\ \quad\ldots\ldots \qquad\qquad \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(1)$$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ & \ldots\ldots & & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \text{ and } B_{aug} = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{in} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ & \ldots\ldots & & \ldots\ldots & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & b_n \end{bmatrix}.$$

The matrix A is called coefficient matrix and the matrix B_{aug} is called augmented matrix. We have (a) The system (1) is consistent if and only if

Rank of A = Rank of B_{aug}

(b) The consistent system (1) has a unique solution if and only if Rank of A = the number of unknowns n.

(c) The consistent system (1) has many solutions if and only if Rank of A < the number of unknowns n.

The system (1) will be inconsistent when Rank of $A \neq$ Rank of B_{aug} and then the system will have no solution.

Here we like to discuss three cases :

Let us consider three sets of non-homogeneous equations with two unknowns :

$$\left.\begin{aligned} 5x-2y&=8 \\ 2x+y&=5 \end{aligned}\right\} \quad ...(i)$$

$$\left.\begin{aligned} x-y&=3 \\ 2x-2y&=6 \end{aligned}\right\} \quad ...(ii)$$

$$\left.\begin{aligned} 2x-y&=6 \\ 4x-2y&=-5 \end{aligned}\right\} \quad ...(iii)$$

Solving equations (i), we get $x=2$, $y=1$, So we have unique solution. The equations (ii) are reduced to a single equation viz. $x-y=3$, *i.e.*, any point on it will satisfy both these equations. So $x=4$, $y=1$ is a solution. In general, $x=3+c$, $y=c$ is a solution of these equations. Since c is arbitrary, the number of solutions is infinite. Lastly we see that the equations (iii) do not give any solution.

So the equations (i) and (ii) are consistent having unique solution or an infinitely many solutions. But the set of equations (iii) is inconsistent. They have no solution.

From Geometrical point of view, the equations (i) represent two intersecting straight lines, intersecting at the point (2,1). The equations (ii) represent two coincident straight lines. The co-ordinates of any point on it is a solution. Since a straight line consists of an infinite number of points, it has infinite number of solutions. The equations (iii) represent two parallel straight lines. They do not meet and hence they have no common point.

8.4.5 Equivalence

Two system using the same set of variables are equivalent if each of the equations in the second system can be derived algebraically from the equations in the first system and vice versa.

8.5 Homogeneous System

A system of linear equations is homogeneous if all of the constant terms are zero:

$$a_{11}x_1+a_{12}x_2+......+a_{1n}x_n=0,$$
$$a_{21}x_1+a_{22}x_2+......+a_{2n}x_n=0,$$
$$\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1+a_{m2}x_2+......+a_{nm}x_n=0$$

A homogeneous system is equivalent to a matrix equation of the form

$$AX = O$$

when A (the coefficient matrix) is an $m \times n$ matrix, X is a column vector with n entries and O is a zero vector with m entries.

8.5.1. Solution Set

Every homogeneous system has at least one solution known as the zero solution (or trivial solution) which is obtained by assigning the value zero to each of the variables.

If the system has a non-singular matrix (det $A \neq 0$), then it is also the only solution.

If the system has a singular matrix, then there is a solution set with an infinite number of solutions.

(i) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors repesenting solutions to a homogeneous system, then the vector sum $\boldsymbol{u} + \boldsymbol{v}$ is also a solution of the system.

(ii) If $\boldsymbol{u}$ is a vector representing a solution to a homogeneous system and r is any scalar, then $r\boldsymbol{u}$ is also a solution to the system.

8.5.2 Relation to non-homogeneous systems

There is a close relationship between the solution to a linear system and the solutions to the corresponding homogeneous system

$$AX = B \text{ and } AX = O.$$

Specially, if p is any specific solution to the linear system $AX = B$, then the entire solution set can be described as $\{p + \boldsymbol{v} : \boldsymbol{v} \text{ is any solution to } AX = O\}$.

8.6 Worked out Examples

Example 1 : Solve the non-homogeneous system of equations

$$x + 2y - z = 6,$$
$$3x - y - 2z = 3,$$
$$4x + 3y + z = 9.$$

Solution : Let us write the equations in matrix notation as

$$AX = B$$

i.e.,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}.$$

So the coefficient matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$ and

the augmented matrix $B_{\text{aug}} = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 3 & -1 & -2 & 3 \\ 4 & 3 & 1 & 9 \end{bmatrix}$.

$$B_{\text{aug}} \sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & -7 & 1 & -15 \\ 0 & -5 & 5 & -15 \end{bmatrix} \text{ by } R_2 \to R_2 - 3R_1 \text{ and } R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -\frac{1}{7} & \frac{15}{7} \\ 0 & -5 & 5 & -15 \end{bmatrix} \text{ by } R_2 \to -\frac{1}{7}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -\frac{1}{7} & \frac{15}{7} \\ 0 & 0 & \frac{30}{7} & -\frac{30}{7} \end{bmatrix} \text{ by } R_3 \to R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -\frac{1}{7} & \frac{15}{7} \\ 0 & 0 & 1 & -1 \end{bmatrix} \text{ by } R_3 \to \frac{7}{30}R_3.$$

This is in echelon form. Since it has 3 non-zero rows, rank of B_{aug} is 3.

Omitting the last column of B_{aug}, the echelon form of A is $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}$. This has three non-zero rows. Therefore rank of A is 3, *i.e.*, rank of A = rank of B_{aug}. So the equations are consistent. The matrix form of the given equations is now,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{15}{7} \\ -1 \end{bmatrix}$$

i.e.,

$$x + 2y - z = 6,$$

$$y - \frac{1}{7}z = \frac{15}{7},$$

$$z = -1.$$

Solving, we get $x = 1$, $y = 2$, $z = -1$.

Example 2 : Solve, if consistent, the system of equations

$$3x + y - z = 1,$$

$$x + y + z = 3,$$

$$7x + 3y - z = 5.$$

Solution : Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 7 & 3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, B_{aug} = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 \\ 7 & 3 & -1 & 5 \end{bmatrix}$.

Now the augmented matrix

$$B_{aug} \sim \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - 2R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 \to R_2 - 3R_1$$

Therefore rank of A is 2 and rank of B_{aug} is 2 < 3 (the number of unknowns). Hence this system of equations has infinitely many solutions.

The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 0 \end{bmatrix}$$

i.e., $$\left.\begin{matrix} x+y+z=3 \\ -2y-4z=-8 \end{matrix}\right\} \text{or, } \left.\begin{matrix} x+y+z=3 \\ y+2z=4 \end{matrix}\right\}$$

Solving, we get $x = z - 1$, $y = 4 - 2z$.

Putting $z = 1$, one set of solution is (0, 2, 1). Putting $z = k$ (any non-zero real number), we get the set of infinite solutions viz. $x = k - 1$, $y = 4 - 2k$, $z = k$.

8.7 Summary and Keywords

Summary :

I. System of Linear Equations written in matrix form : $AX = B$, where

$$A = \left[a_{ij}\right]_{m\times n},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}.$$

A linear system gives any one of three possible cases

(i) The system may have a single unique solution

(ii) The system may have infinitely many solutions

(iii) The system may have no solution.

II. Homogeneous System of Equations

This system is equivalent to a matrix equation of the form $AX = O$ (a null matrix)

Keywords : Linear Equation, Non-Homogeneous, Homogeneous, Solution, Consistency.

8.8 Model Questions

1. Solve $2x + y = 5$, $x - y = 0$ by matrix method.
2. Solve $x + y + z = 4$,

   $2x - y + 3z = 1$,

   $3x + 2y - z = 1$ by matrix method.
3. Test the consistency of the equations

$$x - 5y + 3z = -1,$$
$$2x - y - z = 5,$$
$$5x - 7y + z = 2.$$

4. Test the consistency of the equations and solve if possible

$$2x + y + 4z = 4,$$
$$x - 3y - z = -5,$$
$$-3x + 2y - 2z = 1,$$
$$8x - 3y + 8z = 2.$$

5. Solve the following equations :

(i) $\begin{array}{l} x + y + z = 4, \\ 2x - y + 3z = 1, \\ 3x + 2y - z = 1. \end{array}$ (ii) $\begin{array}{l} x - y = 3, \\ 2x + 3y + 4z = 17, \\ y + 2z = 7. \end{array}$ (iii) $\begin{array}{l} 2x - y + 3z = 9, \\ x + 3y - z = 4, \\ 3x + 2y + z = 10. \end{array}$

6. Find the value of k for which the system of equations $x + y + z = 2$, $2x + y + 3z = 1$, $x + 3y + 2z = 5$, $3x - 2y + z = k$ is solvable and then solve it.
7. Solve the following equations :

$$3x - 3y + 4z = 5,$$
$$2x - 3y + 4z = 4,$$
$$-y + z = 0.$$

8. If $\lambda \neq -14$, then show that the system of equations

$$5x + 2y - z = 1,$$
$$2x + 3y + 4z = 7,$$
$$4x - 5y + \lambda z = \lambda - 5$$

has a unique solution (0, 1, 1).

9. Solve : (i) $\begin{matrix} 2x-3y=0, \\ 5x+2y=0. \end{matrix}$ (ii) $\begin{matrix} 2x+4y+6z=0, \\ 3x+4y+5z=0, \\ 2x+3y+4z=0. \end{matrix}$ (iii) $\begin{matrix} x-y+2z=0, \\ 5x-3y+2z=0. \end{matrix}$

10. Find the values of μ, for which the system of equations

$$\mu x+y+z=1,$$
$$x+\mu y+z=1,$$
$$x+y+\mu z=1$$

will have (i) a unique solution, (ii) many solutions and (iii) no solution. [Hint. Follow Art. 8.4.4.]

8.8.1 Answers

1. $x = y = \frac{5}{3}$. 2. $x = -1$, $y = 3$, $z = 2$.

3. Inconsistent. 4. Consistent, $x = 1$, $y = 2$, $z = 0$.

5. (i) $x = -1$, $y = 3$, $z = 2$. (ii) $x = -2$, $y = -1$, $z = 4$. (iii) $x = 1$, $y = 2$, $z = 3$.

6. $k = -2$, $x = 1$, $y = 2$, $z = -1$.

7. $x = 1$, $y = 2$, $z = 2$.

9. (i) (0, 0) . (ii) (0, 0, 0), (1, –2, 1) and many other solutions.

(iii) (0, 0, 0), $x = 2c$, $y = 4c$, $z = c$, c is any non-zero real number.

10. (i) $\mu \neq 1$, $\mu \neq -2$. (ii) $\mu = 1$. (iii) $\mu = -2$.

Unit-9 ❒ Characteristic Equation of a Matrix

Structure

9.1 Objectives

Here we shall know the following :

- to solve the characteristic equations
- to find the eigen values
- to find the eigen vectors
- to verify every square matrix satisfies its own characteristic equation
- to compute inverse of a matrix using Cayley-Hamilton Theorem.

9.2 Introduction

A problem which arises frequently in application of linear algebra is that of finding values of a scalar parameter λ corresponding to which there exist vectors $\boldsymbol{x} \neq \mathbf{0}$, satisfying $A\boldsymbol{x} = \lambda\boldsymbol{x}$ where A is a given n-th order matrix. Such a problem is called a characteristic value or eigen value problem. Thus characteristic polynomial of a matrix, characteristic equation of a matrix are defined and we shall discuss their different properties. We shall find out the eigen values and eigen vectors of a matrix.

Lastly we shall be acquainted with Cayley-Hamilton Theorem.

9.3 Definition : Matrix Polynomial

A matrix polynomial over a field F of degree m is defined to be an expression of the form $F(x) = A_0 + A_1x + A_2x^2 + ... + A_mx^m$

where A_i (for i = 1, 2,......, m) is a square matrix over the field F of the same order. The symbol x is called indeterminate or unknown.

Example : $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} x + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x^2 + \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} x^3$ is a matrix polynomial of degree 3.

9.3.1 Definition : Equality of Matrix Polynomial

Two matrix polynomials in x over the same field F are said to be equal, if and only if the coefficients of the like powers of x are the same.

Every square matrix whose elements are algebraic polynomials in x can be expressed as a matrix polynomial in x of degree m where m is the highest power of x in any of the elements of the matrix.

Example : Let $A = \begin{bmatrix} 2+3x+x^2 & 3x^2 & 4-5x \\ 3+x^3 & 1+4x^2 & 1-3x+2x^3 \\ 1+2x-3x^3 & 5 & 2x \end{bmatrix}$ be a square matrix whose elements are polynomials. The highest power of x present in the elements is 3. Writing the elements of A as complete polynomials, we get

$$A = \begin{bmatrix} 2+3.x+1.x^2+0.x^3 & 0+0.x+3.x^2+0.x^3 & 4-5.x+0.x^2+0.x^3 \\ 3+0.x+0.x^2+1.x^3 & 1+0.x+4.x^2+0.x^3 & 1-3.x+0.x^2+2.x^3 \\ 1+2.x+0.x^2-3.x^3 & 5+0.x+0.x^2+0.x^3 & 0+2.x+0.x^2+0.x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 3 & 1 & 1 \\ 1 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -5 \\ 0 & 0 & -3 \\ 2 & 0 & 2 \end{bmatrix} .x + \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} .x^2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ -3 & 0 & 0 \end{bmatrix} .x^3,$$

which is a matrix polynomial of A.

9.4 Characteristic Equation of a Matrix

If $A = (a_{ij})_{n\times n}$ be a square matrix of order n over the field F and I be the unit matrix of order n, then

$$A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix},$$

where λ is a scalar, is called the characteristic matrix of A.

The determinant $|A-\lambda I|$ which is an ordinary polynomial in λ of degree n, with scalar coefficients is called the characteristic polynomial of A.

The equation $|A-\lambda I|=0$ is called the characteristic equation of the matrix A.

The roots of the characteristic equation, if any, are called the eigen values or latent roots or characteristic roots of the matrix A.

The roots of the characteristic equation of multiplicity r is called r-fold eigen value of A. The set of eigen values of the matrix A is called the spectrum of the matrix A.

Cor.1 The characteristic roots of a diagonal matrix are the elements of its leading diagonal.

Cor.2 Zero is an eigen value of the matrix A, if and only if A is singular.

9.4.1 Theorem : If λ is an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.

Cor.1 If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an eigen value of A.

Cor.2 If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of Adj A.

9.4.2 Theorem : If λ is an eigen value of a matrix A, then λ^2 is an eigen value of A^2.

9.4.3 Theorem : If k is a non-zero scalar, then λ is an eigen value of A, if and only if $k\lambda$ is an eigen value of $k.A$.

9.4.4 Theorem : For any square matrix A, A and A^T have the same eigen values.

9.4.5 Theorem : If X and X' be any two eigen vectors corresponding to distinct eigen values α, β respectively of a square matrix A, then X and X' are linearly independent.

9.5 Eigen Values and Eigen Vectors of a Matrix

Let A be a square matrix of order n over a field F and V_n (F) denote the vector space of all n-tuples $(x_1, x_2, ..., x_n)$ where $x_i \in F$, for $i = 1, 2, ..., n$.

Then a non-zero vector $\mathbf{X} = (x_1 x_2 .....x_n)^T$ is called an eigen vector or characteristic vector of the matrix A, if there exists an element $\lambda \in F$ such that $A\mathbf{X} = \lambda\mathbf{X}$ holds. That is to say, $\lambda\mathbf{X} = A\mathbf{X}$, where λ is a scalar.

Then, $(A\mathbf{X} - \lambda\mathbf{X}) = \boldsymbol{O}$

or, $(A - \lambda I)\, \mathbf{X} = \boldsymbol{O}$ ... (1)

Therefore, for any value of λ, the null vector $\mathbf{X} = \boldsymbol{O}$ is a solution of equation (1).

A value of λ for which equation (1) has a solution $\mathbf{X} \neq \boldsymbol{O}$, is called an eigen value of A. The corresponding solutions $\mathbf{X} \neq \boldsymbol{O}$ of equation (1) are called the eigen vectors of A corresponding to the eigen value λ.

9.5.1. Definitions

If λ be a root of $|A-\lambda I| = 0$ of multiplicity k, then k is called the algebraic multiplicity of λ and the rank of the characteristic sub-space corresponding to λ, *i.e.*, the number of linearly independent solution of $(A - \lambda I)\, \mathbf{X} = \boldsymbol{O}$, is called the geometric multiplicity of the matrix A.

9.5.2 Theorem :

λ is an eigen value of a matrix A, if and only if there exists a non-zero vector $\mathbf{X}$ such that $A\mathbf{X} = \lambda\mathbf{X}$.

Proof : Let λ be an eigen value of a square matrix A of order n. Therefore $|A-\lambda I| = 0$, *i.e.*, the characteristic matrix $(A - \lambda \mathrm{I})$ is singular. This shows that there exists a non-zero solution of the system $(A - \lambda I)\, \mathbf{X} = \boldsymbol{O}$, or $A\mathbf{X} = \lambda\, I X = \lambda\mathbf{X}$.

Conversely, let there exist a non-zero vector $\mathbf{X}$ such that $A\mathbf{X} = \lambda\mathbf{X}$. Therefore there exists a non-zero solution of $(A - \lambda \mathrm{I})\, \mathbf{X} = \boldsymbol{O} \Rightarrow$ the coefficient matrix $(A - \lambda I)$ is singular $\Rightarrow$ $|A-\lambda I| = 0$,

i.e., λ is an eigen value of A.

9.5.3 Theorem :

The eigen value corresponding to an eigen vector of square matrix is unique.

9.5.4 Theorem :

If **X** is an eigen vector of a square matrix corresponding to an eigen value λ of A, then for any non-zero scalar k, k**X** is an eigen vector of A corresponding to the same eigen value λ.

Note : For any eigen value λ of A,

$1 \leq$ geometric multiplicity of $\lambda \leq$ algebraic multiplicity of λ.

9.6 Cayley-Hamilton Theorem

Statement : Every square matrix satisfies its own characteristic equation.

[If $f(\lambda) = |A - \lambda I| = 0$ denotes the characteristic equation of a square matrix A, then $f(A) = O$ (a null matrix)].

9.7 Worked out Examples

Example 1 : Find the characteristic equation and eigen values of the matrix

$$A = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 3 & -2 \\ 2 & -2 & 3 \end{pmatrix}.$$

Solution : For a real value of λ, we have

$$A - \lambda I = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 3 & -2 \\ 2 & -2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -2 & 2 \\ -2 & 3-\lambda & -2 \\ 2 & -2 & 3-\lambda \end{pmatrix}.$$

The characteristic equation is $|A - \lambda I| = 0$

or,
$$\begin{vmatrix} 3-\lambda & -2 & 2 \\ -2 & 3-\lambda & -2 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$$

or,
$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 1-\lambda \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0.$$

On simplifying, $(1-\lambda)^2(7-\lambda) = 0$

or, $\lambda = 1,\ 1$ and 7

Therefore the eigen values are 1, 1 and 7.

Example 2 : Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Sol. For real λ, we have $A - \lambda I = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$.

The characteristic equation is $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)^2 = 0$.

Roots of the characteristic equation are 1, 1.

Therefore 1 is the only eigen value of A of multiplicity 2.

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to the eigen value 1.

Therefore **X** will be given by a non-zero solution of $(A - 1.I)\,\mathbf{X} = \boldsymbol{O}$

or,
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or,
$$\begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives $x_2 = 0$ and x_1 = any non-zero real number = c (say).

Therefore $$\mathbf{X} = \begin{bmatrix} c \\ 0 \end{bmatrix} \text{ is the eigen vector.}$$

Example 3 : Find the eigen values of $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and verify Cayley-Hamilton theorem.

Solution : We have $|A - \lambda I| = (1-\lambda)\left([2-\lambda]^2 - 1]\right)$

$$= (1-\lambda)(3-\lambda)(1-\lambda).$$

The characteristic equation is $(1-\lambda)^2(3-\lambda) = 0$

or, $$\lambda = 1, 1, 3.$$

Therefore the eigen values are 1, 1, 3.

The characteristic equation can be put as $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

i.e., we are to verify

$A^3 - 5A^2 + 7A - 3I = O.$

Now $$A^2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 4 & 4 & 1 \end{pmatrix}$$

and $$A^3 = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 4 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 13 & 0 \\ 13 & 14 & 0 \\ 13 & 13 & 1 \end{pmatrix}.$$

Therefore $$A^3 - 5A^2 + 7A = \begin{pmatrix} 14 & 13 & 0 \\ 13 & 14 & 0 \\ 13 & 13 & 1 \end{pmatrix} - \begin{pmatrix} 25 & 20 & 0 \\ 20 & 25 & 0 \\ 20 & 20 & 5 \end{pmatrix} + \begin{pmatrix} 14 & 7 & 0 \\ 7 & 14 & 0 \\ 7 & 7 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 14-25+14 & 13-20+7 & 0 \\ 13-20+7 & 14-25+14 & 0 \\ 13-20+7 & 13-20+7 & 1-5+7 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3I.$$

Hence Cayley-Hamilton theorem is verified.

Example 4 : If λ be an eigen value of an orthogonal matrix, then show that $\frac{1}{\lambda}$ is also an eigen value of it.

Solution : Let A be an orthogonal matrix.

Therefore $AA^T = A^T A = I$

i.e., $A^T = A^{-1}$.

If λ be an eigen value of A and $\mathbf{X}$ be the corresponding eigen vector, then we have

$$A\mathbf{X} = \lambda\mathbf{X}$$

or, $$A^{-1}A\mathbf{X} = \lambda A^{-1}\mathbf{X}.$$

Therefore $$A^{-1}\mathbf{X} = \frac{1}{\lambda}I\mathbf{X} = \frac{1}{\lambda}\mathbf{X}$$

i.e., $\frac{1}{\lambda}$ is an eigen value of $A^{-1} \Rightarrow \frac{1}{\lambda}$ is an eigen value of A^T.

But the eigen values of A^T and A are same as $|A^T - \lambda I| = |A - \lambda I|$.

Therefore $\frac{1}{\lambda}$ is an eigen value of A.

Example 5 : Use Cayley-Hamilton theorem to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution : The characteristic equation of A is

$$|A-\lambda I| = 0$$

or, $$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

or, $$(1-\lambda)(\lambda(1+\lambda)-1) = 0$$

or, $$(\lambda-1)(\lambda^2+\lambda-1) = 0$$

or, $$\lambda^3 - 2\lambda + 1 = 0.$$

By Cayley-Hamilton theorem, A will satisfy the equation.

Therefore we get $A^3 - 2A + I = O, \text{(a null matrix)}$. ...(1)

Now $$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1(0-1) = -1 \neq 0.$$

Therefore A^{-1} exists.

Therefore, from equation (1), we get

$$A^3A^{-1} - 2AA^{-1} + IA^{-1} = O$$

or, $$A^2(AA^{-1}) - 2(AA^{-1}) + A^{-1} = O$$

or, $$A^{-1} = 2I - A^2I = 2I - A^2$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

9.8 Summary and Keywords

Summary :

1. Characteristic Equation of a Matrix.

det $(A - \lambda I) = 0$ is called characteristic equation of a square matrix A.

The roots of this equation, *i.e.*, values of λ are called the eigen values of A.

Let A be a square matrix of order n. Then a non-zero vector $\mathbf{X} = [x_1, x_2, \ldots, x_n]^T$ is called an eigen vector of A, if there exists an element λ such that $A\mathbf{X} = \lambda\mathbf{X}$ holds *i.e.*, $A\mathbf{X} = \lambda\mathbf{X}$, where λ is a scalar.

We have $(A - \lambda I)\,\mathbf{X} = \boldsymbol{O}$. ...(1)

For any value of λ, the null vector $\mathbf{X} = \boldsymbol{O}$ is a solution of this equation.

A value of λ for which equation (1) has a solution $\mathbf{X} \neq \boldsymbol{O}$ is called an eigen value of A. The solution $\mathbf{X} \neq \boldsymbol{O}$ of (1) are called the eigen vectors of A corresponding to the eigen value λ.

II. Cayley-Hamilton Theorem.

If $f(\lambda) \equiv \det(A - \lambda I) = 0$ denotes the characteristic equation of a square matrix A, then $f(A) = O$, a null matrix.

KEYWORDS : Characteristic Equation, Eigen value, Eigen vector.

9.9 Model Questions

1. Obtain the characteristic equation of the matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

2. Find the eigen values of the following matrices :

(i) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$ (v) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$.

3. Find the eigen values and the eigen vectors of

(i) $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -1 & -4 & -1 \end{pmatrix}$

(iii) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

4. Verify Cayley-Hamilton theorem for the following matrices and find their inverses :

(i) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ (iii) $\begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$.

5. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$, then use Cayley-Hamilton theorem to show that

$$2A^5 - 3A^4 + A^2 - 4I = 138A - 403I.$$

6. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then verify that A satisfies its own characteristic equation.

Hence find A^9 and A^{-1}.

7. If A is a 3×3 matrix over the field of reals, having eigen vectors $\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

and $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ corresponding to the eigen values 1, –2, 4 respectively, find the matrix A.

8. Use Cayley-Hamilton theorem to express the matrix polynomial

$2A^4 - 7A^3 + 7A^2 - 50A - 12I$ as a linear polynomial in A, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

9.9.1 Answers

1. $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0.$

2. (i) 0. (ii) 1, 1. (iii) 1, 2, 3. (iv) 1, 1, −1. (v) 1, 2, 3.

3. (i) 1, i; $\begin{pmatrix} k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ k \end{pmatrix}$, $k \neq 0$. (ii) 0, 3, 3; $\begin{pmatrix} 0 \\ c \\ -c \end{pmatrix}$, c is non-zero real number.

(iii) 2, 2, 8; $c.\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, c.\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, c.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $c \neq 0$.

(iv) 0, 3, 15; for eigen value 0, eigen vector is $k\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $k \neq 0$.

4. (i) $\frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$. (ii) $\begin{pmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{pmatrix}$. (iii) $\frac{1}{6}\begin{pmatrix} 1 & -3 & 5 \\ 3 & -4 & 3 \\ 1 & 3 & -1 \end{pmatrix}$.

6. $\begin{pmatrix} -1 & -24 & 20 \\ 0 & -55 & 34 \\ 0 & 34 & -21 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. 7. $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$.

8. 86 A + 40.I.

Unit-10 ❐ Relations and Functions

Structure

10.1 Objectives

In this unit, we have defined and discussed the following matters:

- Reflexive Relation
- Symmetric Relation
- Transitive Relation
- Equivalence Relation
- One to One, One to Many, Many to One correspondence between the elements of two sets.

With this idea, we would be able to apply these relations properly.

10.2 Introduction

Relations and functions are two different words having different meaning mathematically. In this section, we shall study both these concepts.

Same as the relations which we have in our daily life, a kind of relation also exists in algebra. In daily life, relations are like brother and sister, friends, students and teacher and many more. In mathematics also, we see some relations like a line parallel or perpendicular to another line, 4 divides 12, 10 is greater than 8, etc. We see that while studying relations, one thing is in common which is that it requires two different objects to different objects via relations.

10.3 Relations

To understand relations, we require basic knowledge of sets.

Definition: A set is a collection of well-defined objects of particular kind.

For example : A set of English alphabets, a set of natural numbers, etc.

If A be the set of all even numbers between 1 and 15, then we write
$A = \{2, 4, 6, 8, 10, 12, 14\}$.

10.3.1 One to One correspondence and Cardinality of a Set

One to one correspondence is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set and each element of the other set is paired with exactly one element of the first set.

The cardinality of a set is the number of distinct elements of a finite set. If a set has 3 elements, then the cardinality of the set is 3. A cardinal number is a number 1, 5, 7 or 9 that tells us how many things there are in a group but not what order they are in.

Examples :

(a) The cardinal number of the null set is 0 (zero).

(b) The cardinal number of the set $\{ 1 \}$ is 1.

(c) The cardinal number of the set $\{2, 4, 5, 9, 10\}$ is 5.

(d) The cardinal number of the set $\{1, 1, 2, 2, 2, 2, 3, 3\}$ is 3.

(e) The cardinal number of the set $\{ \{1, 2\}, \{5, 8, 9\} \}$ is 2.

Two sets A and B are said to be cardinally equivalent, if there exists a mapping from A to B which is one-to-one and onto. A and B are said to be equipotent with each other. A set, which is cardinally equivalent to the set of natural numbers, is called denumerable or enumerable set.

A set is countable, if

(i) it is a finite set, or

(ii) it can be put in one-one correspondence with the set of natural numbers.

A set, which is not countable, is called uncountable set.

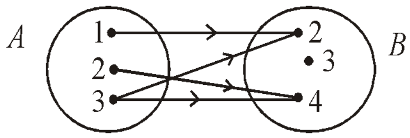
10.3.2 Relation between two sets

If we have two non-void (or null or empty) sets A and B, then the relation R from the set A to the set B is represented by ${}_aR_b$, or ${}_a\rho_b$ where a is the set of elements belonging to the set A while b belongs to the set B.

Relation from a set A to a set B is the sub-set of the cartesian product of A and B, *i.e.*, sub-set of $A\times B$.

Relation, in other way, can also be defined as a collection of ordered pair (a,b) where $a \in A$ and $b \in B$.

Example : Let us consider a set A containing the elements as $\{1, 2, 3\}$ and a set B containing the elements as $\{2, 3, 4\}$. Then the relation between the set A and the set B will be the set of any combination from A to B.



From the above diagram, we see that relation from A to B, *i.e.*, R will be the set of $\{(1, 2), (2, 4), (3, 2), (3, 4)\}$. This relation is a sub-set of the cartesian product of A and B, *i.e.*, sub-set of $A\times B$.

Let us take another example where $A = \{1, 2, 3\}$ and

$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If the relation between A and B is as:

elements of B is the square of the elements of A, then the relation is written as:

$R = \{(a, b)$: where b is the square of a and $a \in A$ and $b \in B\}$.

Here $R = \{(1, 1), (2, 4), (3, 9)\}$.

10.3.3 Total number of relations from A to B

Let the number of relations from A to B be x. Let A contain m elements and B contain n elements, the number of elements in $A\times B$ is $m\times n$.

Therefore the number of non-void sub-sets $= {}^{mn}C_1 + {}^{mn}C_2 + ... + {}^{mn}C_{mn} = 2^{mn} - 1$.

Thus, for $A = \{1, 2, 3\}$ and $B = \{x, y\}$, number of non-void sub-sets or the number of possible relations $= 2^6 - 1$.

In the above definition of relation, if $B = A$, then we say R is a binary relation or a relation on A, *i.e.*, in this case R is a subset of $A\times A$.

A relation R in a set A is said to be the universal relation if $R = A\times A$.

For example, if A = {1, 2, 3}, then

R = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)} is the universal relation R in A.

10.3.4 Different Types of Relations

(a) Reflexive relation :

Let A be a non-empty set and R be the relation defined on A. R is said to be reflexive, if ${}_aR_a$ holds for each $a \in A$.

(b) Symmetric relation :

Let A be a non-empty set and R be the relation defined on A. R is said to be symmetric, if ${}_bR_a$ holds whenever ${}_aR_b$ holds, for any two elements $a, b \in A$.

(c) Transitive relation :

Let A be a non-empty set and R be the relation defined on A. R is said to be transitive, if ${}_aR_c$ holds whenever ${}_aR_b$ and ${}_bR_c$ hold for any $a, b, c \in A$.

(d) Anti-symmetric relation :

Let A be a non-empty set and R be the relation defined on A. R is said to be anti-symmetric, if ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$.

10.3.5 Examples

(i) Consider A = {1, 2, 3}. Then the relation

R_1={(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is reflexive on A.

But R_2= {(1, 1), (2, 1), (2, 2), (3, 2)} is not a reflexive relation on A, since (3, 3) does not belong to R_2.

(ii) Let A = the set of all natural numbers. If a relation R be defined on A by '$x + y = 100$', then this is symmetric in A; for, $a + b = 100 \Rightarrow b + a = 100$. But if the relation R be defined by 'is a divisor of', then R is not symmetric as ${}_4R_{16}$ does not imply ${}_{16}R_4$.

(iii) If a relation R be defined on {1, 2, 3} given by

R = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (1, 3)}, then R is transitive; for, ${}_aR_b$ and ${}_bR_c \Rightarrow {}_aR_c$. But the relation R = {(1, 2), (2, 3), (1,3), (2, 1)} is not transitive as (1, 1) and (2, 2) are missing.

(iv) The relation R defined by '$\leq$' on the set of real numbers is anti-symmetric; for, $a \leq b$ and $b \leq a \Rightarrow a = b$.

10.3.6 Equivalence Relation

A relation R, defined on a non-empty set A, is said to be an *equivalence relation*, if and only if

(i) R is reflexive,
(ii) R is symmetric and
(iii) R is transitive.

Example : Let A be a set of all triangles in a plane. A relation R is defined as "x is similar to y, for all $x, y \in A$". Then

(i) R is reflexive, since every triangle is similar to itself.
(ii) R is symmetric, since if x is similar to y, then y must be similar to x.
(iii) R is transitive, since if x is similar to y and y is similar to z, then x will be similar to z.

Hence R is an equivalence relation.

10.3.7 Equivalence Classes

Let R be a relation on a non-empty set A. Let a be an arbitrary element of A. The elements $x \in A$ which satisfy ${}_xR_a$ form a sub-set of A which is known as an equivalence class of a in A with respect to R. This is written as $\{x : x \in A \text{ and } {}_xR_a\}$.

10.4 Functions or Mappings

Function is one of the most important concepts in Mathematics as every situation in real life is solved and analysed first by writing its mathematical equation of functions. Here we shall discuss various types of functions and their utilities. A function is like a machine which gives unique output for each input that is fed into it.

10.4.1 Function : Its Domain and Range

Functions are defined for certain inputs which are called as its domain and the outputs are called range.

Let A and B be two sets and let there exist a rule or manner or correspondence 'f' which associates to each element of A to a unique element in B, then f is called a function or mapping from A to B. It is denoted by the symbol :

$f : (A, B)$ or $f : A \to B$ or $A \xrightarrow{f} B$ which reads 'f is a function from A to B' or 'f maps A to B'.

(a) If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f-image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the pre-image of b under the function f. We write it as $f:(a,b)$ or $f:a \to b$ or $b = f(a)$.

(b) A is called the domain of f and B is called the co-domain of f.

(c) The set of all f-images of the elements of A, *i.e.*, $f(A)$ is called the range of f. If the range of f is a singleton, say $\{b\}$, then f is called a constant mapping.

(d) For any sub-set C of B, the inverse image of C under f, denoted by $f^{-1}(C)$, defined by the set $f^{-1}(C) = \{x \in A : f(x) \in C\}$.

10.4.2 Summarisation

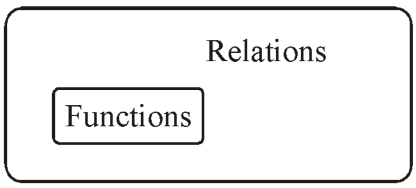
A relation f from a set A to a set B is called as the *function* or the *map* if it satisfies the following conditions :

(i) All the elements of A should be mapped with the elements of B, *i.e.*, $\forall a,\ (a, f(a)) \in f$, where $a \in A$.

(ii) The elements of set A should be uniquely mapped with the elements of set B, *i.e.*, if $(a,b) \in f$ and $(a,c) \in f$, then $b = c$, where $a \in A$ and $b,\ c \in B$.

(iii) Two or more than two elements of A may have the same image in B.

Note : Every function is a relation but every relation is not necessarily a function.



If $A = \{3, 4, 6\}$ and $B = \{5, 7, 8\}$, then $f : x \to 2x - 1$ does not define a mapping as $6 \in A$ has no image in B.

10.4.3 Equality of Mappings

Two mappings f and g from the same domain set A into the same co-domain set B are said to be equal, if $f(a) = g(a),\ \forall a \in A$. We write $f = g$.

10.4.4 Types of Mappings

1. A mapping $f : A \to B$ is called an injective mapping or one-one mapping, if different elements of the set A have different f-images in B, *i.e.*, if $a_1, a_2 \in A$, then $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ in B.

Example 1 : $f(x) = 2x + 3$, from the set of real numbers $\mathbb{R} \to \mathbb{R}$ is an injective. For $x = 1, f(x) = 5$; again for y = 5, we only have x = 1. So $x = y$ when $f(x) = f(y)$.

Example 2. $f(x) = 2x^2 + 1$ from $\mathbb{R} \to \mathbb{R}$ is not an injective. For $f(2) = f(-2) = 9$, but $2 \neq -2$. Again $f(x) = 2x^2 + 1$ from the set of natural numbers $\mathbb{N} \to \mathbb{N}$, is injective.

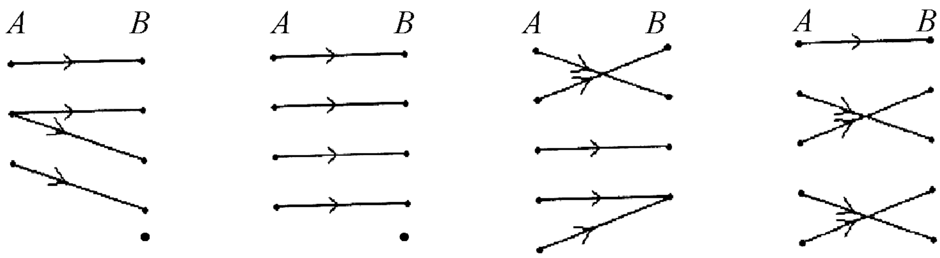
2. A mapping $f : A \to B$ is called a surjective or onto mapping, if for each $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$, *i.e.*, $f(A) = B$.

   Example : The function $f(x) = 2x$ from the set of natural numbers $\mathbb{N}$ to the set of even numbers is a surjective function. But $f(x) = 2x$ from the set $\mathbb{N}$ to $\mathbb{N}$ is not surjective, because no member of $\mathbb{N}$ can be mapped to 5 by this function. In fact, domain and co-domain of each set is very important.

3. A mapping $f : A \to B$ is called a bijective mapping or a one-to-one correspondence, if f is injective as well as surjective.

   Example : Let $\mathbb{Q}$ be the set of all rational numbers and $f : \mathbb{Q} \to \mathbb{Q}$ be the mapping given by $f(x) = 3x - 7$, for $x \in \mathbb{Q}$. Then for all $a, b \in \mathbb{Q}$ and $a \neq b \Rightarrow f(a) = 3a - 7 \neq 3b - 7 = f(b)$ and therefore f is injective. Again, for any $x \in \mathbb{Q}$, we have $\frac{x+7}{3} \in \mathbb{Q}$, such that $f\left(\frac{x+7}{3}\right) = 3.\frac{x+7}{3} - 7 = x$, which shows that f is onto and hence f is a bijective mapping.

   To understand these mappings clearly, we see the following diagram :



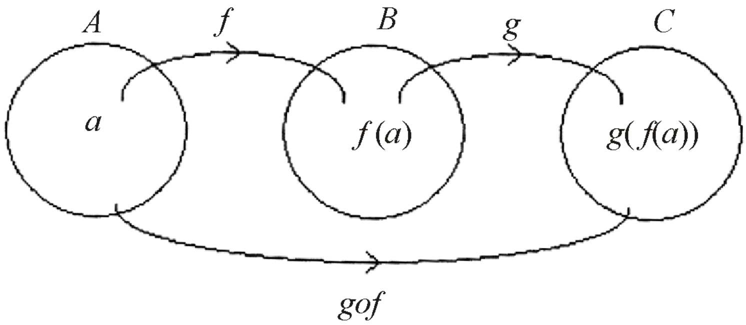
Not a function (*A* has many *B*)	Injective (Not surjective) (*B* cannot have many *A*)	Surjective (Not injective) (Every *B* has some *A*)	Bijective Injective and Surjective (*A* to *B* perfectly)

10.5 Composition of Functions

Composition of functions is the process of combining two functions where one function is performed first and the result of which is substituted in place of each variable in the other.

Let us consider three non-empty sets A, B and C and let $f : A \to B$ and $g : B \to C$ be two mappings. The composition of the mappings f and g (or the product of f and g) is the new mapping from A to C written as gof or gf, given by $(gof)(a) = g(f(a))$, for each $a \in A$.

This composition of two mappings will be clear from the following diagram :



Example : Given $f(x) = 3x + 2$ and $g(x) = 5x + 3$.

We have $f(g(x)) = f(5x+3) = 3.(5x+3) + 2 = 15x + 11$.

Again $g(f(x)) = g(3x+2) = 5.(3x+2) + 3 = 15x + 13$.

Note 1. For the function $f : A \to B$ and $g : B \to A$, $gof : A \to A$ is defined.

Note 2. Product of mappings is not, in general, commutative.

Consider the following example :

Let $f : \mathbb{R} \to \mathbb{R}$ and g : $\mathbb{R} \to \mathbb{R}$ given by $f(a) = a^2 + 1$ and $g(a) = 4a$.

Then $(fog)(2) = f(g(2)) = f(8) = 8^2 + 1 = 65$ and $(gof)(2)=g(f(2))=g(5)=20$.

$\therefore fog \neq gof$.

10.5.1 Now we shall establish the following results :

Let $f : A \to B$ and $g : B \to C$ be two maps; then

(i) If f and g be injective, then gof is injective.

(ii) If f and g be surjective, then gof is surjective.

(iii) If f and g be bijective, then gof is bijective.

Proof of (i) : Since f is injective, for $a_1, a_2 \in A, a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$. Since g is injective, $f(a_1) \neq f(a_2) \Rightarrow g(f(a_1)) \neq g(f(a_2))$, *i.e.*, $(gof)(a_1) \neq (gof)(a_2)$

$$\Rightarrow gof \text{ is injective}.$$

Proof of (ii) : Since g is onto, there exists at least one $b \in B$ such that $g(b) = c \in C$. Again since f is onto and $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. So for each $c \in C$, there exists $a \in A$ such that $c = g(b) = g(f(a)) = (gof)(a)$. Hence gof is onto.

Proof of (iii) follows from (i) and (ii).

Theorem : If $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three mappings defined on the non-empty sets $A, B, C, D,$ then $ho(gof) = (hog)of$.

Proof : As per definition of the composition, functions gof, hog, $ho(gof)$ and $(hog)of$ are defined.

Now for any $x \in A$,

we have $[ho(gof)](x) = h(gof)(x) = h[g(f(x))]$ and

$[(hog)of](x) = (hog)(f(x)) = h[g(f(x))]$.

Hence $ho(gof) = (hog)of$.

This completes the proof.

10.5.2 Identity Mapping

An identity function, also called an identity relation or identity map or identity transformation, is a function that always returns the same value that was used as its argument. In equation, the function is given by $f(x) = x$.

If A be a non-empty set, then the identity mapping is $I_A : A \to A$, *i.e.*, $I_A(a) = a$ for all $a \in A$.

We have the following :

(i) For every non-empty set A, I_A is a bijective map.

(ii) Two sets A and B are equal iff $I_A = I_B$.

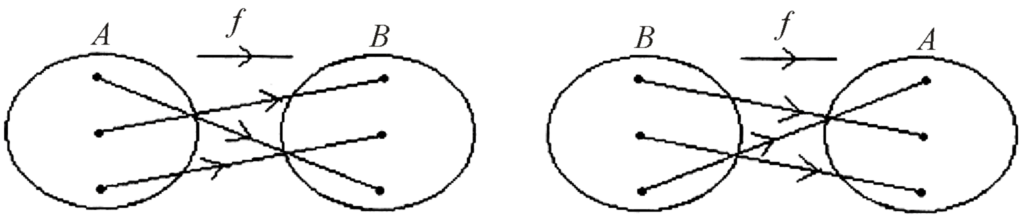
10.6 Inverse Mapping

Let A and B be non-empty sets. Let $f: A \rightarrow B$ be a mapping and an arbitrary element $b \in B$. Then the inverse of the element b is defined as a set consisting of those elements of A which has b as their images. It is denoted by $f^{-1}(b)$ and read as 'f inverse b'.

If f be one-one onto, then $f(a) = b \Leftrightarrow f^{-1}(b) = a$.

As every element $b \in B, f^{-1}$ corresponds to a unique element $a \in A$ such that $f(a) = b, f^{-1}$ is a map. A map is said to be invertible, if it possesses an inverse.

Inverse mapping will be clearly understood from the following diagrams :



Example : If $f(x) = \frac{2x+5}{3}$, find f^{-1}.

Solution : Let $y = f(x) = \frac{2x+5}{3}$ $\qquad \therefore x = \frac{3y-5}{2}$, x in terms of y.

$\therefore f^{-1}(x) = \frac{3x-5}{2}$, y is changed to x.

10.6.1 Some Theorems on Inverse Mappings

Theorem 1. The necessary and sufficient condition that a mapping be invertible is that it is one-one and onto.

Proof : First let $f: A \rightarrow B$ have inverse g. Then $fog = I_B$ and $gof = I_A$. We shall show that f is bijective. In fact $a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow gf(a_1) = gf(a_2)$

i.e., $(gof)(a_1) = (gof)(a_2) \Rightarrow I_A(a_1) = I_A(a_2) \Rightarrow a_1 = a_2 \Rightarrow f$ is one-one.

Again $b \in B \Rightarrow g(b)$ $(= a$ say$) \in A$, such that

$$f(a) = f(g(b)) = (fog)(b) = I_B(b) = b \Rightarrow f \text{, is onto.}$$

Hence f if one-one and onto.

Conversely, let $f : A \to B$ be one-one and onto. Then we have for each $b \in B$, $f(a) = b$ for some $a \in A$. f being one-one, only one such a exists.

Now let $g : B \to A$ be such a mapping that $g(b)=a$ for $b \in B$. We shall show that g is the inverse of f.

We have $(fog)(b) = f[g(b)] = f(a) = b = I_B(b)$. Thus $fog = I_B$.

Again $x \in A \Rightarrow (gof)(x) = g(f(x)) = g(y) = x = I_A(x)$, $[f(x) = y$, say$]$.

Thus $gof = I_A$. Hence g is the inverse of f.

Theorem 2. The inverse of a bijection is also a bijection.

Proof: We are to prove that if $f : A \to B$ be one-one onto map, then $f^{-1} : B \to A$ is also an one-one onto map.

Let $b_1, b_2 \in B$. Then $f^{-1}(b_1) = a_1$ and $f^{-1}(b_2) = a_2$ where $a_1, a_2 \in A$, as f is one-one onto.

$\therefore f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$, f being one-one $\Rightarrow b_1 = b_2 \Rightarrow f^{-1}$ is one-one.

Now, for any element $a \in A$, we can find an element $b \in B$ where $f^{-1}(b) = a$. This shows that f^{-1} is an onto map.

Theorem 3. If f be one-one and onto, then the inverse mapping of f is unique.

Proof : If possible, let $f : A \to B$ have two inverses f_1 and f_2. Therefore $f_1 : B \to A$ and $f_2 : B \to A$. Let b be an arbitrary element of B and let $f_1(b) = a_1$ and $f_2(b) = a_2$. Since f_1 is an inverse of f, $f_1(b) = a_1 \Rightarrow f(a_1) = b$. Similarly f_2 being an inverse of f, $f_2(b) = a_2 \Rightarrow f(a_2) = b$. But f is one-one onto mapping, hence $f(a_1) = b = f(a_2)$ and this implies $a_1 = a_2 \Rightarrow f_1(b) = f_2(b)$, hence $f_1 = f_2$, *i.e.*, f is unique.

Theorem 4. The inverse of the inverse of a function is the function itself.

Proof : Let $f : A \to B$ be invertible. Then there exists a function $g = f^{-1} : B \to A$ such that $f(a) = b \Rightarrow a = f^{-1}(b) = g(b)$ where $a \in A, b \in B$. Obviously, g is invertible *i.e.*, g^{-1} exists. Now $(fog)(b) = f[g(b)] = f(a) = b \Rightarrow fog = I_B$, *i.e.*, f is the inverse of g. So $f = g^{-1} = (f^{-1})^{-1}$.

Theorem 5. If $f : A \to B$ and $g : B \to C$ be two one-one and onto mappings, then $(gof)^{-1} = f^{-1} o\, g^{-1}$.

Proof : Let $c \in C$ be arbitrary. Then $c = g(b)$ where $b \in B$ or $b = g^{-1}(c)$. Let $b = f(a), a \in A$

$\therefore a = f^{-1}(b)$.

We have $(gof)(a) = g[f(a)] = g(b) = c$. $\therefore (gof)^{-1}(c) = a$, since gof is one-one onto map.

Now $(f^{-1} o\, g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a$.

Thus for any arbitrary c of C, $(gof)^{-1}(c) = (f^{-1} o\, g^{-1})(c)$.

Hence $(gof)^{-1} = f^{-1} o\, g^{-1}$.

This completes the proof.

10.7 Worked out Examples

Example 1. Show that the mapping $f : I \to I$, defined by $f(x) = (x+1)^2, x \in I$, where I is the set of positive integers, is one to one into.

Solution : Here domain is $I = \{1, 2, 3, 4, 5, ...\}$ and the range set is $\{4, 9, 16, ...\}$. The range set is a sub-set of the co-domain of I. Clearly it maps I into I. Since any two different elements of I map to the different elements of the range set, it is one to one.

Example 2. If $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 2$, find $f^{-1}(6)$.

Solution : If $f^{-1}(6) = x, f(x) = 6$ or, $x^2 + 2 = 6$ or, $x^2 = 4$ or, $x = \pm 2$.

So $f^{-1}(6) = \{-2, 2\}$.

Example 3. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=2x+3$.

If $A=\{x:1\le x\le 2,\ x\in\mathbb{R}\}$, find $f(A)$.

Solution : Let $a, b\in\mathbb{R}$ such that $a > b$. Since $f(a)-f(b)=2\ (a-b)>0, f$ is a strictly increasing function of x. Now $f(1)$ = 2.1 + 3 = 5 and $f(2)$ = 2.2 + 3 = 7.

$\therefore f(A)=\{5, 7\}$.

Example 4. If $f:\mathbb{R}\to\mathbb{R}$ and $f(x)=2x+3,\ x\in\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ and

$g(x)=3x-2,\ x\in\mathbb{R}$, verify that $(gof)^{-1}=f^{-1}o\ g^{-1}$.

Solution : If $y=2x+3,\ x=\dfrac{y-3}{2}$ and if $y=3x-2,\ x=\dfrac{y+2}{3}$.

$\therefore f^{-1}:\mathbb{R}\to\mathbb{R}$ and $f^{-1}(y)=\dfrac{y-3}{2}$, $g^{-1}:\mathbb{R}\to\mathbb{R}$ and $g^{-1}(y)=\dfrac{y+2}{3}$.

Now $f^{-1}o\ g^{-1}:\mathbb{R}\to R$ and

$$(f^{-1}o\ g^{-1})(y)=f^{-1}[g^{-1}(y)]=f^{-1}\left(\frac{y+2}{3}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{y+2}{3}-3\right)=\frac{y-7}{6} \qquad \text{....(1)}$$

and $(gof)^{-1}:\mathbb{R}\to\mathbb{R}$ and $(gof)(x)=g[f(x)]=g(2x+3)=3.\ (2x+3)-2=6x+7$.

$\because gof$ is one-one and onto, $\therefore (gof)^{-1}:\mathbb{R}\to\mathbb{R}=\dfrac{y-7}{6}$. ...(2)

From (1) and (2), it is verified that $(gof)^{-1}=f^{-1}o\ g^{-1}$.

10.8 Summary and Keywords

Summary

I. **Relations :** (a) Reflexive relation (b) Symmetric relation (c) Transitive relation

II. Equivalence Relation

A relation R is said to be an equivalence relation, iff R is reflexive, symmetric and transitive.

III. Functions

Let A and B be two sets and let there exist a rule or correspondence 'f' which associates to each element of A to a unique element in B, then 'f' is said to be a function from A to B.

It is denoted by $f : A \to B, \forall a \in A, f(a) = b \in B$. Again $f^{-1} : B \to A$, is called inverse of f.

Every function is a relation but every relation is not necessarily a function.

IV. Composition of Functions

This is the process of combining two functions where one function is performed first and the result of which is substituted in place of each variable in the other.

Let A, B and C be three non-empty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings.

The composition of the mappings f and g is the new mapping from A to C written as gof or gf, given by

$(gof)(a) = g\,f(a) \in C$, for each $a \in A$

Keywords : Relation, Function, Mapping, Inverse, Composition, Enumerable

10.9 Model Questions

1. Write the cardinal number of the following sets :
   (i) $\{a, b, c, x, y, z\}$.
   (ii) {the letters of the word CALCUTTA}.
   (iii) {5, 6, 6, 7, 8, 11, 6, 13, 11, 8}.
2. Give an example of a relation on the set of positive integers which is symmetric and reflexive but not transitive.
3. Give an example of a relation on the set of positive integers which is reflexive and transitive but not symmetric.
4. If R is a relation on the set of all integers defined by '$x - y$ is divisible by 6', prove that R is an equivalence relation.

5. If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$, prove that R is symmetric and transitive but not reflexive.
6. If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (1, 2), (2, 3), (3, 3)\}$, prove that R is reflexive but neither symmetric nor transitive.
7. Show that the relation '>' on the set of all integers is transitive but neither reflexive nor symmetric.
8. A relation ρ is defined on the set of all integers Z by "$_a\rho_b$ if and only if $2a + 3b$ be divisible by 5, for all $a, b \in Z$". Prove that ρ is an equivalence relation.
9. On the set of all real numbers, $_a\rho_b$ if and only if $(a + b)$ be even. Show that ρ is an equivalence relation.
10. Show that on the set of all integers the relation $_aR_b$ holds if $a - b = 10$ is neither reflexive nor symmetric nor transitive.
11. Prove that on the set of all integers the relation "$_aR_b$ holds if $ab > 0$", is symmetric and transitive but not reflexive.
12. Prove that on the set of all rational numbers the relation "$_aR_b$ holds if $a - b = 0$ or 5" is reflexive but neither symmetric nor transitive.
13. Given the relation $R = \{(1,1), (2, 2), (3, 3)\}$. Is R symmetric? Transitive?
14. Show that the mapping $f : I \to I$, the set of positive integers, defined by $f(x) = 2x^2 - 1$, is one-one and onto.
15. If the function $f : \mathbb{R} \to \mathbb{R}$, the set of real numbers, be defined by $f(x) = 3x + 1$, then show that $f^{-1}(7) = 2$ and $f^{-1}(-11) = -4$.
16. Show that the mapping $f : \mathbb{R} \to \mathbb{R}$, the set of real numbers, defined by $f(x) = x^3 - x$ is surjective but not injective.
17. Let $A = \{a, b, c\}$ and let $f : A \to A,\ g : A \to A$ be given by $f : a \to b, b \to c, c \to a$; $g : a \to a, b \to c, c \to b$. Show that $fog \neq gof$.
18. If $f : x \to 3x,\ g : x \to 5x + 2$ and $h : x \to 2x - 3$,

    show that (i) $ho(gof) = (hog)of$ (ii) $(gof)^{-1} = f^{-1}o\, g^{-1}$.

19. If the function $f:\mathbb{R}\rightarrow\mathbb{R}$, the set of real numbers, be defined by $f(x)=2x^2-3$, then show that

(i) $f^{-1}(125)=\{-8,8\}$ (ii) $f^{-1}(1)=\{-\sqrt{2},+\sqrt{2}\}$.

20. Is the mapping (i) $f:x\rightarrow 10$, one-to-one?

(ii) $f:x\rightarrow 2x+3$, one-to-one?

(iii) $f:x\rightarrow e^x$, one-to-one?

(iv) $f:x\rightarrow \cos x$, one-to-one?

10.9.1 Answers

1. (i) 6. (ii) 5. (iii) 6.
2. Let A = {1, 2, 3} and R {(1, 1), (2, 2), (1, 2), (2, 3), (3, 3), (3, 2), (2,1)}.
3. A = {1, 2, 3, 4, ...}; R = 'x is a factor of y', x and $y \in A$.

13. Yes, Yes.

20. (i) No. (ii) Yes. (iii) Yes. (iv) No.

Unit-11 ❒ Integers

Structure

11.1 Objectives

Here we shall learn the following properties of integers :

- Well-ordering property of positive integers,
- Division algorithm,
- Euclidean algorithm,
- Method of finding greatest common divisor,
- Congruence relation between integers.

12.2 Introduction

By way of counting, we have come across the numbers 1, 2, 3, ..., naturally. These numbers are called the natural numbers or counting numbers. Gradually, we had to extend the number system while subtracting one number from another and dividing one number by another number. Thus we get negative integers and fractions, besides the positive integers (natural numbers). In real life, the integers play an important role in calculating our daily activities. If there is a profit in a business we have positive numbers and if there is a loss we have negative numbers. In this unit, we shall deal with some basic properties of integers including Well-ordering property, Division Algorithm, Euclidean Algorithm, etc.

11.3 Fundamental Properties of Integers

We like to recall some fundamental properties of integers including definitions. For this purpose, let a, b, c be integers :

(i) Commutative Law of Addition : $a + b = b + a$.

(ii) Commutative Law of Multiplication : $a.b = b.a$.

(iii) Associative Law of Addition : $(a + b) + c = a + (b + c)$.

(iv) Associative Law of Multiplication : $(a.b)c = a.(b.c)$.

(v) Distributive Law : $a. (b + c)= a.b + a.c$ or $(a + b).c = a.c + b.c$.

(vi) Cancellation Law : If $a.b = a.c$ or $b.a = c.a$ then $b = c$ provided $a \neq 0$.

(vii) If $a + 0 = 0 + a = a$, then 0 is said to be the identity element for addition.

(viii) If $a.1 = 1.a = a$, then 1 is said to be the identity element of multiplication.

(ix) Additive inverse : If for an integer a, there exists an integer b such that $a + b = 0$ (the identity element), then b is called the additive inverse of a or negative of a. It is denoted by $-a$.

(x) Multiplicative inverse : A multiplicative inverse or reciprocal for a number a, denoted by a^{-1} or $\frac{1}{a}$ is a number b such that $a.b = 1$ (the multiplicative identity). The multiplicative inverse of a fraction $\frac{a}{b}$ is $\frac{b}{a}$. It is to be noted that 0 has no multiplicative inverse.

(xi) Law of Trichotomy : If a be any integer, then either $a > 0$ or $a = 0$ or $a < 0$.

(xii) Definition : a is said to be greater than b if $a - b$ is a positive integer. We denote this by $a > b$ or $b < a$.

A few important Theorems :

Let a, b, c be integers. Then

(i) $b + a = c + a \Rightarrow b = c$ and $a + b = a + c \Rightarrow b = c$.

(ii) For any integer a, $a.0 = 0.a = 0$.

(iii) If a and b be two integers such that $a.b = 0$, then either $a = 0$ or $b = 0$.

(iv) For any integer a, $-(-a) = +a$.

(v) If $a > b$, then $a + c > b + c$.

(vi) If $a > b$ and d be a positive integer, then $a.d > b.d$, $\frac{a}{d} > \frac{b}{d}$.

(vii) If $a > b$ and d be a negative integer, then $a.d < b.d$, $\frac{a}{d} < \frac{b}{d}$.

11.4 Well-ordering Property of Positive Integers

A non-empty sub-set of the set of natural numbers has a least element, *i.e.*, if S be a non-empty sub-set of the set of natural numbers $\mathbb{N}$, then there exists $n \in S$ such that $n \leq m$ for all $m \in S$. In this case, $\mathbb{N}$ is said to be well-ordered and n is the least element of $\mathbb{N}$.

Note : 1 is the first element of $\mathbb{N}$.

11.5 Divisibility of Integers

Let a ($\neq 0$) and b be two integers. If there exists an integer c such that $b = ac$, then a is said to divide b or a is said to be a divisor of b and we write $a|b$.

11.5.1 Basic Properties of Divisibility

Let a, b, c be integers such that $a \neq 0$; then

(i) $a|a$, $1|a$ and $a|0$.

(ii) If $a|b$ and $b|c$ then $a|c$.

(iii) If $a|b$ and $a|c$, then $a|(bp + cq)$ for any integers p, q.

As $a|b$ and $a|c$, there exist integers m and n such that $b = am$ and $c = an$.

$\therefore$ $bp + cq = amp + anq = a(mp + nq)$. So by definition a divides $bp + cq$.

11.6 Division Algorithm

If a be any integer and b be a positive integer, then there exist two unique integers q and r such that $a = bq + r$, where $0 \leq r < b$. (q is called quotient and r is called remainder)

Proof : Consider the sub-sets of integers $S = \{a - bt | t \in \mathbb{Z}$ and $a - bt \geq 0\}, \mathbb{Z}$ being set of all integers. We have $b \geq 1$. $\therefore$ $|a|.b \geq |a|$. So, $a + |a|.b \geq a + |a| \geq 0$.

Taking $t = -|a|$, we have $a - bt \geq 0$. This shows that S is non-empty with non-negative integers as elements.

Hence, by Well-ordering Principle, S has a least element. Let it be r. Then $r = a - bq$ for some integer q. So we find that $a = bq + r$ such that $0 \leq r$.

If $r \geq b$, then $a - (q + 1)\ b = (a - bq) - b = r - b \geq 0$ which shows that $a - (q + 1)\ b \in S$, *i.e.*, $r - b \in S$.

But $r - b < r$ which is a contradiction (r is the least element of S).

Hence $r < b$ and we thus establish that there are integers q and r such that $a = bq + r$, $0 \leq r < b$. Now we shall prove the uniqueness of q and r.

If possible, let there be another pair u and v such that $a = bu + v$, where $0 \leq v < b$. $\therefore bu + v = a = bq + r$ or $v - r = b\ (q - u)$, *i.e.*, b divides $v - r$ which is impossible, if $v - r < b$, unless $v - r = 0$, *i.e.*, $v = r$. Consequently, $b\ (q - u) = 0$, but $b \neq 0$.

$\therefore\ q - u = 0$ or $u = q$. Hence q and r are unique.

11.7 Greatest Common Divisor

Definition : A non-zero integer d is said to be a common divisor of two integers m and n, if $d | m$ and $d | n$.

A positive integer g is called the greatest common divisor (*g.c.d*) of two integers m and n, if

(i) g be a common divisor of m and n, and

(ii) every common divisor of m and n is a divisor of g.

The greatest common divisor of m and n is usually denoted by gcd (m,n) or simply by (m,n) This is also called highest common factor (hcf).

Note 1. $(m, n) = (-m, n) = (m, -n) = (-m, -n) = (|m|, |n|)$.

Note 2. If $m = 0$, $n \neq 0$, then gcd $(m, n) = n$.
Note 3. If $m = 0$, $n = 0$, then gcd $(m, n) = 0$.
Note 4. If m and n are mutually prime, then gcd $(m, n) = 1$.

11.7.1. Existence of Greatest Common Divisor

Theorem : Let m and n be two integers not both zero. Then (m, n) must exist and if $d = (m, n)$, then there exist integers s and t such that $d = sm + tn$.

Proof: Consider the set $S = \{km + ln \mid k,\ l \in \mathbb{Z} \text{ and } km + ln > 0\}$.

We have $m^2 + n^2 > 0$ and hence $m^2 + n^2 \in S$. Thus S is non-empty containing only positive integers. Hence, by Well-ordering Principle, it contains a least element d (say) so that $d = sm + tn$ for some integers s and t and $0 < d \leq x$ for any $x \in S$.

By division algorithm, there exist integers q and r such that $m = dq + r$ where

$$0 \leq r < d. \qquad ...(1)$$

$$\therefore r = m - dq = m - (sm + tn)q = (1 - qs)m + (-qt)n.$$

If $r > 0$, then r will belong to S which contradicts the fact that d is the least element of S. Hence, from (1), we get $r = 0$. $\therefore d \mid m$. Similarly we can show that $d \mid n$.

We further assume that c belongs to S and $c \mid m$ and $c \mid n$ so that $c \mid sm$ and $c \mid tn$.

Hence $c \mid (sm + tn)$, which implies $c \mid d$ and therefore $d = (m, n)$.

11.8 Euclidean Algorithm (Method of finding G.C.D.)

Since gcd (m, n) = gcd $(|m|, |n|)$, therefore we assume both m and n to be positive integers $(m > n)$.

By division algorithm, we have $m = nq + r$ where $0 \leq r < n$ ...(1)

If $r = 0$, $m = nq$ and hence n is a divisor of m which is gcd of m and n.

$$\therefore (m, n) = n \qquad ...(2)$$

If $r \neq 0$, let $d = (m, n)$ and further suppose that $d' = (n, r)$.

Now $d \mid m$ and $d \mid n$, $\therefore d \mid (m - nq)$, *i.e.*, $d \mid r$, from (1).

Then d is the common divisor of n and r.

$\therefore d \mid d'$, as $d' = (n, r)$. Similarly we see that $d' \mid d$. $\therefore d = d'$.

Hence $(m, n) = (n, r)$.

Again applying division algorithm to n and r, we have $n = rq_1 + r_1$, where $0 \leq r_1 < r$.

If $r_1 = 0$, then $(n, r) = r$.

If $r_1 \neq 0$, we have as before, $(m, n) = (n, r) = (r, r_1)$.

We continue this process so long as the remainder remains non-zero. Thus

$$\begin{aligned} m &= nq + r, &\text{where } &0 \leq r < n \\ n &= rq_1 + r_1, &\text{where } &0 \leq r_1 < r \\ r &= r_1q_2 + r_2, &\text{where } &0 \leq r_2 < r_1 \\ r_1 &= r_2q_3 + r_3, &\text{where } &0 \leq r_3 < r_2 \\ r_2 &= r_3q_4 + r_4, &\text{where } &0 \leq r_4 < r_3 \\ &\ldots\ldots &\ldots\ldots &\ldots\ldots \end{aligned}$$

Since $n > r > r_1 > r_2 > r_3 > r_4 > \ldots\ldots$ and n is a fixed positive integer, this process must terminate after finite number of steps. Therefore, at one stage, let after $(k+1)^{th}$ step, the remainder must be zero. So we get the steps as :

$$\begin{aligned} r_{k-3} &= r_{k-2}\,q_{k-1} + r_{k-1}, &\text{where } &0 \leq r_{k-1} < r_{k-2} \\ r_{k-2} &= r_{k-1}\,q_k + r_k, &\text{where } &0 \leq r_k < r_{k-1} \\ \text{and } r_{k-1} &= r_k q_{k+1} + 0. \end{aligned}$$

Therefore we have $r_k | r_{k-1}$. $\quad \therefore r_k = (r_{k-1}, r_k)$.

Thus we have $(m, n) = (n, r) = (r, r_1) = (r_1, r_2) = \ldots\ldots = (r_{k-1}, r_k) = r_k$.

Hence gcd $(m, n) = r_k$.

Example : Find gcd (360, 75).

Solution : $360 = 75 \times 4 + 60$

$75 = 60 \times 1 + 15$

$60 = 15 \times 4 + 0$

Hence gcd (360, 75) = 15.

Note : Here $15 = 75 - 60 \times 1 = 75 - (360 - 75 \times 4) \times 1 = (-1) \times 360 + 5 \times 75$.

11.9 Congruence relation between Integers

In this section, we shall give the definition of congruence relation between integers and discuss a few basic properties.

11.9.1 Definition

Let m be a positive integer, a and b be integers. We say a is congruent to b modulo m when m is a divisor of $(a - b)$.

We write $a \equiv b \pmod{m}$.

If m does not divide $(a - b)$, we say that a is not congruent to b modulo m which we write $a \not\equiv b \pmod{m}$.

Example : As 7 divides 45 – 3, we write $45 \equiv 3 \pmod{7}$ and we say 45 is congruent to 3 modulo 7. On the other hand, as 7 does not divide 45 – 4, we write 45   4 (mod 7).

11.9.2 Some Important Properties

Let a, b, c be integers and m be a positive integer. Then

(i) $a \equiv a \pmod{m}$.

(ii) If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.

(iii) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

(iv) If $a \equiv b \pmod{m}$, then for any integer c, $a \pm c \equiv (b \pm c) \pmod{m}$ and $ac \equiv bc \pmod{m}$.

11.9.3 Some Important Theorems

Let a, b, c, d be integers and m be a positive integer. Then

(i) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv (b + d) \pmod{m}$.

(ii) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

(iii) If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$, for any positive integer n.

Proof of (i) : Since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then m is a divisor of $a - b$ as well as of $c - d$. Hence there must exist two integers, say, k and t such that $a - b = mk$ and $c - d = mt$.

Then $(a - b) + (c - d) = m(k + t)$

or, $(a + c) - (b + d) = m(k + t)$, *i.e.*, m is a divisor of $(a + c) - (b + d)$ and hence $a + c \equiv (b + d) \pmod{m}$.

Proof of (ii) : we have $a - b = mk$ and $c - d = mt$. $\therefore$ $ac - bc = mkc$ and $bc - bd = mtb$. So $(ac - bc) + (bc - bd) = mkc + mtb = m(kc + tb)$

or, $ac - bd = m(kc + tb)$, *i.e.*, m is a divisor of $ac - bd$. Hence $ac \equiv bd \pmod{m}$.

Proof of (iii) : The theorem is true for $n = 1$. We assume that it is true for some positive integer $k \geq 1$, *i.e.*, we assume $a^k \equiv b^k \pmod{m}$.

We are given $a \equiv b \pmod{m}$. $\therefore$ $a^k.a \equiv b^k.b \pmod{m}$ or $a^{k+1} \equiv b^{k+1} \pmod{m}$.

Thus the theorem is true for the positive integer $k + 1$.

Hence the theorem is true for any positive integer n.

Example : We know that $10 \equiv 1 \pmod 3$. So $10^5 \equiv 1^5 \pmod 3$,
i.e., $100000 \equiv 1 \pmod 3$.

Note : The converse of this theorem is not true.
For, $6^2 \equiv 2^2 \pmod 8$, but $6 \not\equiv 2 \pmod 8$.

Theorem 1. The relation 'congruence modulo m' is an equivalence relation in the set of integers.

Proof : For fixed integer m, $a \equiv b \pmod m$, if $m|(a-b)$.

If a be an integer, $a - a = 0$ and $m|0$. $\therefore a \equiv a \pmod m$. Therefore the relation is reflexive.

If a and b be integers such that $a \equiv b \pmod m$,

then $m|(a-b) \Rightarrow m|-(b-a) \Rightarrow m|(b-a)$.

$\therefore$ $b \equiv a \pmod m$. Therefore the relation is symmetric.

If a,b,c, be positive integers such that $a \equiv b \pmod m$ and $b \equiv c \pmod m$

then $m|(a-b)$ and $m|(b-c) \Rightarrow m|\{(a-b)+(b-c)\} \Rightarrow m|(a-c)$.

$\therefore$ $a \equiv c \pmod m$. So the relation is transitive. This completes the proof.

Theorem 2. $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ is a polynomial where a_i $\{i = 0,1,2......,(n-1),n\}$ are integers. If $a \equiv b \pmod m$, then $f(a) \equiv f(b) \pmod m$.

Proof : Given $a \equiv b \pmod m$. $\therefore$ $a^k \equiv b^k \pmod m$, for any positive integer k.

$\therefore$ $a_ka^k \equiv a_kb^k \pmod m$, where $a_k \in I$.

Putting $k = 0,1,2, ... , n$ and adding the congruence relations, we have

$$\left(a_0 + a_1a + a_2a^2 + ... + a_na^n\right) \equiv (a_0 + a_1b + a_2b^2 + ... + a_nb^n) \pmod m$$

i.e., $f(a) \equiv f(b) \pmod m$.

Theorem 3. Let a,b, c be integers and m be a positive integer. Then

(i) $ac \equiv bc \pmod m$, if and only if $a \equiv b \ (\text{mod}\ \frac{m}{\gcd(c, m)})$.

(ii) If $ac \equiv bc \pmod m$ and $\gcd(c,m) = 1$, then $a \equiv b \pmod m$.

Proof of (i) : First suppose that $ac \equiv bc \pmod{m}$. $\therefore\ ac - bc = qm$, for some integer q. Let $(c, m) = d$. Also let us assume $c = dr$ and $m = ds$ where r and s are such integers that $(r, s) = 1$.

Now $adr - bdr = qds$, or $a - b = \dfrac{qs}{r}$ ...(1)

Since $a - b$ is an integer, $r|qs$. But $(r, s) = 1$. So $r|q$ and hence $\dfrac{q}{r} = k, k$ is an integer. So, from (1), $a - b = ks = k.\dfrac{m}{d}$, *i.e.*, $a \equiv b \left(\text{mod}\,\dfrac{m}{d}\right)$.

This proves $a \equiv b \left(\text{mod}\,\dfrac{m}{\gcd(c,m)}\right)$.

Conversely, let $a \equiv b \left(\text{mod}\,\dfrac{m}{\gcd(c,m)}\right) = b\left(\text{mod}\,\dfrac{m}{d}\right)$.

Then $a - b = k.\dfrac{m}{d}$, for some integer k; or, $ac - bc = k.\dfrac{m}{d}.c = km\dfrac{c}{d} = kmr\ (\because c = dr)$.

Hence m divides $ac - bc$, *i.e.*, $ac \equiv bc \pmod{m}$.
This completes the proof.
Proof of (ii) directly follows from (i).

11.10 Worked out Examples

Example 1 : Prove that gcd (50, 40) = 5 × gcd (10, 8).

Solution : We have $10 = 1 \times 8 + 2$
$8 = 4 \times 2 + 0$. So, gcd (10, 8) = 2.

Again $50 = 1 \times 40 + 10$
$40 = 4 \times 10 + 0$. So, $\gcd(50, 40) = 10 = 5 \times 2 = 5 \times \gcd(10, 8)$.

Example 2 : Find gcd (8470, 945) and find two integers s and t such that $\gcd(8470, 945) = 8470.s + 945.t$.

Solution : We have $8470 = 8 \times 945 + 910$
$945 = 1 \times 910 + 35$
$910 = 26 \times 35 + 0$.

So, gcd (8470, 945) = 35 = 945 – 1 × 910 = 945 – 1 × (8470 – 8 × 945)
= 8470 × (–1) + 945 × 9,
so that $s = -1$ and $t = 9$.

Example 3 : Find all integers , such that $9 \equiv n^2 (\bmod n)$.

Solution : $9 \equiv n^2 (\bmod n)$, then n divides $9 - n^2$. Hence n divides 9. But given that $n \ge 2$.
Therefore $n = 3, 9$.

11.11 Summary and Keywords

Summary

I. Well-ordering property of positive integers.
If S be a non-empty sub-set of the set of natural numbers, then there exists $n \in S$, such that $n \le m, \forall m \in S$.

II. Division Algorithm.
If a be any integer and b is a positive integer, there exist two unique integers q and r such that $a = bq + r$, where $0 \le r < b$.
q is called the quotient and r is called the remainder.

III. Euclidean Algorithm.
This is the method of finding greatest common divisor (gcd) of two positive integers.

IV. Congruence relation between integers.
Let m be a positive integer, a and b are integers. We say 'a is congruent to b modulo m' when m is a divisor of $(a - b)$.
We write $a \equiv b (\bmod\ m)$.

Keywords : Integer, Prime number, Division Algorithm, G.C.D., Congruence.

11.12 Model Questions

1. Prove that gcd (80, 60) = 10 × gcd (8, 6).
2. (i) Find gcd(2169, 135) and find s and t such that
gcd (2169, 135) = $2169s + 135t$.
(ii) Find gcd (792, 385) and express it in the form $792.m + 385.n$.

3. Find all the integers $n \geq 3$, such that $11 = n^2 (\text{mod}\, n)$.

4. Verify : (i) $653 \equiv 5$ (mod 8) (ii) $35 \equiv (-7)(\text{mod } 6)$ (iii) $24 \not\equiv 5(\text{mod } 6)$.

5. Show that (i) $2^{41} \equiv 3(\text{mod } 23)$ (ii) $19^{20} \equiv 1(\text{mod } 181)$.

6. Find all integers k if the following be true :

   (i) $9 \equiv 3k(\text{mod } 5)$ (ii) $3k \equiv k$ (mod 5).

7. Find all the integers $m \geq 2$, such that (i) $11 \equiv (-4)(\text{mod } m)$

   (ii) $-5 \equiv 12(\text{mod } m)$ (iii) $m^2 \equiv 5m(\text{mod } 12)$, $2 \leq m < 15$.

8. Show that $(a+b)^n \equiv (a^n + b^n)$ (mod n), where a, b and n are positive integers.

9. Show that $(n-1) \mid (n^p - 1)$, where p is a positive integer and $n \geq 2$.

10. Prove that $(n-1)^2 \mid (n^p - 1)$, for $n > 2$ and p is a positive integer such that $(n-1) \mid p$.

11.12.1 Answers

2. (i) gcd = 9, s = 1, t = – 16. (ii) gcd = 11; m = –17, n = 35.
3. 11.
6. (i) $k = 5m + 3$, m is any integer. (ii) k = 5m, m is any integer.
7. (i) 3, 5, 15. (ii) 17. (iii) 5, 8, 9, 12.

Unit-12 ❐ Principle of Mathematical Induction And Fundamental Theorem of Arithmetic

Structure

12.1 Objectives

We would be able to prove all mathematical statements which hold for all positive integers n. Also in this unit we shall know the Unique Factorisation Theorem.

12.2 Introduction

If we are asked to prove $1+2+3+.....+n=\frac{n(n+1)}{2}$ for all positive integers n, we can atmost verify its truth for n = 1,2,3,4,....by putting these values on both sides of it. But it is impossible to prove it for all positive integers in this manner. To prove this result for all positive integers n, we follow a method of proof which in known as 'Principle of Mathematical Induction'. In this unit, we shall discuss this method. We consider another problem 'factorise 36'. We easily get 36 = 2.2.3.3, *i.e.*, we write 36 as product of primes. Here we shall state fundamental theorem of arithmetic which proves that any positive integer greater than I can be written as a product of primes uniquely.

12.3 Statement of Principle of Mathemtical Induction

This is nothing but a mathematical technique which is used to prove a statement or a formula or a theorem is true for every natural number. This technique involves two steps to prove a statement.

Step 1. It proves that a statement is true for the initial value.

Step 2. It proves that if a statement is true for the n^{th} iteration then it is also true for the $(n+1)^{th}$ iteration. In some cases, this happens to be the only method.

How to do it :

Let $P(n)$ be a mathematical statement about non-negative integers n and n_0 be a non-negative integer.

Let $P(n)$ be true for $n = n_0$, *i.e.*, $P(n_0)$ is true. Assuming $P(m)$ is true, where $m (\geq n_0)$ is a particular value of n, if we can prove that $P(m+1)$ is also true, then $P(n)$ is true for all integers $n (\geq n_0)$.

12.4 Equivalent Statement of Mathematical Induction

There is an equivalent statement of the Principle of Mathematical Induction. This is called Second Principle of Mathematical Induction which runs as follows :

Let $P(n)$ be a mathematical statement over non-negative integers n and n_0 be a fixed non-negative integer. Suppose $P(n_0)$ is true. If, for any integer $m \geq n_0$, $P(n_0), P(n_0+1), P(n_0+2), \ldots\ldots, P(m)$ are true implies that $P(m+1)$ is ture, then $P(n)$ is true for all $n \geq n_0$.

12.5 Fundamental theorem of Arithmetic (Unique Factorisation Theorem)

Before going to the main theorem, we require some knowledge of prime numbers and a few relevant theorems.

Definition : Prime Numbers.

A prime number is an integer, greater than 1, which has no factors other than itself and 1.

An integer > 1 which is not a prime is called composite.

Note : The smallest prime number is 2.

Theorem 1. If p be a prime and a be a non-zero integer, then $(a, p) = 1$ or $(a, p) = p$.

Proof : By definition of prime, only divisors of p are (± 1), $(\pm p)$.

So, if p does not divide a, then $(a, p) = 1$, on the other hand if p divides a, then $(a, p) = p$.

Theorem 2. (Euclidean Lemma) : If p be a prime integer and $p|a.b$, where a and b are integers, then either $p|a$ or $p|b$.

Proof : If $p \mid a$, the theorem is proved. So let p is not a divisor of a. Let $(p, a) = s \Rightarrow s \mid p$ and $s \mid a$. Since p is a prime integer, the only positive divisors of p are 1 and p. Hence either $s = 1$ or $s = p$. But s cannot be equal to p, because p is not a divisor of a. $\therefore$ $s = 1$. $\therefore$ $(p, a) = 1$, *i.e.*, p and a are mutually prime.

Hence we have $p|a.b \Rightarrow p|b$.

Corollary : If p be a prime and $p|a_1.a_2.a_3....a_n$, then $p|a_k$ for some k, where $1 \le k \le n$.

Theorem 3. (Euclid's Theorem) : The number of primes is infinite.

Proof : If possible, let the number of primes be finite and equal to n. Let them arrange in increasing order of magnitude as $p_1, p_2, p_3, \ldots\ldots, p_n$.

Let $p_1.p_2.p_3.\ldots.p_n = c$ and consider the integer $(c + 1)$. Since none of p_i $(i = 1, 2, 3, \ldots, n)$ is a divisor of $(c + 1)$, we conclude that either $(c + 1)$ is a prime which is greater than p_n or $(c + 1)$ has a prime factor which is greater than p_n. But this contradicts our hypothesis that p_n is the greatest prime.

Hence the number of primes is infinite.

12.5.1 Statement of Fundamental Theorem of Arithmetic

Every integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers and this representation is unique.

Examples :

1. $42 = 2 \times 3 \times 7$
2. 7 already a prime number
3. $22 = 2 \times 11$
4. $12 = 2 \times 2 \times 3$

It is to be noted that all are of unique combination.

Proof of the Theorem : The proof uses Euclid's Lemma : "if a prime p divides the product of two natural numbers a and b, then p divides a or divides b." We need to show that every integer greater than 1 is either prime or a product of primes. For the base case, note that 2 is prime, there is nothing to prove. Otherwise, there are integers a and b where $n = ab$ and $1 < a \le b \le n$. By the induction hypothesis, $a = p_1.p_2.p_3.........p_j$ and $b = q_1.q_2.q_3.......q_k$ are product of primes. But then $n = ab = p_1.p_2.p_3.....p_j.q_1.q_2.q_3....q_k$ is a product of primes.

Proof of uniqueness :

Assume that $s > 1$ is the product of prime numbers in two different ways :

$s = p_1.p_2.p_3.......p_m$

$= q_1.q_2.q_3........q_n.$

We will show $m = n$ and that the q_i are a rearrangement of p_i. As p_1 divides s, Euclid's Lemma implies that p_1 divides one of the q_j, relabeling the q_j, if necessary, say that p_1 divides q_1. But q_1 is prime. So its only divisors are itself and 1.

$$\therefore\ p_1 = q_1, \text{ so that } \frac{s}{p_1} = p_2.p_3......p_m = q_2.q_3.......q_n.$$

Reasoning in the same way, p_2 must equal one of the remaining q_j. Relabeling again, if necessary, say $p_2 = q_2$. Then $\frac{s}{p_1p_2} = p_3.......p_m = q_3.............q_n.$

This can be done for each of the m p_i's showing that $m \le n$ and every p_i is a q_j.

Applying the same argument with the p's and q's reversed, shows $n \le m$.

Hence $m = n$ and every q_j is a p_i.

This completes the proof.

Note : This theorem is one of the main reason why 1 is not considered a prime number. If 1 were prime, then factorisation into primes would not be unique.

For, $6 = 3 \times 2 \times 1$

$= 3 \times 2 \times 1 \times 1 \times 1 \times 1 =$ ....etc.

12.6 Worked out Examples

Example 1. Prove that $3^n - 1$ is divisible by 2 for all positive integers n.

Solution : Step 1. Let $P(n)$ be the given statement. For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is divisible by 2.

So $P(1)$ is true.

Step 2. Let us assume that $P(n)$ is true $n = m$, a particular value of n, *i.e.*, we assume $P(m)$ is true, *i.e.*, $3^m - 1$ is divisible by 2. We have to prove that $P(m+1)$ is true, *i.e.*, $3^{m+1} - 1$ is divisible by 2. Now $3^{m+1} - 1 = 3.3^m - 1 = 2.3^m + (3^m - 1)$. The first part is clearly divisible by 2 and the second part is assumed to be divisible by 2 and therefore $3^{m+1} - 1$ is divisible by 2. So it is proved that $3^n - 1$ is divisible by 2 for all positive integers n.

Example 2. Prove, by method of induction, that $1+3+5+......+(2n-1) = n^2$.

Solution : Step1. Let $P(n)$ be the given statement $1+3+5+......+ (2n-1) = n^2$.

For $n = 1$, L.H.S. =1 and R.H.S $=1^2 = 1$. $\therefore$ $P(1)$ is true.

Step 2. Let us assume $P(m)$ is true where $m (\geq 1)$ is a particular value of n, *i,e.*, we assume $1+3+5+......+ (2m-1) = m^2$. ...(1)

Now we shall prove that $P(m+1)$ is true.

For $n = m + 1$, L.H.S $= 1+3+5+......+ (2m-1) + (2m+1) = m^2 + (2m+1)$, by (1)

$= (m+1)^2$.

Hence $P(m + 1)$ is true. Therefore $P(n)$ is true for all positive integers n.

Example 3. Prove, by method of induction, $n! > 2^n$, for all positive integers $n \geq 4$.

Solution : Step 1. Let $P(n)$ be the given statement $n! > 2^n$, for all psitive integers $n \geq 4$.

For $n = 4$, L.H.S. $= 4! = 1\times2\times3\times4 > 1\times2\times2\times4 = 2^4$ =R.H.S. $\therefore$ $P(4)$ is true.

Step 2. Let us assume $P(m)$ is true where $m (\geq 4)$ is a particular value of n, *i.e.*, we assume $m! > 2^m$. ...(1)

We shall show that $P(m+1)$ is true. Now, for $n = m+1$,

L.H.S. $= (m + 1)! = (m+1).\ m! > (m+1).\ 2^m$ by (1)

But it is obvious that $(m+1) > 2$, for $m \geq 4$. $\therefore$ $(m+1)! > (m+1).\ 2^m > 2.2^m = 2^{m+1}$.

Hence $P(m + 1)$ is true. Therefore $P(n)$ is true for all positive integers $n \geq 4$.

Example 4. Prove, by mathematical induction, that

$$1.2 + 2.3 + 3.4 + ...+ n.\ (n+1) = \frac{1}{3}.n\ (n+1)\ (n+2), \qquad \forall n \geq 1.$$

Solution : Let $P(n)$ be the statement

$$1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{1}{3}.n(n+1)(n+2), \qquad \forall n \geq 1.$$

For $n = 1$, L.H.S. $=1.2 = 2$, R.H.S. $= \frac{1}{3}.1.(1+1).(1+2) = 2$. $\therefore P(1)$ is true.

Let $P(m)$ be true where m is a positive integer

i.e., we assume $1.2 + 2.3 + 3.4 + ... + m(m+1) = \frac{1}{3}m.(m+1)(m+2)$ holds.

Now we have $1.2 + 2.3 + 3.4 + ... + m(m+1) + (m+1)(m+2)$

$$= \frac{1}{3}m(m+1)(m+2) + (m+1)(m+2) = \frac{1}{3}(m+1)(m+2)(m+3).$$

Thus we prove $P(m+1)$ is true. Hence $P(n)$ is true for all integers $n \geq 1$.

Example 5. Show that the integer 2213 is prime.

Solution : Let us find all the prime integers whose squares are ≤ 2213. These are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47. Since none of these is a factor of 2213, it is prime.

Example 6. Show that the integer 2211 is not prime.

Solution : We find all the prime integers whose squares are ≤ 2211.

These are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47. It is seen that 3 divides it. So 2211 is not a prime integer.

12.7 Summary And Keywords

Summary

I. Principle of Mathematical Induction.

This is a mathematical technique which is used to prove a mathematical statement is true for every natural number. This involves two steps to prove a statement.

Step 1. It proves that a statement is true for the initial value.

Step 2. It proves that if a statement is true for the m^{th} iteration, then it is also true for the $(m+1)^{th}$ iteration.

There is an equivalent statement of the Principle of Mathematical Induction known as second principle of mathematical induction which runs as follows :

Let $P(n)$ be a mathematical statement about non-negative integers n and n_0 be a fixed non-negative integer. Suppose $P(n_0)$ is true. If, for any integer $m \geq n_0$,

$m \geq n_0$, "$P(n_0)$, $P(n_o+1)$, $P(n_0+2), \ldots\ldots, P(m)$ are true", it implies that "$P(m+1)$ is true', and then $P(n)$ is true for all $n \geq n_0$.

II. Fundamental Theorem of Arithmetic.

This is unique factorisation theorem.

Statement : Every integer greater than 1 either is a prime number itself or can be represented as the product of prime numbers and this representation is unique.

KEYWORDS : Induction, Equivalent, Unique.

12.8 Model Questions

1. Prove, by mathematical induction :

(i) $1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$.

(ii) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(iii) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$.

(iv) $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

(v) $2.3+3.4+4.5+\ldots\ldots+(n+1)(n+2)=\dfrac{1}{3}(n^3+6n^2+11n)$.

(vi) $\dfrac{3}{1.4}+\dfrac{5}{4.9}+\dfrac{7}{9.16}+\ldots\ldots+\dfrac{2n+1}{n^2(n+1)^2}=1-\dfrac{1}{(n+1)^2}$.

2. If n (> 1) be positive integer, then prove that $n^n > 1.3.5\ldots\ldots\ldots(2n-1)$.
3. If n (≥ 1) be positive integer, then prove that $2^{n+1} < 1 + (n+1).\, 2^n$.
4. If n (≥ 0) be positive integer, then prove that $2^n > n$.
5. If n be any positive integer, then prove that $n(n+1)(n+2)$ is divisible by 6.

6. Show that the square of an odd integer is of the form $8k + 1$.
7. For any integer n, show that $(3n + 1)$ and $(13n + 4)$ are relatively prime.
8. If two integers x and y be relatively prime, then prove that

$$x|m \text{ and } y|m \Rightarrow xy|m.$$

9. Determine which of the following integers are primes: (a) 729 (b) 379.
10. (a) If n is a positive integer such that n^3+1 is a prime, then prove that $n = 1$.

(b) If p is a prime such that $p = n^2 - 4$ for some integer n, then show that $p = 5$.
11. Find all the prime divisors of 35!.
12. If n is a positive integer, show that $n^3 + 8$ is not a prime integer.
13. (a) If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, \forall n \in N.$$

(b) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then show that $(aI + bA)^n = a^n I + na^{n-1} bA,$

where I is identity matrix of order 2.
14. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then prove that

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, \forall n \in N.$$

15. Use mathematical induction to prove that

(i) $2.7^n + 3.5^n - 5$ is divisible by 24 for all $n \in N$.

(ii) $n(n^2 - 1)$ is divisible by 24, if n be any odd integer.

16. Prove, by mathematical induction, that

$$n! < \left(\frac{n+1}{2}\right)^n, \forall n > 1.$$

12.8.1 Answers

9. (b) 379 is prime. 11. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 and 31.

Multiple Choice Questions (MCQ)

1. (i) If $1, \omega, \omega^2, ..., \omega^{n-1}$ be n^{th} roots of unity, then

$(1-\omega)(1-\omega^2)......(1-\omega^{n-1})$ equals to

(a) 0 (b) n (c) 1 (d) n^2

(ii) The value of $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7} - i\cos\frac{2k\pi}{7}\right)$ is

(a) –1 (b) 0 (c) $-i$ (d) i

(iii) The simplest value of $\left(\frac{2i}{1+i}\right)^2$ is

(a) i (b) $2i$ (c) $1-i$ (d) $1+i$

(iv) The simplest value of $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{1000}$ is $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$. It is

(a) True or (b) False.

(v) If the imaginary part of $\frac{2z+1}{iz+1}$ be – 2, then the locus of the point representing z in the complex plane is

(a) a circle (b) a straight line

(c) a parabola (d) a hyperbola.

2. (i) Pick up the correct answer for the statement :

$\log i + \log(-1+i) = \log i(-1+i)$

(a) True (b) False

(ii) If $\pi^i = A + iB$, then value of A is

(a) $\cos(\log\pi)$ (b) $\sin(\log\pi)$

(c) $e^{-2n\pi}\cos(\log\pi)$ (d) $e^{-2n\pi}\sin(\log\pi)$

(iii) The values of i^i form

(a) *A.P.* (b) *G.P.* (c) *H.P.* (d) none of these.

(iv) If $|\sin(\alpha+i\beta)|^2 = A^2 + \frac{1}{4}(e^{\beta} - e^{-\beta})^2$, then A is equal to

(a) 1 (b) cos α (c) sin α (d) tan α

(v) The sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ is

(a) 1 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

3. (i) When $x^3 + 2x^2 - 3x - 4$ is divided by $2x - 1$, the remainder is

(a) – 4 (b) $\frac{13}{2}$ (c) $-\frac{33}{8}$ (d) $-\frac{39}{8}$

(ii) The roots of the equation $x^3 - 7x^2 + 19x - 13 = 0$ are

(a) all real (b) all imaginary

(c) one real and two imaginary (d) two real and one imaginary

(iii) The roots of the equation $x^3 - 3x^2 - 9x + 27 = 0$ are

(a) all equal (b) all different

(c) two of them are equal (d) none of these

(iv) The mumber of imaginary roots of the equation $x^4 + 16x^2 + 7x - 11 = 0$ is

(a) one (b) two (c) three (d) four

(v) The equation $x^3 - 2x - 5 = 0$ has

(a) only one negative real root (b) two negative real roots

(c) three negative real roots (d) no negative real root

4. (i) The roots of the equation $x^3 - 9x^2 + 23x = 15$ are in *A.P.* The mean root is

(a) 1 (b) 2 (c) 3 (d) 5.

(ii) The roots of the equation $x^3 - px^2 + qx - r = 0$ are in G. P. The mean root is

(a) p (b) q (c) $\frac{p}{q}$ (d) $\frac{q}{p}$

(iii) If the roots of the equation $x^3 - ax^2 + bx - c = 0$ are in harmonic progression, then the mean root is

(a) $\frac{3c}{b}$ (b) $\frac{c}{b}$ (c) $\frac{2c}{b}$ (d) $\frac{1}{b}$

(iv) If α, β, γ be the roots of the equation $x^3 + 5x - 1 = 0$, then the value of $\sum \alpha^3$ is

(a) –5 (b) 0 (c) 3 (d) –3

(v) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + 2x^2 + 3x + 4 = 0$, then the value of $\Sigma\alpha^4$ is

(a) –12 (b) –8 (c) 8 (d) 16.

5. (i) The only real root of the equation $x^3 + 72x - 1720 = 0$ is

(a) 5 (b) 8 (c) 10 (d) none of these

(ii) If the roots of the equation $x^3 - 12x + 8 = 0$ be $4\cos\alpha, 4\cos 2\alpha$ and $4\cos 4\alpha$, then the value of α is

(a) $\frac{2\pi}{9}$ (b) $\frac{\pi}{9}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

(iii) The real roots of the equation $x^4 - 4x^2 - 3x + 6 = 0$ are

(a) 1, 3 (b) 1, 2 (c) 2, 3 (d) 3, 6

(iv) The real roots of the equation $x^4 + 2x^3 + 14x + 15 = 0$ are

(a) 1, 3 (b) 1, 5 (c) –1, –5 (d) –1, –3

(v) One imaginary root of the equation $x^4 + 12x = 5$ is

(a) 1 + i (b) 1 – 2i (c) 1 – i (d) 2 + i

6. (i) If the sum of the sides of a quadrilateral be given, then its area is greatest when the quadrilateral is

(a) square (b) parallelogram

(c) rectangle (d) rhombus

(ii) The product of n positive numbers is unity. Then their sum is

(a) a positive integer (b) equal to $n+\frac{1}{n}$

(c) never less than n (d) divisible by n

(iii) The solution set of the inequality $2^x+3^x \geq 2$ is

(a) all natural numbers (b) all real numbers

(c) $0 \leq x \leq \infty$ (d) all negative integers

(iv) If x, y, z be three positive real numbers, not all equal, then the value of $(x+y)(y+z)(z+x)$ is

(a) >8 xyz (b) < $8xyz$ (c) 8 xyz (d) $x^2y^2z^2$

(v) If $a^2+b^2+c^2=1$ and $ab+bc+ca=x$, then

(a) $\frac{1}{2} \leq x \leq 2$ (b) $-1 \leq x \leq 2$ (c) $-\frac{1}{2} \leq x \leq 1$ (d) $-1 \leq x \leq 1$

7. (i) If x be non-zero and the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ be singular, then the value of x is

(a) 1 (b) 2 (c) 3 (d) 4

(ii) If $\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ p & 1 \end{bmatrix}$, then the value of p is

(a) 5 (b) 7 (c) 9 (d) 11

(iii) If the matrix $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & x \end{bmatrix}$ be orthogonal, then x is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

(iv) If the rank of the matrix $\begin{bmatrix} 1 & 3 & -3 & x \\ 2 & 2 & x & -4 \\ 1 & 1-x & 2x+1 & -8-3x \end{bmatrix}$ be 2, then the value of x is

(a) 2 (b) –2 (c) 3 (d) 4

(v) The rank of the matrix $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) 4

8. (i) Pick up the correct answer for the statement :

The system of equations $x+y+z=9, 3x-2y+4z=3$ is consistent

(a) True (b) False

(ii) The value of k for which the system of equations

$x+y+z=2, 2x+y+3z=1, x+3y+2z=5$ and $3x-2y+z=k$ is solvable, is

(a) 1 (b) –1 (c) 2 (d) –2

(iii) The value of λ, for which the system of equations

$\lambda x+y+z=1, x+\lambda y+z=1, x+y+\lambda z=1$ will have an infinite number of solutions, is

(a) –1 (b) 0 (c) 1 (d) 2

(iv) The value of p, for which the system of equations

$x_1 + 2x_2 + 3x_3 = px_1$, $3x_1 + x_2 + 2x_3 = px_2$, $2x_1 + 3x_2 + x_3 = px_3$ has a non-trivial solution, is

(a) 1 (b) 2 (c) 3 (d) 6

(v) The value of μ, for which the system of equations

$x_1 + x_2 + x_3 = 6$, $x_1 + 2x_2 + 3x_3 = 10$, $2x_1 + 4x_2 + 6x_3 = \mu$ has no solution, is

(a) 10 (b) $\neq 20$ (c) 20 (d) 6

9. (i) The statement "The eigen values of the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ defined over the field of real numbers R donot belong to R" is

(a) False (b) Ture

(ii) The eigen values of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ are

(a) 1, 1, 3 (b) 0, 1, 2 (c) 1, 2, 2 (d) 2, 2, 3

(iii) $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ becomes $pA + 5I$, where $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; then the value of p is

(a) – 4 (b) 4 (c) 3 (d) –1

(iv) 1 is an eigen value of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. The geometric multiplicity for 1 is

(a) 0 (b) 1 (c) –1 (d) 2

(v) 8 is an eigen value of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. The algebraic multiplicity for 8 is

(a) 3 (b) 2 (c) 1 (d) 0

10. (i) The statement "If R be an equivalence relation in a set A, then R^{-1} is also an equivalence relation" is

(a) True (b) False

(ii) "${}_x\rho_y$ if and only if $|x-y|$ is of the form $5n$ for some integer n, then ρ is not an equivalence relation." This statement is

(a) True (b) False

(iii) The number of relations which can be defined on a set with n elements is

(a) n^2 (b) 2^n (c) 2^{n^2} (d) $2n^2$

(iv) R is the set of real numbers. A map $f : R \to R$ is defined by $f(x) = x^2 + 1$, $x \in R$; then $f^{-1}(10)$ is

(a) $\{3\}$ (b) $\{-3, 3\}$ (c) $\{-3\}$ (d) $\{x : -3 < x < 3\}$

(v) Given $f : N \to N$, defined by $f(x) = x - 1$ if $x \geq 2$

$= 1$ if $x = 1$.

The range of f is

(a) set of real numbers (b) set of all integers

(c) set of all rational numbers (d) set of all positive integers

11. (i) Tick (✓) the correct answer :

$-75 \equiv 5 \pmod 8$

(a) True (b) False

(ii) Tick (✓) the correct answer :

$512 \not\equiv 2$ mod (13)

(a) True (b) False

(iii) The gcd of 385 and 792 is

(a) 55 (b) 33 (c) 22 (d) 11

(iv) The gcd of 2274 and 174 is expressed in the form $2274p + 174q$, where p and q are integers. The value p is

(a) 14 (b) 15 (c) –14 (d) –16

(v) The sum of the first $n + 1$ odd integers is

(a) $(n - 1)^2$ (b) n^2 (c) $(n + 1)^2$ (d) $(n + 2)^2$

12. (i) Pick the correct answer :

2217 is a prime integer.

(a) True (b) False

(ii) Pick the correct answer :

$n^3 - n$ is divisible by 6 for all $n \in N$.

(a) True (b) False

(iii) For all $n \in N$, $10^n + 3.4^{n+2} + 5$ is divisible by

(a) 2 (b) 8 (c) 9 (d) 5

(iv) If $x + \frac{1}{x}$ be an integer, then $x^n + \frac{1}{x^n}$ is an integer for all $n \in N$.

It is

(a) correct (b) not correct

(v) If $u_n = \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$, $\forall n \in N$, then u_n *is*

(a) never an integer (b) an integer

(c) a surd (d) multiple of 5

Answers :

1. (i) (b)	(ii) (d)	(iii) (b)	(iv) (a)	(v) (b)
2. (i) (b)	(ii) (c)	(iii) (b)	(iv) (c)	(v) (c)
3. (i) (d)	(ii) (c)	(iii) (c)	(iv) (b)	(v) (d)
4. (i) (c)	(ii) (d)	(iii) (a)	(iv) (c)	(v) (b)
5. (i) (c)	(ii) (a)	(iii) (b)	(iv) (d)	(v) (b)

6. (i) (a)	(ii) (c)	(iii) (c)	(iv) (a)	(v) c)
7. (i) (c)	(ii) (c)	(iii) (a)	(iv) (b)	(v) (b)
8. (i) (a)	(ii) (d)	(iii) (c)	(iv) (d)	(v) (b)
9. (i) (b)	(ii) (a)	(iii) (a)	(iv) (b)	(v) (c)
10. (i) (a)	(ii) (b)	(iii) (c)	(iv) (b)	(v) (d)
11. (i) (a)	(ii) (a)	(iii) (d)	(iv) (c)	(v) (c)
12. (i) (b)	(ii) (a)	(iii) (c)	(iv) (a)	(v) (b)

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