PREFACE

In a bid to standardise higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific generic elective, ability* and *skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry acquired credits. I am happy to note that the University has been accredited by NAAC with grade 'A'.

UGC (Open and Distance Learning Programmes and Online Learning Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the *six* semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

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Vice-Chancellor

Netaji Subhas Open University

Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Sub: Honours in Mathematics (HMT)
Course Code : CC-MT-03
Course : Calculus

First Print : December, 2021

Printed in accordance with the regulations of the
Distance Education Bureau of the University Grants Commission.

Netaji Subhas Open University

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Netaji Subhas Open University

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(HMT)

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Unit-1 ❑ Hyperbolic Functions

Structure

1.1 Objectives

We all know about trigonometric functions. In this lesson we will know about hyerbolic functions and their relations to the trigonometric functions. After studying this chapter the learner should be :

- understanding the meaning of hyperbolic functions and inverse hyperbolic functions.
- able to derive the hyperbolic functions in terms of exponential functions.
- mighty to realize the infinite series expansion of hyperbolic functions.
- aware of some important identities.
- skilled to express the inverse hyperbolic functions in terms of logarithmic functions.

1.2 Introduction

We have seen the shape of a heavy cable suspended between pylons. Is there any mathematical function corresponding the shape of the cable ? Yes, there is a function named hyperbolic function which has great applications in mathematics, physics and engineering. The hyperbolic functions have similar names to the trigonometric functions but they are defined in the forms of exponential functions. In this chapter we will derive the expressions of hyperbolic functions and express them in logarithmic forms. The first systematic development of hyperbolic functions was implemented by Swiss Mathematician Johann Heinrich Lambert (1728-1777).

1.3 Derivation of Hyperbolic Functions

Hyperbolic functions can be derived mathematically in various ways. We will derive the hyperbolic functions graphically. In this method an analogous relation can be found between the circular (trigonometric) functions and the hyperbolic functions. We elaborate this by starting first with the unit circle $u^2 + v^2 = 1$. Consider x as an angle forming a circular sector MOP of area C (see Fig. 1.1). Now the area C of this circular sector MOP is $\frac{1}{2}x$. Then twice C (the area of the circular sector MOP) is equal to circular angle x in radians.

For the unit circle $u^2 + v^2 = 1$, where $OM = 1$, we see that $\sin x = v/OM = v$, and $\cos x = u/OM = u$.

We can now develop analogously for the hyperbolic functinos. Suppose that H is the area of hyperbolic sector MOP (see Fig. 1.2) of the unit rectangular (equilateral) hyperbola $u^2 - v^2 = 1$, or $v = \sqrt{u^2 - 1}$.

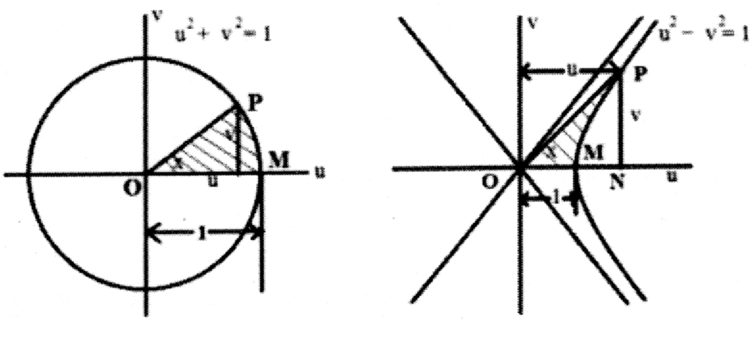

Fig. 1.1 Fig. 1.2

Then twice H (the area of the hyperbolic sector MOP) is equal to the hyperbolic angle x in radians. Now, from Fig. 1.2 we see that H (the area of the hyperbolic sector MOP) is the area of NOP less the area NMP, where area $NOP = \frac{1}{2}uv$ and area

$NMP = \int_1^u v\,du$. Then we obtain.

$$x = 2H \text{ (twice the area of MOP)} = 2\left[\frac{1}{2}uv - \int_1^u v\,du\right]$$

$$= \left[uv - 2\int_1^u \sqrt{u^2-1}\,du\right]$$

$$= \left[uv - u\sqrt{u^2-1} + \log\left(u + \sqrt{u^2-1}\right)\right]$$

$$= \left[uv - uv + \log\left(u + \sqrt{u^2-1}\right)\right]$$

$$= \log\left(u + \sqrt{u^2-1}\right).$$

From this we have

$$\left(u + \sqrt{u^2-1}\right) = e^x \text{ and } \left(u - \sqrt{u^2-1}\right) = e^{-x}$$

From this we get $u = \frac{e^x + e^{-x}}{2}$ and $v = \frac{e^x - e^{-x}}{2}$. These last two expressions are the familiar formulas "hyperbolic cosine" and "hyperbolic sine" denoted by cosh x and sinh x respectively. So we have

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2},$$

where x is the twice of the area of the hyperbolic sector.

Remark : We need to point out that graphically it is not possible to draw the hyperbolic angle x in the same way that the circular angle x is drawn, for x has no such reality. It only exists as a function of the hyperbolic sector area H. It is important to avoid attempting to interpret x as an angle meeting at a point on the hyperbola.

1.3.1. Infinite Series Expansion of Hyperbolic Functions

Expanding e^x and e^{-x}, we get the expansions of cosh x and sinh x as

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

1.3.2. Periodicity of Hyperbolic Functions

Using the definition of hyperbolic functions we can easily prove that
sinh ($2n\pi i$ + x) = sinh x, cosh ($2n\pi i + x$) = cosh x, tanh ($2n\pi i + x$) = tanh x.
Thus hyperbolic functions are periodic functions of imaginary periods.

1.3.3. Some Important Identities

(i) $\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = 1.$

(ii) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x+e^{-x}}{e^x-e^{-x}}.$

(iii) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}}, \quad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x-e^{-x}}.$

(iv) $\operatorname{sech}^2 x + \tanh^2 x = \left(\frac{2}{e^x+e^{-x}}\right)^2 + \left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$

(v) $\coth^2 x - \operatorname{cosech}^2 x = \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2 = 1.$

(vi) cosh $(-x)$ = cosh x. sinh $(-x)$ = – sinh x.
(vii) tanh $(-x)$ = – tanh x, coth $(-x)$ = – coth x.
(viii) sech $(-x)$ = sech x, cosech $(-x)$ = – cosech x.
(ix) cos (ix) = cosh x, sin (ix) = i sinh x.
(x) tan (ix) = i tanh x, cot (ix) = – i sinh x.
(xi) cosh 0 = 1, sinh 0 = 0, tanh 0 = 0.

Example 1.3.1 Show that cosh $(x - y)$ = cosh x cosh y – sinh x sinh y.

Solution : From the definition of hyperbolic functions we have

cosh x cosh y – sinh x sinh y

$$= \frac{1}{4}\left\{\left(e^x+e^{-x}\right)\left(e^y+e^{-y}\right) - \left(e^x-e^{-x}\right)\left(e^y-e^{-y}\right)\right\}$$

$$= \frac{1}{4}\left\{\left(e^{x+y}+e^{x-y}+e^{-(x-y)}+e^{-(x+y)}\right) - \left(e^{x+y}-e^{x-y}-e^{-(x-y)}+e^{-(x+y)}\right)\right\}$$

$$= \frac{1}{2}\left\{e^{x-y}+e^{-(x-y)}\right\}$$

= cosh $(x - y)$.

Example 1.3.2 Show the equation sinh $x = \frac{3}{4}$.

Solution : $\sinh x = \frac{3}{4}$

$$\Rightarrow \frac{e^x - e^{-x}}{2} = \frac{3}{4}$$

$$\Rightarrow 2e^x - 3 - 2e^{-x} = 0$$

$$\Rightarrow 2e^{2x} - 3e^x - 2 = 0$$

$$\Rightarrow (e^x - 2)(2e^x + 1) = 0$$

$$\Rightarrow e^x = 2 \quad or \; 2e^x = -1.$$

But e^x is always positive, so $e^x = 2 \Rightarrow x = \log 2$.

1.4 Inverse Hyperbolic Functions

The inverse hyperbolic function $\sinh^{-1} x$, $\cosh^{-1} x$ are written as

$y = \sinh^{-1} x \Rightarrow \sinh y = x$ with $x \in (-\infty, \infty)$.

$y = \cosh^{-1} x \Rightarrow \cosh y = x$ with $x \geq 1$ and $y \geq 0$.

$y = \tanh^{-1} x \Rightarrow \tanh y = x$ with $|x| < 1$ and $y\ (-\infty, \infty)$.

1.4.1. Logarithmic Interpretation of Inverse Hyperbolic functions

Suppose

$$y = \sinh^{-1} x$$

$$\Rightarrow x = \sinh y$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \text{ as } e^y > 0$$

$$\Rightarrow y = \log\left(x + \sqrt{x^2 + 1}\right).$$

Thus $y = \sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$.

Similarly we see that

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right).$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

1.5 Summary

In this unit we derived hyperbolic functions in terms of exponential functions and their important identities. We also defined Inverse hyperbolic functions and their logarithmic expressions. We learned about the relations between hyperbolic functions and trigonometric functions. We expressed the infinite series expansion of hyperbolic functions and discussed the periodicity property of this functions.

1.6 Exercises

1. Prove the following identities.

(i) $\sinh(x + y) = \sinh x \cosh y + \cosh y \sinh x$

(ii) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 - \tanh x \tanh y}$

(iii) $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(iv) $\cosh 2x = 1 + 2\sinh^2 x$.

2. Solve the following equations.

(i) $2\cosh 2x + 10\sinh 2x = 5$ (ii) $\sinh x = \dfrac{5}{12}$

(iii) $4\cosh x + \sinh x = 4$ (iv) $\tanh x = \dfrac{1}{2}$

(v) $9\cosh x - 5\sinh x = 15$ (vi) $3\cosh^2 x + 11\sinh x = 17$.

3. Express the followings in logarithmic form

(i) $\sinh^{-1}\dfrac{3}{4}$ (ii) $\text{sech}^{-1} x$ (iii) $\tanh^{-1} x$.

1.7 References

1. J.G. Chakravorty, P.R. Ghosh : Advanced Higher Algebra, U.N. Dhar & Sons Private Limited.
2. W. K. Robinson, Slide Rules with Hyperbolic Function Scales, The Journal of the Oughtred Society, Vol. 14, No. 1, 2005.

Unit-2 ❑ Higher Order Derivatives

Structure

2.1 Objectives

In this unit the readers will learn the followings.

- The definition of higher order derivatives.
- Leibnitz's formula.
- The differentiation of power functions.
- The higher order derivatives of product functions and quotient functions.

2.2 Introduction

The derivative is the first of the two main tools of calculus which was discover independently by Issach Newton and Gottfried Leibniz in the mid 17th century. The derivative of a function of real variable measures the rate of change of function value with respect to the change of independent variable. In this chapter we will see how to compute higher order derivatives and will explore some of their applications.

2.3 Higher Order Derivatives

Successive differentiation is the process of differentiating a function repeatedly n times and the results of such differentiation are called successive derivatives. The higher order derivatives have most importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its first derivative be $f'(x)$. If $f'(x)$ itself differentiable, we denote the derivate of $f'(x)$ by $f''(x)$ and call $f''(x)$, the second order derivative of $f(x)$. Continuing in this manner, we obtain the functions $f'(x)$, $f''(x)$, $f'''(x)$, $f^{iv}(x)$, ... $f^{(n)}(x)$, each of which is the derivative of previous one. We call $f^{(n)}(x)$, the nth derivative of $f(x)$ or the derivative of order n of the function $f(x)$.

2.3.1. Notations of Higher Order Derivatives

1st order derivative : $f'(x)$ or $\dfrac{dy}{dx}$ or Dy or y_1 or y'

2nd order derivative : $f''(x)$ or $\dfrac{d^2y}{dx^2}$ or D^2y or y_2 or y''

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n^{th} order derivative : $f^{(n)}(x)$ or $\dfrac{d^ny}{dx^n}$ or D^ny or y_n or $y^{(n)}$

Clearly $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right) = \dfrac{d^2}{dx^2}\left(\dfrac{d^{n-2}y}{dx^{n-2}}\right) = \ldots,$

and so on.

2.4 Calculation of n^{th} Order Derivatives

2.4.1. n^{th} derivative of x^k

Let $y = x^k$, k being any real number.

Then $y_1 = kx^{k-1}$

$y_2 = k(k-1)x^{k-2}$

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$y_n = k(k-1)(k-2)...(k-n+1)x^{k-n}$, for all positive integer n.

If k be positive integer, then $y_k = k(k-1)(k-2)...(k-k+1)x^{k-k} = k!$

If k be positive integer but n is positive integer greater than k, then $y_n = 0$.

If $y = x^{-k}$, k being positive real number, then

$$y_n = -k(-k-1)(-k-2)...(-k-n+1)x^{-k-n}, \text{ for all positive integer } n$$

$$= (-1)^n \frac{k(k+1)(k+2)...(k+n-1)}{x^{n+k}}$$

$$= (-1)^n \frac{(k+n-1)!}{(k-1)!x^{n+k}}.$$

e.g., $y = x^{-1}$, $y_n = (-1)^n \dfrac{n!}{x^{n+1}}$.

$y = x^{-2}$, $y_n = (-1)^n \dfrac{(n+1)!}{x^{n+2}}$.

$y = \log x \ (x > 0)$, $y_1 = \dfrac{1}{x}$.

Hence y_n = nth derivative of $\log x$

= $(n-1)$th derivative of $\dfrac{1}{x}$

$$= (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

2.4.2. n^{th} derivative of e^{ax}.

Let $y = e^{ax}$.

Then $y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

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$y_n = a^ne^{ax}$.

2.4.3. n^{th} derivative of $\sin(ax + b)$

Let $y = \sin(ax + b)$.

Then $y_1 = a\cos(ax + b) = a\sin\left(ax + b + \dfrac{\pi}{2}\right)$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{2\pi}{2}\right)$$

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$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

$$y_{2n} = a^{2n} \sin\left(ax + b + \frac{2n\pi}{2}\right) = a^{2n}(-1)^n \sin(ax + b) = (-a^2)^n \sin(ax + b).$$

Similarly, if $y = \cos(ax + b)$,

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

$$y_{2n} = (-a^2)^n \cos(ax + b).$$

2.4.4. n^{th} derivative of $e^{ax} \sin(bx + c)$

Let

$$y = e^{ax} \sin(bx + c).$$

$$y_1 = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$$

$$= e^{ax}\left[a \sin(bx + c) + b \cos(bx + c)\right]$$

$$= e^{ax}\left[r \cos\alpha \sin(bx + c) + r \sin\alpha \cos(bx + c)\right]$$

[Putting $a = r\cos\alpha$, $b = r\sin\alpha$]

$$= re^{ax} \sin(bx + c + \alpha), \text{ where } r^2 = a^2 + b^2, \tan\alpha = \frac{b}{a}.$$

Similarly, $y_2 = r^2 e^{ax} \sin(bx + c + 2\alpha)$

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$$y_n = r^n e^{ax} \sin(bx + c + n\alpha).$$

Similarly, if $y = e^{ax} \cos(bx + c)$,

$$y_n = r^n e^{ax} \cos(bx + c + n\alpha).$$

Example 2.4.1 If $y = \frac{1}{x^2 - a^2}$, then find y_n.

Solution : Here

$$y = \frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

Thus, $$y_n = \frac{1}{2a}(-1)^n n!\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right].$$

Example 2.4.2 If $y = \sin^3 x$, then find y_n.

Solution : We know that $\sin 3x = 3\sin x - 4\sin^3 x$. Hence

$$y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

and $$y_n = \frac{1}{4}\left[3\sin\left(x + \frac{n\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right].$$

Example 2.4.3 If $y = \sin 3x \cos 2x$, then find y_n.

Solution : $$y = \sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x).$$

Therefore, $$y_n = \frac{1}{2}\left[5^n \sin\left(5x + \frac{n\pi}{2}\right) + \sin\left(x + \frac{n\pi}{2}\right)\right].$$

Example 2.4.4 If $y = \sqrt{x}$, then find y_n.

Solution : Here

$$y = \sqrt{x} = x^{1/2}.$$

Thus, $$y_1 = \frac{1}{2}x^{-1/2}$$

$$y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$$

$$y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} = (-1)^2 \frac{1.3}{2^3}\frac{1}{x^{3-1/2}}.$$

Differentiating continuously, we get

$$y_n = (-1)^{n-1}\frac{1.3.5.....(2n-3)}{2^n}\frac{1}{x^{n-1/2}}.$$

2.5 Leibnitz's Theorem on Successive Differentiation

If u and v are two functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = {}^nC_o u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n,$$

where u_r and v_r *represent* r^{th} derivatives of u and v respectively.

Proof :

By differentiating directly, we get

$$(uv)_1 = u_1 v + u v_1$$

$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + u v_2) = u_2 v + 2u_1 v_1 + u v_2$$

$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2.$$

Thus, the theorem is true for $n = 1$ and $n = 2$.

Now we assume that the theorem is true for a certain positive integer m $(m < n)$.

Then $(uv)_m = {}^mC_o u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots + {}^mC_r u_{m-r} v_r + \ldots + {}^mC_m u v_m.$

Differentiating both sides once more, we obtain

$$(uv)_{m+1} = {}^mC_0(u_{m+1} v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1} v_2) + \ldots$$

$$+ {}^mC_r(u_{m-r+1} v_r + u_{m-r} v_{r+1}) + \ldots + {}^mC_m(u_1 v_m + u v_{m+1})$$

$$= {}^mC_0 u_{m+1} v + ({}^mC_0 + {}^mC_1) u_m v_1 + ({}^mC_1 + {}^mC_2) u_{m-1} v_2$$

$$+ \ldots + ({}^mC_{r-1} + {}^mC_r) u_{m-r+1} v_r + \ldots$$

$$+ ({}^mC_{m-1} + {}^mC_m) u_1 v_m + {}^mC_m u v_{m+1}$$

$$= {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + \ldots + {}^{m+1}C_r u_{m-r+1} v_r$$

$$+ \ldots + {}^{m+1}C_{m+1} u v_{m+1} \quad (\text{using } {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r).$$

Thus, the theorem is true for $n = m + 1$ if it is true for $n = m$. Thus, by Mathematical induction the theorem is true for all positive integers n.

2.5.1. n^{th} derivative of $e^{ax+b} \sin x$

Let $y = e^{ax+b} \sin x = uv$, where $u = e^{ax+b}$ and $v = \sin x$.

Then $u_k = a^k e^{ax+b}$ and $v_k = \sin\left(x + \frac{k\pi}{2}\right)$.

Thus, by Leibnitz's rule

$$y_n = a^n e^{ax+b}\sin x + {}^nC_1 a^{n-1}e^{ax+b}\sin\left(x+\frac{\pi}{2}\right)+\ldots+e^{ax+b}\sin\left(x+\frac{n\pi}{2}\right)$$

$$= e^{ax+b}\left\{a^n\sin x + {}^nC_1 a^{n-1}\sin\left(x+\frac{\pi}{2}\right)+\ldots+\sin\left(x+\frac{n\pi}{2}\right)\right\}.$$

Similarly, if $y = e^{ax+b}\cos x$, then

$$y_n = e^{ax+b}\left\{a^n\cos x + {}^nC_1 a^{n-1}\cos\left(x+\frac{\pi}{2}\right)+\ldots+\cos\left(x+\frac{n\pi}{2}\right)\right\}.$$

2.5.2. n^{th} derivative of $(ax + b)^n \sin x$.

Let $y = (ax + b)^n \sin x = uv$, where $u = (ax + b)^n$ and $v = \sin x$.

Then $u_k = a^k\dfrac{n!}{(n-k)!}(ax+b)^{n-k}$ and $v_k = \sin\left(x+\dfrac{k\pi}{2}\right)$.

Thus, by Leibnitz's theorem, we obtain

$$y_n = \frac{n!}{0!}a^n\sin x + \frac{{}^nC_1 n!}{1!}a^{n-1}(ax+b)\sin\left(x+\frac{\pi}{2}\right)$$

$$+\ldots+(ax+b)^n\sin\left(x+\frac{n\pi}{2}\right)$$

$$= n!\left\{\frac{1}{0!}a^n\sin x + \frac{{}^nC_1}{1!}a^{n-1}(ax+b)\sin\left(x+\frac{\pi}{2}\right)\right.$$

$$\left.+\ldots+\frac{1}{n!}(ax+b)^n\sin\left(x+\frac{n\pi}{2}\right)\right\}.$$

Similarly, if $y = (ax + b)^n \cos x$, then

$$y_n = n!\left\{\frac{1}{0!}a^n\cos x + \frac{{}^nC_1}{1!}a^{n-1}(ax+b)\cos\left(x+\frac{\pi}{2}\right)\right.$$

$$\left.+\ldots+\frac{1}{n!}(ax+b)^n\cos\left(x+\frac{n\pi}{2}\right)\right\}.$$

Example 2.5.1 Find the n^{th} derivative of $y = x^3 \log x$.

Solution : Let $u = \log x$ and $v = x^3$.

Then $u_k = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$ and $v_k = 0$ for $k \geq 4$.

By Leibnitz's theorem, we have

$$(uv)_n = {}^nC_0u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \ldots + {}^nC_ru_{n-r}v_r + \ldots + {}^nC_nuv_n.$$

Thus,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}.x^3 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.3x^2$$

$$+\frac{n(n-1)}{2!}.\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}.6x + \frac{n(n-1)(n-2)}{3!}.\frac{(-1)^{n-4}(n-4)!}{x^{n-3}}.6$$

$$= (-1)^n\frac{6(n-4)!}{x^{n-3}}.$$

Example 2.5.2 Find the n^{th} derivative of $y = x^2e^{3x}\sin 4x$.

Solution : Let $u = e^{3x}\sin 4x$ and $v = x^2$.

Then $u_k = e^{3x}5^k\sin\left(4x + k\tan^{-1}\frac{4}{3}\right)$ and $v_k = 0$ for $k \geq 3$.

By Leibnitz's theorem, we have

$$(uv)_n = {}^nC_0u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \ldots + {}^nC_ru_{n-r}v_r + \ldots + {}^nC_nuv_n.$$

Thus,

$$y_n = e^{3x}5^n\sin\left(4x + n\tan^{-1}\frac{4}{3}\right).x^2 + n.e^{3x}5^{n-1}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right).2x$$

$$+\frac{n(n-1)}{2!}.e^{3x}5^{n-2}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right).2$$

$$= e^{3x}5^n\left\{x^2\sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right)\right.$$

$$\left. + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right\}.$$

Example 2.5.3 If $y = \sin(m\sin^{-1}x)$, then show that

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n.$$

Also find $y_n(0)$.

Solution : $y = \sin(m\sin^{-1}x)$ (2.5.1)

$$\Rightarrow \quad y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x) \tag{2.5.2}$$

$$\Rightarrow \quad (1-x^2)y_1^2 = m^2\cos^2(m\ \sin^{-1}x)$$

$$\Rightarrow \quad (1-x^2)y_1^2 = m^2(1-y^2)$$

$$\Rightarrow \quad (1-x^2)y_1^2 + m^2y^2 = m^2.$$

Differentating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + y_1^2(-2x) + m^2 2yy_1 = 0$$

$$\Rightarrow \quad \left(1-x^2\right)y_2 - xy_1 + m^2y = 0. \tag{2.5.3}$$

Using Leibnitz's theorem, we get

$$\left[y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2)\right] - \left(y_{n+1}x + {}^nC_1 y_n.1\right) + m^2y_n = 0$$

$$\Rightarrow \quad \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow \quad \left(1-x^2\right)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n. \tag{2.5.4}$$

Putting $x = 0$ in (2.5.1), (2.5.2) and (2.5.3), we get

$$y(0) = 0,\ y_1(0) = m,\ \ and\ y_2(0) = 0.$$

Putting $x = 0$ in (2.5.4), we have

$$y_{n+2}(0) = (n^2-m^2)y_n(0).$$

Putting $n = 1, 2, 3 \ldots$ in the above equation we have

$$y_3(0) = \left(1^2-m^2\right)y_1(0)$$

$$= \left(1^2-m^2\right)m$$

$$y_4(0) = \left(2^2-m^2\right)y_2(0)$$

$$= \left(2^2-m^2\right).0$$

$$= 0$$

$$y_5(0) = \left(3^2-m^2\right)y_3(0)$$

$$= m\left(1^2-m^2\right)\left(3^2-m^2\right).$$

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Therefore

$$y_n(0)=\begin{cases}0, \text{ if } n \text{ is even}\\ m\left(1^2-m^2\right)\left(3^2-m^2\right)...\left((n-2)^2-m^2\right), \text{ if } n \text{ is odd.}\end{cases}$$

2.6 Summary

After studying this unit we have seen that we can derive a general formula of nth order derivative of a function without computing intermediate derivatives or by Leibnitz's Rule. To derive a general formula of nth order derivative of a function, it is better to differentiate again and again until it is clear.

2.7 Exercises

1. Find n^{th} order derivative of the following functions :

(i) $e^x \sin x \sin 2x$ (ii) $\tan^{-1}\dfrac{x}{a}$ (iii) $\dfrac{x^2}{x-1}$

(iv) $\dfrac{a-x}{a+x}$ (v) $\sin x \sin 2x \sin 3x$ (vi) $\tan^{-1}\dfrac{1+x}{1-x}$.

2. Use Leibnitz's formula to find the n^{th} derivative of the following functions :

(i) $e^x \log x$ (ii) $x^2 \tan^{-1} x$

(iii) $\log(ax + x^2)$ (iv) $x^3 \sin x$.

3. If $y = e^m \sin^{-1} x$, then show that

$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+m^2\right)y_n=0$. Also find $y_n(0)$.

4. If $y = \tan^{-1} x$, then show that

$\left(1-x^2\right)y_{n+2}+2(n+1)xy_{n+1}+n\left(n+1\right)y_n=0$. Also find $y_n(0)$.

5. If $y = (\sin^{-1} x)^2$, then show that

$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$. Also find $y_n(0)$.

2.8 References

1. B.C. Das, B.N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
2. R.K.Ghosh, K.C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.
4. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-3 ❑ Curvature

Structure

3.1 Objectives

After going through this lesson the readers will learn :

- the definition of curvature.
- to derive the formula of radius of curvature.
- to find out the centre of curvature.

3.2 Introduction

In this chapter we will describe the nature of bending of a curve at a particular point and its numerical measurement. The curvature measures how fast a curve is changing direction at a given point.

3.3 Definitions

Suppose the tangents at two points P and Q on a curve make angles ψ and $\psi + \Delta\psi$ with positive x-axis. Suppose that are $AP = s$, arc $AQ = s + \Delta s$ so that are $PQ = \Delta s$, A being fixed point on the curve from which the length of ares are measured. We then construct the following definitions.

The angle $\Delta\psi$ between the tangents at P and Q is called the total curvature of the arc PQ.

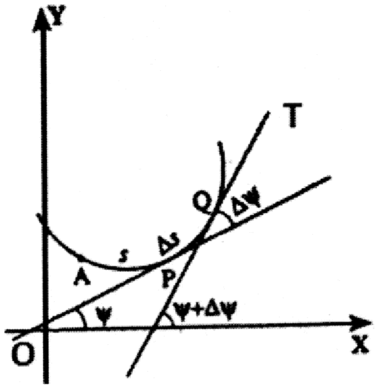


Fig. 3.1

The mean or average curvature of the arc PQ is defined as the ratio $\frac{\Delta\psi}{\Delta s}$.

The curvature (k) at a point P of the curve is defined as the limiting value of mean curvature when the arc $\Delta s \to 0$: that is

$$\text{Curvature } (k) \text{ at } P = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}.$$

Thus the curvature is the rate fo change of direction of the curve with respect to the arc length.

Suppose that $k \neq 0$ and $\rho = \frac{1}{k} = \frac{ds}{d\psi}$. Now construct a circle of radius ρ and a center C so that the circle and the curve Γ have the same tangent at P. The circle is drawn in such a way that it lies on the same side of the tangent as the curve. This circle has the same curvature as the given curve at P. We call this circle as the circle of curvature at P; its centre C is the center of curvature for the curve at P and its radius ρ, normal to curve at P is the radius of curvture of the curve at P. Thus the reciprocal of the curvature at any point P is called the radius of curvature at P, and is denoted by $\rho = \frac{1}{k} = \frac{ds}{d\psi}$.

3.4 Formulae for the Radius of Curvature

3.4.1. For the Intrinsic Equation $s = f(\psi)$.

The relation between the length of the arc (s) of a given curve, measured from a given fixed point on the curve and the angle between the tangents at its end (ψ)

is called the intrinsic equation of the curve and the formula of radius of curvature for this equation is

$$\rho = \frac{ds}{d\psi}.$$

For example, the intrinsic equation of Catenary is $s = c \tan \psi$ and $\rho = \frac{ds}{d\psi} = c \sec^2 \psi.$

3.4.2. For the Cartesian Equation (Explicit Function) $y = f(x)$ or $x = f(y)$.

In a rectangular Cartesian co-ordinates system, we have

$$\tan \Psi = \frac{dy}{dx} = y_1.$$

Therefore

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan \psi) = \frac{d}{d\psi}(\tan \psi).\frac{d\psi}{dx}$$

$$= \sec^2 \psi.\frac{d\psi}{ds}.\frac{ds}{dx}$$

$$= \sec^2 \psi.\frac{d\psi}{ds} \quad \left[\text{since } \frac{ds}{dx} = \sec\psi\right].$$

Since sec $\psi = \left(1+\tan^2 \psi\right)^{\frac{1}{2}} = \left(1+y_1^2\right)^{\frac{1}{2}}$, we have

$$\rho = \frac{ds}{d\psi} = \frac{\sec^3 \psi}{\frac{d^2y}{dx^2}} = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}, \text{ where } y_2 \neq 0. \qquad (3.4.1)$$

Similarly, for the equation $x = f(y)$,

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}, \ [x_2 \neq 0]$$

where x_1 and x_2 denote $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ respectively.

Note 3.4.1 Since ρ is always positive the root of numerator will be taken positive when y_2 is positive and negative when y_2 is negative.

Example 3.4.1 Find the radius of curvature of the parabola $y^2 = 4ax$ at the point $(a, 2a)$.

Solution : Here $y^2 = 4ax$.

Differentiating w.r.t x, we obtain

$$2yy_1 = 4a.$$

Again differentiating we have

$$2y_1^2 + 2yy_2 = 0.$$

Thus, at $(a, 2a)$, $y_1 = 1$ and $y_2 = -\frac{1}{2a}$.

Hence the required radius of curvature at $(a, 2a)$ is

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{-(1+1)^{\frac{3}{2}}}{-1/2a} = 4\sqrt{2a}.$$

3.4.3. For the Cartesian Equation (Implicit Function) $f(x, y) = 0$.

For the implicit equation, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}\left(f_y \neq 0\right),$$

$$\text{i.e., } f_x + f_y\frac{dy}{dx} = 0.$$

Differentiating again, we have

$$f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\frac{d^2y}{dx^2} = 0 \text{ [taking } f_{xy} = f_{yx}] \qquad (3.4.2)$$

Putting the value of $\frac{dy}{dx}$ in (3.4.2), we get

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}.$$

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (3.4.1), we obtain

$$\rho = -\frac{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}, \qquad (3.4.3)$$

where denominator is not equal to zero.

Example 3.4.2. Find the radius of curvature of the ellipse $9x^2 + 4y^2 = 36x$ at the point (2, 3).

Solution : We have $f(x, y) = 9x^2 + 4y^2 - 36x$.

Differentation $f(x, y)$ partially w.r.t. x, we get $f_x = 18x - 36$.

Similarly, we obtain

$$f_y = 8y,\ f_{xx} = 18,\ f_{yy} = 8,\ f_{xy} = 0.$$

Now at (2, 3),

$$f_x = 0,\ f_y = 24,\ f_{xx} = 18,\ f_{yy} = 8,\ f_{xy} = 0.$$

Thus, using the formula (3.4.3), we get

$$\rho = -\frac{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

$$= -\frac{\left[0 + (24)^2\right]^{\frac{3}{2}}}{18.(24)^2 - 0 + 0} = \frac{4}{3}$$

which is the required radius of curvature.

3.4.4. For the Parametric Equation $x = f(t)$, $y = \phi(t)$.

Here

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{y'}{x'}\ (x' \neq 0)$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

$$= \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{x'^2}\frac{1}{x'}$$

Thus, using the formula (3.4.1), we get

$$\rho = \frac{\left(1 + \left(\frac{y'}{x'}\right)^2\right)^{\frac{3}{2}}}{\frac{x'y'' - y'x''}{x'^3}}$$

$$=\frac{\left(x'^2+y'^2\right)^{\frac{3}{2}}}{x'y''-y'x''},\ x'y''-y'x''\neq 0. \tag{3.4.4}$$

Example 3.4.3 Find the radius of curvature of $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0$.

Solution : We have

$$x = a(\theta+\sin\theta),\ y = a(1-\cos\theta).$$

Differentiating w.r.t. θ, we obtain

$$x' = a(1+\cos\theta),\ y' = a\sin\theta.$$

Again differentiating w.r.t. θ, we obtain

$$x'' = -a\sin\theta,\quad y'' = a\cos\theta$$

Now at $\theta = 0$, we have

$$x' = 2a,\quad x'' = 0,\quad y' = 0,\quad y'' = a.$$

Thus, using the formula (3.4.4), we get the radius of curvature at $\theta = 0$ as

$$\rho=\frac{\left(x'^2+y'^2\right)^{\frac{3}{2}}}{x'y''-y'x''}=\frac{\left[(2a)^2+0\right]^{\frac{3}{2}}}{2a.a-0}=4a.$$

3.4.5. For the Polar Equation $r = f(\theta)$.

We know $\psi=\theta+\phi=\theta+\tan^{-1}\frac{r}{r_1}$, where $r_1=\frac{dr}{d\theta}$.

Thus,

$$\frac{d\psi}{d\theta}=1+\frac{1}{1+\frac{r^2}{r_1^2}}.\frac{r_1^2-rr_2}{r_1^2}$$

$$=\frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}.$$

Again

$$\frac{ds}{d\theta}=\sqrt{r^2+r_1^2}.$$

Thus

$$\rho=\frac{ds}{d\psi}=\frac{ds}{d\theta}\frac{d\theta}{d\psi}=\frac{\left(r^2+r_1^2\right)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}. \tag{3.4.5}$$

Corollary 3.4.1 For the polar equation $u = f(\theta)$, where $u = \frac{1}{r}$.

Since $r = \frac{1}{u}$, we have

$$r_1 = \frac{dr}{d\theta} = -\frac{1}{u^2}.\frac{du}{d\theta} = -\frac{u_1}{u^2},\ r_2 = -\frac{uu_2 - 2u_1^2}{u^3},$$

where u_1 and u_2 denote $\frac{du}{d\theta}$ and $\frac{d^2u}{d\theta^2}$ respectively.

Thus

$$\rho = \frac{\left(u^2 + u_1^2\right)^{\frac{3}{2}}}{u^3\left(u + u_2\right)}. \tag{3.4.6}$$

Example 3.4.4 Find the radius of curvature of the curve $r = a(1 - \cos\theta)$ at the point (r, θ).

Solution : We have $r = a(1 - \cos\theta)$.

Differentiating w.r.t. θ, we get $r_1 = a\sin\theta$. Again differentiating w.r.t. θ, we have $r_2 = a\cos\theta$.

Thus the radius of curvature is

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} = \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta).a\cos\theta}$$

$$= \frac{a^3(2-2\cos\theta)^{\frac{3}{2}}}{a^2(3-3\cos\theta)} = \frac{2\sqrt{2a}}{3}\sqrt{1-\cos\theta}.$$

3.4.6. For the Pedal Equation $p = f(r)$.

We know the pedal equation as $p = r\sin\phi$.

Differentiating w.r.t. r, we get

$$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}\ \left[\tan\phi = r\frac{d\theta}{dr}\right]$$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{ds}$$

$$= r\frac{d}{ds}(\theta+\phi)$$

$$= r\frac{d\psi}{ds}.$$

Thus

$$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp}.$$

Example 3.4.5 Find the radius of curvature of the ellipse $\frac{a^2b^2}{p^2} = a^2+b^2-r^2$.

Solution : We have $\frac{a^2b^2}{p^2} = a^2+b^2-r^2$.

Differentiating w.r.t. p, we get $\frac{-2a^2b^2}{p^3} = -2r\frac{dr}{dp}$.

Therefore

$$\rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3}.$$

3.4.7. For the Tangential Polar Equation $p = f(\psi)$.

We have

$$\frac{dp}{d\psi} = \frac{dp}{dr}\frac{dr}{ds}\frac{ds}{d\psi} = \frac{dp}{dr}\cos\phi . r\frac{dr}{dp} = r\cos\phi$$

Thus,

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2.$$

Differentiating w.r.t. p, we get

$$2p + 2\frac{dp}{d\psi}\frac{d^2p}{d\psi^2}\frac{d\psi}{dp} = 2r\frac{dr}{dp}.$$

Thus,

$$\rho = p + \frac{d^2p}{d\psi^2}. \qquad (3.4.8)$$

Example 3.4.6 Find the radius of curvature of the epicycloid $p = a \sin b\psi$.

Solution : We have $p = a \sin b\psi$.

Differentiating w.r.t. ψ, we get

$$\frac{dp}{d\psi} = ab\cos\psi, \quad \frac{d^2p}{d\psi^2} = -ab^2 \sin b\psi = -b^2 p.$$

Thus, the radius of curvature is

$$\rho = p + \frac{d^2p}{d\psi^2} = p - b^2 p = p(1 - b^2).$$

3.5 Radius of Curvature : Newton's Approach

I. If a curve passes through the origin and the axis of x is tangent at the origin, then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

gives the radius of curvature at the origin.

II. If a curve passes through the origin and the axis of y is tangent at the origin, then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

gives the radius of curvature at the origin.

III. If a curve passes through the origin and $ax + by = 0$ be the tangent at the origin, then

$$\frac{\sqrt{a^2+b^2}}{2} \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2+y^2}{ax+by}$$

gives the radius of curvature at the origin.

Example 3.5.1 Find the radius of curvature at the origin for the curve
$x^2 + y^3 - 2x^2 + 6y = 0$.

Solution : Here $y = 0$, the x-axis is the tangent at the origin. Thus by Netwon's formula, the radius of curvature is given by $\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$, i.e., $2\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y}$.

Now dividing the given equation by y and making $x \to 0$ and $y \to 0$, we get

$$0.\rho + 0 - 4\rho + 6 = 0.$$

Therefore

$$\rho = \frac{3}{2},$$

which is the required radius of curvature.

3.6 Centre of Curvature

The co-ordinates $C(\bar{x}, \bar{y})$ of centre of curvature at a point $P(x, y)$ of a curve $y = f(x)$ is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}, \ (y_2 \neq 0). \tag{3.6.1}$$

Proof : We use the fact that the centre of curvature at P is the limit point of intersection of the normal at P and the normal at a neighbouring point Q when $Q \to P$ along the curve. The equation of the normal at P is

$$(Y - y)\phi(x) + (X - x) = 0, \tag{3.6.2}$$

where the slope of the tangent at $P(x, y)$ is $y_1 = \frac{dy}{dx} = \phi(x)$ and X, Y are the current co-ordinate of any point on the normal.

The normal at a neighbouring point $Q(x + h, y + k)$ is

$$(Y - y - k)\phi(x + h) + (X - x - h) = 0. \tag{3.6.3}$$

At their point of intersection, the ordinate is given by,

$$(Y - y)\{\phi(x+h) - \phi(x)\} - k\phi(x+h) - h = 0. \tag{3.6.4}$$

Dividing by h and making $h \to 0$, we get

$$(\bar{y} - y)\left\{\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}\right\} - \left\{\lim_{h\to 0}\frac{k}{h}\right\}\lim_{h\to 0}\phi(x+h) - 1 = 0.$$

$$\text{or,} \quad (\bar{y} - y)\phi'(x) - \phi(x)\phi(x) - 1 = 0,$$

$$\text{or,} \quad \bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}.$$

As $(\bar{x}, \bar{y})$ is a point in (3.6.2), we get

$$(\bar{y} - y)\phi(x) + (\bar{x} - x) = 0$$

or, $\dfrac{\left(1+y_1^2\right)}{y_2}.y_1+(\overline{x}-x)=0$

or, $\overline{x}=x-\dfrac{y_1\left(1+y_1^2\right)}{y_2}.$

Example 3.6.1 Find the centre of curvature at any point (x, y) on the parabola $y^2=4ax$.

Solution : Here $y^2 = 4ax$. Differentiating w.r.t. x,

$$2yy_1 = 4a.$$

Thus,

$$y_1=\frac{2a}{y}=\sqrt{\frac{a}{x}}.$$

Again differentiating we get.

$$yy_2+y_1^2=0.$$

Thus,

$$y_2=-\frac{y_1^2}{y}=-\frac{1}{2}\frac{\sqrt{a}}{x\sqrt{x}}.$$

Hence

$$y_1\left(1+y_1^2\right)=\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)=\frac{\sqrt{a}(x+a)}{x\sqrt{x}}.$$

Thus,

$$\overline{x}=x-\frac{y_1\left(1+y_1^2\right)}{y_2}=x+2(x+a)=3x+2a$$

$$\overline{y}=y+\frac{1+y_1^2}{y_2}=y-\frac{2\sqrt{x}(x+a)}{\sqrt{a}}=-\frac{2x\sqrt{x}}{\sqrt{a}}.$$

3.7 Summary

In this chapter we have discussed the curvature of a smooth curve. We have also derived several formula for determining the radius of curvature for a curve and centre of curvature.

3.8 Exercises :

1. Find the radius of curvature at any point (s, ψ) on the following curves:

(i) $s = 8a\sin^2\frac{1}{6}\psi$ (ii) $s = a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$

(iii) $s = c\log\sec\psi$ (iv) $s = a\left(e^{mx}-1\right)$.

2. Find the radius of curvature at any point (x, y) on the following curves :

(i) $e^{y/a} = \sec(x/a)$ (ii) $xy = c^2$

(iii) $x^{2/3}+y^{2/3}=a^{2/3}$ (iv) $x^3 + y^3 = 3axy$

(v) $y = 4\sin x - \sin 2x$ (vi) $\sqrt{x}+\sqrt{a}=\sqrt{a}$.

3. Find the radius of curvature at any point t on the following curves :

(i) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

(ii) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

(iii) $x = a\sin 2t(1 + \cos 2t)$, $y = a\cos 2t(1 - \cos 2t)$

(iv) $x = at^2$, $y = 2at$

(v) $x = ae^t(\sin t - \cos t)$, $y = ae^t(\sin t + \cos t)$.

4. Find the radius of curvature at any point (r, θ) on the following curves :

(i) $r^2 = a^2\cos 2\theta$ (ii) $r = ae^{\theta\cot\alpha}$

(iii) $r^2\cos 2\theta = a^2$ (iv) $r = a\sin n\theta$.

5. Find the radius of curvature at any point on the following curves :

(i) $p = a(1 + \sin\psi)$ (ii) $p^2 + a^2\cos 2\psi = 0$

(iii) $p = r\sin\alpha$ (iv) $p^2 = ar$.

6. Find the radius of curvature at the origin on the following curves :

(i) $y = x^4 - 4x^3 - 18x^2$ (ii) $3x^2 + xy + y^2 - 4x = 0$

(iii) $x^2 + 6y^2 + 2x - y = 0$ (iv) $x^4 + y^2 = 6a(x + y)$

(v) $3x^4 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$.

7. Find the centre of curvature of the following curves :

(i) $a^2y = x^3$ (ii) $2y = a(e^{x/a} + e^{-x/a})$

(iii) $y = x^3 + 2x^2 + x + 1$ at (0, 1) (iv) $y = \sin^2 x$ at (0, 0)

(v) $x^2 = 4ay$ (vi) $xy = x^2 + 4$ at (2, 4).

8. Find the radius of curvature of the curve $r = \dfrac{1}{1+e\cos\theta}$ **at** $\theta = \pi$.

9. Find the centre of curvature of the curve $x^3 + 2x^2 + x + 1$ **at (0, 1).**

3.9 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental differential Calculus, Santra Publication Pvt. Ltd., India, 2018.
4. D. Sengupta, Application of Calcus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-4 ❑ Concavity, Convexity and Points of Inflection

Structure

4.1 Objectives

We all have intuitive concepts of concavity and convexity. After reading this lesson the students will learn :

- the definition of concavity and convexity.
- the criterion for convexity and concavity.
- the meaning of points of inflection.
- to determine the points of inflection.

4.2 Introduction

In this unit we shall discuss about the sense of concavity and convexity at a special point of a curve $y = f(x)$. This special point is called a point of inflection.

4.3 Concavity and Convexity w.r.t. to a Line and Points of Inflection

Let P be a point on a plane curve. Let l be a straight line not passing through P. Then the curve is

(i) concave at P w.r.t. the line l if a sufficiently small arc containing P lies within the acute angle formed by l and the tangent to the curve at P.

(ii) convex at P w.r.t. the line l if a sufficiently small arc containing P lies outside the acute angle formed by l and the tangent to the curve at P.

On the other hand, if the curve is concave on one side of P and convex on other side w.r.t. l, then evidently the curve crosses its tangent at P. This point P is called a point of inflection.

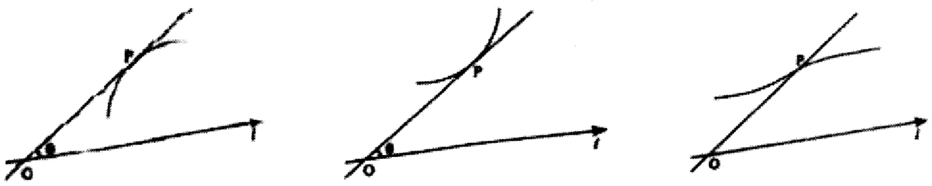


Fig. 4.1

4.4 Criterion of Concavity or Convexity w.r.t. x-axis

Let P (x, y) be a point of a curve $y = f(x)$ and $Q(x+\Delta x,\ y+\Delta y)$ be a neighbouring point of P. Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R. The equation of PT is

$$Y - y = f'(x)(X - x).$$

Since the abscissa X of R is $x + \Delta x$, its ordinate is

$$RM = Y = y + \Delta x f'(x).$$

Also the ordinate of Q is

$$QM = f(x+\Delta x)$$

$$= f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x+\theta\Delta x),\ 0<\theta<1.$$

[Using Taylor's theorem]

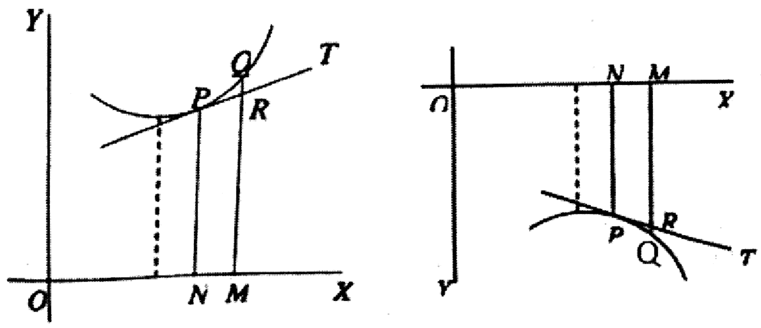


Fig. 4.2

Therefore

$$QM - RM = \frac{(\Delta x)^2}{2!} f''(x + \theta\Delta x).$$

If $f''(x)$ does not vanish and is continuous at x, $f''(x + \theta\Delta x)$ has the same sign as that of $f''(x)$ when $|\Delta x|$ is sufficiently small. Hence $QM - RM$ has the same sign as $f''(x)$ for sufficiently small values of $|\Delta x|$.

Thus, if $f''(x)$ is positive, then $QM - RM$ is positive for sufficiently small values of $|\Delta x|$ i.e., the small arc of the curve in a small neighborhood of P will be situated outside the acute angle formed by the tangent at P to the x-axis if the curve lie in the upper-side of x-axis.

Again if $f''(x)$ is negative, then $QM - RM$ is negative for sufficiently small values of $|\Delta x|$ i.e., the small arc of the curve in a small neighborhood of P will be situated outside the acute angle formed by the tangent at P to the x-axis if the curve lie in the lower-side of x-axis.

Combining this two situation we can say that the curve is convex at P to the x-axis if $yf''(x)$ at P is positive.

Analogously, if $yf''(x)$ at P is negative, then the curve at P is concave to the x-axis.

Note 4.4.1. The curve at P is convex or concave w.r.t. the y-axis according as $xf''(y)$ is positive or negative at P.

Example 4.4.1 Prove that the curve $y = e^{2x}$ is convex to the x-axis at every point.

Solution : Here

$$\frac{dy}{dx} = 2e^{2x}$$

$$\frac{d^2y}{dx^2} = 4e^{2x}.$$

Hence

$$y\frac{d^2y}{dx^2} = 4e^{4x} > 0, \text{ for all values of x.}$$

Thus, the curve is convex to the x-axis at every point.

4.5 Criterion for Points of Inflection

We have defined a point of inflection on the curve $y = f(x)$ as a point where the curve crosses its tangent. We have shown that such a point can only exist if $f'(x) = 0$. The abscissa of the points of inflection are therefore the roots of the equation

$$f''(x) = 0.$$

But the converse is not true.

From the previous discussion we see that if $f''(x) = 0$ and $f'''(x) \neq 0$.

$$QM - RM = \frac{(\Delta x)^3}{3!} f'''(x + \theta \Delta x).$$

This gives opposite sign for positive and negative value of Δx. Hence in order that the abscissa x corresponds to a point of inflection.

$$f''(x) = 0 \text{ and } f'''(x) \neq 0.$$

More general form :

Suppose that at P, $f''(x) = f'''(x) = \ldots = f^{(n-1)}(x) = 0$ *and* $f^{(n)}(x) \neq 0$.

Then by Taylors theorem,

$$QM - RM = \frac{(\Delta x)^n}{n!} f^{(n)}(x + \theta \Delta x).$$

If n is even, then the curve is convex at P to the the x-axis when $yf^{(n)}(x)$ at P is positive and concave at P to the the x-axis when $f^{(n)}(x)$ at P is negative.

If n is odd, then the point of inflection are the roots of the equation $f^{(n)}(x) = 0$.

Example 4.5.1 Show that the curve $y = x^3$ has a point of inflection at $x = 0$.

Solution : Here $\frac{dy}{dx} = 3x^2$ and $\frac{d^2y}{dx^2} = 6x$.

At $x = 0, \frac{d^2y}{dx^2} = 0$.

When $x < 0$ (sufficiently near to zero) $\frac{d^2y}{dx^2}$ remains negative so that the curve is concave downwards there. But when $x > 0$ (sufficiently near to zero) $\frac{d^2y}{dx^2}$ becomes positive so that the curve is concave upwards there. Hence $x = 0$ is a point of inflection.

Example 4.5.2 Examine the curve $y = \sin x$ regarding its concavity or convexity to the x-axis, and determine its point of inflection, if any.

Solution : Here $\frac{dy}{dx} = \cos x$ and $\frac{d^2y}{dx^2} = -\sin x$.

Hence $y\frac{d^2y}{dx^2} = -\sin^2 x$ which is negative for all values of x excepting those which make $\sin x = 0$, i.e., for $x = k\pi$, k being any integer.

Thus the curve is concave w.r.t. x-axis at every point except at points where the curve crosses the x-axis.

Hence these points given by $x = k\pi$, where $\frac{d^2y}{dx^2} = 0$, crosses the x-axis are points of inflection.

4.6 Summary

After reading this lesson we came to know a very important significance of second derivatives. It's change of values determines the concavity, convexity and point of inflection of a curve at a point.

4.7 Exercises

1. **Find the points of inflection, if any on the following curves.**

   (i) $y = \frac{x}{(x+1)^2 + 1}$ (ii) $y^2 = x(x+1)^2$

   (iii) $x = 3y^4 - 4y^3 + 5$ (iv) $y(x-a)^2 = a^2x$.

2. Prove that the curve $y = \cos^{-1} x$ is everywhere convex to the y-axis excepting where it crosses the y-axis.

3. Show that the curve $(y - a)^3 = a^3 - 2a^2x + ax^2$, $(a > 0)$ is concave to the x-axis.

4. Show that the curve $y = \log x$ is convex everywhere to the y-axis.

4.8 References

1. B.C. Das, B.N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India. 1949.
2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-5 ❑ Asymptotes

Structure

5.1 Objectives

In this chapter the students will learn the followings :

- definition of asymptote.
- type of asymptotes.
- the method for finding asymptotes of a curve.

5.2 Introduction

The concept that a curve may come arbitrary close to a line may introduce the word 'Asymptote' which was introduced by Apollonius of Perga in his work on conic sections. Asymptotes of a curve are very important to sketch its graph.

5.3 Definition

A point P with co-ordinate (x, y) on an infinite branch of a curve is said to tend to infinity ($P \to \infty$) along the curve if either x or y or both tend to $\pm\infty$ as P traverses along the branch of the curve.

5.4 Asymptotes

A straight line is said to be a rectilinear asymptote of an infinite branch of a curve if as a point P of the curve tends to infinity along the branch, the perpendicular distance of P from the straight line tends to zero.

Example 5.4.1 For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the straight lines $y = \pm x$ are two asymptotes as shown in the Fig. 5.1.

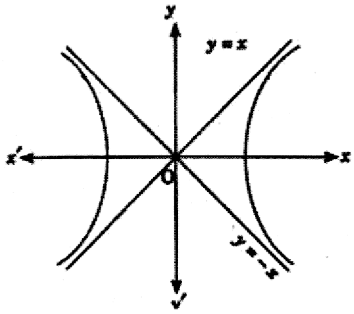


Fig. 5.1

5.5 Asymptotes Parallel to Axes

5.5.1. Asymptotes parallel to y-axis for the curve $y = f(x)$:

Theorem 5.5.1 A necessary and sufficient condition that the line $x = a$ may be an asymptote to the curve $y = f(x)$ is that $|f(x)| \to \infty$ as $x \to a+0$ or $x \to a-0$ or $x \to a$.

Proof : First suppose that $x \to a - 0$. Let $P(x, y)$ be a point on an infinite branch of the curve $y = f(x)$. As $|f(x)| \to \infty$, i.e., $y \to +\infty$ or $-\infty$ for $x \to a-0$, it immediately follows that $P \to \infty$. As $x \to a - 0$, the perpendicular distance PT of P from the line $x = a$ is $|x-a|$ which tends to zero. Hence $x = a$ is an asymptote.

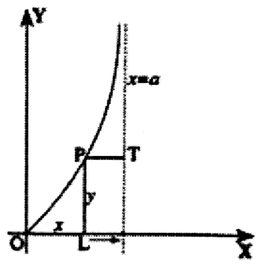


Fig. 5.2

Similar arguments follows for the cases $x \to a + 0$ or $x \to a$.

Conversely, let $x = a$ is an asymptote. Now as $x \to a + 0$ or $x \to a - 0$ or $x \to 0$, we must say $|f(x)| \to \infty$, otherwise P can not tend to ∞ which is essential for obtaining the asymptote.

In the same ideas we obtain the following :

5.5.2. Asymptotes parallel to *x*-axis for the curve $x = g(y)$

Theorem 5.5.2 A necessary and sufficient condition that the line $y = b$ may be an asymptote to the curve $x = g(y)$ is that $|g(y)| \to \infty$ as $y \to b+0$ or $y \to b-0$ or $y \to b$.

Example 5.5.1 Find the asymptotes parallel to the axes of the following curves :

(i) $y = f(x) = \dfrac{3x}{x-5}$, (ii) $y = f(x) = xe^{\frac{1}{x}}$, (iii) $x = \dfrac{t^2+1}{t^2-1}$, $y = \dfrac{t^2}{t-1}$.

Solution : (i) Here

$$\lim_{x\to 5+0} f(x) = \lim_{x\to 5+0} \frac{3x}{x-5} = +\infty$$

$$\lim_{x\to 5-0} f(x) = \lim_{x\to 5-0} \frac{3x}{x-5} = -\infty.$$

Hence $x = 5$ is the asymptote parallel to y-axis of the given curve.

(ii) Here

$$\lim_{x\to 0+0} f(x) = \lim_{x\to 0+0} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = +\infty.$$

Hence the curve has a vertical symptote $x = 0$.

(iii) As $t \to 1+0,\ x \to \infty,\ y \to \infty$ *and as* $t \to -1-0,\ x \to \infty,\ y \to -\frac{1}{2}$.

Hence $y = -\frac{1}{2}$ is an asymptote parallel to x-axis.

5.5.3. Asymptotes parallel to the axes for the rational algebraic curve $f(x, y) = 0$

Let the equation $f(x, y) = 0$, when arranged in descending powers of y be represented by

$$f(x,y) = y^n\phi_0(x) + y^{n-1}\phi_1(x) + y^{n-2}\phi_2(x) + \ldots + \phi_n(x) = 0, \tag{5.5.1}$$

where $\phi_0(x), \phi_1(x), \phi_2(x), \ldots\ldots, \phi_n(x)$ are polynomials in x.

Rearranging (5.5.1), we get

$$\phi_0(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots + \frac{1}{y^n}\phi_n(x) = 0. \tag{5.5.2}$$

If now there exists an asymptote parallel to y-axis, say $x = \lambda$, (a finite real number), then as $x \to \lambda$, $y \to \infty$ and so (5.5.2) gives

$$\phi_0(\lambda) = 0 \tag{5.5.3}$$

i.e., λ is a root of the equation

$$\phi_0(x) = 0, \tag{5.5.4}$$

where $\phi_0(x)$ is the coefficient of the highest degree terms in $f(x, y) = 0$.

If it so happens that $\lambda_1, \lambda_2, \ldots$ are the real roots of $\phi_0(x) = 0$, then $x = \lambda_1$, $x = \lambda_2, \ldots$ are the asymptotes parallel to y-axis provided the infinite branches of the curve corresponding to the asymptotes actually exist.

We now from algebra if $\lambda_1, \lambda_2, \ldots\ldots$ are the real roots of $\phi_0(x) = 0$, then $\phi_0(x) = (x - \lambda_1)(x - \lambda_1)(x - \lambda_2) \ldots\ldots$ Hence we have the following rules :

I. The asymptotes parallel to y-axis are determined by equating to zero the real linear factors in the coefficient of the highest power of y present in the equation $f(x, y) = 0$.

Note 5.5.1 No such vertical asymptotes exist if the coefficient of the highest power of y is a constant or not resolvable into real linear factors.

In similar manner we have the following rule :

II. The asymptotes parallel to x-axis are determined by equating to zero the real linear factors in the coefficient of the highest power of x present in the equation $f(x, y) = 0$.

Note 5.5.2 No such horizontal asymptotes exist if the coefficient of the highest power of x is a constant or not resolvable into real linear factors.

Example 5.5.2 Find the asymptotes of the curve

$$x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0,$$

which are parallel to axes.

Solution : This equation is in algebraic form. Here the highest of x is x^2 and its coefficient is $(y^2 - a^2)$. Hence the asymptotes parallel to x-axis are $y = a$, $y = - a$.

Similarly, we see that the asymptotes parallel to y-axis are $x = a$, $x = - a$.

Example 5.5.3 Find the asymptotes, if any, parallel to the co-ordinate axes to the curve $x^3 - 2x^2y + xy^2 + x - xy + 2 = 0$.

Solution : The coefficient of highest degree of x i.e., of x^3 is constant. Hence there is no asymptote parallel to x-axis.

The highest degree term in y is y^2 and its coefficient is x. Hence the asymptote parallel to y-axis is $x = 0$.

5.6 Oblique Asymptotes

Theorem 5.6.1 If an infinite branch of a curve possesses an asymptote $y = mx + c$, (m and c being finite), then

$$m = \lim_{|x|\to\infty} \frac{y}{x}; \quad c = \lim_{|x|\to\infty} (y - mx)$$

and conversely.

Proof Let $P(x, y)$ be a point on an infinite branch of a curve. The perpendicular distance of P from the line $y = mx + c$ is

$$d = \left| \frac{y - mx - c}{\sqrt{1+m^2}} \right|.$$

If the line $y = mx + c$ is an asymptote then d should tend to zero as $P \to \infty$,

$$i.e., \; d = \left| \frac{y - mx - c}{\sqrt{1+m^2}} \right| \to 0 \text{ as } |x| \to \infty,$$

$$\text{i.e., } |y - mx - c| \to 0 \text{ as } |x| \to \infty.$$

Hence

$$c = \lim_{|x|\to\infty}(y - mx).$$

Again

$$\lim_{|x|\to\infty}\left(\frac{y}{x} - m\right) = \lim_{|x|\to\infty}(y - mx).\lim_{|x|\to\infty}\frac{1}{x} = c.0 = 0.$$

Therefore

$$m = \lim_{|x|\to\infty}\frac{y}{x}.$$

Conversely, if the given condition holds, them $y - mx - c \to 0$ as $P \to \infty$ which means that $d \to 0$ as $P \to \infty$. Hence $y = mx + c$ is an asymptote.

Example 5.6.1 Examine the asymptotes of the curve $y = \frac{3x}{x-1} + 3x.$

Solution : Since

$$\lim_{x\to 1+0} y = \lim_{x\to 1+0}\left(\frac{3x}{x-1} + 3x\right) = +\infty$$

$$\lim_{x\to 1-0} y = \lim_{x\to 1-0}\left(\frac{3x}{x-1} + 3x\right) = -\infty,$$

the curve has a vertical asymptote $x = 1$.

Moreover for the oblique asymptotes

$$m = \lim_{|x|\to\infty}\frac{y}{x} = \lim_{|x|\to\infty}\left(\frac{3}{x-1} + 3\right) = 3,$$

$$c = \lim_{|x|\to\infty}(y - mx) = \lim_{|x|\to\infty}\left(\frac{3x}{x-1} + 3x - 3x\right) = 3.$$

Therefore the straight line $y = 3x + 3$ is an oblique asymptote.

5.7 Asymptotes Non-parallel to y-axis of the Rational Algebraic Curve $f(x, y) = 0$

Let the equation of the curve $f(x, y) = 0$ be arranged in groups of homogeneous terms as

$$f(x,y) = \left(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n\right)$$
$$+\left(b_1x^{n-1} + b_2x^{n-2}y + b_3x^{n-3}y^2 + \ldots + b_ny^{n-1}\right)$$
$$+ \ldots + \left(l_{n-1}x + l_ny\right) + k_n = 0,$$

which may be written as

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+\ldots+x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0, \tag{5.7.1}$$

where $\phi_r\left(\frac{y}{x}\right)$ denotes an algebraic polynomial in $\left(\frac{y}{x}\right)$ of degree r.

Dividing by x^n and then make $|x| \to \infty$, we suppose that $\lim\limits_{|x|\to\infty}\frac{y}{x}=m$. We then obtain

$$\phi_n(m)=0,$$

which gives the slopes of asymptotes correponding to different branches of the curve. To get c corresponding to m, we put $y - mx = k$, where $k \to c$ as $|x| \to \infty$.

Putting $\frac{y}{x}=m+\frac{k}{x}$ in (5.7.1) we obtain

$$x^n\phi_n\left(m+\frac{k}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{k}{x}\right)$$
$$+\ldots..+x\phi_1\left(m+\frac{k}{x}\right)+\phi_0\left(m+\frac{k}{x}\right)=0. \tag{5.7.2}$$

Expanding Taylor's expansion, we have

$$x^n\left\{\phi_n(m)+\frac{k}{x}\phi_n'(m)+\frac{k^2}{2x^2}\phi_n''(m)+\ldots\right\}$$
$$+x^{n-1}\left\{\phi_{n-1}(m)+\frac{k}{x}\phi_{n-1}'(m)+\frac{k^2}{2x^2}\phi_{n-1}''(m)+\ldots\right\}$$
$$= \ldots = 0 \tag{5.7.3}$$

Arranging (5.7.3) we get

$$x^n\phi_n(m)+x^{n-1}\left\{k\phi_n'(m)+\phi_{n-1}(m)\right\}$$
$$+x^{n-2}\left\{\frac{k^2}{2}\phi_n''(m)+k\phi_{n-1}'(m)+\phi_{n-2}(m)\right\}$$
$$+ \ldots = 0 \tag{5.7.4}$$

Since $\phi_n(m)=0$, dividing (5.7.4) by x^{n-1} and making $|x| \to \infty$, we get

$$c\phi_n'(m)+\phi_{n-1}(m)=0, \ \text{[since } k\to c \text{ as } |x|\to\infty] \tag{5.7.5}$$

$$\text{or, } c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)} \text{ provided } \phi'_n(m) \neq 0. \tag{5.7.6}$$

$$\text{Thus } y = mx - \frac{\phi_{n-1}(m)}{\phi'_n(m)}$$

is the asymptote corresponding to the slope m, provided $\phi'_n(m) \neq 0$.

Note 5.7.1 If $\phi'_n(m) = 0$ but $\phi_{n-1}(m) \neq 0$, then there is no asymptote corresponding to the slope m.

Note 5.7.2 If $\phi'_n(m) = 0 = \phi_{n-1}(m)$, then (5.7.5) becomes an identity and then dividing (5.7.4) by x^{n-2} and making $|x| \to \infty$, we have

$$\frac{c^2}{2}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2} = 0, \text{ [since } \to c \text{ as} |x| \to \infty] \tag{5.7.7}$$

which gives two values of c, in general, provided $\phi''_n(m) \neq 0$. Thus, we have two parallel asymptotes.

Note 5.7.3 The cases explained in Note 5.7.1 and Note 5.7.2. be treated similar manner for the next terms in the equation (5.7.4).

Remark 5.7.1 The polynomial $\phi_n(m)$ can be obtained by putting $x = 1$ and $y = m$ in the n^{th} degree homogeneous polymonial of $f(x, y)$.

Example 5.7.1 Find the asymptotes of the curve $xy^2 - y^2 - x^3 = 0$.

Solution : The coefficient of highest power of y is $(x - 1)$. Hence $x = 1$ is a vertical asymptote of the curve. The cofficient of highest power of x is constant. Hence there are no horizontal asymptotes.

Putting $x = 1$ and $y = m$ in highest degree (third degree) terms $(xy^2 - x^3)$ to get

$$\phi_3(m) = m^2 - 1; \quad \phi'_3(m) = 2m$$

$$\phi_3(m) = 0 \text{ gives } m = 1, -1.$$

Putting $x = 1$ and $y = m$ in the terms of 2nd degree $(- y^2)$, we have

$$\phi_2(m) = -m^2.$$

Now for $m=1, c=-\frac{\phi_2(1)}{\phi_3'(1)}=\frac{1}{2}$. Hence $y=x+\frac{1}{2}$ is an asymptote of the curve.

For $m=-1, c=-\frac{\phi_2(-1)}{\phi_3'(-1)}=-\frac{1}{2}$. Hence $y=-x-\frac{1}{2}$ is another asymptote of the curve.

Example 5.7.2 Find the asymptotes of the curve $x^3+x^2y-xy^2-y^3+x^2-y^2=2$.

Solution : The coefficients of highest power of x and y are constants. Hence there are no horizontal and vertical asymptotes.

Putting x = 1 and y = m in the highest degree (third degree) terms $\left(x^3+x^2y-xy^2-y^3\right)$ to get

$$\phi_3(m)=1+m-m^2-m^3;\quad \phi_3'(m)=1-2m-3m^2$$

$$\phi_3(m)=0 \text{ gives } m=1,\ -1,\ -1.$$

Putting x = 1 and y = m in the terms of 2nd degree ($x^2 - y^2$), we have

$$\phi_2(m)=1-m^2.$$

Now for $m=1, c=-\frac{\phi_2(1)}{\phi_3'(1)}=0$. Hence y = x is an asymptote of the curve.

For m = –1, since $\phi_3'(-1)=0=\phi_2(-1)$, the value of c can be obtained from

$$\frac{c^2}{2}\phi_3''(-1)+c\phi_2'(-1)+\phi_1(-1)=0$$

$$\text{or, } \frac{c^2}{2}.4+c.2+0=0.$$

Thus, c = 0, –1.

Hence y = –x and y = –x –1 are two parallel asymptotes of the curve.

5.8 An Alternative Method of Finding Asymptotes of Algebraic Curves

Let the equation of an rational algebraic curve of n^{th} degree be represented by

$$P_n+Q_{n-1}=0, \qquad (5.8.1)$$

where P_n is homogeneous polynomials in x and y of degree n and Q_{n-1} contain the terms of degree not higher thaan $n - 1$.

I. Let $y - m_1 x$ be a non-repeated factor of P_n. Then the equation (5.8.1) can be written as

$$(y - m_1 x)F_{n-1} + Q_{n-1} = 0, \qquad (5.8.2)$$

where F_{n-1} is homogeneous polynomials in x and y of degree $n - 1$.

Clearly m_1 is a root of $\phi_n(m) = 0$. Hence there exist an asymptote $y = m_1 x + c_1$ provided we can determine the value of c_1. Using art. 5.6 and the equation (5.8.2) we obtain

$$c_1 = \lim_{|x|\to\infty}(y - m_1 x) = -\lim_{|x|\to\infty}\frac{Q_{n-1}}{F_{n-1}},$$

where to determine the limiting value, we use $\lim_{|x|\to\infty}\frac{y}{x} = m_1$. Thus, the asymptote under this discussion is

$$y = m_1 x - \lim_{|x|\to\infty}\frac{Q_{n-1}}{F_{n-1}}.$$

For each non-repeated linear factor of the n^{th} degree homogeneous terms we may proceed in a similar manner.

II. If the n^{th} degree homogeneous terms in the equation of the curve contain $(y - m_1 x)^2$ as a factor and $(n - 1)^{th}$ degree homogeneous terms do not contain the factor $y - m_1 x$, then there is no asymptote corresponding to the slope m_1.

III. On the other hand, we could write the equation of the curve in the form

$$(y - m_1 x)^2 F_{n-2} + (y - m_1 x)P_{n-2} + Q_{n-2} = 0, \qquad (5.8.3)$$

where F_{n-2} contain the terms of degree not higher than $n - 3$. Then on similar arguments as in case I,

$$(y - m_1 x)^2 + (y - m_1 x).\lim_{|x|\to\infty}\frac{P_{n-2}}{F_{n-2}} + \lim_{|x|\to\infty}\frac{Q_{n-2}}{F_{n-2}} = 0$$

will be the pair of parallel asymptotes.

IV. We can proceed exactly in a similar manner if the n^{th} degree terms contain $(y - m_1 x)^3$ or higher power of $(y - m_1 x)$ as factor.

V. If in I, we have the factor ($ax + by + c$) instead of ($y - m_1x$), then the asymptote will be

$$ax+by+c+\lim_{|x|\to\infty}\frac{Q_{n-1}}{F_{n-1}}=0,$$

where $\lim_{|x|\to\infty}\frac{y}{x}=-\frac{a}{b}$.

Example 5.8.1 Find all the asymptotes of $x^3-2x^2y+xy^2+x^2-xy+2=0$.

Solution : The coefficient of highest power of y is x. Hence the asymptote parallel to y-axis is $x = 0$.

The coefficient of highest power of x is constant. Hence there is no asymptote parallel to x-axis.

Factorizing the terms of third degree, the given equation becomes

$$x(y-x)^2-x(y-x)+2=0.$$

Hence the parallel asymptotes will be given by

$$(y-x)^2-(y-x)\lim_{|x|\to\infty}\frac{x}{x}+\lim_{|x|\to\infty}\frac{2}{x}=0,$$

i.e., $(y-x)^2-(y-x)=0,$

or, $y - x = 0;\ y - x - 1 = 0.$

Thus, the three asymptotes are $x = 0;\ y - x = 0;\ y - x - 1 = 0$.

5.9 Asymptotes by Inspection

If the equation of a curve be of the form

$$F_n+F_{n-2}=0,$$

where F_n is a polynomial of degree n and F_{n-2} is a polynomial of degree ($n - 2$) at the most and if F_n can be broken up into n distinct linear factors so that when equated to zero they represent n straight lines, no two of which are parallel, then all the asymptotes of the curve are given by $F_n = 0$,

e.g., the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ have asymptotes $\frac{x}{a}\pm\frac{y}{b}=0$.

5.10 Summary

Throughout this unit we have learnt the meaning of asymptotes of a curve. We also studied several types of asymptotes and discussed the method to find out the equation of asymptote of a curve.

5.11 Exercises

Find the asymptotes of the following curves :

(i) $y^3 - x^2y - 2xy^2 - 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

(ii) $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y = 3$

(iii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

(iv) $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + 5x - 6y + 7 = 0$

(v) $y^3 = x(a^2 - x^2)$

(vi) $4x^3 - 3x^2y - y^3 + 2x^2 - xy - y^2 - 1 = 0$

(vii) $y(x - y)^3 - y(x - y) - 2 = 0$

(viii) $(x^2 - y^2)(x^2 - 9y^2) + 3xy - 6x - 5y + 2 = 0$

(ix) $(y + x + 1)(y + 2x + 2)(y + 3x + 3)(y - x) + x^2 + y^2 - 8 = 0$

(x) $(x + y)^2(x + 2y + 2) = x + 9y + 2.$

5.12 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.

2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.

3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.

4. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-6 ❑ Envelope

Structure

6.1 Objectives

After going through this unit, the learners will be able to :

- understand the definition of envelopes.
- determine the envelopes of family of curves.

6.2 Introduction

A curve which touches each member of a given family of curves is called envelope of that family. In this chapter we shall study the idea of envelope and its determination.

6.3 Family of Curves

Let $(x-\alpha)^2+y^2=a^2$, where a and α are fixed in a certain moment, but if we allow α to take a series of values keeping a fixed, then we have a series of cricles of equal radii a. A system of curves fromed in this way is called family of curves and the quantity which takes a series of values is called a parameter. We write the equation of a one parameter family of curves by a symbol

$$f(x,y,\alpha)=0.$$

We may think a two or three or more parameter family of curves. The equation of a two parameter family of curves is of the form

$$f(x, y, \alpha, \beta) = 0,$$

where α and β are arbitrary parameters : e.g., $(x-\alpha)^2 + (y-\beta)^2 = 1$ is a two parameter family of circles of radii 1. Also $(x-\alpha)^2 + (y-\beta)^2 = c^2$ gives the three parameter family of circles with center at any point of the plane and with any radius, i.e., the family of all circles on the plane.

6.4 Definitions

Definition 6.4.1 A point $P(a, b)$ is a singular point of a curve

$$f(x, y, \alpha) = 0 \ (\alpha \text{ is fixed}),$$

if it satisfies the curve as well as the two equations

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0.$$

The point P is said to be ordinary point if at least one of the two partial derivatives f_x, f_y is not zero at (a, b).

Definition 6.4.2 The characteristic points of a family of curves

$$f(x, y, \alpha) = 0 \ (\alpha \text{ is arbitrary})$$

are those points of the family where the two equations

$$f(x, y, \alpha) = 0 \ \textit{and} \ \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$$

simultaneously hold.

Note 6.4.1 If $f(x, y, \alpha) = 0$ and $\frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$ both hold for a point where $f_x = 0$ and $f_y = 0$ then the point is a singular point and therefore not a characteristic point.

Example 6.4.1 Find the characteristic points of the family of circles

$$(x-\alpha)^2 + y^2 = a^2 \ (\alpha \text{ is arbitrary}).$$

Solution : Solving the equations

$$f(x, y, \alpha) \equiv (x-\alpha)^2 + y^2 - a^2 = 0$$

$$\text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) \equiv -2(x-\alpha) = 0,$$

we get the points $(\alpha, \pm a)$. It can be easily shown that these points do not satisfy $f_x = f_y = 0$. Hence $(\alpha, \pm a)$ are the characteristic point of the family.

Definition 6.4.3 The envelope of a family of curves $f(x, y, \alpha) = 0$ (α is arbitrary) is the locus of their isolated characteristic points.

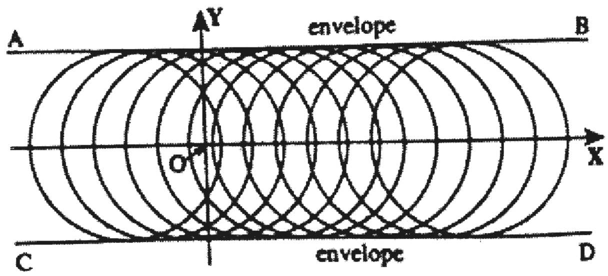


Fig. 6.1

Thus, envelope of a family of curves $f(x, y, \alpha) = 0$, (α being parameter) is a curve which touches every member of the family i.e., each point on the curve is touched by some member of the family.

6.5 Envelopes of One Parameter Family of Curves

If there exists an envelope of a family of curves, its equation may be obtained in either of the following ways :

I. Eliminate α between

$$f(x,y,\alpha) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad (6.5.1)$$

The elimination (an expression in x and y) is the envelope.

II. Solve for x and y in terms of α from the equation (6.5.1). It will give the parametric representation of the envelope.

III. For an algebraic curve, the equation of envelope obtained by eliminating α between

$$f(x,y,\alpha) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad (6.5.2)$$

is exactly the condition that the relation $f(x, y, \alpha) = 0$, considered as an equation in α, has a repeated root. Thus if

$$f(x,y,\alpha) = A(x,y)\alpha^2 + B(x,y)\alpha + C(x,y) = 0,$$

then the envelope is given by $B^2 - 4AC = 0$.

Example 6.5.1 Obtain the envelope of the family of ellipses $\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1$, α being the parameter.

Solution : We have

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1. \qquad (6.5.3)$$

Differentiating w.r.t. α, we obtain

$$\frac{-2x^2}{\alpha^3} + \frac{2y^2}{(a-\alpha)^3} = 0.$$

Thus,

$$\frac{x^2}{\alpha^3} = \frac{y^2}{(a-\alpha)^3}.$$

Therefore

$$\frac{\frac{x^2}{\alpha^2}}{\alpha} = \frac{\frac{y^2}{(a-\alpha)^2}}{(a-\alpha)} = \frac{\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2}}{a} = \frac{1}{a} \text{ [by (6.5.3)]}$$

Hence

$$\alpha = a^{\frac{1}{3}}x^{\frac{2}{3}}; \ (a-\alpha) = a^{\frac{1}{3}}y^{\frac{2}{3}}.$$

Putting this values in (6.5.3), we obtain

$$\frac{x^2}{a^{2/3}x^{4/3}} + \frac{y^2}{a^{2/3}y^{4/3}} = 1$$

i.e., $$x^{2/3} + y^{2/3} = a^{2/3}.$$

This is the required envelope.

Example 6.5.2 Find the envelope of the system of parabolas $\lambda x^2 + \lambda^2 y = 1$, being parameter.

Solution : Since the equation of family is $\lambda^2 y+\lambda x^2-1=0$, the quadratic form of parameter, the equation of envelope of the family is $x^4+4y=0$.

6.6 Envelopes of Two Parameter Family of Curves

Let

$$f(x, y, \alpha, \beta)=0 \tag{6.6.1}$$

be the family of curves involving two parameters α and β connected by the relation

$$\phi(\alpha, \beta) = 0. \tag{6.6.2}$$

We can find out the envelope by two methods.

I. First we express β in terms of α from the equation (6.6.2) and then substituting it in the equation (6.6.1) to obtain the equation (6.6.1) in one parameter family of curves. Hence as before the envelope of the family of curves will be determined.

II. Differentiating both of (6.6.1) and (6.6.2) partially with respect to α (α being regarded as independent variable whereas β is dependent variable) we obtain

$$\frac{\partial f}{\partial \alpha}+\frac{\partial f}{\partial \beta}\frac{d\beta}{d\alpha}=0 \text{ and } \frac{\partial \phi}{\partial \alpha}+\frac{\partial \phi}{\partial \beta}\frac{d\beta}{d\alpha}=0.$$

Eliminating $\frac{d\beta}{d\alpha}$ from the above equations we get

$$\frac{\partial f}{\partial \alpha}\Big/\frac{\partial \phi}{\partial \alpha}=\frac{\partial f}{\partial \beta}\Big/\frac{\partial \phi}{\partial \beta}. \tag{6.6.3}$$

Now eliminating α and β from (6.6.1), (6.6.2) and (6.6.3) we get the required envelope of the family of curves.

Example 6.6.1 Find the envelope of the family of ellipes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters are connected by $a + b = c$, c being constant.

Solution : Let the family of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{6.6.4}$$

where parameters are connected by the relation

$$a + b = c, \quad c \text{ being constant.} \tag{6.6.5}$$

Differentiating (6.6.4) and (6.6.5) w.r.t. a, we get

$$x^2(-2/a^3)+y^2(-2/b^3)\frac{db}{da}=0$$

and $$1+\frac{db}{da}=0.$$

Eliminating $\frac{db}{da}$, we obtain

$$\frac{x^2}{a^3}=\frac{y^2}{b^3}$$

i.e., $$\frac{x^2/a^2}{a}=\frac{y^2/b^2}{b}=\frac{x^2/a^2+y^2/b^2}{a+b}=\frac{1}{c}.$$ [from (6.6.4) and (6.6.5)]

Therefore $a=(cx^2)^{1/3}$ and $b=(cy^2)^{1/3}$. Hence from (6.6.5), we get

$$(cx^2)^{1/3}+(cy^2)^{1/3}=c$$

i.e., $$x^{2/3}+y^{2/3}=c^{2/3},$$

which is the required envelope.

Example 6.6.2 Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters are connected by $a + b = c$, c being constant.

Solution : Let the family of lines be

$$\frac{x}{a}+\frac{y}{b}=1, \qquad (6.6.6)$$

where parameters are connected by the relation

$$a + b = c, \text{ } c \text{ being constant.} \qquad (6.6.7)$$

From (6.6.7), we get $b = c - a$. Then the family of curves becomes

$$\frac{x}{a}+\frac{y}{c-a}=1, \qquad (6.6.8)$$

where a is only one parameter.

From (6.6.8), we get

$$x(c-a)+ya-a(c-a)=0$$

or, $$a^2+a(y-x-c)+cx=0,$$

which is a quadratic equation of the parameter a, Hence the required envelope is

$$(y-x-c)^2=4cx.$$

6.7 Summary

In the unit we discuss the definition of envelope of a family of curves and the method to work out the equation of envelope of the family of curves.

6.8 Exercises

Find the envelopes of the following families of curves :

(i) $y = mx + am^3$, m being parameter

(ii) $(x-a)^2 + (y-a)^2 = 2a$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^n + b^n = c^n$, c is constant

(iv) $\frac{x}{a} + \frac{y}{b} = 1$, where $\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1$, l, m are constants

(v) $y = mx + \sqrt{a^2m^2 + b^2}$

(vi) $x \cos \alpha + y \sin \alpha = 4$.

6.9 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.
4. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-7 ❑ Curve Tracing

Structure

7.2 Objectives

After reading this chapter, the learners will be able to :

- draw graph of a curve.
- know various properties of a curve.

7.1 Introduction

In this unit we study the systematic procedure to draw or sketch the graphs of curves given by the equations which are either in Cartesian Co-ordinates system or in Polar Co-ordinates system. With the knowledge of tangents, normals, curvature, asymptotes, singular points, extreme points, symmetry of curves etc., we may obtain a good idea to trace the shape of a curve.

7.3 Procedure of Curve Tracing in Cartesian Co-ordinate System

To draw a curve we need the following observations :

I. Symmetry : A curve is symmetric w.r.t.

(i) the x-axis, if its equation contains only even power of y and hence remains unchanged if y is replaced by $-y$.

(ii) the y-axis, if its equation contains only even power of x and hence remains unchanged if x is replaced by $-x$.

(iii) the line $y = x$, if its equation remains unchanged when x and y are interchanged.

(iv) the line $y = -x$, if its equation remains same when (x, y) is replaced by $(-y, -x)$.

(v) the origin, if its equation remains unchanged when (x, y) is replaced by $(-x, -y)$.

II. Intercepts : To obtain the points where the curve intersects the co-ordinate axes

(i) put $y = 0$ in the equation to get x intercepts.

(ii) put $x = 0$ in the equation to get y intercepts.

III. Passes through origin : The curve passes through the origin if its equation satisfy $x = 0$ and $y = 0$ simultaneously. If the curve passes through the origin, write down the equation of tangents at origin. If the origin is singular point, find the nature of singularity, cusp of various species, node, or isolated. Also check the origin is whether multiple point of higher order than two or not.

IV. Concavity, convexity or point of inflection : We determine the points where the curve has concavity, convexity or point of inflection.

V. Extreme points : We determine the points where the curve has extermum. We also determine the intervals where the curve increases or decreases. In fact, y increases or decreases for those values of x where $\frac{dy}{dx}$ is positive or negative respectively.

VI. Region of existence : By solving the equation of curve for one variable in terms of other and thereby we can find out the set of values of one variable which make the imaginary. In this way we can find the region of existence of the concerned curve to be traced.

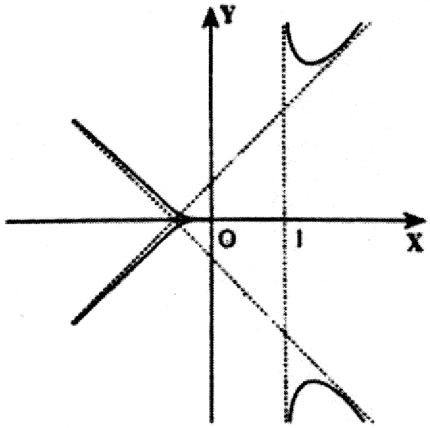
VII. Asymptotes : We determine the vertical or horizontal or oblique asymptotes if, any, to the curves having infinite branches. Also we determine the points where these meet the curve and the sides of the curves towards which this lie.

VIII. Periodicity : For trigonometric functions we check it is whether periodic or not. It will enable us greatly to sketch the graph of the curve.

Example 7.3.1. Trace the curve $y^2(x-1) = x^3$.

Solution : The curve is symmetric with respect to x-axis. The intercepts are $x = 0$ and $y = 0$. The curve exists in the range $-\infty < x \leq 0$ and $x > 1$ and for all values of y.

For the branch $y = x\sqrt{\frac{x}{x-1}}$, the point $x = 3/2$ gives the minimum point.



$y^2(x-1)-x^3=0$

Fig. 7.1

There is no point fo inflection. This branch is convex to the x-axis. The lines $x = 1$, $y = x+\frac{1}{2}$, $y = -x-\frac{1}{2}$ are asymptotes. The origin is a cusp of first species. $y = 0$ is the cuspidal tangent. Thus, the graph of the given curve is shown in Fig. 7.1.

7.4 Procedure of Curve Tracing in Polar Coordinate System

We also observe the following characteristics in tracing a curve $r = f(\theta)$ or $f(r, \theta) = 0$.

I. The curve passes through the pole if $r = 0$ for some values of θ.

II. If the values of r does not exist or becomes imaginary for some values of θ, say θ_1, θ_2 where $\theta_1 < \theta < \theta_2$, then the curve has no portion between the lines $\theta = \theta_1$ and $\theta = \theta_2$.

III. If a and b are respectively the minimum and maximum values of r, then the curve lies wholly within the circles $r = a$ and $r = b$.

IV. Observe the variation of the values of r for increasing and decreasing values of

θ from 0° in the anticlockwise and clockwise senses. In fact, if $\frac{dr}{d\theta} > 0$, r increases as θ increases and if $\frac{dr}{d\theta} > 0$, r decreases as θ increases.

V. When θ is replaced by $-\theta$, if it is observed that the equation remains unchanged, we say that the curve is symmetric about the line $\theta = 0$. If θ is replaced by $\pi - \theta$, and the equation remains unchanged, the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

Also the curve is symmetric about the pole if the equation of the curve does not change when θ is replaced by $\pi + \theta$.

VI. Let ϕ be the angle between the radius vector and the tangent to the curve at a point (r, θ). Then we know that

$$\tan\phi = r\frac{d\theta}{dr}.$$

If $\phi = 0$ for some values of θ, say θ_1, then the line $\theta = \theta_1$ is a tangent to curve at $\theta = \theta_1$ and if $\phi = \pi/2$ for some values of θ, say θ_2, then at the point $\theta = \theta_2$, the tangent is perpendicular to the line $\theta = \theta_2$.

Example 7.4.1 Trace the curve $r = a \sin 3\theta$, $a > 0$ (Rose-petal).

Solution : We observed the followings :

(i) $r = 0$ for $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

Here sin 3θ is periodic function of period 2π, hence we consider only those values of θ ranges from 0 to 2π, whreas the remaining values of θ yield no new branches of the curve.

(ii) We also observe that the curve passes through the pole.

(iii) Here $\tan\phi = r\frac{d\theta}{dr} = \frac{r}{3a\cos 3\theta}$.

Hence $\phi = 0$ for $r = 0$ and the corresponding values of θ are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Thus $\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \pi, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}$ are the tangents to the curve at pole.

(iv) Replacing θ by $\pi - \theta$, it follows that the equation of the curve remains unchanged and hence the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

(v) As $-1 \leq \sin 3\theta \leq 1$, the maximum value fo r is a. Consequently, the curve lies wholly within a circle of radius a.

(vi) Table of variation of the values of r and θ :

θ :	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r :	a	0	$-a$	0	a	0	$-a$

Thus r increases from 0 to a when θ increases from 0 to $\pi/6$; r then decreases from a to 0 when θ increases from $\pi/6$ to $\pi/3$ and r increases numerically from 0 to a, when θ increases from $\pi/3$ to $\pi/2$, the portion of which lies in the third quadrant.

The curve being symmetric about the line $\theta = \pi/2$, we consider the table of variations for r as θ increases from $-\pi/2$ to $\pi/2$.

With all these facts which we trace the curve as given in Fig. 7.2.

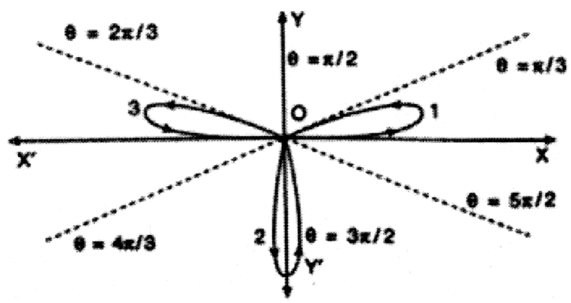


Fig. 7.2

Note 7.4.1 The curve $r = a \sin n\theta$, $a > 0$, n = an integer (rose-petals), traces is similar loops as given in Fig. 7.2 lying wholly within a circle of radius a with center at the origin and are symmetric about the pole. In case n be odd, there are n-loops and if n be even, the number of loops are $2n$.

The order in which loops occurs as θ increases from 0 to 2π are mentioned in the figure by numbers.

In the following Fig. 7.3 we trace the curve $r = a$ ain 2θ, $a > 0$.

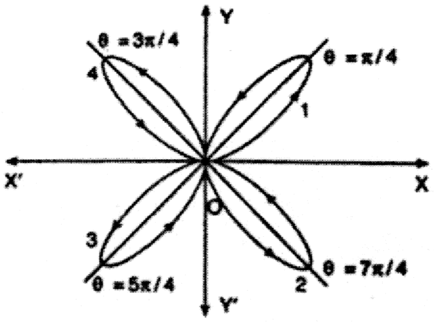


Fig. 7.3

Example 7.4.2 Trace the curve $r^2 = a^2 \cos 2\theta$ (Lemniscate of Bernouli).

Solution : Here the curve satisfies the following conditions :

(i) Replacing θ by $-\theta$ and $\pi - \theta$, it is observed that the equation remains unchanged. Hence the curve is symmetric about the intial line and the line $\theta = \pi/2$.

(ii) When $r = 0$, $\theta = \pm\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$.

(iii) Table of variations fo r and θ :

θ :	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	π
r :	0	$\pm\frac{a}{\sqrt{2}}$	$\pm a$	$\pm\frac{a}{\sqrt{2}}$	0	$\pm a$

(iv) As maximum of $\cos 2\theta$ is 1, maximum value of r is a and the curve lies wholly within the circle fo radius a with center at the pole.

(v) Here $\cos 2\theta$ is positive for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$ and we get real values of r there at. But for θ satisfying $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $\cos 2\theta$ becomes negative and as such r becomes imaginary.

Also r increases for $-\frac{\pi}{4} < \theta < 0$ and r decreases for $0 < \theta < \frac{\pi}{4}$.

Thus the curve has two similar loops and we trace the curve as given in the Fig. 7.4.

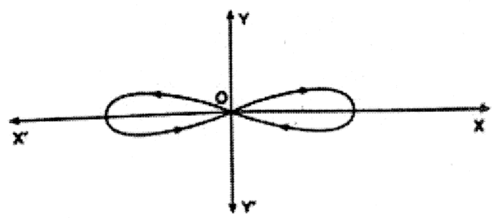


Fig. 7.4

7.5 Some Well Known Curves

1. Cycloid : $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

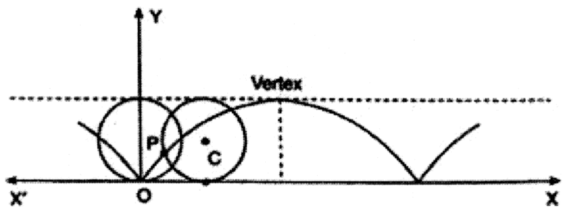
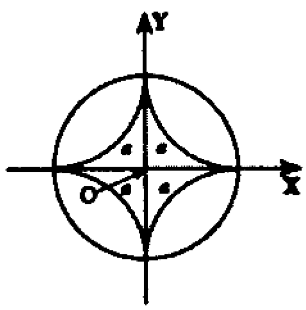


Fig. 7.5

2. Astroid : $x = a\cos^3\theta$, $y = a\sin^3\theta$ $\left(x^{2/3} + y^{2/3} = a^{2/3}\right)$.



$x^{2/3} + y^{2/3} = a^{2/3}$

Fig. 7.6

3. Cardioid : $r = a(1 - \cos\theta)$.

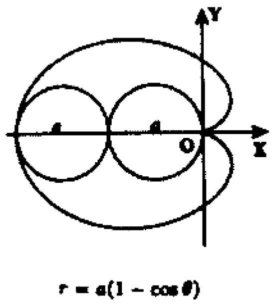


Fig. 7.7

4. Follium of Descartes : $x^3 + y^3 = 3axy$.

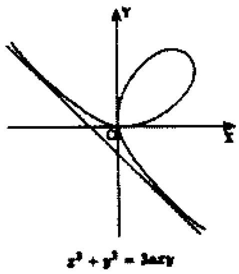


Fig. 7.8

5. Equiangular Spiral : $r = ae^{\theta \cot \alpha}$.

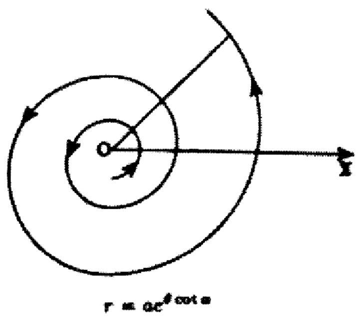


Fig. 7.9

7.6 Summary

In this unit we discuss the procedure to draw a graph of a given curve in Cartesian or Polar co-ordinate system. We draw the graph of curves using their properties. We sketched also some well known curves.

7.7 Exercises

Trace the following curves :

(i) $x^3 + y^3 = 3axy$ (Folium of Descartes)

(ii) $\left(a^2 + x^2\right)y = a^2 x$

(iii) $x^5 + y^5 = 5a^2 x^2 y$

(iv) $x = a(t + \sin t),\ y = a(1 - \cos t)$ (Cycloid)

(v) $r = a + b \cos \theta$, $(a < b)$

(vi) $y = \cosh x/c$

(vii) $r = a\sin \theta \tan \theta$.

7.8 References

1. B.C. Das, B.N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India, 1949.

2. R.K. Ghosh, K.C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.

3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.

4. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd. Kolkata, India, 2012.

Unit-8 ❑ L'Hospital's Rule

Structure

8.1 Objectives

After going through this unit, the learners will be able to :

- understand the L'Hospital's Rule.
- determine the limits of indeterminate forms.

8.1 Introduction

In this unit we investigated a very important application of mean value theorem.

In the case $\lim\limits_{x\to a}\dfrac{\phi(x)}{\psi(x)}=\dfrac{\lim\limits_{x\to a}\phi(x)}{\lim\limits_{x\to a}\psi(x)}$, if both the limits $\lim\limits_{x\to a}\phi(x)$ and $\lim\limits_{x\to a}\psi(x)$ are zero then we face with the problem like 0/0 which is meaningless. Such a case is known as Indeterminate form.

Other indeterminate forms are ∞/∞, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. For evaluation of indeterminate forms ∞/∞ or 0/0 we shall use a particular device known as L'Hospital's Rule.

8.3 L'Hospital's Rule

8.3.1. 0/0 form :

The quotient $\dfrac{f(x)}{g(x)}$ of two functions $f(x)$ and $g(x)$ is undefined at $x = a$ if $g(a) = 0$. But if $f(a) = 0 = g(a)$, then the ratio is of the indeterminate form 0/0 and we can determine the limit of the ratio $\frac{f(x)}{g(x)}$ at $x = a$ by the conception of derivatives. In this connection we state a basic theorem known as L'Hospital's Rule.

* **L' Hospital's rule :** If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a + h]$,

(ii) derivable in the open interval $(a, a + h)$ and

(iii) $\lim_{x\to a+0} f(x) = 0 = \lim_{x\to a+0} g(x)$, $h > 0$ is a suitably small number,

then,

$$\lim_{x\to a+0} \frac{f(x)}{g(x)} = \lim_{x\to a+0} \frac{f'(x)}{g'(x)},$$

provided the limit of the right hand side exists.

Proof : From Cauchy mean value theorem we obtain

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}, \ a < c < x < a+h.$$

$$\text{i.e.,} \quad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \text{ as } f(a) = 0 = g(a).$$

Therefore $\lim_{x\to a+0} \frac{f(x)}{g(x)} = \lim_{x\to a+0} \frac{f'(c)}{g'(c)}$.

Again since $a < c < x$, $c \to a + 0$ when $x \to a + 0$, we get

$$\lim_{x\to a+0} \frac{f(x)}{g(x)} = \lim_{x\to a+0} \frac{f'(c)}{g'(c)} = \lim_{x\to a+0} \frac{f'(x)}{g'(x)}.$$

Note 8.3.1 It can be similarly shown that

$$\lim_{x\to a-0} \frac{f(x)}{g(x)} = \lim_{x\to a-0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note 8.3.2 If $\lim_{x\to a+0} \frac{f'(x)}{g'(x)}$ does not exist, then we should not conclude that $\lim_{x\to a+0} \frac{f(x)}{g(x)}$ does not exist.

Note 8.3.3 L'Hospital's Rule also holds when $a = \infty$.

***Generalization of L'Hospital's Rule :** If also $\lim_{x\to a} f'(x) = 0 = \lim_{x\to a} g'(x)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)},$$

provided the last limit exists.

We continue in this manner until one of the derivative $g^{(r)}(a) \neq 0$.

Example 8.3.1 Evaluate the limit $\lim_{x\to 0}\frac{\log(1+x)}{x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\log(1+x)}{x} \left[\frac{0}{0}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{1}{1+x}$$

$$= 1.$$

Example 8.3.2 Evaluate the limit $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\tan x - x}{x - \sin x} \left[\frac{0}{0}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{1 - \cos x} \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2\sec^2 x \tan x}{\sin x}$$

$$= \lim_{x\to 0} 2\sec^3 x$$

$$= 2.$$

8.3.2. ∞/∞ form :

If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a + h]$,

(ii) derivable in the open interval $(a, a + h)$ and

(iii) $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$, where $h > 0$ is a suitably small number,

then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit of the right hand side exists.

Example 8.3.3 Evaluate the limit $\lim_{x\to 0}\frac{\log x^2}{\log\cot^2 x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\log x^2}{\log\cot^2 x}\left[\frac{\infty}{\infty}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{f'(x)}{g'(x)}\\ &= \lim_{x\to 0}\frac{\frac{1}{x^2}.2x}{\frac{1}{\cot^2 x}.2\cot x(-\operatorname{cosec}^2 x)}\\ &= -\lim_{x\to 0}\frac{\sin x\cos x}{x}\left[\frac{0}{0}\text{ form}\right]\\ &= -\lim_{x\to 0}\frac{\cos^2 x-\sin^2 x}{1}\\ &= -1.\end{aligned}$$

8.3.3. ∞ – ∞ from :

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$. To find $\lim_{x\to a}\{f(x) - g(x)\}$, we take

$$f(x) - g(x) = \frac{1/g(x) - 1/f(x)}{1/\{f(x)g(x)\}},$$

which is of the form 0/0 and can be evaluated by the method discussed in 8.3.1.

Example 8.3.4 Evaluate the limit $\lim_{x\to 0}\left\{\frac{1}{x} - \frac{2}{x(e^x+1)}\right\}$.

Solution : Here

$$\lim_{x\to 0}\left\{\frac{1}{x} - \frac{2}{x(e^x+1)}\right\} [\infty - \infty \text{ form}]$$

$$= \lim_{x\to 0}\frac{e^x - 1}{x(e^x+1)} \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{e^x}{e^x + 1 + xe^x} [\text{using L' Hospital's Rule}]$$

$$= \frac{1}{2}.$$

8.3.4. 0 × ∞ form :

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to\alpha} f(x) = 0$ and $\lim_{x\to\alpha} g(x) = \infty$.

To find $\lim_{x\to\alpha}\{f(x)g(x)\}$, we take

$$f(x)g(x) = \frac{f(x)}{1/\{g(x)\}},$$

which is of the form 0/0 and can be evaluated by the method discussed in 8.3.1.

Example 8.3.5 Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$.

Solution : Here

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} \quad [0\times\infty \text{ form}]$$

$$= \lim_{x\to 1}\frac{1-x}{\cot\frac{\pi x}{2}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}} \qquad [\text{by L' Hospital's Rule}]$$

$$= \frac{2}{\pi}.$$

8.3.5. 0^0, ∞^0, $1^{\pm\infty}$ forms :

The three exponential forms 0^0, ∞^0, $1^{\pm\infty}$ are dealt with by taking their logarithms and each of the forms is reduced to the form $0 \times \infty$ already discussed in 8.3.4.

Example 8.3.6 Find the limit $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

Solution : Here the limit is of the form 1^∞.

Let $y=\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. Then $\log y=\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$. Thus

$$\lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\text{ form}\right].$$

Using L'Hospital's Rule we get

$$\lim_{x\to 0}\log y$$

$$= \lim_{x\to 0}\frac{\frac{x}{\tan x}\cdot\frac{x\sec^2 x-\tan x}{x^2}}{2x}=\lim_{x\to 0}\frac{x-\frac{1}{2}\sin 2x}{x^2\sin 2x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{2x\sin 2x+2x^2\cos 2x} \quad \left[\frac{0}{0}\text{ form}\right] \; [\text{using L'Hospital's Rule}]$$

$$= \lim_{x\to 0}\frac{2\sin 2x}{2\sin 2x+4x\cos 2x+4x\cos 2x-4x^2\sin 2x} \quad [\text{using L'Hospital's Rule}]$$

$$= \lim_{x\to 0}\frac{\sin 2x}{\sin 2x+4x\cos 2x-2x^2\sin 2x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{2\cos 2x}{2\cos 2x + 4\cos 2x - 8x\sin 2x - 4x\sin 2x - 4x^2\cos 2x}$$

[using L'Hospital's Rule]

$$= \frac{2}{2+4} = \frac{1}{3}.$$

Thus, $\lim_{x\to 0}\log y = \frac{1}{3}$. We know that $\lim_{x\to 0}\log y = \log\left(\lim_{x\to 0} y\right)$. Therefore, $\log\left(\lim_{x\to 0} y\right) = \frac{1}{3}$.

Hence $\lim_{x\to 0} y = e^{1/3}$ *or*, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}$.

8.4 Summary

In this unit we have learnt a very important technique to evaluate the limits in the indeterminate forms. We gave some examples to understand the technique.

8.5 Exercises

Evaluate the following limits :

(i) $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x}$

(ii) $\lim_{x\to 0}\frac{e^{2x} - 1}{\log(1+x)}$

(iii) $\lim_{x\to \infty}\frac{x + \log x}{x\log x}$

(iv) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$

(v) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$

(vi) $\lim_{x\to 0}\left(\frac{x - \sin x}{\tan^3 x}\right)$

(vii) $\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}\right)$

(viii) $\lim_{x\to 1}\left(\frac{\log(1-x)}{\cos(\pi x)}\right)$.

8.6 References

1. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
2. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.

Unit-9 ❑ Applications in Business Economics and Life Sciences

Structure

9.1 Objectives

After studying this lesson the readers should be able to :

- understand the concepts related to business economics and life sciences with the applications of calculus.
- apply the techniques of differentiation to solve business economics and life science problems.

9.2 Introduction

Calculus is a very important part of our real life. But many of us ask how calculus help us in real life. In this unit we are going to discuss how it is useful in our real life. Calculus is used to determine the right time for buying and selling of products. It helps economists to grow up their business economics.

Biologists also make use of calculus to determine the growth rate of bacteria, modeling population growth and so on. In medical field calculus is also useful.

Calculus is required by architects, engineers to build roads, bridges, tunnels, building etc. and without the use of calculus our real life is unsafe.

9.3 Definitions Related to Business Economics

Total Cost (TC) : Total cost is the combinations of fixed cost and variable cost of output. If the production increases, only variable cost will increase in direct proportion but the fixed cost will remain unchanged within a relevant range.

Total Revenue (TR) : Total revenue is the product of price/ demand functions and output.

Profit (P) : Profit are defined as the excess total revenue over total cost. Symbolically it can be expressed as

$$P = TR - TC.$$

The rules for finding a maximum point tell us that P is maximized when the derivative of the profit function is equal to zero and te second derivative is negtive. If we denote the derivatives of the total revenue and total cost functions by dTR and dTC, we have P will be maximum when $dTR - dTC = 0$.

Hence the derivative of the total revenue function must be equal to the derivative of the total cost function for profit to be maximized. Hence

$$\text{Profit maximizing output} = \frac{d(\text{profit function})}{dx}.$$

Therefore in case of maximization, the conditions are $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$.

Similarly, we have

$$\text{Cost minimizing output} = \frac{d(\text{total cost function})}{dx},$$

and the conditions of minimizing are $\frac{dTC}{dx} = 0$ and $\frac{d^2TC}{dx^2} > 0$.

Merginal Cost (MC) : Marginal cost is the extra cost for producing one additional unit when the total cost at certain level of output is known. Hence it is the rate of change in total cost with respect to the level of output at the point where the total cost is known. Therefore we have $MC = \frac{dTC}{dx}$.

Marginal Production (MP) : Marginial production is the incremental production i.e., the additional production added to the total production (TP), i.e.,

$$MP = \frac{dTP}{dx}.$$

Marginal Revenue (MR) : Marginl revenue is defined as the change in the total revenue for the sale of an extra unit. Hence it is the rate of change in total revenue with respect to the quantity demanded at the point where total revenue is known. Therefore we have

$$MR = \frac{dTR}{dx},$$

where total revenue is the function of x, the quantity demanded.

Example 9.3.1 Let the profit function of a company is given by $P = f(x) = x - 0.00001x^2$, where x is units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

Solution : Here $P = f(x) = x - 0.00001x^2$. The profit will be maximum if $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$.

Now

$$\frac{dP}{dx} = 1 - 0.00002x.$$

Hence

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 1 - 0.00002x = 0$$

$$\Rightarrow \quad x = 50000 \text{ units.}$$

Also $\frac{d^2P}{dx^2} = -0.00002 < 0$. Hence The profit will be optimum for the sales of 50000 units and the profit will be

$$P = 50000 - 0.00001.(50000)^2$$

$$= 25000 \text{ currency units.}$$

9.4 Applications in Life Sciences

9.4.1. Density Dependent (Logistic) Growth in a Population

Biologists have seen that the growth rate of a population depends not only on the size of the population but also on how crowded it is. Constant growth is not sustainable.

When individuals have to compete for resources, nesting sites, mates, or food, they cannot invest time or energy in reproduction, leading to a decline in the rate of growth of the population. Such population growth is called density depndent growth.

We suppose that the growth rate of the population is G, which depends on the density of the population, N as follows :

$$G(N) = rN\left(\frac{K-N}{K}\right).$$

Here N is the independent variable and $G(N)$ is the function of interest. All other quantities are constant :

- $r > 0$ is a constant, called the intrinsic growth rate.
- $K > 0$ is a constant, called carrying capacity. It represents the population density that a given environment can sustain.

Example 9.4.1 (i) Find the population density N that leads to the maximal growth rate $G(N)$.

(ii) Find the value of the maximum growth in terms of r and K.

(iii) For what population size is the growth rate zero ?

Solution : We can rewrite $G(N)$ as

$$G(N) = rN\left(\frac{K-N}{K}\right) = rN - \frac{r}{K}N^2,$$

from which it is apparent that $G(N)$ is a polynomial in powers of N, with constant coefficients r and r/K.

(i) To find critical points of $G(N)$, we find N such that $G(N) = 0$, and then test for maxima :

$$G'(N) = r - 2\frac{r}{K}N = 0 \quad \Rightarrow N = \frac{K}{2}.$$

Hence $N = \frac{K}{2}$ is a critical point, but is it a maximum ? We check this as follows :

$$G''(N) = -2\frac{r}{K} < 0.$$

Thus $N = \frac{K}{2}$ is the maximum point. Therefore the population density with the greatest growth rate is $K/2$.

(ii) The maximal growth is give by

$$G(K/2) = r.K/2 - r/K.(K/2)^2 = \frac{rK}{2}.$$

(iii) To find out the population size at which the growth rate is zero, we solve the equation

$$G(N) = rN\left(\frac{K-N}{K}\right) = rN - \frac{r}{K}N^2 = 0.$$

There are two solution, one is $N = 0$ and other is $N = K$. The solution $N = 0$ is biologically interesting in the sense that life can arise on its own. So no population arises to logistic growth. The solution $N = K$ means that the population is at its carrying capacity.

9.4.2 Cell Size for Maximum Nutrient Accumulation Rate

The nutrient absorption and consumption rates, $A(r)$ and $C(r)$, of a simple spherical cell of radius r are

$$A(r) = k_1 S = 4k_1\pi r^2,\ C(r) = k_2 V = \frac{4}{3}\pi k_2 r^3,$$

for k_1, $k_2 > 0$ constants,

The net rate of increase of nutrients, which is the difference of the two is

$$N(r) = A(r) - C(r) = 4k_1\pi r^2 - \frac{4}{3}\pi k_2 r^3.$$

This quantity is the function of radius r of the cell.

Example 9.4.2 Determine the radius of the cell for which the net rate of increase of nutrients $N(r)$ is largest.

Solution : We know

$$N(r) = A(r) - C(r) = 4k_1\pi r^2 - \frac{4}{3}\pi k_2 r^3.$$

Differentiating w.r.t. r we get

$$N'(r) = 8k_1\pi r - 4\pi k_2 r^2.$$

To find the larget nutrients rate the condition of critical points is $N'(r) = 0$. Hence $8k_1\pi r - 4\pi k_2 r^2 = 0 \Rightarrow r = 0,\ 2\frac{k_1}{k_2}$. To test the critical points for extereme we differentiate again to have

$$N''(r) = 8k_1\pi - 8\pi k_2 r.$$

Now at $r = 2\frac{k_1}{k_2}$,

$$N''\left(2\frac{k_1}{k_2}\right) = 8k_1\pi - 8\pi k_2 . 2\frac{k_1}{k_2} = -8\pi k_1 < 0.$$

Hence the net rate of increase of nutrients $N(r)$ is largest for $r = 2\frac{k_1}{k_2}$.

9.5 Summary

Calculus was invented from the visions of master minds. It took little time to break through the bridge of theoretical inquiry to practical skills of human activities. The application of the novel methods of calculus enabled to determine the timing of buying, selling the products and to help us to know how much units should be sold to maximize profit. Calculus also determines the activities in our human body.

9.6 Exercises

1. If the total cost y of manufacturing x units of a production is given by $y = 20x + 5000$, then

(i) What is the variable cost per unit ?

(ii) What is the fixed cost ?

(iii) What is the total cost of manufacturing 4000 units ?

(iv) What is the marginal cost of producing 2000 units ?

2. The total cost of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ and market demand is $P = 2530 - 5x$, where x is the no. of units of production. Find the profit maximizing price.

9.7 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India, 1949.

2. L. E. Keshet, Differential Calculus for the life sciences, University of British Columbia, 2017.

Unit-10 ❑ Reduction Formula

Structure

10.2 Objectives

After going through this chapter, the learners will be able to :

- derive the reduction formula of some standard integral problems.
- understand the technique of integration to derive the reduction formula.

10.1 Introduction

A Reduction formula is one that enables us to solve an integral problem by reducing it to an easier integral problem, and then reducing that to the more easier integral problem, and so on. Reduction formulae are mostly obtained by the process of integration by parts.

10.3 Derivation of Reduction Formulae

10.3.1. Reduction Formula for $\int x^n e^{ax} dx$, n being a positive integer :

Let

$$I_n = \int x^n e^{ax} dx$$

$$= x^n . \frac{e^{ax}}{a} - \int n x^{n-1} . \frac{e^{ax}}{a} dx. \quad \text{[Int. by parts]}$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx.$$

Thus,

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}.$$

Example 10.3.1 Find $\int x^4 e^{ax} dx$.

Solution : Applying reduction formula, we have

$$I_4 = \frac{x^4 e^{ax}}{a} - \frac{4}{a} I_3, \quad I_3 = \frac{x^3 e^{ax}}{a} - \frac{3}{a} I_2$$

$$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1, \quad I_1 = \frac{x e^{ax}}{a} - \frac{1}{a} I_0$$

and $$I_0 = \int x^0 e^{ax} dx = \frac{e^{ax}}{a}.$$

Hence

$$I_4 = \frac{x^4 e^{ax}}{a} - \frac{4x^3 e^{ax}}{a^2} + \frac{12x^2 e^{ax}}{a^3} - \frac{24x e^{ax}}{a^4} + \frac{24 e^{ax}}{a^5} + c.$$

10.3.2. Reduction Formula for $\int sin^n x\, dx$ and $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$, *n* being a positive integer greater than 1

Here

$$I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx$$

$$= \sin^{n-1} x(-\cos x) - \int (n-1)\sin^{n-2} x \cos x(-\cos x)dx \quad \text{[Int. by parts]}$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n.$$

Simplifying the above we get the reduction formula

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}.$$

Furthermore, we take

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$= -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} J_{n-2}.$$

Thus, in this case we have the reduction formula as

$$J_n = \frac{n-1}{n} J_{n-2}.$$

Example 10.3.2 Find $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$.

Solution : Applying reduction formula, we have

$$J_5 = \frac{4}{5} J_3, \quad J_3 = \frac{2}{3} J_1$$

$$\text{and} \quad J_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = -[\cos x]_0^{\frac{\pi}{2}} = 1.$$

Hence

$$J_5 = \frac{4}{5} . \frac{2}{3} . 1 = \frac{8}{15}.$$

10.3.3. Reduction Formula for $\int \cos^n x \, dx$ and $\int_0^{\frac{\pi}{2}} \cos^n x \, dx$, n being a positive integer greater than 1

Taking $I_n = \int \cos^n dx = \int \cos^{n-1} x. \cos x \, dx$ and proceeding as in the previous article, we may find

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{and} \quad J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{n-1}{n} J_{n-2}.$$

10.3.4. Reduction Formula for $\int \sin^m x \cos^n x \, dx$, m, n being positive integers greater than 1

Let

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \cos^{n-1} x (\sin^m x \cos x) dx$$

$$= \cos^{n-1} x . \frac{\sin^{m+1} x}{m+1} - \int (n-1) \cos^{n-2} x (-\sin x) . \frac{\sin^{m+1}}{m+1} dx$$

[Int. by parts, taking $u = \cos^{n-1} x$, $dv = \sin^m x \cos x \, dx$]

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \sin^2 x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \left(1 - \cos^2 x\right) dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}.$$

Simplifying we obtain

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}. \qquad (10.3.1)$$

In the similar manner, if we take

$$I_{m,n} = \int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x (\cos^n \sin x) dx$$

integrating by parts, taking $u = \sin^{m-1} x$, $dv = \cos^n x \sin x \, dx$, we obtain

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}. \qquad (10.3.2)$$

10.3.5. Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$, *m, n* being positive integers greater than 1

Take

$$J_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

$$= \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n} \right]_0^{\frac{\pi}{2}} + \frac{n-1}{m+n} J_{m,n-2} = \frac{n-1}{m+n} J_{m,n-2}$$

$$= -\left[\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right]_0^{\frac{\pi}{2}} + \frac{m-1}{m+n} J_{m-2,n} = \frac{m-1}{m+n} J_{m-2,n}.$$

Therefore

$$J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} = \frac{m-1}{m+n} J_{m-2,n}. \qquad (10.3.3)$$

Example 10.3.3 Find $\int \sin^4 x \cos^2 x\, dx.$

Solution : Applying reduction formula, we have

$$I_{4,2} = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6} I_{2,2} \quad \text{[from (10.3.2)]}$$

$$= -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left(-\frac{\sin x \cos^3 x}{4} + \frac{1}{4} I_{0,2} \right) \quad \text{[from (10.3.2)]}$$

$$= -\frac{\sin^3 x \cos^3 x}{6} - \frac{\sin x \cos^3 x}{8} + \frac{1}{8}\left(\frac{\sin x \cos x}{2} + \frac{1}{2} I_{0,0} \right) \quad \text{[from (10.3.1)]}$$

Also

$$I_{0,0} = \int \sin^0 x \cos^0 x\, dx = x.$$

Therefore

$$I_{4,2} = -\frac{\sin^3 x \cos^3 x}{6} - \frac{\sin x \cos^3 x}{8} + \frac{\sin x \cos x}{16} + \frac{x}{16} + c.$$

Example 10.3.4 Find $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x\, dx.$

solution : By (10.3.3), we get

$$I_{4,8} = \frac{7}{12} I_{4,6} = \frac{7}{12} \cdot \frac{5}{10} I_{4,4} = \frac{7}{12} \cdot \frac{5}{10} \cdot \frac{3}{8} I_{4,2} = \frac{7}{12} \cdot \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} I_{4,0}$$

$$= \frac{7}{12} \cdot \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} I_{2,0} = \frac{7}{12} \cdot \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_{0,0} = \frac{7}{1024} I_{0,0}.$$

Also

$$I_{0,0} = \int_0^{\frac{\pi}{2}} \sin^0 x \cos^0 x\, dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

Therefore

$$I_{4,8} = \frac{7\pi}{2048}.$$

10.3.6. Reduction Formula for $\int \sin^m x \cos^n x\, dx$, where either *m* or *n* or both are negative integers

Let

$$I_{m,n} = \int \sin^m x \cos^n x \, dx.$$

We have from (10.3.1),

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$$

Changing n to $n + 2$,

$$I_{m,n+2} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n+2} + \frac{n+1}{m+n+2} I_{m,n}$$

and transposing,

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2} \,(n+1 \neq 0). \qquad (10.3.4)$$

Similarly, from (10.3.2), we can find

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n} \,(m+1 \neq 0). \qquad (10.3.5)$$

Example 10.3.5 Find $\int \frac{\cos^4 x}{\sin^2 x} dx.$

Solution : Applying (10.3.5), with $m = -2$, $n = 4$, we get

$$I_{-2,4} = \frac{(\sin x)^{-1} x \cos^5 x}{-1} + \frac{4}{-1} I_{0,4}.$$

Now applying (10.3.1) with $m = 0$, $n = 4$, we obtain

$$I_{-2,4} = -\frac{\cos^5 x}{\sin x} - 4\left(\frac{\sin x \cos^3 x}{4} + \frac{3}{4} I_{0,2}\right)$$

$$= -\frac{\cos^5 x}{\sin x} - \sin x \cos^3 x - 3\left(\frac{\sin x \cos x}{2} + \frac{1}{2} I_{0,0}\right)$$

Also

$$I_{0,0} = \int \frac{\cos^0 x}{\sin^0 x} dx = \int dx = x.$$

Therefore

$$\int \frac{\cos^4 x}{\sin^2 x} dx = -\frac{\cos^5 x}{\sin x} - \sin x \cos^3 x - \frac{3}{2}\sin x \cos x - \frac{3}{2}x + c.$$

10.3.7. Reduction Formula for $\int \tan^n x dx$ and $\int_0^{\frac{\pi}{4}} \tan^n x\, dx$, n being a positive integer greater than 1

Let

$$I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} \text{ [substituting } \tan x = z].$$

Thus,

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}. \tag{10.3.6}$$

Using (10.3.6) and taking $J_n = \int_0^{\frac{\pi}{4}} \tan^n x dx,$ we get

$$J_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2}.$$

Therefore

$$J_n = \frac{1}{n-1} - J_{n-2}. \tag{10.3.7}$$

Example 10.3.6 Find $\int \tan^4 x\, dx.$

Solution : Applying (10.3.6), we get

$$I_4 = \frac{\tan^3 x}{3} - I_2 = \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - I_0\right)$$

where $I_0 = \int \tan^0 x \, dx = x$.

Therefore

$$I_4 = \frac{\tan^3 x}{3} - \tan x + x + c.$$

Example 10.3.7 Find $\int_0^{\frac{\pi}{4}} \tan^6 x \, dx$.

Solution : Applying (10.3.7), we get

$$J_6 = \frac{1}{5} - J_4 = \frac{1}{5} - \frac{1}{3} + J_2$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - J_0,$$

where $J_o = \int_0^{\frac{\pi}{4}} \tan^0 x \, dx = [x]_0^{\pi/4} = \frac{\pi}{4}$.

Thus

$$J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}.$$

10.3.8. Reduction Formula for $\int \cot^n x \, dx$ and $\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cot^n x dx$, n being a positive integer greater than 1

Proceeding similar as in the art. 10.3.7 and expressing $\cot^n x = \cot^{n-2} x(\operatorname{cosec}^2 x - 1)$, we see that

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2} \text{ and } J_n = -\frac{1}{n-1} - J_{n-2}.$$

10.3.9. Reduction Formula for $\int \sec^n x \, dx$, n being a positive integer greater than 1

Let

$$I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x \, dx \quad \text{[Int. by parts]}$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)(I_n - I_{n-2}),$$

and transposing

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

10.3.10. Reduction Formula for $\int \text{cosec}^n x\,dx$, n being a positive integer greater than 1

As in art. 10.3.9 we may find

$$I_n = -\frac{\text{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

Example 10.3.8 Find $\int \sec^4 x\,dx$.

Solution : Here

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\left(\frac{\tan x}{1} + 0\right)$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x + c.$$

10.3.11. Reduction formula for $\int x^m (\log x)^n dx$, n being a positive integer

Let

$$I_{m,n} = \int x^m (\log x)^n dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1}.\frac{1}{x}.\frac{x^{m+1}}{m+1} dx \text{ [Int. by parts]}$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}.$$

Thus, the reduction formula is

$$I_{m,n} = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1} I_{m,n-1}.$$

10.3.12. Reduction formula for $\int \cos^m x \sin nx\, dx$, *m, n* being a positive integer

Let

$$I_{m,\,n} = \int \cos^m x \sin nx\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx\, dx \text{ [int. by parts]}$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin nx \cos x\, dx$$

$$+ \frac{m}{n} \int \cos^{m-1} x \sin(n-1)x\, dx$$

[since cos nx sin x = sin nx cos x – sin $(n - 1)x$]

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,\,n-1}.$$

Thus, the reduction formula is

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}.$$

10.4 Summary

In this unit we have learnt the reduction formula of several functions. These fomula give us to find out the integrals easily.

10.5 Exercises

1. Evaluate the following integrals :

(i) $\int x^3 e^a x\,dx$ (ii) $\int x^3(\log x)^2 dx$

(iii) $\int \sin^8 x\,dx$ (iv) $\int \sin^8 x \cos^2 x\,dx$

(v) $\int \frac{\sin^5 x}{\cos^4 x} dx$ (vi) $\int \tan^5 x\,dx$

(vii) $\int \sec^5 x\,dx$ (viii) $\int \text{cosec}^3 x\,dx$

(ix) $\int \tan^5 x\,dx$ (x) $\int \cot^3 2x\,dx$.

2. Find the value of the following ingegrals

(i) $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x\,dx$

(iii) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x\,dx$.

10.6 References

1. D. Chatterjee, B. K. Pal, Integral Calculus and Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 2019.
2. D. C. Das, B. N. Mukherjee, Integral Calculus : Differential Equations, U. N. Dhur & sons Private Ltd. Kolkata;, India, 1938.
3. R. K. Ghosh, K. C. Maity, An introduction to analysis : Integral Calculus, NCBA, India, 1959.

Unit-11 ❑ Arc Length

Structure

11.1 Objectives

After going through this chapter, the readers will be able to :

- understand the formula of arc length.
- determine the length of an arc of a curve.

11.2 Introduction

In this unit, we use definite integral to find the arc length of a curve.

11.3 Length of an Arc of a Curve

Let the given arc AB of a curve $y = f(x)$ between $x = a$ and $x = b$ be divided into n parts by points $P_1, P_2, \ldots, P_{r-1}, P_r, \ldots, P_{n-1}$ as shown in the Fig. 11.1. Suppose that the corredponding abscissae of these points are

$$x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}$$

and ordinates are

$$y_1, y_2, \ldots, y_{r-1}, y_r, \ldots, y_{n-1}.$$

We draw chords $AP_1, P_1P_2, \ldots, P_{r-1}P_r, \ldots, P_{n-1}B$ through consecutive points. Then the sum of lengths of these chords is

$$AP_1 + P_1P_2 + \ldots + P_{r-1}P_r + \ldots + P_{n-1}B.$$

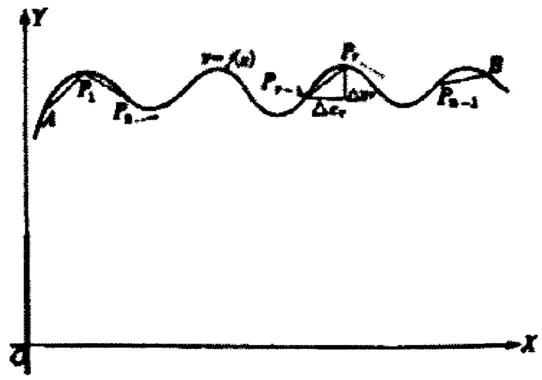


Fig. 11.1

When n becomes infinite, the length of each chord tends to zero. Hence the limiting value of the sum of the length of chords will be the length of the given are AB.

Now

$$\text{length of chord } P_{r-1}P_r = \sqrt{(\Delta x_r)^2 + (\Delta y_r)^2} = \sqrt{1+\left(\frac{\Delta y_r}{\Delta x_r}\right)^2}\,\Delta x_r.$$

Now if the curve be continuous and derivable at every point on $[a, b]$, from the mean value theorem of the differential calculus, there exists at least one point, say $x = \xi_r$ on the are $P_{r-1}P_r$ at which the slope of the tangent, $f'(\xi_r)$ is equal to $\frac{\Delta y_r}{\Delta x_r}$.

Thus the length of the chord $P_{r-1}P_r$ becomes $\sqrt{1+\{f'(\xi_r)\}^2}\Delta x_r$ and, consequently, by the fundamental theorem of integration, the total length (s) of the are AB is

$$s = \lim_{n\to\infty}\sum_{r=1}^{n}\sqrt{1+\{f'(\xi_r)\}^2}\,\Delta x_r = \int_a^b \sqrt{1+\{f'(x)\}^2}\,dx,$$

where a, b are respectively the abscissae of A, B.

I. Therefore the length of the arc of the courve $y = f(x)$ between the points whose abscissae are a and b is given by

$$\int_a^b \sqrt{1+\{f'(x)\}^2}\,dx \quad \text{or} \quad \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

II. If the curve be given in the form $x = f(y)$, the length of the arc between two points (a, c) and (b, d), may similarly be given by

$$\int_c^d \sqrt{1+\{f'(y)\}^2}\,dy \quad \text{or} \quad \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

III. If a curve be given in the parametric from $x = f(t)$ and $y = \phi(t)$ and if t_1 and t_2 be the corresponding points of a and b respectively, then the length of the arc of the curve be derived from I as

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

IV. If the polar equation of the curve be $r = f(\theta)$, then as $x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$, the length of the arc between two points whose vectorial angles are θ_1 and θ_2 is given by

$$\int_{\theta_1}^{\theta_2} \sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2}\,d\theta \quad \text{or} \quad \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

V. If the pedal equation of a curve be $p = f(r)$, then the length of an arc of the curve from $r = r_1$ to $r = r_2$ may similarly be given by

$$\int_{r_1}^{r_2} \frac{r\,dr}{\sqrt{r^2-p^2}} \quad \left[\frac{ds}{dr} = \sec\phi = \frac{1}{\cos\phi} = \frac{r}{\sqrt{r^2-p^2}}\right]$$

Example 11.3.1 Find the length of the perimeter of the circle $x^2 + y^2 = 25$.

Solution : Using the formula I and Fig. 11.2, we see that

the perimeter of the circle = 4 × the perimeter of the circle in the first quadrant

$$= 4\int_0^5 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^5 \frac{5}{\sqrt{25-x^2}}\,dx$$

$$= 20\left[\sin^{-1}\frac{x}{5}\right]_0^5 = 10\pi.$$

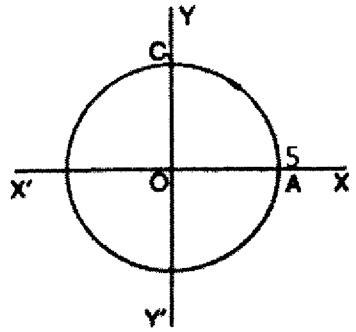


Fig. 11.2

Example 11.3.2 Determine the length of one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

Solution : Referring to Fig. 11.3 and applying formula III, we see that the length of on arch of the cycloid

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} 2a \sin\frac{\theta}{2}\, d\theta = 8a.$$

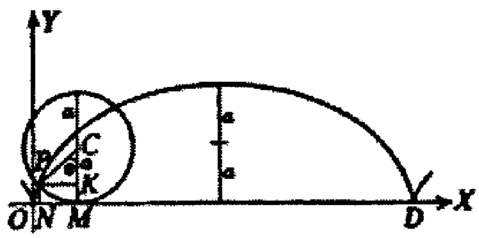


Fig. 11.3

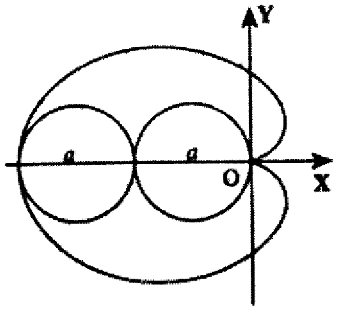
Example 11.3.3. Determine the perimeter of the cardioid $r = a(1 - \cos\theta)$.

Solution : Referring to Fig. 11.4 and applying formula IV, we see that the total length of the cardioid

$$= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\; d\theta$$

$$= 4a\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 8a.$$



$r = a(1 - \cos\theta)$

Fig. 11.4

11.4 Summary

In this unit we see that the formula of arc length comes from the approximating the curve by straight lines connecting successive points on the curve using Pythagorean theorem. An integral fromula is developed to compute the arc length of a curve.

11.5 Exercises

1. Find the length of the followings :

(i) $y = \log(1 - x^2)$ between $x = 0$ and $x = \frac{1}{3}$.

(ii) $ay^2 = x^3$ from $x = 0$ to $x = 5a$.

(iii) $r = \theta^2$; $\theta = 0$ and $\theta = \sqrt{5}$.

(iv) $x = e^{\theta} \sin\theta$, $y = e^{\theta} \cos$; $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(v) the perimeter of the astroid $x^{2/3} + y^{3/2} = a^{3/2}$.

2. Find the length of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$.

3. Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

11.6 References

1. D. Chatterjee, B. K. Pal, Integral Calculus and Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 2019.

2. D. C. Das, B. N. Mukherjee, Integral Calculus : Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 1938.

3. R. K. Ghosh, K. C. Maity, An introduction to analysis : Integral Calculus, NCBA, India, 1959.

Unit-12 ❑ Volume and Area of urface of Revolution

Structure

12.1 Objectives

After reading this text, the students should be able to :

- find the volume of solid by revolving a curve around a line.
- determine the area of surface of revolution.

12.2 Introduction

In this unit we shall discuss a very important process to find out the volume of solid and area of surface of revolution. The method of definite integration enabled us to find these. The process of finding the area of plane figure will be extended to determine the volume of solid and area of surface.

12.3 Volume of Solid of Revolution

Let V be the volume formed when an area $ABCD$ in Fig. 12.1, under the curve $y = f(x)$ between $A(x = a)$ and $B(x = b)$ is revolved about the x-axis. We divide the interval $[a, b]$ into n parts by means of the arbitrary set of points

$$x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}.$$

Let

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_r, \ldots, \Delta x_n$$

be the length of respective sub-intervals into which $[a, b]$ is subdivided. As the entire area about the x-axis being perpendicular to it, a general infinitesimal strip of area *PQRS*, of base Δx_r likewise revolves and generates an infinitesimal disc of volume ΔV_r (say). The entire volume can now be thought of as composed of the set of discs generated by the revolution of the various strips of the area *ABCD*.

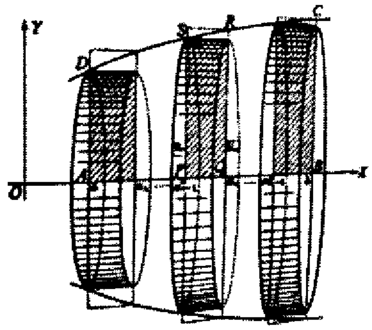


Fig. 12.1

Next let $\underline{y_r}$ and $\overline{y_r}$ denote respectively the least and greatest value of $y = f(x)$ in Δx_r. Then since volume = face area × thickness, we have

$$\pi \underline{y_r}^2 \Delta x_r \leq \Delta V_r \leq \pi \overline{y_r}^2 \Delta x_r.$$

By addition and nothing that $\sum_{r=1}^{n} V_r = V$, we have

$$\sum_{r=1}^{n} \pi \underline{y_r}^2 \Delta x_r \leq V \leq \sum_{r=1}^{n} \pi \overline{y_r}^2 \Delta x_r.$$

Now if the manner of subdivision of $[a, b]$ be such that the greatest of $\Delta x_r \to 0$ as $n \to \infty$, both sums approach the same limit. Hence

$$V = \lim_{n\to\infty} \sum_{r=1}^{n} \pi y_r^2 \Delta x_r = \lim_{n\to\infty} \sum_{r=1}^{n} \pi \{f(\xi_r)\}^2 \Delta x_r$$

in which $y_r = f(\xi_r)$ is the ordinate of an arbitrary point $x = \xi_r$, in Δx_r. Applying the fundamental theorem to the last sum

$$V = \pi \int_a^b y^2 dx = \pi \int_a^b \{f(x)\}^2 dx.$$

I. Thus the volume generated by revolving an area bounded by the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is expressed by the integral

$$V = \pi\int_a^b y^2 dx = \pi\int_a^b \{f(x)\}^2 dx.$$

II. If however the curve be expressed by $x = f(t)$, $y = \phi(t)$

$$V = \pi\int_a^b y^2 dx = \pi\int_{t_1}^{t_2} \{\phi(t)\}^2 f'(t)dt,$$

where t_1, t_2 are values of t that correspond to $x = a$ and $x = b$ respectively.

III. If again the curve $x = \phi(y)$ bounded by $y = c$ and $y = d$ be revolved about the y-axis, the volume is given by

$$V = \pi\int_c^d x^2 dy = \pi\int_c^d \{\phi(y)\}^2 dy.$$

Example 12.3.1 Find the volume of a sphere of radius a.

Solution : Let the equation of the circle in Fig. 12.2 be $x^2 + y^2 = a^2$. The center is at the origin and radius $OA = a$. Let the quadrant OAB be rotated about OX. Then a hemisphere will be created.

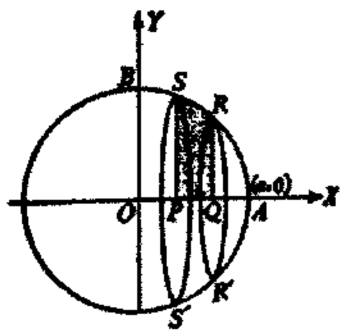


Fig. 12.2

Thus, the volume of sphere will be

$$V = 2\pi\int_0^a y^2 dx$$

$$= 2\pi\int_0^a (a^2 - x^2)dx = \frac{4}{3}\pi a^3.$$

Example 12.3.2 Find the solid formed by the rotation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution :

(i) About the major axis :

Referring to Fig 12.3 and rotating about x-axis, we see that the volume obtained by the rotation of the upper half of the ellipse

= 2 × volume obtained by the rotation of the quadrant OAB

$$=2\times\int_0^a \pi y^2 dx$$

$$=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx=\frac{4}{3}\pi ab^2.$$

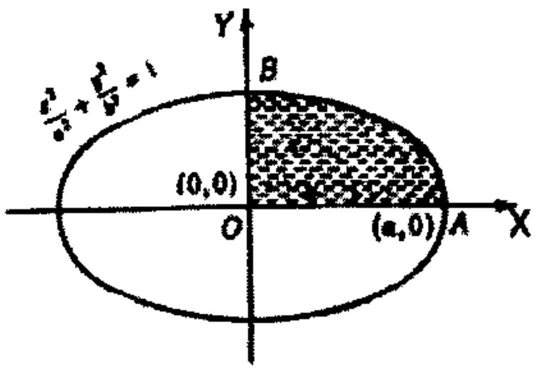


Fig. 12.3

(ii) About the minor axis :

The rotation being about y-axis, the volume of the whole ellipsoid

$$=2\times\int_0^b \pi x^2 dy$$

$$=2\pi\int_0^b \frac{a^2}{b^2}(b^2-y^2)dx=\frac{4}{3}\pi a^2 b.$$

Example 12.3.3 Find the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.

Solution : Referring to Fig 12.4 and applying Rule II we find the required volume

= 2 × volume generated by half of the arch

$$=2\times\int_0^\pi \pi a^3(1-\cos\theta)^3 d\theta$$

$$=2\pi a^3\int_0^\pi 8\sin^6\frac{\theta}{2}d\theta$$

$$=5\pi^2 a^3.$$

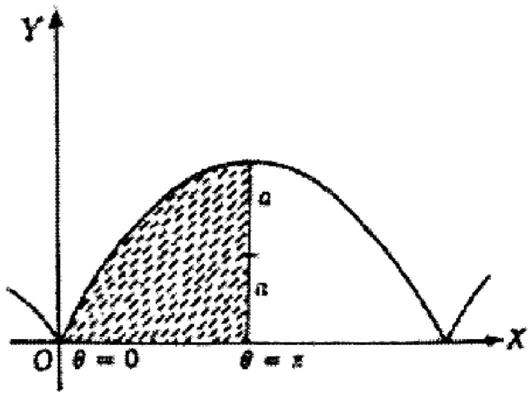


Fig. 12.4

Example 12.4 Find the volume of the solid generated by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.

Solution : Referring to Fig 12.5 the required volume by rule I is $\pi\int y^2dx$.

Now changing the variables from x and y to r and θ, we observed that

$x = r\cos\theta = a(1 - \cos\theta)\cos\theta$, thereby $dx = a(-\sin\theta + 2\sin\theta\cos\theta)d\theta$

$y = r\sin\theta = a(1 - \cos\theta)\sin\theta$.

and the volume becomes

$$\pi\int_{\pi}^{0} a^2(1-\cos\theta)^2\sin^2\theta . a\sin\theta(2\cos\theta-1)d\theta$$

$$= \pi a^3\int_{-1}^{1}(1-z)^2(1-z^2)(1-2z)dz \text{ [putting } \cos\theta = z]$$

$$= \frac{8}{3}\pi a^3.$$

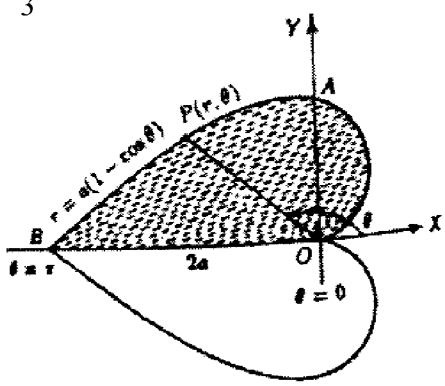


Fig. 12.5

12.4 Area of Surface of Revolution

When an arc of a plane curve is revolved about an axis in its plane, a surface of revolution is generated. The area of such a surface is defined and calculated as follows.

We can derive the formula of the area of surface from the formula of arc length. Let's look at rotating the continuous function $y = f(x)$ in the interval $[a, b]$ about the x-axis.

Let the generating arc be the portion DC of the curve $y = f(x)$ between $x = a$ and $x = b$ and let the axis of revolution be the x-axis. We divide the interval $[a, b]$ into n parts by means of the arbitrary set of points

$$x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}$$

and let

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_r, \ldots, \Delta x_n$$

be the length of respective sub-intervals into which $[a, b]$ is subdivided. Let the arc DC be divided into n parts by means of the arbitrary set of points

$$P_1, P_2, \ldots, P_{r-1}, P_r, \ldots, P_{n-1},$$

the corresponding ordinates being

$$y_1, y_2, \ldots, y_{r-1}, y_r, \ldots, y_{n-1},$$

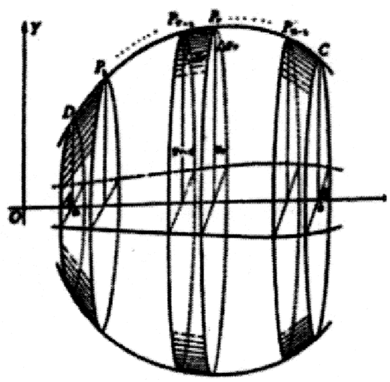


Fig. 12.6

We next draw chords through consecutive points

$$DP_1, P_1P_2, \ldots, P_{r-1} P_r, \ldots, P_{n-1}C$$

and consider a typical one $P_{r-1}P_r$ corresponding to Δx_r and $\Delta y_r (= y_r - y_{r-1})$ revolving about the x-axis. This chord generates and infinitesimal frustum of a cone, whose surface are ΔS_r (say) is given by

$$\Delta S_r = \pi \times \text{sum of the radii of the two bases} \times \text{slant height}$$

$$= \pi(y_{r-1} + y_r) \times P_{r-1}P_r = \pi(y_{r-1} + y_r)\sqrt{(\Delta x_r)^2 + (\Delta y_r)^2}$$

$$= 2\pi \frac{y_{r-1} + y_r}{2} \sqrt{1 + \left(\frac{\Delta y_r}{\Delta x_r}\right)^2} . \Delta x_r .$$

Now if the curve be continuous and has a derivative at every point, then by the mean value theorem fo the differential calculus, there exists at least one point on the arc $P_{r-1}P_r$, at which the slope of the tangent $f'(\xi_r)$ is equal to the slope of the secant $\frac{\Delta y_r}{\Delta x_r}$. Moreover $\frac{1}{2}(y_{r-1} + y_r)$ is just the average height, or height at the middle point of the chord $P_{r-1}P_r$ and from the continuity of the curve $y = f(x)$, there must exist at least one point between P_{r-1} and P_r say the point $x = \eta_r$ at which the ordinate is equal to the average height.

Hence

$$\Delta S_r = 2\pi f(\eta_r)\sqrt{1 + \{f'(\xi_r)\}^2}\, \Delta x_r .$$

Defining the area of the entire surface to be the limit of the sum of this typical areas when $n \to \infty$ in such a way that the length of each chord approaches zero, we have

$$S = \lim_{n\to\infty} \sum_{r=1}^{n} S_r = \lim_{n\to\infty} 2\pi \sum_{r=1}^{n} f(\eta_r)\sqrt{1 + \{f'(\xi_r)\}^2}\, \Delta x_r$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

or briefly,

$$S = 2\pi \int_a^b y \frac{ds}{dx} dx ,$$

in which y and ds are to be replaced by their equals in terms of x.

Cor. 1. In the case of the curve $x = f(t),\ y = \phi(t)$

$$S = 2\pi\int_{t_1}^{t_2} y\frac{ds}{dt}dt = 2\pi\int_{t_1}^{t_2} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Cor. 2. When the axis of revolution is the y-axis the corresponding formula will be (taking $y = c$ and $y = d$)

$$S = 2\pi\int_{c}^{d} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi\int_{c}^{d} x\frac{ds}{dy}dy.$$

Cor. 3. The area may also be found out in terms of polar coordinates when an equation is given in the polar form by the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

Example 12.4.1 Find the surface of a sphere generated by the circle $x^2 + y^2 = a^2$ about the x-axis.

Solution : To find out the area of surface of sphere we consider to Fig 12.7 and apply the result of art. 12.4. Since $x^2 + y^2 = a^2$, $\frac{dy}{dx} = -\frac{x}{y}$ and $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{y^2} = \frac{a^2}{y^2}$.

Therefore

$$S = 2\times \text{ surface area generated by arc } AB$$

$$= 2\times 2\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi\int_0^a a\,dx = 4\pi a^2.$$

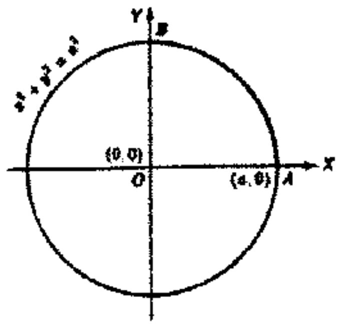


Fig. 12.7

Example 12.4.2 Find the area of the surface generated by revolving about y-axis that part of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, that lies in the first quadrant.

Solution : Using Cor. 1, of art. 12.4, with

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

we have from Fig 12.8,

$$S = 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2} 3a\sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta}.\ a\cos^3\theta d\theta$$

$$= 6\pi a^2\int_0^{\pi/2} \sin\theta\cos^4\theta d\theta = \frac{6}{5}\pi a^2.$$

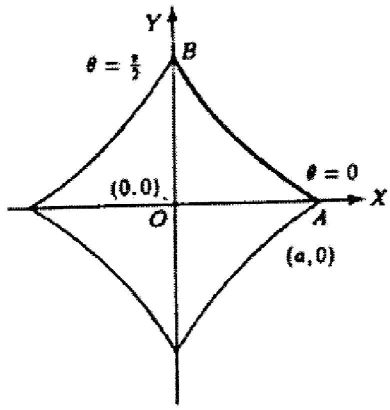


Fig. 12.8

Example 12.4.3 What is the area of the entire surface formed when the cardioid $r = a(1 + \cos\theta)$ is revolved about the initial line ?

Solution : Using Cor. 3 of art. 12.4, with

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} = 2a\cos\frac{\theta}{2},$$

we have from Fig. 12.9,

$$S = 2\pi\int_0^{\pi} y\frac{ds}{d\theta}d\theta$$

$$= 2\pi\int_0^\pi a(1+\cos\theta)\sin\theta.2a\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\int_0^\pi \sin\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta = \frac{32}{5}\pi a^2.$$

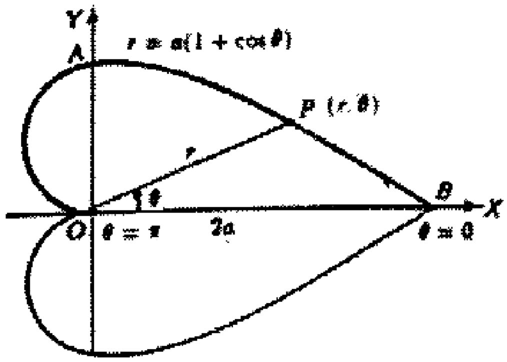


Fig. 12.9

12.5 Summary

In this unit we have learnt how a solid is formed by revolution of arc of a curve about a line and determined the formula of volume and the area of surface of that solid.

12.6 Exercises

1. Find the volume of solid generated by revolving the following curve :

(i) $y = \sqrt{x}$ between the lines $x = 1$ and $x = 4$ about x-axis,

(ii) $y = 5x - x^2$ between the lines $x = 0$ and $x = 5$ about x-axis,

(iii) $y^2 = 4ax$ between the lines $x = 0$ and $x = 2a$ about x-axis,

(iv) $x^2 - y^2 = a^2$ between the lines $x = 0$ and $x = 2a$ about x-axis,

(v) $xy = 2$ between the lines $y = 1$ and $y = 4$ about y-axis.

2. Find the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2(2 - x)$.

3. Find the volume of the solid generated by the revolution of the loop of the curve $y^2(a + x) = x^2(a - x)$ about the x-axis.

4. Find the area of surface generated by revolving the following curve :

(i) $y = x^2$ between the lines $x = 0$ and $x = \sqrt{2}$ about y-axis,

(ii) $r = 2a \cos \theta$ about the initial line,

(iii) an arc of $y = \sin x$ about x-axis,

(iv) $2y = x + 1$ between the lines $x = 1$ and $x = 3$ about x-axis.

5. Find the area of surface generated by revolving the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.

12.7 References

1. R. K. Ghosh, K. C. Maity, An introduction to analysis : Integral Calculus, NCBA, India, 1956.
2. D. Chatterjee, B. K. Pal, Integral Calculus and Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 2019.
3. D. C. Das, B. N. Mukhejee, Integral Calculus : Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 1938.

Unit-13 ❑ Vector Valued Functions of Scalar Variables

Structure

13.1 Objectives

In this unit the readers will learn the followings :

- Definition of vector valued functions of real variable.
- Limits and continuity of vector valued functions.
- Differentiation and integration of vector valued functions.
- Tangent and normal components of acceleration.

13.2 Introduction

In ordinary Calculus we have learn the concepts of real valued functions of real variables and their limits, continuity, differentiability and integrability. In this unit we will discuss about vector valued functions of scalar variables and their limits, continuity, differentiability and integrability. We shall also study differential geometry using vector calculus in brief.

13.3 Vector Valued Functions of Scalar Variable

If by some law $\vec{f}$, for each value of a scalar variable t in some interval $[a, b]$, there corresponds a definite unique vector $\vec{r}$, then $\vec{f}$ is called a single-valued vector function of the scalar variable t and is denoted by $\vec{r} = \vec{f}(t)$.

$\vec{f}(c)$ denotes the particular vector for some fixed value c of t.

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along three mutually perpendicular fixed directions, then $\vec{f}(t)$ can be expressed as $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ or simply $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$, where $f_1(t), f_2(t), f_3(t)$ are the scalar functions of t along $\hat{i}, \hat{j}, \hat{k}$ respectively.

If in particular the scalar t represents time, then $\vec{r}$ represents the position vector of any moving point at time t w.r.t. a certain vector origin. Then the velocity and acceleration of that moving point are also another vector functions of the same scalar variable t.

13.4 Limits of Vector Functions

A vector function $\vec{f}(t)$ is said to tend to the limit $\vec{l}$ when $t \to c$, if for any preassigned positive number ε, however small, there corresponds a positive number δ such that

$$\left|\vec{f}(t) - \vec{l}\right| < \varepsilon \text{ when } 0 < |t - c| \leq \delta$$

and it will be expressed as $\lim_{t \to c} \vec{f}(t) = \vec{l}$, provided that such limit exists.

If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and $\hat{l} = l_1(t)\hat{i} + l_2(t)\hat{j} + l_3(t)\hat{k}$, then when $\lim_{t \to c} \vec{f}(t) = \vec{l}$, then $\lim_{t \to c} f_1(t) = l_1$, $\lim_{t \to c} f_2(t) = l_2$ and $\lim_{t \to c} f_3(t) = l_3$.

13.4.1. Some standard results on limits

If $\vec{f}(t)$ and $\vec{g}(t)$ are two vector functions of scalar variable t and if $\lim_{t\to c}\vec{f}(t)=\vec{l}$ and $\lim_{t\to c}\vec{g}(t)=\vec{m},$ then

(i) $\lim_{t\to c}\left[\vec{f}(t)\pm\vec{g}(t)\right]=\vec{l}\pm\vec{m}$ (ii) $\lim_{t\to c}\left[\vec{f}(t).\vec{g}(t)\right]=\vec{l}.\vec{m}$

(iii) $\lim_{t\to c}\left[\vec{f}(t)\times\vec{g}(t)\right]=\vec{l}\times\vec{m}$ (iv) $\lim_{t\to c}\left|\vec{f}(t)\right|=\left|\vec{l}\right|$

(v) $\lim_{t\to c}\left[\phi(t)\vec{f}(t)\right]=p\vec{l},$ where $\lim_{t\to c}\phi(t)=p.$

13.5 Continuity of Vector Functions

A vector function $\vec{f}(t)$ of a scalar variable t is said to be continuous at $t = c$, if for any preassigned positive number ε, however small, there corresponds a positive number δ such that

$$\left|\vec{f}(t)-\vec{f}(c)\right|<\varepsilon \text{ when } 0\le\left|t-c\right|\le\delta$$

and it will be denoted by $\lim_{t\to c}\vec{f}(t)=\vec{f}(c).$

Also a function $\vec{f}(t)$ is said to be continuous in an interval $[a, b]$ of t, if it is continuous for every value of t in $[a, b]$.

In the same way, if $\vec{f}(t)$ and $\vec{g}(t)$ are two vector functions of scalar variable t are continuous, then the functions expressed in art. 13.4.1 are also continuous.

13.6 Differentiation of Vector Functions

A vector function $\vec{f}(t)$ of a scalar variable t is said to have a derivative at c, if

$$\lim_{h\to 0}\frac{\vec{f}(c+h)-\vec{f}(c)}{h}$$

exists. Then that limiting value is said to the derivative of $\vec{f}(t)$ at $t = c$ and is denoted by $\vec{f}'(c)$ or by $\frac{d\vec{f}}{dt}$ at $t = c$.

This process of finding the derivative of functions is known as differentiation or derivation.

Another definition of differentiation of $\vec{f}(t)$ is given as

$$\vec{f}'(c) = \lim_{t \to c} \frac{\vec{f}(t) - \vec{f}(c)}{t - c},$$

provided that limit exists.

For any differentiable function $\vec{f}(t)$, we may write

$$\frac{d\vec{f}}{dt} = \vec{f}'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}.$$

By writing $h = \Delta t$, we get

$$\frac{d\vec{f}}{dt} = \vec{f}'(t) = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}.$$

A function is said to be derivable in an interval if it is derivable at every point in that interval.

13.6.1. Higher Order Derivatives

If the vector function $\vec{f}(t)$ of scalar variable t has a derivative $\vec{f}'(t)$ in a given interval, and $\vec{f}'(t)$ is itself derivable in some interval, then the second order derivative of $\vec{f}(t)$ is defined by

$$\frac{d^2\vec{f}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{f}}{dt}\right) = \vec{f}''(t) = \lim_{\Delta t \to 0} \frac{\vec{f}'(t+\Delta t) - \vec{f}'(t)}{\Delta t}.$$

Similarly, we can define more higher order derivatives of vector functinos.

13.6.2. Differentials

From the definition of derivative, we can write

$$\frac{\Delta\vec{f}}{\Delta t} = \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t} = \vec{f}'(t) + \vec{\delta},$$

where $\vec{\delta}$ is a vector such that $\vec{\delta} \to 0$ as $\Delta t \to 0$.

Then $\vec{f}$ is said to be differentiable at t if we can express

$$\Delta\vec{f} = \vec{f}(t+\Delta t) - \vec{f}(t) = \vec{f}'(t)\Delta t + \vec{\delta}\Delta t,$$

where $\vec{\delta}$ is a vector such that $\vec{\delta} \to 0$ as $\Delta t \to 0$.

The part $\vec{f}'(t)\Delta t$ is called the diferential of the vector function $\vec{f}$ and is denoted by

$$d\vec{f} = \vec{f}'(t)\Delta t.$$

Since the above expression is true for every diferentiable vector function $\vec{f}(t)$, we may take $\vec{f}(t) = t\hat{i}$. Then we have $\Delta t = dt$. Hence the differential of the vector function $\vec{f}$ is

$$d\vec{f} = \vec{f}'(t)dt.$$

If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$d\vec{f}(t) = df_1\hat{i} + df_2\hat{j} + df_3\hat{k}.$$

Theorem 13.6.1 Every derivable function is continuous but the converse may not be true.

Proof : For any value c of t, we may write

$$\vec{f}(t) - \vec{f}(c) = \frac{\vec{f}(t) - \vec{f}(c)}{t - c}.(t - c).$$

So

$$\begin{aligned} \lim_{t\to c}[\vec{f}(t) - \vec{f}(c)] &= \lim_{t\to c}\frac{\vec{f}(t) - \vec{f}(c)}{t - c}.\lim_{t\to c}(t - c) \\ &= \vec{f}'(c).0 = 0 \quad [\text{since } \vec{f} \text{ is derivable}]. \end{aligned}$$

Hence $\lim_{t\to c}\vec{f}(t) = \vec{f}(c)$.

Therefore the function $\vec{f}(t)$ is continuous at $t = c$.

Let $\vec{f}(t) = |t|\hat{i}$, which is continuous for every value of t, but it is not derivable at $t = 0$, because

$$\lim_{t\to 0}\frac{\vec{f}(t) - \vec{f}(0)}{t - 0} = \lim_{t\to 0}\frac{|t|\hat{i}}{t} = \hat{i} \text{ or } -\hat{i},$$

according as $t \to 0$ from positive side or negative side. So $\vec{f}'(0)$ does not exist.

Theorem 13.6.2 If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ is a derivable function then $f_1(t), f_2(t), f_3(t)$ are also derivable functions and $\frac{df}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$.

Proof : we can easily see that

$$\frac{d\vec{f}}{dt} = \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

$$= \lim_{h\to 0}\frac{f_1(t+h) - f_1(t)}{h}\hat{i} + \lim_{h\to 0}\frac{f_2(t+h) - f_2(t)}{h}\hat{j} + \lim_{h\to 0}\frac{f_3(t+h) - f_3(t)}{h}\hat{k}$$

$$= \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$ [since $f_1(t), f_2(t), f_3(t)$ are derivable functions].

Theorem 13.6.3 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}[\vec{f}(t) \pm \vec{g}(t)] = \frac{d\vec{f}(t)}{dt} \pm \frac{d\vec{g}(t)}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t) \pm \vec{g}(t)$.

Then

$$\frac{d}{dt}[\vec{f}(t) \pm \vec{g}(t)] = \frac{d\vec{G}}{dt} = \lim_{h\to 0}\frac{\vec{G}(t+h) - \vec{G}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h} \pm \lim_{h\to 0}\frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \frac{d\vec{f}(t)}{dt} \pm \frac{d\vec{g}(t)}{dt}.$$

Theorem 13.6.4 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}[\vec{f}(t).\vec{g}(t)] = \frac{d\vec{f}}{dt}.\vec{g}(t) + \vec{f}(t).\frac{d\vec{g}}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t).\vec{g}(t)$.

Then

$$\frac{d}{dt}\left[\vec{f}(t).\vec{g}(t)\right] = \frac{d\vec{G}}{dt} = \lim_{h\to 0}\frac{\vec{G}(t+h) - \vec{G}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\vec{f}(t+h).\vec{g}(t+h) - \vec{f}(t)\vec{g}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}.\vec{g}(t) + \lim_{h\to 0} \vec{f}(t+h).\frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}.\vec{g}(t) + \lim_{h\to 0} \vec{f}(t+h).\lim_{h\to 0} \frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \frac{d\vec{f}}{dt}.\vec{g}(t) + \vec{f}(t).\frac{d\vec{g}}{dt}.$$

Theorem 13.6.5 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}\left[\vec{f}(t)\times\vec{g}(t)\right] = \frac{d\vec{f}(t)}{dt}\times\vec{g}(t) + \vec{f}(t)\times\frac{d\vec{g}(t)}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t)\times\vec{g}(t)$.

Then

$$\frac{d}{dt}\left[\vec{f}(t)\times\vec{g}(t)\right]$$

$$= \frac{d\vec{G}}{dt} = \lim_{h\to 0}\frac{\vec{G}(t+h) - \vec{G}(t)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\vec{f}(t+h)\times\vec{g}(t+h) - \vec{f}(t)\times\vec{g}(t)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\vec{f}(t+h) - \vec{f}(t)}{h}\times\vec{g}(t) + \vec{f}(t+h)\times\frac{\vec{g}(t+h) - \vec{g}(t)}{h}\right]$$

$$= \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h}\times\vec{g}(t) + \lim_{h\to 0}\vec{f}(t+h)\times\lim_{h\to 0}\frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \frac{d\vec{f}(t)}{dt}\times\vec{g}(t) + \vec{f}(t)\times\frac{d\vec{g}(t)}{dt}.$$

Theorem 13.6.6 (Differentiation of a function of a function) :

If $\vec{f}(u)$ is a differentiable vector function of a scalar variable u and u itself is a differentiable function of another scalar variable t, then $\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}.\frac{du}{dt}$.

Proof : Since u is a differential function of t, we may write

$$\Delta u = \frac{du}{dt}\Delta t + \alpha\Delta t, \tag{13.6.1}$$

where $\alpha \to 0$ as $\Delta t \to 0$.

Again since $\vec{f}(u)$ is a differentiable function of a scalar variable u, we have

$$\Delta\vec{f} = \frac{d\vec{f}}{du}\Delta u + \vec{\beta}\Delta u, \tag{13.6.2}$$

where $\vec{\beta} \to \vec{0}$ as $\Delta u \to 0$.

Now Putting the value of Δu from (13.6.1) in (13.6.2), we get

$$\begin{aligned}\Delta\vec{f} &= \frac{d\vec{f}}{du}\left(\frac{du}{dt}\Delta t + \alpha\Delta t\right) + \vec{\beta}\left(\frac{du}{dt}\Delta t + \alpha\Delta t\right)\\ &= \left(\frac{d\vec{f}}{du}\cdot\frac{du}{dt}\right)\Delta t + \left(\alpha\frac{d\vec{f}}{du} + \vec{\beta}\frac{du}{dt} + \vec{\beta}\alpha\right)\Delta t.\end{aligned} \tag{13.6.3}$$

Since both $\frac{d\vec{f}}{du}$ and $\frac{du}{dt}$ exist, $\vec{\beta} \to \vec{0}$ as $\Delta u \to 0$ and $\alpha \to 0$ as $\Delta t \to 0$, we obtain

$$\vec{\gamma} = \alpha\frac{d\vec{f}}{du} + \vec{\beta}\frac{du}{dt} + \vec{\beta}\alpha \to \vec{0} \text{ as } \Delta t \to 0.$$

Using this in (13.6.3), we obtain

$$\Delta\vec{f} = \left(\frac{d\vec{f}}{du}\cdot\frac{du}{dt}\right)\Delta t + \vec{\gamma}\Delta t,$$

where $\vec{\gamma} \to \vec{0}$ as $\Delta t \to 0$.

So we may say that $\vec{f}(t)$ is a differentiable function of scalar variable t and the differential of $\vec{f}$ is

$$d\vec{f} = \left(\frac{d\vec{f}}{du}\cdot\frac{du}{dt}\right)\Delta t. \tag{13.6.4}$$

Since (13.6.4) is true for any differentiable function $\vec{f}(t)$ of t, taking $\vec{f} = u\hat{i}$ and $u = t$, we get $\Delta t = dt$ and then from (13.6.4), we obtain

$$\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}\cdot\frac{du}{dt}.$$

Theorem 13.6.7 If $\vec{f}(t)$ is a vector function of scalar variable t and $\phi(t)$ is a scalar function and they are derivable, then $\frac{d}{dt}[\phi\vec{f}] = \frac{d\phi}{dt}\vec{f} + \phi\frac{d\vec{f}}{dt}$.

Proof : Let $\vec{G}(t) = \phi(t).\vec{f}(t)$.

Then

$$\frac{d}{dt}\left[\phi(t).\vec{f}(t)\right] = \frac{d\vec{G}}{dt}$$

$$= \lim_{h\to 0}\frac{\phi(t+h).\vec{f}(t+h) - \phi(t).\vec{f}(t)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\phi(t+h)-\phi(t)}{h}.\vec{f}(t) + \phi(t+h).\frac{\vec{f}(t+h)-\vec{f}(t)}{h}\right]$$

$$= \lim_{h\to 0}\frac{\phi(t+h)-\phi(t)}{h}.\vec{f}(t) + \lim_{h\to 0}\phi(t+h).\lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}$$

$$= \frac{d\phi}{dt}.\vec{f} + \phi.\frac{d\vec{f}}{dt}.$$

Theorem 13.6.8 If $\vec{f}(t)$, $\vec{g}(t)$ and $\vec{h}(t)$ are three vector functions of scalar variable t, then

$$\frac{d}{dt}\left[\vec{f}\vec{g}\vec{h}\right] = \left[\frac{d\vec{f}}{dt}\vec{g}\vec{h}\right] + \left[\vec{f}\frac{d\vec{g}}{dt}\vec{h}\right] + \left[\vec{f}\vec{g}\frac{d\vec{h}}{dt}\right].$$

Proof :

$$\frac{d}{dt}\left[\vec{f}\vec{g}\vec{h}\right] = \frac{d}{dt}\left[\vec{f}.(\vec{g}\times\vec{h})\right] = \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\frac{d}{dt}(\vec{g}\times\vec{h})$$

$$= \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\left(\frac{d\vec{g}}{dt}\times\vec{h} + \vec{g}\times\frac{d\vec{h}}{dt}\right)$$

$$= \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\left(\frac{d\vec{g}}{dt}\times\vec{h}\right) + \vec{f}.\left(\vec{g}\times\frac{d\vec{h}}{dt}\right)$$

$$= \left[\frac{d\vec{f}}{dt}\vec{g}\vec{h}\right] + \left[\vec{f}\frac{d\vec{g}}{dt}\vec{h}\right] + \left[\vec{f}\vec{g}\frac{d\vec{h}}{dt}\right].$$

Theorem 13.6.9 If $\vec{f}(t)$, $\vec{g}(t)$ and $\vec{h}(t)$ are three vector functions of scalar variable t, then

$$\frac{d}{dt}\left(\vec{f}\times(\vec{g}\times\vec{h})\right)=\frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}\right)+\vec{f}\times\left(\vec{g}\times\frac{d\vec{h}}{dt}\right).$$

Proof :

$$\frac{d}{dt}\left(\vec{f}\times(\vec{g}\times\vec{h}\right) =\frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\frac{d}{dt}(\vec{g}\times\vec{h})$$

$$=\frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}+\vec{g}\times\frac{d\vec{h}}{dt}\right)$$

$$=\frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}\right)+\vec{f}\times\left(\vec{g}\times\frac{d\vec{h}}{dt}\right)$$

Theorem 13.6.10 A necessary and sufficient condition that a vector $\vec{u}$ has a constant length is that $\vec{u}.\frac{d\vec{u}}{dt}=0$.

Proof : We know that $\vec{u}^2=|\vec{u}|^2=\vec{u}.\vec{u}$.

So

$$\frac{d}{dt}(\vec{u})^2=\frac{d}{dt}\left(|\vec{u}|^2\right)$$

i.e., $$2\vec{u}.\frac{d\vec{u}}{dt}=2|\vec{u}|\frac{d|\vec{u}|}{dt}.$$

Thus,

$$\vec{u}.\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}.$$

The condition is necessary

When $\vec{u}$ is a proper vector of constant length, then $|\vec{u}|=\text{constant}\neq 0$.

Thus,

$$\vec{u}.\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}=|\vec{u}|.0=0.$$

Therefore the condition is necessary.

The condition is sufficient

If $\vec{u}.\frac{d\vec{u}}{dt}=0,$ then $|\vec{u}|\frac{d|\vec{u}|}{dt}=0.$ But since $\vec{u}$ is proper vector, $|\vec{u}|\neq 0.$ Hence $\frac{d|\vec{u}|}{dt}=0,$ i.e., $|\vec{u}|=$ constant. Thus $\vec{u}$ is a proper vector of constant length.

Note 13.6.1 The derivative of a vector of constant length is perpendicular to that vector.

Theorem 13.6.11 A necessary and sufficient condition that a proper vector $\vec{u}$ always remains parallel to a fixed line i.e., to have a constant direction is $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}.$

Proof : Let $\vec{u}=|\vec{u}|\hat{u},$ where $\hat{u}$ is the unit vector in the direction of $\vec{u}$.

Now

$$\vec{u}\times\frac{d\vec{u}}{dt}=|\vec{u}|\hat{u}\times\frac{d}{dt}\left(|\vec{u}|\hat{u}\right)=|\vec{u}|\hat{u}\times\left(\frac{d|\vec{u}|}{dt}\hat{u}+|\vec{u}|\frac{d\hat{u}}{dt}\right)$$

$$=|\vec{u}|^2\hat{u}\times\frac{d\hat{u}}{dt}\ [\text{since } \hat{u}\times\hat{u}=\vec{0}].$$

The condition is necessary

When $\vec{u}$ remains parallel to a fixed direction then $\hat{u}$ = constant and hence $\frac{d\hat{u}}{dt}=\vec{0}.$

Thus

$$\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}.$$

Therefore the condition is necessary.

The condition is sufficient

Since $\vec{u}\neq\vec{0},$ so the condition

$$\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}\Rightarrow\hat{u}\times\frac{d\hat{u}}{dt}=\vec{0}. \qquad (13.6.5)$$

Also since $\hat{u}$ is a proper vector of constant length of unity, we have

$$\hat{u}.\frac{d\hat{u}}{dt}=0. \qquad (13.6.6)$$

The conditions of (13.6.5) and (13.6.6) will be simultaneously satisfied if $\frac{d\hat{u}}{dt} = \vec{0}$ i.e., $\hat{u}$ is constant i.e., $\vec{u}$ is parallel to a fixed line.

Theorem 13.6.12 A necessary and sufficient condition for the vector $\vec{u}(t)$ to be constant is $\frac{d\vec{u}}{dt} = \vec{0}$,

Proof : The condition is necessary

Let $\vec{u}(t)$ be a constant vector. Then for an increment Δt in the scalar variable t, there will be no change in $\vec{u}$, i.e., $\Delta\vec{u} = \vec{0}$.

So $\frac{\Delta\vec{u}}{\Delta t} = \vec{0}$.

Now taking $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{u}}{\Delta t} = \vec{0} \text{ i.e., } \frac{d\vec{u}}{dt} = \vec{0}.$$

So the condition is necessary.

The condition is sufficient

Let $\frac{d\vec{u}}{dt} = \vec{0}$.

Now if $\vec{u}(t) = u_1(t)\hat{i} + u_2(t)\hat{j} + u_3(t)\hat{k}$, then

$$\frac{d\vec{u}}{dt} = \frac{du_1}{dt}\hat{i} + \frac{dy_2}{dt}\hat{j} + \frac{du_3}{dt}\hat{k}.$$

So

$$\frac{d\vec{u}}{dt} = \vec{0} \Rightarrow \frac{du_1}{dt} = 0, \frac{du_2}{dt} = 0\ \frac{du_3}{dt} = 0.$$

Therefore u_1= constant, u_2= constant, u_3= constant, Thus $\vec{u}(t) =$ constant. So the condition is sufficient.

Example 13.6.1 If $\vec{r} = t^2\hat{i} + \cos t\,\hat{j} + \sin^2 t\,\hat{k}$, find the derivative of $\vec{r}$ w.r.t. t.

Solution : Here $\vec{r} = t^2\hat{i} + \text{cost}\ \hat{j} + \sin^2 t\hat{k}$.

Therefore

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(t^2\hat{i} + \cos t\hat{j} + \sin^2 t\,\hat{k}\right)\\ &= 2t\hat{i} - \sin t\,\hat{j} + \sin 2t\,\hat{k}.\end{aligned}$$

Example 13.6.2 If $\hat{r} = (5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}$, find $\frac{d^2\vec{r}}{dt^2}$.

Solution : Here $\vec{r} = (5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}$.

Therefore

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left\{(5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}\right\}$$

$$= 3\hat{i} - 2\hat{j} + (1-32t)\hat{k}$$

$$\text{and } \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left\{3\hat{i} - 2\hat{j} + (1-32t)\hat{k}\right\} = -32\vec{k}.$$

Example 13.6.3 If $\vec{a} = t^2\vec{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{b} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then show that (i) $\frac{d}{dt}(\vec{a}.\vec{b}) = -6$ at $t = 1$, (ii) $\frac{d}{dt}(\vec{a}\times\vec{b}) = 7\hat{j} + 3\hat{k}$ at $t = 1$.

Solution : Here $\frac{d\vec{a}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k}$ and $\frac{d\vec{b}}{dt} = 2\hat{i} - \hat{k}$.

(i) $\frac{d}{dt}(\vec{a}.\vec{b})$

$$= \frac{d\vec{a}}{dt}.\vec{b} + \vec{a}.\frac{d\vec{b}}{dt}$$

$$= (2t\hat{i} - \hat{j} + 2\hat{k}).\{(2t-3)\hat{i} + \hat{j} - t\hat{k}\} + \{t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\}.(2\hat{i} - \hat{k})$$

$$= 4t^2 - 6t - 1 - 2t + 2t^2 - 2t - 1 = 6t^2 - 10t - 2.$$

Hence $\frac{d}{dt}(\vec{a}.\vec{b}) = -6$ at $t = 1$.

(ii) $\frac{d}{dt}(\vec{a}\times\vec{b})$

$$= \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2t-3 & 1 & -t \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (t-2)\hat{i} + (2t^2+4t-6)\hat{j} + (4t-3)\hat{k} + t\hat{i} + (t^2+4t+2)\hat{j} + 2t\hat{k}$$

$$= (2t-2)\hat{i} + (3t^2+8t-4)\hat{j} + (6t-3)\hat{k}.$$

Therefore at $t = 1$,

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = 7\hat{j} + 3\hat{k}.$$

Example 13.6.4 If $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then find $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$ at $t = 2$.

Solution : Here $\frac{d\vec{\alpha}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k}$, $\frac{d\vec{\beta}}{dt} = 2\hat{i} - \hat{k}$ and $\frac{d^2\vec{\beta}}{dt^2} = \vec{0}$.

$$\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$$

$$= \frac{d\vec{\alpha}}{dt}\times\frac{d\vec{\beta}}{dt} + \vec{\alpha}\times\frac{d^2\vec{\beta}}{dt^2}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \hat{i} + (2t+4)\hat{j} + 2\hat{k}.$$

Hence at $t = 2$,

$$\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right) = \hat{i} + 8\hat{j} + 2\hat{k}.$$

13.7 Geometrical and Physical Interpretation of Derivative of Vector Functions

Let $\vec{r} = \vec{f}(t)$ be a continuous and single valued vector function of a scalar variable t. Let $\vec{r}$ and $\vec{r}+\Delta\vec{r}$ be the position vectors of two neighbouring points P and Q respectively on teh continuous curve $\vec{r} = \vec{f}(t)$ w.r.t. origin O. Then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}.$$

So $\frac{\Delta\vec{r}}{\Delta t}=\frac{\overrightarrow{PQ}}{\Delta t}$. When $Q\to P$, $\Delta t\to 0$, the chord PQ tends to the tangent PT to the curve at P.

Hence $\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t}=\frac{d\vec{r}}{dt}$ represents a vector along the tangent to the curve $\vec{r}=\vec{f}(t)$ at P.

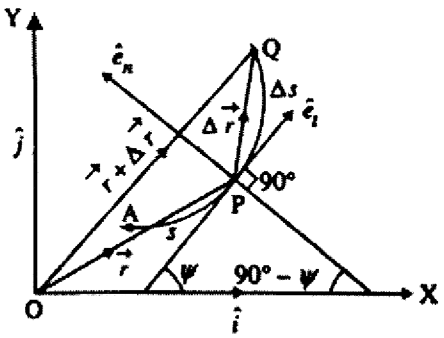


Fig. 13.1

If t represents time then $\Delta\vec{r}$ represents displacement during time Δt . Hence $\frac{d\vec{r}}{dt}$ represents the rate of change of displacement of the point P which is called velocity vector at P along tangential direction. Thus the velocity vector is

$$\vec{\upsilon}=\frac{d\vec{r}}{dt}.$$

Similarly if $\Delta\vec{\upsilon}$ be the increment of $\vec{\upsilon}$ in time Δt, then the rate of change of velocity of the point P which is acceleration of the point P is

$$\vec{a}=\lim_{\Delta t\to 0}\frac{\Delta\vec{\upsilon}}{\Delta t}=\frac{d\vec{\upsilon}}{dt}=\frac{d^2\vec{r}}{dt^2}.$$

13.7.1. Tangential and Normal Components of Velocity and Acceleration

Let $\vec{i}$ and $\vec{j}$ be two unit vectors along two rectangular axes $\overline{OX}$ and $\overline{OY}$ respectively. Let a particle be moving along a plane curve $\vec{r}=\vec{f}(s)$, where s is the length of an arc AP of the curve where A be the fixed point from which the length of the arc be measured and $P(\vec{r})$ be the position of the particle on the curve at time

t. Let $Q(\vec{r}+\Delta\vec{r})$ be the positon at time $t+\Delta t$ where $AQ = s+\Delta s$.

Then

$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = \vec{r}+\Delta\vec{r}-\vec{r} = \Delta\vec{r}$$

$$\text{and are } PQ = \text{arc } AQ - \text{arc } AP = s+\Delta s - s = \Delta s.$$

If $\vec{\upsilon}$ be the velocity of the paricle along tangential direction at P, then $\vec{\upsilon} = |\vec{\upsilon}|\hat{e}_t$ where $\hat{e}_t$ is the unit vector along tangent at P.

So

$$|\vec{\upsilon}| = \left|\lim_{Q\to P}\frac{\overrightarrow{PQ}}{\Delta t}\right| = \lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ}\cdot\frac{arc\ PQ}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is the tangential component of the velocity on the curve $\vec{r} = \vec{f}(s)$ in the sense of s increasing.

Since the particle does not leave the curve, there is no displacement in normal direction and so the velocity along normal direction is zero. Hence we can write

$$\vec{\upsilon} = |\vec{\upsilon}|\hat{e}_t + 0\hat{e}_n,$$

where $\hat{e}_n$ is the unit vector along normal direction.

Now

$$\hat{e}_t = \cos\psi\hat{i} + \sin\psi\hat{j} \text{ and } \hat{e}_n = -\sin\psi\hat{i} + \cos\psi\hat{j}.$$

So

$$\frac{d\hat{e}_t}{dt} = \left(-\sin\psi\hat{i} + \cos\psi\hat{j}\right)\frac{d\psi}{dt} = \hat{e}_n\frac{d\psi}{dt}$$

and

$$\frac{d\hat{e}_n}{dt} = \left(-\cos\psi\hat{i} - \sin\psi\hat{j}\right)\frac{d\psi}{dt} = -\hat{e}_t\frac{d\psi}{dt}.$$

Now acceleration vector at P will be given by

$$\begin{aligned}\vec{a} &= \frac{d\vec{\upsilon}}{dt} = \frac{d}{dt}\left(|\vec{\upsilon}|\hat{e}_t + 0\hat{e}_n\right)\\ &= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\frac{d\hat{e}_t}{dt}\\ &= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\hat{e}_n\frac{d\psi}{dt}\end{aligned}$$

$$= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\hat{e}_n \frac{d\psi}{ds}.\frac{ds}{dt}$$

$$= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + \frac{|\vec{\upsilon}|^2}{\rho}\hat{e}_n \left[\text{sinc}\,\rho = \frac{ds}{d\psi} \text{ and } |\vec{\upsilon}| = \frac{ds}{dt}\right]$$

$$= a_t\hat{e}_t + a_n\hat{e}_n,$$

where ρ is the radius of curvature at P.

Therefore the tangential component of acceleration at P is $a_t = \frac{d|\vec{\upsilon}|}{dt} = \frac{d^2s}{dt^2}$ and normal component at P is $a_n = \frac{|\vec{\upsilon}|^2}{\rho}$.

13.8 Integration of Vector Functions

We have already seen in art. 13.6 that for any vector function $\vec{f}(t)$ of a scalar variable t, $d\vec{f} = \vec{f}'(t)dt$.

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, then

$$d\vec{f} = df_1\hat{i} + df_2\hat{j} + df_3\hat{k}.$$

Now if $\vec{F}(t)$ be the derivative of a vector function $\vec{f}(t)$ of a scalar variable t i.e., if $\vec{F} = \frac{d\vec{f}}{dt}$, then $d\vec{f}(t) = \vec{F}(t)dt$.

In this case $\vec{f}(t)$ is called the indefinite integral of $\vec{F}(t)$ and is written as $\int \vec{F}(t)dt = \vec{f}(t)$ where $\vec{f}(t)$ is called the integral and $\vec{F}(t)$ is called the integrand.

As in the case of integration of any scalar function, here we also say that the derivative of the integral is equal to the integrand or, the integration is the inverse process of differentiation.

As in the case of integration of a scalar function we may write $\int \vec{F}(t)dt = \vec{f}(t) + \vec{c}$, where $\vec{c}$ is an arbitrary constant vector of integration and this can be calculated from some given condition.

13.8.1. Some Important Formulae of Integration

Let $\vec{r}(t)$ and $\vec{s}(t)$ be two vector functions of scalar variable t then

(i) $\int\left(\frac{d\vec{r}}{dt}.\vec{s}+\vec{r}.\frac{d\vec{s}}{dt}\right)=\vec{r}.\vec{s}+c.$

(ii) $\int\left(2\vec{r}.\frac{d\vec{r}}{dt}\right)dt=\vec{r}^{\,2}+c.$

(iii) $\int 2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}dt=\left(\frac{d\vec{r}}{dt}\right)^2+c.$

(iv) $\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\vec{r}\times\frac{d\vec{r}}{dt}+\vec{c}.$

(v) $\int\left(\frac{d\vec{r}}{dt}\times\vec{s}+\vec{r}\times\frac{d\vec{s}}{dt}\right)dt=\vec{r}\times\vec{s}+\vec{c}.$

(vi) $\int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{dr}{dt}\frac{\vec{r}}{r^2}\right)dt=\frac{\vec{r}}{r}+\vec{c}.$

(vii) $\int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt=\vec{a}\times\vec{r}+\vec{c},$ $\vec{a}$ being a constant vector.

(viii) If $\vec{f}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$, then

$$\int\vec{f}(t)dt=\left(\int f_1(t)dt\right)\hat{i}+\left(\int f_2(t)dt\right)\hat{j}+\left(\int f_3(t)dt\right)\hat{k}+\vec{c}.$$

Example 13.8.1 If $\vec{f}(t)=(t^2+1)\hat{i}+(t+1)\hat{j}-3\hat{k}$, then find $\int\vec{f}(t)dt$ and $\int_2^3\vec{f}(t)dt.$

Solution :

$$\int\vec{f}(t)dt = \int\left\{(t^2+1)\hat{i}+(t+1)\hat{j}-3\hat{k}\right\}dt$$

$$= \hat{i}\int(t^2+1)dt+\hat{j}\int(t+1)dt+\hat{k}\int(-3)dt$$

$$= \left(\frac{t^3}{3}+t\right)\hat{i}+\left(\frac{t^2}{2}+t\right)\hat{j}-3t\hat{k}+\vec{c}.$$

Hence $\int_2^3 \vec{f}(t)dt = \hat{i}\left[\frac{t^3}{3}+t\right]_2^3 + \hat{j}\left[\frac{t^2}{2}+t\right]_2^3 + \hat{k}\left[-3t\right]_2^3$

$= \frac{22}{3}\hat{i} + \frac{7}{2}\hat{j} - 3\hat{k}.$

Example 13.8.2 Evaluate $\int_2^3 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$ where $\vec{r} = t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}.$

Solution : We have $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2}.$

Therefore $\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + \vec{c}.$

Now $\frac{d\vec{r}}{dt} = 3t^2\hat{i} + 4t\hat{j} + 3\hat{k}.$

Hence $\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & 2t^2 & 3t \\ 3t^2 & 4t & 3 \end{vmatrix}$

$= -6t^2\hat{i} + 6t^3\hat{j} - 2t^4\hat{k}.$

Therefore $\int_2^3 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \left[-6t^2\hat{i} + 6t^3\hat{j} - 2t^4\hat{k}\right]_2^3$

$= -30\hat{i} + 114\hat{j} - 130\hat{k}.$

Example 13.8.3 If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2 j + 4\sin t\,\hat{k}$ and $\vec{r} = 2\hat{i} + j,\ \frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ when $t = 0$, then show that $\vec{r} = \left(t^3 - t + 2\right)\hat{i} + \left(1 - 2t^4\right)\hat{j} + (t - 4\sin t)\hat{k}.$

Solution : We have $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}.$

Integrating w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} + \vec{c}_1, \text{ where } \vec{c}_1 \text{ is a integrating constant.}$$

Since $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ when $t = 0$, we get $-\hat{i} - 3\hat{k} = -4\hat{k} + \vec{c}_1$ i.e., $\vec{c}_1 = -\hat{i} + \hat{k}$.

Therefore $\frac{d\vec{r}}{dt} = (3t^2 - 1)\hat{i} - 8t^3\hat{j} + (1 - 4\cos t)\hat{k}$.

Integrating again w.r.t. t, we obtain

$$\vec{r} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + \vec{c}_2,$$

where $\vec{c}_2$ is a integrating constant.

Now since $\vec{r} = 2\hat{i} + j$ when $t = 0$, we get $\vec{c}_2 = 2\hat{i} + j$.

Hence $\vec{r} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}$.

13.9 Summary

In this chapter we have studied about vector valued function of scalar variable and their limits, continuity, differentiability and integrability. We have also discussed about geometrical significance of vector differentiation and tangential and normal components of velocity and acceleration vectors. We also worked out some example to understand the differentiation and integration of vector functions.

13.10 Exercises

1. If $\vec{r} = t\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$, then find $\frac{d\vec{r}}{dt}$.

2. If $\vec{a} = 3t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{b} = \sin t\,\hat{i} - 2\cos t\,\hat{j}$ then find $\frac{d}{dt}(\vec{a} \times \vec{b})$ *and* $\frac{d}{dt}(\vec{a}.\vec{b})$.

3. If $\vec{a} = \sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k}$, $\vec{b} = \cos t\,\hat{i} - \sin t\,\hat{j} - 3\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} - \hat{k}$ then find the value of $\frac{d}{dt}\left\{\vec{a} \times (\vec{b} \times \vec{c})\right\}$ at $t = \frac{\pi}{2}$ and $\frac{d}{dt}\left\{\vec{a}.(\vec{b} \times \vec{c})\right\}$ at $t = 0$.

4. Evaluate $\int_0^{\pi/2} (5\cos t\,\hat{i} - 7\sin t\,\hat{j})\,dt$.

5. If $\vec{a} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{b} = 2t^2\hat{i} + 6t\hat{k}$ then find the value of $\int_0^2 \vec{a}.\vec{b}\,dt$ and $\int_0^2 (\vec{a}\times\vec{b})\,dt$.

13.11 References

1. U. Chatterjee, N. Chatterjee, Vector and tensor analysis, Academic Publishers, Kolkata, India, 2018.
2. A. K. Mukherjee, N. K. Bej, Vector analysis, Books and Allied (p) Ltd., Kolkata, India, 2018.