NETAJI SUBHAS OPEN UNIVERSITY

Choice Based Credit System

(CBCS)

SELF LEARNING MATERIAL

HPH

PHYSICS

CC-PH-03

Under Graduate Degree Programme

PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses *viz. core, discipline specefic, generic elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade ''A''.

UGC (Open and Distance Learning Programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for UG programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University

Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Physics (HPH)
Course : Mechanics and General Physics
Course Code : CC-PH-03

First Print : December, 2021

Printed in accordance with the regulations of the Distance Education Bureau of the University Grants Commission.

Netaji Subhas Open University

Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Physics (HPH)
Course : Mechanics and General Physics
Core Course : CC-PH-03

: Writer :

Dr. Rupayan Bhattacharya
Retd. Principal
Gurudas College

: Editor :

Dr. Shib Kumar Chakraborty
Associate Professor of Physics
Netaji Subhas Open University

:Format Editor :

Dr. Gahul Amin
Assistant Professor of Physics
Netaji Subhas Open University

Netaji Subhas
Open University

UG : Physics
(HPH)

Course : Mechanics and General Physics

Unit-1 ❑ Laws of Motion

Structure :

3.1.1 Proposal

Description of motion of any object always needs a frame of reference. But frames of reference are not unique. Observation on position, velocity, acceleration are dependent on the observer and also on the nature of the frames of reference. The description can be simplified for the motion of a group of particles if reference is made to the centre of mass of the moving system. Motion of a group of particles with variable mass generates lot of interest as it is related to rocket motion under different situations. Whether the force, the prime cause of motion, can be obtained from a potential or not is of general interest. During the course of motion two particles may come in contact causing a collision. Theory of collision needs an in-depth discussion.

❑ Outcome

After reading this chaper you will be able to

(i) understand how the form of Newton's law changes from inertial to non-inertial frames of reference,

(ii) understand the physics behind Galilean transformation and Galilean invariance,

(iii) appreciate the logic behind the dynamics of a system of particles and will be able to describe the motion of a system of particles having different symmetries,

(iv) describe the motion of a rocket in various circumstances,

(v) understand the nature of forces, both conservative and non-conservative and

(vi) describe the motion of paricles undergoing elastic and inelastic collisions.

3.1.2 Description of Motion

In the universe motion is everywhere. Only at a temperature of absolute zero the motion in any body is truly absent. If motion exists then so also does energy. To the delight of the physicist the tools that were invented by Galileo, Newton and others 200 years ago to descibe motion apply everywhere in the known universe, from electrons in our own bodies to the farthest galaxy. The study of motion and of energy is at the heart of physics.

The subject of motion is divided into two parts, namely kinematics and dynamics. Kinematics is concerned with the aspects of motion which exclude the forces which cause motion ; thus, in a manner of speaking, kinematics is focussed on the development of definitions ; position, displacement, velocity, acceleration and on the relationships between them. Dynamics widens the study of motion to include force and energy.

Kinenatics begins with the idea of position. Suppose that we photograph an object moving to the left along a horizontal path at two instants of time and superimpose the images for study. We examine one image with a ruler and mark off the number of units which separates the object from the ruler's zero. The zero is a reference or origin at a position of zero units by definition while the object is at another position, say x units. X is an instantaneous quantity since it applies to a specific clock time—the instant of the taking of the photograph. Position like length is a basic quantity being dependent only on the unit adopted. But position involves also direction : in principle the object could be to our right or to our left. To include the information of direction we use a vector. The magnitude of length of the vector, say r, is x (or r), while the direction is to the right, meaning the object is to the right of the reference point. We could also agree that, by convention, the sign of x is to be positive here.

The two position of the object in the photographs can be said to show two events, an initial "i"event and a final "f" event. There is now an elapsed time between the events given by :

$\Delta t = t_f - t_i$ 3.1.1

(unit seconds). Keep in mind the difference between the two concepts of time ; an elapsed time is the difference between two clock times.

❑ Displacement

Displacement differs from position. In the interval of time between the events the object moves from one position to another. The displacement is the difference between the two vectors describing the two positions.

$\Delta \vec{r} = \vec{r}_1 - \vec{r}_2$ ....(3.1.2)

(unit meters). Displacement, being the difference between two vectors, is a vector.

❑ Velocity

Another quantity in kinematics in the average velocity, or the displacement an object undergoes in one second of elapsed time. This is the ratio

$$\frac{\Delta\vec{r}}{At} = \left(\frac{\vec{r}_f - \vec{r}_f}{\Delta t}\right) = \vec{v} \qquad (3.1.3)$$

(unit meters per second). The average velocity, being a vector divided by a scalar, is a vector. The average velocity is negative here, since it points towards the origin, it smagnitude is the speed. The elapsed time in eqs. (3.1.1) and (3.1.3) is a finite interval. What if it is infinitesimally small? Mathematically, this amounts to taking the limit of eq (3.1.3) as $\Delta t \to 0$. The increments Δ are replaced by the differentials d. Eq. (3.1.3) then becomes what is known as the instantaneous velocity

$$\frac{d\vec{r}}{dt} = \vec{v} \quad (3.1.4)$$

❑ Acceleration

The velocity of any object may not be uniform. It may change with time. The velocity could decrease due to a force of friction with the path. Or the velocity could increase if the path were not horizontal and a component of the force of gravity acts on the object. The time rate of change of the average velocity is called the average acceleration and the time rate of change of the instantaneous velocity is called the instantaneous acceleration. Both types of acceleration are defined as in eqs (3.1.3) and (3.1.4) with "v" substituted for "r" and "a" substituted for "v".

❑ Worked out examples :

(1) If the displacement vector is given by $\dot{\mathbf{r}} = \hat{i}\, a \sin\omega t + \hat{j}\, 0 + \hat{k}0$, prove that the acceleraiton is always proportional to the displacement and acts in the opposite direction.

Solution : Acceleration is given $\bar{a} = \dfrac{d^2\bar{r}}{dt^2} = -a\omega^2 \text{Sin}\, \omega t\, \hat{j} = -\omega^2\, \bar{r}$. So, accleration is proportional to the displacement and is directed against the displacement.

(2) An object is moving in a straight line and its displacement is given by

$x = P - Qe^{-\alpha t}$, where P, Q and α are all constants. Find out the velocity, acceleration of the object. What will be the terminal position of the object ?

Solution : Velocity $\vec{x} = \frac{d\hat{i}}{dt} = \hat{\imath}\,\alpha Q\, e^{-\alpha t}$, Acceleration $\hat{a} = \frac{d^2x}{dt^2} = -\,\hat{\imath}\,\alpha^2 Q\, e^{-\alpha t}$. The terminal position is obtained from the expression of the displacement by putting $t \rightarrow \propto$, which gives $x_f = P - Q$.

3.1.3 Inertial Frames of Reference

We must consider one important point that the motion of an object is meaningful with respect to a frame of reference. As the frame of reference changes the idea about the motion has to change—for example, a man standing on the platform appears moving in the direction opposite to the direction of motion to a passenger of a moving train while a co-passenger appears static to the same person. Therefore, frames of reference occupies a vital role in the discussion of motion.

Let us recall certain basic concepts of motion, namely Newton's first two Laws of Motion, which are presumably as basic and fundamental as any nature law can be:

(1) The Law of Inertia : A body which has no force acting on it will remain stationary for ever or continue to move with uniform motion (that is, with constant speed and direction).

We get ideas about three things from the law : (i) inertia of rest, which imples static object remains static unless affected by external influrence, (ii) inertia of motion, i.e. continuation of uniform velocity in absence of external influence and (iii) force or the external influence itself which change the inertia rest of a body to inertia of motion and vice-versa. The concept of force is explained in second law.

(2) **The force Law :** Momentum will change in the direction of force.

Mathematically, one may say, if $\vec{F}$ be the force and $\vec{p}$ be its momentum (mass times velocity), then $\vec{F}\alpha\frac{d\vec{p}}{dt}$, or $\vec{F}=k\frac{d\vec{p}}{dt}$, where k is a constant. In S.I system one

Newton is that force which generates one kilogram-meter per second change in momentum in one second. So, in the case one can write $\vec{F}=\frac{d\vec{p}}{dt}$. Under non-relativistic (velocity much less compared to the velocity or light in vacuum) condition $\vec{p} = m\vec{v}$. Considering mass to be constant we can write $\vec{F} = m\vec{a}$

Now, these two laws seem very obvious, and perfectly reasonable and correct. So much so, that if we see a unformly moving object, we presume that it is not under any force (for at least, any net force) acting on it, whereas if we see an object which is accelerating, we presume it must have some force acting on it, in the direction of its acceleration. However, we often find ourselves in a situation in which bodies appear to be accelerating under the influence of some force, even though there is actually no force acting on them.

To understand the truth in such a statement, we need to discuss frames of reference. A frame of reference is that section of the world around us, which we utilise to measure the motion of moving bodies. For all practical purposes, the world around us appear to be at rest, and insofar as that statement is true, then any motion we measure relative to our surroundings is correcly observed, and if a motion appears uniform, it must be uniform, and if the motion appears to be non-uniform, then there is some reason behind that type of motion.

But suppose that instead of using whole of the world around us, we use some particular portion of the world, such as a railway car, which is moving relative to the rest of the world. As long as the car moves along its tracks uniformly, the laws of motion will remain unaltered and we can predict the future of the kinematic variables correctly. This is an example of the inertial froame of reference. But in practice there can be non-inertial frame of reference also.

3.1.4 Motion in a Non-Inertial Reference Frame

In inertial reference frames the Newton's laws of motion are valid. However, it is difficult sometimes to express the motion of interest in an inertial reference frame. For example, consider the motion of a clock lying on top of a table. In a reference frame which is fixed with respect to the Earth, if the clock is at rest, it will remain at rest for long long time (here our assumption is that the surface of the table is horizontal). But we know that the frame fixed to the earth is not an inertial frame. For description of the motion of the clock in an inertial frame, we need to take into account the rotation of the Earth around its own axis, the rotation of the Earth around the Sun, the rotation of our solar system around the center of our galaxy, etc., etc. The motion of the clock will all of a sudden be a lot more complicated! For many experiments, the effect of the Earth not being an inertial reference frame is too small to be observed, so one can safely ignore that. So, the frame of reference fixed to the earth can be taken as an inertial frame of reference. Example of a non-inertial frame is a rotating frame of reference which is rotating with a constant angular velocity with respect to an inertial frame of reference.

Let us see whether there is any importance of a reference frame in Newton's laws. Let the law be valid in reference frame S. We consider another frame of reference S' which is moving with a velocity v with respect to S. If the position of a particle in S be r and in S' be r', then

$$\vec{r} = \vec{r} + \int v dt$$

We have assumed here that at t = 0 the frames S and S' were coincident. By differentiating the above equation with respect to r we get

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dt} | v$$, Differentiating once again we get

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{v}}{dt}$$

This equation shows that if $d\vec{v}/dt = 0$, i.e., if there is no acceleration between the frames, = valid, but what happens in frame S'. Using the expression for acceleration we get

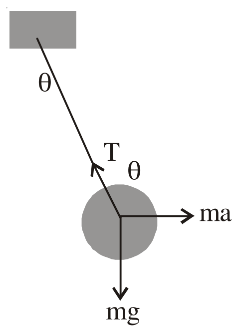
$$\vec{F} = m\left(\vec{a} + \frac{d\vec{v}}{dt}\right)$$

or $$\vec{F} - m\frac{d\vec{v}}{dt} = m\vec{a};$$

a new force called 'Pseudo force' arises due to relative movement of the reference frame. So, reference frame S' is not inertial. We shall come to this point later.

❑ Worked out Example 3.1.1

A small weight of mass m hangs from a string in a car which accelerates with an acceleration a toward left. What is the static angle of the string with the vertical and what is its tension?



We plot the free body diagram of the bob here and resolve the force in horizontal and vertical direction then,

$$T \cos \theta - mg = 0 \text{ (vertical equilibrium)}$$

$$T \sin \theta = ma \text{ (horizontal equilibrium)},$$

Then we have $\tan \theta = \tan^{-1}\left(\frac{a}{g}\right)$

or $\theta = \tan^{-1}\left(\frac{a}{g}\right)$

and $T = m(g^2 + a^2)^{1/2}$

3.1.5 Rotating Coordinate System

In order to find out the effect of the rotation of a frame of reference on the motion of a particle observed from another which is not-rotating. we shall have to describe the motion of a particle from two different frames of reference at the same time. Let us consider the two coordinate systems as shown in Fiture 1. The non-primed coordinates are the coordinates in the rotating frame, and the primed coordinates are the coordinates in the fixed coordinate system (fixed to the earth say). The vector $\vec{R}$ indicates the origin of the rotating coordinate system from the point of view of the fixed frame of reference. Now, let us consider the motion of a particle represented by the point P. In order to consider motion of the particlev, we have to find out the changes occuring to the position vector of the particle with time. In the fixed coordinates system, the position of P is denoted by the position vector $\vec{r}$ and in the rotation coordinate system, the position is denoted by the position vector $\vec{r}$. These two vectors are related by the relation :

$$\vec{r} = \vec{r} + \vec{R}$$

We shall now consider the situation when the rotating coordinate system rotates by an infinitestial angle dθ. If point P is at rest in the rotating coordinate system, we will see the position of P in our fixed coordinate system change by an amount dr, where

$$(d\vec{r})_r = d\vec{\boldsymbol{\theta}} \times \vec{r} \qquad (3.1.5.1)$$

If the rotation of the frame of reference happens during a period dt, we can find out the rate of change of the position vector as

$$\left(\frac{d\vec{r}}{dt}\right) = \frac{d\vec{\theta}}{dt} \times \vec{r} = \vec{\omega} \times \vec{r}$$

While deriving this relation our assumption is that point P remains at rest in the rotating coordinate system. If the point P moves with respect to the rotating coordinate system, this contribution must be added to the expression of the velocity of *P* in the fixed coordinate system. The above relation can be generalised to have

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_{rotating} + \vec{\omega} x$$

In this equation any kinematic variable can be inserted and corresponding changes can be found out

3.1.5.1 Galilean transformation

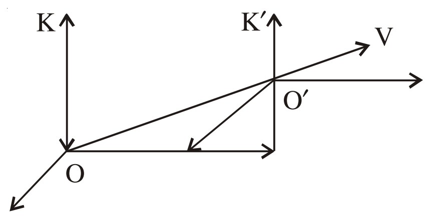
We mainly use inertial flames in which a free body (no forces applied) move with a constant velocity. A frame moving with a constant velocity with respect to an inertial frame is inertial, too. Thus there is an infinite number of inertial frames. Let frame K' is moving with a constant velocity V with respect to frame K', then we have

$$\vec{v} = \vec{v} - \vec{V},\ \vec{r}' = \vec{r} - \vec{V}t, \text{ and } t' = t. \qquad (3.1.5.4)$$

This is known as Galilean transformation. We now use Language function of Lagrangian L, which actually characterises the motion of a particle under different conditions, to describe the kinetic properties of the particle. It is given by the expression for free particle as

$$L = \frac{1}{2}mv^2 \qquad (3.1.5.5)$$

The equations of motion are invariant with respect to transformations from one inertial frame to another, and the transformed Lagrange function can differ from the intial one only by an irrelevant full derivative. This is the principle of the Galilean invariance, i.e., invariance with respect to Galilean transformations, that is valid in the classical mechanics.



Transformation of the Lagrange function of a free particle gives.

$$L' = \frac{m}{2}v^2 = \frac{m}{2}\left(\vec{v} - \vec{V}\right)^2 = \frac{m}{2}v^2 - m\vec{v}.\vec{V} + \frac{m}{2}V^2$$

$$= L + \frac{d}{dt}\left(-m\vec{r}.\vec{V} + \frac{m}{2}V^2t\right) \qquad (3.1.5.6)$$

The second term is an irrelevant full time derivative. Thus the forms of Lagrangian are the same in both frames of referenec. This shows Galilean invariance of Lagrangian. The true check of the Galilean invariance should be the identical forms of the Lagrange equations, that is

$$\frac{d}{dt}\frac{\partial L}{\partial V}=0\rightarrow\frac{d}{dt}\frac{\partial L}{\partial \vec{V}'}=0$$

The above relation can be proved easily as $L=\frac{1}{2}mv^2$, one obtains

$$0=\frac{d}{dt}\frac{\partial L}{\partial V}=\frac{d}{dt}(m\vec{v})=\frac{d}{dt}\left[m\left(\vec{v}'+\vec{V}\right)\right]=\frac{d}{dt}(m\vec{v}')=\frac{d}{dt}\frac{\partial L'}{\partial \vec{V}'}$$

where $L'=\frac{1}{2}mv'^2$ (3.1.5.7)

This proves the invariance relation.

3.1.6 System of Particles

3.1.6.1 Centre of mass

The motion of a system of particles can be described in terms of motion of a single point. This special point is called the centre of mass of the system of particles.

The position co-ordinate of the centre of mass of a system of two particles with mass m_1 and m_2, placed at position x_1 and x_2. respectively with respect to a particular co-ordinate system, is defined as

$$x_{cm}=\frac{m_1x_1+m_2x_2}{m_1+m_2}=\frac{1}{M}\sum_i m_ix_i \qquad (3.1.6.1)$$

We can define the origin of our coordinate system to be at centre of the left most object (see Figure 3.1.6.1). The position of the centre of mass is now

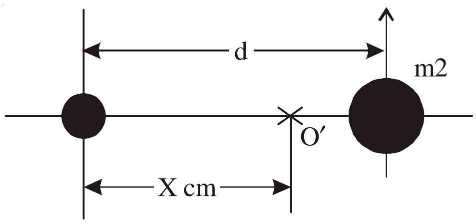


Figure 3.1.6.1 Position of the centre of mass in 1 dimension

$$x_{cm} = \frac{m_2 d}{m_1 + m_2}$$

From this equation it is clear that the center of mass lies between the two masses. It is closeex to the heavier mass. In general, for a system with more than two particles, the position of the centre of mass will satisfy the following relation

$$x_{min} \le x_{cm} \le x_{max}$$

The center of mass in one dimension for any number of particles can be easily geoneralized to three dimensions.

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i \tag{3.1.6.3}$$

$$y_{cm} = \frac{1}{M}\sum_i m_i y_i \tag{3.1.6.4}$$

$$z_{cm} = \frac{1}{M}\sum_i m_i z_i \tag{3.1.6.5}$$

or in vector notation

$$\bar{r}_{cm} = \frac{1}{M}\sum_i m_i \bar{r}_i$$

For a homogeneous rigid body, the summation can be replaced by an integral

$$\bar{r}_{cm} = \frac{1}{M}\int_V r dm \qquad (3.1.6.7)$$

Suppose we are dealing with more than two number of objects. Figure 3.1.6.2 shows a system consisting of 4 masses, m_1, m_2, m_3 and m_4. located at x_1, x_2, x_3 and x_4, respectively. The x-co-ordinate of the center of mass of m_1 and m_2 is given by

$$x_{cm}^{1,2} = t\frac{m_1x_1 + m_2x_2}{m_1 + m_2} \qquad (3.1.6.8)$$

Similarly, the x-co-ordinate of the centre of mass of m_3 and m_4 is given by

$$x_{cm}^{3,4} = \frac{m_3x_3 + m_4x_4}{m_3 + m_4} \qquad (3.1.6.9)$$

Now, the x-co-ordinate of the centre of mass of the system of four particles is given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_2x_2 + m_4x_4}{m_1 + m_2 + m_3 + m_4} \qquad (3.1.6.10)$$

Another representation of this is

$$x_{cm} = \frac{1}{m_1 + m_2 + m_3 + m_4}\left((m_1 + m_2)\frac{m_1x_1 + m_2x_2}{m_1 + m_2} + (m_3 + m_4)\frac{m_3x_3 + m_4x_4}{m_3 + m_4}\right)$$

Using the centre of mass of m_1 and m_2 and of m_3 and m_4 we can express the centre of mass of the whole system as follows :

$$x_{cm} = \frac{(m_1 + m_2)x_{cm}^{1,2} + (m_1 + m_2)x_{cm}^{3,4}}{(m_1 + m_2) + (m_3 + m_4)} \quad (3.1.6.11)$$

Figure 3.1.6.2 Location of 4 masses

This shows that the center of mass of a system can be calculated from the position of the centre of mass of all objects that make up the system. For example, the position of the centre of mass of a system consisting out of several spheres can be calculated by assuming that the mass of each sphere is concentrated in the centre of that sphere (its centre of mass).

❑ Important points :

* The centre of mass of an object always lies on a point/ line/plane of symmentry (for homogeneous objects).
* The center of mass of an object does not need to lie within the body of that object (for example : the center of a doubt is its center of mass even though there is no mass at that point).

❑ Worked out Example 3.1.6.1

A circular metal plate of radius 28 from which a disk of radius R has been removed is shown in fig. 3.1.6.3. Let us call the portion as object X. We need to locate the center of mass of object X.

Let us consider a co-ordinate frame with its origin at the centre of the circular plate. Suppose the hole in object X is filled with a disk of radius R. The new object (object C. Figure 3.1.6.3b) is symmetric around the origin of our coordinate system and that point is therefore the centre of mass of object C. However, object C consist out of object X and a disk with radius R centered on the x-axis at x = – R (this disk is called object D). The center of mass of this system (consisting out of object X and object D) can be easily calculated :

$$x_{cm,c} = \frac{x_{cm},Xm_X + x_{cm,D}m_D}{m_X + m_D} = 0$$

The equation can be rewritten as

$$x_{cm,x} = \frac{x_{cm,D}m_D}{m_x} = \frac{Rm_D}{m_x}$$

Since we are discussing a homogeneous disk (with mass per unit area σ) the masses of object X and D can be calculated.

$m_D = \sigma\pi R^2$

$m_x = \sigma\pi(2R)^2 \cdot m_D = \sigma 3\pi R^3$

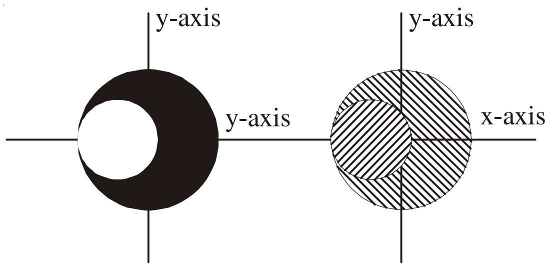


Figure : 3.1.6.3 Sample Problem 3.1.6.1

The position of the centre of mass of object X is given by

$$x_{cm,x} = \frac{1}{3}R$$

❑ Worked out Example 3.1.6.2

A one dimensional rod is shown in fig.3.1.6.4. The density of the rod is not a constant. It depends on position : $\lambda(x) = a + bx + cx^2$ λ = mass/unit length. Determine the location of the center-of-mass of the rod.

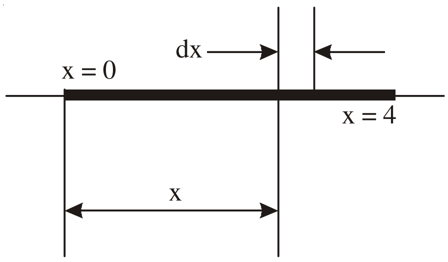


Figure : 3.1.6.4 Position Dependent Density

The mass of a small element of the rod (length dx) is given by

dm = λ (x) dx

The position of the center of mass of the rod can be determined as follows

$$x_{cm} = \frac{1}{M}\int_{x_i}^{x_t} x\,dm = \frac{1}{M}\int_0^d x\sigma(x)dx = \frac{1}{M}\int_0^d (ax + bx^2 + cx^3)dx$$

After evaluating the integral we obtain

$$x_{cm} = \frac{1}{M}\left(\frac{a}{2}d^2 + \frac{b}{3}d^3 + \frac{c}{4}d^4\right)$$

3.1.7 Uniqueness of the position of the center of mass

We would like to show that the position of the center of mass does not depend on the origin of the co-ordinate frame chosen. Let us take another point O' as the new origin.

Let $OO' = \vec{d} + O'P_i = \vec{r}_i = \vec{d} + \vec{r}_i$

Now with O′ as origin, the center of mass G′ will be given by

$$O'G' = \frac{\sum m_i r_i}{M} = \sum m_i \frac{(\vec{d}_i + r_i)}{M} = \frac{\sum m_i \vec{d}}{M} + \frac{\sum m_i r_i}{M} = d + OG = O'O + OG = O'G$$

The above steps show that G′ and G are identical points, So, the center of mass is a unique point irrespective whatever co-ordinates system is chosen.

3.1.8 Motion of the Center of Mass

The centre of mass of a system of particles can be expressed as

$$\mathbf{M\vec{r}_{cm} = \sum_i m_i \vec{r}_i} \quad \text{— (3.1.8.1)}$$

Where M is the total mass of all the particles. From this equation by differentiation with respect to time we get

$$\mathbf{M\vec{v}_{cm} = \sum_i m_i \vec{v}_i} \qquad (3.1.8.2)$$

Here $\vec{V}_{cm}$ is the velocity of the center of mass and $\vec{V}_i$ is the velocity of mass m_i. From Newton's 2nd Law we can write for the i-th particle of the system.

$$\vec{F}_{iex} + \sum_i \vec{F}_{ij} = \frac{d\vec{P}_i}{dt}, (i \neq j)$$

Here $\vec{F}_{iex}$ is the external force on the i-th particle and $\vec{F}_{ij}$ is the external force exerted by the j-th particle on the i-th particle, only j is not equal to i. The total linear momentum of the system with respect to any arbitrary point B can be expressed as

$$\vec{P}_n = \sum_i \vec{P}_{iB} = \sum_i m_i \left(\dot{\vec{r}}_i - \dot{\vec{r}}_B\right) = \sum_i \vec{P}_i - M\dot{\vec{r}}_B \qquad (3.1.8.4)$$

here r_i and r_B refers to the position vector of the i-th and the reference point B. By differentiating both sides of the above equation w.r.t t we get

$$\frac{d\vec{P}_B}{dt} = \sum_i \frac{d\vec{P}_i}{dt} = M\ddot{\vec{r}}_B \qquad (3.1.8.5)$$

If the distances are taken from the origin of the inertial frame, then we can write

$$\frac{d\vec{P}_B}{dt} = \vec{F}_{ext} + \sum_i \sum_j \vec{F}_{ij} - M\ddot{\vec{r}}_B \qquad (3.1.8.6)$$

here $\vec{F}_{ext}$ is the net external force on the system , $\ddot{\vec{r}}_B$ vanishes as B is the origin. Moreover

$$\sum_i \sum_j \vec{F}_{ij} = 0$$

as the sum of all internal forces vanishes because of Newton's third law. So, in an inertial frame of reference.

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext} \qquad (3.1.8.7)$$

As a consequence of this we can easily say that when $\vec{F}_{ext}$ = O, the net linear momentum $\vec{p}$ = constant. This is the conservation of linear momentum of a system of particles.

3.1.9 Angular Momentum of a System of Particles

We now calculate the angular momentum about any point P associated with a system of N point particles. We characterize each individual particle by the index j, j = 1,2,..., N. Let the j-th particle have mass m and velocity $\vec{v}_j$; The momentum of an individual particle is then $\vec{p}_j$; = m_j; $\vec{v}_j$;. Let $\vec{r}_j$ be the vector from the point P to the jth particle, and let θ be the angle between the vectors ; $\vec{r}_p$; and $\vec{p}_j$. The angular momentum Lp.j of the j-th particle is

$$\vec{L}_{p.j} = \vec{r}_{pg} \times \vec{p}_j \qquad (3.1.9.1)$$

The angular momentum for the system of particles is the vector sum of the individual angular momenta of all the particles.

$$\vec{L}_P^{sys} = \sum_{j=1}^{j=N} \vec{L}_{Pj} = \sum_{j=1}^{j=N} \vec{r}_{P.j} \times \vec{p}_j \qquad (3.1.9.2)$$

The rate of change of the angular momentum of the system of particles about a point P is given by $\frac{d\vec{L}_P^{sys}}{dt} = \frac{d}{dt}\sum_{j=1}^{j=N}\vec{L}_{P,J} = \sum_{j=1}^{j=N}\left[\frac{d\vec{r}_{p,j}}{dt}\times\vec{p}_j\right] + \sum_{j=1}^{j=N}\left[\vec{r}_{p,j}\times\frac{d\vec{p}_j}{dt}\right]$ (3.1.9.3)

Because the velocity of the j_{th} particle is $\vec{v}_{p,j} = d\vec{r}_{p/t}$, the first term in the parentheses vanishes (the cross product of a vector with itself is zero because they are parallel to each other). As we know that

$$\vec{F}_j = \frac{dp_j}{dt},$$

immediately we get

$$\frac{d\vec{L}_P^{sys}}{dt} = \sum_{j=1}^{j=N}\left[\vec{r}_j\times\frac{d\vec{p}_j}{dt}\right] = \sum_{j=1}^{j=N} r_{p,j} = \vec{r}_P^{ext} \quad (3.1.9.4)$$

Therefore equation (3.1.9.3) becomes

$$\tau_P^{ext} = \frac{d\vec{L}_P^{ext}}{dt} \quad (3.1.9.5)$$

It is clear that the external torque about the point P is equal to the time derivative of the angular momentum of the system of particles about the same point P.

3.1.10 Angular Momentum of a System of Particles about Different Points

Let us consider a system of N number of particles moving around two points A and B. The angular momentum of the i-th particle about one of the points A is given by

$$L_{A.i=}\vec{r}_{A,i}\times m_i\vec{v}_i \quad (3.1.10.1)$$

About the point A the net angular momentum of the system of particles is given by $\vec{L}_A = \sum_{i=1}^{i=N}\vec{L}_{A,i} = \sum_{i=1}^{i=N}\vec{r}_{A,i}\times m_i\vec{v}_i$ (3.1.10.2)

Similarly, the angular momentum about the point B can be calculated in a similar way which is given by

$$L_B = \sum_{j=1}^{j=N} L_B, j = \sum_{j=1}^{j=N} r_{B,j} \times m_j \vec{v}_j \qquad (3.1.10.3)$$

but $\vec{r}_{A,j} = \vec{r}_{B,j} + \vec{r}_{A,B}$ (3.1.10.4)

Here after substituting eq. (3.1.10.4) into eq. (3.1.10.2) we get

$$\vec{L}_A = \sum_{j=1}^{j=N} \left(\vec{r}_{B,j} + \vec{r}_{A,B}\right) \times m_j \vec{v}_j = \sum_{j=1}^{j=N} \vec{r}_{B,j} \times m_j \vec{v}_j + \sum_{j=1}^{j=N} \vec{r}_{A,B} \times m_j \vec{v}_j \qquad (3.1.10.5)$$

In this equation $\vec{r}_{A,B}$ is a constant , so, it can be taken out the summation. Therefore, we get

$$\vec{L}_A = \vec{L}_B + \vec{r}_{A,B} \times \sum_{j=1}^{j=N} m_j v_j \qquad (3.1.10.6)$$

The sum in the second term represents the momentum of the system. Thus we can conclude that if the momentum of the system is zero, the angular momentum is the same about any point.

$$\vec{L}_A = \vec{L}_n \text{ if } (p_{sys} = 0) \qquad (3.1.10.7)$$

3.1.11 Variable Mass Systems : The Rocket Equation

In this section, we shall consider a dynamical problem in which the mass of the experimental body changes during the motion, which means m is a function of t, i.e, m(t). Although there are many cases for which this particular model is applicable, one of obvious importance to us are rockets. A significant fraction of the mass of a rocket is the fuel, which is expelled during flight at a hight velocity and thus, provides the propulsive force for the rocket. To analyse this question we must consider a system of variable mass, and the process by which it gains velocity as a result of ejecting mass. Let us consider a body of variable mass, with velocity $\vec{v}$ and external forces $\vec{F}$. The said body is gaining mass at a rate dm/dt. Let us look at the process of gaining a small amount of mass dm. Let $\vec{v}_0$ be the velocity of dm before it is captured by m, and let $\vec{F}$ represent the average value of the impulse forces that dm exerts on m. during the short interval dt. in which the capturing takes place. By Newton's third law, dm will experience a force $\vec{f}$, exerted by m, over the same dt. We can now examine the capture process from the point of view of dm and equate the the impulse, -$\vec{f}$dt, to the change in linear momentum of dm.

$$\vec{F}dt + \vec{f}\,dt = m(\vec{v} + d\vec{v}) - m\vec{v} = md\vec{v} \tag{3.1.11.1}$$

Here, $\vec{v} + d\vec{v}$ is the velocity of m (and dm) after impact. Analogously, from the point of view of m.

we write

$$\vec{F}dt + \vec{f}\,dt = m(\vec{v} + d\vec{v}) - m\vec{v} = md\vec{v} \tag{3.1.11.2}$$

As the term dmd$\vec{v}$ in equation (3.1.11.1) is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated

$$\vec{F}dt - dm(\vec{v} + \vec{v}') = md\vec{v}$$

or, dividing through by dt,

$$m\frac{d\vec{v}}{dt} = \vec{F} - (\vec{v} - \vec{v}')\frac{dm}{dt} = \vec{F} + (\vec{v}' - \vec{v})\frac{dm}{dt} \tag{3.1.11.3}$$

Here, $\vec{v}' - \vec{v}$ is the velocity of dm relative to m. This expression is valid when $\frac{dm}{dt} > 0$ (mass gain) and when $\frac{dm}{dt} < 0$ (mass loss). If we compare the expression to the more familiar form of Newton's law for a particle of fixed mass $m\frac{d\vec{v}}{dt} = \vec{F}$, we see that the term $(\vec{v}' - \vec{v})\frac{dm}{dt}$ is an additional force on m which is due to the gain (or loss) or mass. Equation (3.1.11.3) can also be written as

$$\frac{d(m\vec{v})}{dt} = \vec{F} + \vec{v}'\frac{dm}{dt}$$

where $\vec{v}'$ is the velocity of the captured (or expelled) mass relative to the velocity of the mass m. This shows that, for systems involving variable mass, the usual expression stating conservation of linear momentum. $\frac{d(m\vec{v})}{dt} = \frac{dm}{dt}$, is only applicable when the initial (final) velocity of the captured (expelled) mass, $\vec{v}$, is zero. The behaviour of m(t) is not an unknown, but is specified according to the characteristics of the rocket.

In most cases $\frac{dm}{dt}$ is a constant and negative. In some cases, the behaviour of m(t) may be determined by a control system. In any case, it is a given quantity.

❑ Worked out Example : Conveyer Belt

Let us consider a situation where sand particles are dropping from a stationary hopper at a rate $\frac{dm}{dt}$ onto a coneyer belt which is moving with a velocity $\vec{v}$ with respect to the reference frame fixed to the Laboratory. We want to find out the force required to keep the above conveyer belt moving with the same velocity.

As the velocity is constant, we get $\frac{d\vec{v}}{dt}$ = 0. According to the problem sand is dropping from a stationary hopper, so, $\vec{u}$ = 0. Therefore we get, $\vec{F} = \vec{v}\frac{dm}{dt}$. Remember here $\frac{dm}{dt} \neq 0$ as the system gains mass with time

3.1.12 The Rocket Equation

Next we consider a rocket or mass m, moving with a velocity $\vec{v}$ and subject to external forces $\vec{F}$ (typically gravity and drag). The rocket mass changes at a rate dm/dt, with a velocity vector $\vec{c}$ relative to the rocket. We shall assume that the magnitude of $\vec{c}$ is constant.

The velocity of the gas observed from a stationary co-ordinate frame will be v′ = $\vec{v}$ + $\vec{c}$. In this frame, $\vec{c}$ is a vector aligned along the flight path in a negative direction c = – $c\hat{n}$, whre $\hat{n}$ is the unit direction along the flight path.

Thus $$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{C}\frac{dm}{dt} \qquad (3.1.12.1)$$

The term T = $\vec{c}\frac{dm}{dt}$ is called the thrust of the rocket and can be interpreted as an additional force on the rocket due to the gas expulsion. Equation (3.1.11.1) is a vector equation which can be projected along the direction of $\vec{v}$ (tangent to the path). Thus,

$$m\frac{d\vec{v}}{dt} = \vec{F}_t - \vec{c}\frac{dm}{dt} = \vec{F}_t + \vec{T}$$

where $\vec{F}_t$ is the tangential component of $\vec{F}$, v and c are the magnitudes of $\vec{v}$ and $\vec{c}$ respectively, and we have assumed that $\vec{c}$ is parallel and has opposite direction to $\vec{v}$. The magnitude of the thrust is $\vec{T}$ = – $\vec{c}$ dm/dt. Note that for a rocket, m will be negative (mass is lost). If the force $\vec{F}_t$ is known, this equation can be integrated in time to yield an expression for the velocity as a function of time. Let us consider some simple cases :

❑ No External Forces : $F_t = 0$

If gravity and drug effects are neglected, we have,

$$m\frac{d\vec{v}}{dt} = -\vec{C}\frac{dm}{dt}$$

or, integrating between an intial time t_0, and a final time t,

$$\Delta v = \vec{v} - \vec{v}_0 = -\vec{c}(\ln m - \ln m_0) = -\vec{c}\ln\frac{m}{m_0} = \vec{c}\ln\frac{m_0}{m_f} \qquad (3.1.12.3)$$

Alternatively, this expression can be east as the well known rocket equation.

$$m(t) = m_0 e^{-\Delta v/c} \qquad (3.1.12.4)$$

which gives the mass of the rocket at a time t, as a function of the initial mass m_0, $\Delta\vec{v}$, and $\vec{c}$. The mass of the propellant, $m_{propellant}$ is given by,

$$m_{propellant} = m_0 - m = m_0 (1-e^{-\Delta v/c}) \qquad (3.1.12.5)$$

From the above equations, we see that for a given Δv and m_0, increasing c increases m (payload plus structure) and decreases $m_{propellant}$. Unfortunately, we can only choose $\vec{c}$ as high as the current technology will allow. For current chemical rockets, $\vec{c}$ ranges from 2500 – 4500 m/sec. Ion engines can have $\vec{c}$'s of roughly 10^5 m/sec.

❑ Gravity : $F_t = -mg$

A constant gravitational field acting in the opposite direction to the velocity vector can be easily incorporated. In this case, equation (3.1.11.2) becomes

$$m\frac{d\vec{v}}{dt} = -m\vec{g} - \vec{c}\frac{dm}{dt}$$

which can be integrated to give

$$\vec{v} = v_0 - \vec{c}\ln\frac{m}{m_0} - gt = \vec{v}_0 - \vec{c}\ln\frac{m}{m_0} - \vec{g}\frac{m - m_0}{\dot{m}} \qquad (3.1.12.6)$$

The solution assumes that $\vec{c}\frac{dm}{dt} > m_0 g$ at t = 0. If this is not true, the rocket will sit on the pad. burning fuel until the remaining mass satisfies this requirement.

3.1.13 Conservative and non-conservative force-fields

Suppose that a non-uniform force-field f(r) acts on an object in such a way that the object moves along a curved trajectory, from point A to point B. See fig.1. The work W_1 performed by the force field on the object can be written as a line-integral along this trajectory :

$$W_i = \int_{(A\to B)_{path_1}} \underset{\sim}{f} \cdot d\vec{r} \tag{3.1.13.1}$$

Let us suppose that the same object moves along a different trajectory, (say path 2), between the same two points. In this case, the work W_2 performed by the force-field is

$$W_2 = \int_{A\to B\,path_2} \vec{f}.d\vec{r} \tag{3.1.13.2}$$

Basically, there are two possibilities. Firstly, the line integrals (3.1.13.1) and (3.1.13.2) might depend on the end points, A and B, but not on the path taken between them, in which case $W_1 = W_2$. Secondly, the linc integrals (3.1.13.1) and (3.1.13.2) might depend both on the points, A and B, and the path taken between them, in which case $W_1 \neq W_2$ (in general). The first possibility corresponds to what physicists term a conservative force-field. whereas the second possibility corresponds to a non-conservative force field.

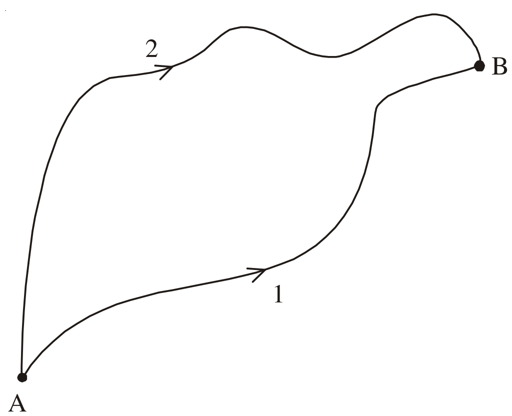


Figure : 1. Two alternative paths between point A and B

What is the physical distinction between a conservative and a non-conservative force-field? Well, the easiest way of answering this question is to slightly modify the problem discussed above. Suppose, now, that the object moves from point A to point B along path 1, and then from point B back to point A along path 2. What is the total work done on the objet by the forec-field as it executes this closed circuit? Incidentally, one fact which should be clear from the definition of a line-integral is that if we simply reverse the path of a given integral then the value of that integral picks up a minus sign : in other words,

$$\int_A^B \vec{f}.d\vec{r} = -\int_B^A \vec{f}.d\vec{r},$$

where it it understood that both the above integrals are taken in opposite directions along the same path. Recall that conventional 1-dimensional integrals obey an analogous rule : i.e., if we swap the limits of integration then the integral picks up a mimus sign. It follows that the total work done on the object as it executes the circuit is simply

$$\Delta W = W_1 - W_2$$

where W_1 and W_2 defined in Eqs. (3.1.13.1) and (3.1.13.2), respectively. There is a minus sign in from W_2 of because we are moving from point B to point A, instead of the other way around. For the case of a conservative field, we have $W_1 = W_2$. Hence, we conclude that

$$\Delta W = 0 \qquad (3.1.13.4)$$

In other words, the net work done by a conservative field on an object taken around a closed loop is zero. This is just way of saying that a conservative field stores energy without loss ; i.e., if an object gives up a certain amount of energy to a conservative field in travelling from point A to point B, then the field returns this energy to the object—without loss—when it travels back to point B. For the case of a non-conservative field. $W_1 \neq W_2$. Hence, we conclude that

$$\Delta W \neq 0 \qquad (3.1.13.5)$$

In other words, the net work done by a non-conservative field on an object taken around a closed loop is non-zero. In pratice, the net work is invariabley negative. This is just another way of saying that a non-conservative field dissipates energy ; i.e., if an object gives up a certain amount of energy to a non-conservative field in traveling from point A to point B, then the field only returns part, or, perhaps, none, of this energy to the object when it travels back to point B. The remainder is usually dissipated as heat.

What are typical examples of conservative and non-conservative fields? Well, a gravitioinal field is probably the most well-known example of a conservative field. A typical example of a non-conservative field might consist of an object moving over a rough horizontal surface.

We have seen that the work done by a conservative force on a particle as it moves from point A to a point B does not depend on the path chosen and therefore it depends only on the position co-ordinates of the terminal points (A, B). Then the work done can be expressed as the difference of a function of position variables only :

$$\int_A^B \vec{F}.d\vec{r} = V(r_A) - V(r_B) = -\int_A^B dV(\vec{r}) \qquad (3.1.13.6)$$

Where $V(\vec{r})$ is known as potential energy function $V(\vec{r})$ is a scalar function of position co-ordinates. So, we arrive as a relation.

$$\vec{F}.d\vec{r} = -\,dV\tfrac{1}{2} \qquad (3.1.13.7)$$

Upon integration of the Eq. (3.1.12.7) we get

$$V(\vec{r}) = -\int \vec{F}.d\vec{r} + C \qquad (3.1.13.8)$$

which means that the absolute value of the potential function V at any point is uncertain by a constant C.

From Eq. (3.1.12.6) we get

$$\int_A^B \vec{F}.d\vec{r} = -\int_A^B dV(\vec{r}) = -\int_A^B \overrightarrow{\nabla V}.d\vec{r} \qquad (3.1.13.9)$$

or

$$-\int_A^B \left(\vec{F}+\overrightarrow{\nabla V}\right).d\vec{r}=0 \qquad (3.1.13.10)$$

This Eq. (3.1.12.10) is true for any path connecting two arbitrary points A an B in a conservative force field. So, we can conclude.

$\vec{F} + \overrightarrow{\nabla V} = 0$

or $\vec{F} = -\overrightarrow{\nabla V}$

Since $\vec{\nabla} \times \overrightarrow{\nabla V}$ is always zero so in case of conservative field $\nabla\times\vec{F}=0$.

❑ Worked out example :

Prove that $\vec{F} = r^2\vec{r}$ is conservative. Find the corresponding scalar potential.

Solution : $\nabla \times \vec{F} = \nabla \times (r^2\vec{r}) = r^2 \nabla \times \vec{r} + (\nabla \vec{r}^{\,2}) \times \vec{r} = 0 + \frac{d}{dr}(r^2)\, \hat{r} \times \vec{r} = 0$

Therefore, $\vec{F}$ is conservative.

So, $\vec{F} = -\overrightarrow{\nabla V}$. Now $dV = \overrightarrow{\nabla V}.\, d\vec{r} = -\vec{F}.\, d\vec{r} = -r^3\, dr$

Hence, $V = -\frac{1}{4}(r^4) + c$ (c is a constant of integration)

3.1.14 Path Integral of a force : The Work-Energy Theorem :

The work done by a force $\vec{F}$ acting on a particle which moves in a trajestory from a point A to another point B is certain region of space is given b

$$W_{AB} = \int_A^B \vec{F}(r)\cdot d\vec{r} \qquad (3.1.14.1)$$

This can be expressed as

$$W_{AB} = \int_A^B m\frac{d\vec{v}}{dt}\cdot\vec{v}dt = \int_A^B \frac{1}{2}m\frac{d}{dt}\left(v^2\right)dt$$

$$= \frac{1}{2}mv_B{}^2 - \frac{1}{2}mv_A{}^2 \qquad (3.1.14.2)$$

This constitutes the general statement of the work energy theorem. The work done by a force acting on a particle of mass m is equal to the change in its kinetic energy, the difference between the final and initial values of kinetic energy.

But we have seen that in a conservative force field the work done by a force is equal to the change in potential energy of the particle. So, one can write.

$$\frac{1}{2}m\,v_B^2 - \frac{1}{2}mv_A^2 = V_A - V_B \qquad (3.1.14.3)$$

Since A and B are quite aribtrary points in space, this can only be true if and only if, each side of the equation is equal to a constant E, known as total mechanical energy. In other words the total mechanical energy is conserved in a conservative force field.

❑ Worked out examples :

(a) Inverse square law type force field

Consider a system consisting of two object of masses m_1 and m_2 that are separated by a center to center distance $r_{2,1}$. The internal gravitational force on object 1 due to the interaction between the two objects is given by

$$F_{21}^G = \frac{-Gm_1m_2}{r_{21}^2}\hat{r}_{21}$$

The displacement vector is given by $d\vec{r}_{21} = d\hat{r}_{21}r_{21}$. So we get for the scalar product

$$\vec{F}_{21}^G.d\vec{r}_{21} = \frac{-Gm_1m_2}{r_{21}^2}d\vec{r}_{21}$$

From our definition of potential energy we have mentioned earlier that the change in potential energy of a system depends on the work done in moving the system from an initial position of the center of mass of the two objects apart by a distance r_i to a final position of the center of mass of the same two objects apart by a distance r_f is given by

$$\Delta U^G = \int_A^B \vec{F}_{21}^G \cdot d\vec{r}_{21} = -\int_i^f \frac{-Gm_1m_2}{r_{21}^2} dr_{21} = \frac{-Gm_1m_2}{r_f} + \frac{-Gm_1m_2}{r_i}$$

The reference point for the zero of the potential energy is chosen to be at infinity, $r_i = \propto$, with the choice that $U^G \propto \alpha\, 0$. By making this choice, the term $\frac{1}{r}$ in the exprssion for the change in potential energy vanishes when $r_i = \propto$. The gravitional potential energy as a function of the relative distance r between the two objects is given by

$$U_G\ (r) = -\frac{Gm_1m_2}{r},\ \text{with}\ U^G\ (\propto) = 0$$

(b) Hooke's law type force field

Let us consier a spring-object system lying on a horizontal surface which is frictionless. One and of the spring-object system is fixed to a wall and the other end is attached to an object of mass m. The spring force is an internal conservative force. The wall exerts an external force on the spring object system but since the point of contact of the wall with the spring undergoes no displacement, no work is done by this external force.

We choose the origin at the position of the center of the object when the spring is relaxed (the equilibrium position). Let x be the displacement of the object from the origin. We choose the $+\hat{i}$ unit vector to point to the direction the object moves when the spring is being stretched (to the right of x = 0 on the figure). The spring force on a mass is then given by

$$\vec{F} = F_x\hat{i} = kx\hat{i} + 0\hat{j} + 0\hat{k}\vec{r}\,.$$

The displacement is $d\vec{r} = \hat{i}\,dx$. The scalar product is

$\vec{F}.d\vec{r} = -kx\hat{i}\cdot\hat{i}dx = kx\,dx$. The work done by the spring force on the mass is

$$W = \int_{x=x_i}^{x=x_f} \vec{F}.d\vec{r} = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

We can now define a change in potential energy in the same spring object system in moving the object from an initial position x_i from equilibrium to a final position s_i by.

$$\Delta U = U(x_f) - U(x_i) = -W = \frac{1}{2}\ (x_f^2 - x_i^2)$$

So, an arbinary stretch or compression of a spring-object system obeying Hook's Law, from an equilibrium position at $x_i = 0$ to a final position $x_f = x$ changes the potential energy by

$$\Delta U = U(x) - U(0) = -\frac{1}{2}kx^2$$

If we take U(0) = 0, then with this choice of zero reference potential, the expression for potential energy is given by $U(x) = \frac{1}{2}kx^2$, with U(0) = 0.

3.1.15 Collisions

Any collision between two or more particles can be characterised by three stages : 1) before the collisions—particles are free and moving in straight lines with constant velocities. 2) collisions take place in the interaction zone with large force but for a very short time interval during which change in momentum and energy take place. 3) After the collisions the particles move freely in straight lines with constant velocity.

In any collision of two bodies, their net momentum is conserved. That is, the net momentum vector of he bodies just after the collision is the same as it was just before the collision.

$$\vec{P}_{net} = m_1\vec{v}_1' + m_2\vec{v}_2' = m_1\vec{v}_1 + m_2\vec{v}_2$$

So, if we know the velocity vectors of both bodies before the collision and if we also know the velocity vector of one body after the collision, then using this formula we may find out the velocity vector of the other body after the collision. But if we only know the initial velocities of the two bodies and we want to find out their velocities after the collision, we need to invoke additional physics. In particular, we need to know what happens to the net kinetic energy of the two bodies,

$$K_{net} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \qquad (3.1.15.2)$$

Let us now recognize this net kinetic energy into two terms, one due to the net momentum (3.1.15.1) of two particles and the other due to their relative velocity $\vec{v}_{ret} = \vec{v}_t - \vec{v}_2$.

$$K_{net} = \frac{P_{net}^2}{2(m_1 + m_2)} + \frac{m_1 m_2}{2(m_1 + m_2)} \times v_{rel}^2 \qquad (3.1.15.3)$$

In Eq. (3.1.15.3) the first term denotes the kinetic energy due to motion of the centre of mass of the two-body system.

$$K_{cm} = \frac{P_{net}^2}{2(m_1 + m_2)} = \frac{m_1 + m_2}{2} \times v_{cm}^2 \qquad (3.1.15.4)$$

In any two-body collision this term is conserved as the net momentum P_{net} is conserved in such collisions.

In Eq. (3.1.14.3) the second term represents the kinetic energy due to relative motion of the two colliding bodies, i.e.,

$$K_{rel} = \frac{m_1 m_2}{2(m_1 + m_2)} \times v_{rel}^2 = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} \times (v_1 - v_2)^2 \qquad (3.1.15.5)$$

We are interested in what happens during the collision. There are three possibilities :

(a) The collision is elastic—in an elastic collision the kinetic energy of the relative motion is converted into the elastic energies of the two colliding and compressed bodies. It is then converted back into the kinetic energy. Therefore, the kinetic energy of relative motion before collision is equal to the kinetic energy of relative motion after the collision and as a result the net kinetic energy of the two colliding bodies is conserved.

$$\left(\frac{1}{2}\right)m_1 v_2'^2 \times \left(\frac{1}{2}\right)m_2 v_2'^2 = \left(\frac{1}{2}\right)m_1 v_1^2 + \left(\frac{1}{2}\right)m_2 v_2^2 \qquad (3.1.15.6)$$

and also $v_1' - v_2' = v_1' - v_2'$ (3.1.15.7)

(b) The collision is inelastic—a part of the kinetic energy of relative velocity K_{ret} is converted into elastic energy and then back intothe kinetic energy of changed relative velocity. The rest of the initial kinetic energy is converted into heat or other form of non-mechanical energy. So, we get,

$$0 < K'_{rel} < K_{rel}$$

(c) Totally inelastic collision—conversion of all the kinetic energy of relative motion into heat or any other non-mechanical energies takes place. Therefore, we see $K_{rel} = 0$ after collision and there is no-relative motion.

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = v_B / v_A \qquad (3.1.15.8)$$

If the magnitude of the relative velocity does not change during a collision, e = 1, then the change in kinetic energy is zero. Collisions in which there is no change in kinetic energy are called elastic collisions.

$\Delta K = 0$, elastic collision (31.15.9)

If the magnitude of the final relative velocity is less than magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy changes are called inclastic collisions.

$\Delta K \neq 0$, inclastic collision (3.1.15.10)

If the trwo objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called totally inelastic. The change in kinetic energy can be found as

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2} m_1 m_2 v_A^2 (m_1 + m_2), \text{ total inelastic collision.} \quad (3.1.15.11)$$

3.1.16 One-Dimensional Collision Between Two Objects—Center of Mass Reference Frame

Now let's view the collision from the center of mass (CM) frame. The x-component of velocity of the center of mass is

$$v_{x,cm} = \frac{m_1 v_{1x,i} + m_1 v_{2x,i}}{m_1 + m_2} \quad (3.1.16.1)$$

The x-components of the velocities w.r.t the center of mass are

$$v_{1x,i} = v_{1x,i} - v_{x,cm} = \left(v_{1x,i} - v_{2x,i}\right)\frac{m_2}{m_1 + m_2}$$

$$v'_{2x,i} = v_{2x,i} - v'_{x,cm} = \left(v_{2x,i} - v_{2x,i}\right)\frac{m_1}{m_1 + m_2} \quad (3.1.16.2)$$

In the CM frame the momentum of the system is zero before the collision and hence the momentum of the system is zero after the collision. For an elastic collision, the only way for both momentum and kinetic energy to be the before and after the collision is either the objects have the same velocity (a miss) or to reverse the direction of the velocities.

In the CM frame, the final x components of the velocities are

$$v'_{1x,f} = v_{1x,i} = \left(v_{2x,i} - v_{1x,i}\right)\frac{m_1}{m_1 + m_2}$$

$$v'_{2x,f} = v_{2x,i} = \left(v_{2x,i} - v_{1x,i}\right)\frac{m_2}{m_1 + m_2}$$

The final x components of the velocities in the "laboratory frame" are then given by

$$\vec{v}_{1x,f} = \vec{v}_{1x,f} + v_{x,cm}$$

$$= \left(\vec{v}_{2x,i} - \vec{v}_{1x,i}\right)\frac{m_2}{m_2 + m_1} + \frac{m_1 i_{1x} + m_2 v_{2x,i}}{m_1 + m_2}$$

$$= \vec{v}_{1x,i}\frac{m_1 - m_2}{m_1 + m_2} + \vec{v}_{2x,t}\frac{2m_2}{m_1 + m_2} \qquad (3.1.16.4)$$

❑ Worked out Example :

Show that equal mass particles in a two dimensional elastic collision emerge at right angles.

In this problem there is no mention of external forces acting on the two objects during the collision. All forces are internal which means momentum is conserved.

$$\vec{p}_r = \vec{p}_t \qquad \text{(Ex-1)}$$

which implies (initially the second mass is at rest)

$$m_1 . \vec{v}_{1,i} = m_1 . v_{1\cdot f} + m_1 . v_{2,f} \qquad \text{(Ex-2)}$$

or, $\vec{v}_{1,i} = \vec{v}_{1,f} + \vec{v}_{2,f}$

(Ex. 3)

We take the dot product of each side of eq. (Ex.3) with itself

$$\vec{v}_{1,i} \cdot \vec{v}_{1,i} + 2 = (\vec{v}_{1,f} + \vec{v}_{2,f}) \cdot (\vec{v}_{2,f} + \vec{v}_{2,f}) \qquad \text{(Ex. 4)}$$

Now we invoke the condition for elastic collision to which kinetic energy is the same before and after the collision. As because the objects have equal masses, we have

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2 \qquad \text{(Ex.5)}$$

Comparing sq. (Ex.4) with eq. (Ex. 5) one can see that

$$\vec{v}_{1,f} . \vec{v}_{2,f} = 0 \qquad \text{(Ex,6)}$$

The dot product of two non-zero vectors is zero when the two vectors are at right angles to each other which means after the collision the particles emerge at right angles to each other.

3.1.17 Substance of the chapter :

Just look at what we have discussed in this chapter.

1. You have learned about the kinematics of motion, like definitions of position, velocity, acceleration.
2. You have learned how to get other kinematic variables if the information about time dependence of one variable is given.
3. You have learned about inertial and non-inertial frames of reference and their importance in the context of study of motion.
4. The concept of force as if appears in Newton's laws of motion has been discussed.
5. Galilean invariance and Galilean transformation were explained.
6. Dynamics of a system of particles has been discussed. The idea of center of mass and its relevance has been discussed. How to locate center of mass of different objects having some symmetries were discussed.
7. You have learned about the motion of a system with variable mass. In this context you have also leaned about the motion of a rocket.
8. The idea of a conservative force field was introduced. How a conservative force can be expressed in terms of a potential has been discussed.
9. Physics behind collisions was discussed along with different types of two body collisions.

3.1.18 Questions (short answer type) :

1. Define force, acceleration, velocity.
2. What do you mean by conservation of linear momentum?
3. What is center of mass ? What is the difference from center of gravity.
4. Under what condition angular momentum of an object remains unaltered?
5. Why three stage rockets are used for lifting a satellite to the orbit?
6. When a group of forces will be in equilibrium?

7. How many kinds of collisions are there ?
8. Both kinetic energy and momentum are conserved in all collisions—True or False?

3.1.19 Questions

1. A particle is travelling in a circular path with uniform speed. How the path and speed of the particle will appear to another particle moving with a uniform speed?
2. Write down Newton's first law of motion. Explain inertial and non-inertial frames of reference giving examples.
3. The bob of a 1m long pendulum is released from the horizontal position while the string being taut. What will be its speed at its lowest position?
4. Show that the motion of center of mass of n-particle system due to internal forces remain unchanged.
5. A boat of mass 75 kg and of length 5m is at rest in still water. If a man walk from the front to the back of the boat. What will be the displacement of the boat? Resistance due to water can be neglected.
6. A rocket with an initial velocity $v_1 = \left(\frac{3}{4}\right)\left(\frac{2GMr}{R_E}\right)\frac{1}{2}$ is projected. M_E and R_E are the mass and radius of the earth respectively. Air resistance and effect of the rotation of the earth can be neglected. Applying conservation of mechanical energy, find out the highest distance of the rocket from the center of the earth for vertical projection.
7. A steel ball of radius 6 cm is static on a horizontal smooth plane. Another steel ball of radius 3 cm moving with a velocity of 450 cm/s collies with the first ball. Assuming perfectly elastic collision determine the velocities of each ball.

3.1.20 Answers for the short questions

1. See text.
2. When the resultant (external) force acting on a particle is zero, the total linear momentum of the particle remains constant in time $F = \frac{dp}{dt} = 0 \Rightarrow p = \text{constant}$.
3. The point in a rigid body or system of discrete particles where a point mass equalling the mass of the body or system of particles can be placed for all motion related matters is known as center of mass. The center of gravity of a body or a system of particles is the point about which the vector sum of the torques due to gravity vanishes.
4. If the total external torque (about a specific point O, say) vanishes, the total vector angular momentum of the system (about the same point O) will remain constant in time.
5. Every stage of a rocket, when separates, gives thrust in the forward direction thus increasing the velocity of the system. Three successive stages provides three successive boosts so that the rocket attains the escape velocity from the gravitational attactive field of the earth to reach its designated orbit.
6. When the resultant of all the forces is equal to zero and the resultant torque produced by the forces is zero, the forces will be in equilibrium.
7. Check the text.
8. Only in perfectly elastic collision both kinetic energy and linear momentum are conserved. So, the statement is false.

3.1.21 Answers for the general questions :

1. Let the particle moves in the $x - y$ plane. If one takes the origin at the center fo the circle then at any point on the path of the particle will have co-ordinates $x = R \cos \omega t$ and $y = R \sin \omega t$, $z = 0$. The velocity components are $dx/dt =$

$- \omega R \sin \omega t$, $dy/dt = \omega R \cos \omega t$, $dz/dt = 0$.

The co-ordinates of the particle moving with a uniform velocity $\vec{v}$ are $x' = v_x t$, $y' = v_y t$, $z' = v_z t$. Therefore, the relative co-ordinates fo the particle moving in a circular path are $x'' = R \cos \omega t - v_x t$, $y'' = R \sin wt - v_y t$, $z'' = -v_z t$. So, the relative path is a circle whose centre is moving with a uniform velocity $-\vec{v}$.

2. See text.

3. When the bob and the taut string are in horizontal position, the height of the bob from equilibrium position is 1m. i.c. the length of the string. So, the potential energy of the bob w.r.t the equilibrium position where potential energy can be taken to the zero. is P. E = mgh = m x 9.8 x 1 = 9.8 m J where m is the mass of the bob.

   At the horizontal position before the release of the bob, its kinetic energy is zero. At the equilibrium position when the string is vertical, its potential energy is zero, the total energy is kinetic. If at that point v be the velocity, then due to conservation of energy principle. $\frac{1}{2} mv^2 = mgh$, or, $v^2 = 2gh$. So. $\vec{v} = (2gh)^{\frac{1}{2}}$

   $= (2 \times 9.8 \times 1)^{\frac{1}{2}} = 4.43$ m/s.

4. It can be shown for a group of n-particles that the equation of motion of the system is $M \frac{d^2\vec{R}}{dt^2} = \vec{F}^{ext}$, where $M\vec{R} = \Sigma m_i F_i$, and $\vec{F}^{ext}$ is the net external force on the system. So, this equation shows that the motion of the center of mass remains unchanged due to internal forecs operating in the system.

5. The external force on the boat is absent here. Therefore the motion of the center of mass will be uniform. In this case, the boat is stationary, so, its speed is zero. Therefore, if the man moves from front side to back side the center of mass will not move.

6. For vertical projection h = 0, now from conservation of energy

$$\frac{1}{2}m_1^2 - \frac{GM_Em}{R_E} = \frac{1}{2}m\,v_A^2 - \frac{GM_Em}{r_A},$$ substituting the value of v_i we get $r_A = \frac{16}{7} R_E$.

7. The mass of the stationary ball $m_1 = C\,6^3$ (as volume is proportional to radius^3). C is a constant.

   The mass of the moving ball = $C.3^3$. Let u_1 and u_2 be the velocities of the balls before collision and v_1, v_2 be the velocities after the collisions, Now. $u_1 = 0$ and $u_2 = 450$ cm/s.

   Applying conservation of momentum for the collision.

   $C3^3 \text{ X } 450 = C\ 3^3 \text{ X } v_1 + C\ 6^3 \text{ X } v_2$. Again for elastic collision $u_1 - u_2 = v_1 - v_2$

   Now $u_1 = 0$, So, after solving these two equations,
   we get $v_t = 100$ cm/s and $v_2 = - 350$ m/s.

Unit–2 ❑ Rotational Dynamies

Structure

3.2.1 Proposal

Angular motion is a nother kind of motion which is a part of our daily life. Description of angular motion needs the help angular kinematics as well as angular kinetics. Similarities between linear motion an angular motion helps in understanding of the plays the similar role of mass in angular motion. The axis of rotation plays a vital role in angular motion. When frame of reference in which the motion is studied in non-inertial, new concepts of pseudoi force come into play. These pseudo forces have geographical implication on the flow frivers.

❑ Outcome

after reading this chapter you shall have a clear understanding about

(i) angular motion in a plane (2D) and angular motion in space. (3D)

(ii) angular kinemtics and angular kinetics.

(iii) conservation of angular momentum

(iv) moment of inertia, theorems of moment of inertia, calculation of moment of inertia of different objects.

(v) equation of motion in an accelerated frame, pseudo force.

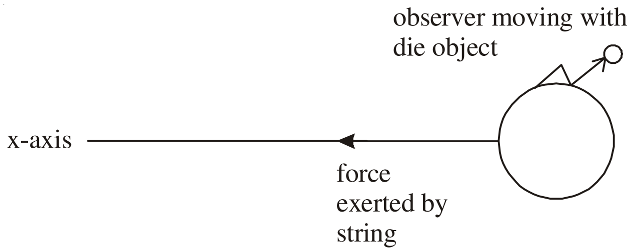
(vii) effect of rotation of the earth on the motion of particles on its surface.

❑ Facts about Angular Motion

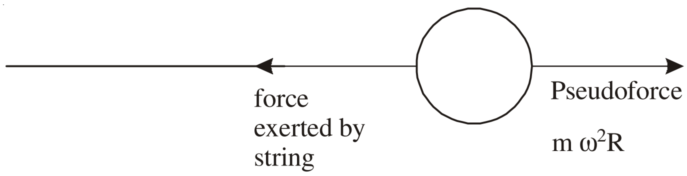
So far we have discussed linear motion of a particle or of a group of particles. We shall now turn to the angular motion of a group of particles. It will be examined in a restricted form that of rigid body rotating about a fixed axis. For such a system the particles follow circular trajectories in a plane perpendicular to the rotational axis. In other words every particle will rotate in the plane, i.e. it executes two dimensional rotational motion. In the absence of any external forces the axis of rotation must pass through the centre of mass of the body.

If we use the origin for all our measurements of internal motion such as rotational motion, located at the centre of mass, we can safely ignore the motion of the centre of mass itself.

Imagine an object being swung around in an approximately horizonatal circular path (See figure below). Now imagine an oberver is riding on the object. The frame of reference for this observer is accelerating. This observer knows that the string exerts a force to the left, but cannot explain why the object does not go off in that direction, although as per the equation of motion is concered the object should move in that direction. To make the equation of motion work the observer introduces a pseudo-force in this case a force to the right in order to counteract the force exerted by the string.



We know that for the object moving in the circle, the force exerted by the string is equal to $m\omega^2R$. Therefore the moving observer's pseudo-force must also be equal to $m\omega^2R$, but it act in the opposite direction outwards. This kind of pseudo-force is called centrifugal force.



So, we see that the equation of motion works properly only in a frame of reference that is not accelerating. However it is often convenient, to use a acclerated reference frame and non-physical pseudo-forces which are invented only in order to preserve the equation of motion. We say that the pseudo-force are non-physical firstly because they violate the law that force always occure in pairs are secondly because is it not possible to identify a physical object which is the source of the force.

There is a very important concept in the case angular motion–the notion of moment of ineria. We are already familar with two kinds of intertia, namely inertia of rest and inertia of motion. We have also learned that acceleration produced on an object by a force depends on the inertial mass of the object. Higher the mass–lower the acceleration and vice versa. In the case of angular motion we shall see that angular acceleration produced by a torque, not a force, will depend on moment of inertia, not on mass only. Equations of linear and angular motIon are very similar: $\vec{F} = M\vec{a}$ and $\vec{T} = I\vec{\alpha}\cdot$

So, we see that for angular motion, moment of inertia I plays a similar role like mass in the case of linear motion. Details will be provided later.

3.2.2 Angular Kinematics

We now develop the kinematic equation for circular motion in order to be able to describe the dynamics of angular motion. The rectilinear co-ordinates (x, y, z) are not useful for this purpose. See figure 3.2.1.

Since the angular motion takes place about a fixed axis the co-ordinates r an θ will be helpful to describe a particle's angular kinematics. As we consider only a rigid body, only the angle θ will very with time. The rate of angular displacements, ω

$$\omega = \frac{d\theta}{dt} \tag{3.2.2.1}$$

and the rate of change of angular velocity, α

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{3.2.2.2.}$$

In order to describe the kinematics of angular motion θ is measured usually in radians. Then ω, the analogue of velocity, has units of radians per second and α, the analogue of acceleration, has units of radians per second squared.

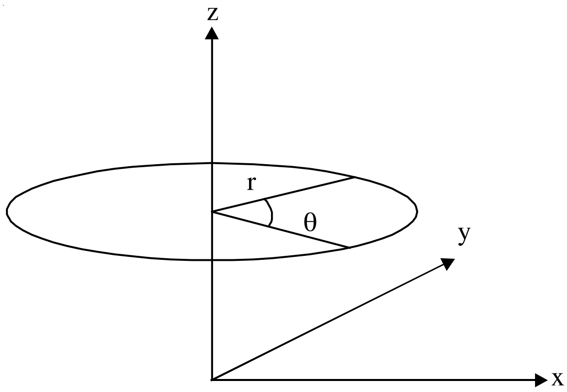


Figure 3.2.1 Co-ordinate system of cylindrical symmetry suitable for description of angular motion

The angular displacement θ vis-a-vis linear displacemment x, angular veocity ω versus linear velocity v and angular acceleration α versus linear acceleration a, helps to write down the kinematic equations for motion under constant angular acceleration.

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$\omega = \omega_0 + \alpha t \qquad v = v_0 + at$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad v^2 = v_0^2 + 2a(x - x_0)$$

(3.2.2.3)

where the subscript zero (0) indicates the value of appropriate quantity measured t t = 0.

θ, ω and α are actually vectors as they must be able to indicate the direction of the angular motion. We put the vectors $\vec{\theta}, \vec{\omega}$ and $\vec{\alpha}$ on the axis of rotation. Now we have to specify in which direction the vectors should point. By convention we use the right hand screw rule determine the direction in which the vectors point.

In the figure 3.2.2 the direction of $\vec{\theta}$ and $\vec{\omega}$ have been determined as follows. Let us assume that the particle is roating in the anti-clockwise direction. If we were to turn a right handed screw in the same sense the tip of it of it would travel in the positive z direction. This is the direction in which the vectors must therefore point by the right hand screw rule. We show $\vec{\alpha}$ pointIng in the opposIte direction, which means that the particle's angular velocity is slowing down.

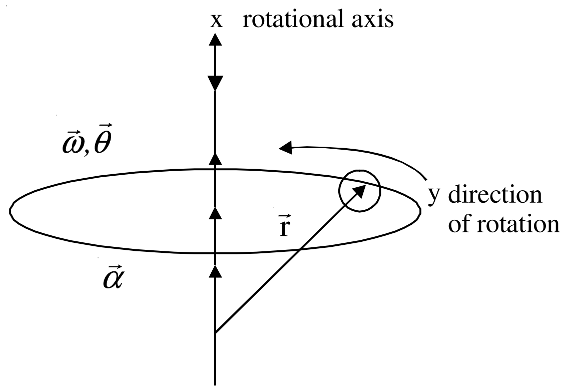


Figure 3.2.2 determintion of the direction of rotion vectors.

In all calculations for angular motion we shall be able to relate the angular velocity and acceleration to the instantaneous linear velocty and accelertion of a particle. For that look at the figure 3.2.3:

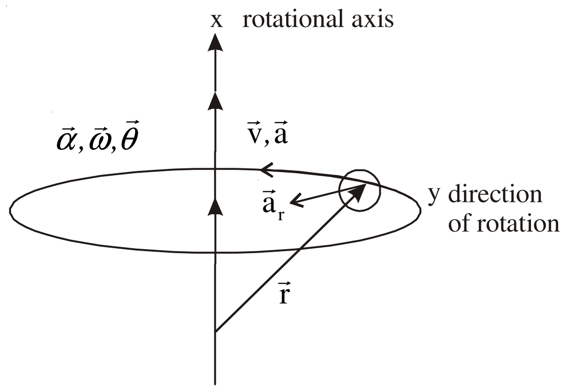


Figure 3.2.3 diagram to relate angular and linear kinematic quantities.

The derivation of the relationship between $\vec{v}, \vec{a}_t, \vec{a}_r$ and $\vec{\omega}$ and $\vec{\alpha}$ is a given in 3.2.2.3, we use the results here and explaIn their meaning. The inatantaneous velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{r} \tag{3.2.2.4}$$

The order of the cross product, ensures that $\vec{v}$ points in the correct direction. Remember that the magnitude of v = ωrsinθ, where θ is the angle between the vectors $\vec{\omega}$ and $\vec{r}$. This means that magnitude of v is simply ω times the prependicular distance from the axis of rotation to the particle. So even though it make sense to put the origin for the position vector at the centre of mass, for the purposes of these calculations, it suffices to ensure that the origin is somewhere along the axis of rotation.

The radial and tangential accelerations are given by

$$\dot{\vec{v}} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a}_t + \vec{a}_r$$

$$\vec{a}_t = \vec{\alpha} \times \vec{r}; \vec{a}_r = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{3.2.2.5}$$

Again we find that the magnitude of the tangential acceleration is equal to α times the perpendicular distance from the rotation axis to the particle and the radial or centripetal (towards the centre) acceleration is equal to ω^2 times the perpendicular distance from the axis of ration to the particle. More importantly, even if the rate of rotation is constant there is still a radial or centripetal acceleration.

3.2.3 Relation Between Angular and Linear Velocity and Acceleration.

In order to examine the dynamics of angular motion of a rigid body we need to determine the instantaneous acceleration and velocity of the constituent atoms. We find the appropriate relations in the following manner.

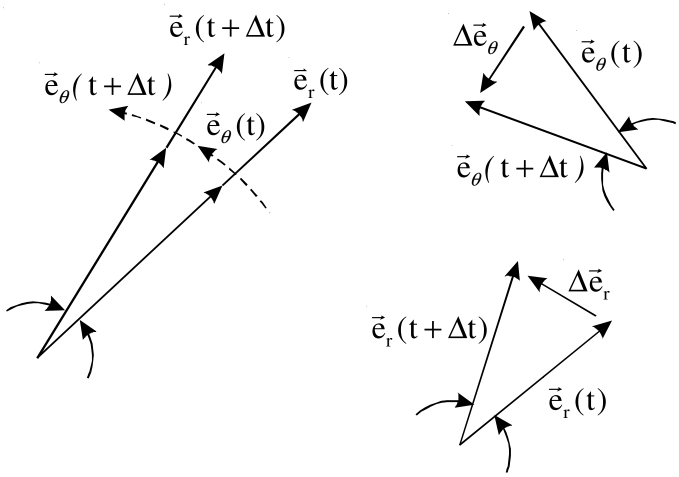


Figure 3.2.4 Radial and transverse components in a rotating system.

Figure 3.2.4 shows a particle moving in a circular path in the x–y plane at a constant radius r rotating about the z axis. We define two units vectors $\vec{e}_r$ and $\vec{e}_\theta$ pointing in the radial and transverse or tangential direction. So the rectilinear position vector $\vec{r}$, is given by

$$\vec{r} = \vec{e}_r r \tag{3.2.3.1}$$

and hence the instantaneous velocity $\vec{v}$ is given by

$$\frac{d\vec{r}}{dt} = \frac{d\vec{e}_r}{dt} r \tag{3.2.3.2}$$

Now looking at our vector diagram we an see that to convert this into an angular velocity $\vec{\omega}$, we can take advantage of the fact that

$$\Delta \vec{e}_r = \Delta \vec{\theta} \tag{3.2.3.3}$$

where we define angles in radians, and the angle is small enough. Now examination of the vector diagram reveals that this change in the radial vector actually points along the transverse direction and hence we get

$$\frac{d\vec{e}_r}{dt} = \vec{e}_\theta \frac{d\theta}{dt} = \vec{e}_\theta \omega \tag{3.2.3.4}$$

and hence

$$\vec{v} = \vec{e}_\theta \ \omega r \tag{3.2.3.5}$$

To find an expression for the acceleration this expression is differentiated

$$\frac{dv}{dt} = \vec{e}_\theta \frac{d\omega}{dt} r + \omega \frac{d\vec{e}_\theta}{dt} r \qquad (3.2.3.6)$$

but $\frac{f\vec{\omega}}{dt}$ is simply the angular acceleration $\vec{\alpha}$ and you can show, in the same way that we previouly did for $d\vec{e}_r/dt$ that

$$\frac{d\vec{e}_\theta}{dt} = -\vec{e}_r \omega \qquad (3.3.3.7)$$

which gives us

$$\vec{a} = \vec{e}_\theta \alpha r - e_r \omega^2 r \qquad (3.2.3.8)$$

The acceleration has a tangential and radial component. The tangential acceleration $\vec{a}_t$ is simply proporation to the angular acceleration. However, even if $\vec{\alpha}$ is zero there is a radial acceleration $\vec{a}_r$, directed towards the axis of rotation (the negative sign). The acceleration is called centripetal acceleration and is present even at constant angular velocity, because we have to apply a force in order to make a particle deviate from straight line motion (Newton's first law). In the case of circular motion at constant angular velocity, the instantaneous rectilinear velocity is changing in direction, but not magnitude.

We have yet to express the angular kinematic terms as vectors, which we will need to do in order to understand forces and momenta in rotating systems, since these are of course vectorial. Let us deal with the instantaeous velocity and angular velocity first.

The magnitude of the velocity is given by

$$v = \omega r \qquad (3.2.3.9)$$

By convention the right hand rule is used, giving us

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad (3.2.3.10)$$

Remember that this makes the magnitude of $\vec{v}$ times the sine of the angle between the vectors ω and r. This means that the magnitude of the velocity is equal to the angular velocity times the radial distance between the axis of rotation and the moving particle. The radial and tangential acceleration then follow.

$$\vec{a}_t = \vec{\alpha} \times \vec{r} \qquad (3.2.3.11)$$

$$\vec{a}_r = \vec{\omega} \times \vec{v} \qquad (3.2.3.12)$$

We will find that there are direct parallels between the equations of linear dynamics an those for rotational dynamics in a rotating frame. Those analogues are summarised below.

❑ Workedout Example

The transverse component of acceleration of a particle is zero but $d\theta/dt$ is not zero. Find out the radial component of acceleration eliminating the variable t.

❑ Solution

$$a_t = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \text{ means } r^2 \frac{d\theta}{dt} = \text{constant} = h \text{ (say)}$$

$$\text{Then } \frac{dr}{dt} = \frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{d\theta},$$

$$\text{So, } \frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{d\theta}\left(\frac{h}{r^2 \frac{dr}{d\theta}}\right)\frac{d\theta}{dt}$$

$$\text{Now, } \frac{d\theta}{dt} = \frac{h}{r^2}, \text{ so, } a_r = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) - \frac{h^2}{r^3}.$$

❑ Linear Dynamics vs. Angular Dynamics

v <--> ω

a <--> α

m <--> I

$\vec{F} = m\vec{a}$ <--> $\left(\frac{d\vec{V}}{dt}\right)_r = I\vec{\alpha}$

$\vec{P} = m\vec{v}$ <--> $\vec{L} = I\vec{\omega}$

$dW_{lin} = \vec{F}.d\underline{\vec{s}}$ <--> $dW_{rot} = \vec{\tau}.d\vec{\theta}$

$K_{lin} = \frac{1}{2}\ mv^2$ <--> $K_{rot} = \frac{1}{2}\ I\omega^2$.

Let us start by looking at the rotational analogue of force, the torque.

3.2.5 torque and the Moment of Inertia

Everyday experience tells us that the action which causes an extended object to move in an angular motion depends on a couple which actually is a pair of oppositely directed parallel forces whose lines of action are separated by a distance. It depends both on the magnitude of either of the applied forces and the distance from the axis of rotation. If you want to push open a door, you apply the force at the edge of the door furthest from the hinge which provides the axis of angular motion. In this way you maximise the turning effectiveness of your push or pull. If you try opening the door by applying a force close to the hinge you will find the effort required is far greater. It is clear that further away you are from the axis of rotation the greater the effect of a given force.

So we would guess that the torque, $\vec{\tau}$, is related to the force, $\vec{F}$, by

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where r is the perpendicular distance from the axis of rotation to the point at which the force was applied.

We note that only the component of force acting tangential, F_t to the direction of rotation is used to cause the rotation. If the force is applied along the door face you will not make it turn at all, whereas if you apply the force at right angles to the door face you maximise the efficiency with which you turn it.

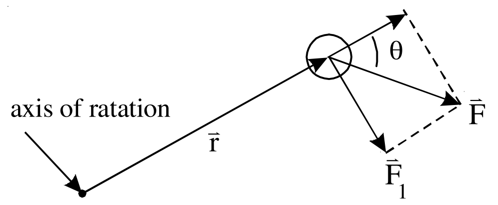


Figure 3.2.5. Turning of the particle about the axis of rotation

This mens that

$$\tau = rF \sin\theta \qquad (3.2.5.2)$$

where θ is the angle between the position vector and the force vector (See Figure 3.2.5).

Then $\vec{\tau} = \vec{r} \times \vec{F}$

and we once again follow the right hand rule. In a rigid body calculation the total torque, $\vec{T}$, is The sum of the torque on each particle.

$$\vec{T}=\sum_{d} r_j \times \vec{F}_j \quad \vec{T}=\sum_{d} r_j \times \vec{F}_j \qquad (3.2.5.4)$$

where the subscript j identifies the jth particle. Clearly the value of the torque will in general depend on the location we choose as the origin of our co-ordinate system. It is best therefore to choose the centre of mass as the origin for our measurements, although it is well to note that in conditions where the net force on the rigid body is zero, i.e. in conditions where the external forces only cause rotational motion, the value of the total torque is equivalent from whatever point the measurements are made.

We shall now derive the equation which would be analoguesa of $\vec{F} = m\vec{a}$. Let us consider the case of a torque applied to a mass, m, at a distance r from the axis of rotation, where $\theta=\frac{\pi}{2}$.

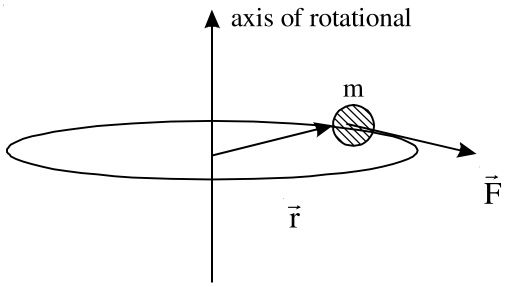


Figure 3.2.6. Moment of inertia of inertia of a particle.

In this case the magnitude of the torque is

$$\tau = rF \qquad (3.2.5.5)$$

but the applied foce, F, i equal to the particle's mass times the tangential acceleration (since r is constant). Therefore

$$\tau = r\, ma_t \quad (3.2.5.6)$$

but $a_t = \alpha r$ and therefore

$$\tau = mr^2\alpha \quad (3.2.5.7)$$

Now mr^2 is called the moment of inertia, I, which is the rotational analogue of the inertial mass in linear motion. In other words the larger the moment of inertia the harder it is to get the object to change its angular velocity. In full vectorial form we have

$$\vec{\tau} = I\vec{\alpha} \quad (3.2.5.8)$$

and if you check you can see that this expression is consistent with the direction of vector cross product $\vec{r} \times \vec{F}$.

In all the cases of interest to us there will be more than one mass to consider in calculations of the effect of a torque applied to a molecule, so we must generalise the moment of inertia for a system of connected masses (atoms)

$$\sum_j m_j r_j^2 \quad (3.2.5.9)$$

where r_j is the perpendicular distance of the jth particle to the axis of rotation.

3.2.6 Energy due to Rotation

3.2.6.1 Rotational Work, Potential and Kinetic Energy

Finally we will look at the energetics involved in rotating a multi-atom molecule. Continuing our analogies to liner dynamics, the work done in rotating any rigid body through an angle dθ by the action of an external torque will be given by

$$dW_{rot} = \vec{\tau}.d\vec{\theta} \quad (3.2.6.1)$$

which is directly analogous to the definition of work for a linear displacement, with the force being replaced by torque and linear is placement being replced by angular displacement.

By extension we may then also deduce that th potential energy assosiacted with rotating the rigid body is given by

$$dU_{rot} = -\vec{\tau}.d\vec{\theta} \qquad (3.2.6.2)$$

The kinetic energy of a particle of mass m moving in a straight line is 1/2 mv^2. If the particle moves in a circle we know that v = ωr and hence we would expect the rotational kinetic energy to be

$$K.E._{rot} \frac{1}{2}m\omega^2 r^2 = \frac{1}{2} = I\omega^2 \qquad (3.2.6.3)$$

and once again the moment of inertia replaces the inertial mass and ω replces v. For a system with n number of partilces, the above equation can be generalised to

$$K.E._{rot} = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 = \frac{1}{2}I\omega^2, \text{ where now } I = \sum_i m_i r_i^2 \qquad (3.2.6.4)$$

Here one can see that in the expression for kinetic energy for rotational motion, the moment of inertia plays the role of mass in case of linear motion and angular velocity plays the role of linear velocity.

We now have the means to solve dynamical problems in rotation using conservation of energy and the conservation of angular momentum.

Let us take the case of a rigid body, one in which the distances $\left|\vec{r}_i - \vec{r}_j\right|$ between points are held fixed. A general rigid body will have six degrees of freedom (but not always, see below). In order to spefify the position of all points in the body with only six parameters, let us first fix some point r_1 of the body, which is to be treated as its "centre" or origin from which all other points in the body can be referenced from ($\vec{r}_1$ can be, but not necessarily, the centre of mass). Once the coordinates of $\vec{r}_1$ are speified (in relation to some origin of a co-ordinate system outside of the body), we have used up three degrees of freedom. With $\vec{r}_1$ fixed, the position of any other $\vec{r}_2$

can be specified using only two coordinates since it is constrained to move on the surface of sphere centered on $\vec{r}_1$. We have now used up five degrees of freedom. If we now consider any other third point $\vec{r}_3$ not located on the axis joining $\vec{r}_1$ and $\vec{r}_2$, its position can be specified using and degree of freedom (or co-ordinate) for it can only rotate about the axix connecting $\vec{r}_1$ and $\vec{r}_2$. We thus have used up the six degrees of freedom. It is interesting to note that in the case of a linear rod, any point $\vec{r}_3$ must lay on the axis joining $\vec{r}_1$ and $\vec{r}_2$; hence a linear rod has only five degrees of freedom. Usually, the six degrees of freedom are divided in two groups : three degrees for translation (to specify the position of the "center" $\vec{r}_1$ an three rotational agles to specify the orientation of the rigid body (normally taken to be the so-called Euler angles)

3.2.7 The Ineria Tensor

Let us consider a rigid body containing n particle of mass m_i, i = 1, n. If the body rotates with an angular velocity ω about some point fixed with respect to the body coordinates (this "body" coordinate system is what we used to refer to as "non-inertial" or "rotating" coordinate system and if this point moves linearly with a velocIty V with respect to a fixed (i.e. inertial) coordinate system, then the velocity of the particle is given by equation

$$\vec{v} = \vec{V} + \vec{\omega} \times \vec{r}_i \qquad (3.2.7.1)$$

The total kinetic energy of the body is given by

$$T = \sum_i T_i = \frac{1}{2}\sum_i m_i v_i^2$$

$$= \frac{1}{2}\sum_i m_i \left(\vec{V} + \vec{\omega} \times \vec{r}_i\right)^2$$

$$= \frac{1}{2}\sum_i m_i V^2 + \sum_i m_i \vec{V}.(\vec{\omega} x \vec{r}_i) + \frac{1}{2}\sum_i m_i (\vec{\omega} x \vec{r}_i)^2 \qquad (3.2.7.2)$$

Although this equation for the total kinetic energy is perfectly general, considerable simplification will result if we choose the origin of the body coordinte system to coincide with the center of mass. With this choice, the second term on the right hand side of the last of equations (3.2.7.2) can be seen to vanish from

$$\sum_i m_i \vec{V}.(\vec{\omega} x \vec{r}_i) = \vec{V}.\left[\vec{\omega} x \left(\sum_i m_i \vec{r}_i\right)\right] = 0 \qquad (3.2.7.3)$$

since the centre of mass $\vec{R}$ of the body, of mass M, is defined such that

$$\sum_i m_i \vec{r}_i = 0 \qquad (3.2.7.4)$$

The total kinetic energy can then be broken into two components: one for the translational kinetic energy and another for the rotational kinetic energy. That is,

$$T = T_{trans} + T_{rot} \qquad (3.2.7.5)$$

With

$$T_{trans} = \frac{1}{2}\sum_i m_i V^2 = \frac{1}{2} M V^2$$

and as seen earlier,

$$T_{rot} = \frac{1}{2}\sum_i m_i (\vec{\omega} \times \vec{r}_i)^2$$

The expression for T_{rot} can be further modified, but to do so we will now take resort to tensor (or index) notation. So, let's consider the following vector equation

$$(\vec{\omega} \times \vec{r}_i)^2 = (\vec{\omega} \times \vec{r}_i).(\vec{\omega} \times \vec{r}_i) \qquad (3.2.7.7)$$

and rewrite it using the Levi-Civita and the Kroncker tensors

$$\left(\varepsilon_{ijk}\omega_j x_{\alpha,k}\right)\left(\varepsilon_{imn}\omega_m x_{\alpha,n}\right) = \varepsilon_{ijk}\varepsilon_{imn}\omega_j x_{\alpha,k}\omega_m x_{\alpha,n}$$

$$= \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)\omega_j x_{\alpha,k}\omega_m x_{\alpha,n}$$

$$= \omega_j\omega_j x_{\alpha,k} x_{d,k} - \omega_j x_{\alpha,j}\omega_k x_{\alpha,k} \qquad (3.2.7.8)$$

Inserting this result in the equation for T_{rot} in equation (3.2.7.6) we get

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha \left(|\vec{\omega}|^2 |\vec{r}_\alpha|^2 - (\vec{\omega}.\vec{r}_\alpha)^2\right) \qquad (3.2.7.9)$$

Alternatively, keeping with the tensor notation we have

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha\left[\omega_j\omega_j x_{\alpha,k} x_{\alpha,k} - \omega_i x_{\alpha,i}\omega_j x_{\alpha,j}\right]$$

$$= \frac{1}{2}\sum_{\alpha} m_\alpha\left[\left(\omega_i\omega_j\delta_{ij}\right) x_{\alpha,k} - \omega_i x_{\alpha,i}\omega_j x_{\alpha,j}\right]$$

$$= \frac{1}{2}\left(\omega_i\omega_j\right)\sum_{\alpha} m_\alpha\left[\delta_{ij} x_{a,i} x_{\alpha,k} - x_{\alpha,i} x_{\alpha,j}\right]$$

(3.2.7.10)

We now define the components I_{ij} of the so-called inertia tensor {I} by

$$\boxed{I_{ij} = \sum_{\alpha} m_\alpha\left[\delta_{ij} x_{\alpha,k} x_{a,k} - x_{\alpha,i} x_{\alpha,j}\right]}$$

(3.2.7.11)

and the rotational kinetic energy becomes

$$\boxed{T_{rot} = \frac{1}{2} I_{ij}\vec{\omega}_i\vec{\omega}_j} \qquad (3.2.7.12)$$

or in vector notation $$\boxed{T_{rot} = \frac{1}{2}\vec{\omega}.\{I\}.\vec{\omega}} \qquad (3.2.7.13)$$

For our purposes it will be sufficient to treat the intertia tensor as a regular 3 × 3 matrix.

Indeed, we can explicitly write {I} using equation (3.2.7.11) as

$$\{I\} = \left\{ \begin{array}{ccc} \sum_\alpha m_\alpha\left(x_{\alpha.2}{}^2 + x_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,1}x_{\alpha,2} & -\sum_\alpha m_\alpha x_{\alpha,3}x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2}x_{\alpha,1} & \sum_\alpha m_\alpha\left(x_{\alpha.1}{}^2 + x_{\alpha,3}{}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,2}x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3}x_{\alpha,1} & -\sum_\alpha m_\alpha x_{\alpha,3}x_{\alpha,2} & \sum_\alpha m_\alpha\left(x_{\alpha.1}{}^2 + x_{\alpha,2}{}^2\right) \end{array} \right\}$$

It is easy to see from equaton (3.2.7.14) that the inertia tensor is symmetric, that is,

$$I_{ij} = I_{ji} \qquad (3.2.7.15)$$

The diagonal elements I_{11}, I_{22}, I_{33} are called he Principal Moments of Inertia about the x_1–, x_2– and x_3– axes, respectively. The negatives of the off-diagonal elements are the Products of Inertia. Finaly, in most cases the rigid body is continuous and not made up of discrete particles as was assumed so far, but the results are results are easily generalized by replacing the summation by a corresponding integral in the expression for the components of the inertia tension.

$$I_{ij} = \int_V \rho(\vec{r})\left(\delta_{ij}x_k x_k - x_i x_j\right)dx_1 dx_2 dx_3 \qquad (3.2.7.16)$$

where $\rho(\vec{r})$ is the nass density at the position $\vec{r}$, and the integral is to be performed over the whole volume V of the rigid body.

❑ Worked out Example

Calculate the inertia tensor for a homogeneous cube of density ρ mass M, and side length b. Let one corner be at the origin, and three adjacent edges lie along the coordinate axes (see Fig. 3.2.1).

Solution.

We use equation (3.2.7.16) to calculate the components of the inertia tensor. Because the symmetry of the problem. It is easy to see that the three moments of inertial I_{11}, I_{22}, and I_{33} are equal and that same holds for all of the products of inertia. So,

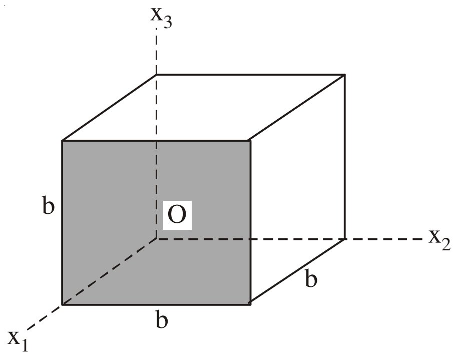


Fig. 3.2.1 Homogeneous cube

$$I_{11} = \int_0^b \int_0^b \int_0^b \rho\left(x_2^2 + x_3^2\right) dx_1 dx_2 dx_3$$

$$= \rho\int_0^b dx_3 \int_0^b dx_2 \left(x_2^2 + x_3^2\right) \int_0^b dx_1$$

$$= \rho b\int_0^b dx_3 \left(\frac{b^3}{3} + bx_3^2\right) = \rho b\left(\frac{b^4}{3} + \frac{b^4}{3}\right)$$

$$= \frac{2}{3}\rho b^5 = \frac{2}{3}Mb^2. \qquad (3.2.ex,1)$$

it should be noted that in this example the origin of the coordinate system is not located at the centre of mass of the cube.

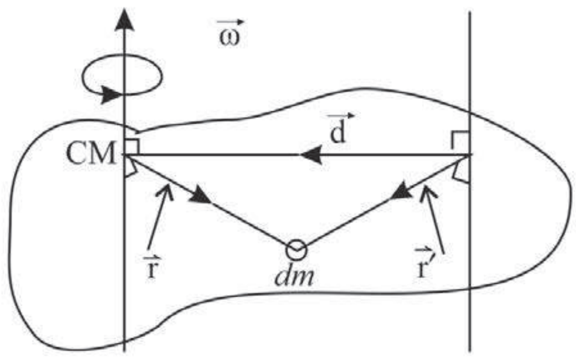
The products of inertia of the cube (–negative) are

$$I_{12} = -\int_0^b\int_0^b\int_0^b \rho x_1 x_2 dx_1 dx_2 dx_3$$

$$= -\rho\left(\frac{b^2}{2}\right)\left(\frac{b^2}{2}\right)(b)$$

$$= -\frac{1}{4}\rho b^5 = -\frac{1}{4}Mb^2.$$

3.2.8 Parallel Axis Theorem



Statement: The moment of inertia (I) of a body about any axis is the sum of its moment of inertia (I_{cm}) about a parallel axis through the centre of mass and the product of the mass (M) of the body by the square of the distance (d^2) between the two axes.

Proof: Let I_{cm} be the moment of inertia of a body of mass M about an axis passing through its centre of mass. Let I be the moment of inertia of the same body about an axis parallel to the previous one and at a distance d from it. The moment of inertia of the body about the axis passing through centre of mass is given by

$I_{CM} = \int r^2 dm$

But according to the problem $\vec{r}' = \vec{r} + \vec{d}$. So, we get $r'^2 = d^2 + 2\vec{d}\cdot\vec{r} + r^2$

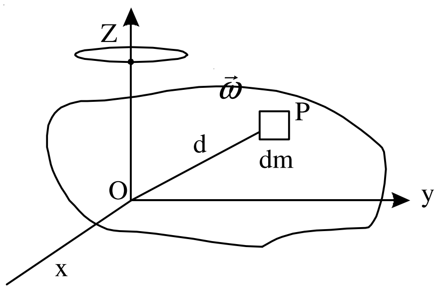
Now, the moment of inertia of the body the parallel axis is

$1 = \int r'^2 dm = \int d^2 dm + 2\vec{d}.\ \int \vec{r}\, dm + \int r^2 dm = d^2M + \int r^2 dm$ as the second term is zero because of the definition of the center of mass.

Hence, $I = I_{cm} + Md^2$. If Now d = 0, then we get $I = I_{cm}$.

3.2.9 Perpendicular axes theorem

This therorem is applicable to planar objects, Let us consider a rigid object that lies entirely within a plane (X – Y). The perpendicular axes theorem links I_z (moment of inertia about an axis perpendIcular to the plane) with I_x, I_y (moment of inertia about two perpendicular axes lying within the plane). Now look at the figure: the body lies in the x–y plane on which at O, the origin, three mutually perpendicular axes meet.



The moment of inertia of the body about the z-axis is given by $I_z = \int d^2\, dm$. or

$$I_z = \int \left(x^2 + y^2\right) dm = \int x^2 dm + \int y^2 dm = I_x + I_y$$

which proves the theorem.

3.2.10 Radius of gyration

The moment of inertia of an object can be equated to the product of the mass M of the object and square of some appropriate length K, i.e. $I = MK^2$. The quantity K is called the radius of gyration of the given object about the given axis.

3.2.11 Moment of Inertia of objects of different shapes :

(a) A unifom thin rod about an axis passing though the center of mass

We shall take the case of a uniform (density and shape) thin rod of mass M and length L such that one can assume the cross-section area of the rod is small and the rod can be thought of a string of masses along a one-dimensional straight line. The axis of rotation is perpendicular to the rod and passes through the center of mass, i.e., the midpoint of the rod. Our task is to calculate the moment of inertia about this axis. We take the z-axis as the axis of rotation and the x-axis passes through the length of the rod, as shown in the figure. This is to facilitate integration along the x-axis.

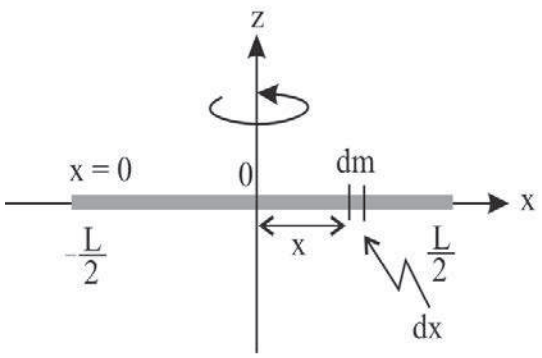


Fig. Calculation of the moment of inertia I for a uniform thin rod about an axis through the center of the rod.

We define dm to be a small element of mass making up the rod. The moment of inertia is an integral over the mass distribution. We need to find a way to relate mass to spatial variables. We do this using the linear mass density λ of the object, which is the mass per unit length. Since the mass density of thi object is uniform, we can write.

$$\lambda = \frac{M}{L} \qquad (3.211.1)$$

Note that a piece of the rod dl lies completely along the x-axis and has a length dx; in fact, dl = dx in this situation. We can therefore write dm = λ dx. The distance of each piece of mass dm from the axis is given by the variable x, as shown in the figure. Putting this all together, we obtain $I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx.$ (3.2.11.2)

In the last step we have to be careful about our limits of integration. The rod extends from $x = -\frac{L}{2}$ to $+\frac{L}{2}$, since the axis is in the midle of the rod at x = 0. So, after integreation we get

$$I = \frac{1}{12} ML^2 \tag{3.2.11.3}$$

where M is the mass of the rod and L its length.

(b) A uniforem Thin Rod with Axis at the End

Now consider the same uniform thin rod of mass M an length L, but this time we move the axis of rotation to the end of the rod. We wish to find the moment of inertia about this axis (Figure).

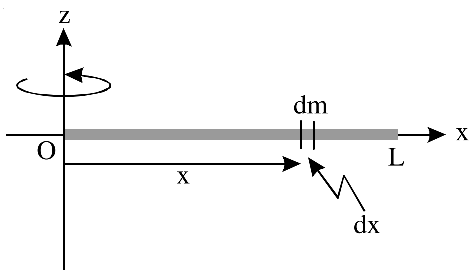


Figure moment of inertia I for a uniform thin rod about an axis through the end of the rod.The quantity dm is again defined to be a small element of mass making up the rod. Just as before, we obtain

$$I = \int r^2 dm = x^2 dm = \int x^2 \lambda dx \tag{3.2.11.4}$$

However, this time we have different limits of integration. The rod extends from x = 0 to x = L, since the axis is at the end of the rod at x = 0. Therefore we find.

$$I = \frac{1}{3}ML^2 \qquad (3.2.11.5)$$

Note the rotational inertia of the about its endpoint is larger than the rotational inertia about it centre (consistent with the barbell example) by a factor of four.

(c) A Uniform Thin Disk about an Axis through the center

In this problem we want to find out the moment of inertia of a two-dimensional object-a uniform thin disk about an axis through its centre (Figure)

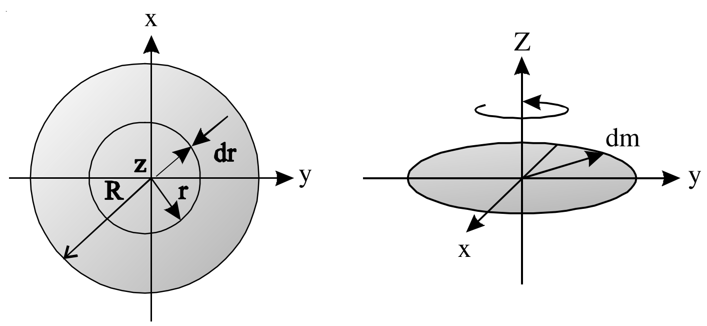


Figure Calculating the moment of inertia for a thin disk about an axis through its center.

Since the disk is thin, we can take the mass to be distributed entirely in the xy-plane. We start with the surface mass density, which is the mass per unit surface area. Since it is uniform, the surface mass density σ is constant.

$$\sigma = \frac{M}{A}, \text{ where M is the mass of the disk and A is its area.}$$

The area can be throught of as made up of a series of thin rings of increasing radii, where each ring is a mass increment dm of radius r equidistant from the axis, as shown in part (b) of the figure. The infinitesimal area of each ring dA is therefore given by the length of each ring ($2\pi r$) times the infinitesimal width of each ring dr:

$$dA = 2\pi r\ dr. \qquad (3.2.11.6)$$

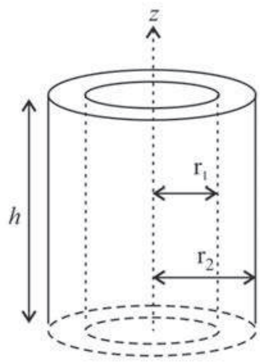
So, the mass of the elementary ring is

$$dm = \sigma\ 2\pi r\ dr = 2\pi M r\ \frac{dr}{A}. \qquad (3.2.11.7)$$

Therefore, the moment of inertia of the whole disk will be

$$I = \int_0^R dm r^2 = \int_0^R 2\pi \frac{M}{A} r^3 dr = \frac{1}{2} MR^2 \qquad (3.2.11.8)$$

(d) A hollow cylinder having mass M, an inner radius r_1, outer radius r_2 and length L, about its central axis.



Let us consider a situation where the cylinder is cut into infinitesimally thin rings centered at the middle. The thickness of each ring is dr, with length L. The moment of inertia of the elementary ring

$$dI = r^2\ dm \qquad (3.2.11.9)$$

Now, we have to find dm, (which is just density multiplied by the volume occupied by one ring

$$dm = \rho\ dV \qquad (3.2.11.10)$$

We have introduced dV in the above equation, so, we have to find out what dV is:

$$dV = dA\ L \qquad (3.2.11.11)$$

Here dA is the area of the top of the ring, which is the area of a rectangular strip of length 2πr and width dr. We have:

$$dA = 2\pi r dr \qquad (3.2.11.12)$$

Substituting dA into dV,

$$dV = 2\pi r L dr \qquad (3.2.11.13)$$

Finally, we have an expression for dm. We substitute that into the dl equation,

$$dI = 2\rho\pi r^3\ Ldr \qquad (3.2.11.15)$$

to get the final form of moment of inertia

$$I = \int_{r_1}^{r_2} 2\pi\rho L r^3 dr \qquad (3.2.11.16)$$

Now, we can find the expression for density.

$$\rho = M\ /\ \{\pi\ (r_2^2 - r_1^2)L\} \qquad (3.2.11.17)$$

Substituting this back into the integrated solution, we have:

$$I = \frac{1}{2}M\left(r_2^2 + r_1^2\right) \qquad (3.2.11.18)$$

Special Cases:

Hoop or thin cylindrical shell: ($r_1 = r_2 = r$)

$$I = Mr^2 \qquad (3.2.11.19)$$

Disk or solid cylinder: ($r_1 = 0$, $r_2 = r$)

$$I = \frac{1}{2} \qquad (3.2.11.20)$$

(e) A rectangular plate about an axis passing through its center and perpendicular to its plane.

The moment of inertia for the rectangular plate of sides ‘a’ and ‘b’ can be found by using the expression for momen of inertia of a rod about an axis passing through its center of mass and perpendicular to its length and the parallel axis theorem. The moment o inertia of a rod of mass M and length L, with axis separated dy distance x from the original one (through the center of mass), is $I_x = I_{CM} + Mx^2 = \frac{1}{2}\ ML^2 + Mx^2$. (3.2.11.21)

Now we replace M → a, M → dm = σ a dx, where σ is the surface mass density.

Integrating over x from – b/2 to b/2, one obtains

$$I = \int_{-b/2}^{b/2} \left(\frac{1}{12} a^3 \sigma + a x^2 \sigma \right) dx = \frac{1}{12} \sigma \left(a^3 b + a b^3 \right) = \frac{M}{12} \left(a^2 + b^2 \right) \qquad (3.2.11.22)$$

where M = σ ab has been used.

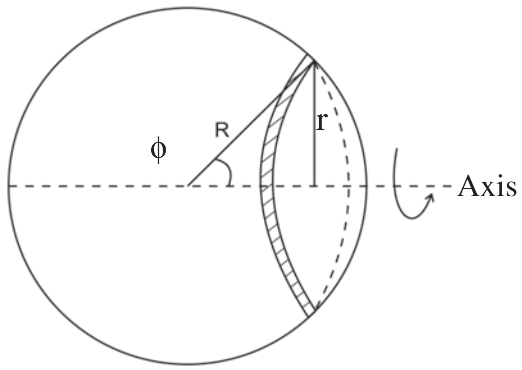


Fig. Thin spherical shell with rotation axis

(f) Thins pherical shell (Please speciy the axis and give fig.)

Let us consider a thin spherical shell of radius R and mass M. We take spherical coordinates with azimuthal angle θ and zenith angle ϕ. On the spherical shell the mass element is

$$dm = \sigma R \sin \frac{M}{4\pi R^2}, \qquad (3.2.11.23)$$

where $\sigma = M/4\pi R^2$ is the surface mass density as we are considering a shell, and the distance from the rotationl axis is r = R sin ϕ. Hence the moment of inertia to be calculated is

$$I = \int r^2 dm = 2\pi\sigma R^4 \int_0^{\pi} \sin^3 \phi \, d\phi \qquad (3.2.11.24)$$

Noting that

$$\int_0^{\pi} \sin^3 \phi \, d\phi = \int_0^{\pi} \sin\phi \left(1 - \cos 2\phi \right) d\phi$$

$$= \int_0^{\pi} \sin\phi d\phi - \int_{-1}^{1} u^2 du$$

$$= [-\cos\phi]_0^{\pi} - \left[\frac{1}{2}u^3\right]_{-1}^{1}$$

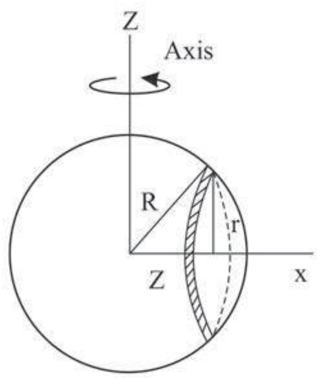
$$= \frac{4}{3} \qquad (3.2.11.25)$$

(the variable has been changed as u = cos ϕ and du = d cos ϕ = – sinϕ dϕ), we now find

$$I = \frac{2}{3}MR^2. \qquad (3.2.11.26)$$

Solid sphere

The moment of inertia for a solid sphere of radius R and mass M can be obtained by integrating the result for the disk $(I_{CM} = \frac{1}{2}MR^2)$ over changing distance from the axis. Choosing the z-axis as the axis of rotation and letting the distance from it to the mass element on the shell as r, we have



$$r^2 = R^2 - z^2. \qquad (3.2.11.27)$$

Now $M \to dm = \to \pi r^2 \rho dz$ and

$R^2 \to r^2$,

we have

$$I = \int_{-R}^{R} \frac{1}{2}\pi r^2 \rho . r^2 dz$$

$$= \frac{1}{2}\pi\rho\int_{-R}^{R}\left(R^4 - 2R^2z^2 + z^4\right)dz$$

$$= \frac{2}{5}MR^2. \qquad (3.2.11.28)$$

where the mass of the sphere is

$$M = \frac{4}{3}\pi R^3 \rho. \qquad 3.2.11.29)$$

❑ Worked out Example:

Find the distance travelled by the axis of a solid right circular of radius r and mass m after it has rolled from rest without slipping for time t on a plane inclined at an angle θ with the horizontal.

Solution

Let $\vec{F}$ = frictional force and $\vec{N}$ = normal reaction. The equation of motion is mg sinθ – F = ma and F r = I α (about the c.m. of the circulr face of the cylinder), where θ is th angle of the plane or the angle between the vertical and the normal reaction, r is the radius of the cylinder.

Now a = αr, so, $F = \dfrac{mk^2}{\frac{r.a}{r}} = ma\dfrac{k^2}{r^2} = ma\dfrac{k^2}{r^2}$, k is the radius of gyration of the cylinder about its axis.

From the equation for normal reaction, mg sin θ – $ma\dfrac{k^2}{r^2}$, = ma

or, mg sinθ = ma $\left(1+\dfrac{k^2}{r^2}\right)$. therefore, one gets a = g sin θ $\left(1+\dfrac{k^2}{r^2}\right)$.

For a right circular cylinder $I = mk^2 = \frac{1}{2}mr^2$. So, $\frac{k^2}{r^2} = \frac{1}{2}$, Therefore, a = $\frac{2}{3}$ g sinθ.

Hence in time t the cylinder has travelled a distance s given by

$$s = \frac{1}{2}at^2 = \frac{1}{3}g\sin\theta\, t^2.$$

3.2.12 Euler Angles and Euler Equations

In this section, we set to determine the set of angles that can be use to specify the rotation of a rigid body. We know that the transformation from one coordinate system to another can be represented by a matrix equation such as

$$\vec{x} = \lambda\vec{x}' \qquad (3.2.12.10)$$

If we identify the inertial (or fixed) system with x′ and the rigid body coordinate system with x then, the rotation matrix λ describes the relative orientation of the body in relation to the fixed system. Since there are three rotational degrees of freedom, λ is actually a product from three individual rotation matrices; one for each independent angles. Although there are many possible choices for the selection of these angles, we will use the so-called Euler angles φ, θ and ψ.

The Euler angles are generated in the following series of rotation that takes the fixed $\vec{x}'$ system to the rigid body $\vec{x}$ system (see figure)

1. First of the rotations is taken counter clockwise through an angle φ about the x_3 – axis. It transforms the inertial system into an intermediate set of x_1''–axes. The transformation matrix is.

$$\lambda_\phi = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (3.2.12.2)$$

with $0 \le \phi \le 2\pi$, and $x'' = \lambda_\phi\, x'$ (3.2.12.3)

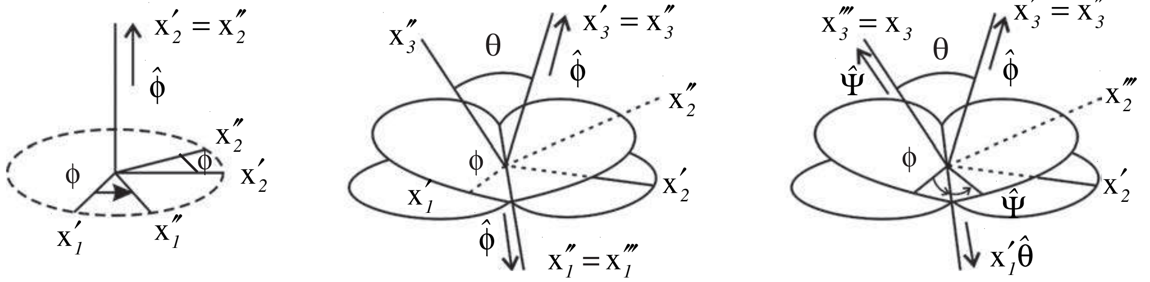


Figure 3.2.12.1 The Euler angles are use to rotate the fixed x_1' system to the rigid body $\vec{x}$ system. (a) The first rotation is counter clockwise through an angle ϕ about the x_3' –axis. (b) The second rotation is counter clockwise through an angle θ about the x_1'' –axis. (c) The third rotation is counter clockwise through an angle ψ about x_3''' –axis.

2. The second rotation is counter clockwise through an angle θ about the x_1'' –axis (also called the line of nodes). It transforms the inertial system into an intermediate of x_1''' –axes. The transformation mtrix is

$$\lambda_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{3.2.12.4}$$

with $0 \le \theta \le \pi$ and $x''' = \lambda_\theta x''$ (3.2.12.5)

3. The third rotation is counter clockwise through an angle ψ about the x_3''' –axis. It transforms the inertial system into the final set rigid body x_1–axes. The transformation matrix is

$$\lambda_\Psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.12.6}$$

with $0 \le \psi \le 2\pi$ and $x = \lambda_\psi x'''$ (3.2.12.7)

One can combine the three rotations using equations (3.2.12.3), (3.2.12.5) and (3.2.12.7) to get finally the complete transformation geven by

$$x = \lambda_\psi \lambda_\theta \lambda_\phi x' \qquad (3.2.12.8)$$

and the rotation matrix for the complete transformation is

$$\lambda = \lambda_\psi \lambda_\theta \lambda_\phi. \qquad (3.2.12.9)$$

3.2.13 Euler's Equations

the equation of circular motion of a rigid body is given by

$$\left(\frac{d\vec{L}}{dt}\right)_f = \vec{N} \qquad (3.2.13.1)$$

where $\vec{L}$ is the angular momentum and $\vec{N}$ is the net torque operting on the body. From the study on non-inertial frame of reference we know that

$$\left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} \qquad (3.2.13.2)$$

We can rewrite this equation using tensor notation as

$$\frac{d\vec{L}_i}{dt} + \varepsilon_{ijk}\, \omega_j L_k = N_i \qquad (3.2.13.3)$$

Now these equations can be modified further if we choose the co-ordinate axes for the body frame of reference to coincide with the principal axes of the rigid body, then

$$L_1 = I_1\omega_1,\ L_2 = I_2\omega_2,\ L_3 = I_3\omega_3. \qquad (3.2.13.4)$$

But we know that the principal moments of inertia are constant in time, so by combining (3.2.13.4) we get

$$I_1 \frac{d\omega_1}{dt}(I_2 - I_3)\omega_2\omega_3 = N_1$$

$$I_2 \frac{d\omega_2}{dt}(I_3 - I_1)\omega_3\omega_1 = N_2 \qquad (3.2.13.5)$$

$$I_3 \frac{d\omega_3}{dt} - (I_1 - I_2)\,\omega_1\omega_2 = N_3$$

Above three equations can be combined together in to one equation as

$$\left(I_i - I_j\right)\omega_i\omega_j - \sum_k \in_{ijk}\left(I_k \frac{d\omega_k}{dt} - N_k\right) = 0 \qquad (3.2.13.6)$$

These equations are known as Euler's equations of motion for a rigid body.

3.2.14 Motion in a Non-Inertial Frame

3.2.14.1 Time derivatives in fixed and rotatIng frames:

The time derivative of an arbitrary rotating-frame vector $\vec{A}$ in a fixed frame is expressed as shown earlier

$$v_f = \left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega}\times\vec{A} \qquad (3.2.14.1)$$

where the time derivative as observed in the fixed (f) frame is expressed as $\left(\frac{d}{dt}\right)^f$, and the time deivative as observed in the rotating (r) frame is expressed as $\left(\frac{d}{dt}\right)_r$. An importnt appliction of the above formula can be found when $\vec{A}$ is replaced by the angular velocity ω. It can easily be seen that $(d\omega/dt)^f = (d\omega/dt)_r$ as the second term in eq (3.2.14.1) vanishes when $\vec{A} = \vec{\omega}$.

3.2.14.2 Accelertion in rotating frames:

We shall now discuss the rotational motion of a particle considering the general case of a rotating frame associated with the particle and a fixed frame being related by translation and rotation.

Let is consider the present position of the particle be P. According to the fixed frame of reference the position vector is $\vec{r}'$ (see fig.) while the position vector of the same point P with reference to the rotating frame of reference is $\vec{r}$, while $\vec{r}'$ and $\vec{r}$ are related by

$$\vec{r}' = \vec{R} + \vec{r} \qquad (3.2.14.2)$$

where $\vec{R}$ is the position vector of the origin of the rotating frame corresponding to the fixed frame.

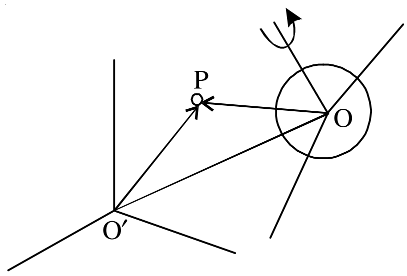


Figure 3.2.14.1 Vectorial relation between fixed frame and rotating frame of reference

The velocities of point P as observed in the fixed and rotating frames are expressed as

$$v_f \left(\frac{d\vec{r}'}{dt}\right)_f \text{ and } \vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_r \qquad (3.2.14.3)$$

respectively.

$$\text{But,} \quad \vec{v}_f = \left(\frac{d\vec{R}}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}, \qquad (3.2.14.4)$$

where $\vec{V} = \left(\frac{dR}{dt}\right)_r$ denotes the translation velocity of the rotating-frame origin (as observed in the fixed frame).

Using Eq. (3.2.14.4), we are now in position to evalute expression for the acceleration of point P as oberved in the fixed and rotating frames of reference

$$\vec{a}_f = \left(\frac{d\vec{v}_f}{dt}\right)_f \text{ and } \vec{a}_r = \left(\frac{d\vec{v}_r}{dt}\right)_r$$

respectively. (3.2.14.5)

Hence using Eq. (3.2.14.4) we get

$$\vec{a}_f \left(\frac{d\vec{V}}{dt}\right)_f + \left(\frac{d\vec{v}_r}{dt}\right)_f + \left(\frac{d\vec{\omega}}{dt}\right)_f \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_f$$

$$= \vec{A} + (\vec{a}_r + \vec{\omega} \times \vec{v}_r) + \left(\frac{d\vec{\omega}}{dt}\right)_f \times \vec{r} + \vec{\omega} \times (\vec{v}_r + \vec{\omega} \times \vec{r}) \qquad (3.2.14.6)$$

or, $$\vec{a}_f + \vec{A} \times \vec{a}_r + 2\vec{\omega} \times \vec{v}_f + \left(\frac{d\vec{\omega}}{dt}\right) \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad (3.2.14.7)$$

where $\vec{A} = \left(\frac{d\vec{V}}{dt}\right)_f$ denotes the translational acceleration of the rotating-frame (as observed in the fixed frame of reference). We can now write an expression for the acceleration of point P as observed in the rotating frame as

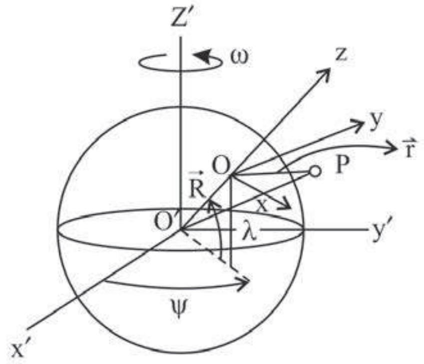
$$\vec{a}_r = \vec{a}_f - 2\vec{\omega} \times \vec{v}_f - \left(\frac{d\vec{\omega}}{dt}\right) \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad (3.2.14.8)$$

which represents the sum of the net inertial acceleration ($\vec{a}_f - \vec{A}$), the centrifugal acceleration – × ($\vec{\omega} \times \vec{r}$) and the Coriolis acceleration $-2\vec{\omega} \times \vec{v}_r$ and an angular acceleration $-\left(\frac{d\vec{\omega}}{\delta\tau}\right) \times \vec{r}$ which epends explicity on the time dependence of the angular velocity $\vec{\omega}$.

3.2.15 Motion relative to Earth

These expressions can now be applied to the important case of the fixed frame of reference having its origin at the centre of Earth (point O′ in the Figure below) and the rotating frame of reference having its origin at latitude λ and longitude ψ (point

O in the Figure below). We note that the rotation of the Earth is now represented as $\frac{\delta\vec{\Psi}}{\delta\tau} = \vec{\omega}$ and that dω/dt = 0.



In this diagram the (x –, y–, z–) axis of the rotating frame have been arranged so that the z-axis is a continution of the position vector $\vec{R}$ of the rotating-frame origin, i.e., R = R$\hat{z}$ in the rotating frame (where R = 6378 km is the Earth's radius assuming a spherical Earth). When expressed in terms of the fixed-frame latitude angle λ and the azimuthal angle ψ, the unit vector $\hat{z}$ is

$$\hat{z} = \cos\lambda\left(\cos\psi\,\hat{x}' + \sin\psi\,\hat{y}'\right) + \sin\lambda\,\hat{z}'$$

Likewise, we choose the x-axis to be tangent to great circle passing through the North and South poles, so that

$$\hat{x} = \sin\lambda\left(\cos\psi\,\hat{x}' + \sin\psi\,\hat{y}'\right) - \cos\lambda\,\hat{z}'.$$

Lastly, the y–axis is chosen such that

$$\hat{y} = -\sin\psi\,\hat{x}' + \cos\psi\,\hat{y}'$$

We now consider the acceleration of point P as observed in the rotating frame O by writing Eq. (3.2.14.8) as

$$\frac{d^2\vec{r}}{dt^2} = \vec{g} - \frac{d^2\vec{R}_f}{dt^2} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{2\vec{\omega}}{\frac{d\vec{r}}{dt}} \quad (3.2.15.1)$$

The first term represents the pure gravitational acceleration due to the gravitational pull of the Earth on point P (as observed in the fixed frame located at Earth's centre)

$$\vec{g} = \frac{-GM\vec{r}'}{|r'|^3}$$

where $\vec{r}' = \vec{R} + \vec{r}$ is the position of point P in the fixed frame and $\vec{r}$ is the location of P in the rotating frame. When expressed in terms of rotating-frame spherical coordinates (r, θ, ϕ):

$$\vec{r} = r\left[\sin\theta\left(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right) + \cos\theta\,\hat{z}\right];$$

and

$$|r'|^3 = \left(R^2 + 2Rr\cos\theta + r^2\right)^{\frac{3}{2}}$$

The pure gravitational acceleration is, therefore, expressed in the rotating frame of the Earth as

$$g = -g_0\left[\frac{(1+\in\cos\theta)\hat{z} + \in\sin\theta(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})}{\left(1+2\in\cos\theta + \in^2\right)^{\frac{3}{2}}}\right] \quad (3.2.15.2)$$

where $g_0 = \frac{GM}{R^2} = 9.789\frac{m}{s^2}$ and $\in = r/R << 1$.

From the above discussion it is clear that the centrifugal force

$\vec{F}_{cent} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is directed perpendicularly away from the rotation axis of the Earth. At latitude λ its value is $m\omega^2 r\cos\lambda$.The acceleration due to gravity which is measured in the laboratory is actually the effective acceleration

$$\vec{g}_e = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) \quad (3.2.15.3)$$

Thus, we can conclude that centrifugal force exists when viewed rom a rotating frame of reference.

❑ Worked out Example :

What will be the shape of the surface of water in a bucket which is rotating with an angular velocity $\vec{\omega}$?

The water surface appears static to an observer rotating with the bucket. As equilibrum is maintained all over the water, for a small of mass m of the water, net force must be zero. If the contact force $\frac{\omega^2 r}{g}$ act at the contact point making an angle θ with the vertical, then one can write

$$N \cos\theta - mg = 0$$

and $$-N \sin\theta + m\omega^2 r = 0 \qquad (3.2.\text{Ex } 1)$$

N must be normal to the liquid surface.

The slope of the surface is $\frac{dz}{dr} = \tan\theta = \frac{\omega^2 r}{g}$ (from Eq. 3.2.Ex 1) (3.2.Ex 2)

On integration we get $z = \frac{1}{2}\frac{\omega^2 r^2}{g}$

Thus the surface will be parabolic.

3.2.16 Coriolis Force

This force arises due to the rotation of the frame of reference in which the particle itself is moving. The particle has to move in some direction other than the one parallel to the rotation axis,

$$\vec{F}_{cor} = -2m\ \omega \times \frac{dr'}{dt} \qquad (3.2.16.1)$$

As is evident from the above equation the coriolis force is always perpendicular to the motion of the particle. No work can be done by this force. Only the force can change the direction of motion of the particle. There are numerous effects of this force on the Earth's surface, for example, for the soldier to hit a long distance target, correction should be made for the coriolis deflection. Rivers in the northern hemisphere flowing in the north-south direction towards the sea, deviates to the right of motion due to corioli force. This results in a greater erosion of the right bank. In the southern hemisphere the erosion takes place in the opposite direction.

❑ Worked out Example :

1. Using the expressing for the coriolis force show that fallin from a height h the deflection of a particle on the ground will be $\frac{\omega}{3}\left(\frac{8h^3}{g}\right)^{\frac{1}{2}}$, where g is the local acceleration due to gravity. (Assume latitude of the place to be zero)

<u>Solution</u> : The particle at the height h was moving with the same angular velocity as that of the earth $\vec{\omega}$. So, it had alinaer velocity (R + h) $\vec{\omega}$ towards east, whereas the linear velocity on the surface of he earth was Rω. For this reason the particle falls a little bit on the east.

As the latitude is zero, vertical direction and the rotational axis are perpendicular to each other. The coriois force will be 2m (v × ω) towards east. Therefore, the equation of motion for the x-component (towards east) will be

$$\frac{md^2}{dt^2} = 2m\,(\vec{v}\times\vec{\omega}) = 2\ mv\omega.$$

Now, v = gt and h= $\frac{1}{2}$ gt^2. So, we have m $\frac{d^2x}{dt^2}$ = 2 mωgt.

Integrating once we get $\frac{dx}{dt}$ = $\omega g t^2$. Then integrating once again, x = $\frac{1}{3}$ $\omega g t^3$ = $\frac{\omega}{3}\left(\frac{8h^3}{g}\right)^{\frac{1}{2}}$.

2. A sphere full of particles is rotating around an axis passing through its center (diameter). The sphere shrank to $\frac{1}{8}^{th}$ of its original volume. If it is assumed that during

this shrinking the distance of every particle from the axis of rotation shrank by the same ratio, then how the angular velocity changed? In what ratio the rotational kinetic energy changed?

Solution : If the changed volume became $\frac{1}{8}^{th}$ of the original volume, it means radius has become $\frac{1}{2}$ of its original value ($V \propto r^3$). As the moment of inertia is proportional to r^2, it has reduced to $\frac{1}{4}^{th}$ of its original value. In absence of external torque the angular momentum remains, constant, so , it has changed by $\frac{1}{4} \times 4^2 = 4$ times.

3.2.17 Substance

- For the analysis of curvilinear motion, use of polar co-ordinates in place of Cartesian co-ordintes facilitate solution easily.
- The ideas aout angular displacement, angular velocity and angular acceleration have been learned. Angular momentum and its conservtion have also been discussed.
- Angular motion of rigid bodies and the importance of moment of inertia have been discussed.
- Motion in inertial and non-intertial frames of references have been explained.
- How pseudo forces moe into play and their importance in life have beeen shown.

3.2.18 Last Questions

1. The angular velocity of a particle is 30º per minute. If the radius of the circular path of the particle be 1m, what is its linear velocity?
2. If both the radial and transverse acceleration of a particle be zero, what will be its path? Find the equation of the path using plane polar co-ordinates.
3. Find out the centrifugal acceleration on a ball of mass 1 kg due to the annual rotation of the earth about the Sun.
4. What is the angle between the vertical line with the rotation vector of the Earth at the equator? A particle has a velocity in the perpendicular direction of the rotation vector of the Earth at the equator. What should be the velocity of the particle so that the coriolis force becomes equal to the weight of the particle?

3.2.19 Answers

1. $30^\circ = \frac{\pi}{6}$ radian. So, angular velocity = $\frac{\pi}{360}$ radian/s. Linear velocity v = $\frac{\pi}{360}\times 1=\frac{\pi}{360}$ m/s.
2. As per condition there is no acceleration. So, the motion will be uniform velocity along a straight line. Using the expression for radial acceleration we get $r^2 \frac{d\theta}{dt}$ = h (constant) and $a_r = -h^2u^2 \frac{d^2u}{d\theta} - h^2 u^3 = 0$, or

   $d\frac{d^2u}{d\theta} + u = 0 -> u = a \cos(\theta - \alpha)$. This represents a straight line.

3. Magnitude of the centrifugal force is $m\omega^2 r$. Here the time period is one year,

   so, $\omega = \frac{2\pi}{(3\times10^7)}$ radian/s.

   r = The distance between the sun and the earth = 1.5×10^{11} m. So, the attractive

   force on a 1kg ball = $\left[\frac{2\pi}{(3\times10^7)}\right]^2 \times 1.5 \times 10^{11}$ N ~ 6×10^{-3}N.

4. The rotation vector is along North-South in the horinzontal direction. So, the angle is 90º. If the velocity is along the East-West direction, the coriolis force will be along the vertical direction. If the magnitude of the coriolis force is equal to the weight of the particle, then $2mv\omega = mg$, or

$$v = \frac{g}{2\omega \sim 6800\,m/s.}$$

Unit–3 ❑ Gravitation

Structure :

3.3.1 Proposal

3.3.2 Newton's Law of Gravitation

3.3.3 Gravitational Field

3.3.4 Calculation of Gravitational Field Intensity

3.3.5 Gravitational Field Due To Uniform Ring

3.3.6 Field Due To A Thin Rod

3.3.7 Field Due To Uniform Disc.

3.3.8 Gravitational Potention

3.3.9 Calculation of Gravitational Potential And Hence Intensity.

3.3.9.1. Due To Ring

3.3.9.2. Potential And Hence Intensity Due To A Spherical Hollow Shell

3.3.10 Gravitational Potential And Hence Intensity Due To Thick Shell.

3.3.11 Gauss's Law

3.3.11.1 Flsux of Gravitational Field

3.3.11.2 Surface Area And Solid Angle

3.3.11.2 Gauss's Law

3.3.11.3 Proof of Gauss's Theory

3.3.12 Poission's Equation

3.3.13 Short Questions

3.3.1 Proposal

Right from the ancient days man has marvelled about the twinkling stars hanging over the night sky, the moon, the shooting stars–the eclipse all bounded by invisible threads, roaming in the dark bed of sky at night and disappearing at day time, in the fathomless blue–ocean overhead.

However these threads are not really there, but, there exists an attraction force between the so called heavenly bodies. This atraction in called 'GRAVITATION'.

The subject gravitation was in its rudimentary state until 1559–the brilliant astronomer Tycho Brahe published his observation of planetary motion, establishing the Heleocentric solar system without any telescopic aid. His assistant Johannes Kepler a German astronomer succeded in formulation of three of planetary motion, which goes as Kepler's laws of planetary motion. The first two laws were published in 1609 and the third law in 1619.

After about more than half a century Sir Isaac Newton, one of the greatest human mind, justified Kepler's experimental law in generalized mathematical frame of Central Conservative force field; which ultimately led Sir Newton to formulate laws of gravitation – the greatest production of human mind as said by Language– an eminent mathematician and physicist.

In 1917 another break through was initiated again by a German scientist Albert Einstein – a great scientific mind, who amalgamated gravitation with space-time curvature property showing gravitation causes a curvature in space-time. So a falling body apparently falling in straight path really follows a curved space-time path. However, we will confine here in Newtons law of gravitation – where space and time are not interrelated.

❑ Outcome

After reading this chapter you will have a definite idea about one of most important forces in the nature, namely gravitation. You will be able

- to understand the Newton's law of gravitation.

- to find solutions of different problems related to gravitation.
- to understand what is meant by gravitational potential, intensity and equipotential surfaces.
- to explain Gauss's law in case of gravitation and how to apply it in different cases.
- to determine expressions for gravitational potential and intensity due to point and spherical objects like solid sphere, spherical shells both thick and thin.
- to find out the effect of coriolis force on a falling body
- to explain escape velocity and how artificial satellites are placed in their orbits.

3.3.2 Newton's Law of Gravitation

Statement :

Every particle in the universe attracts every other particle with a force which is

i) directly proportional to the product of their masses,

ii) inversely proportional to the square of the distance of separation between the particles,

iii) acting along the line joining the particles.

If m_i and m_j are the masses of ith and jth particle at position vectors $\vec{r}_i$ and $\vec{r}_j$ respectively, then according to the Newton's law of gravitation the force of jth particle due to ith particle is

$$\vec{F}_{ji} = -\frac{Gm_i m_j}{r^2_{ij}}\hat{r}_{ij} \;....... \;(3.3.2.1)$$

Where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_j$, $r_{ij} = \left|\vec{r}_j - \vec{r}_i\right|$ $\hat{r}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{\left|\vec{r}_j - \vec{r}_i\right|}$

Using Newton's third law, the force on ith particle due to jth particle

$$\vec{F}_{ij} = -\vec{F}_{ji} = -\frac{Gm_i m_j}{r_{ij}^2}\hat{r}_{ji} \text{ .... } (3.3.2.2)$$

So the total force on ith particle due to a cluster of particles j = 1, 2, 3....N will be

$$\vec{F}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_i m_j}{r_{ij}^2}\hat{r}_{ji} \text{ .... } (3.3.2.3)$$

The eqn. (3.3.2.3) shows that the principle of superposition is applicable in case of gravitation. G is called universal gravitational constant as its value is spece-time independent i.e. G remains same for all space points at all time and does not depend on intervening medium. It is also independent of temperature, pressure and the presence of other force fields.

3.3.3. Gravitational Field

Gravitational force is a distant force, which means gravitational interaction can migrate through space without any material interaction. The questions thus arises :

i) How the interaction migrate through space

and

ii) With what speed this field migrates?

To answer to this poser scientists introduced the field conception, here known as gravitational field. This field is a quality developed in space due to the presence of mass and with this quality the material interaction takes place. [The intraction travels with speed of light in free space and the carrier is known as 'graviton' like that photon which migrate electric-and-magnetic interaction].

The field is defined as,

a space is said to posses gravitational field if a mass undergoes gravitational interaction when placed in that space.

Obivously this definition is qualitative. The quantitative definition goes with a term called gravitational field intensity.

The gravitational field intensity at a point is defined as gravitational force experienced by a unit mass placed at that point. It is denoted by $\vec{E}$. Thus the force on a mass m placed in a gravitation field $\vec{E}$, $\vec{F} = m\vec{E}$

3.3.4. Calculation of Gravitational Field Intensity

1. For a point mass

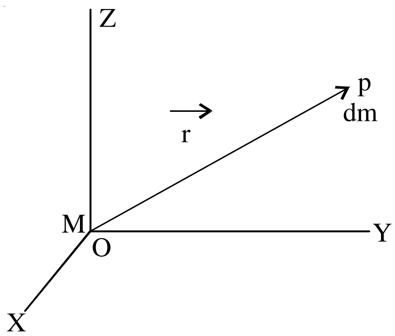


Figure : 3.3.4.1

The fig. (3.3.4.1) shows a point mass M at the origin of reference frame. To calculate the gravitational field intensity at 'p' due to mass M, we place a test mass dm at p which is so small that it does not put any change to the field pattern produced by M. Then from Newton's law the force on mass dm

$$d\vec{F} = -\frac{GMdm}{r^2}\hat{r} \quad (\hat{r} \text{ is unit vector along } \vec{r})$$

The gravitational field intensity at p is

$$\vec{E} = \frac{d\vec{F}}{dm} = -\frac{GM}{r^2}\hat{r} \text{ ....} \qquad (3.3.4.1)$$

3.3.5. Gravitational Field Due to Uniform Ring

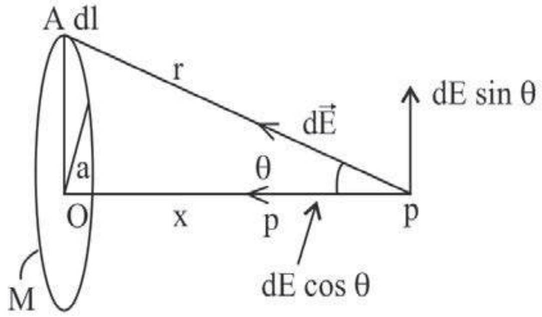


Figure : 3.3.5.1

The fig (3.3.5.1) shows a ring of mass M and radius 'a'. In the mode of calculating the field at p at a distance x from center O of the ring, consider an elemental length 'dl' at point A on the ring (AP = r)

The mass of the element = $\lambda\,dl\ \left(\lambda = \frac{M}{2\pi a}\right)$

Field due to this element at p $d\vec{E} = \frac{G\lambda dl}{r^2}$ along PA

Resolving $d\vec{E}$ as dE cos θ and dE sin θ as in fig (3.3.5.1), the component dE sin θ yields to a null value when summed over the whole ring. So the net field is

$$\vec{E} = \int dE\cos\theta \text{ along pO}$$

$$= \int \frac{G\lambda dl}{r^2}\cos\theta \text{ along pO} = -\frac{G\lambda 2\pi a}{r^2}\cos\theta\ \hat{x}$$

$$= -\frac{GM}{r^3}x\,\hat{x}\ \ (\hat{x} \equiv \text{unit vector along Op})$$

$$\vec{E} = -\frac{GM\bar{x}}{(a^2+x^2)^{3/2}} \quad .... \qquad (3.3.5.1)$$

3.3.6. Field Due to a Thin Rod

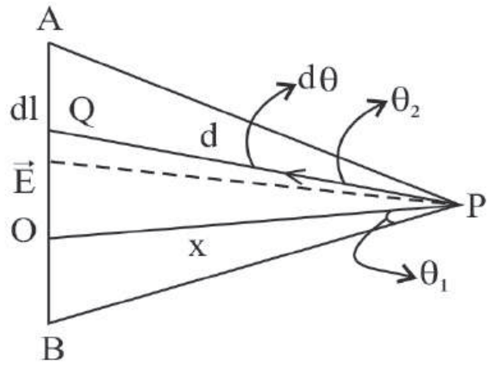


Figure. 3.3.6.1

The fig. (3.3.6.1) shows a thin rod of mass M. We have to find out the intensity at P at a distance x from a point O on the rod. θ_2 and θ_1 be the angular elevations of the rod from the point P. The gravitational intensity at P due to the elemental length dl as in figure $d\vec{E} = \frac{G\lambda dl}{r^2}$ along PQ. Resolving $d\vec{E}$ as dE $\cos\theta$ and dE $\sin\theta$ along and perpendicular to OP, we have the horizontal x component of net field

$$E_x = G\int_{-\theta_1}^{\theta_2} \frac{\lambda dl}{r^2}\cos\theta = +\frac{\lambda G}{x^2}\int_{\theta_1}^{\theta_2} \cos^3\theta\, dl = +\frac{\lambda G}{x^2}\int \cos^3\theta\, x \sec^2\theta\, d\theta$$

$$\text{Putting } x = x\tan\theta = +\frac{\lambda G}{x}\int_{-\theta_1}^{\theta_2}\cos\theta d\theta = +\frac{\lambda G}{x}(\sin\theta_2 + \sin\theta_1)\ldots.. \qquad (3.3.6.1)$$

$$\text{Similarly } E_y = \lambda G\int\frac{dl\sin\theta}{r^2} = \frac{\lambda G}{x^2}\int_{-\theta_1}^{\theta_2}\cos^2\theta\, x \sec^2\theta\ \sin\theta\, d\theta$$

$$= \frac{\lambda G}{x}(\cos\theta_2 - \cos\theta_1)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.3.6.2)$$

3.3.7. Field Due to Uniform Disc

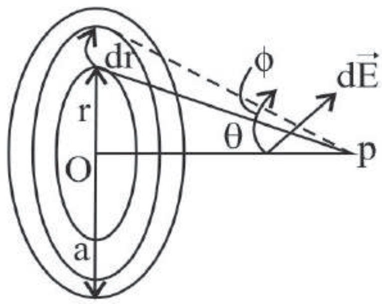


Figure 3.3.7.1

The fig (3.3.7.1) shows a uniform disc of mass M and radius 'a'. To calculate the gravitational field intensity at p, we consider an elemental ring of radius r and thickness dr. σ be the mass per unit area (σ = $M/\pi a^2$). The field intensity at 'p' due to this elemental ring.

$$d\bar{E} = -G \frac{2\pi r dr \sigma}{(r^2+x^2)^{3/2}} \bar{x} \quad \text{(ref. eqn. (3.3.5.1)}$$

So the magnitude of field

$$d\bar{E} = \left| \frac{G 2\pi r dr \sigma x}{(r^2+x^2)^{3/2}} \right|$$

$$\text{So} \quad dE = \left| G 2\pi\sigma x \frac{rdr}{(r^2+x^2)^{3/2}} \right| = \left| \frac{2\pi\sigma G}{r^2} \cos^3\theta rdr \right|$$

Now $r = x \tan\theta$, $\therefore$ $dr = x \sec^2\theta \, d\theta$

$$dE = \left| \frac{G 2\pi\sigma}{x^2} \cos^3\theta . \; x \tan\theta . \, x \sec^2\theta d\theta \right|$$

$$= |2\pi G\sigma \sin\theta \, d\theta|$$

$$\therefore\ E = \left|2\pi\sigma G \int_0^{\phi} \sin\theta\, d\theta\right| = 2\pi\sigma G\ (1 - \cos\phi)$$

$$E = \frac{2GM}{a^2}\ (1 - \cos\phi) = \frac{2GM}{a^2}\left(1 - \frac{x}{\sqrt{a^2 + x^2}}\right) \ldots\ldots. \qquad (3.3.7.1)$$

So, $$\vec{E} = -\frac{2GM}{a^2}\ (1 - \cos\phi)\,\hat{x} = -\frac{2GM}{a^2}\left(1 - \frac{x}{\sqrt{a^2 + x^2}}\right)\hat{x} \ldots.. \qquad (3.3.7.2)$$

❑ Worked out Example

Show that gravitational field intensity is maximum for a uniform ring of radius 'a' on the axis of the ring lies at a distance $x = \frac{a}{\sqrt{2}}$ from the centre of the ring.

The gravitational field on the axis of the ring is given by (Ref. eqn. no. 3.3.5.1)

$$\vec{E} = -\frac{GM\bar{x}}{(a^2 + x^2)^{3/2}}$$, for $\vec{E}$ to be maximum.

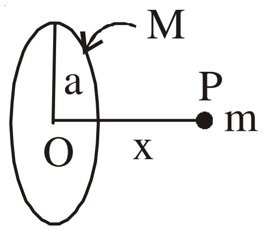
$$\frac{d}{dx}|\vec{E}| = \frac{d}{dx}\left|\frac{GMx}{(a^2 + x^2)^{3/2}}\right| = 0$$

or, $$\frac{1}{(a^2 + x^2)} + \frac{x}{(a^2 + x^2)^{5/2}}\left(-\frac{3}{2}.2x\right) = 0 \text{ or } x = \frac{a}{\sqrt{2}} \ \ldots\ldots \qquad (3.3.7.3)$$

The double differentiation of $\vec{E}$ is found to be negative at $x = \frac{a}{\sqrt{2}}$, thus $\vec{E}$ is maximum at $x = \frac{a}{\sqrt{2}}$.

❑ Worked out Example :

Show that if a particle is released from a point on the axis of a fixed ring (x<<a), it will undergo simple harmonic motion. Find the expression of time period of motion.



Let M be the mass of the fixed ring and a be its radius. Then the field intensity at point P at a distance x from centre O of the ring $\vec{E} = -\frac{GM\vec{x}}{(a^2+x^2)^{\frac{3}{2}}}$

Then the force on the mass m,

$$\vec{F} = m\vec{E} = -\frac{GM\vec{x}}{(a^2+x^2)^{\frac{3}{2}}}$$

If x <<a

$$F = -\frac{GMm}{a^3}\vec{x} \quad .... \qquad (3.3.7.4)$$

So, $\vec{F} \ \alpha - \vec{x}$, the motion is simple harmonic and

$$\omega = \sqrt{\frac{GM}{a^3}} \text{ and } T = 2\pi\sqrt{\frac{a^3}{GM}}$$

3.3.8 Gravitational PotentiAL

We have already seen in previous discussion that the gravitational field intensity at a point due to a point mass M is given by

$$\vec{E} = -\frac{GM}{r^2}\hat{r}$$

So, $\vec{\nabla} X \vec{E} = -GM\nabla X\left(\frac{\vec{r}}{r^3}\right)$

$$= 0 \quad .... \qquad (3.3.8.1)$$

So, gravitational field is a conservative field. As curl of gradient of a scalar is always zero, we can write

$\bar{E} = -\Delta \bar{V}$ (the –ve sign is a logical convention)........ (3.3.8.2)

V is called gravitational potential. Now in eqn. (3.3.8.2) $\bar{E}$ remains same for any additive constant with V. So, it is preferable to set gravitational potential to be zero at infinity, where there is no sense of gravitational field.

The formal definition of gravitational potential goes as ;

The gravitational potential at a point is the work done to bring a unit mass from infinity to the point quasistatically due to gravitational field.

from eqn. (3.3.8.2)

$$\bar{E}.d\bar{r} = -\nabla\bar{V}\cdot d\bar{r} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right)$$

or $\bar{E}\cdot d\bar{r} = -dv$ .... (3.3.8.3)

The –ve sign thus indicates that as the system does work its potential energy decreases.

3.3.9 Calculation of Gravitational Potential and Hence Intensity.

3.3.9 1. DUE TO RING

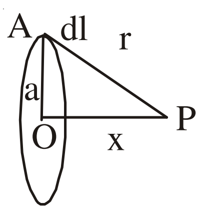


Figure. 3.3.9.1

The fig. (3.3.9.1) shows a ring of mass M and radius 'a'. To calculate the potential at the point P, we consider an elemental length dl at A as in fig. (3.3.9.1). The potential due to this elemental length at P is

$$dV = -G\frac{\lambda dl}{r} \quad (\lambda = \text{mass/unit length of ring})$$

Then the total potential at P is

$$V = -\frac{G\lambda}{r}\oint dl = -\frac{G\lambda 2\pi a}{r} - \frac{GM}{r}$$

$$V = -\frac{GM}{(a^2+x^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots \qquad (3.3.9.1)$$

The intensity at point P

$$E = -\frac{\partial V}{\partial x} \quad \text{(as V is a function of x only)}$$

$$= \frac{GM}{(a^2+x^2)^{\frac{3}{2}}} \times \left(\frac{-1}{2}.2x\right) = -\frac{GMx}{(a^2+x^2)^{\frac{3}{2}}}$$

Considering the direction.

$$\vec{E} = -\frac{GMx}{(a^2+x^2)^{\frac{3}{2}}} \quad \ldots\ldots\ldots \qquad (3.3.9.2)$$

3.3.9.2. Potential and Hence Intensity Due to a Spherical Hollow Shell

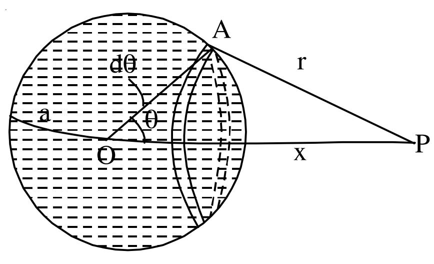


Figure 3.3.9.2

The fig. (3.3.9.2) shows a speherical hollow shell of radius a and mass M. To calculate the potential at point P at a distance x from centre 'O' of the shell, we consider an elemental ring within the angle θ and $\theta + d\theta$. Then,

The radius of ring = a $\sin\theta$

Area of the ring = $2\pi a\sin\theta \cdot a d\theta$

Mass of ring = $2\pi a^2 \sin\theta \, d\theta \, \sigma$

where σ = mass per unit area of the shell. The potential at point P due to this elemental ring.

$$dV = -\frac{G2\pi a^2 \sin\theta d\theta\sigma}{r} \quad .... \qquad (3.3.9.1)$$

Now from the triangle OAP

$$r^2 = a^2 + x^2 - 2ax\cos\theta$$

$$2rdr = 2ax\,\sin\theta d\theta \Rightarrow \frac{\sin\theta d\theta}{r} = \frac{dr}{ax}$$

Putting this value in above eqn.

$$dV = -G\,2\pi a^2\sigma\,\frac{dr}{ax}$$

$$dV = -2\pi G\sigma\,\frac{a}{x}\,dr$$

When the point P is out side the shell x > a

$$V_0 = -2\pi G\sigma\,\frac{a}{x}\int_{x-a}^{x+a} dr = -\frac{4\pi a^2 G\sigma}{x}$$

$$V_0 = -\frac{GM}{x} \quad .... \qquad (3.3.9.2)$$

When the point p is inside the shell

$$V_i = -2\pi G\sigma \int_{a-x}^{x+a} = -4\pi a G\sigma$$

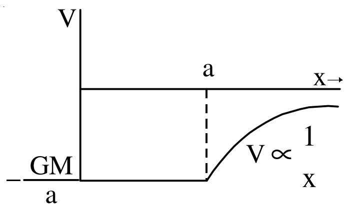
$$= -\frac{4\pi a^2 \sigma G}{a} = -\ GM/a............ \qquad (3.3.9.3)$$

So the intensity of field outside the shell is

$$\vec{E} = -\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\frac{GM}{x} = -\frac{GM}{x^2}\hat{x}$$

Inside the shell as the potential is constant throughout, hence the intensity is zero throught x < a.

for x = a the intensity



$$\vec{E} = -\ \frac{GM}{a^2} \text{ towards the center O......... } \qquad (3.3.9.4)$$

3.3.10. Gravitational Potential and Hence Intensity Due to Thick Shell.

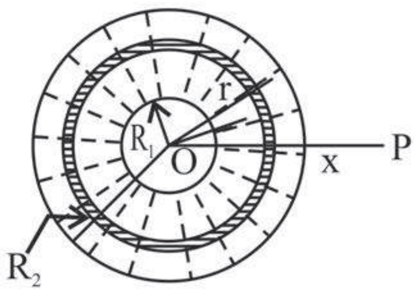


Figure : 3.3.10.1

The fig. (3.3.10.1) shows a spherical shell of uniform density P and radius R_1 and R_2 ($R_2 > R_1$). We have to calculate potential at point P at a distance x from centre O.

(a) **When P is outside the shell.** We consider an elemental speherical shell of radius r and thickness dr.

The mass of the elemental shell $dm = 4\pi r^2 dr \rho$.

$$dV = -\frac{Gdm}{x} = -G\frac{4\pi r^2 dr\rho}{x}$$

So the total potential

$$V_0 = \int dV = \frac{-G4\pi\rho}{x}\int_{R_1}^{R_2} r^2 dr = -\frac{G4\pi\rho}{3x} \quad - \frac{3}{5}\frac{GM^2}{R}. \quad (R_2^3 - R_1^3)$$

$$\boxed{V_0 = -\frac{GM}{x}} \text{ ....} \qquad (3.3.10.1)$$

So the intensity $\vec{E}_0 = -\frac{\partial V_0}{\partial x}\hat{x} = -\frac{GM}{x^2}\hat{x}$ .... (3.3.10.2)

(b) **When P is inside the shell :**

We consider a shell of radius x ; V_1 and V_2 be the potential due to portions inside and outside of the shall. Then the total potential at P, when $R_1 < x < R_2$ is

$$V_i = V_1 + V_2 \text{ ....} \qquad (3.3.10.3)$$

To calculate V_1 we proceed as previous and get

$$V_1 = -G\,\frac{\text{(mass inside the radius x)}}{x}$$

$$= -G\,\frac{4}{3}\pi\rho\,\frac{(x^3 - R_1^3)}{x} \text{ ....} \qquad (3.3.10.4)$$

To calculate V_2 we proceed as follows :

We consider an elemental shell of radius r (r > x) and thickness dr. Then the potential at P due to this elemental shell.

$$dV_2 = -G\,\frac{4\pi r^2 dr\rho}{r} \text{ (since it is inside the shell)}$$

So $V_2 = -G\pi\rho \int_x^{R_2} r dr = -G4\pi P \frac{\left(R^2{}_2 - x^2\right)}{2}$ .... (3.3.10.5)

So the total potential

$$V_i = V_1 + V_2 = -G\frac{4}{3}\pi\rho\frac{\left(x^3 - R_1{}^3\right)}{x} - G2\pi\rho\,(R_2^2 - x^2) \qquad (3.3.10.6)$$

$$= G\frac{2}{3}\pi\rho\, x^2 - G\ \frac{4}{3}\pi\rho\frac{R_1^3}{x} - G\ 2\pi\rho\ R_2^2$$

So the intensity at the point

$$E_i = -\ \frac{\partial V_i}{\partial x} = -G\frac{4}{3}\pi\rho x + G\frac{4}{3} r\rho\frac{R_1^3}{x^2}$$

$$= -\ G\frac{4\pi\rho}{3}\frac{\left(x^3 - R_1^3\right)}{x^2} = -\frac{GM_i}{x^2}$$

$$\therefore \overline{E_i} = -\frac{GM_i}{x^2}\hat{x} \ \ .... \qquad (3.3.10.7)$$

M_i = Core mass.

So, the effect on gravitational intensity is due to core mass only, the exterior mass does not contribute to gravitational intensity.

❑ Worked out Examples :

1. Find the expression of gravitational potential due to a uniform solid disc of mass M and radius 'a' on the axis of the disc.

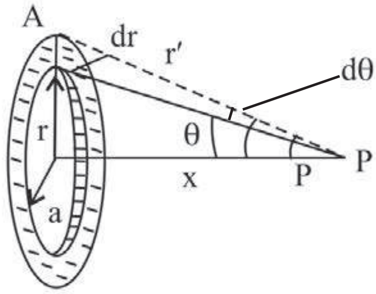


Figure : 3.3.13.1

The fig. (3.3.13.1) shows a disc of radius a and mass M. We have to find out the potential at point P at a distance x from the center O of the disc.

We consider an elemental ring of radius r and thickness dr.

The area of the ring = $2\pi rdr$

Mass of the ring = $2\pi rdr\sigma$ (σ = mass/unit area of disc)

Potential at point P due to this ring

$$dV = -\frac{G2\pi rdr\sigma}{r'} \quad \text{(Ref. Eq. 3.3.10.1)}$$

$$= -\frac{G2\pi\sigma}{x}\cos\theta\, rdr$$

We put $r = x\tan\theta$, so $dr = x\sec^2\theta\, d\theta$

$$\therefore\ dV = -G2\pi\sigma x\,\frac{\sin\theta}{\cos^2\theta}d\theta$$

$$V = -G2\pi\sigma x\int_0^{\phi}\frac{\sin\theta d\theta}{\cos^2\theta}$$

$$= -G\,2\pi\sigma x\left(\frac{1}{\cos\phi}-1\right)$$

$$= -G2\pi\sigma x\left(\frac{\sqrt{a^2+x^2}}{x}-1\right)$$

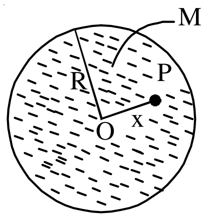
$$= G2\pi\sigma\left(\sqrt{a^2+x^2}-x\right)$$

$$= -\frac{2GM}{a^2}\left(\sqrt{a^2+x^2}-x\right)$$

As V is a function of x only so the intensity will be along $\bar{x}$.

$$\vec{E} = -\frac{\partial V}{\partial x}\text{ along PO} = -\frac{2GM}{a^2}(1-\cos\phi)\,\hat{x}.$$

2. Considering Earth to be a perfect sphere of uniform density, of mass M and radius R, find the gravitational potential at a distance x from the centre of Earth (x < R).



SOLN Please ref : (3.3.10.6)

Put $R_1 = 0$ and $R_2 = R$

$V_i = V_1 + V_2$

$$= -G\ \frac{4}{3}\frac{\pi\rho x^3}{x} - G4\pi\rho\frac{(R^2 - x^2)}{2}$$

$$= G\ \frac{2}{3}\pi\rho\, x^2 - G\ 2\pi\rho\, R^2$$

$$= -\ \frac{2\pi\rho}{3} G\ (3R^2 - x^2)$$

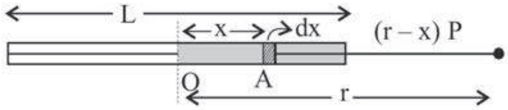
$$V_i = -\ \frac{GM}{2R^3}\ (3R^2 - x^2) \ .... (3.3.10.8)$$

So the gravitational field intensity at ρ will be

$$E_i = -\ \frac{\partial V}{\partial x} = -\frac{GM}{R^3}\vec{x}$$

3. Find the expression of gravitational potential (V) and hence the gravitational field ($\vec{E}$) for a thin rod of length L on its axial point at a distance r from centre O of the road.

SOLUTION:



The fig. shows a rod of length L and mass m. The potential at the axial point P is to be calculated. We consider an elemental length dx at a distance x from center O. The potential due to this element dx at P.

$$dv = -\frac{G\lambda dx}{r-x} \text{ where } \lambda = \text{mass/unit length of the rod.}$$

So the total potential at.

$$V = -G\lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{r-x} = G\lambda \ln(r-x)\Big|_{-\frac{1}{2}}^{+\frac{1}{2}} = -G\lambda \left(\ln\frac{r+\frac{1}{2}}{r-\frac{1}{2}}\right)$$

$$= -G\lambda \ln\left(1+\frac{L}{r-\frac{L}{2}}\right)$$

$$E = -\frac{\partial V}{\partial r}$$

3.3.11 Gauss's Law

3.3.11.1 Flux of gravitational field

In science, flux, usually concerns to some flow of physical quantity or property. In case of 'fluid flux' it refers to the amount of fluid crossing a specific area per unit time. However, in case of gravitational flux no such transport physically, exist. Gravitational flux through an ele ment of area $d\bar{s}$ placed at a point where the gravitational field intensity is $\bar{E}$, is defined as,

$$d\phi = \bar{E}\cdot d\bar{s}\ .$$

The total flux through a finite surface S is

$$\phi = \int_S \bar{E}\bullet d\bar{s} \text{ (where the integration is carried over the whole surface).}$$

3.3.11.2 Surface area and solid angle

Surface area is treated as a vector whose magnitude is the area of the surface considered and direction is specified as follows.

i) For closed surface the direction is +ve in outwards normal to the surface.

ii) In case of open surface the direction is specified by right hand screw rule as illustrated in the adjoining figure.

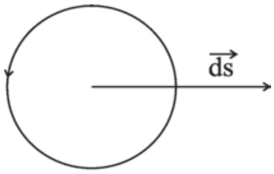


Fig. 3.3.11.1a

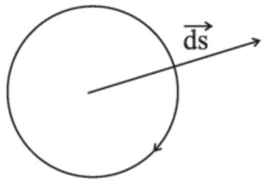


Fig. 3.3.11.1b

The **solid angle** is three dimensional analogues to that of an **angle** in two dimension. Now an angle can be visualized physically as a two dimensional peeping from a point. Mathematically it is defined as the ratio of the arc by radius of a circle.

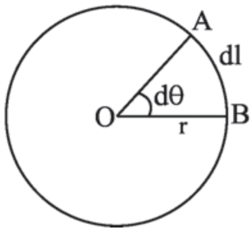


Figure : 3.3.11.2

In fig. the angle subtended

$d\theta = \frac{arc\,AB}{r} = \frac{dl}{r}$, which is a dimensionless quantity. Its **units** taken as radian which is defined as angle subtended by an arc of an circle of unit length and unit radius at the center of the circle, so, the total angle about a point is $\frac{2\pi r^2}{r^2} = 2\pi$ radian. Similarly, a solid angle can be visualized as a three dimensional peeping through a point and it is mathematically defined as three dimensional angle produced at the center of a sphere due to an area boundary on the surface of the sphere.

If ds be the elemental area in the surace of a sphere then, the solid angle suspended at the centre O is,

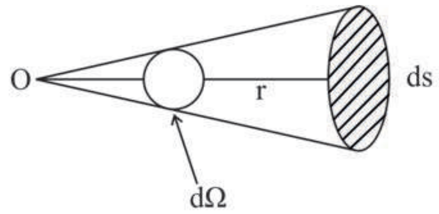


Fig. 3.3.11.3

$$d\Omega = \frac{ds}{r^2} \text{..........} \qquad (3.3.11.1)$$

If the area plane makes an angle θ with the tangent to the sphere at that point then solid angle $d\Omega = \frac{ds\cos\theta}{r^2} = \frac{d\vec{s}.\hat{r}}{r^2} = \frac{ds\vec{n}.\hat{r}}{r^2}$ ...... (3.3.11.2)

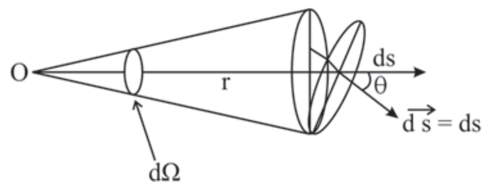


Fig. 3.3.11.4
$\hat{n}_2$

Where

$\hat{n}$ and $\hat{r}$ are unit vectors along $d\vec{s}$ and $\vec{r}$. as explained in the figure. **Unit of solid angle** is named steradian, one steradian is the solid angle subtended at the centre of a sphere of unit radius by a unit surface on the sphere.

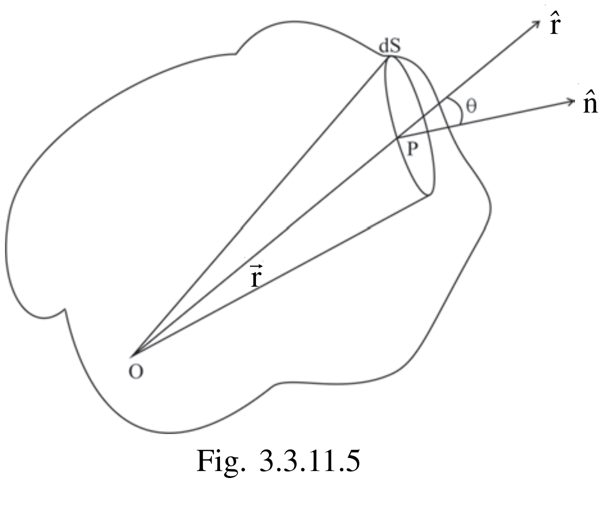


Fig. 3.3.11.5

Obviously the total solid angle about a point will be

$$\Omega = \oint \frac{ds\hat{n}.\hat{r}}{r^2} \oint \frac{ds\cos\theta}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradian........} \quad (3.3.11.3)$$

When O is outside please refer to the fig. Due to the orientation of $\hat{n}_1$ and $\hat{n}_2$ of the surface area vectors the total solid angle subtended at O will obviously be zero, because they will cast solid angles of same magnitude but in opposite sense and will cancel each other to make the yield null.

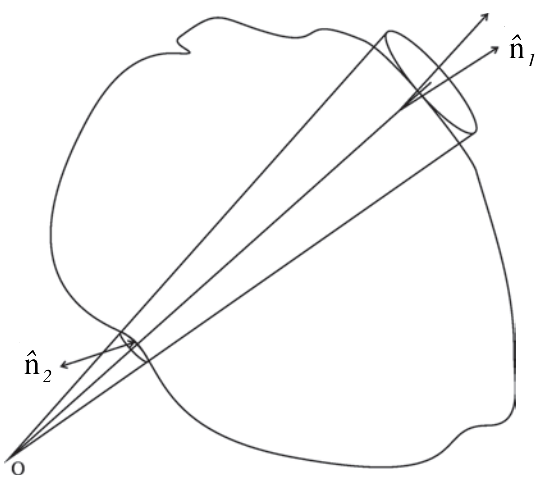


Fig. 3.3.11.6

3.3.11.2 Gauss's Law

The law states that 'the total gravitational flux through a closed surace is $-4\pi G$ times the mass enclosed by the surface.

mathematically it is represented as,

$$\oint \overline{E} \cdot d\overline{s} \text{ closed surface's} = \oint \overline{E} \cdot d\overline{s} = -4\pi G \text{ (total mass enclosed in S)}$$

$$= -4\pi G \sum_{i=1}^{N} m_i = -4\pi G \int dm = -4\pi G \int \rho dv \quad (3.3.11.4)$$

where ρ is the density at the point), for a continuous distribution of mass.

3.3.11.3 Proof of Gauss's theoem

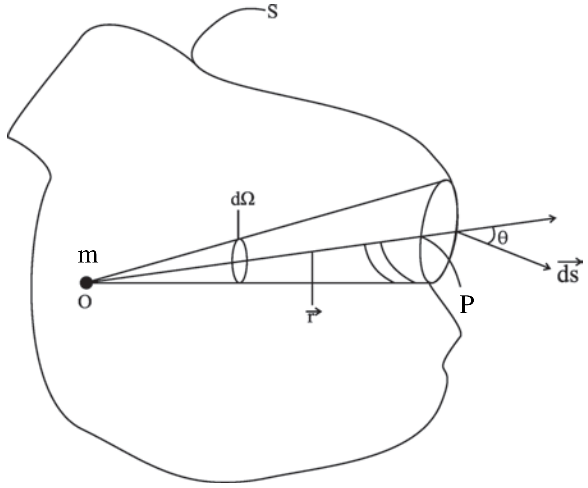


Figure : 3.3.11.7

The fig. (3.3.11.7) shows a point mass 'm' enclosed within a closed surace S. We consider an elemental area $d\bar{s}$ at point P at a position vector $\bar{r}$ from mass point m at O. $\bar{E}$ be the gravitational field intensity at 'P', then flux through the elemental surface $d\bar{s}$,

$$d\phi = \bar{E}\cdot d\bar{s} = -\frac{GM}{r^2}\hat{r}\cdot d\bar{s} = -Gm\frac{ds\cos\theta}{r^2}$$

$= -Gm d\Omega$ ($d\Omega$ is the solid angle. So the total flux through the formed of O by element ds).

$$\phi = -\int Gm d\Omega = -4\pi Gm \;......... \quad (3.3.11.3.1)$$

The eqn. (3.3.11.3.1) holds for any mass point independent of position, so if there are N number of particles of masses m_1, m_2.......m_i.......m_N then the eqn. (3.3.11.3.1) takes the form

$$\phi = -4\pi G\sum m_i \;.......... \quad (3.3.11.3.2)$$

Thus for a continuous distribution of mass we can write

$$\phi = -4\pi G\int dm = -4\pi G\int_v pdv \quad .... \quad (3.3.11.3.3)$$

Where ρ is the density of the medium as the point concerned, here it is the point ρ. The integration is to be carried over the entire volume enclosed by the surface S. Thus the integral form of Gauss's law can be written as

$$\int_S \bar{E}\cdot d\bar{s} = -4\pi G \int_V \rho dv \; .... \qquad (3.3.11.3.4)$$

3.3.12 Poission's Equation

Now using Gauss's divergence theorem, for the integral form of Gauss's theorem, we can write

$$\oint_v \nabla\cdot\bar{E}\cdot dV = -\,4\pi G \oint_v \rho dv$$

$$\oint_v (\nabla\cdot\bar{E} + 4\pi G\rho)\, dv = 0$$

or, $\nabla\cdot\bar{E} = -4\pi G\rho$ .... (3.3.12.1)

which is the differential form of Gauss's law, applied to gravitational field.

We have already deduced form the conservative nature of gravitational field that

$$\vec{E} = -\nabla V \text{ (Ref. eqn. (3.3.8.2)}$$

using eqn ( ) in diferential form of Gauss's law we have

$$\nabla\cdot(-\nabla V) = -\,4\pi G\rho$$

or, $\nabla^2 V = 4\pi G\rho$ .... (3.3.12.2)

The above equation is known as Poission's equation applied to gravitational field. In cartesian co-ordinates it takes the form,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial Z^2} = 4\pi G\rho$$

In free space, where the space does not contain matter, Poission's equation yield to

$$\nabla^2 V = 0, \ .... \qquad (3.3.12.3)$$

which is called Laplace's equation.

❑ Worked out Examples :

1. Find the expression of potential due to a hollow hemisphere of mass M and radius R at the centre of its base.

The fig. (Ex. 1) shows a hemisphere of radius R and mass M. We consider an elemental ring on the surface of hemisphere within the angle θ and $\theta + d\theta$ as in fig.

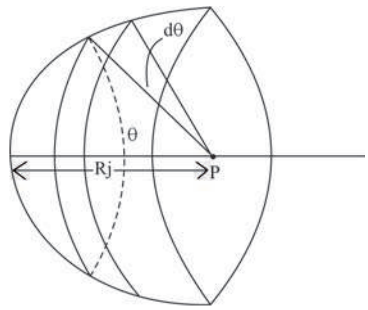


Figure : (Ex. 1)

Then the radius of ring = R $\sin\theta$

Area of the ring = $2\pi R \sin\theta R d\theta$

Mass of the ring = $2\pi R^2 \sin\theta\, d\theta\, \sigma$

where σ = mass/unit area.

Potential due to the ring at the centre P,

$$dV = -G\frac{2\pi R^2 \sin\theta d\theta\sigma}{R}$$

$$\text{Total potential at PV} = \int dV = -G\ 2\pi\sigma R\ (\cos\theta)\Big|_o^{\frac{\pi}{2}}$$

$$= -G2\pi\sigma R = -G\ \frac{2\pi R^2\sigma}{R}$$

$$V = -\frac{GM}{R}$$

2. Calculate the gravitational potential at the centre of base due to a solid hemisphere of radius R and Mass M.

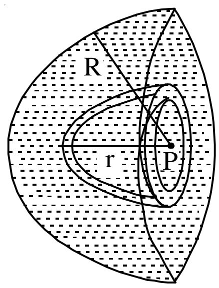


Figure : Ex. 2

The fig (Ex. 2) shows a solid hemisphere of mass M and radius R. We have to find out the potential at point 'P'.

We consider a thin hemispherical shell of radius r and thickness dr. Mass of the elemental shell dm = $2\pi r^2 dr \rho$ (ρ = density of the material of hemisphere)

Potential at P due to this shell,

$$dV = -G\frac{dm}{r} = -G\,2\pi r^2\,dr\frac{\rho}{r}$$

$$= -G2\pi\rho\,rdr$$

So the total potential

$$V = -G2\pi\rho\int_0^R rdr = -G\pi\rho R^2 = -\frac{3}{2}\frac{GM}{R}$$

Exercise

1. Two particles of masses m_1 and m_2 start falling to each other due to their mutual interaction. Initially they are at large separation and were at rest. Calculate their relative velocity when they reach the separation D.

For any Separation r, force of interaction $\vec{F}_1 = \frac{Gm_1m_2}{r^2}\hat{r}$

So acceleration of m_1, $\vec{a}_1 = \frac{Gm_2}{r^2}\hat{r}$

Similarly, acceleration of m_2, $\vec{a}_2 = -\frac{Gm_1}{r^2}\hat{r}$

Relative acceleration of m_1 w.r. to $\vec{a}_2$

$$(\vec{a}_2 - \vec{a}_1) = -\frac{G(m_1 + m_2)}{r^2}\hat{r}$$

$$\frac{dv_{rel}}{dt} = -\frac{G(m_1 + m_2)}{r^2}$$

$$v_{rel}\frac{dv_{rel}}{dr} = -\frac{G(m_1 + m_2)}{r^2}$$

$$\frac{{v_{rel}}^2}{2} = +\frac{G(m_1 + m_2)}{r} + C$$

As $r \to \infty$, $v_{rel} = 0$, $\therefore C = 0$

$$v_{rel} = \frac{\sqrt{2G(m_1 + m_2)}}{r}$$

2. Find the experession of gravitational field intensity at the base centre of a hollow hemispherical shell

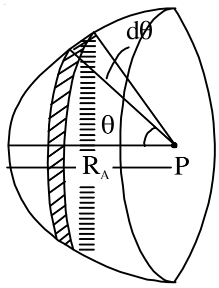


Figure : Ex. 3

The fig. (3) shows a hemispherical shell of radius R and mass M. We consider an elemental ring within the angle θ and $\theta + d\theta$. The gravitational field intensity at ρ due to this elemental ring

$$d\bar{E} = \frac{Gdm}{R^2} \cos \text{ along PA.}$$

$$= G \ \frac{2\pi R \sin\theta R d\theta \cos\theta\sigma}{R^2}$$

$$\bar{E} = G2\pi\sigma \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \text{ along pA}$$

$$= G\pi\sigma \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = -G\pi\sigma \frac{\cos 2\theta}{2}\Bigg|_o^{\frac{\pi}{2}}$$

$$= -\frac{G\pi\sigma}{2} \ [-1-1] \text{ along pA}$$

$$= G\pi\sigma \text{ along pA} = \frac{G\cdot 2\pi R^2\sigma}{2R^2} \text{ along pA}$$

or, $\vec{E} = \dfrac{GM}{2R^2}$ along pA

3. Find the expression of gravitational field intensity due to a solid hemisphere at its base centre.

HINTS/SOLN

The fig. (4) shows a hemisphere of radius R and mass M.

To calculate the field intensity at P, we consider an elemental disc within the angular elevation $d\theta$ at θ. P is the density of solid in hemisphere.

Mass of the disc = r $(R\ \sin\theta)^2$ Rdθ SinθP

$= \pi R\rho\ \sin^3\theta d\theta$

$$dE = \frac{2G\pi R^3\rho\sin^3\theta d\theta}{R^2\sin^2\theta}\ (1 - \cos\theta)$$

$$= 2G\pi R\rho\ \sin\theta\ (1 - \cos\theta)\ d\theta$$

$$= 3\frac{GM}{R^2}\sin\theta(1 - \cos\theta)\ d\theta$$

$$\therefore\quad E = \frac{3GM}{R^2}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{3GM}{R^2}\left[1 - \frac{1}{2}\right] = \frac{3GM}{2R^2}$$

4. Show that field intensity inside a solid sphere remain same if the density of the material of the solid varies inversely as the distance from its origin.

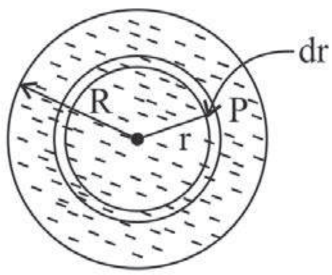


Figure : Ex. 5

Hints : The fig. (5) shows a sphere of mass M and radius R. Consider an elemental shell of radius r and thickness dr. ρ be the density of the shell. By the problem $\rho = \frac{K}{r}$.

Field at a point P at a distance r (r < R)

$$E_p = -\frac{GM_i}{r^2} \quad (M_i = \text{Mass inside radius r})$$

$$= -G\frac{\int_0^r 4\pi r^2 dr\rho}{r^2}$$

$$= -G\frac{\int_0^r 4\pi r^2 dr \frac{K}{r}}{r^2} = -G\frac{2\pi K r^2}{r^2} = -G2\pi K$$

= constant at all points inside the sphere.

❑ Worked out examples

1. The masses of the planet A and planet B have a ratio 2 : 3, while the ratio of their radii is 1 : 2. The weight of an object on planet A is found to be w, what is the weight of the object on the planet B?

Weight of any mass m on any planet (say A) = $Gm\frac{M_A}{r_{A^2}}$, where G is the universal gravitational constant, M_A is the mass of the planet and r_A is the radius of the planet.

Let the weight of the same mass on the planet B be w', then $w' = Gm\frac{M_B}{r_{B^2}}$. So,

$$\frac{w'}{w} = \frac{M^B}{M^A} \times \frac{r_{A^2}}{r_{B^2}} = \frac{3}{2} \times \frac{1}{4} = \frac{3}{8}.$$

Therefore, $w' = \frac{3}{8}w$

2. An object from a spaceship is dropped from rest on a planet A, and travels a distance 22.5 meters in 5 seconds. The radius of planet A is 5.82×10^6 meters.

a) Find out the acceleration of the falling object.

b) Find also the mass of planet A.

From laws of motion, distance travelled $s = \frac{1}{2}at^2$. So, $a = \frac{2s}{t^2} = 1.8s$

Again, $a = \frac{GM_p}{r_p^2}$. So, $M_p = \frac{ar_p^2}{G} = 1.8 \times (5.82)^2 \times \frac{10^{12}}{(6.674\times10^{-11})} = 9.24 \times 10^{23}$ kg.

3. Infinite number of masses each of 1 kg are placed along a straight line at the distances of 1m, 2m, 4m, 8m,.....from a point O on the same line. If G is the universal gravitational constant, then what would be the gravitational field intensity at O?

From the law of gravitation, I $= =GM\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+...\right)$

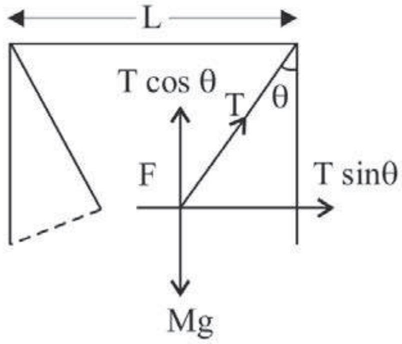
$$=GM\left(1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+...\right)$$

$$=GM\left\{\frac{1}{(1-\frac{1}{4})}\right\}=\frac{4GM}{3}$$

4. Two spheres each of mass M are suspended by two strings each of length L. The distance between the upper ends of strings is also L. Find the angle which the strings will make with the vertical due to mutual attraction.

From the figure below, it is clear that for equilibrium if T is the tension in the string, then equating the forces on the second sphere we get T $\sin\theta = =\frac{GM^2}{L^2}$ and T $\cos\theta =$ Mg.

Therefore, $\tan\theta = \frac{GM}{gL^2}$.



❑ Substance :

In this unit you have learnt about universal gravitational force operating between objects which is attractive in nature. The observations made by Johaness Kepler on the movement of the planets which led to Kepler's laws of planetary motion were finally explained by Newton when he propounded his law of gravitation. In the analysis based on the law of gravitation, ideas about gravitational potential and intensity and the calculation of the aforementioned quantities were discussed in details. Acceleration due to gravity and its implication on escape velocities of objects were analysed. Gauss's law and its importance in the discussion of gravitational potential and intensity were highlighted..

3.3.13 Short Questions.

1. What sould be the duration of the year if the distance between the earth and the sun gets doubled?

2. The value of 'g' at a particular point is 9.8 m/sec^2. Suppose the earth suddenly shrink uniformly to half its present size without losing any mass. What will be the value of 'g at the same point (assuming that the distance of the point from the center of the earth does not shrink)?

3. If the value of gravitational acceleration at a height h above the Earth's surface is the same as that at a depth d below the Earth's surface (with both h and d small

compared to the Earth's radius R), then what is the relation between h and d?

4. The weight of a body is not a fixed quantity. It depends upon the location as well as upon?

5. Find out the gravitational intensity of an infinite plane of uniform mass distribution.

Answers :

1. $T^2 \propto R^3$, so time period of revolution of the earth will be 8 times the original time.

2. $g = \frac{GM}{R^2}$. So, if mass and the distance do not change, there will be no change in 'g'.

3. $g_h = g_o \frac{R^2}{(R+h)^2}$ and $g_d = g_0 \frac{(R-d)}{R}$, So, if $g_h = g_d$, then $\frac{R^2}{(R+h)^2} = \frac{R-d}{R}$.

4. The motion of the frame of reference.

5. Let us consider a Gaussian pill-box of height r above and below the plane and ds is the corss-sectional area. The intensity $\mathbf{E} = E\,\hat{\mathbf{r}}$ and $d\mathbf{s} = ds\,\hat{\mathbf{r}}$. Applying Gauss' theorem on the pill-box, we get $\oint \mathbf{E}.d\mathbf{s} = 2E\,ds = -4\pi G\,(ds\,\sigma)$. Or, $\mathbf{E} = -2\pi G\sigma\,\hat{\mathbf{r}}$, where σ is the surface density of mass.

Numerical Problems :

1. Three identical balls, each of mass M are placed at the vertices of an equilateral triangle of side L. What should be the speed of their movement if they move under the action of one another's gravitational pull in a circular orbit circumscribing the triangle while maintaining the shape of equilateral triangle?

2. How does the acceleration due to gravity vary with latitude?

3. Establish the Kepler's laws of planetary motion from Newton's law of gravitation.

4. Three equal masses are placed at the vertices of a Cartesian right angled triangle in x-y plane. The co-ordinates of the vertices are (L, 0), (–L, 0) and (0, 2L). Calculate the gravitational potential and intensity at (0, L).

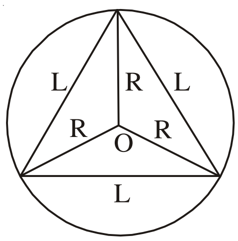
5. It is found that the acceleration due to gravity decreases with the depth inside a homogeneous solid sphere, but the acceleration due to gravity inside a mine is more than that at the surface of the earth, why?

6. Using Gauss's theorem find out the value of the gravitational intensity due to an infinitely long cylindrical wire.

7. Find out the gravitational self-energy of a homogeneous sphere.

Answers.

1. The resultant force on any one of the masses will be directed towards O, the circumcentre of the triangle ABC. The centripetal force is



$= 2G \frac{m^2}{L^2} \cos 30^0 = G \frac{m^2}{L^2} \sqrt{3}$. The equation of motion of any one of the masses

is $G \frac{m^2}{L^2} \sqrt{3} = m \frac{v^2}{R}$

where R can be obtained from cos 30^0 = L/2R, or R = L/$\sqrt{3}$. Now, we get

v^2 = GM/L, so v = $\left(\frac{GM}{L}\right)$

2. Let an object of mass m is placed at a place on the surface of the earth where latitude is λ. With the rotation of the earth, the object travels in a circle of radius r (different from the radius R of the earth with an angular velocity ω same as that of the earth. Then, $mg_\lambda = mg_0 - m\omega^2 r \cos\lambda$.

or, $g\lambda = g_o - \omega^2 R \cos^2\lambda$, because $r = R \cos\lambda$.

3. See text

4. Let the masses be equal to M. Gravitational potential

$$V = -\frac{GM}{L} - \frac{\sqrt{2}\,GM}{L} = -GM\frac{(1+\sqrt{2})}{L}$$

$$\text{Inensity}\,E_X = -\frac{GM}{2L^2}\times\frac{1}{\sqrt{2}} + \frac{GM}{2L^2}\times\frac{1}{\sqrt{2}} = 0$$

$$E_y = \frac{GM}{2L^2}\;(1 - 1\sqrt{2}).$$

5. It can be shown that the acceleration due to gravity at a depth d is $g_d = \frac{4}{3rG\rho}$ (R – d). So, we see that g should decrease with the depth. But for the Earth density is not uniform, it increases inside the earth. That is why the acceleration due to gravity increases inside the earth.

6. We assume a co-axial cylindrical surface (Gussian surface) of radius r and height h around the wire. Gravitational intensity **f** at any point on the curved surface of the outer cylinder will be same everywhere and will be directed inward. The

assumed Gaussian surface has three parts – i) top, ii) curved side and iii) bottom. The normal to these surfaces are a) vertically upward, b) radially outward and c) vertically downward. Therefore,

$\iint f.ds = \int_{curvedsurface} f.ds$, as there is no contribution from the top and bottom surface as **f** and **ds** are perpendicular to each other. So, we get

$\iint$ f.ds = f 2 πrh. Now, applying Gauss's theorem

f 2πrh = –4πG (hm), where m is the mass per unit length of the wire,

Therefore, $f = -\frac{2GM}{r}$, or $f = -\frac{2GM}{r\,\hat{r}}$

7. Let us consider a homogeneous sphere of mass M, radius R and density ρ. By self energy of this sphere we mean the work done by the attractive forces of the particles constituting the sphere, when they are brought from an infinite distance to the particular position in the system.

Suppose at some point in the formation of the sphere, a small sphere of radius r has already formed. The potential V_r on its surface is $-\frac{GM_1}{r} = -G\left(\frac{4}{3}\right)\pi\rho r^2$. The amount of work to be done in bringing in a further mass dm from infinity on the surface of the existing sphere

dW = $V_r$dm. Let us assume this dm forms a thin shell of thickness dr on the surface of our old sphere of radius r. Then dm = $4\pi\rho r^2$ dr. Then,

$$dW = V_r dm = -G\left(\frac{4}{3}\right)\pi\rho r^2\, 4\pi\rho r^2\, dr = -\frac{1}{3}\, G(4\pi\rho)^2 r^4\, dr$$

Thereore, total work in forming a sphere of radius R is

$$W = \int_{r=0}^{r=R} V dm = -\left(\frac{1}{3}\right) G(4\pi\rho)^2 \int_0^R r^4\, dr = -\frac{3}{5}\frac{GM^2}{R}.$$

Unit-4 ❑ Central Force Motion

Structure :

3.4.1 Proposal

From his understanding of the theory of motion Sir Issac Newton felt that the Sun occupies the central role for governing the motion of planets. He proved the very fact that constancy of the areal velocity is a direct consequence of the idea that all the forces are directed exctly toward the Sun or in other words the force behind the motion of the planets is central in character. By analysing Kepler's third law it is possible to prove that the strength of the force weakens with distance – larger the distance, weaker the force becomes. By combining these two laws Newton came to a conclusion that there must be a force inversely proportional to the square of the distance, directed in a line between two objects.

Now we all know that the above situation culminated into proclamation of Universal Law of Gravitation which is basically an example of central force problem. Later we came across another famous example of central force problem – Coulomb's law, characterising the force operating between two differently charged bodies – different in amount and different in amount and different in nature. These phenomena drew attention of the physicists working all over the world about the features of particles moving under a central force.

In the discussion on Central Force problem, we will examine a mathematically tractable and physically useful problem – that of two bodies interacting wih each other through a force that has two characteristics: (a) it depends only on the separation between the two bodies, and (b) it points along the line connecting the two bodies. Such a force is called a central force.

❑ Outcome

By learing this chapter, you shall be able to

- Understand the nature of central forces.
- Understand the physics behind the motion of particles moving under the action of a central force – like potential, intensity of the force field.
- Derive the equation of the orbit of a particle moving in a central force field.
- Explain the conservation principles relevant to the central force field.
- Derive Kepler's laws from Newton's law of gravitation and vice-versa.

3.4.1 Central Force

Central Forc is that force which is always acting along some space point and whose magnitude is a function of distance from the space point.

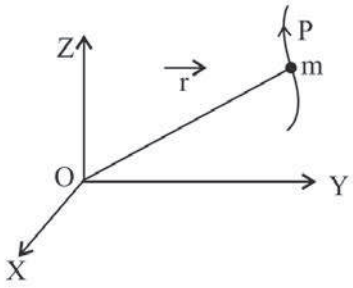


Figure : (3.4.1)

The fig. (3.4.1) shows a particle of mass m undergoing motion under a central force abour the point 'O' called **Center of force**, chosen here to be at the origin of reference frame. P be the position of particle at time t. Then following the definition of central force, the force equation of the particle can be written as,

$\vec{F} = f(r)\hat{r}$ .... (3.4.1)

If f(r) is positive the force on the particle is repulsive and it is attractive when f(r) is negative.

For example in case of gravitation $f(r) = -\frac{GMm}{r^2}$ represents an attractive force.

For electrostatic force $f(r) = k\frac{q_1q_2}{r^2}$ can be both attractive and repulsive depending on the sign of charges.

3.4.2 Some importnt properties of Central Force

1. <u>Central force is conservative</u>

To show this we workout $\vec{\nabla}\times\vec{F}$. Now $\vec{\nabla}\times\vec{F}$

$$= \vec{\nabla}\times(f(r)\hat{r}) = \vec{\nabla}\times\left(\frac{f(r)}{r}\vec{r}\right) = \vec{\nabla}\times\left[G(r)\vec{r}\right]$$

$$= G(r)\vec{\nabla}\times\vec{r} + \vec{\nabla}G(r)\times\vec{r}.$$

$$\vec{\nabla}\times\vec{F} = G(r)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} + \left(\hat{i}\frac{\partial G}{\partial x} + \hat{j}\frac{\partial G}{\partial y} + \hat{k}\frac{\partial G}{\partial z}\right)\times\vec{r}$$

$$= 0 + \frac{\partial G}{\partial r}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]\times\vec{r}$$

$$= \frac{\partial G}{\partial r}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]\times\vec{r}$$

$$\therefore\ \vec{\nabla}\times\vec{F} = \frac{1}{r}\frac{\partial G}{\partial r}(\vec{r}\times\vec{r}) = 0 \text{ ....(3.4.2)}$$

Thus central orce is curlfree, hence conservative.

2. Angular momentum in Central Force is conserved

Let a particle of mass is undergoing C.F. motion,

then the force equation

$$\vec{F} = f(r)\vec{r}$$

$$\vec{r}\times\vec{F} = 0, \text{ so, } \vec{r}\times m\frac{d\vec{v}}{dt} = 0$$

or, $$\frac{d}{dt}(\vec{r}\times m\vec{v}) = 0$$

since $\frac{d\vec{r}}{dt}\times m\vec{v}=\vec{v}\times m\vec{v}=0.$

So, $\vec{r}\times m\vec{v}=\vec{L}$ ...... is a constant vector. (3.4.3)

The equation (3.4.3) shows that angular momentum remains conserved in C.F. motion.

3. The motion is planar.

From equation (3.4.3), $\vec{r}\times\vec{v}=\frac{\vec{L}}{m}=\vec{h}$ (say)

So, $\vec{r}.(\vec{r}\times\vec{v})=0$, which implies that the motion is always confined in a plane perpendicular to the direction of $\vec{r}\times\vec{v}$.

4. Areal Velocity is constant in C.F. motion

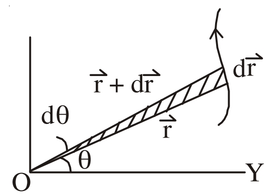


Figure : (3.4.2)

We have already deduced that in C.F. motion

$$\vec{r}\times\vec{v}=\vec{h}$$

Or, $\vec{r}\times\frac{d\vec{r}}{dt}=\vec{h}$

$$\frac{1}{2}\vec{r}\times\frac{d\vec{r}}{dt}=\frac{\vec{h}}{2}.$$

$\frac{1}{2}\vec{r}\times d\vec{r}$ represents vectorial area of the shaded region.

or, $\frac{d\vec{A}}{dt} = \frac{\vec{h}}{2}$ = constant vector. .... (3.4.4)

which shows that area velocity i.e. area swept out by radius vector is a constant of motion, which is the Kepler's second law in mathematical form.

3.4.3 Reduction of two body problem to One body problem

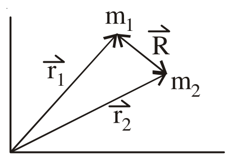


Figure : 3.4.3

We consider two particle of masses m_1 and m_2 at position vectors $\vec{r}_1$ & $\vec{r}_2$ as in Fig. Then force equation of masses are, $m_1\ddot{\vec{r}}_1 = \vec{F}_{12} + \vec{F}_{1e}$ and $m_2\ddot{\vec{r}}_2 = \vec{F}_{21} + \vec{F}_{2e}$

where $\vec{F}_{12}$ and $\vec{F}_{21}$ are the force of interaction between the masses and $\vec{F}_{1e}, \vec{F}_{2e}$ are external forces acting on masses m_1 and m_2.

So acceleration of m_2 with respect to m_1

$$\left(\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1\right) = \frac{\vec{F}_{21}}{m_2} + \frac{\vec{F}_{2e}}{m_2} - \frac{\vec{F}_{12}}{m_1}\frac{\vec{F}_{1e}}{m_1}$$

$$= \vec{F}_{21}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \left(\frac{\vec{F}_{2e}}{m_2} - \frac{\vec{F}_{1e}}{m_1}\right).$$

New if the external force is proportional to the mass then the last term within braces vanishes. Thus we have $\mu\ddot{\vec{R}} = \frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{R}} = \vec{F}_{21}$ = force on particle 2 due to particle 1. $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass.

3.4.4 Expression of Velocity & Acceleation in Polar Co-ordinate (A recapitulation)

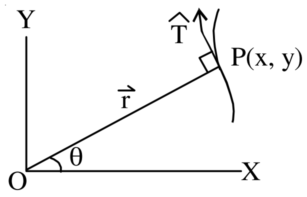


Figure : (3.4.4)

The Fig. (3.4.4) show a particle p undergoing a planar motion (Remember C.F. motion is planar) in x, y plane. $\vec{r}$ be the position vector at any instant t.

Then, $\vec{r} = r\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right) = r\hat{r}$ .... (3.4.5)

($\vec{r}$ i the unit vector along $\vec{r}$).

So, the velocity at that instant,

$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + \dot{\hat{r}} \quad \ldots\ldots\ldots \qquad (3.4.6)$$

Since $\dot{\hat{r}} = \left(-\hat{i}\sin\theta + \hat{j}\cos\theta\right)\dot{\theta} = \left[\hat{i}\cos\left(\frac{\pi}{2}+\theta\right) + \hat{j}\sin\left(\frac{\pi}{2}+\theta\right)\right]\dot{\theta}$

= $\dot{\theta}\hat{T}$ (where $\hat{T}$ is a unit vector perpendicular to $\hat{r}$) ....(3.4.7)

So, $\boxed{\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{T}} = \vec{v}_r + \vec{v}_T$ [velocity has both radial and transver components] (3.4.8)

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{T} + r\ddot{\theta}\hat{T} + r\dot{\theta}\dot{\hat{T}}$$

Now, $\hat{T} = -\hat{i}\sin\theta + \hat{j}\cos\theta$, so, $\dot{\hat{T}} = \left(-\hat{i}\sin\theta + \hat{j}\sin\theta\right)\dot{\theta} - = -\dot{\theta}\hat{r}$

Using the values of $\hat{T}$ and $\dot{\hat{r}}$, we have,

$$\ddot{\vec{r}} = \boxed{\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{T}} \quad \ldots\ldots\ldots \qquad (3.4.9)$$

$$= a_r\,\hat{r} + a_T\,\hat{T}$$

($a_r \equiv$ radial component of acceleration ; $a_T \equiv$ transverse component of acceleration).

3.4.5 Equation of Motion in Central Force

In central field $\vec{F} = f(r)\hat{r}$, i.e. force is always along $\vec{r}$. So from equation (3.4.9) we can write.

$$m\left(\ddot{\vec{r}} - r\dot{\theta}^2\right) = f(r) \quad \ldots\ldots\ldots \qquad (3.4.10)$$

$$2m\dot{r}\dot{\theta} + mr\ddot{\theta} = 0 \;=\; \ldots\ldots\ldots \qquad (3.4.11)$$

Equations (3.4.10) and (3.4.11) represent the eqn. of a particle undergoing C.F. motion in polar co-ordinate.

Now, $2m\dot{r}\dot{\theta} + mr\ddot{\theta} = 0$ implies

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$$

Or, $\dfrac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

So, $mr^2\dot{\theta} = L = \text{constant.} \quad \ldots\ldots\ldots \qquad (3.4.12)$

The equation 8 is the scalar equivalent of th conservation of angular momentum and L stands for angular momentum.

3.4.6 The energy of a particle in Central Force

Since the field is conservative the total energy

$$E = \frac{1}{2}mv^2 + \phi = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \phi$$

$$= \frac{1}{2}m\dot{r}^2 + \phi_e \text{ where } \phi_e = \frac{1}{2}mr^2\dot{\theta}^2 + \phi$$

$$= \frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} + \phi\right), \ \phi_e = \frac{L^2}{2mr^2} + \phi$$

$$E = \frac{1}{2}m\dot{r}^2 + \phi_e \ .... \quad (3.4.13)$$

Now if we confine to the inverse square field only,

then $\phi = \frac{K}{r}\left(\text{As } F = \frac{K}{r^2}\right)$, If K is + ve the force is repulsive and – ve if the force is attractive.

For positive K, K > 1 the total energy is positive. $E_T = E + \phi_e = + v_e$ constant. ϕ_e is zero at infinity and gradually increases as r decreases.

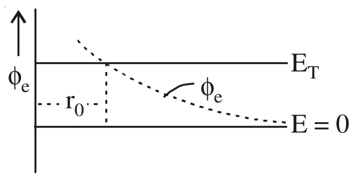


Figure : (3.4.5)

Here in the plot the bold line represents the total energy and the dotted line shows the variation of ϕ_e.

The graph shows the particle can come upto r_0 because if $r < r_0$, ϕ_e will be greater than E_T, which will violate the energy conservation. The orbit is unbounded and r_0 is the turning point, for K = 0, ϕ_e and E is + Ve. So here also the orbit is unbounded and the sketch of ϕ_e and E is similarto that of k > 0`

When K < 0, we can consider three cases of total energy.

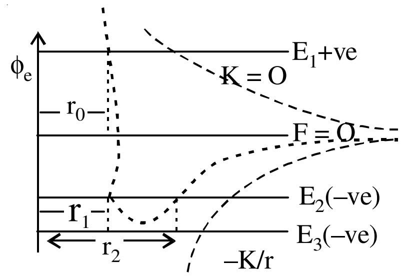


Figure : 3.4.5

for E_1 + Ve, there is a turning point at $r = r_0$ and the orbit is unbounded.

When E is – Ve, say E_2, then there are two turning points so the orbit is bounded between r_1 and r_2.

When E = E_3, then also the orbit will be bounded and circular and of radius $r = \frac{L^2}{mK}$.

3.4.7 The equation of Orbit

To find the equation of orbit we have to find the relation between r and θ in polar co-ordinates. In the process of findng the relation we put,

$$r = \frac{1}{u},$$

So, $\dot{r} = -\frac{1}{u^2}\frac{du}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{L}{mr^2}$ (using equation 3.4.8)

i.e. $\dot{r} = -\frac{L}{m}\frac{du}{d\theta}$, L is the angular momentum of the orbital particle.

So, $\ddot{r} = \frac{-L}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right) = \frac{-L}{m}\frac{d^2u}{d\theta^2}\cdot\frac{L}{r^2} = \frac{-L^2u^2}{m}\frac{d^2u}{d\theta^2}$

Using the above equation in equation (3.4.6) we have

$$m\left(\frac{-L^2u^2}{m^2}\frac{d^2u}{d\theta^2}-\frac{1}{u}\frac{L^2u^4}{m^2}\right)=-f\left(\frac{1}{u}\right)$$

$$\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=-f\left(\frac{1}{u}\right)$$

or, $$\frac{d^2u}{d\theta^2}+u=-\frac{m}{L^2u^2}\ f\left(\frac{1}{u}\right)\ \ldots\ldots \qquad (3.4.14)$$

This equation is known as differential equation of orbit.

Now if we confine to the inverse square attractive field only, then F can be written

as $F=-\dfrac{K}{r^2}=-Ku^2$ (In case of gravitation K = GMm)

So, $$\frac{d^2u}{d\theta^2}+=\frac{+mK}{L^2}$$

i.e. $\dfrac{d^2u}{d\theta^2}=-\left(u-\dfrac{mK}{L^2}\right)$, we put $U=u\dfrac{-mK}{L^2}$

So, the equation yields to the form,

$$\frac{d^2U}{d\theta^2}=-U\ \ldots\ldots\ldots(3.4.15)$$

which is a equation of simple harmonic function of U.

So the soln. can be written as

$$U=u-\frac{mK}{L^2}=a\cos(\theta-\theta_0)\quad \text{(a is a constant)}$$

or, $\frac{1}{r} = \frac{mK}{L^2} + a\cos(\theta - \theta_0)$.

By a suitable choice of co-ordinates we can make $\theta_0 = 0$.

So, the equation boils down to

$$\frac{d^2u}{d\phi^2} = 0 .$$ where is a constant.

So, $$r = \frac{1}{\frac{mK}{L^2} + a\cos\theta} = \frac{\frac{L^2}{mk}}{1 + \frac{aL^2}{mK}\cos\theta}$$

$$\boxed{r = \frac{l}{1 + \in \cos\theta}} \quad \ldots\ldots\ldots \qquad (3.4.16)$$

Where $l = \frac{L^2}{mk}$ and $\in = \frac{L^2}{mK}$. .... (3.4.17)

The equation (3.4.16) is a general equation of a conic section whose shape obviouly depends on $\in$ called the eccentricity.

When (1) $\in = 0$, the path is a circle, as $r = l$ = constant.

(2) $\in = 1$, the path is a parabola, as $r \rightarrow \infty$, as $\theta \rightarrow \infty$

(3) $\in > 1$, the path is a hyperbola as $r \rightarrow \infty$ for a value of $\theta < \pi$.

(4) $\in < 1$, r remains finite form $O < \theta < 2\pi$,

so the path is in general an ellipse.

Now, let us confine to the study of elliptic orbit and the factors which determine the shape of ellipse.

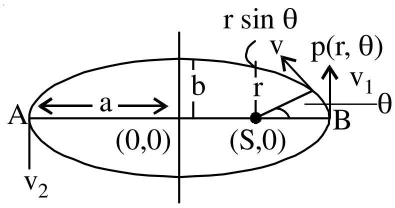


Figure : (3.4.7)

The figure shows a particle of mass m undergoing an elliptic path in a force field acting along pS. The force is a inverse square force. So, this study gives us the study of planetary motion. Now in Fig. (3.4.7)

A and B are called two apse points, where the velocity is perpendicular to the radius vector. Now $r_1 = SB = \frac{l}{(1+\epsilon)}$ and $SA = \frac{l}{(1-e)}$, so the major axis AB = SB + SA = $\frac{2l}{(1-e^2)}$ = 2a (say)

So the semi-major axis

$$a = \frac{\ell}{(l-\varepsilon^2)} \quad \ldots\ldots \tag{3.4.18}$$

Now the semi minor axis OB = b, corresponds to the maximum value of r sinθ.

So, in this case, $\frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}\left(\frac{l}{1+\epsilon\cos\theta}\cdot\sin\theta\right) = 0$

$$\therefore \quad \frac{l\cos\theta}{1+\epsilon\cos\theta} - \frac{l\sin\theta}{(1+\epsilon\cos\theta)^2}(-\epsilon\sin\theta) = 0$$

Or, $\quad l\cos\theta + l\epsilon\cos^2\theta + l\epsilon\sin^2\theta = 0$

Or, $\cos\theta = -\in$. Putting this value of cosθ.

we have, $b = r\sin\theta = \frac{l}{1-\in^2}\sqrt{\left(1-\in^2\right)} = \frac{l}{\sqrt{1-\in^2}}$ ....(3.4.19) Comparing this with equation (3.4.18) We have

$$a = b\sqrt{1-\in^2} \quad ...... \qquad (3.4.20)$$

Since it is a conservative field the total energy will remain conserved. To calculate the total energy, we focus on the Apse point B; where $\dot{r} = 0$, The total energy,

$$E = \frac{1}{2}mr_1^2\dot{\theta}^2 - \frac{K}{r_1} = \frac{1}{2m}m^2\frac{r_1^4\dot{\theta}^2}{r_1^2} - \frac{K}{r_1}$$

$$= \frac{1}{2m}\frac{L^2}{l^2}(1+\in)^2 - \frac{K}{l}(1+\in) = \frac{1}{2}\cdot\frac{K}{l}(1+\in)^2 - \frac{K}{l}(1+l)$$

$$= \frac{1}{2}\frac{K}{l}(1+\in)\left[(1+\in)-2\right] = \frac{K}{2l}\left(\in^2 - 1\right)$$

$\therefore\ E = \frac{K}{2l}\left(\in^2 - 1\right)$, as $\in \ll 1$, So E is negative

$$\in = \sqrt{1+\frac{2lE}{K}} = \sqrt{1+\frac{2L^2E}{mK^2}} \quad .... \qquad (3.4.21)$$

3.4.8 The Kepler's Laws

We have alredy seen that when E is negative and $\in < 1$, then the motion is elliptical, which obviouly states the Kepler's first Law.

Planets move around its own Sun in elliptic orbits with Sun at one of its foci.

Again we have already proved that

$\frac{1}{2}(\bar{r}\times\bar{v}) = \frac{d\vec{A}}{dt}$ = areal velocity is constant for a planet, which proves the Kepler's second law.

The areal velocity of a planet is a constant of motion. Now the period of rotation of a planet.

$$T = \frac{\text{area of the elliptic path}}{\text{areal velocity}}$$

$$= \frac{\pi ab}{\frac{h}{2}} \quad \{\text{eqn. .... (3.4.4)}\}$$

$$= \frac{2\pi ab}{\frac{h}{2}} = \frac{2\pi mab}{L} = \frac{2\pi ma\sqrt{al}}{L}$$

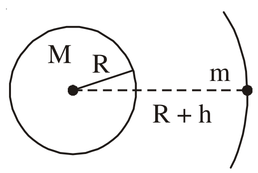
$$= 2\pi m a^{\frac{3}{2}} \frac{\sqrt{l}}{L} = \frac{2\pi m a^{\frac{3}{2}}}{\sqrt{Km}}$$

$$\therefore\ T^2 = \frac{4\pi m}{K} a^3 \ \therefore\ T^2 \propto a^3 \quad .... \qquad (3.4.22)$$

Which is Kepler's 3rd law, which states that, the square of the time period of the planet is proportional to the cube of the semi-major axis.

3.4.9 Artificial Satellite

A satellite launched from the Earth to orbit around the earth in specific orbit is known as artificial satellite.



We consider a satellite of mass m revolving around the Earth's center. Earth is taken to be a huge mass M compared to the satellite of mass m. (Otherwise m should be replaced by $\left(1+\frac{m}{M}\right)$. h be the height of satellite then, considering a circular orbit we can write, the centripetal force.

$$\frac{GMm}{(R+h)^2} = ma_c = \frac{mv_s^2}{(R+h)} = m\omega_s^2 (R+h)$$

Where V_s and W_s represent the linear speed and angular speed of satellite.

So $v_s = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{gR^2}{R+h}}$ (g = acceleration due to gravity on Earth's surface)

$$\omega_s = \sqrt{\frac{gR^2}{(R+h)^3}} \quad .... \tag{3.4.23}$$

from the above equations it is obvious that v_s, ω_s and thus obviously the period of satellite is independent of its mass.

A satellite which remains at a fixed point from the Earth's surface is called geostationary satellite. It is obvious that it will be on an equitorial satellite with same angular velocity as that of Earth.

If H be the height of a geostationary satellite

$$\text{Wearth} = \frac{2\pi}{24\,\text{hrs}} = \frac{2\pi}{86400} = \sqrt{\frac{gR^2}{(R+h)^3}}$$

or, $$(R + h)^3 = gR^2 \frac{(86400)^2}{4\pi^2}$$

$$\therefore \quad h = \left[\frac{gR^2(86400)^2}{4\pi^2}\right]^{\frac{1}{3}} - R \simeq 3.6\times10^4 \text{km}.$$

This satellite plays an important role in telecommunication, weather forecasting and geographical survey.

3.4.10 Escape Velocity

Escape Velocity of a body is the minimum velocity with which a second body is to be projected from the surface of the first body so that it can just reach infinity.

To calculate the escape velocity of a planet (say Earth), we consider Earth to be a perfect sphere of radius R and mass M. Let m be the second mass which has to be projected from the planet surface so that it can just reach infinity.

Now the total energy of mass m at infinity will be E = 0, since both the potential energy and kinetic energy are zero.

Now if V_e is the velocity with which the mass is thrown, then V_e will be the escape velocity. Then the total energy on planet's surface.

$$E = \frac{1}{2}mV_e^2 - \frac{GMm}{R},$$ and as the field is conservative E = 0, so

$$V_e = \sqrt{\frac{2GM}{R}} \quad ....$$

Altenative method :

Force of gravitation acting on the mass m when it is at a distance r from the center of Earth is

$$\vec{F} = \frac{-GMm}{r^2}\hat{r}$$

So the work done to displace it by $d\vec{r}$

$$dw = \frac{-GMm}{r^2}\hat{r}.d\vec{r} = +\frac{GMm}{r^2}dr$$

(since angle between $\vec{E}$ & $d\vec{r}$ is π)

So the work done in moving the mass from the surface of planet to infinity

$$W = \int_R^{\infty} \frac{GMm}{r^2}dr \text{ = change in kinetic energy.}$$

$$\frac{GMm}{R} = \frac{1}{2}mV_e^2 \text{ or, } V_e = \sqrt{\frac{2GM}{R}}.$$

3.4.11 Weightlessness in satellite

In case of a satellite (say artificial satellite) the gravitational pull due to satellite itself is negligible with any mass component attached to it. Obviously that mass and the satellite it self orbiting about the Earth will have centripetal acceleration a_c such that

$$\frac{GMm}{r^2} = ma_c$$

So $a_c = \frac{GM}{r^2}$ = acceleration due to gravity of Earth at planet point. So we can visualize that a setellite or a mass attatched to it, is a freely galling body a lways missing the center of earth.

As a freely falling body is weightless, so the sattellite and a mass attached to it weightless. However, if it is a massive satellite like moon, the body will have a sense of weight with respect to moon.

3.4.12 Numerical Problems

1. A spherical planet starts rotating faster. Find the limit of frequency of rotation upto which the surface particles will not fly off the planet.

   Solution

   If R is the radius of planet and p be its average density, then for a particle on the surface to be intact

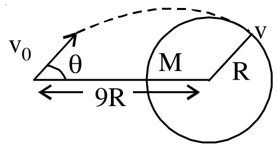
   $mg \geq m\omega^2R = m\ 4\pi^2\upsilon R$ (m = mass of particle)

   or, $$\upsilon \leq \sqrt{\frac{g}{4\pi^2 R}} = \frac{GM/r^2}{4\pi^2 R} = G\frac{4}{3}\frac{\pi R^3 p}{4\pi^2 R^3}$$

   So, $$\upsilon \leq \sqrt{\frac{Gp}{3\pi}}$$

2. A particle of mass m is thrown towards a spherical planet of radius R and mass M from a distance 9R from the center of the planet making an angle θ with the line joining the particle and the center of planet. v_0 be the velocity of projection. For what angle of projection the particle will just graze the planet?

   Solution



   From conservation of angular momentum $mv_0\ 9R \sin\theta = mvR$ ....(Ex. 2.1)

   From conservaion of energy $$\frac{-GMm}{9R} + \frac{1}{2}mv_0^2 = \frac{-GMm}{R} + \frac{1}{2}mv^2$$ .....(Ex. 2.2)

   or, $v^2 - v_0^2 = \frac{16GM}{9R}$ or, $\frac{v}{v_0} = \sqrt{1 + \frac{16GM}{9Rv_0^2}}$

Using in equation (Ex. 2.1 $\theta = \sin^{-1}\frac{1}{9}\sqrt{1+\frac{16GM}{9Rv_0^2}}$

3. The orbit of a particle is given by r = a(1 + cosθ).

   Find the nature of force

   Solution

$$u = \frac{1}{a(1+\cos\theta)}, \quad \therefore \frac{du}{d\theta} = \frac{1}{a}\frac{\sin\theta}{(1+\cos\theta)^2}$$

$$\therefore \frac{d^2u}{d\theta^2} = \frac{1}{a}\left[\frac{\cos\theta}{(1+\cos\theta)^2} + \frac{2\sin^2\theta}{(1+\cos\theta)^3}\right]$$

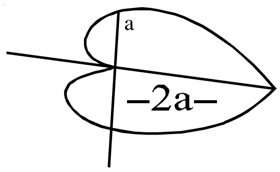
$$= \frac{1}{a(1+\cos\theta)^3}\left[\cos\theta + \cos^2\theta + 2\sin^2\theta\right] = \frac{1+\cos\theta+\sin^2\theta}{a(1+\cos\theta)^3}$$

No The radial component of acceleration

$$a_r = \frac{-L^2}{m^2}u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{-L^2}{m^2}\cdot\frac{1}{a^2(1+\cos\theta)^2}\left[\frac{1+\cos\theta+\sin^2\theta}{a(1+\cos\theta)^3} + \frac{1}{a(1+\cos\theta)}\right]$$

$$= \frac{-L^2}{m^2}\frac{1}{a^3(1+\cos\theta)^5}\left[1+\cos\theta+\sin^2\theta+1+2\cos\theta+\cos^2\theta\right]$$

$$= \frac{-L^2}{m^2}\frac{1}{a^3}\frac{3(1+\cos\theta)}{(1+\cos\theta)^5} = \frac{-3L^2}{m^2}\frac{a}{r^4}$$

So, $F = \frac{-K}{r^4}$. The trajectory 

4. Consider a pair of stars of equal mass M rotating about their common center of mass. The attraction between the stars is gravitational and they keep a separation ℓ between them. Show that the time period of rotation of this double star system is $\pi l\sqrt{\frac{2l}{GM}}$ [C.U.]

Solution

$$\frac{GMM}{l^2} = \mu a = \frac{M.M}{M+M}\omega^2 l = M\omega^2 \frac{l}{2}$$

$$\omega = \sqrt{\frac{2GM}{l^3}} \quad \therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l^3}{2GM}}$$

$$= \pi l\sqrt{\frac{2l}{GM}}.$$

5. A particle of mas m describes an elliptic orbit about a central field $\vec{F} = -\frac{k}{r^2 \hat{r}}$.

Show that the speed of the particle at any point of orbit is given by $v = \sqrt{\frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)}$, where a = semi major axis.

Solution

Total energy $E = \frac{-k}{2a} = \frac{1}{2}mv^2 - \frac{K}{r}$

Or, $v = \sqrt{\frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)}$

6. Show that total energy of a particle undergoing an elliptic orbit in an inverse square c.f force $\vec{F} = \frac{-K}{r^2}\hat{r}$.

Solution

As the field is conservative the total energy E will be conserved through out the trajectrory. For simplicity we calculate the energy at Apse.

$$E = \frac{1}{2}mr_1^2\dot{\theta}^2 - \frac{k}{r_1} = \frac{1}{2}\frac{L^2}{mr_1^2} - \frac{k}{r_1} = \frac{1}{2}\frac{L^2}{l^2}(1+\in)^2 - \frac{k}{l}(1+\in)$$

$$= \frac{1}{2}\frac{mk}{ml}(1+\in)^2 - \frac{k}{l}(1+\in)$$

$$= \frac{1}{2}\frac{k}{l}(1+\in)(\in -1) = \frac{1}{2}\frac{k}{l}\left(\in^2 -1\right) = -\frac{k}{2a}$$

7. A planet revolves around a star in an elliptic orbit. The ratio of the farthest distance to the closest one of the planet from the star is 4:1. Find the ratio of the kinetic energy of the planet at the furthest to the closest positions. (Gate)

Solution

$$K.E = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}\frac{mr^4\omega^2}{r^2} = \frac{1}{2}\frac{L^2}{mr^2}$$

$$(K.E)_f : (K.E)_c = \frac{r_c^2}{r_f^2} = 1:16$$

8. If a particle follows a spiral orbit given by $r = c\theta^2$ under an unknown force. Prove that such an orbit is possible in central field. Also find the form of the force law. (Cal Univ.)

Solution

$$u = \frac{1}{c\theta^2} \quad \therefore \quad \frac{du}{d\theta} = -\frac{2}{c\theta^3} \quad \text{and} \quad \frac{d^2u}{d\theta^2} = \frac{6}{c\theta^4}$$

If it follows the central field motion then

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} F\left(\frac{1}{u}\right)$$

$$F(r) = F\left(\frac{1}{u}\right) = \frac{-L^2u^2}{m}\left(\frac{6}{c\theta^4} + \frac{1}{c\theta^2}\right) = \frac{-L^2}{m}\left(\frac{1}{r^3} + \frac{6c}{r^4}\right)$$

This is the nature of force.

9. A particle moves in circular orbit obeying the inverse square law. Show that the orbits of different radii, the angular momentum of the particle about the center of mass varies as the square root of the radius and the total energy varies inversely as the radius. (Cal Univ.)

Solution

We consIder a particle of mass m circling about a mass M in a radius r. If μ is the reduced mass of m, then

$$\frac{\mu v^2}{r} = \frac{k}{r^2}$$

or, $v = \sqrt{\frac{k}{\mu r}}$, so the angular momentum

$$L = \mu r v = \sqrt{\frac{k\mu^2 r^2}{\mu r}} = \sqrt{\mu k r}, \text{ so } L \propto \sqrt{r}$$

So, total energy $E = \frac{1}{2}\mu v^2 - \frac{k}{r} = \frac{k}{2r} - \frac{K}{r} = \frac{-k}{2r}$

$\therefore E \propto \frac{1}{r}$ proved.

10. A particle moving under a central force describes a spiral orbit given by $r = ae^{b\theta}$, where a and b are constant. Obtain the force law. (Guru Nanak Univ.)

Solution

$$u = \frac{1}{a}e^{-b\theta}, \text{ so } \frac{du}{d\theta} = \frac{-b}{a}e^{-b\theta}, \therefore \frac{d^2u}{d\theta^2} = \frac{b^2}{a}e^{-b\theta}$$

from the differential equation of orbit (3.4.14)

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F\left(\frac{1}{u}\right)$$

$$F(r) = F\left(\frac{1}{u}\right) = \frac{-L^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{-L^2}{mr^2}\left(\frac{b^2}{a}e^{-b\theta} + \frac{1}{a}e^{-b\theta}\right)$$

$$= \frac{-L^2}{mr^2}\left[\frac{b^2}{r} + \frac{1}{r}\right] = \frac{-L^2}{m}\frac{\left(1+b^2\right)}{r^3}$$

$$F(r) = \frac{-L^2}{m}\left(1+b^2\right)\frac{1}{r^3} \quad \therefore \; F \times \frac{1}{r^3}$$

11. The motion of a particle under the influence of a central force is described by : $r = a \sin\theta$. Find the expression of force.
(Guru Nanak Univ.)

Solution

we put $u = \frac{1}{r} = \frac{1}{a}\operatorname{cosec}\theta$, so $\frac{du}{d\theta} = -\frac{\operatorname{cosec}\theta\cos\theta}{a}$

and $\frac{d^2u}{d\theta^2} = -\frac{1}{a}\left[-\operatorname{cosec}\theta\cot^2\theta - \operatorname{cosec}^3\theta\right]$

$$= +\frac{1}{a}\operatorname{cosec}\theta\left(\cot^2\theta + \operatorname{cosec}^2\theta\right)$$

So using equation (3.4.14)

$$F(r)=F\left(\frac{1}{u}\right)=-\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$= F(r)=F\left(\frac{1}{u}\right)=-\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)$$

$$= \frac{-L^2}{mr^2}\left[\frac{1}{a}\cos ec\theta\left(\cot^2\theta+\cos ec^2\theta\right)+\frac{1}{a}\cos ec\theta\right]$$

$$= \frac{-L^2}{mar^2}\cdot\left[\cos ec\theta\left(1+\cos ec^2\theta+\cot^2\theta\right)\right]$$

$$= \frac{-L^2}{mar^2}\cdot\left[2\cos ec^3\theta\right]=\frac{-2L^2a^2}{mr^2}\frac{1}{r^3}$$

$$F=\frac{-2L^2a^2}{m}\frac{1}{r^5} \qquad F\propto\frac{1}{r^5}.$$

12. A particle moves along an orbit r = A cos ϕ under the influence of a central field F(r). Find the r dependence of force.
(Cal. Univ.)

Solution See problem – 11 Ans. $F(r)\propto\frac{1}{r^5}.$

13. Calculate th maximum velocity with which a body may be projected so that it may beome a satellite of Earth. Show that it is $\sqrt{2}$ times the minimum velocity of projection for a circular orbit close to the earth.
(Pune Univ.)

Solution The total energy of a satellite at a height h from earth's surface,

$$E=\frac{1}{2}mv^2-\frac{GMm}{R+h}, \text{ R = radius of earth,}$$

M = mass of Earth, m = mass of satellite. Now to be a satellite E must be – ve.

$$\text{So} \quad v < \sqrt{\frac{2GM}{R+h}} = \sqrt{\frac{2gR^2}{R+h}}$$

So, $v_{max} = \sqrt{2gR}$ ....

Now the minimum velocity of a satellite is such that

$$\frac{mv_{min}^2}{R} = \frac{GMm}{R^2}$$

Or, $v_{min} = \sqrt{2g}$

Sso, $v_{max} : v_{min} = \sqrt{2} : 1$

14. If v_A be the velocity of a planet at its perihebion what will be its velocity at its aphelion.

Solution

From conservation of angular momentum

$$mr_A v_A = m.r_B v_B$$

$$\frac{1}{1+\in} v_A = \frac{1}{1-\in} v_B \quad \therefore \quad v_B = \frac{1-\in}{1+\in} v_A$$ [Ret. 4 (vii)

Substance :

the study of this chapter has enabled you to understand the characteristics of a central foce. The particles moving under a central force obey Kepler's laws and some physical quantities are conserved for this kind of motion. There are many areas left to explore if you are interested: questions of the stability of orbits under perturbations, the precession of the orbit, and whether it is open or closed. There are many interesting examples, even within our solar system, that show the varied and unique outcomes of central force interctions. Central forces can be attractive or repulsive in nature.

3.4.13 Short Questions :

1. Show that the effective potential energy of a particle of mass under the action of a central force is given by $V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$, wher L is the angular momentum of the particle.
2. Prove that the total energy of a particle of mass m moving under the action of a central force is given by $E = \frac{L^2}{2m}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right]$ + V (r), where V(r) is the potential energy and L is the angular momentum of the body. $U = \frac{1}{r}$, r and ϕ are the polar co-ordinates of the particle.
3. Find the effective potential and equation of the orbit for a particle moving under the action of an attractive central force field given by $F(r) = -\frac{k}{r^3}$. What happens if L^2 becomes equal to mk?

3.4.14 Solution :

1. Eq. of motion gives $m\frac{d^{2r}}{dt^2} = f(r) + \frac{L^2}{mr^2} = -\frac{d}{dr}\left(V + \frac{L^2}{2mr^2}\right) = -\frac{dV_{eff}}{dr}$ which shows the value of V_{eff}.

2. $r = r = \frac{1}{u}, \frac{dr}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\phi}\right)\left(\frac{d\phi}{dt}\right) = -\frac{L}{m}\left(\frac{du}{d\phi}\right)$

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V(r) = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right] + V(r)$$

Using the value of $\left(\frac{dr}{dt}\right)$, we get E = $\frac{L^2}{2m}\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right] + V(r)$

3. Potential V(r) = – $\frac{k}{2r^2}$, Effective potential $V_{eff}(r) = \frac{1}{2}\left[\frac{L^2}{m-k}\right]\frac{1}{r^2}$

Eq.of the orbit is $\frac{d^2u}{d\phi^2} + u = -\left[\frac{m}{L^2u^2}\right]$ [– ku^3]

or, $\frac{d^2u}{d\phi^2} + \left[1 - \frac{mk}{L^2}\right]u = 0.$

In case of L^2 = mk, $\frac{d^2u}{d\phi^2} = 0.$ or, u = Aϕ + B.

Unit-5 ❑ Elasticity

Structure :

3.5.1 Proposal

In general we talk about rigid bodies while considering their motion. Elasticity is that property of rigid bodies by which they regain or try to regain its original shape or volume or length when the balanced forces causing the change in shape, volume or length are withdrawn. Obviously the forces must be small enough to make the changes temporary. This chapter deals with the elastic properties of different materials under different force conditions and finds interrelationships amongst the different elastic constants of solid bodies. In connection with this the torsional rigidity of materials and bending of beams of different materials under various force conditions will also be discussed.

❑ Outcome

After reading this chapter you should be able:

- to know about one of the most important properties of matter, namely elasticity.
- to learn about elastic bahaviour of those substances which have the property of recovering their size and shape when the forces producing the deformation are withdrawn.
- to discover the relationships among various elastic constants of different materials.
- to understand the physics behind the torsion of a cylinder.
- to develop the logic of calculating bending of beams of different objects.

3.5.2 Elasticity, Stress and Strain

When a piece of material is under the action of balanced forces, the material is deformed. If the forces are small, if the relative displacements of the various points in the material are proportional to the forces we say the behaviour is elastic. If all the parts of the material have identical properties in all respects–the material is said to be homogeneous. Again if the properties of the material are same in all directions–the material is isotropic. We shall consider only homogeneous and isotropic materials which are in stable equilibrium, i.e. the net force acting on the body is zero and net moment of the forces is also zero.

It we apply two equal but oppositely directed forces on a solid homogeneous and isotropic cylinder of length L_0, we find that there can be either expansion by the amount ΔL depending on the direction of the forces as can be seen by the diagram 3.5.1. Here the only point to be considered is that the forces must be small enough causing no damage of the material.

The ratio $\frac{\Delta V}{V}$, is known as longitudinal strain.

By similar fashion we can find out the ratio of change in volume of the material to its original volume $\frac{\Delta V}{V}$, which is known as volume strain.

This strain (either longitudinal or volume) is caused by the external force and it generates an internal force due to intermolecular interaction which actually brings the object to its original length, shape or volume when the external force is removed. Under equilitbrium condition, the external force is equal to the internal force and oppositely directed. This internal force developed within the materials per unit area is known as stress. Stress is determined by the following equation:

Stress = Opposing force of intermolecular origin, Area = $\frac{F}{A}$, where A is the area of cross-section of the mater

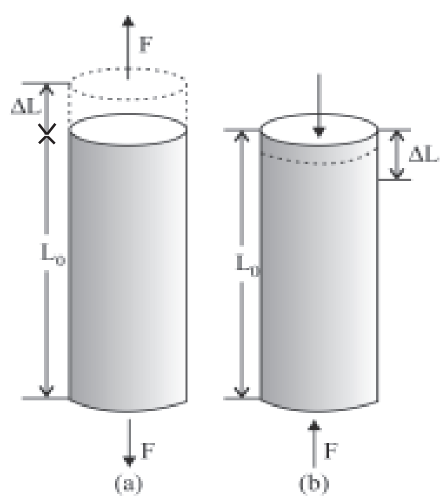


Figure : 3.5.1 Effect of tensile or longitudinal stresses on a cylindrical system

The strain has no dimension as it is a ratio of change in length over original length or change in volume over original volume. The stress has a dimension of $ML^{-1}T^{-2}$ and the unit of stress in SI system is Nm^{-2}.

When the deforming forces produce an actual change in the shape of the body, then the strain produced in the body is called shear strain. Shear strain is defined as the ratio of relative displacement of any layer to its perpendicular distance from the fixed layer. $\tan\theta = \frac{w}{L}$ (Fig.3.5.2). In passing it may be mentioned here that shearing stress is equivalent to an equal linear tensile stress and an equal compression stress at right angles to each other.

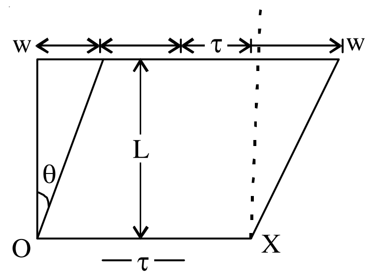


Figure : 3.5.2 Shear strain produced in a square object

3.5.3 Elastic Limit

The question that comes to our mind is whether every object regains its original length, shape or volume when the deforming stresses are removed. The answer to this question can be obtained by studying the stress-strain curve of that sample as shown in Fig. 3.5.3.

Under the action of a gradually increasing stress which is equal to external force developed per unit area at equilibrium, the behaviour of a substance is represented by its nominal stress-strain curve. In the nominal stress-strain curve the reduction of the cross-section of the material is neglected. The nominal stress-strain curve for different materials is different from each other. The strain is shown as the percent elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behaviour with stress directly proportional to strain. This straight-line ends at point A ; the stress at this point is called the proportional limit. From A to B, stress and strain are no longer proportional, and Hooke's law is *not* obeyed. If the load is gradually removed, starting at any point between O and B, the curve is retraced until the material returns to its original length. The deformation is reversible, and the forces are conservative; the energy put into the material to cause the deformation is recovered when the stress is removed. In region OB we say that the material shows elastic behaviour. Point B, the end of this region, is called the yield point; the stress at the yield point is called the elastic limit.

When we increase the stress beyond point B, the strain continues to increase. But now when we remove the load at some point beyond B, say C, the material does not come back to its original length. Instead, it follows the dotted line in Fig. 3.5.3. The length at zero stress is now greater than the original length; the material has undergone an irreversible deformation and has acquired what we call a 'permanent set'. Further increase of load beyond C produces a large increase in strain for a relatively small increase in stress, until a point D is reached at which fracture takes place. The behaviour of the material from B to D is called plastic flow or plastic deformation. A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state. For some materials, such as the one whose properties are graphed in Fig. 3.5.3, a large amount of plastic deformation takes place between the elastic limit and the fracture point. Such a material is said to be ductile. The stress required to cause actual fracture of a material is called the breaking stress, the ultimate strength, or (for tensile stress) the tensile strength. Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses.

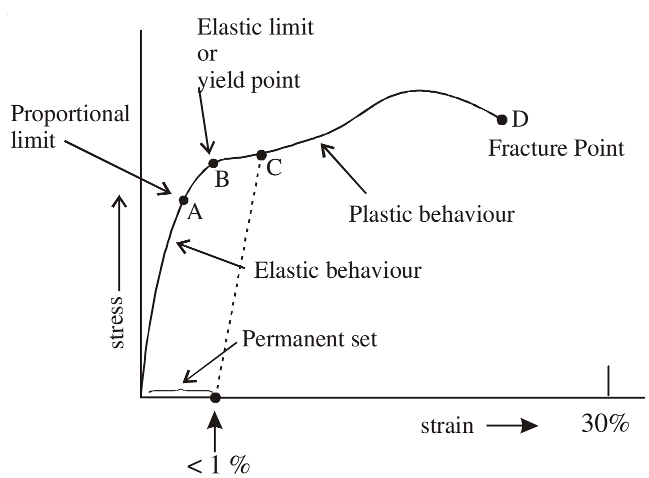


Figure : 3.5.3 Stress-Strain curve of a solid material

❑ Hooke's Law :

Within elastic limit it has been found through several experiments that stress and strain are proportional to each other and $\frac{\text{Stress}}{\text{Strain}}$ = Constant.

This is known as Hooke's law.

This proportionality constant is called modulus Strain of elasticity. This constant depends on the properties of the material. Temperature and formation history of the material has some influence on the elastic properties of the material. It has been observed that elasticity of any material decreases with temperature.

3.5.4. Different Elastic Constants

To determine the elastic behaviour of homogeneous, isotropic bodies we need to specify five quantities as described below:

(a) Young's Modulus (Y)
(b) Bulk Modulus(K)
(c) Modulus of rigidity (G)
(d) Axial Modulus (χ)
(e) Poisson's Ratio(σ)

(a) Young's Modulus

When the deforming force is applied to the body only along a particular direction, the change per unit length in that direction is called longitudinal, linear or elongation strain, and the force applied per unit area of cross-section at equilibrium is called longitudinal or linear stress. The ratio of longitudinal stress to linear strain, within the elastic limit, is called Young's Modulus, and is usually denoted by the letter Y. Thus, if F be the force applied normally to a cross-sectional area A, the stress is $\frac{F}{A}$. And, if there be change ℓ produced in the original length L, the strain is given by $\frac{\ell}{L}$. So that, Young's Modulus, $Y = \frac{\frac{F}{A}}{\frac{l}{L}} N/m^2$ within elastic limit.

(b) Bulk Modulus

Here, the force is applied normally and uniformly to the whole surface of the body; so that, while there is a change of volume, there is no change of shape. Geometrically speaking, therefore, we have here a change in the scale of the coordinates of the system of the body. The force applied per unit area, (or pressure), gives the Stress, and the change per unit volume, the Strain, their ratio giving the Bulk Modulus for the body. It is usually denoted by the letter K.Thus, if F be the force applied uniformly and normally on a surface area A, the stress, or pressure, is F/A or P; and, if v be the change in volume produced in an original volume V, the strain is v/V. and, therefore, Bulk

Modulus, $K = \frac{\text{Volume Stress}}{\text{Volume Strain}} = \frac{F}{\frac{A}{\theta}} N/m^2$

The reciprocal of bulk modulus is called compressibility.

(c) Modulus of Rigidity or Shear Modulus

If a force F tangential to the surface of area A is applied, tangential stress F/A is generated which gives rise to an angle of shear θ, then

Modulus of Rigidity (G) = $K = \frac{\text{VolumeStress}}{\text{VolumeStrain}} = \frac{F}{\frac{A}{\theta}} N/m^2$

(d) Axial Modulus

The axial modulus is defined as the ratio of the longitudinal stress to the corresponding strain when there are other stresses present which prevent any lateral change of dimensions.

Axial Modulus (χ) = $\left[\frac{\text{Longitudinal stress}}{\text{Longitudinal strain}}\right]$lateral strain $= 0$

(e) Poisson's Ratio

For any material wire at constant temperature the ratio of lateral strain to the longitudinal strain within elastic limit is a constant. This constant is known as Poisson's ratio. If L and R be the oiginal length and radius before straining and 1 and r be corresponding changes after straining, then Poisson's ratio

$$(\sigma) = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\frac{r}{R}}{\frac{1}{L}}, \sigma < \frac{1}{2}.$$

3.5.5 Relation among Elastic constants:

Relation between Y, G and σ :

We have seen earlier that there are several elastic constants which depict the behaviour of the substance under different stressed condition. All of them are not independent. Let us first try to establish a relation among the elastic constants Y, G and μ : Let us consider a cube of material of side 'a'. It has been subjected to the action of the shearing stressT. The result of the shear is shown below.

We assume that the strains are small and the angle A C B may be taken as 45º.

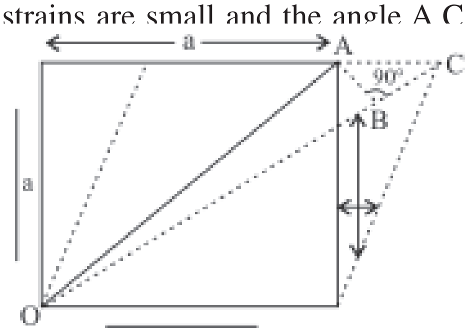


Figure 3.5.4 Shear and shearing strain

The diagonal OA is under a strain and that strain = change in length/original length

The diagonal OA due to shearing strain has been changed to OB. Since angle between OA and OB is very small hence OA ≈ OB, therefore BC can be taken as the change in the length of the diagonal OA. Thus the strain on OA is = $\frac{BC}{OA} = \frac{AC\cos 45^{o}}{OA}$. Now, OA = $\sqrt{2}a.$ Therefore, the strain on OA is $\frac{AC}{\left[\sqrt{2}\sqrt{2}a\right]} = \frac{AC}{2a}$, But AC = γ a, so, the strain is $\frac{\tau}{2G}$, which is in turn equal to $\tau/2G,$ where τ is the stress and G is the shear modulus.

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45º as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain. Thus, for the direct state of stress system which applies along the diagonals, we see equivalence of shearing strain with longitudinal strains, both compressive and expansive.

Strain on diagonal = $\frac{\frac{\sigma_1}{Y}}{\frac{-\sigma\sigma_2}{Y}} = \frac{\tau}{\frac{Y-\sigma(-\tau)}{Y}} = \frac{\tau}{Y(1+\sigma)}$, equating the two strains we get $\frac{\tau}{2G} = \frac{\tau}{Y(1+\sigma)}$, or $Y = \frac{Y}{2G\,(1+\sigma)}$, where σ is the Poisson's ratio. (3.5.5.1)

We have introduced a total of four elastic constants, i.e Y, G, K and σ. It turns out that not all of these are independent of the others. In fact given any two of then, the other two can be found. We know $Y = 3K(1-2\sigma)$ (shown below) irrespective of the stresses i.e, the material is incompressible.

When σ = 0.5 Value of K is infinite, rather than a zero value of Y and volumetric strain is zero, or in other words, the material is incompressible.

❑ Relation between Y,K and σ :

Consider a cube subjected to three equal stresses E_s as shown in the figure below

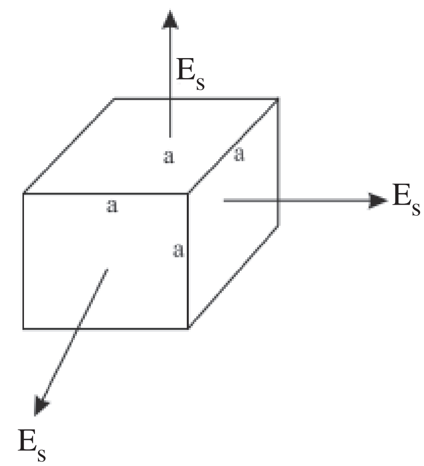


Figure 3.5.5 Cube under stress in all directions

The total linear strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress E_s is given as

$$E_S = \frac{s}{Y} - \sigma\frac{s}{Y} - \sigma\frac{s}{Y}$$

$$= \frac{s}{y}\ (1-2\sigma)$$

Bulk modulus = (volumetric stress/(volumetric strain). or

K= E_s, (volumetric strain), so, volumetric strain = $E_s K$, which is 3 times linear strain.

Therefore, equating the two strains we may write

$$\frac{E_s}{K} = \frac{3E_s(1-20)}{Y} \text{ or}$$

$$Y = 3K(1-2\sigma). \qquad (3.5.5.3)$$

❑ Relation between Y, G and K :

The relationship between Y, G and K can be easily determined by eliminating μ from the already derived relations

$Y = 2G(1+\sigma)$ and $Y = 3K(1-2\sigma)$ (3.5.5.4)

Thus, the following relationship may be obtained

$$Y = \frac{9GK}{(3K+G)} \text{ or, } \frac{9}{Y} = \frac{3}{G} + \frac{1}{K}$$

From the already derived relations, Y can be eliminated

$Y = 2G(1+\sigma)$, and again $Y = 3K(1-2\sigma)$.

Thus, we get

$3K(1-2\sigma) = 2G(1+\sigma)$,

therefore $\phi = \frac{(3K-2G)}{2(G+3K)}$• (3.5.5.6)

hence if σ = 0.5, the value of K becomes infinite and the volumetric strain is zero or in other words, the material becomes incompressible

Further, it may be noted that under condition of simple tension and simple shear, all real materials tend to experience displacements in the directions of the applied forces and under loading they tend to increase in volume. In other words the value of the elastic constants Y, G and K cannot be negative

Therefore, the relations

Y = 2G(1+ σ)

Y = 3K (1–2 σ) ... ... (3.5.5.7)

yields $-1 \le \sigma \le 0.5$

In actual practice no real material has value of Poisson's ratio negative. Thus, the value of σ cannot be greater than 0.5.

❑ Worked out Examples

1. Show that a small and uniform volume strain v is equivalent to three linear strains $\frac{v}{3}$, in any three perpendicular directions

Solution:

Imagine a unit cube to be compressed equally and uniformly from all sides, so that the length of each edge is decreased by a length x,i.e. the side becomes 1–x. Then, clearly, decrease in volume of the cube, i.e., $v=1-(1-x)^3 = 1-1 + 3x - 3x^2 + x3 = 3x$,

i.e.. $x = \frac{v}{3}$, neglecting x^2 and x^3, the value of x being small.

Thus, a small uniform volume strain v is equal to three linear strains, each equal to $\frac{v}{3}$, in three perpendicular directions.

❑ Torsion of a cylindrical rod:

Let us consider a cylindrical rod with length L and radius R, fixed and rigidly supported at one end, and loaded at the other end with an axial torque. For rotational equilibrium of this rod, the external torque is balanced internally by the tongue generated by shear stress. The shear stress may be seen as acting in each imaginary perpendicular cut with a torque equal but opposite to the external torque (Fig. 3.5.6).

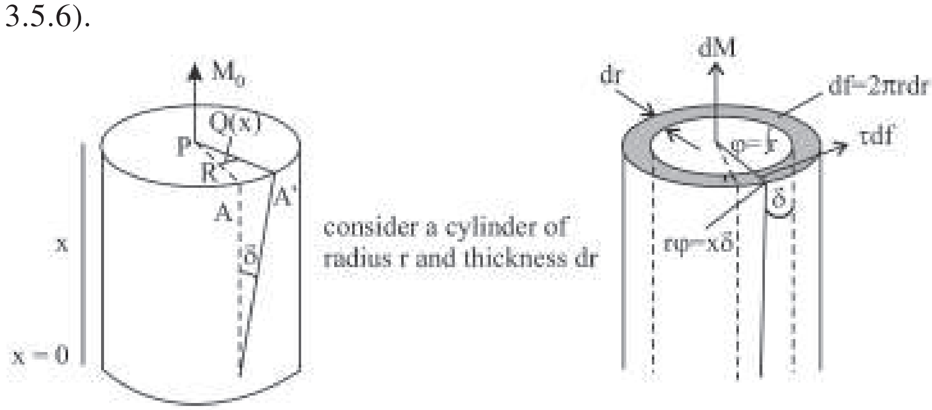


Figure 3.5.6 Torsion of a cylindrical rod under axial stress.

From the fig. 3.5.6. it is clear that the following relations hold

$$AA' = x\delta = r\phi, \text{ but } \delta = \frac{\tau}{G} \text{ so, we get } \tau(r,x) = G\frac{r\phi}{x} \qquad (3.5.5.9)$$

The total torque produced by the shear stress can be calculated by integrating over the cross-sectional area with respect to the center P of the cross-section (see Fig. 3.5.6)

$$M_0 = \int_p r\tau df = \frac{G\phi(x)}{x}\int_p r^2 df = \frac{G\phi(x)}{x}\int_0^R r^2 2\pi r dr = \frac{G\phi(x)\pi R^4}{x2} \quad \dots \quad \dots \qquad (3.5.5.9)$$

The angle of torsion at the end of the rod (x=L) then becomes

$$\phi(L) = \frac{2LM_0}{\pi GR^4} \quad (3.5.5.10)$$

Thus we see that for a given cylinder or wire, the angle of twist is proportional to the torque. The torque per unit twist is given by

$$c = \frac{M_0}{\phi} = \frac{\pi GR^4}{2L}. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (3.5.5.11)$$

It is called torsional rigidity.

❑ Worked out Example :

A gold wire 0.32×10^{-3} m in diameter, elongates by 10^{-3} m, when stretched by a force of 330 $\times 10^{-3}$ kg wt., and twists through 1 radian, when equal and opposite torques of 145×10^{-7} N-m are applied at its ends. Find the value of Poisson's ratio for gold.

Solution :

$Y = \frac{FL}{a1}$, here $F = 330 \times 10^{-3} \times 9.81$ N, $1 = 10^{-3}$ m and $a = \pi x\left(0.16 \times 10^{-3}\right)^2$ sq.m

Therefore, $Y = \frac{330 \times 10^{-3} \times 9.81 \times L}{[\pi \times (0.16 \times 10^{3}) \times 10^{-7}\ N - m}$ The angle of twist is 1 radian, / then,

couple per unit twist = 145×10^{-7} N–m.

This must be equal to $\frac{\pi N r^4}{2L}$, so we have $\frac{\pi N r^4}{2L} = 145 \times 10^{-7}$.

Therefore, $N = \dfrac{145\times10^{-7}\times2L}{\left[\pi\times\left(0.16\times10^{-3}\right)^2\right]}$ So, we get $\dfrac{Y}{N}$ =2.858.

Since $\dfrac{Y}{N} = 2(\sigma + 1)$, this leads to $\sigma = 0.429$, this is the value for Poisson's ratio for gold.

❑ Torsional Oscillation :

A torsional pendulum, or torsional oscillator, consists of a disk-like mass suspended from a thin rod or wire. When the mass is twisted about the axis of the wire, the wire exerts a torque on the mass, tending to rotate it back to its original position. If twisted and released, the mass will oscillate back and forth, executing simple harmonic motion for small torsion

$$\frac{NGr^4\theta}{2L} \text{ v.}$$

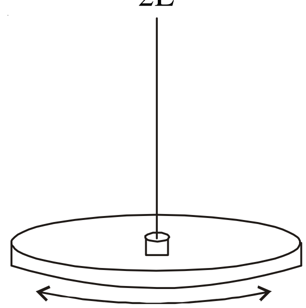


Figure 3.5.7 Instrument to measure torsional oscillation in the Laboratory

Consider a thin rod with one end fixed in position and the other end twisted through an angle θ about the rod's axis. If the angle θ is sufficiently small that the rod is not plastically deformed, the rod exerts a torque τ proportional to the angle θ,

$$\tau = -K\theta \text{ (similar to F= –kx for a harmonic oscillator)} \qquad (3.5.5.12)$$

The –ve sign indicates that τ and θ are oppositely directed where k (Greek letter kappa) is called the torsion constant. The minus sign indicates that the direction of the torque vector τ is opposite to the angle vector θ, so the torque tends to undo the twist. This is just like Hooke's Law for springs.

If a mass with moment of inertia I is attached to the rod, the torque will give the mass an angular acceleration α according to

$$\tau = I \propto = I\frac{d^2\theta}{dt^2}\left(\text{like } F = ma = m\frac{d^2x}{dt^2}\right) \tag{3.5.5.13}$$

Combining (3.5.5.12) and (3.5.3.1.3) yields the equation of motion for the torsional pendulum,

$$I\frac{d^2\theta}{dt^2} = -k\theta \tag{3.5.5.14}$$

$$\text{or,} \quad \frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta \tag{3.5.5.15}$$

The solution to this differential equation is

$$\theta(t) = \theta_m \cos(\omega t + \phi), \tag{3.5.5.16}$$

$$\text{where,} \quad \omega = \left(\frac{K}{1}\right)^{\frac{1}{2}} \tag{3.5.5.17}$$

θ_m and ϕ are constants which depend on the initial position and angular velocity of the mass. (The equation of motion is a second order differential equation so its solution must have two constants of integration.) θ_m is the maximum angle; θ oscillates between $+\theta_m$ and $-\theta_m$.

The constant ω is related to the frequency f and the period T of the simple harmonic motion by

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{3.5.5.18}$$

So, we conclude that the time period T is given by

$$T = 2\pi\left(\frac{I}{K}\right)^{\frac{1}{2}} \qquad (3.5.5.19$$

The torsion constant can be determined from measurement of T if I is known, conversely if K is known the moment of inertia can be determined from measurement of T. K for a cylindrical rod is given by

$$K = \frac{\pi G R^4}{2I},$$

Where G = modulus of rigidity of the material of the rod. R is the radius and L is its length. Therefore, G of the material can be experimentally determined by measuring the time period of oscillation, radius and length of the rod. By the way, the moment of inertia of the disk is given by $I=\frac{1}{2}\ mr^2$ about the axis of rotation. This type of measurment of modulus of rigidity is known as "Dynamical Method of Determination of Modulus of Rigidity".

3.5.6 Potential Energy due to Strain

When the shape or volume of a body changes due to the action of an external force, the internal parts of the body, i.e., the molecules of the system develops an internal stress due to which the object regains its shape or volume when the external force is withdrawn. Certain amount of work is done by the external force while changing the initial state. This work done is stored within the system as potential energy, if elastic limit is not crossed, which gets converted to kinetic energy as soon as the external force is withdrawn. A compressed spring is in ideal example of the above phenomenon. In the calculation for potential energy gained by strain, it is assumed that equilibrium has been maintained during the process

3.5.6.1 Potential energy stored due to tensile strain

Maintaining equilibrium, the tensile stress on a thin wire of length L, cross-sectional area A having Young's modulus Y fixed at one end be slowly increased to a value $\delta 1$(say).

If at any instant of time t, the elongation is x, the stress due to which this elongation is produced is $\frac{Yx}{L}$ and the corresponding force applied at the free end is $F=\frac{YAx}{L}$. Now the force applied be slowly increased to a value such that the increase in length changes to x + dx. Then the work done is F dx. In the same token, the total work done by the external force as the elongation of the wire reaches the final value δ1 is

$$W = \int_{x=0}^{x=\delta l} F dx = \int_{0}^{\delta l} \frac{YAx}{L} dx = \frac{1}{2} YA \frac{(\delta 1)^2}{L} \qquad (3.5.6.1)$$

Now the tensile force at the end of extension is $F=YA\left(\frac{\delta 1}{L}\right)$. So, the strain energy can be written as

$$W = \frac{1}{2} YA\left(\frac{\delta 1}{L}\right).\delta 1 = \frac{1}{2} F.\delta 1 \qquad (3.5.6.2)$$

Because the wire is uniform and cylindrical, its volume is AL. Therefore the energy density due to strain is $\frac{W}{V} = \frac{1}{2}\frac{YA}{AL}\left(\frac{\delta 1}{L}\right).\delta 1 = \frac{1}{2} Y\left(\frac{\delta 1}{L}\right)^2$.

3.5.6.2 Potential energy stored due to volume strain

The change in volume occurs due to volume stress which is actually the pressure acting on the system. We follow the same line of argument as in the case of tensile strain. If δ p be the increase in pressure due to which we find the decrease in volume to be δv. Then,

$$W = \frac{1}{2} K(\delta v / V) x \delta v = \frac{1}{2} \delta p \times \delta v.$$

Now, energy density =

$$\frac{W}{V} = \frac{1}{2} K\left(\frac{\delta v}{V}\right) = \frac{1}{2} K\left(\frac{\delta v}{V}\right) \times \left(\frac{\delta v}{V}\right) = \frac{1}{2}(\text{stress})_{\text{final}} \times (\text{strain})_{\text{final.}} \qquad (3.5.6.3)$$

3.5.6.3 Potential energy stored due to shear

The procedure for calculating the potential energy due to shear is similar to above. We know that if a tangential force F acting on opposite faces of a parallellopiped of area A separated by a distance L produces a shearing strain θ, then we have,

$W = \frac{1}{2} F x L\theta$ and work done per unit volume $\frac{W}{V} = \frac{1}{2} G\theta^2$,

or, $\frac{W}{V} = \frac{1}{2} = \frac{1}{2}\left(\frac{F}{A}\right)\theta = \frac{1}{2} \times (\text{stress})_{\text{final}} \times (\text{strain})_{\text{final.}}$

❑ Worked out Examples :

Find the work done in Joules in stretching a wire of cross-section 1 sq. mm. and length 2 metres through 0.1 mm., if Young's modulus for the material of the wire is $2 \times 10^{11} \frac{N}{m^2}$.

Solution :

Work done = $\frac{1}{2} F.1 = \frac{1}{2} Ya \frac{1}{L}$ where F is the stretching force.

Here, $Y = 2x10^{11} \frac{N}{m^2}, a = 1\text{sq.mm.} = \frac{1}{10^6} = 0.000001$ sq.m.,

1=0.1 mm = 0.0001 m; and L = 2m

Therefore, work done = $\frac{\frac{1}{2} Yal^2}{2} = \frac{\frac{1}{2} x 2x10^{11} x10^{-6} x10^{-8}}{2}$

Thus, work done in stretching the wire is 5×10^{-3} Joules.

3.5.7 Bending of beams

We start with a question: what is a beam? –a beam is usually a metallic solid rod of uniform cross-section. It may be circular, rectangular or any other regular geometric shaped rod whose length is much larger compared to its cross-section so that the shearing stresses over any section are small and may be ignored

Often we come across a situation where a beam is bent due to some reasons or other. When a beam is fixed or supported at one end and loaded at the other, it bends due to the moment created by the weight of the load. The plane of bending is the same as that of the couple produced. As discussed earlier restoring forces come into play and in the equillibrium state, the restoring couple is equal and opposite to the external bending couple, both being in the plane of bending.

After bending of the beam, its filaments on the inner or the concave side get shortened or compressed and those on the outer or the convex side get lengthened or extended.

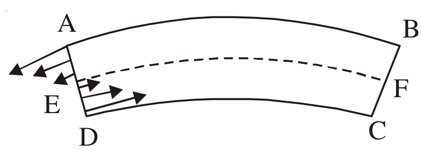


Figure 3.5.7.1 Bending effect on the filaments of a beam

In between these two portions there lies a layer or a surface in which the filaments are neither compressed nor extended. This surface is called the neutral surface and its section (EF) by the plane of bending which is perpendicular to it is called the neutral axis

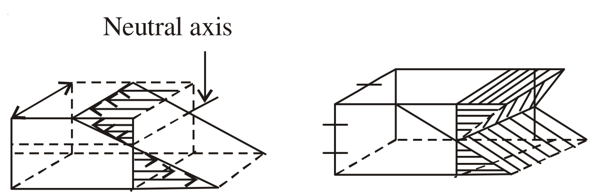


Figure 3.5.7.2 Neutral axis for a rectangular beam

In absence of any strain of the beam, the neutral surface becomes a plane surface, and the filament of this unstrained or unstretched layer or surface, lying in the plane of symmetry of the bent beam, is refered to as the neutral filament. It passes through the c.g. (or the centroid) of every transverse section of the beam. The change in length of any filament is proportional to its distance from the neutral surface. Let a small part of the beam be bent, as shown in Fig. 3.5.7.3 in the form of a circular arc, subtending an angle at the centre of curvature O. Let R be the radius of curvature of this part of the neutral axis, and let 'ab' be an element at a distance z from the neutral axis.

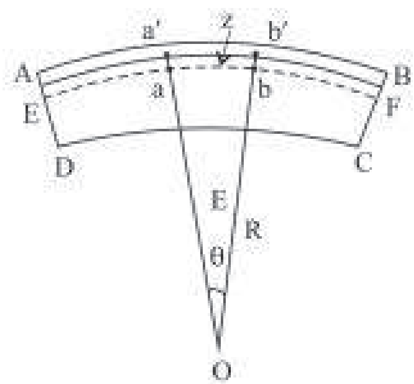


Figure 3.5.7.3 Bending strain of a beam

Then, $a'b' = (R+z).\theta$

and its original length $ab = R\theta$.

Therefore, increase in length of the filament

$= a'b' - ab = (R+z)\theta - R\theta = z\theta$. (3,5,7.1)

The original length of the filament = $R\theta$, then we have

$$\text{strain} = \frac{z\theta}{R\theta} = \frac{z}{R}, \qquad (3.5.7.2)$$

i.e. the strain is proportional to the distance from the neutral axis.

Since there are no shearing stresses, nor any change of volume, the contractions and extensions of the filaments are purely due to forces acting along the length of the filaments.

Let PQRS (Fig. 3.5.7.4), be a section of the beam at right angles to its length and the plane of bending. Then, the forces acting on the filaments are perpendicular to this section, and the line MN lies on the neutral surface.

Let the breadth of the section be PQ = b and its depth, QR = d.

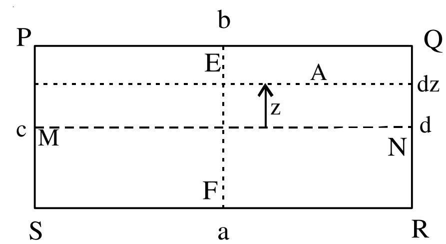


Figure 3.5.7.4 Cross-section of the beam

The forces producing elongations and contractions in all filaments act perpendicularly to the upper and the lower halves, PQNM and MNRS respectively, of the rectangular section PQRS, their directions being opposite to each other.

Let us consider a small area δa about a point A, distant z from the neutral surface. The strain produced in a filament passing through this area will be $\frac{z}{R}$, (shown above).

Now, $Y = \frac{\text{stress}}{\text{strain}}$, therefore, stress = Y x strain, where Y is the Young's modulus of the material of the beam.

Therefore, stress about the point A = $\frac{Y.z}{R}$, and, the force on the area

$$= \delta a.\ \frac{Y.z}{R}. \tag{3.5.7.3}$$

and, moment of this force about the line MN= $\frac{Y.z.\delta a}{R}.z = \frac{Y.\delta a.z^2}{R}$. (3.5.7.4)

Since the moments of the forces acting on both the upper and the lower halves of the section are in the same direction, the total moment of the forces acting on the filaments in the section PQRS is given by

$$\sum_{1}\frac{Y.\delta a_i z_i^2}{R} = \frac{Y}{R}\sum_{i}\delta a_i\, z_i^2 \qquad (3.5.7.5)$$

Now, $\sum_{1}\delta a_i z_i^2$ is the geometrical moment of inertia (I_g) of the section about MN, and, therefore $\frac{Y}{R}$, equal to ak^2, where a is the whole area of the surface PQRS and k, its radius of gyration about MN.

Hence, the moment of the forces about MN= $\frac{Y}{R}ak^2 = YI_g$ (3.5.7.6)

For rotational equilibrium, this moment balances the couple of the bending moment M, acting on the beam due to the load. In other words, it is the moment of the stress set up in the beam or the moment of resistance to bending, as it is usually called in engineering practice, and is also of the nature of a couple, for only a couple can balance a couple. Obviously, it acts in the plane of bending and is equal to the bending moment at the section due to the load.

The quantity $YI_g = Yak^2$ is called the flexural rigidity of the beam. (3.5.7.7)

So, bending moment = $\frac{Y}{R}$ x geometric moment of inertia of the section.

$= \frac{\text{flexural rigidity}}{R}$, whatever the shape of the cross-section of the beam.

For a rectangular cross-section, $a = bd$, and $k^2 = \frac{d^2}{12}$. $I_g = ak^2 = \frac{bd^2}{12}$. Hence, bending moment for a rectangular cross-section = $\frac{Y.b.d^2}{12R}$. (3.5.7.8)

For a circular section, $a = \pi r^2$ and $k^2 = \frac{r^2}{4.1_g}$. $I_g = ak^2 = \frac{\pi r^4}{4R}$. (3.5.7.9)

3.5.8 The Cantilever

A cantilever is a beam fixed horizontally at one end and loaded at the other.

❑ Cantilever loaded at the free end :

Here, two cases arise, viz., (a) when the weight of the beam itself produces no bending, and (b) when it does so. Let us consider both the cases.

(a) When the weight of the beam is ineffective.

Let AB, (Fig. 3.5.8.1) represent the neutral axis of a cantilever, of length L fixed at the end A, and loaded at B with a weight W, such that the end B is deflected or depressed into the position B' and the neutral axis takes up the position AB', it being assumed that the weight of the beam itself produces no bending. Consider a section l of the beam at a distance x from the fixed end A. The moment of the external couple at this section, due to W or the bending moment acting on it

$$= W \times PB' = W(L-x) \tag{3.5.8.1}$$

$$\frac{W(L-x)^2}{2}$$

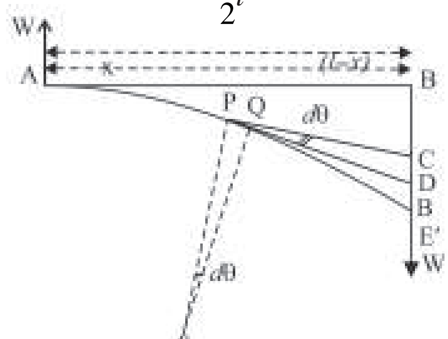


Figure 3.5.8.1 Loading of a cantilever

As the beam is in equilibrium, this must be equal to $\frac{YI_g}{R} = \frac{Yak^2}{R}$, where R is the radius of curvature of the neutral axis at P

$$\text{Therefore, } \frac{YI_g}{R} = \frac{Yak^2}{R} \tag{3.5.8.2}$$

Since the moment of the load increases as we proceed towards the fixed end A, the radius of curvature is different at different points and decreases as we approach the point A. For a point Q, however, at a small distance dx from P, it is practically the same as at P.

So that,

PQ = R.dθ. Or, dx = R.dθ, Where dθ is the angle POQ

whence, $R = \dfrac{dx}{d\theta}$

Substituting the value fo R in (3.5.8.1) above, we have

$$W(L-x) = Y\ ak^2 \frac{d\theta}{dx}$$

$$\text{or, } d\theta = \frac{W(L-x)dx}{(Yak^2)} \qquad (3.5.8.3)$$

If tangents are drawn to the neutral axis at P and Q, meeting the vertical line through BE' in C and D respectively. Then, the angle subtended by them is also equal to dθ, the radi at P and Q being perpendicular to the tangents there.

Now, clearly, the depression of Q below P is equal to CD, equal to dy, (say)

$$\text{Then, } dy = (L-x)d\theta = \frac{(L-x)W(L-x)dx}{(Yak^2)}$$

$$= \frac{W(L-x)^2 dx}{(Yak^2)} \qquad (3.5.8.4)$$

Therefore, the depression y = BB' of the loaded end B below the fixed end A, is obtained by integrating the expression for dy between the limits, x = 0 and x =1

$$\text{or, } y = \int_0^L W\frac{(L-x)^2}{Yak^2}dx = \frac{W}{Yak^2}\int_0^L \left(L^2 - 2xL + x^2\right)dx \qquad (3.5.8.5)$$

$$y = \frac{W}{Yak^2} \cdot \frac{L^3}{3} = \frac{WL^3}{3YI_g} \qquad (3.5.8.6)$$

Thus, the free end of the cantilever is depressed by

$$y = \frac{W}{Yak^2} \cdot \frac{L^3}{3} = \frac{WL^3}{3YI_g} \qquad (3.5.8.7)$$

(b) the total bending moment of the beam

$$= \frac{W(L-x) + w(L-x)}{2} = W(L-x) + \frac{W(L-x)^2}{2}$$

Imposing the condition for equilibrium we get,

$\frac{W(L-x) + w}{2(L-x)^2} = \frac{Yak^2}{R} = Yak^2 \frac{d\theta}{dx}$, Obtaining $d\theta$ from here we can find out dy, where dy = (L–x)dθ

$$= \frac{W(1-x)^2 + \frac{w}{2}(L-x)^2}{Yak^2}$$

Therefore,

$$y = \frac{WL^3}{3Yak^2} + \frac{wL^4}{8YaK^2} \qquad (3.5.8.8)$$

3.5.9 Depression of a Beam supported at the ends

(i) When the beam is loaded at the centre

We consider a uniform beam supported on two knife edges symmetrically placed at its two ends A and B, as show in Fig. 3.5.6.1, and let it be loaded in the middle at C with a weight W.

The reaction at each knife edge will clearly be W/2, in the upward direction (Fig. 3.5.9.1). Since the middle part of the beam is horizontal, the beam may be considered as equivalent to two inverted cantilevers, fixed at C, the bending being produced by the loads W/2, acting upwards, at A and B.

If, therefore, L be the length of the beam AB, the length of each cantilever (AC and BC) is $\frac{L}{2}$, and the elevation of A or B above C or equivalently the depression of C below A or B is given by

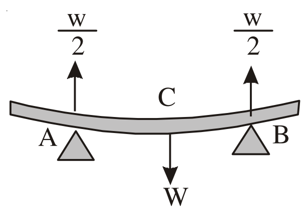


Figure 3.5.9.1 Beam loaded at the center

$$y = \frac{WL^3}{48Yak^2} = \frac{WL^3}{48YI_a} \qquad (3.5.9.1)$$

If the beam is of circular cross-section, we have $ak^2 = \frac{\pi r^4}{4}$ where r is the radius of the beam. For this kind of beam the depression at the center of the beam is

$$y = \frac{WL^3}{(12\,Y\,\pi r^4}. \qquad (3.5.9.2)$$

And, if the beam is of rectangular cross-section of breadth 'b' and depth 'd', we have $ak^2 = \frac{bd^2}{12}$.

For such a beam $y = \frac{WL^3}{(4\,Y\,bd^4)}$ (3.5.9.3)

Thus we see that knowing the shape of the beam and its loading pattern one can find out the depression at a particular chosen point.

❑ Worked out Examples :

brass bar 1 cm. square in cross-section is supported on two knife edges 100 cm. apart. A load of 1 kg. at the centre of the bar depresses that point by 2.51 mm.What is Young's modulus for brass?

Solution :

We know that the depression of the mid-point of the bar is given by $y = Wl^3/48Yl_g$

[See text]

Now, for a bar of rectangular cross-section,

$$I_g = \frac{b.d^2}{12}$$

Here, b = d = 1 cm., because the bar is 1 cm. square in cross-section.

$b.d^3 = 1\times1 = 1$; W = 1 kg. wt. = 1×9.81 N.

1= 100 cm = 1m. and y = 2.51 mm = 0.00251 m.

$$\text{Therefore, } Y = \frac{Wl^3}{4ybd^3}$$

Or, the value of Young's Modulus for brass is 9.77×10^{10} N/m^2

3.5.10 Questions :

1. Explain the stress-strain curve. From the curve, explain elastic limit, plasticity

2. Prove that (a) $\frac{9}{Y} = \frac{1}{K} + \frac{3}{G}$ (b) $\sigma = \frac{(3K - 2G)}{(6 + 2G)}$

3. If the volume of a thin rubber string remains unchanged after a little elongation, what is its Poisson's ratio?

4. A thin uniform brass rod of length 1 and mass m rotates uniformly with an

angular velocity ω in a horizontal plane about a vertical axis passing through one of its ends. Determine the tension in the rod as a function of the distance from the rotation axis. Find the elongations of the rod.

5. Calculate the geometrical moment of inertia of a) thin rectangular sheet, (b) thin hollow circular section.

6. A cantilever beam of rectangular section has breadth 'b' and depth 'd'. If d = 2b, find the ratio of depression at the free end when (i) d is vertical and (ii) b is vertical.

7. Find the potential energy due to twist in a wire of circular cross-section.

3.5.11 Answers :

1. See section 3.5.3

2. See section 3.5.5

3. $V = \pi r^2 1$, as V is constant $\delta V = 2\pi r l\ \delta r + \pi r^2 \delta 1 = 0$

So, $\sigma = -\dfrac{\frac{\delta r}{r}}{\frac{(\delta 1)}{1}} = \dfrac{1}{2}$.

4. Let us consider a section of the rod at a distance x from the axis of rotation when the rod is not in a condition of stretching. Let the tension at that point be T when the rod is rotating. T arises due to centripetal force on the remainisng portion of length (1–x) $T = \frac{m}{1}(1-x)\omega^2.\frac{1}{2}(1+x) = \frac{1}{2}m\omega^2 1\frac{(1-r^2)}{1^2}$. If the elongation be $\delta\xi$ at length δx, then $Y = \dfrac{\left(\frac{T}{A}\right)}{\left(\frac{\delta\xi}{\delta x}\right)}$, So, $\delta\xi\ \frac{T\ \delta x}{(A\ Y)}$, A = cross-sectional area of the element.

Therefore, total elongation $\delta 1= \int_0^1 \left(\frac{T}{AY}\right)dx = \frac{\left(m\omega^2 1\right)}{2AY} / \int_0^1 \frac{1-x^2}{1^2} dx$

$$= \frac{1}{3}\left(\frac{m\omega^2 1^3}{AY}\right)$$

5. Geometrical moment of inertia of a thin rectangular sheet of breadh b and width w about an axis passing through its center and parallel to its side of length w

$$I = \int y^2 dA = \int y^2 dxdy = \frac{bw^3}{12}.$$

(b) the geometrical moment of inertia of the hollow circular cross-section of internal and external radii r_1 and r_2 respectively is given by

$$I = \int\int (r\sin\theta)^2 \, r d\theta dr = \int_{r_1}^{r_2} r^3 dr \int_0^{2\pi} \sin^2\theta d\theta = \frac{\pi}{4}\left(r_2^4 - r_1^4\right)$$ about any diameter.

6. The depression of the free end of a cantilever is given by

$\delta = 41^3 \frac{(W + 3W_0/8)}{Ybd^3}$, where 1=length of the cantilever, b=breadth, d=depth,

$\frac{1}{4}$ W = weight at the end of the free end of the beam, W_0 = its own weight and Y = young's modulus of the material of the beam. Therefore, the depression of a beam will be $\frac{1}{4}$th of the depression of the same beam when its depth and breadh are interchanged.

7. If θ is the twist at any time, the internal torque $C = \frac{NGr^4\theta}{2L}$. If the twist is further increased by by dq, the work done due to this twist is Cdθ. As θ increases from 0 to θ_0, the total work done is given by

$$W = \int_0^{\theta_0} C d\theta = \pi G r^4 \theta_0^2 / 4L = \frac{1}{2} \text{x (applied torque)}_f \text{x (twist)}_f.$$

Unit-6 ❑ Viscosity and fluid dynamics

Structure :

3.6.1 Proposal

Viscosity is the property that describes a fluid's resistance to flow like frictional force opposing motion between two solid surfaces in contact. Fluids try to resist the relative motion of objects through them as well as to the motion of layers with differing velocities within them.

❑ Outcome

After reading this chapter you should be able to:

- think enthusiastically about analysing movements of fluids having different kinds of viscosity.
- explain the effect of temperature and pressure on the coefficient of viscosity of different fluids.
- describe different methods of determining the coefficient of viscosity of fluids in the laboratory.

3.6.2 Motion of fluids

Fluid dynamics is the study of motion of fluids (gases and liquids). As this study is a macroscopic one, fluid can be regarded as a continuous media. However, we must be aware of the fact that even a tiny volume element of fluid contains very many number of molecules. So, when we consider fluid particle we are actually talking about the motion of a cluster of fluid molecules represented by a point inside the fluid.

There are types of fluid motion: (a) steady or laminar flow and (b) turbulent flow.

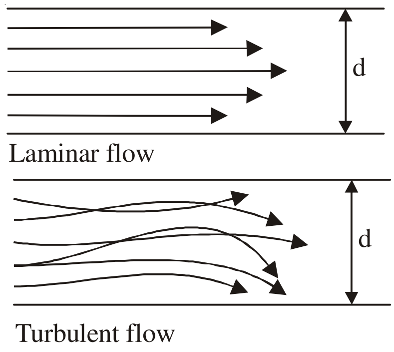


fig. 3.6.2.1

Steady flow is the flow in low speed such that its adjacent layers slide smoothy with respect to each other, Streamline is an imaginary line showing the path of any part of the fluid during its steady flow inside the tube. Particles of a continuous fluid can be considered to travel along smooth continuous paths named streamlines. These streamlines can be curved or straight, depending on the lateral pressure on fluid. This type of motion is also called laminar flow. If a tangent is drawn at any point on the streamline, it gives the direction of the fluid motion at that point.

A transition from laminar flow to turbulent flow occurs very suddenly as the flow rate increases. The flow becomes unstable at some critical speed. Turbulent flow occurs when there are abrupt boundary surfaces. The flow of blood through a normal artery is laminar. However, when irregularities occur the flow becomes turbulent. The noise generated by the turbulent flow can be heard with a stethoscope. When the flow becomes turbulent there is a dramtic decrease in the volume flow rate as eddies and vortices are formed.

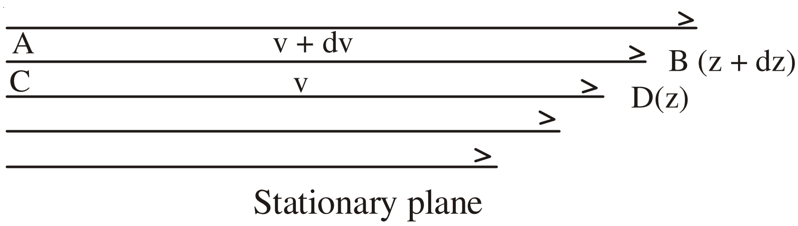


Fig. 3.6.2.2

Any slowly moving liquid over a stationary plane can be subdivided in to several horizontal layers of liquids moving with different velocities in the vertical directions. The layer of liquid in contact with the stationary plane will be at rest due to adhesive forces. So, a velocity gradient dv/dz is established within the moving liquid and successive two layers try to reduce the relative velocity between these layers. In the figure 3.6.2.2 layer CD applies a tangential force F against the motion whereas the layer AB applies a tangential force F in the forward direction. Now, let us discuss Newton's law for this liquid movement.

$F \propto A$,

Where A is the area of the tangential layer

$\propto \frac{dv}{dz}$,

Where $\frac{dv}{dz}$ is the velocity gradient.

So, $F \propto A \frac{dv}{dz}$,

or $F = \eta A \frac{dv}{dz}$

Therefore, $$\eta = \frac{F/A}{\frac{dv}{dz}}. \qquad (3.6.2.1)$$

This η is called co-efficient of viscosity and it depends on the nature of the liquid (gas). It is dependent on the temperature and pressure on the liquid (gas), For an ideal gas it depends on the temperature only. From the expression of co-efficient of viscosity one can derive an alternative definition: Co-efficient of viscosity is the force acting on unit area of a fluid moving steadily with unit velocity gradient as $\eta = F$, if $A = 1$ and $\frac{dv}{dz} = 1$. The dimension of η is $\left[ML^{-1}T^{-1}\right]$. In CGS system unit of η $gcm^{-1}\,s^{-1}$ or Poise and in SI system it is $Kgm^{-1}s^{-1}$ or 10 Poise.

3.6.3 Newtonian and Non-Newtonian Fluid :

The fluids for which Eq. 3.6.2.1 is applicable are said to be Newtonian fluid and those fluids for which $\frac{F}{A}$ is not proportional to $\frac{dv}{dz}$ at some definite temperature and pressure are known as Non-Newtonian fluid.

3.6.4 Critical velocity and Reynold's number:

As the velocity of the fluid rises the motion turns from laminar flow to turbulent flow after becoming more than a particular velocity known as critical velocity. Reynold after doing an exhaustive study on fluid motion has shown that critical velocity of the fluid

$V_c \propto \frac{\eta}{pr}$, where η = Co-efficient of viscosity of the fluid, p = density of the fluid and r = radius of the tube through which fluid is moving. By dimensional analysis one can show that

$V_c = \frac{\eta}{pr}$, where N is Reynold's number (dimensionless) for the fluid. (3.6.4.1)

N is 1000 (approx) for water flowing through a capillary tube. Actually, one should consider G.N, where G.N is the ratio of pressure drag to the viscous drag and G~0.01 for spherical bodies.

3.6.5 Poiseuille's Equation:

Let us consider an elementary tube of liquid having length L and radius r moving towards right of the diagram. Let P and P + ΔP be the pressures on the right and left of the tube.

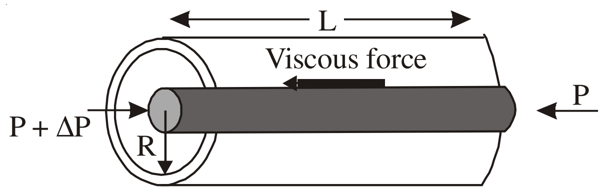


Fig. 3.6.2.3

The driving force on the liquid cylinder of radius r due to the pressure difference is:

$$F_{pressure} = \Delta P\left(\pi r^2\right) \quad (3.6.5.2)$$

There will be a viscous drag force opposing the motion towards right, which depends on the surface area of the cylinder (length L and radius r):

$$F_{viscosity} = -\eta(2\pi rL)\frac{dv(r)}{dr} \quad (3.6.5.3)$$

In developing the Poiseuille's equation one must make sure that

(a) The liquid must be Newtonian.

(b) The liquid layer in contact with the surface of the tube must be stationary.

(c) The liquid flow should be steady.

and (d) The pressure is same at all points in any cross-section of the tube.

After satisfying all the above conditions for constant speed, as the net force goes to zero, we have

$$F_{pressure} + F_{viscosity} = 0$$

$$\Delta P\left(\pi r^2\right) = \eta(2\pi rL)\frac{dv}{dr}$$

So

$$\frac{dv}{dr} = \frac{\Delta P\left(\pi r^2\right)}{\eta(2\pi rL)} = \left(\frac{\Delta P}{2\eta L}\right).r \quad (3.6.5.4)$$

We know that at the centre of the tube through which the liquid is flowing towards right

r = 0

$$\frac{dv}{dr} = 0$$

v is at its maximum, at the edge and

r = R

v = 0

$$\frac{dv}{dr} = \left(\frac{\Delta P}{2\eta L}\right).r$$

Integrating with proper limits,

$$\int_v^0 dv = \left(\frac{\Delta P}{2\eta L}\right).\int_r^R rdr$$

$$v(r) = \left(\frac{\Delta P}{4\eta L}\right)\left[R^2 - r^2\right] \qquad (3.6.5.5)$$

Using the equation of continuity which gives the volume flux for a variable speed, we get:

$$\frac{dv}{dt} = \int v.dA \qquad (3.6.5.6)$$

In the above equation we substitute the velocity profile equation and the surface area of the moving cylinder:

$$\frac{dv}{dt} = \int v.dA = \int_0^R \left(\frac{\Delta P}{4\eta L}\right)\left[R^2 - r^2\right].(2\pi rdr)$$

$$= \left(\frac{\pi.\Delta P}{2\eta L}\right)\int_0^R \left(R^2 r - r^3\right) dr$$

$$= \left(\frac{\pi.\Delta P}{2\eta L}\right)\left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{\pi.\Delta P.R^4}{8\eta L} \qquad (3.6.5.7)$$

This is Poiseuille's equation.

Is this derivation for Poiseuille's equation correct? No, there are two corrections needed for completing this derivation of Poiseuille's eq. : 1) Correction for kinetic energy–this correction arises due to the assumption that the force due to pressure difference is expended against viscous force. But actually the liquid coming out of the tube has a kinetic energy and the effective pressure operating for the movement of the liquid is less than the actual pressure difference. (2) In our derivation we have not considered any acceleration of the liquid as it enters the tube. However, the acceleration vanishes after the liquid travels a little distance within the tube. So, we have to take this acceleration into account. In order to consider the effect of acceleration usually the length is modified to have a larger value.

3.6.6 Determination of Co-efficient of viscosity by Poiseuille's method :

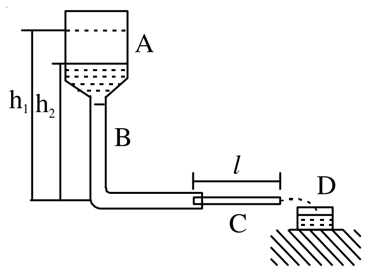


Fig. 3.6.6.1

Initially the water level in the container is maintained at height h_1 from the central plane of the capillary tube held horizontally.

The average radius 'R' of the capillary tube is determined by measuring the mass 'm' of a mercury thread of length 'L' and using the formula

$$R = \sqrt{\frac{m}{r\rho L}}$$

The length 'I' of the capillary tube is measured by a meter rule.

If the height difference between the liquid columns in the container be 'h' where $h = h_1 - h_2$, the pressure acting at the entry point of the capillary tube is

$P = h\rho g$.

The volume of water V collected by the container at D over a time 'T' measured by a stop watch provides 'v' which gives the volume of liquid flowing through the capillary tube per second.

Then the co-efficient of viscosity of water at the temperature of the laboratory can be calculated via the formula

$$\eta = \frac{\pi P R^4}{8vl}$$

❑ Worked out Examples:

1. Water is pumped steadily out of a flooded basement at a speed of 5.30 m/s through a uniform hose of radius 9.70 mm. The hose passes out through a window 2.90 m above the water line. How much power is supplied by the pump?

Solution:

The kinetic energy of the water per unit mass when it leaves from the uniform hose through the window is

$$K = \frac{1}{2}v^2$$

Here, mass of the flow of water is 1 and speed of the water flow is υ.

The corresponding potential energy per unit mass of the flow of water through the window is U = gh

Here, acceleration due to gravity of the Earth is g and height of the window from the basement is h. The volume rate of the flow of water from the hose through the window is

$$\vec{R} = \bar{v}A$$

Here, cross-sectional area of the hose is A and speed of the water flow is $\bar{v}$.

The mass rate of the flow of water is

$$\mathbf{R_m} = \rho\mathbf{R}$$

Here, density of the water is ***p***.

The power supplied by the pump is given by

$$\mathbf{P = (K + U)\ R_m}$$

Substituting the values, the power comes out to be P = 66.49 Watt.

2.An intravenous (IV) system is supplying saline solution to a patient at the rate of 0.120 cm^3/s through a needle of radius 0.150 mm and length 2.50 cm. What pressure is needed at the entrance of the needle to cause this flow, assuming the viscosity of the saline solution to be the same as that of water? The gauge pressure of the blood in the patient's vein is 8.00 mm Hg. (Assume that the temperature is 20ºC.)

❑ Solution :

Assuming laminar flow, Poiseuille's law applies. This is given by

$$Q = \frac{\pi r^4 (P_2 - P_1)}{8\eta^1}$$

where P_2 is the pressure at the entrance of the needle and P_1 is the pressure in the vein. The only unknown is P_2.

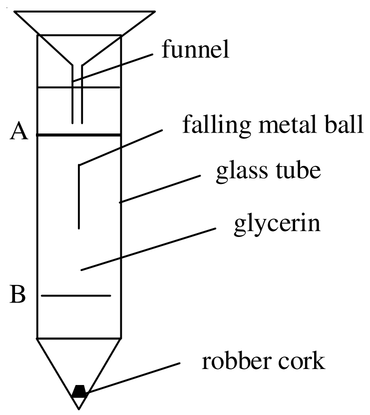
Solving for P^2 yields

$$P_2 = \left(\frac{8\eta\ell}{\pi r^4}\right)(Q + P_1)$$

P_1 is given as 8.00 mm Hg, which converts to $\frac{1.066\times10^2 N}{m^2}$. Substituting this and the other known values yields

$$P_2 = \frac{1.62\times10^4 N}{m^2}.$$

3.6.7 Stokes' method of determination of co-efficient of viscosity of a liquid and Terminal velociy



When a metallic ball of spherical shape falls down through a less denser long liquid column, it's velocity increases due to gravitational force, and the opposing frictional force due to viscosity also increases. A stage is reached at a particular downward velocity at which the viscous force is just equal in magnitude but opposite in direction to the gravitational force. Under this condition, the falling metallic sphere moves with a constant velocity called terminal velocity v_t. The British scientist, Sir George G. Stokes (1819-1903) showed that the retarding force F_v due to viscosity acting upwards on a spherical body of a radius 'r' falling through a medium of viscosity η is

$$F_v = 6\pi\eta r v_t \qquad (3.6.7.1)$$

The eq (3.6.7.1) can also be derived from dimensional analysis as given below

$$F_V \propto \eta^x r^y v_t^z$$

$$M\,LT^{-2} = \left(ML^{-1}T^{-1}\right)^x \left(LT^{-1}\right)^y L^z$$

$$MLT^{-2} = M^x L^{-x+y+z} T^{-x-y} \qquad (3.6.7.2)$$

From eq (3.6.7.2), one can easily obtain x =1, y=1, z =1. Therefore, after substituting the values

$$F_v \propto \eta v_t r$$

or $F_v = k\eta v_t r$ (3.6.7.3)

According to Stokes the above constant $k = 6\pi$

The weight of the spherical ball is,

$$\frac{4}{3}\pi r^3 \sigma g \qquad (3.6.7.4)$$

Where σ is the density of the spherical ball.

The upward thrust exerted by the liquid medium on the spherical body is equal to the weight of the medium displaced by the body.

The weight of the displaced liquid

$$= \frac{4}{3}\pi r^3 \rho g \qquad (3.6.7.5)$$

where ρ is the density of the liquid.

The net downwards force acting on the metallic sphere is,

$$F = \frac{4}{3}\pi r^3 (\sigma - \rho) g \qquad (3.6.7.6)$$

At equilibrium, i.e., when the two body moves with terminal velocity, eqn (3.6.7.1) and (3.6.7.6) can be equated as follows,

$$\frac{4}{3}\pi r^3 (\sigma - \rho) g = 6\pi\eta r v_t$$

or

$$v_t = \frac{2}{9}\frac{r^2}{\eta}(\sigma - \rho) g \qquad (3.6.7.7)$$

So, from experiment if the values of v_t, r, ρ, σ are determined one can find out the value of η.

Procedure:

- The least count and zero correction of the given screw gauge are to be found.
- The diameter (d) of the ball using the screw guage is to be found. Now, the radius (r) of ball can be calculated as; $r = \frac{d}{2}$.
- The inner diameter of the jar is to be measured using a vernier calipers. Hence the inner radius of the jar R can be found.
- Two reference points A and B on the jar are to be marked using two threads. The marking A is made well below the free surface of liquid, so that by the time when the ball reaches A, it would have acquired terminal velocity v_t.
- The position the thread B is to be set so that the distance between A and B is of the order of 60cm.
- The ball of known diameter is to be dropped gently in the liquid. It falls down in the liquid with accelerated velocity for about 30% of the height. Then it falls with a uniform terminal velocity.

- When the ball crosses the point A, the stop watch should be started and the time taken by the ball to reach the point B is noted.
- If the distance moved by the ball is d and the time taken to travel is t, then velocity,

$$v_t = \frac{d}{t}$$

- The terminal velocity of the ball v_t is calculated using the relation,

$$\eta = \frac{2}{9} r^2 \frac{(\sigma - \rho)}{v_t}$$

- Now, the experiment is repeated by changing the diameter of the ball. The value of $\frac{r^2}{v_t}$ in each time is to be noted.

A graph is to be plotted with r^2 along X axis and terminal velocity along Y axis. The coefficient of viscosity of the liquid is calculated by using the slope of the graph.

$$\eta = \frac{2}{9}(\sigma - \rho) g \frac{1}{\text{slope}}$$

❑ Worked out Examples :

1. A ball of copper of density 8960 kg/m^3 and of radius 1 mm has been allowed to fall through a column of castor oil having density 956 kg/m^3 and co-efficient of viscosity 0.65 Pa.s. Find the terminal velocity of the ball.

Solution : Terminal velocity v_t is given by $v_t = \left(\frac{2}{9}\right) \times \frac{r^2 g(\rho - \sigma)}{\eta}$

$$= \left(\frac{2}{9}\right) \times \left(10^{-3}\right)^2 \times 9.81 \left(\frac{8960 - 956}{0.65}\right)$$

$$= 0.027 \text{ m/s}$$

2. In the oil-drop experiment Robert Millikan observed that an oil drop having density $\rho = 0.851\,gm/cm^3$ obtained the terminal velocity in air $\left(\eta = 171\times10^6\ P\ and\ \sigma = \frac{0.001293\,gm}{cm^3}\right)$. He obtained the radius of the drop to be 1.64 $\times10^{-4}$ cm using Stoke's law. Justify his arguments.

Solution :

The terminal velocity obtained by the drop $v_t = \frac{2r^2 g(\rho - \sigma)}{9\eta} = 0.029\,cm/s$ after putting the values mentioned in the problem.

The Reynold's number for the situation is $N = \frac{\sigma r v_t}{\eta} = 3.6\times10^{-5} << 1$.

But G.N (G=0.01 for spherical bodies) = 3.6 $\times10^{-7}$ <<1. So, Millikan's action is justified.

3.6.8 Euler's equation of motion for fluids :

Now let us consider the motion of a fluid moving with a steady flow. We want to tell about the changes the motion introduces in the system. If one wishes to describe mathematically the state of a moving fluid, he or she should use some mathematical functions which give the distribution of the fluid velocity **v** =**v** (x,y,z,t) can be one and the other can be the density $\rho(x, y, z, t)$. From the discussion on the thermodynamic properties cf systems we know that all the thermodynamic quantities are determined by the values of any two of them, together with the equation of state which connects several of them. So, if we get hold of five quantities like the three componens of the velocity $\vec{v}$, the pressure p and the density of the fluid ρ, the state of the moving fluid is completely determined for any observation.

Let us fix our attention on the state of some elementary volume element dV bounded by a surface dA. The force acting on this volume element is p dA. Therefore, the total force on this volume is

$$-\int_s pdA$$

The integral is taken over the surface bounding the volume. Using Gauss theorem on integrals we have

$$\int pda = -\int \nabla pdv$$

Hence we observe that the fluid surrounding any volume element within it exerts on that particular element a force $-dV\nabla p$. So, we may conclude that a force $-\nabla p$ acts on unit volume of the fluid. The equation of motion of a volume element in the fluid can now be written by equating the volume force to the product of the mass per unit volume (ρ) and the acceleration $d\bar{v}/dt$:

$$\rho\frac{dv}{dt} = -\nabla\rho \qquad (3.6.8.1)$$

The total time derivative $d\bar{v}/dt$ which appears in eqn. (3.6.8.1) has a special significance: it denotes not only the rate of change of the fluid velocity $\bar{v}$ at a fixed point in space, but also the rate of change of the velocity of a given fluid particle as it moves about in space. We notice here that the total change $d\bar{v}$ in the velocity of the given fluid particle during the time interval dt is made up of two components, namely the change during dt in the velocity at a point fixed in space and the difference between the velocities (at the same time) at two points $d\vec{r}$ apart, where dr is the distance moved by the given fluid particle during the time dt. The first part is $\left(\frac{\partial v}{\partial t}\right)$ keeping x,y,z constant. The second part is

$$dx\frac{\partial v}{\partial x} + dy\frac{\partial v}{\partial y} + dz\frac{\partial v}{\partial z} = \left(d\bar{r}.\bar{\nabla}\right)\bar{v}.$$

Thus, $d\bar{v} = \left(\frac{\partial \bar{v}}{\partial t}\right)dt = \left(d\bar{r}.\bar{\nabla}\right)\bar{v}$

dividing both sides by dt we get,

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v}.\bar{\nabla})\bar{v} \quad (3.6.8.2)$$

Substituting this in eqn, (3.6.8.1) we get

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}.\bar{\nabla})\bar{v} = -\frac{1}{\rho}\nabla p \quad (3.6.8.3)$$

This is Euler's equation of motion for fluids. If the fluid moves in a gravitational field, there will be an extra force ρg where g is the local acceleration due to gravity. In that case the equation of motion takes the form

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}.\bar{\nabla})\bar{v} = -\frac{1}{\rho}\nabla p + g \quad (3.6.8.4)$$

3.6.9 Equation of continuity :

We shall now discuss one of the fundamental equations of fluid dynamics. Equation of continuity deals with conservation of matter. Let us consider some volume V_0 of thefluid. The mass of fluid contained in this volume is $\int_{V_0} \rho dV$, where ρ is the density of the fluid. The mass of fluid flowing in unit time through an element $d\vec{A}$ of the surface bounding this volume is $\rho\, \bar{v}\, .\, d\vec{A}$. Conventionally the vector $d\vec{A}$ is taken along the outward normal to have positive contribution. Then for fluid flowing out of the volume $\rho\, \bar{v}\, .d\vec{A}$ is positive and it becomes negative if fluid goes in the volume. Then the total mass of fluid flowing out of the volume V_0 in unit time is

$$\int_s \rho \bar{v}.d\vec{A}$$

where the integration has to be done over the bounding surface. One can find out the decrease in the mass per unit time inside the volume V_0

$$-\frac{\partial}{\partial t}\int \rho dV$$

As mass is conserved, we have

$$\frac{\partial}{\partial t}\int_{v} \rho dv = \int_{s} \rho\vec{v}.d\vec{A} \qquad (3.6.9.1)$$

The surface integral on the right hand side can be transformed to a volume integral by Green's formula

$$\int_{s} \rho\vec{v}.d\vec{A} = \int_{v} \vec{\nabla}.(\rho\vec{v})dV$$

Thus we get,

$$\int\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\rho\vec{V})\right]dV = 0$$

Since the volume element is quite arbitrary, the integrand must vanish for all dV. Thus,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0 \qquad (3.6.9.2)$$

This is the equation of continuity in differential form for moving fluid.

3.6.10 Bernoulli's theorem and it's application :

Let us consider a fluid moving with a steady flow . For this system we can have the following thermodynamic relation

dw = Tds + Vdp, where 'w' is the heat function per unit mass, 'S' is the entropy, 'V' is the specific volume given in terms of density as $V = \frac{1}{\rho}$ and 'T' is the temperature. For isentropic system (Entropy 's' is constant) we get

$$dw - Vdp = \frac{dp}{\rho},$$

So, $$\nabla w = \frac{\nabla p}{\rho}$$

Therefore, from the Euler's equation one gets,

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v} = -\nabla w \qquad (3.6.10.1)$$

Using a vector identity

$$\frac{1}{2}\nabla v^2 = \vec{v} \times \text{curl}\vec{V} + \left(\vec{v}.\vec{\nabla}\right)\vec{v},$$

we arrive at $\frac{\partial v}{\partial t} + \frac{1}{2}\nabla v^2 - v \times \text{curl } \vec{v} = -\nabla W$ (3.6.10.2)

When fluid motion is said to be in steady flow, the equations describing the steady flow get simplified.The fluid velocity $\vec{v}$ is a function of position coordinates only, there is no time vairation.

So, we have $\frac{\partial v}{\partial t} = 0$. Under this condition eqn. (3.6.10.2) reduces to

$$\frac{1}{2}\nabla v^2 - \vec{v} \text{ x } \text{cur}\vec{l}v = -\nabla w \qquad (3.6.10.3)$$

Now we introduce the idea of streamlines. These lines are such that tangent at any point along this line gives the direction of flow at that point. Mathematically these are given by

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \qquad (3.6.10.4)$$

One can further simplify the equation (3.6.10.3) by taking the scalar product with the unit vector $\hat{n}$ tangent to the streamline at each point. The projection of the gradient on any direction is the derivative in that direction. So, for the projection of ∇w is $\frac{\partial w}{\partial n}$.

For the vector $\bar{v} \times \overline{curl} v$ we get the projection on the direction of $\hat{n}$ to be zero as the vector is perpendicular to $\bar{v}$.

Thus we arrive at after simplification $\frac{\partial\left(\frac{1}{2}v^2 + w\right)}{\partial n} = 0$.

Therefore we can conclude that should right justified

along a streamline $\frac{1}{2}v^2 + w$ = constant.

This equation is known as Bernoulli's equation.

If now we consider the fluid motion in the gravitational field, then eqn. (3.6.10.3) gets modified. One has to add local acceleration due to grarvity $\bar{g}$ on the right hand side of the eqn. The projection of the acceleration due to gravity on the unit vector $\bar{n}$ is $-\bar{g}\ \frac{\delta w}{\delta n}$, if the acceleration due to gravity acts in the direction of z-axis. So, we ultimately have

$$\frac{\partial\left(\frac{1}{2}v^2 + w + gz\right)}{\partial n} = 0$$

Thus on a streamline, Bernoulli's equation takes the form,

$$\frac{1}{2}v^2 + w + gz = \text{cons tan t} \qquad (3.6.10.5)$$

Sometimes we use slightly different form for Bernoulli's equation, where we use $\frac{P}{\rho}$ for 'w' and h for 'z'. Then the form of Bernoulli's equation changes to

$$\frac{1}{2}v^2 + \frac{P}{\rho} + gh = \text{constant}$$

$$\text{or, } \frac{v^2}{2g} + \frac{P}{\rho g} + h = \text{constant} \qquad (3.6.10.6)$$

here $\frac{P}{\rho g}$ is known as pressure head, $\frac{v^2}{2g}$ is known as velocity head, h is known as elevation head and finally the sum of the above three is known as total head. One can say from this equation that if velocity increases for a fluid the pressure decreases while increase in pressure reduces the velocity of the fluid to keep total energy conserved.

❑ Worked out Example :

Water is flowing through a tapering pipe having diameters 200 mm and 100 mm at sections 1 and 2 respectively. The discharge through the pipe is 20 liters/s. The section 1 is 10m above datum and section 2 is 5m above datum. Find the the pressure at section 2, if that at section 1 is $\frac{400\text{kN}}{\text{m}^2}$.

Solution :

Velocity of fluid at section 1, $v_1 = \frac{Q}{\text{area}} = \frac{20\times10^{-3}\ \text{m}^3/\text{s}}{\frac{\pi(0.2)^2}{4}} = 0.6366\text{m/s}$

Velocity of fluid at section 2, $v_2 = \frac{Q}{\text{area}} = \frac{20\times10^{-3}}{\frac{\pi(0.1)^2}{4}} = 2.546 = \text{m/s}.$

Substituting the above values in Bernoulli's equation

$$\frac{p_1}{w} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{v_2^2}{2g} + z_2$$

$$\frac{400}{(9.81\times1000)}+\frac{(0,6366)^2}{2\times9.81}+10=\frac{p_2}{(9.81\times1000)}+\frac{(2.546)^2}{(2\times9.81)}+5$$

or, $p^2 = 9.809\ kN/m^2$

3.6.11 Application of Bernoulli's principle:

It has been observed that the thatched roofs or roofs made from asbestos sheets, of village houses get blown off during severe storms. When wind blows above the roof-tops with high speed, the local pressure decreases. However, the air inside the room being static generates high pressure zone. This pressure difference causes blowing up of the roof-tops.

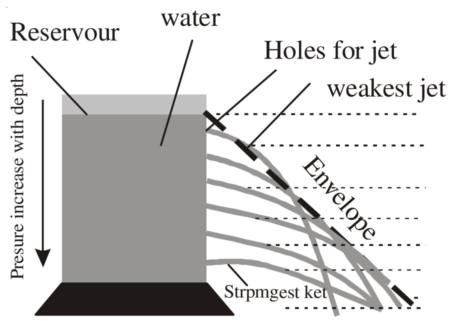
When an explosive bomb detonates in a section of the city, the air near the site of explosion moves with high speed while the air inside the rooms of houses cannot move so fast. As a result the pressure inside the room increases causing splintering of window glasses outside the rooms.

A fast moving train drags the adjacent air layers along with it thus causing a drop in the air pressure. The air layers adjacent to a standing passenger near the edge of the platform are static and as a result the pressure at that are a is high. So when the train passes by the platform a difference in pressure happens which has a tendency of pushing the passenger towards the moving train. Hence it is advisable not to stand near the edge of the platform.

It is a general observation that a table-tennis ball clings to the water jet moving upwards. The side of the ball near the edge of moving jet of water faces low pressure zone while the other face being adjacent to static air layers is at a high presure area. The change in air pressure pushes the ball towards the moving water jet.

3.6.12 Torricelli's theorem :

Torricelli's theorem, is a theorem which relates the speed of fluid flowing out of an orifice to the height of fluid above the opening.



Bernoulli's principle states that for an incompressible fluid with negligible viscosity.

$$\frac{v^2}{2g}+\frac{P}{\rho g}+h = \text{constant}$$

where v is fluid speed, g is the local acceleration due gravity (9.81 m/s^2), *h* is the fluid's height above a reference point, P is pressure, and ρ is density. Let us define the opening to be at h=0. At the top of the tank, P is equal to the atmospheric pressure. v can be considered 0 because the fluid surface drops in height extremely slowly compared to the speed at which fluid exits the tank. At the opening, h=0 and P is again atmospheric pressure. Eliminating the constant and solving gives:

$$gh+\frac{P_{atm}}{\rho}=\frac{v^2}{2}+\frac{P_{atm}}{\rho}$$

or, $v^2 = 2gh$

or, $v = \sqrt{2gh}$

This is Torricelli's theorem.

3.6.13 Venturi meter :

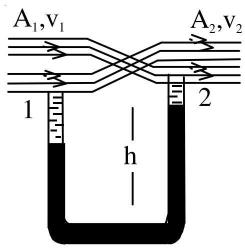
Venturi meter is an instrument used for measuring the speed and flow rate of liquid through a pipe. It is made up of a U-shaped tube filled partially with mercury. The venturi meter is connected to a pipe at two points as shown in the adjacent figure.

The area of cross sections of pipe at ends where the venturi meter has been connected are A_1 and A_2 respectively. The corresponding speed of fluid are v_1 and v_2. Let P_1 and P_2 are the pressure of fluid at the two ends 1 and 2.

Bernoulli's equation can be written as,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

The vertical height of pipe is same, so there is no contribution from the term ρgh



Here, $P_1 > P_2, P_1 - P_2 = h\rho g$, where ρ is the density of the liquid. Therefore, v_1 is less than v_2. As the amount of liquid flowing through different sections of the tube is same, hence $A_1v_1 = A_2v_2$,

$$\text{or, } v_2 = \frac{v_1 A_1}{A_2}.$$

$$P_1 - P_2 = \frac{1}{2}\rho\left[\left(v_1\frac{A_1}{A_2}\right)^2 - v_1^2\right]$$

$$= \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

From here one can easily find out the velocity of the liquid at the entry point (v_1).

$$v_1 = \left[2\frac{gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}\right]^{1/2}$$

3.6.14 Effect of temperature and pressure on the viscosity of liquids:

In most cases, a fluid's viscosity increases with increasing pressure.Compared to the temperature influence, liquids are influenced very little by the applied pressure. The reason is that liquids (other than gases) are almost non-compressible at low or medium pressures.

We can summarize the most different forms of temperature dependence of viscosity proposed under correlation methods by the following equation:

$$\text{In }(\eta) = A + \frac{B}{T+C} + a.\log T + b.T + c.T^2 + \frac{D}{T^2} + \frac{E}{T^3} + \frac{F}{T^n}$$

So, generally one can make a comment that viscosity of liquid decreases with rise in temperature.

3.6.15 Important points :

- Due to viscosity the relative velocity between successive layers of moving fluids is hindered. Co-efficient of viscosity is a measure of this hindrance. It's dimension is $ML^{-1}T^{-1}$.

- If the rate of flow is small the flow is laminar. If the rate of flow goes beyond critical velocity it turns turbulent. Critical velocity depends on Reynolds' number.

- The amount of fluid crossing a capillary tube in unit time is given by $v = \frac{\pi P r^4}{8\eta l}$.

  This is known as Poiseuille's equation.

- Stokes' law determines the fall of a solid sphere through a viscous liquid

  $$\eta = \frac{2}{9}\left[\frac{gr^2(\rho-\sigma)}{v}\right].$$

3.6.16 Questions (short answer type) :

1. State the difference between laminar and turbulent flow. What is critical velocity?
2. What are differences between Newtonian and non-Newtonian fluids?
3. What is the importance of Reynolds' number?
4. What are basic assumptions of Poiseuille's equation?
5. What conditions must be satisfied in Stokes' experiment?
6. State and explain Bernoulli's theorem.
7. How viscosity of a fluid vary with temperature and pressure?

3.6.17 Numerical Problems :

1. Water is in streamline flow through two capillaries, one of which is 1 m long and 1 mm in radius while the other is 60 cm long and 0.6 mm in radius. What is the pressure difference between the two ends of the second tube if that between the ends

of the combination is 20 cm of water?

2. Two tubes of equal lengths but of different radii are connected in series. Use Poiseuille's formula without correction to obtain an expression for the volume of liquid flowing through the tube per second when the pressure difference between the two ends of the series is P.

3. Through a glycerine column a steel ball of density 7.8×10^3 kg/m^3 and of radius 2 mm is falling. Glycerine has a co-efficient of viscosity 0.83 Pas and its density is 1.2×10^3 kg/m^3. Find the terminal velocity of the steel ball.

4. 850 cc water has flowed in 12 minutes through a horizontal capillary tube of length 20 cm and radius 0.08 cm, under 20 cm water pressure. Find the viscosity of water.

5. There is a hole in the vertical wall of a reservoir. If the depth of the hole is 2.7 cm from the top water level of the reservoir, what will be the velocity of efflux through the hole?

3.6.18 Answers to short questions:

1. See section 3.6.2 and section 3.6.4
2. See section 3.6.3
3. See section 3.6.4
4. See section 3.6.5
5. See section 3.6.7
6. See section 3.6.10
7. See section 3.6.14

3.6.19 Answers to numerical problems:

1. Poiseuille's formula is similar to Ohm's law, $Z = \frac{P}{V} = \frac{8\eta l}{\pi r^4}$. If the capillary tubes

are connected in series, the rate of flow V of water will be the same for all of them. Then $V=\frac{P_1}{Z_1}=\frac{P_2}{Z_2}=\frac{(P_1+P_2)}{(Z_1+Z_2)}=\frac{P}{Z}$, where P is the pressure difference between the extreme ends and $Z=Z_1+Z_2$

$$Z_1=\frac{8\times171\times10^{-6}\times1}{(3.141\times0.001^4}=435.5\times10^6$$

$$Z_2=\frac{8\times171\times10^{-6}\times0.6}{(3.141\times0.0006^4}=2020\times10^6$$

$Z = Z_1 + Z_2 = 2455.5 \times 10^6$. Then,

$$P_2=\frac{P\times Z^2}{Z}=\frac{20\times2020}{2455.5}=16.45\,\text{cm}$$

2. See solution 1.

3. Terminal velocity $v_t=\frac{2r^2g(\rho-\sigma)}{9\eta}=0.07\,\text{m/s}$

4. Rate of flow of water = $\frac{850}{(12\times60)}=1.18\,\text{cc/s}=V$

$$\eta=\frac{\pi Pr^4}{8Vl}=\frac{3.141\times20\times(0.08)4}{(8\times1.18\times20)}=0.136\times10^{-6}\,P$$

5. $V^2=2\times g\times h$, then V = 2×9.81×2.7, which gives V = 7.28 cm/s

Unit–7 ❑ Special Theory of Relativity

Structure :

3.7.1 Proposal :

Newton's three Laws of Motion along with the ideas about the properties of space and time provided a basis on which the motion of matter could be completely understood. However, the ideas propounded by Maxwell of a unified theory of electreomagnetism completely shttered the columns of superstructer of Physics. The theory of Maxwell was extraordinaryly successful, yet at a fundamental level it appered to be inconsistent with certain aspects of the Newtonian ideas of space and time.

A radical modification of these letter concepts, and consequently of Newton's equations themselves, was found to be order of the day. It was the genius of Albert Einstein that combined the experimental results and physical arguments of others with his own unique insights and formulated the new principles of mechanics in terms of which space, time, matter and energy were to be understood. These principles along with their consequences constitute the Special Theory of Relativity. According to Special Theory of Relatively all laws of nature appear to have the same mathematical from in all inertial frames of reference and the speed of light is constant in all directions. Latter, Einstein was able to further develop this theoy, leading to what is known as the General Theory of Relativity.

❑ Outcome

This chapter gives an overview of the Special Theory of Relativity. After reading this chapter you will be able to

(i) learn the deficiencies in ideas of space and time prevalent in the Newtonian Mechanics.

(ii) learn about the unity of space and time in description of motion of a particle.

(iii) get an idea about Lorentz transformation, Lorentz invariance.

(iv) understand time dilation and space contraction and velocity addition theorem.

(v) Lern about the relativistic Doppler effect.

(vi) understand the equivalence of mass and energy : how mass varies with velocity.

(vii) calulate energy, momentum in relativistic form and their transformations.

(viii) write and interpret Newton's laws in relativistic covariant form

3.7.2 The Michelson-Morley experiment

The Newton's laws of motion and consequently the relativity principle derived from it were quit successful till the advent of Maxwell's mathematical theory of electromagnetism which, amongst other things, provied a successful physical theory of light. It was anticipated that the equations of Maxwell should also obey the Newtonian principle of relativity, or in other words Maxwell's equations should also have the same in all inertial frames of reference. Unfortuntely, it was found that this was not case. Maxwell's equation were found to assume completely different forms in different inertial frames of reference. But in the theory of Newton a tacit assumption about a special frame of reference was made. This 'special frame' S was assumed to be the one that deined the state of absolute rest as postulated by Newton, and that stationary relative to it was a most unsual entity, the ether. The ether was a substance that was supposedly the medium in which light waves were trnsmitted in a way something like the way in which air carries sound waves.

Consequently it was belived that the velocity of light, as measured form a frame of reference moving relative to the ether would be diufferent from its value as measured from a frame of reference stationary with respect to the ether. This was the famous experiment of Michelson and Morley. It was 18 years later before the negative results of the experiment were finally explained, by Einstein.

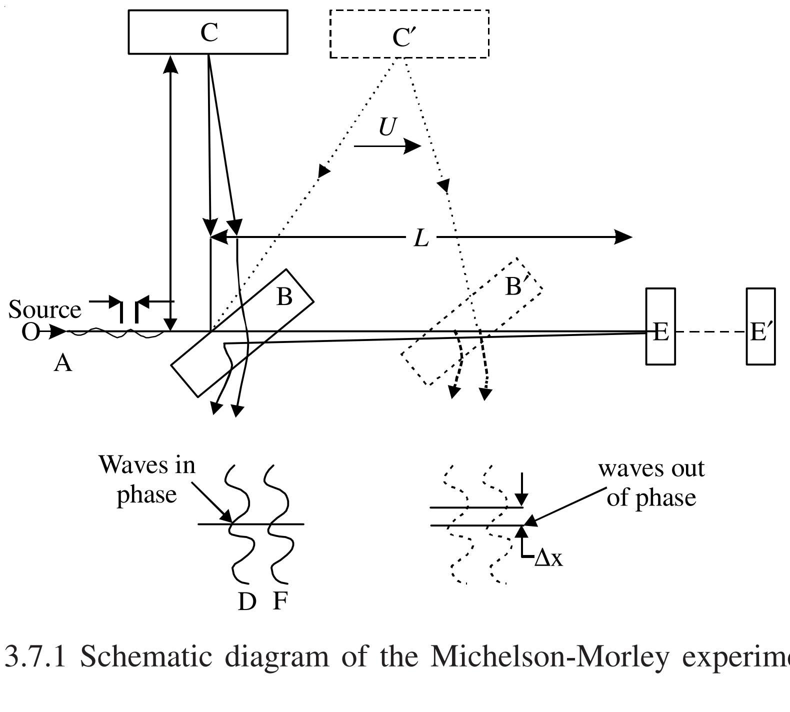


Fig 3.7.1 Schematic diagram of the Michelson-Morley experiment.

The Michelson-Morley experiment was performed with an apparatus like that shown schematically in Fig. 3.7.1. The apparatus essentially comprises of a light source A, a partially silvered glass plate B, and two mirrors C and E, mounted on a rigid base. The mirrors are placed at equal distances (L) from B. The purpose of the plate B is to split an incoming beam of light, and the two resulting beams continue in mutually perpendiculard directions to the mirrors, from where they are reflected back to B. On arrival back at B, the two beams are combined as two superposed beams, D and F. If the time taken for the light to go from B to E and back is the same as the time from B to C back, the emerging beams D and F will be in phase and will reinforce each other, but if the two times differ slightly, the beam will be slightly out of phase and interference will result. If the apparatus is "at rest" in the ether, the times should be presisely equal, but if it is moving towards the right with a velocity $\bar{u}$, there shoul be difference in the times.

In carrying out the experiment, Michelson and Morley set the apparatus in such way that the line BE was nearly parallel to the earth's motion in its orbit (at certain times of the day and night). The apparatus was amply sensitive to observe an effect of interference due to diference in arrival time, but no time difference wa found–the velocity of the earth through the ether could not be detected. The result of the experiment was null. Poincare then proposed that there is such a law of nature, that it is not possible to discover an ether wind by *any* experiment; that is, there is no way to determin an absolute velocity.

Their experimental result appeared to say that the earth was not moving relative to the ether, which was obviously wrong since the earth was moving in a circular path around the Sun, so at some particular point in time it had to be moving relative to the ether. Many theoretical attempts were put forward to patch things up while still retaining the same Newtonian ideas of space and time. It was also suggested that the earth dragged the ether in its immediate vicinity along with it. Someone proposed that objects contracted in length along the direction parallel to the direction of motion of the object relative to the ether. This suggestion, due to Fitzgerald and elaborated on by Lorentz, known as the Lorentz-Fitzgerald contraction, 'explained' the negative results of the Michelson-Morley experiment, but failed in part because no physical mechanism could be conceived that would be responsible for the contraction. It was Einstein who pointed the way out of impasse that required a huge revision of our concepts of space, and particularly of time.

3.7.3 Einstein's Postulates

The difficulty of explaining null resut of Michelson-Morley experiment that had to be resolved amounted to choosing among three alternatives:

1. The Galilean transformation was correct but there were some problems in Maxwell's equations.

2. The Galilean transformation was applicable to Newtonian mechanics only.

3. The Glilean transformation, and the Newtonian principle of relativity based on this transformation were wrong and there mut be some other principle of relativity which consistently combines Maxwell's equations and Galilean transformations.

The first possibility was thrown out as Maxwell's equations proved to be totally successful in application. The second was unacceptable as it preaches subject of non-universality of physical phenomena. The third was all that was left, so Einstein set about trying to uncover a new principle of relativity. His investigations led him to make two postulates:

1. All the laws of physics are the same in every inertial frame of reference. This postulate implies that there is no experiment from which it is possible to determin whether or not a frame of refence is in a state of uniform motion.

2. The speed of light in free space independent of the motion of its source.

Einstein made those postulates through his study of the properties of Maxwell's equations. It is these potulates that force us to reconsider what we understnd by space and time. One immediate consequence of these two postulates is that speed of light is the same in all inertial frames of reference. We can see this by considering a source of light and two frames of reference, the first frame of reference S_0 stationary relative to the source of light and the other, S, moving relative to the source of light. In both of these frames the velocity of light, irrespective of the dynamical status of the frames, is found to be c.

3.7.4 Lorentz Transformations

The constancy of the speed of light, independent of the motion of the emanating source, gives rise to the relations between time and space coordinates in different inertial frames of reference known as Lorentz transformations.

Let us consider two inertial reference frames S and S′ with a relative velocity v_x between them. The time and space coordinates of a point under consideration are (t, x, y, z) and (t′, x′, y’, z′) in the frames S and S′, respectively. All the coordinate axes in the two frames mentioned above are parallel and oriented such that the frame S′ is moving in the positive x-direction with speed v_x, as viewed from S. Let the origins of the coordinates in S and S′ be coincident at t = t′ = 0. If a light source at rest at the origin in S (and so moving with a speed v_x in the negative x-direction as seen from S′) flickers on and off rapidly t = t′ = 0, Einstein’s second postulate implies that observers in both S and S′ will see a spherical shell of radiation with increasing radius moving outward from the respective origins with speed c. The wave front reaches a point (x, y, z) in the frame S at a time t given by the equation.

$$c^2t^2 - (x^2 + y^2 + z^2) = 0 \qquad (3.7.3.1)$$

Similarly, in the frame S′ the same wave front is specified by

$$c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0 \qquad (3.7.3.2)$$

We assume that space-time is homogeneous and isotropic, as implied by the first postulate. Then the connection between the two sets of coordinates is linear. The events defined by equations (3.6.3.1) and (3.6.3.2′) are then related by

$$c^2t'^2 - (x'^2 + y'^2 + z'^2) = k[c^2t^2 - (x^2 + y^2 + z^2)] \qquad (3.7.3.3)$$

where k = $k(v_x)$ is a possibl change of scale between frames. With the choice of orientation of axes and considerations of the inverse transformation from S′ to S it is straightforward to show that $k(v_x) = 1$ for all v_x and that the time and space coordinates in S′ are related to those in S by the Lorentz transformation

$$x_0' = \gamma(x_0 - \beta x_1)$$

$$x_1' = \gamma(x_1 - \beta x_0)$$

$x_2' = x_2$

$x_3' = x_3$ (3.7.3.4)

where we have used the notation $x_0 = ct$, $x^1 = x$, $x_2 = y$, $x_3 = z$ and also the symbols,

$$\vec{\beta} = \frac{\vec{v}_x}{c}, \beta = |\beta| \text{ and } \gamma = \left[\frac{1 - v_x^2}{c^2}\right]^{-\frac{1}{2}} = \left[1 - \beta^2\right]^{-\frac{1}{2}} \quad (3.7.3.5)$$

The inverse Lorentz transformation is given by

$x_0 = \gamma(x_0' + \beta x_1')$

$x_1 = \gamma(x_1' + \beta x_0')$

$x_2 = x_2'$

$x_3 = x_3'$

This can also be obtained by replacing v_x by $- v_x$ and swapping primed and unprimed symbols in the first set of equations (3.7.3.4). This is how it must turn out, since if S′ has velocity $\vec{v}_x$ in S,then S has velocity $- \vec{v}_x$ in S′ and both are equaly valid inertial frames.The coordinates perpendicular to the direction of relative motion are unchanged while the parallel coordinate and the time are transformed. This can be contrasted with the Galilean trnsformations

$x' = x - vt$, $y' = y$, $z' = z$ and $t' = t$, where time is taken to be absolute.

❑ Worked out Example :

Suppose an event A takes place in frame S at $x_a = 0$ and $t_a = 0$ and another event B takes lace at $x_B = b$ and $t_B = 0$. These two events are simultaneous in S. Will they be simultaneous in frame S′ which is moving with a velocity v along x?

From the Lorentz transormations we get $x_A' = 0$, $t_A' = 0$, but $x_B' = \gamma b$ and $t_B' = \gamma(v/c^2)b$.

Now, according to S′ clocks event B has occurred before event A. So, they are not simultaneous in frame S′.

Exercise :

Find the inverse Lorentz transformations from Eq. (3.7.3.4)

3.7.5 Length Contraction

Let us extract from the Lorentz transformation the phenomenon of Lorentz contraction first. For Lorentz contraction, one should consider not two different events but two different world lines. They are the world lines of the two ends of some object in the *x* direction, fixed in S,. Now, we place the origin of the frame of reference S on one of these world lines, and then the other end lies at $x = L_0$ for al t, where L_0 is the rest length. Let us consider these world lines in the frame S′ and pick the time t′ = 0. At this moment, the world line passing through the origin of S is also at the origin of S′, i.e. at x′ = 0. From the lorentz transformation, the other world line can be found at

$$\gamma\left(t - \frac{vL_0}{c^2}\right); \quad x' = \gamma(-vt + L_0) \qquad (3.7.4.1)$$

Since we are considering the situation at t′ = 0 we deduce from the first equation that

$$x' = \quad x' = \gamma L_0\left(1 - \frac{v^2}{c^2}\right) = \frac{L_0}{\gamma}$$

Thus in the frame S′ at a given instant the two ends of the object are at x′ = 0 and $x' = \dfrac{L_0}{\gamma}$.

Therefore the length of the object is reduced from L_0 by a factor γ. This is Lorentz contraction.

❑ Worked out Example:

At what speed does a meters tick move if its length is observed to shrink to 0.5 m?

Our assumption is that the stick is at rest in S′. In S the meter stick is moving in the positive x direction with a speed of v. Now, we know that $x' = \gamma(x - vt)$. Let $\Delta x'$ be the length of the meter stick measured at ret in S′. Then $\Delta x' = \gamma\Delta x$, as $\Delta t = 0$. i.e. the measurements are done at the same time. So, it can be shown that

$$\beta^2 = 1 - \left(\frac{\Delta x}{\Delta x'}\right)^2.$$

Now, $\Delta x = \left(\frac{\Delta x'}{2}\right)^2$ which gives $\beta = 0.866$ or $v = 0.866$ c.

3.7.6 Time dilation :

Our concept of time has to be drastically modified, as one considers the unexpected consequences of the Lorentz transformation. In a frame of reference S′, let us consider a clock C′ placed at rest at some point x′ on the X axis. Let us suppose that this frame is moving with a velocity $\bar{v}_x$ relative to some other frame of reference S. At a time t_1' registered by clock C′ there will be a clock C_1 in he S frame of reference passing the position of C′

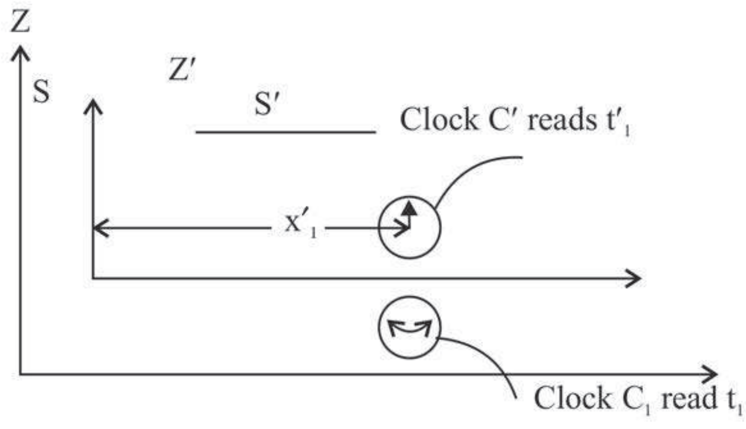


Figure 3.7.5.1 In frame S′ the stationary clock C′ reads t_1' while passing a stationary clock in the frame S, which reads t_1 at the same instant.

The time registered by C_1 will be given according to Lorentz Transformation as

$$t_1 = \gamma\left(\frac{t_1' + v_x x'}{c^2}\right) \qquad (3.7.5.1)$$

After some time the clock C′ will register the time t_2' at which instant a different clock C_2 in the frame S will pass the position x_1' in S′.

Now, this clock C_2 will show a time $t_2 = \gamma\left(\frac{t_2' + v_x x'}{c^2}\right)$ (3.7.5.2)

Thus one get from eq. (3.6.5.1) and (3.6.5.2)

$$\Delta t = t_2 - t_1 = \gamma(t_2' + t_1') = \gamma\Delta t' \qquad (3.7.5.3)$$

It appears from the eq. (3.6.5.3) that time is passing slowly in he frame S′ as observed from the frame S. This phenomenon is known as time dilation.

Another important aspect of Lorentz Transformation is that events which take place simultaneously in one frame of reference will not appear simultaneous in another frame of reference which is moving with a velocity v with respect to the former one.

To prove the above statement let us consider two events A and B which are taking places at x_A and x_B at the same time, i.e. $t_A = t_B$. Then according to Lorentz Transformations the time interval between these two events in S′ will be

$$t_2' - t_1' = \gamma\left(t_2 - \frac{v_x x_2}{c^2}\right) - \gamma\left(t_1 - \frac{v_x x_1}{c^2}\right)$$

$$= \gamma\left(\frac{t_1 - x_2}{c^2}\right) v_x \qquad (3.7.5.4)$$

is not zero as x_1 is not equal to x_2.

Thus events which are simultaneous in frame S are not simultneous in S′.

❑ Worked out Example :

1. At what speed does a clock move if it runs at a rate which is one-half the rate of a clock at rest?

Let us assume in the frame S′ the clock is at rest. For an observer who is stationary in S, The same clock is moving in the positive x direction with a speed v. Now.

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Let $\Delta t'$ = time interval measured in S′ (proper time) when $\Delta x' = 0$ and let Δt be the time interval measured at rest in S. Then $\Delta t = \gamma\Delta t'$ and therefore

$$\beta^2 = \left[1 - \left(\frac{\Delta t'}{\Delta t}\right)^2\right]$$

Then $\beta = 0.866$, when $\frac{\Delta t}{\Delta t} = 2$.

2. An atomic clock is placed in a jet airplane. The clock meaures a time interval of 3600s when the jet moves with a speed 400m/s. What will be the time interval recorded by an identical clock held by an observer at rest on the ground?

Let us take the S frame to be attached to the Earth and the S′ frame to be the rest frame of the atomic clock. Now,

$$\gamma \simeq 1 + \frac{\beta^2}{2}$$

$$\delta t = \Delta t - \Delta t' \simeq \frac{\beta^2 \Delta t'}{2}$$

So, we get $\delta t = 3.2$ ns when $v = 400$ m/s and $\Delta t' = 3600$ s.

3. The average lifetime of a π meson in its own frame of reference is 26.0 ns. (This is its proper lifetime.)

(a) If the π meson moves with speed 0.95c with respect to the Earth, what is its lifetime as measured by an observer at rest on the Earth?

(b) What is the average distance it travel before decaying as measured by an observer at rest on the Earth?

Let the S frme to be attached to the Earth and the S′ frame to be the rest frme of the π meson.

We get from eq. (3.7.5.3) that $\Delta t' = 26.0$ ns and $v = 0.95c$.

The average distance travelled before decaying as measured by an observer at rest on the Earth is $v\,\Delta t = 24.0$ m

4. The muon is an unstable particle that spontaneously decays into an electron and two neutrinos. If the number of muons at $t = 0$ is N_0, the number N at time t is

$$N = N_0 e^{-t/\tau}$$

where $\tau = 2.20\ \mu s$ is the mean lifetime of the muon. Suppose the muons move at speed 0.95c.

What is the observed lifetime of the muons? How many muous remain after travelling a distance of 3.0 km?

We take the S frame to be attached to the Earth and the S′ frame to be the rest frame of the muon. It follows from Eq. (3.7.5.3) that $\Delta t = 7.046 \times 10^{-6}$s when $\Delta t' = 2.2 \times 10^{-6}$s and

$\beta = 0.95$.

A muon at this speed travels 3.0 km in 10.53×10^{-6} s. After travelling this distance, N muons remain from an initial population of N_0 muons where

$N = N_0\, e^{-t/\tau} = N_0\, e^{-10.53/7.046} = 0.225N_0$.

Exercise 2 :

A rod of length L_0 moves with speed v along the horizontal direction. The rod makes an angle θ_0 with respect to the x′ axis.

(a) Deternine the length of the rod as measured by a stationary observer.

(b) Determine the angle the rod makes with the x axis.

3.7.7 Lorentz Invariance :

Lorentz invariance demands that the laws of physics are the same for different observers moving with constant different velocities like an observer who is rotated through some angle, or traveling at constant speed relative to the observer at rest. Before any discussion on Lorentz invariance it is advisable to discuss about the nature of vectors involved. So we start from a discussion on four-vectors.

3.7.7.1 Four Vectors :

Because (ct, x, y, z) and have the similar transformations under changes of coordinate, we call them both 4-vectors. The vectors with which we are familiar can be defined as objects possessing a magnitude and a direction, or objects that transform in a well-defined way under rotations of coordinate system. 4-vectors are difined as objects that transform under the Lorentz transformations when converting between the measurements made by two different inertial (non-accelerating) observers.

In the Lorentz transformation we get the description of the transformation of the coordinates of a point from one inertial frame to another. For rotation in three dimensions, the basic transformation law is defined in terms of the coordinates of a point. In three dimmensions we designate x_1, x_2, x_3 as the components of any vector. We describe by the same name any three physical quantities that transform under rotations in the same way as the components of x. We therefore anticipate that there are many physical quantities that trnform under Lorentz transformations in the same manner as the time and space coordinates of a point. By analogy we speak of 4-vectors. The coordinate 4-vector is (x_0, x_1, x_2, x_3). Similarly the components of an arbitary 4-vector is (A_0, A_1, A_2, A_4) where A_1, A_2, A_3 are the components of a 3-vector A. The Lorentz transformation law for an arbitary 4-vector is

$$A_0' = \gamma(A_0 - \vec{\beta}.\vec{A}) \qquad (3.7.6.1)$$

$$A_{\parallel}' = \gamma\,(A_{\parallel} - \beta\, A_0) \qquad (3.7.6.2)$$

$$A_{\perp}' = A_{\perp} \qquad (3.7.6.3)$$

here the parallel and perpendicular signs indicate components relative to the velocity $\vec{v} = \beta\vec{c}$.

The invariance from one inertial frame to another can be shown in the form

$$A_0'^2 - |\vec{A}'|^2 = A_0^2 - |\vec{A}|^2 \qquad (3.7.6.4)$$

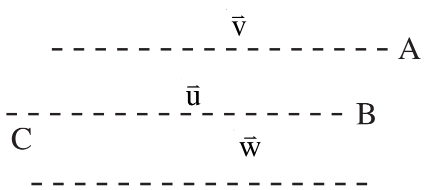
where the components (A_0', A′) and (A_0, A) refer to any two inertial reference frames. For two 4-vectors (A_0, A_1, A_2, A_3) an (B_0, B_1, B_2, B_3) the "scalar product' is an invariant, that is,

$$A_0'B_0' - \vec{A}' \cdot \vec{B}' = A_0B_0 - \vec{A}\cdot\vec{B} \qquad (3.7.6.5)$$

This result can be verified by explicit construction of the left-hand side. It is the Lorentz transformation analogue of the invarince of $\vec{A}\,.\,\vec{B}$ under rotation in three dimensions.

3.7.8 Addition of Velocities :

Suppose an object A is moving in the positive x-direction with a velocity $\vec{v}$ relative to an object B, and B is moving with a velocity $\vec{u}$ (in the same direction) relative to an object C. What will be the velocity of A relative to C?



In non-relativistic case, the velocities are simply added and the answer is that A is moving with a velocity $\vec{w} = \vec{u} + \vec{v}$ relative to C. But in special relativity the velocities must be combined using the formula

$$w = 1 + \cfrac{u+v}{\cfrac{uv}{c^2}} \tag{3.7.8.1}$$

If $\vec{u}$ and $\vec{v}$ are both small compared to the speed of light c, then the answer is approximtely the same as the non-relativistic theory. In the limit where $\vec{u}$ is equal to $\vec{c}$ (in the case C is a massless particle moving to the left at the speed of light), the sum gives $\vec{c}$. This proves that anything going at the speed of light does so in all inertial reference frames.

This change in the velocity addition formula the non-relativistic to the relatives theory is not due to making measurements without taking into account light-travel times, or the Doppler effect. Rather, it is what is observed after such effects have been accounted for. It is an effect of special relativity which cannot be accounted for using Newtonian mechanics.

The formula can also be applied to velocities in opposite directions by simply changing signs of velocity values, or by rearranging the formula and solving for $\vec{v}$. In other words, if B moving with velocity $\vec{u}$ relative to C and A is moving with velocity w relative to C then the velocity of A relative to B is given by

$$v = \cfrac{w-u}{1-\cfrac{wu}{c^2}} \tag{3.7.8.2}$$

Notice that the only case with velocities less than or equal to $\vec{c}$ that is singular is $\vec{w} = \vec{u} = \vec{c}$, which gives the indeterminate value zero divided by zero. In other words, it is meaningless to ask for the relative velocity of two photons that are moving in the same direction.

Originally we wanted to know the velocity of C as measured relative to A, and not the speed at which B observes A and C to aproach each other. The rulers and clocks set up by B cannot be used to measure distances and times correctly by A, since for A the clocks do not even show the same time. To go from the reference frame of A to the reference frame of B, a Lorentz transformation must be applied on co-ordinates in the following way (taking the x-axis parallel to the direction of travel and the space-time origins to coincide):

$$x_B = \gamma(v)(x_A - vt_A) \qquad (3.7.7.3)$$

$$t_B = \gamma(v)\left(t_A - \frac{v.x_A}{c^2}\right) \qquad (3.7.7.4)$$

$$(v) = \sqrt[\gamma]{1-\frac{v^2}{c^2}} \qquad (3.7.7.5)$$

To go from the frame of B to the frame of C we should apply a similar transformation

$$x_c = \gamma(u)(x_B - u\ t_B) \qquad (3.7.7.6)$$

$$t_c = \gamma(u)\left(t_B - \frac{u\,x_B}{c^2}\right) \qquad (3.7.7.7)$$

These two transformations can be combined to give a transformation which simplifies to

$$x_C = \gamma(w)(x_A - wL_A) \qquad (3.7.7.8)$$

$$t_C = \gamma(w)\left(t_A - \frac{w\,x_A}{c^2}\right) \qquad (3.7.7.9)$$

$v = \dfrac{u+v}{1-\dfrac{uv}{c^2}}$, this proves the velocity addition theorem for relativistic motions.

A novel feature of the veloity addition formula is that if two velocities less than the speed of light are combined, one always get a result that is still less than the speed of light. This means that no amount of combining velocities can take any one beyond the speed of light.

❑ Worked out Examples :

Two meteorites approach each other, each moving with the same speed as measured by a stationary observer on the Earth. Their relative speed is 0.70c,

Determine the velocities each meteorite as measured by the stationary observer on Earth.

Lorentz velocity transformation gives

$$u'_x = \frac{u_x - v}{1 + \frac{u_x v}{c^2}}$$

where u_x is the velocity of an object measured in the S frame, u_x' is the velocity of the object measured in the S′ frame and v is he velocity of the S′ frame along the x-axis of S.

Let us take the S frame to be attached to the Earth and the S′ frame to be attached to the meteorite moving to the right velocity v. The other meteorite has velocity $u_x = -v$ in S and velocity $u_x' = -0.70c$ in S′.

We get $0.70 = \frac{2\beta}{\left(1+\beta^2\right)}$, solving which yields $\beta = 0.41$. As measured by the stationary observer on Earth, the meteorites are moving with velocities ± 0.41c.

Exercise 3:

Two space ships approach each other with velocities of 0.9c. According to an observer on the space ship, what is the velocity of the other ship.

3.7.9 The Relativistic Deppler Effect

There is a shift in frequency in the sound of a train's horn as the train passes by due to the relative motion of the train and the audience. Similarly, there is a shift in frequency of light due to relative motion of the source and observer. This is known as Doppler Effect. Relaivity modifies this Dopplerr Effect due to time dilation.

Let us consider a source of sound at rest at the origin with an observer moving in the positive x-direction. We shall consider the possibility that the observer is located at some distance in y. The beginning of one wavelength is at $t_1 = 0$ and $x_1 = y_1 = 0$. The end of the wave is emitted at $t_2 = \tau$ and still at $x_2 = y_2 = 0$. This transform to the observers frame to be at

$$ct_1{}' = \gamma(ct_1 - \beta x) = 0,\ x_1{}' = \gamma(x - \beta ct) = 0,\ y_1{}' = y_1 = 0 \qquad (3.7.8.1)$$

$$ct_2{}' = \gamma(c\tau - \beta x) = \gamma c\tau \qquad (3.7.8.2)$$

$$x_2{}' = \gamma(x - \beta c\tau) = -\beta\gamma c\tau \qquad (3.7.8.3)$$

$$y_2{}' = y_2 = 0 \qquad (2.7.8.4)$$

$$\tau' = \gamma\tau \qquad (3.7.8.5)$$

The time to emit the wave in the observer frame is dilated which decreases the frequency. If the wave travels to the observer in the y direction, the travel time is essentially the same for the beginning and the end of the wave so the frequency is not affected. That is the transverse Doppler effect gives a red-shift

$$v_\perp{}' = \frac{v}{\gamma} \qquad (3.7.8.6)$$

which is entirely a relativistic effect. (3.7.8.6)

If the observe is moving directly away from the source we have the additional effect of the distance to the observer increasing with time which gives rise to the parallel Doppler effect. The time at which the beginning and end of the wave arrive at the observer is

$$t_{10}{}' = \frac{t_1{}' - x_1{}'}{c} = 0 \qquad (3.7.8.7)$$

$$t_{20}' = \frac{t_2' - x_2'}{c} = \gamma\tau + \beta\gamma\tau = \gamma\ (1 + \beta)\tau \tag{3.7.8.8}$$

$$\tau_0' = (1+\beta)\tau = \frac{(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}}\tau = \sqrt{\frac{1+\beta}{1-\beta}}\tau \tag{3.7.8.9}$$

$$v_{||}' = \sqrt{\frac{1-\beta}{1+\beta}}v \tag{3.7.8.10}$$

β is positive for the observer moving away from the source and negative if the observer is moving towards the source.

❑ Worked out Example :

How fast and in what direction must galaxy A be moving if an absorption line found at wavelength 550nm (green) for a stationary galaxy is shifted to 450 nm (blue) (a "blue-shift") for galaxy A?

Galaxy A is approaching since an absorpton line with wavelength 550nm for a stationary galaxy is shifted to 450 nm. To find the speed v at which A is approaching, we use

$$v_{obs} = \sqrt{\frac{1+\beta}{1-\beta}}v_{source}$$

As $\lambda = \frac{c}{v}$,

$$\lambda_{obs} = \sqrt{\frac{1-\beta}{1+\beta}}\lambda_{source}$$

from which

$$\beta = (\lambda_{source}^2 - \lambda_{obs}^2)/(\lambda_{source}^2 + \lambda_{obs}^2)$$

We get that $\beta = 0.198$ when $\lambda_{source} = 550$ nm and $\lambda_{obs} = 450$ nm.

3.7.10 Relativistic mass

The mass or inertia of body is too affexted when measured in different inertial frames. To establosh the relation between how the mass varies with velocity, we consider two references frames S and S′ as in fig. 3.7.10.1 . S′ is moving with a velocity v with respect to S-frame, x, x′ coincident.

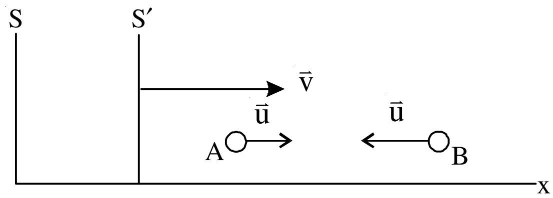


Figure : 3.7.9.1

We consider two identical masses A and B each of mass m measured in frame S′ moving in opposite direction, collide and coalesce to form a single mass 2m.

The from conservation of motion in S′- frame

$$mu - mu = 0 = 2mV \text{ i.e } V \overset{\bar{v}}{=} 0 \quad \ldots \quad \ldots \qquad (3.7.10.1)$$

Now we view the collision from S-frame of reference. Let u_1 and u_2 be the velocities of the masses A and B respectively with respect to S-frame. Then using relativistic velocity transformation equation, we can write,

$$u_1 = \frac{v+u}{\frac{1+uv}{c^2}} \text{ and } u_2 = \frac{v-u}{\frac{1-uv}{c^2}} \quad \ldots \ldots \qquad (3.7.10.2)$$

Then from conservation of momentum in S-frame,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

$$m_1 \frac{v+u}{\frac{1+uv}{c^2}} + m_2 \frac{v-u}{\frac{1-uv}{c^2}} = (m_1 + m_2)v,$$ as they becomes at rest after ins S′-frame.

Thus $$m_1\left\{\frac{v+u}{1+\frac{uv}{c^2}}-v\right\}=m_2\left\{v-\frac{v-u}{1-\frac{uv}{c^2}}\right\}$$

or, $$\frac{m_1u\left(1-\frac{v^2}{c^2}\right)\left(1+\frac{uv}{c^2}\right)=m_2u\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{uv}{c^2}\right)}$$

or, $$m_1\left(1-\frac{uv}{c^2}\right)=m_2\left(1+\frac{uv}{c^2}\right)\ \ldots\ \ldots \qquad (3.6.10.3)$$

Now $$\frac{1-\frac{u_1^2}{c^2}}{1-\frac{u_2^2}{c^2}}=\frac{\dfrac{1-\frac{(u+v)^2}{c^2}}{1+\frac{(uv)^2}{c^2}}}{1-\dfrac{\frac{(v-u)^2}{c^2}}{1-\frac{(uv)}{c^2}}}=\frac{1-\frac{(uv)^2}{c^2}}{1+\frac{(uv)^2}{c^2}} \text{ or, } \frac{\sqrt{1-\frac{u_1^2}{c^2}}}{\sqrt{1-\frac{u_2^2}{c^2}}}=\frac{1-\frac{(uv)}{c^2}}{1+\frac{(uv)}{c^2}} \qquad (3.7.10.4)$$

Using equation (3.7.10.4) in (3.7.10.3) we have

or, $$m_1\sqrt{1-\frac{u_1^2}{c^2}}=m_2\sqrt{1-\frac{u_2^2}{c^2}}\ \ldots\ \ldots \qquad (3.7.10.7)$$

Obviously, eqn. (3.7.10.5) is applicable in all inertial rames for any values of u. If m_0 is the mass of a particle measured in a frame at rest with the body and m is the mass of the particle in reference frame moving with velocity $\bar{u}$ then from eqn. (3.7.10.5) then, $$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \quad \ldots \ldots \qquad (3.7.10.7)$$

Here m is referred as relativistic mass m_0 as rest mass.

3.7.11 Mass–Energy Equivalence

To establish mass-energy equivalence in Einsten's special theory of relativity, we consider a mass be acted by an external force. In the cours of its motion the force applied on the mass m at instant t be $\vec{F}$ when its velocity is $\vec{v}$. Let d$\vec{v}$ be the change of velocity in time interval dt. Then the change in kinetic energy

$$dT = \vec{F}.d\vec{r} = \frac{d(m\vec{v})}{dt}.d\vec{r} = \vec{v}.d(m\vec{v}) = vd(mv)$$

so the total kintic energy acquired by the body starting from rest and acquiring the velocity $\vec{v}$ is, $$T = \int_0^v \vec{v}.d(m\vec{v}) = \int_0^v \left[v^2 dm + mv\ dv\right] \quad \ldots \ldots \qquad (3.7.11.1)$$

Now from Einstein's mass variation equation $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

We have $2m\ dmc^2 = 2m\ dmv^2 + m^2\ (2v\ dv)$

$$dmc^2 = dm\ v^2 + mv\ dv \quad \ldots \quad \ldots \quad \ldots \qquad (3.7.11.2)$$

Using equations (3.7.11.1) and (3.7.11.2)

$$T = \int_{m_0}^{m} dm c^2 = (m - m_0)c^2, \text{ or, } mc^2 = T + m_0c^2 \quad \ldots\ldots. \qquad (3.7.11.3)$$

We can visualize equation (3.7.11.3) as, that due to the application of force the energy increases from energy possesssed by the body at rest (m_0c^2) or rest energy, to the energy possessed by the body at motion (mc^2) with respect to rest frame. Thus from equation (3.7.11.3) we can conclude the total energy possessed by the body

$$E = mc^2 \qquad (3.7.10.4)$$

This equation relates mass-energy and is known as relativistic mass-energy equivalence.

3.7.12 Relativistic energy and momentum transformation

We consider two reference frames S and S′. S′ is moving with respect to S frame with a velocity $\vec{v}$ in + x-direction with x and x′ axis coincident (fig-3.7.8.9). A particle of mass m is moving with a velocity $\vec{u}$ in S-frame.

Then total energy in S-frame, $E = mc^2 = \dfrac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}}$ ... ... (3.7.12.1)

and momentum $\dfrac{\vec{p} = m\vec{u} = m_0\vec{u}}{\sqrt{1-\dfrac{v^2}{c^2}}}$, with its components are

$$\frac{p_x = mu_x = m_0u_x}{\sqrt{1-\dfrac{v^2}{c^2}}},\quad \frac{p_y = mu_y = m_0u_y}{\sqrt{1-\dfrac{v^2}{c^2}}},\quad \frac{p_z = mu_z = m_0u_z}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (3.7.12.2)$$

Let us view the corresponding energy and momentum from S′-frame. $\vec{u}'$ be the velocity of the particle in S′-frame. From Lorentz transformation,

$$u'_x = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)}; u'_y = \frac{u_y\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 - \frac{vu_x}{c^2}\right)} \text{ and } u'_z = u_z\sqrt{1 - \frac{vu_x}{c^2}} \quad \ldots \quad \ldots \quad \ldots \quad (3.7.12.3)$$

Equation (13) yields to $\left(1 - \frac{u'^2}{c^2}\right) = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{vu_x}{c^2}\right)}$ (3.7.12.4)

$$E' = m'c^2 = \frac{m_0c^2}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 - \frac{u^2}{c^2}}} = \frac{mc^2\left(1 - \frac{vu_x}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{E - vP_x}{\sqrt{1 - \frac{v^2}{c^2}}},$$

using eqn. (3.7.12.3)

where p_x = x component of momentum with respect to S–frame.

The inverse transformation equation will be, $E = \frac{E' + vp'_x}{\sqrt{1 - \frac{v^2}{c^2}}}$,

Similarly the momentum in S′-frame,

$$p'_x = m'u'_x = \frac{\frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}}(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}\frac{(u_x - v)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{p_x - \frac{vE}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \ldots \quad (3.7.12.5)$$

$$p'_y = m'u'_y = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}u'_y = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}\frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)} = \frac{m_0 u_y}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\sqrt{1-\frac{v^2}{c^2}}\left(1-\frac{vu_x}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}\left(1-\frac{vu_x}{c^2}\right)} = mu_y = p_y$$

(3.7.12.6

Similartly $p'_z = p_z$ ...

Set of equations (3.7.12.4), (3.7.12.5) *and* (3.7.12.6) constitute momentum transformation equation.

3.7.12.1 Relation between Energy and Momentum of a Particle:

We have seen earlIer the equivalence of mass and energy as

$E = mc^2$. So, one gets $E^2 = m^2c^4$, or one can write $E^2 = m^2c^2 - m_0^2c^4 + m_0^2c^4$.

So, $$E^2 = \frac{m_0^2c^4[1]}{\frac{(1-v^2)}{c^2}} - [1] + m_0^2c^4 = m_0^2c^4\left[\frac{\frac{(v^2)}{c^2}}{\frac{(1-v^2}{c^2}}\right] + m_0^2c^4 = p^2c^2 + m_0^2c^4.$$

(3.7.11.7)

3.7.13 The Lorentz transformation equation for Newton's Laws of motion

Consider a particle of mass m moving in S-frame with a velocity $\vec{u}.(p_x, p_y, p_z)$ are the (x, y, z) component of momentum.

Then in S-frame, $F_x = \frac{dp_x}{dt}$, $F_y = \frac{dp_y}{dt}$ and $F_z = \frac{dp_z}{dt}$. In S′-frame the corresponding components are,

$$F'_x = \frac{dp'_x}{dt'}, \quad F'_y = \frac{dp'_y}{dt'} \text{ and } F'_z = \frac{dp'_z}{dt'}.$$

Now from Lorentz transformation

$$\frac{dt}{dt'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)} \quad \dots \quad \dots \tag{3.7.12.1}$$

$$F'_x = \frac{dp'_x}{dt'} = \frac{dp'_x}{dt}\frac{dt}{dt'} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{\sqrt{1-\frac{v^2}{c^2}}} \frac{\sqrt{1=\frac{v^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{\left(1-\frac{vu_x}{c^2}\right)} \tag{3.7.12.2}$$

$$F'_y = \frac{dp'_y}{dt'} = \frac{dp'_y}{dt}\frac{dt}{dt'} = \frac{dp_y}{dt}\frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)} \quad \dots \dots \tag{3.7.12.3}$$

$$F'_y = \frac{dp'_z}{dt'} = \frac{dp'_z}{dt}\frac{dt}{dt'} = \frac{dp_z}{dt}\frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)} \quad \dots \quad \dots \tag{3.7.12.4}$$

3.7.14 Substance

After a thorough learning of this important chapter you must have understood that our common sense always does not work. Both time and displacement have equal roles in understanding the motion of any object. Through Lorentz transformations we can transform respect to the other and vice-versa. Length contraction and time dilations are major fallouts of these transformations. Equvalence of mass and energy shows clearly the importance of reltivistic kinematics. Newton's laws motion can be interpreted in terms of relativistic transformations. The outbound velocities of distant stars can be found from the relativistic Doppler Effect.

3.7.15 Short Quations:

1. State the differences between inertial an non-inertial frames of reference.
2. Explain why any object cannot move with a speed more than the speed of light.
3. Calculate the rest energy of electron and proton in electron Volt. Given m_e = 9.11 x 10^{-31} kg and m_p = 1.673 × 10^{-21} kg.
4. A cubical shape of body with 1m length on each side when it is rest, moves with a velocity 0.6c along x – direction. What is the shape and dimension of the body noted by an observer on the ground.
5. Two β particles move in opposite direction with velocity 0.6c in the laboratory, S′ frame. Calculate the velocity of one β particle in the moving frame attached to the other β particle by applying relativistic transformation.

3.7.16 Answer to Exercises and short questions :

Ex 1. Use eq. (3.6.3.4)

Ex. 2. Let us take the S′ frame to be the rest frame of the rod. A rod of length L_0 in S′ makes an angle θ_0 with the x' axis. Its projected length Δx_0 and $\Delta y_0 = L_0 \cos\theta_0$ and $\Delta y_0 = L_0 \sin\theta_0$. In a frame S in which th rod moves at speed v along the x axis, the projected lengths Δx and Δy are given by $\Delta x' = \gamma\, \Delta x$ and $\Delta y' = \Delta y$, as $y' = y$.

The length L of the rod as measured by a stationary observer in S is

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2} = L_0\left(1-\beta^2\cos^2\theta_0\right)^{\frac{1}{2}}$$

The rod makes an angle θ with the x axis in S where

$$\tan\theta = \frac{\Delta y}{\Delta x} = \gamma \tan\theta_0.$$

The rod in S appears contracted and rotated.

Ex. 3 Use the velocity addition formula, $v' = \dfrac{(u+v)}{\dfrac{(1+uv)}{c^2}}$. Both $\bar{u}$ and $\bar{v}$ are 0.9c.

3.7.17 Short Question :

1. See text.
2. Use velocity addition theorem.
3. Use $E = mc^2$.
4. $x_0 = y_0 = z_0 = 1$m. v = 0.6c, $x = x_0\sqrt{1-\dfrac{v^2}{c^2}}$, $y = y_0$ and $z = z_0$.

   We get x = 0.8m, y = 1m and z = 1m.

5. $u_1 = 0.6c$, $u_2 = -0.6c$ in S-frame and $u_2' = \dfrac{(u_2 - v)}{\dfrac{(1+u_2 v)}{c^2}}$ in S′ frame.

   $u_2 = -0.6c$ and $v = +0.6c$ which gives $u_2' = -0.88c$.

Notes

Notes

Notes