PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses *viz. core, generic, discipline specific, elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade "A".

UGC (Open and Distance Learning Programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for UG programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed by the Teachers, Officers, Staff of the University and I heartily congratulate all concerned in the preparation of these SLMs.

I wish you all a grand success.

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Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Physics (HPH)
Course : Waves and Optics
Course Code : GE-PH-31

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Unit - 1 □ Simple Harmonic Motion

Structure

1.0 Objectives

After studying this unit you will be able to

- learn the basic idea for the somple harmonic motion.
- establish the differential equation for the system executing SHM and how to solve it.
- define the terms amplitude, phase, time period, frequency, velocity and acceleration.
- compute potential, kinetic and total energies of a body executing SHM.

1.1 Introduction

In your school science courses, you must have learnt about different types of motions. As for example, translational motion is the motion of a free-falling body under the action of gravity and the periodic motion, which is the motion of a particle, traces and the same path again and again and comes back to a given point on the path at a regular internal of time.

A particle in periodic motion performs a vibratory or oscillatory motion if it moves to and forepeatedly over the same path at regular intervals of time e.g. the motion of a simple pendulum is clearly vibratory or oscilallatory. Again when a body is moving uniformly in a circular path executes a periodic motion, but not the vibratory or oscillatory e.g. the motion of a ceiling fan.

The simplest kind of oscillatory motion which can be analyzed mathematically easily is the Linear or simple Harmonic Motion (SHM).

In this unit, we will study oscillatory systems using simple mathematics. We calculate the kinetic energy, potential energy and hence we see the conservation of energy of SHM. We also study the different types of cases of SHM as we see in nature are very important.

1.2. Definition and Basic characteristics of Simple Harmonic Motion (SHM)

Definition : When a body or a particle moves to and fro along a straight line such that the restoring force acting on it is always directed towards a fixed point on its path and is proportional to its distance measured from that fixed point, the motion of the body or particle is called simple harmonic motion (SHM).

Characteristics :

(i) The motion is linear and periodic.

(ii) The motion is always directed towards a fixed point on its path

(iii) The restoting force (which bring back the body to its equilibrium position) acting on the body is proportional to the distance from the fixed point i.e. displacement.

1.2.1 Set up Differential equation of SHM :

Let x be the displacement of any instant of time t of a particle of mass m executing the simple harmonic motion. Then for small displacements, the restoring force F acting on it to bring back the particle to its equilibrium position is proportional to the displacement.

Hence we can write.

$F = -sx$ where s is a constant, called force constant, spring constant or stiffness constant, defined as restoring force per unit displacement. The negative sign of the above equation indicates that F and x are oppositely directed.

Equating the restoring force to the force of inertia, we get

$$m\frac{d^2x}{dt^2} = -8x$$

$$\text{or,}\quad \frac{d^2x}{dt^2} + \frac{s}{m}x = 0$$

$$\therefore\quad \frac{d^2x}{dt^2} + w^2x = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

where $\omega^2 = \frac{s}{m}$, ω is called the angular frequency.

Note that the quantity $\frac{s}{m}$ has units of $Nm^{-1}kg^{-1} = (kg.ms^{-2})kg^{-1}m^{-1} = s^{-2}$

The equation (1.1) is the differential equation of motion of Simple Harmonic Oscillator.

1.2.2. Solution of the differential equation of SHM :

Equation (1.1) is a second order differential equation, the general solution of which will contain two arbitrary constants.

Let $x = Ae^{\alpha t}$, (α and A are constants) be the trial solution of the equation (1.1.).

$$\text{Now,}\quad \frac{dx}{dt} = A\alpha e^{\alpha t} = \alpha x \quad \textit{and}$$

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t} = \alpha^2 x$$

Putting these in equation (1.1). We get,

$$\left(\alpha^2 + \omega^2\right)x = 0$$

or, $\alpha^2 + \omega^2 = 0$ since, x cannot be equal to zero.

$\therefore\ \alpha = \pm\, i\omega$ where $i = \sqrt{-1}$

Hence, the solution of the differential eqution (1.1) is $x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$ ... (1.2) where A_1 and A_2 are two orbitrary constants.

Using $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$. we can write the above equation as

$x = A_1(\cos \omega t + i \sin \omega t) + A_2 (\cos \omega t - i \sin \omega t)$

or, $x = (A_1 + A_2) \cos \omega t + i(A_1 - A_2) \sin \omega t$ ... ... (1.3)

Since, x is real, the aceptable physical solution is either real or the imaginatory part of the euation (1.3)

$\therefore\ x = a \cos \omega t + b \sin \omega t$ ... (1.4)

where, $a = A_1 + A_2$ and $b = i (A_1 - A_2)$ are two real arbitrary constants.

Now, Let $a = A \cos \varphi$ and $b = A \sin \varphi$.

$\therefore\ A = \sqrt{a^2 + b^2}$ and $\tan \varphi = \frac{b}{a}$

So, the equation (1.4) becomes

$x = A \cos (\omega t - \varphi)$ ... (1.5)

Putting $\varphi = \frac{\pi}{2} - \theta$, the equation (1.5) changes into

$x = A \sin (\omega t + \theta)$ ... ... (1.6)

Each of equation (1.2), (1.4), (1.5) and (1.6) gives the general physical solution of the differential equation (1.1).

1.2.3 Some parameters of SHM :

In general, the solution of differential equation of SHM takes as $x = A \sin (\omega t + \theta)$

(i) Amplidute and Phase :

The quantity A in the above equation is the maximum displacement of the particle from the mean position and is known as the amplitude of the oscillation.

The angle $(\omega t + \theta)$ is called the phase angle or phase. At $t = 0$, the phase angle is θ, called the epoch.

(ii) Time period and frequency of oscillation :

You know that the displacement x at time t and t + T must be some, if T be the time period of oscillation.

$\therefore$ sin (ωt + θ) = sin (2π + ωt + θ) = sin [ω(t + T) + θ] = sin (ωT + ωt + θ)

Thus, ωT = 2π

$$\therefore \text{ Time period } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}} \quad \left[\therefore\ w^2 = \frac{s}{m}\right]$$

And the frequency of oscillation

$$n = \frac{1}{T} = \frac{\omega}{2\pi}$$

(iii) Velocity and Acceleriaton :

We know the velocity $(v) = \frac{dx}{dt} = \frac{d}{dt}$ [A sin (ωt + θ)] = Aω cos (ωt + θ)

or, $v = A\omega\sqrt{1 - \frac{x^2}{A^2}}$ $[\because\ x^2 = A^2 \sin^2(\omega t + \theta) = A^2 [1 - \cos^2(\omega t + \theta)]$

or, $\cos(\omega t + \theta) = \sqrt{1 - \frac{x^2}{A^2}}$]

$$\therefore\ v = \omega\sqrt{A^2 - x^2} \quad \ldots \quad (1.7)$$

$\therefore$ Velocity of the particle is SHM $(v) = \omega\sqrt{A^2 - x^2}$

Again we see that at $x = 0$, $v_{max} = A\omega$, i.e, the velocity of the particle is maximum, when the particle is at the equilibrium position during oscillation.

Now acceleration, $f = v = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t + \theta)$

$$\therefore\ f = -\omega^2 x \quad \ldots \quad (1.2)$$

Maximum acceleration, $f_{max} = A\omega^2$ and it occurs when $x = x_{max} = A$, i.e, the acceleration is maxmimum when the particle is at the two extreme positions during the oscillation.

The negative sign in the equation (1.8) indicates that the acceleration is always directed opposite to the displacement.

1.2.4. Evaluation of Constants a and b :

To evaluate the constants a and b of equation (1.4). we consider the particle is brought to a distance x_0 and then released. Thus x_0 is the maximum displacement a time t = 0.

Now equation (1.4) i.e., x = a cos ωt + b sin wt can be written as

x_0 = a cos 0 + b sin 0 = a ,

$\therefore$ a = x_0

Differentiating equation (1.4) with respect to t. we get the velocity of the particle at any instant t as :

$$v = \frac{dx}{dt} = -a\omega \sin \omega t + bw \cos \omega t$$

: Let, at t = 0, velocity, v = v_0, then from above equation.

we get $v_0 = b\omega$

$$\therefore \ b = \frac{v_0}{\omega}$$

Putting these values of a = x_0 and $b = \frac{v_0}{\omega}$ in equation (1.4) we get,

$$x = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.9)$$

Case-1 : If the motion starts with initial displacement (x_0) but velocity is zero.

Then from equation (1.9) we get,

$x = x_0 \cos \omega t$, as in the case of simple harmonic motion.

Case-II : If the motion starts with initial velocity (v_0) only then

$$x = \frac{v_0}{\omega} \sin \omega t$$

This type of motion takes place, when the pendulum bob is struck by a hammer to give the initial velocity v_0.

Exercise-1

A particle executing simple harmonic motion has displacements x_1, x_2 and velocity v_1, v_2 at t_1, t_2 respectively. Calculate the amplitude and time period of oscillation.

Exercise-2

The displacement of a simple harmonic oscillator is given by x = a sin (ωt + θ). If the oscillations started at time t = 0 from a position x_0 with velocity v_0, show that $\tan\theta = \frac{\omega x_0}{v_0}$

and $a = \left(x_0^2 + \frac{v_0^2}{\omega^2}\right)^{\frac{1}{2}}$

1.3 Energy of a simple harmonic oscillator

Let the displacement at any time t of a particle executing simple harmonic motion from the mean position $x = A\sin(\omega t + \varphi)$. If E_k and E_p are the kinetic energy and potential energy of the oscillator respectively at that instant (t). Then the total mechnical energy of the oscillator is $E = E_k + E_P$

Kinetic energy :

The instantaneous kinetic energy of the simple harmonic oscillatory of mass m is

$$E_K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\left[\frac{d}{dx}\left[\sin\left(\omega t + \phi\right)\right]\right]$$

$$= \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \varphi) \quad ...(1.10)$$

$$= \frac{1}{2}mA^2\omega^2\left\{1 - \sin^2(\omega t + \varphi)\right\}$$

$$= \frac{1}{2}m\omega^2A^2 - \frac{1}{2}m\omega^2A^2(\omega t + \varphi)$$

$$= \frac{1}{2}sA^2 - \frac{1}{2}Sx^2 \quad \left[\because \omega^2 = \frac{s}{m} \text{ and } x = A\sin\left(\omega t + \varphi\right)\right]$$

$$\therefore\ E_k = \frac{1}{2}s\left(A^2 - x^2\right) \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (1.11)$$

Potential Energy :

The potential energy is given by the amount of work required to move the system from $x = 0$ to x by applying a force.

Here, the force must be just enough to oppose the restoring force sx., i.e the forec to be applied is eaqual to sx. Thus, the work done against the resorting force for the displacement dx is sxdx.

Hence the total workdone to displace the particle from 0 to x is

$$\omega = \int_0^x sx\,dx = \frac{1}{2}sx^2$$

$\therefore$ The potential energy, $E_P = \frac{1}{2}sx^2 = \frac{1}{2}\ m\omega^2x^2$ ... ... (1.12)

or, $E_p = \frac{1}{2}\ m\omega^2A^2 \sin^2(\omega t + \varphi)$ ... (1.13)

$\therefore$ Total energy $E = E_k + E_P$

Now from equation (1.10) and (1.13) we can write

$$E = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \varphi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \varphi)$$

$$= \frac{1}{2}m\omega^2A^2 = \text{constant}$$

Again, from equation (1.11) we see that the maximum kinetic energy,

$$E_{kmax} = \frac{1}{2}m\omega^2A^2 = E$$

and from equation (1.12) maximum potential energy

$$E_{pmax} = \frac{1}{2}m\omega^2A^2 = E$$

Thus, the maximum kinetic and the maximum potential energies are equal and is equal to total mechanical energy of the oscillator, which is constant.

Average kinetic energy :

Average kinetic energy over a time period T, is given by

$$E_{kav} = \frac{1}{T}\int_0^T \frac{1}{2} m\left(\frac{dx}{dt}\right)^2 dt$$

$$\left[= \therefore \int_0^T \cos^2(\omega t+\phi)dt = \int_0^T \frac{1+\cos^2(\omega t+\phi)dt}{2}\right.$$

$$= \frac{1}{2T} m\omega^2 A^2 \frac{T}{2}(\omega t+\varphi)dt$$

$$= \frac{1}{2}T + \frac{1}{2}\cdot\left[\frac{\sin 2(\omega t+\varphi}{2\omega}\right]_0^T$$

$$= \frac{1}{2}\cdot\frac{1}{2} m\omega^2 A^2 \frac{T}{2}$$

$$= \frac{T}{2} + \frac{1}{4\omega}\{\sin 2(\omega t+\varphi) - \sin 2\varphi\}$$

$$= \frac{1}{2}\cdot\frac{1}{2} m\omega^2 A^2$$

$$= \frac{T}{2} + \frac{1}{4\omega}\{\sin 2(\omega t+\varphi) - \sin 2\varphi\} \quad [\because \omega T = 2\pi]$$

$$\left. = \frac{T}{2} \right]$$

$$\therefore E_{kev} = \frac{E_{k\,max}}{2} = \frac{1}{2}E \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots(1.14)$$

Average potential energy :

Average potential energy over a time period T, is given by

$$E_{pav} = \frac{1}{T}\int_0^T \frac{1}{2} sx^2 dt = \frac{1}{2T} m\omega^2 A^2 \int_0^T \sin^2(\omega t+\varphi)dt$$

$$\frac{1}{27} m\omega^2 A^2 \cdot \frac{T}{2} \quad \left[\because \int_0^T \sin^2(\omega t+\varphi)dt = \frac{T}{2}\right]$$

$$\frac{1}{2}\cdot\frac{1}{2} n\omega^2 A^2$$

$$E_{pav} = \frac{Ep\,max}{2} = \frac{1}{2}E \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.15)$$

Equations 1.14 and 1.15 we see that the average kinetic energy and average potential energy of a harmonic oscillator are equal and each is equatation to half of the corresponding maximum energies and also half of the total energy.

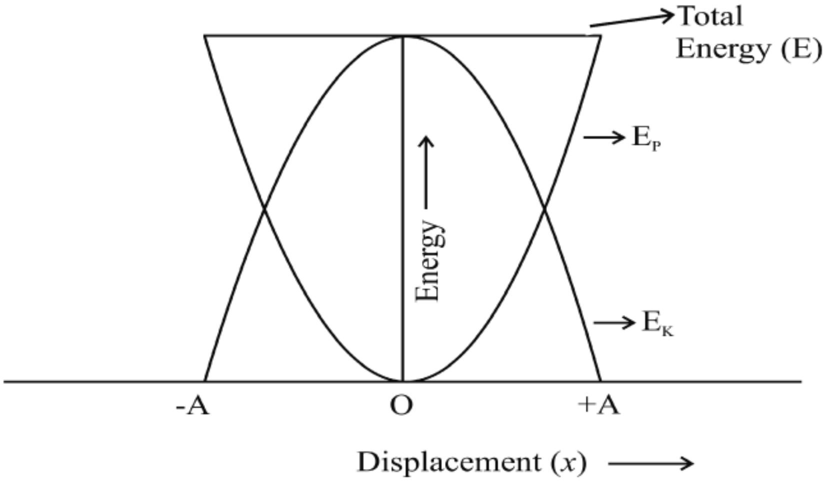


Figure 1.1 shows how the kinetic energy, potential energy and total energy of the harmonic oscillator vary with displacement.

From figure it is clear that as the particle oscillates, its energy continuously changes between the kinetic energy and potential energy, i.e. we can say one increase at the cost of he other. But total energy is always constant.

1.4 Examples of some physical systems executing SHM.

You have got some idea on simple harmonic motion. Now, we will study some physical systems executing simple harmonic motion.

1.4.1 Simple Pendulum :

An ideal simple pendulum consists of a small mass (bob) suspended by a light inextensible string from a fixed point. As the bob of mass m is displaced by small angle θ from its quilibrium position (A) the restoring force is provided by the tangential component of the weight mg along the arc as shown in fig 1.2, is given by

$F = -mg\sin\theta$

The equation of motion of the bob is

$$m\frac{d^2x}{dt^2} = -mg\sin\theta = -mg\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.16)$$

When θ is very small. The bob is moving along the arc (PA) whose length at any instant is given by x, when the angular displacement is θ.

Then, $x = l\theta$ .. (1.17)

where, l is the length of the pendulum, which is the sum of the length of the string by which the bob is suspended and the radius of the bob.

Now, from equations (1.16) and (1.17) we get,

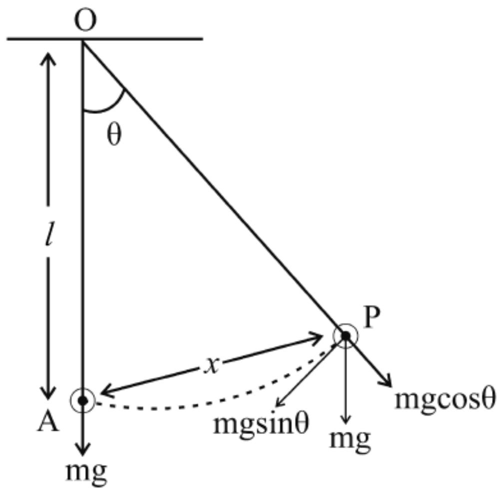


Figure 1.2

$$ml\frac{d^2\theta}{dt^2} = -mg\theta$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\therefore \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad \text{... (1.18)}$$

$$\text{where, } \omega = \sqrt{\frac{g}{l}}$$

Equation (1.18) is exactly of the standard from of equation (1.1) showing that the pendulum executes the simple harmonic motion, when θ is very small..

The time period of oscillation is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \quad \text{.... (1.19)}$$

By analogy, we can write the general solution of equation (1.18) as

$$\theta = \theta_0 \sin(\omega t + \varphi) \quad \text{... ... ...} \quad (1.20)$$

From equation (1.19) you have noted that for small angular displcement, the time period of oscillation of the pendulum depends on l and g, but not on the mass of the bob. So,due to variation of acceleration due to gravity (g) with lalitude of the earth a pendulum clock will move slower near the equator than at the pole of the earth.

1.4.2 Oscillation of Srping mass system :

(a) Horizontal oscillations of spring mass system :

Consider one end of a massless ideal spring is fixed to a wall and the other end is attached to a body of mass m, which is free to move on a frictionless horizontal surface (fig. 1.3). Figure 1.3a represents the position of equilibrium. When no force is acting on the body. Now, if the body is pulled to right (fig. 1.3b) through a small distance x, then the force exerted by the spring on the body is directed towards the left, which is the restoring force and is given by F = – kx, where k is the spring constant or stiffness constant.

Since, the restoring force is proportional is the displacement and is opposite to the direction of displacement, so the motion is simple harmonic

∴ We can write

$$m\frac{d^2k}{dt^2} = -kx$$

$$\text{or, } \frac{d^2k}{dt^2} + \omega^2 x = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1.21)$$

Here, $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of oscillation.

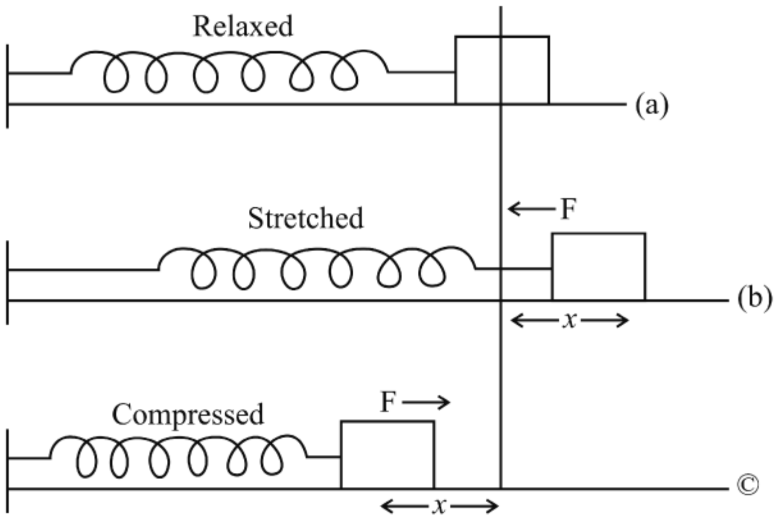


fig. 1.3

The time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

Vertical oscillations of spring mass system :

Consider a massless spring is suspended vertically from a rigid support and the lower and is attached to a mass m. In th equilibrium position, the mass is at A (fig. 1.4).

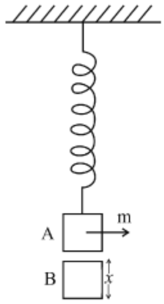


fig. 1.4

Now, if the mass is displaced through a small distance AB = x from the equilibrium position and released, it starts oscillation with simple harmonic motion.

Since the restoring force is given by F = – kx,

where k is the spring constant,

Thus we can write

$$F = m\frac{d^2x}{dt^2} = -kx$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1.22)$$

$$\because \quad \omega = \sqrt{\frac{k}{m}}$$

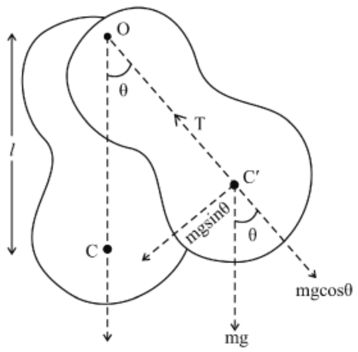


fig. 1.5

Which is the equation of simple harmonic motion, whose time period of oscilllation is

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$, which is same as that for horizontal motion.

1.4.3 Compound Pendulum :

A rigid body of any shape capable of oscillating in a vertical plane about a horizontal axis passing through a point near one end is called compound pendulum.

Fig. 1.5 represents the vertical section of a rigid body free to rotate about the point of suspension (0), called centre of suspension, c is the centre of mass of the body. When the rigid body is in equilibrium the line OC remains vertical as shown by dotted line in the figure 1.5. The distances OC = l, called the length of the pendulum.

When the pendulum is given a small angular displacement θ and released, it begins to oscillate about the point O.

The weight mg of the body acts vertically downwords at C′ in displaced position. This force mg is, when resolved we get mgcos θ along OC′ and mgsin θ perpendicular to OC[1]. The component mgcos θ is balanced by the tension (T) along CO and the component mgsin θ tends to bring back the body to its equilibrium position.

The moment of this force about θ is equal to mgsin $\theta.\ell = mg\ell\theta$. When θ is small.

If I is the moment of inertia of the body about the axis of oscillation, the eqution of motion of the body is given by

$$I\frac{d^2\theta}{dt^2} = -mg\ell\theta$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \frac{mg\ell}{I}\theta = 0$$

$$\therefore \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \ldots (1.23)$$

where angular frequency $\omega = \sqrt{\frac{mg\ell}{I}}$

This equation (1.23) is the equtqion of simple harmonic motion and its time period.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mg\ell}} \ldots (1.24)$$

If I_g is the moment of inertia of the pendulum about an axis passing through its centre of mass C and parallel to its axis of rotation. Then we have

$I_g = mk^2$, where k = radius of gyration

Then usig the parallel axis thorem, wet get

$I = I_g + m\ell^2 = mk^2 + m\ell^2 = m(k^2 + \ell^2)$

$$\therefore T = 2\pi\sqrt{\frac{m\left(k^2+\ell^2\right)}{mg\ell}} = 2\pi\sqrt{\frac{m\left(\frac{k^2}{\ell}+\ell\right)}{g}} \ldots \quad \ldots \quad \ldots \quad \ldots(1.25)$$

The equivalent simple pendulum :

If the time period of oscillation of a compund pendulum as equal to the time period of oscillation of simple pendulum, then that compound pendulum is called an equivalent simple pendulum.

From equation (1.25) we have

$$T = 2\pi\sqrt{\frac{\frac{k^2}{\ell}+\ell}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

where, $L = \frac{k^2}{\ell} + \ell$ ... ... (1.26)

Now, the time period of a simple pendulum of effective length L is

$$T' = 2\pi\sqrt{\frac{L}{g}}$$

∴ If $T' = T$ and L is the length of the equivalent simple pendulum, then

$$L = \frac{k^2}{\ell} + \ell$$

or, $\ell^2 - L\ell + k^2 = 0$... (1.27)

This is a quadratic equqion of ℓ.

Let ℓ_1 and ℓ_2 are two values of ℓ which satisfies the equation (1.27)

Then, $(\ell - l\ell + \ell_2) + \ell_1\ell_2 = \ell^2 - L.\ \ell + k^2$

Comparing both sides of this equation, we get,

$\ell_1 + \ell_2 = L$ and $\ell_1\ell_2 = k^2$

∴ The time period of compiound pendulum will be same as that of simple pendulum when, $OC = \ell_1$ or, ℓ_2.

Now, we consider $OC = \ell_1$ and $O'C = \ell_2$

Since, $L = \ell_1 + \ell_2$

Then we may also prove that O and O′ points are interchangable in the following way :

When O is the point of suspension and O′ is the point of oscillation (as in figure 1.6).

Then from equation ... ... (1.2.5)

$$T' = 2\pi\sqrt{\frac{k^2+\ell_1^2}{g\ell_1}} = 2\pi\sqrt{\frac{\ell_1\ell_2+\ell_1^2}{g\ell_1}} = T' = 2\pi\sqrt{\frac{\ell_2+\ell_1}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

Again when O′ is the point of suspensioni and O is the point of oscillation, then

$$T'' = 2\pi\sqrt{\frac{\ell_1\ell_2+\ell_2^2}{g\ell_2}} = 2\pi\sqrt{\frac{\ell_1\ell_2+\ell_2^2}{g\ell_2}} = 2\pi\sqrt{\frac{L}{g}}$$

$\therefore T' = T''$

Thus the point of suspension O and point of oscillation O′ are interchangable.

Maximum and minimum time periods of compund pendulum can be calculated from the equation of time period of oscillation of a compound pendulum.

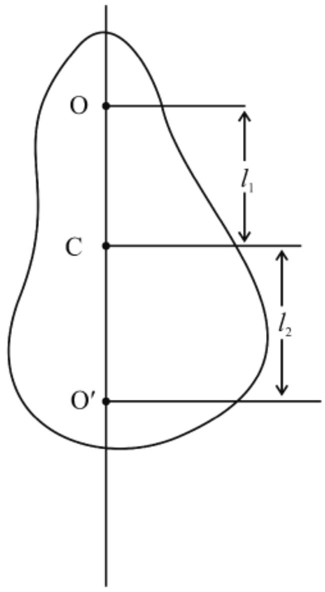


Fig. 1.6

$T = 2\pi\sqrt{\frac{k^2+\ell^2}{\ell g}}$, we see that T depends only on ℓ, because for a particular rigid body k is constant.

Now, $$\frac{dT}{dt} = 2\pi\cdot\frac{1}{2}\left(\frac{k^2+\ell^2}{\ell g}\right)^{-\frac{1}{2}}\frac{2\ell^2 - 1\left(k^2+\ell^2\right)}{g\ell^2}$$

$$= \pi\left(\frac{k^2+\ell^2}{\ell g}\right)^{-\frac{1}{2}}\frac{\ell^2-k^2}{g\ell^2}$$

From maxima or mimima

$$\frac{dt}{d\ell} = 0$$

$$\therefore \frac{\ell^2-k^2}{\ell^2} = 0 \text{ or, } k = \ell$$

$\therefore$ For $k = \ell$, $\dfrac{k^2 + \ell^2}{\ell}$ is minimum.

$\therefore$ Minimum time period, $\therefore$ $T_{min} = 2\pi\sqrt{\dfrac{2k}{g}}$ ... (1.28)

Again, from equation (1.25) we see that

as Lt $\ell \to 0$ or Lt $\ell \to \infty$, $T \to \infty$

and for $\ell = k$, T is minimum.

Therefore, if there be a little change in the value of ℓ, the change in the time period of the pendulum becomes negligibly small.

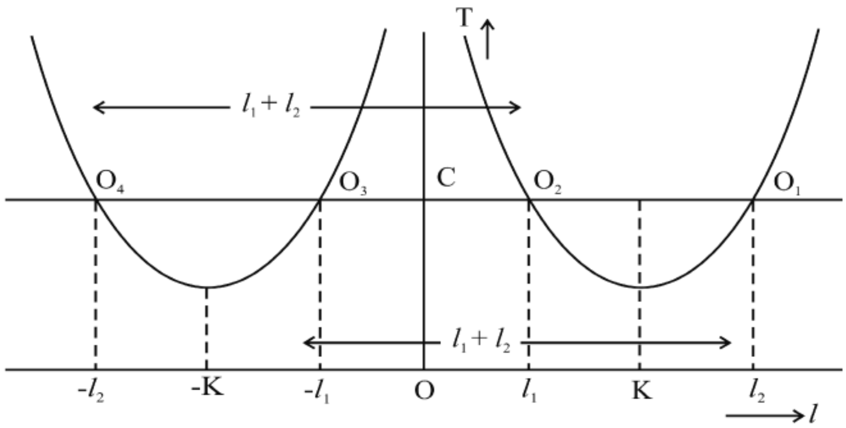


Fig. 1.7

figure 1.7 shows how the period of a compound pendulum varies with the distance of its centre of suspension from the centre of mass C.

It is seen from the graph that the time period increases enormouly when the point of suspension approches the centre of mass and it is minimuim when $\ell = k$.

Again, in the graph

$CO_2 = CO_3 = \ell_1$ and $CO_1 = CO_4 = \ell_2$

$\therefore$ $\ell_1 + \ell_2 = CO_3 + CO_1 = O_3O_1$

Also, $\ell_1 + \ell_2 = CO_2 + CO_4 = O_2O_4$

Thus, knowing the values of time period (T) and $\ell_1 + \ell_2$ from the graph, we can determine the value of g from the equation.

$T = 2\pi\sqrt{\frac{\ell_1+\ell_2}{g}}$ or, knowing k and T_{min} we can also determine g from equation (1.28)

1.4.4 Torsional Pendulum :

When a rigid body like a disc, cylinder, sphere or rod is suspended from one end of a thin vertical wire and the other end is clamped to a fixed support (fig.1.8), then this arrangement is called a torsional pendulum. Here the axis of the wire passed through the centre of gravity of the body.

If the body is given a slight rotation about the suspension and then released, it executes osciallations about its axis. This type of oscillation is called torsional oscillation or angular SHM.

Now if the vertical wire is twisted by applying a torque to the body, then the wire tends to untwist itself exerting a restorting torque on the body. Within the elastic limit, this restoring torque is proportional to the angle of twist (θ).

Let c be the restoring torque per unit twist, then the restoring torque for θ rotation is Cθ.

It I be the moment of inertia of the body about the axis of rotation, then the equation of motion of torsional oscillation is

$$I = \frac{d^2\theta}{dt^2} = -C\theta$$

$$\therefore \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad ... \quad ... \quad ... \quad ..(1.29)$$

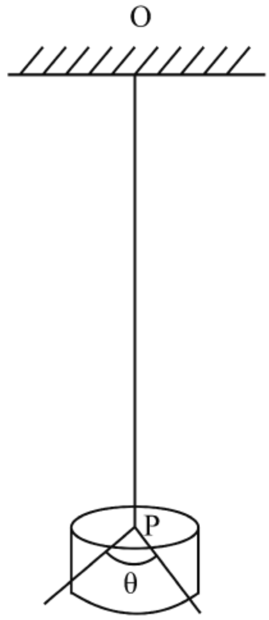


Fig. 1.8

where $\omega = \sqrt{\frac{C}{I}}$. This equation (1.29) is exactly of the form of SHM.

Hence, the motion is simple harmonic and the time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{C}} \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (1.20)$$

1.4.5 Kater's Pendulum

Kater, constructed a special type of compound pendulum to determine the accurate value of acceleration due to gravity 'g'. The principle of this pendulum is based on the reversibility of centre of suspension and centre of oscillation.

It consists of a metallic rod carrying a heavy bob (W) and two adjustable weights A and B, as shown in figure 1.9. A being larger then B. Two knife edges K_1 and K_2 are fixed, so that they face each other and are either side of the centre of gravity (C.G.) of the pendulum. The object of this experiment is to arrange that the time periods of oscillation about each of these knife edges are equal by adjusting the two masses A and B.

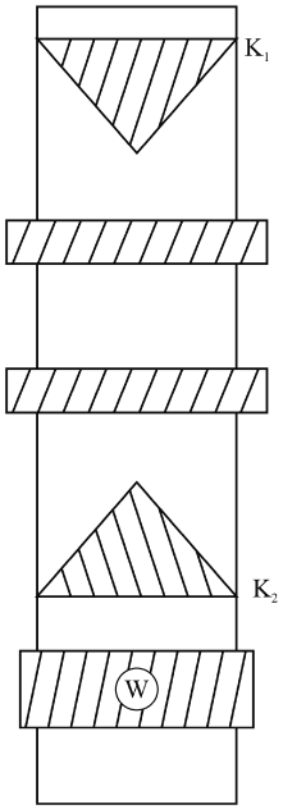


Fig. 1.9

Let, ℓ_1 and ℓ_2 are the distances of the knife edges from C.G. of the pendulum in the final adjustment positions.

Let T_1 and T_2 be the corresponding time periods about the two knife edges k_1 and k_2

Then from equation (1.25) we can write

$$T_1 = 2\pi\sqrt{\frac{k^2+\ell_1^2}{\ell_1 g}} \quad \ldots \quad \ldots \quad \ldots \quad (1.31)$$

and $$T_2 = 2\pi\sqrt{\frac{k^2+\ell_2^2}{\ell_2 g}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (1.32)$$

Squaring equation (1.31) and (1.32) we get

$$\ell_1 T_1^2 = 4\pi^2 \frac{k^2+\ell_1^2}{g} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.33)$$

and $$\ell_2 T_2^2 = 4\pi^2 \frac{k^2+\ell_1^2}{g} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.34)$$

Subtracting above two equations, we get

$$1\ell T_1^2 - \ell_2 T_2^2 = \frac{4\pi^2}{g}\left(\ell_1^2 - \ell_1^2\right) = \frac{4\pi^2}{g}(\ell_1+\ell_2)(\ell_1-\ell_2)$$

or, $$\frac{4\pi^2}{g} = \frac{\ell_1 T_1^2 - \ell_2 T_2^2}{(\ell_1+\ell_2)(\ell_1-\ell_2)} = \frac{A}{\ell_1+\ell_2} + \frac{B}{\ell_1-\ell_2} \text{ (say)} \quad \ldots \quad \ldots \quad \ldots (1.35)$$

Now, we can write

$$\frac{4\pi^2}{g} = \frac{\ell_1 T_1^2 - L_1 T_2^2}{(\ell_1+\ell_2)(\ell_1-\ell_2)} = \frac{(\ell_1-\ell_2)A+(\ell_1+\ell_2)B}{(\ell_1+\ell_2)(\ell_1-\ell_2)} = \frac{\ell_1(A+B)-\ell_2(A-B)}{(\ell_1+\ell_2)(\ell_1-\ell_2)}$$

∴ From above equation, we have

$A + B = T_1^2$ and $A - B = T_2^2$

$\therefore\ A = \frac{1}{2}\left(T_1^2 + T_2^2\right)$ and $B = \frac{1}{2}\left(T_1^2 - T_2^2\right)$

Putting the values of A and B in eqution (1.35), we get,

$$\frac{4\pi^2}{g} = \frac{T_1^2 + T_2^2}{2(\ell_1+\ell_2)} + \frac{T_1^2 - T_2^2}{2(\ell_1-\ell_2)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.36)$$

As the quantities ℓ_1, ℓ_2, T_1 and T_2. are known by the experiment, the value of g can be determined, provide the position of the c.g. is accuarely known.

Since, it is very difficult to docate the position of c.g.. So, the time periods T_1 and T_2 are adjusted to be very nearly equal, i.e, $T_1 = T_2$ T (say).

Hence, the term $\frac{T_1^2 - T_2^2}{2(\ell_1-\ell_2)}$ is the above equation is negligibly small.

Then, the equation (1.36) takes the form

$$\frac{4\pi^2}{g} = \frac{2T^2}{2(\ell_1+\ell_2)} = \frac{T^2}{L}, \text{ Let } \ell_1 + \ell_2 = L$$

$$\therefore\ g = 4\pi^2 \frac{\ell}{T^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.37)$$

Here, L is the distance between the two knife edges. Equation (1.37) is similar to the equation of a simple pendulum.

1.4.6 Two masses connected by a spring

Let two masses m_1 and m_2 are connected by a massless spring of length l and spring constant k. The masses are constrained to move on a frictionless floar along x axis, i.e. along the axis of the spring as shown in figure 1.10.

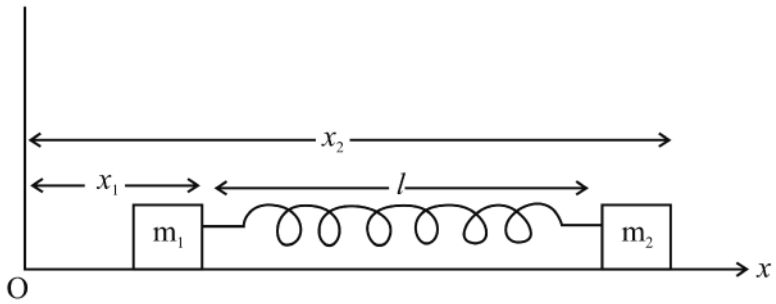


Fig. 1.10

If x_1 and x_2 are the co-ordinate of the two ends of the spring at any time t, then the change in length of the spring is giveb by

$$x = (x_2 - x_1) - \ell \quad \ldots \quad (1.38)$$

From equation (1.38) it is clear the when $x > 0$, $x = 0$ and $x < 0$, the spring is in extended, normal and in compressed conditions respectively.

Now, in the extended condition ($x > 0$), if the spring exerts a forec $F_1 = kx$ on the mass $m_1$then the opposing force $F_2 = - kx$ will be exerted on m_2 .

Hence, the can write the equation of motion of two masses are :

$$F_1 = m_1 \frac{d^2 x_1}{dt^2} = kx$$

$$\text{or, } \frac{d^2 x_1}{dt^2} = \frac{k}{m} x \quad \ldots \quad (1.39)$$

$$\text{and } \frac{d^2 x_2}{dt^2} = -\frac{k}{m_2} x \quad \ldots \quad (1.40)$$

Subtracting equations (1.39) and (1.40) we get

$$\frac{d^2}{dt^2}(x_2 - x_1) = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)x$$

From equation (1.38) we have $\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(x_2 - x_1)$

Since, the length of the spring ℓ is constant.

Therefore, the equation of motion of the system is

$$\frac{d^2x}{dt^2} + \frac{k}{\mu}x = 0$$

or, $\frac{d^2x}{dt^2} + \omega^2 x = 0$ ... ... (1.41)

where $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}$

or, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass of the system. That means, the system behaves, like a single object of mass μ connected by the spring.

Equation, (1.41) is the general form of SHM of angular frequency $\omega = \sqrt{\frac{k}{\mu}}$ and frequency of oscillation $f = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$

1.4.7 An inductance-capacitance circuit :

You have got an idea abou the SHM of mechanical systems. Now we will discuss the simple harmonic oscillations of changes in an idea. L. C circuit (when resistance is zero).

Let a condenser (C) be first connected by the two way key to the cell E. When it is fuly changes, then the key is thrown to the position is remove the cell out of the circuit. (as shown in figure 1.11), so that the condenser gradually gets dischanged through the inductance (l).

Let at any instant the change of the condenser is q and the current in the circuit is $i = \frac{dq}{dt}$. then the potential difference across the inductance $V_L = -L\frac{di}{dt}$ and potential difference across the condenser is $V_C = \frac{q}{C}$.

Since, there is no sourec of emf in the circuit, so we can write

$V_C = V_L$ or, $\frac{q}{c} = -L\frac{di}{dt}$

or, $L\frac{di}{dt} + \frac{q}{c} = 0$

or, $L\frac{d^2q}{dt^2} + \frac{q}{c} = 0$ $\left[\because i = \frac{dq}{dt}\right]$

$\therefore$ $\frac{d^2q}{dt^2} + \omega^2 q = 0$ ... (1.42)

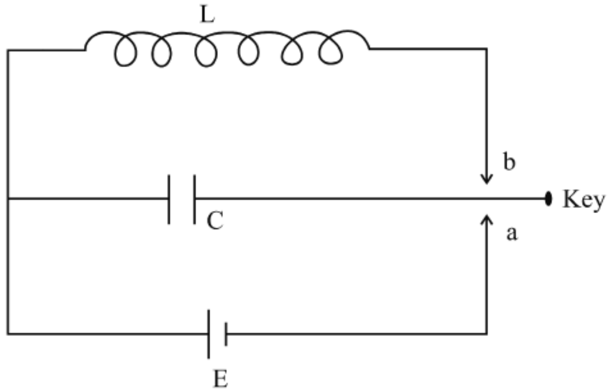


Fig. 1.11

where, $\omega = \frac{1}{\sqrt{LC}}$. equation (1.42) represents and equation of SHM of time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$ and frequency $f = \frac{1}{2\pi\sqrt{LC}}$

The solution of equation (1.42) is

$q = q_0 \cos(\omega t + \varphi)$ ... ... (1.43)

where, q_0 is the maximum charge in the capacitor.

If we consider at t = 0, i.e. when the key is released to the position tot he charge is q_0, then we can write the equation (1.43) as

$q = q_0 \cos \omega t$ and the current in the circuit as

$$i = \frac{dq}{dt} = -q_0\omega \sin\omega t = q_0\omega\cos\left(\omega t + \frac{\pi}{2}\right)$$

Thus we see that both charge and current in the circuit are oscillatory, the phase difference between them is $\frac{\pi}{2}$.

Now you can have a question, that how oscillation take place without any mechanical energy.

As you know that the energy stored within the capacitor is $E_C = \frac{1}{2}\frac{q^2}{C}$ and that of within the inductances is $E_L = \frac{1}{2}Li^2$.

Now we can write,

$$E_C = \frac{1}{2C}q_0^2 \cos^2 \omega t \text{ and } E_L = \frac{1}{2}Lq_0^2\omega^2 \sin^2 \omega t$$

$$= \frac{1}{2}\cdot Lq_0^2 \frac{1}{LC}\sin^2 \omega t$$

$$= \frac{1}{2}\frac{q_0^2}{C}\sin^2 \omega t$$

$\therefore$ Total energy (E) = $E_C + E_L = \frac{1}{2}\frac{q_0^2}{C}$ $(\cos^2 \omega t + \sin^2 \omega t)$

$$\therefore \; E = \frac{1}{2}\frac{q_0^2}{C}$$

From the above equation we see that

when $\omega t = 0$, i.e. at $t = 0$, $E_C = \frac{1}{2}\frac{q_0^2}{C} = E_{C_{max}} = E$ amd $E_2 = 0$

and when $\omega t = \frac{\pi}{2}$

$E_C = 0$ and $E_L = \frac{1}{2}\frac{q^2 0}{C} = E_{L\,max} = E$

Again total energy $E = E_l + E_2 = \frac{1}{2}Li^2 + \frac{1}{2}\frac{1}{C}q^2$

similar to the mechanical oscillator total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2$

Thus, we can say as charge and current very with time, the inductor and capacitor exchange their energy periodically.

Exercise 3

The springs S_1 and S_2 each of length ℓ_1, have spring constants k_1 and k_2. Calculate the spring constant of the spring, system when connected in parallel as shown in figure.

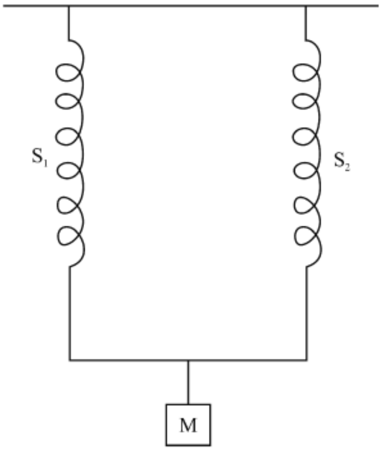


fig. 1.12

Exercise-4

What is the length of the equivalent pendulum which consists of a sphere of radius 10 cm suspended by a light sprig of length 50 cm?

Exercise -5

Show that a particle allowed to slide down without friction through a tunnel bored through earth will undergo simple harmonic motion. Also show that the time period is independent of the direction of the tunnel and radius of the sphere.

1.5 Summary

(i) Simple harmonic motion : An oscillatory motion will be simple harmonic, when the restoring foce is proportiona to the displacement and is always directed against it .

(ii) Differential equation of SHM is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ where } \omega = \sqrt{\frac{5}{m}} \text{ and } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

(iii) The most general solution of the differential euation of SHM is $x = A\sin(\omega t + \theta)$

(iv)Velocity and energy of oscillator :

$$v = \omega\sqrt{A^2 - x^2}$$

Kineti energy (Ex) $= \frac{1}{2}s(A^2 - x^2)$, potential energy ($E_P$) $= \frac{1}{2}sx^2$

Total energy(E) $= \frac{1}{2}sA^2 = \frac{1}{2}m\omega^2A^2$

Table of Analogies

System	Differential Equation	Inertial factor	Spring factor	W	T
Simple pendulum	$\ddot{\theta} + \omega^2\theta = 0$	m	mg/ℓ	$\sqrt{\frac{g}{l}}$	$2\pi\sqrt{\frac{l}{g}}$
Spring mass system	$m\ddot{x} + kx = 0$	m	k	$\sqrt{\frac{k}{m}}$	$2\pi\sqrt{\frac{m}{k}}$
Compound Pendulum	$I\ddot{\theta} + mgl\theta = 0$	I	$mg\ell$	$\sqrt{\frac{mg\ell}{I}}$	$2\pi\sqrt{\frac{I}{mg\ell}}$
Torsional Pendulum	$i\ddot{\theta} + C\theta = 0$	I	C	$\sqrt{\frac{C}{I}}$	$2\pi\sqrt{\frac{I}{C}}$
Two body system	$\mu\ddot{x} + kx = 0$	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	k	$\sqrt{\frac{k}{\mu}}$	$2\pi\sqrt{\frac{\mu}{k}}$
L. C. circuit	$L\ddot{q} + \frac{1}{c}q = 0$	L	$\frac{1}{C}$	$\frac{1}{\sqrt{LC}}$	$2k\sqrt{LC}$

(vi) Determination of g by Kater's pendulum :

$$\frac{1}{g} = g^{-1} = \frac{1}{8\pi^2}\left[\frac{T_1^2 + T_2^2}{\ell_1 + \ell_2} + \frac{T_1^2 - T_2^2}{\ell_1 - \ell_2}\right]$$

1.6. Questions and Problems

1.6.1 Show that the momentum of a particle executing SHM is plotted against its displacement will be elliptic.

1.6.2 Two springs S_1 and S_2 each of length l, have spring constants, K_1 and K_2.Calculate the spring constant of the spring system when connected in series.

1.6.3 Calculate the ratio of the amplitude to displacement for a simple harmonic motion when the kinetic energy is 90% of the total energy.

1.6.4 A U tube of constant cross-sectional area A of the limbs is filled with a liquid of density ρ upto a length l in each limb when the tube is vertical The tube is slightly tilted and then again made vertical . Supposing the force is only gravitational, calculate the time period of oscillations of the liquid.

1.6.5 A heavy uniform rod of length 90 cm swings in a vertical plane about a horizontal axis (called bar pendulum) passing through its one ends.

Calculate the position at which a concentrated mas may be plced so that the time of swing remains unaltered.

1.6.6 Calculate the frequency of electrical oscillation, when an inductor of 30 mH is connected with a capacitor of 3μF. If the maximum potential difference across the capacitor is 10 volt, Calculate the energy of oscillation.

1.7 Solutions

Exercise : 1

Let $x = a \sin(\omega t + \theta)$, then $v = \frac{dx}{dt} = a\,\omega \cos(\omega t + \theta)$

$$\therefore \frac{x}{a} = \sin(\omega t + \theta) \text{ and } \frac{v}{a\omega} = \cos(\omega t + \theta)$$

$$\text{or, } \frac{x^2}{a^2} + \frac{v^2}{a^2\omega^2} = \sin^2(\omega t + \theta) + \cos^2(\omega t + \theta) = 1$$

$$\text{Now, } \frac{x_1^2}{a^2} + \frac{v_1^2}{a^2\omega^2} = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (1)$$

$$\text{and, } \frac{x_2^2}{a^2} + \frac{v_2^2}{a^2\omega^2} = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (2)$$

Subtracting equation (1) and (2) we get

$$\frac{x_1^2 - x_2^2}{a^2} + \frac{v_1^2 - v_2^2}{a^2\omega^2} = 0$$

$$\text{or, } \frac{v_1^2 - v_2^2}{\omega^2} = x_2^2 - x_1^2$$

or, $\omega = \sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

$\therefore$ Time period of oscillation (T) = $\dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

Again putting ω^2 in equation (1) we get,

$$x_1^2 + \frac{v_1^2\left(x_2^2 - x_1^2\right)}{v_1^2 - v_2^2} a^2$$

or, $a^2 = \dfrac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}$

$\therefore$ Amplitude (a) = $\sqrt{\dfrac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$

Exercise – 2 :

Here, $x = a \sin(\omega t + \theta)$ ... (1)

$\therefore$ Velocity (v) = $\dfrac{dx}{dt} = a\omega \cos(\omega t + \theta)$ ... ... (2)

Now, at t = 0, $x = x_0$

$\therefore$ From equation (1) we have

$x_0 = a \sin\theta$ .... (2)

And at t = 0, $v = v_0$

$\therefore$ From equation (2) we have

$v_0 = a\omega \cos\theta$ ... ... (4)

$\therefore \dfrac{x_0}{v_0} = \dfrac{\sin\theta}{\omega\cos\theta}$

$\therefore \tan\theta = \dfrac{\omega x_0}{v_0}$ ... (5)

Again from equation (3) and (4) we can get,

$$\frac{x_0^2}{a^2}+\frac{v_0^2}{a^2\omega^2}=\sin^2\theta+\cos^2\theta=1$$

or, $a^2 = x_0^2 + \frac{v_0^2}{\omega^2}$

$$\therefore \ a=\left(x_0^2+\frac{v_0^2}{\omega^2}\right)^{1/2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Exercise-3 :

Let x be the extension at the lower end of the spring system, when mass m is attached is the system as shown in figure 1.12.

Now, the tension in S_1 is $F_1 = K_1x$ and that in S_2 is $F_2 = K_2x$.

So, the resultant tension, $F = F_1 + F_2 = (k_1 + k_2)\,x$ is balanced by the weight mg of mass m.

$\therefore$ mg = $(k_1 + k_2)x$... (1)

Let k be the spring constant of the combination then F = mg = kx ... ... (2)

$\therefore$ From equation (1) and (2) we get

k = $k_1 + k_2$

Exercise-4

The length of the equvalent simple pendulum is $L=\frac{k^2+l^2}{l}$ where k is the radius of gyration of sphere, i.e, $k^2=\frac{2}{5}R^2$, R = radius of the sphere =10 cm.

$$=\frac{2}{5}10^2 = 40 \text{ cm}^2$$

l = effective length of the pendulum = 50 + 10 = 60 cm.

$$\therefore \ L=\frac{40+60^2}{60}=60.67 \text{ cm}$$

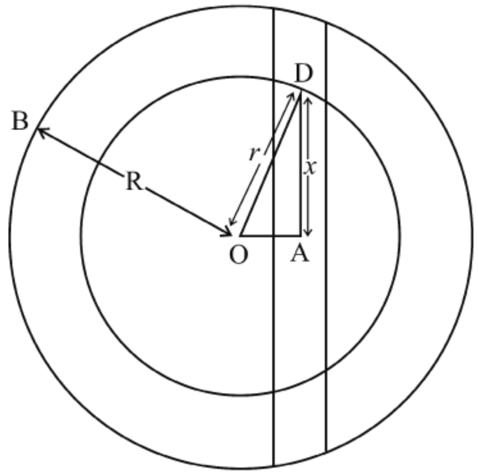


Fig. 1.13

Exercise-5 :

Let D be the position of the particle inside the tunnal at any instant and A is the middle point of the tunnel as in figure1.13. AD = x.

Force F on the particel is along $\overrightarrow{DO}$ where O is the centre of the earth of radius R and is given by

F = GmM/r^2, where m = mass of the particle and M = mass of the concentric sphere of radius OD = r.

$$= \frac{4}{3}\pi r^3 \rho \text{ , } \rho = \text{density of earth}$$

$$\therefore \ F = \frac{Gm\left(\frac{4}{3}\pi r^3 \rho\right)}{r^2} = \frac{4}{3}\pi r \rho m G$$

Component of F along $\overrightarrow{DA} = \frac{4}{3}\pi r\rho mg(\cos\angle ODA)$

from fig.

$$= \frac{4}{3}\pi r\rho mG.\frac{x}{r} \qquad \left[\because \cos\angle ODA = \frac{x}{r}\right]$$

$$= \frac{4}{3}\pi r\rho mG.x$$

Hence, the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{4}{3}\pi\rho mGx$$

or, $$\frac{d^2x}{dt^2} + \frac{4}{3}\pi\rho Gx = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{... (1)}$$

where $\omega^2 = \frac{4}{3}\pi\rho G$ or, $\omega = \sqrt{\frac{4\pi\rho G}{3}}$

∴ The particle executes SHM with a period

$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3}{4\pi\rho G}} = \sqrt{\frac{3\pi}{\rho G}}$, which is independent of the direction of the tunnel and the size of the sphere (R).

Solutions of problems :

1.6.1 As you know that the total energy

E = kinetic energy + potential energy

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2$$

Now, momentum, $P = m\frac{dx}{dt}$ or, $\frac{dx}{dt} = \frac{P}{m}$

and $s = m\omega^2$

∴ We can write $\frac{1}{2}m\frac{p^2}{m^2} + \frac{1}{2}m\omega^2 x^2 = E$

or, $$\frac{p^2}{2m} + \frac{x^2}{\frac{2}{m\omega^2}} = E$$

$$\text{or, } \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{m\omega^2}} = 1$$

$$\therefore \frac{p^2}{\left(\sqrt{2mE}\right)^2} + \frac{x^2}{\left(\sqrt{\frac{2E}{m\omega^2}}\right)^2} = 1$$

which represents an equation of elliptic curve in $p - x$ plane.

1.6.2 Here a mass m is attached at the lower end of the spring system. the series combination at s_1 and s_2 (as shown in fig).

Let x_1 and x_2 are the extentions of s_1 and s_2 respectively.

Then the total extension is $x_1 + x_2$.

Now, we have, $mg = k_1x_1$

$$\text{or, } x_1 = \frac{mg}{k_1}$$

and $mg = k_2x_2$ or, $x_2 = \dfrac{mg}{k_2}$

where k_1 and k_2 are spring constants of springs S_1 and S_2 respectively.

$$\text{Again, } mg\ (x_1 + x_2)k = \left(\frac{mg}{k_1} + \frac{mg}{k_2}\right)k = mg\left(\frac{1}{k_1} + \frac{1}{k_2}\right)k$$

$$\text{or, } \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\therefore k = \frac{k_1k_2}{k_1 + k_2}$$

here, k = spring constant of the system.

You know the total energy $E = \frac{1}{2}m\omega^2a^2$.

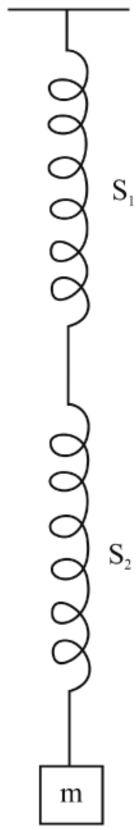


Fig. 1.14

Let x be the displacement when kinetic energy is 90% of total energy.

Since, the kinetic energy is 90% of total energy, then the potential and energy, E_P = 10% of total energy $= \frac{10}{100}E$.

But potential energy $E_p = \frac{1}{2}m\omega^2 x^2$

$$\therefore \frac{E}{E_P} = \frac{\frac{1}{2}m\omega^2 a^2}{\frac{1}{2}m\omega^2 x^2} = \frac{a^2}{x^2}$$

$$\text{or, } \frac{100E}{10E} = \frac{a^2}{x^2} \text{ or, } \frac{a^2}{x^2} = 10$$

$$\text{or, } \frac{a}{x} = \sqrt{10} = 3.16$$

∴ The ratio of the amplitude to displacement is equal to 3.16.

1.6.4 Suppose that at any instant, the level in one arm rises by x and in the other falls by x (as in fig).

Let us take gravitational energy at the bottom of the U tube as zero.

Then from the principle of conservation of energy.

Initial potential energy = sum of potential and kinetic energies at any instant.

Since, the centre of mass of a height ℓ of the liquid is at $\frac{1}{2}$.

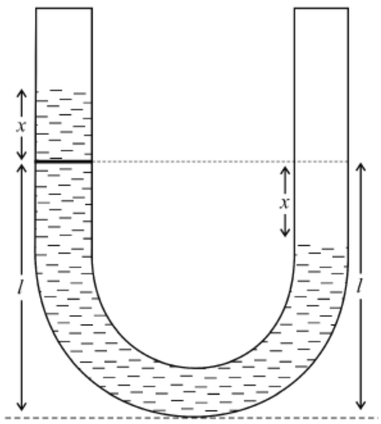


Fig. 1.15

∴ Total potential energy at the displaced position x is

$$= A\rho\left(\frac{\ell + x}{2}\right)g(\ell + x) + A\rho\left(\frac{\ell - x}{2}\right)g(\ell - x)$$

$[\because \text{P.E} = \text{mgh}]$

and kinetic energy = $\frac{1}{2} \cdot 2A\ell\rho\left(\frac{dx}{dt}\right)^2$

$\therefore$ Total energy = $\frac{1}{2} \cdot Ag\rho(\ell + x)^2 + \frac{1}{2} \cdot Ag\rho(\ell + x)^2 + A\ell\rho\left(\frac{dx}{dt}\right)^2$ = constant

Now, differentiating the above equation

$$\frac{1}{2}A\rho g\left\{2(\ell + x)\frac{dx}{dt} + 2(\ell - x)\left(-\frac{dx}{dt}\right)\right\} + 2A\ell\rho \cdot \frac{d^2x}{dt^2} = 0$$

or $A\rho(g\ell + gx - \ell g + gx)\frac{dx}{dt} + 2A\ell\rho\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} = 0$

or, $2gx + 2.l.\frac{d^2x}{dt^2} = 0$

or, $\ell\frac{d^2x}{dt^2} + gx = 0$

or, $\frac{d^2x}{dt^2} + \frac{g}{\ell}x = 0$

$\therefore$ $\frac{d^2x}{dt^2} + \omega^2 x = 0$

where $\omega = \sqrt{\frac{g}{\ell}}$ is the angular frequency

$\therefore$ Time period, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$

1.6.5 Here, the length of the rod, L = 90 cm

Square of the radius of gyration of the rod, $K^2 = \frac{L^2}{\ell 2}$

Distance of the point of suspension from the centre of graviting, $\ell_1 = 45$ cm

Let the distance of the point of oscillation from the centre of gravity = ℓ_2

Then, $K^2 = \ell_1 \ell_2$

or, $\ell_2 = \dfrac{K^2}{\ell_1} = \dfrac{90 \times 90}{12 \times 45} = 15$ cm

∴ The distance of the point of oscillation from the point of suspension = $\ell_1 + \ell_2 = 45 + 15 = 60$ cm

∴ A concentrated mass has to be placed at a distance of 60 cm from the point of suspension, so that the time of swing remains unaltered.

1.6.6. Here, inductor, $L = 30$ mH $= 30 \times 10^{-3}$ H

Capacitor, $C = 3\mu F = 3 \times 10^{-6}$ F

We know the frequency of oscillation

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{30 \times 10^{-3} \times 3 \times 10^{-6}}} = \frac{1}{2\pi \times 3 \times 10^{-4}} = 530.79 \text{ HZ.}$$

Again, here maximum potential difference across the capacitor is V = 10 volt.

∴ we have the energy $E = \frac{1}{2}CV^2 = \frac{1}{2} \times 3 \times 10^{-6} \times (10)^2 = 1.5 \times 10^{-4}$ Joule.

Unit : 2 ❒ Superposition of Simple Harmonic Oscillations

Structure

2.0 Objectives :

After studying this unit you will be able to—

- state the principle of superposition of harmonic motion.
- apply the principle of superposition of two harmonic oscillation of the same frequency or different frequencies along a line or perpendicular to each other.
- Use the methods of vector addition and complex numbers for superposition of two or amny simple harmonic oscillations.
- learn about Lissajous figures.

2.1 Introduction

In unit 1, we studied the simple harmomic motion with a number of examples from different branches of physics. We observed that in each case the motion in goverened by a harmogeneous second order differential equation, $\frac{d^2x}{dt^2}+\omega^2 x=0$. The solution of this equation x= a sin (ωt + q) or a cos (ωt + q) gives us information regarding the displacement of the body as a function of time. But in many situations, we have to deal with a combination of two or more simple harmonic oscillation. So, in this unit we discuss the principle of superposition and then we apply this principle to different situations, where two or more harmonic oscillations, are superposed along a line or in perpenducular directions.

2.2. Principle of Superposition

The principle of superposition states that, when two or more waves of same type meet at a point, the resultant displacement at that point is equal to the vector sum of the displacements due to each individual wave at the same point.

In unit -1 we observed that the differential equation of Simple Harmonic motion is

$$\frac{d^2x}{dt^2}+\omega^2 x=0 \quad \ldots \quad (2.1)$$

This is a linear homogeneous equation of second order and x is the displacement of vibration of any time t.

Now, consider x_1(t) and x_2(t) respectively satisfy the equation (2.1). Then,

$$\frac{d^2x_1}{dt^2}+\omega^2 x_1=0 \text{ and } \frac{d^2x_2}{dt^2}+\omega^2 x_2=0$$

Additing these two equations, we get

$$\frac{d^2(x_1+x_2)}{dt^2}+\omega^2(x_1+x_2)=0 \quad \ldots \quad (2.2)$$

According to the principle of superposition, the sum of two displacements is given by

x (t) = x_1(t) + x_2(t) ... (2.3)

also satisfies equation (2.1). Thus we can say, the superposition of two displacements satisfies the same linear homogeneous differential equation which is satisfied individually by $x_1(t)$ and $x_2(t)$.

Using this principle, now we shall discuss about the superposition of a different types of vibrations.

2.3 Superposition of two colinear simple harmonic motions of same frequency but different amplitudes and phases.

Let the displacements of two simple harmonic vibrations are represented by

$x_1 = a_1 \cos(\omega t + \phi_1)$ ... (2.4)

and $x_2 = a_2 \cos(wt + \phi_1)$ ... (2.5)

where a_1 and a_2 are the amplitudes, ϕ_1 and ϕ_2 are the initial phase of the two simple harmonic motions of same angular frequency ω.

By the principle of superposition, the resultant displacement is given by

$x = x_1 + x_2$

$= a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2)$

$= a_1 (\cos \omega t \cos \phi_1 - \sin \omega t \sin \phi_1) + a_2 (\cos \omega t \cos \phi_2 - \sin \omega t \sin \phi_2)$

or, $x = (a_1 \cos \phi_1 + a_2 \cos \phi_2) \cos \omega t - (a_1 \sin \phi_1 + a_2 \sin \phi_2) \sin \omega t$

$= A \cos \omega t \cos \delta - A \sin \omega t \sin \delta$

$\therefore x = A \cos(\omega t + \delta)$ ... (2.6)

where $A \cos \delta = a_1 \cos \phi_1 + a_2 \cos \phi_2$ ... (2.7)

$A \sin \delta = a_1 \sin \phi_1 + a_2 \sin \phi_2$ ... (2.8)

From equations (2.7) and (2.8) we get

$A^2 = (a_1 \cos \phi_1 + a_2 \cos \phi_2)^2 + (a_1 \sin \phi_1 + a_2 \sin \phi_2)^2$

$= a_1^2 + a_2^2 + 2a_1a_2(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)$

$\therefore a^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2)$ ... (2.9)

and $\tan \delta = \dfrac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}$ ... (2.10)

Equation (2.6) shows that the resultant motion is also a SHM of same frequency with amplitude A and initial phase δ.

Special cases :

(1) If $\phi_1 - \phi_2 = 2n\pi$ where n = 0, 1, 2 .... etc.i.e, when the phase difference is even multiple of π (same phase) then

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 = (a_1 + a_2)^2$$

$\therefore A = a_1 + a_2$

(ii) If $\phi_1 - \phi_2 = (2n + 1)\pi$ i.e., the component vibrations one in opposite phase, then

$$A^2 = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$$

$\therefore A = a_1 - a_2$

Again, if $a_1 = a_2$, then A = 0, i.e., the vibrating particle with remain at rest.

2.3.1 Method of vector addition :

The rotating vector method is explained in figure 2.1, to obtain the resultant of two simple harmonic motions of same frequencies. In figure $\vec{OA}$ is a rotating vector of constant length a rotating anticlockwise with a constant angular velocity ω and making an angle $(\omega t + \phi_1)$ with respect to x-axis at any time t. The projection ON_1 of this vector of x-axis gives the displacement x_1 at time t. Similarly the component ON_2 of the relating vector $\vec{OB}$ in figure 2.1 will represent the displacement x_2.

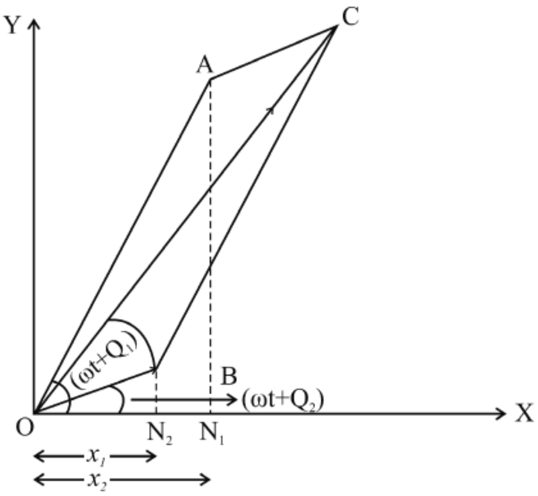


Fig. 2.1

Now, the resultant motion will be given by the vector sum of $\vec{OA}$ and $\vec{OB}$. By the parallelogram law of vector addition the magnitude A of the resultant $\vec{OC}$ is given by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2) \quad [\therefore \phi_1 - \phi_2 \text{ is the angle between } \vec{OA} \text{ and } \vec{OB}]$$

which is same as equation (2.9)

If the resultant $\vec{OC}$ makes an angle $(\omega t + \delta)$ will x axis, then from figure 2.1, $\omega t + \delta = \omega t + \phi_2 + \alpha$

$\therefore\ \delta = \phi_2 + \alpha$

$$\therefore\ \tan\delta = \tan(\phi_2 + \alpha) = \frac{\tan\phi_2 + \tan\alpha}{1 - \tan\phi_2 \tan\alpha}$$

Again, $\tan\alpha = \dfrac{a_1 \sin(\phi_1 - \phi_1)}{a_2 + a_1 \cos(\phi_1 - \phi_2)}$

Putting tan α in the above equation and simplifying we get,

$\tan\delta = \dfrac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \sin\phi_1 + a_2 \cos\phi_2}$ same as equation ... ... (2.10)

$\therefore$ The projection of $\vec{OC}$ on x-axis is

$x = A\cos(\omega t + \delta)$ represent the simple harmonic motion with frequency ω.

2.3.2 Method of Complex Numbers :

You know that any complex number

$z = x + iy$ can be expressed as $z = Ae^{i\theta}$

if $x = A\cos\theta$ and $y = A\sin\theta$.

Again $Zz^* = Ae^{i\theta}.Ae^{-i\theta} = A^2 = x^2 + y^2$

z^* is the complex conjugate of z and tan $\tan\theta = \dfrac{y}{x}$

Now, we can write the quations (2.4) and (2.5) as the real part of

$x_1 = a_1 e^{i(\omega t + \phi_2)}$ and $x_2 = a_2 e^{i(\omega t + \phi_2)}$

$\therefore$ we have $x = x_1 + x_2 = A\cos(\omega t + \delta)$

Let, $x = Ae^{i(\omega t + \delta)}$ then we can write

$$Ae^{i(\omega t + \delta)} = a_1 e^{i(\omega t + \phi_2)} + a_2 e^{i(\omega t + \phi_2)}$$

or, $Ae^{i\omega t}.e^{i\delta} = \left(a_1 e^{i\phi_1} + a_2 e^{i\phi_2}\right)e^{i\omega t}$

or, $Ae^{i\delta} = a_1 e^{i\phi_1} + a_2 e^{i\phi_2}$ ... (2.11) or, $A^2 = AA^* = \left(a_1e^{i\phi_1} + a_2e^{i\phi_2}\right)\left(a_1e^{-i\phi_1} + a_2e^{-i\phi_2}\right)$

$= a_1^2 + a_2^2 + a_2a_2\left\{e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)}\right\}$

$= \left\{a_1^2 + a_2^2 + a_1a_2\left\{\cos(\phi_2 - \phi_2) + i\sin(\phi_1 - \phi_2) + \cos(\phi_1 - \phi_2) - i\sin(\phi_1 - \phi_2)\right\}\right.$

or, $A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)$ same as equation (2.9)

Again from equation (2.11) we get

$Ae^{i\delta} = a_1e^{i\phi_1} + a_2e^{i\phi_2}$

or $A(\cos\delta + i\sin\delta) = (a_1\cos\phi_1 + a_2\cos\phi_2) + i(a_1\sin\phi_1 + a_2\sin\phi_2)$ equating real parts and imaginary party from both sides we can write.

$A\cos\delta = a_1\cos\phi_1 + a_2\cos\phi_2$

and $A\sin\delta = a_1\sin\phi_1 + a_2\sin\phi_2$

$\therefore \tan\delta = \dfrac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}$ same as equation ... ... (2.10)

Thus we can find the equation of superposition of two simple harmonic vibration by vector addition method or complex number method easily,

2.4 Superposition of many harmonic oscillation of same frequencies

Instead of two vibrations, it these are several vibration of different amplitudes and phases but same frequency, the resultant vibration can be deduced in the same way as discussed in the previous article.

The resultant displacement of large number of simple harmonic vibration can be written as

$x = a_1\cos(\omega t + \phi) + a_2\cos(\omega t + \phi_2) + a_3\cos(\omega t + \phi_2) + ...$

$= (a_1\cos\phi_1 + a_2\cos\phi_2 + a_1\cos\phi_3 + ...)\cos\omega t$

$(a_1\sin\phi_1 + a_2\sin\phi_2 + a_3\sin\phi_3 + ...)\sin\omega t$

Let $A\cos\phi_1 = a_1\cos\phi_1 + a_2\cos\phi_2 + a_3\cos\phi_3 + ... = \sum_i a_i\cos\phi_i$

A sin δ $= a_1 \cos \phi_1 + a_2 \cos \phi_2 + a_3 \cos \phi_3 + ... = \sum_i a_i \cos\phi_i$

Then we have x = A cos (ωt + δ)

where $A^2 = (a_1 \cos\phi_1 + a_2 \cos\phi_2 + a_3 \cos\phi_3 + ...)^2$

$$+ (a_1 \sin\phi_1 + q_2 \sin\phi_2 + a_3 \sin\phi_3 + ...)^2 = \left(\sum_i a_1 \cos\phi_j\right)^2 + \left(\sum_i a_1 \sin\phi_i\right)^2$$

$$\text{and } \tan\delta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2 + ...}{a_1 \cos\phi_1 + a_2 \sin\phi_2 + ...} = \frac{\sum_i a_i \cdot \sin\phi_i}{\sum_i a_i \cdot \sin\phi_i}$$

Exercise—1

Two simple harmatic motions acting simultaneously on a particle are given by the equations

$x_1 = 2\sin\left(\omega t + \frac{\pi}{6}\right)$ and $x_2 = 3\sin\left(\omega t + \frac{\pi}{3}\right)$ calculate ampliude and phase of the resultant vibration.

2.5 Superposition of two SHMS of slightly different frequencies along the same straight line : Beats :

Let two simple harmonic motions (SHMs) of angular frequencies ω and ω + Δω, (Δω << ω) are

$x_1 = a_1 \cos(\omega t + \phi)$

and $x_2 = a_2 \cos\{(\omega t\ \Delta\omega)t + \phi_2\}$

or, $x_2 \cos(\omega t + \Delta\omega t + \phi_2)$

$= a_2 \cos(\omega t + \phi'_2)$ where $\phi'_2 = \Delta\omega t + \phi_2$

∴ After superposition the resultant displacement is given by

$x = x_1 + x_2$

$= a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi'_2)$

$= (a_1 \cos\phi_1 + a_2 \cos\phi'_2) \cos\omega t - (a_1 \sin\phi_1 + a_2 \sin\phi'_2) \sin\omega t$

$= A \cos(\omega t + \delta_)$ .... (2.12)

Here, $A \cos \delta = a_1 \cos \phi_1 + a_2 \cos \phi_2'$

$A \sin \delta = a_1 \sin \phi_1 + a_2 \sin \phi_2'$

and $A^2 = (a_1 \cos \phi_1 + a_2 \cos \phi_2')^2 + (a_1 \sin \phi_1 + a_2 \sin \phi_2')^2$

$$= a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2') \quad \text{... (2.13)}$$

$$\tan \delta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2'}{a_1 \cos\phi_1 + a_2 \sin\phi_2'} \quad \text{... (2.14)}$$

The resultant motion described by equation (2.12) is not simple harmonic, for both.

The amplitude A and initial phase angle δ very with time, because $\phi_2' (= \Delta\omega t + \phi_2)$ is function of time.

Thus, when $\phi_1 - \phi_2' = \phi_1 - \phi_2 - t\Delta\omega = (2n + 1)\pi$

where n = 0, 1, 2, 3 ..... etc

Then from equation (2.13) we get

$A^2 = a_1^2 + a_1^2 - 2a_1a_2 = (a_1 - a_2)^2$ [$\because \cos(2n + 1)\pi = -1$]

$\therefore A = a_1 - a_2$

and when $\phi_1 - \phi_2' = \phi_1 - \phi_2' - t\Delta\omega = 2n\pi$

$A = a_1 + a_2$

Hence, the amplitude of the resultant vibration changes between $a_1 - a_2$ to $a_1 + a_2$ with time

If $a_1 = a_2$ then the limits would have been 0 to 2a. Thus the amplitude of the resultant vibration changes periodically with a frequency equal to $\frac{\Delta\omega}{2\pi} = \Delta\vartheta$, the frequency difference of the component of vibrations.

This phanomenon is known as the beats.

You can produce the beats by using two tuning torks or any two sources at sound at nearly equal frequencies are sounded together. The method of beats is very important to measure the unknown frequency.

Figure 2.2 shows the variation of resultants displacement with time after superposition of two SHMs which produces beats.

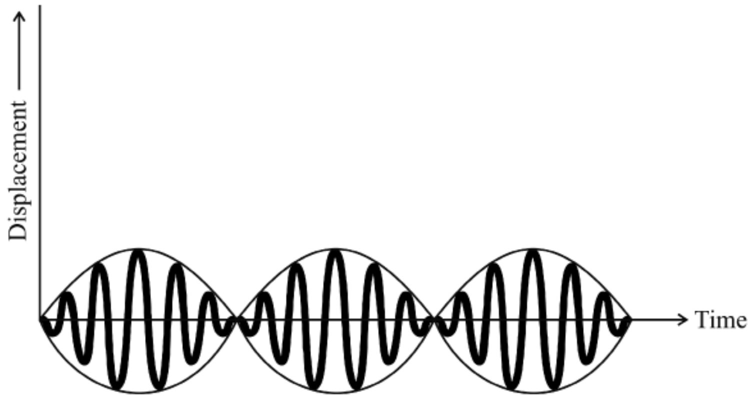


Fig. 2.2

Exercise-2

A note produces 20 beats/see with a tuning fork of frequency 512Hz and 4 beats/see. with a tunning fork of frequency 514 Hz. Find the frequency of the note.

2.6 Oscillation in two dimensions

So far our discussions were confined to harmonic oscillations in one dimension. Now we see when a pendulum oscillates in x – y plane, we call it spherical pendulum.

Now we apply the principle of superposition to the case where two harmonic oscillation are mutually perpendicular.

2.6.1 Superposition of two mutually perpendicular harmonic oscillations of same frequency :

Consider two mutually perpendicular oscillation having amplitudes a and b of same frequency w are described by equations.

$x = a \cos \omega t$ .... (2.15)

and $y = b \cos (\omega t + \phi)$ ... (2.16)

here ϕ is the phase difference between two vibrations

Now, $y = b(\cos \omega t + \cos \phi - \sin \omega t \phi \sin \phi)$

From equation (2.15) we get

$$\frac{x}{a} = \cos \omega t$$

$$\frac{x^2}{a^2} = 1 - \sin^2 \omega t$$

$$\text{or, } \sin \omega t = \sqrt{1 - \frac{x^2}{a^2}}$$

$$\therefore \quad y = b\left(\frac{x}{a}\cos\phi - \sqrt{1 - \frac{x^2}{a^2}}\sin\phi\right)$$

$$\text{or, } \sqrt{1 - \frac{x^2}{a^2}}\sin\phi = \frac{x}{a}\cos\phi - \frac{y}{b}$$

$$\text{squaring } \left(1 - \frac{x^2}{a^2}\right)\sin^2\phi \; - \; \frac{y^2}{b^2} + \frac{x^2}{a^2}\cos^2\phi - 2\frac{x}{a}\frac{y}{b}\cos\phi$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2xy}{ab}\cos\phi = \sin^2\phi \qquad \text{... (2.17)}$$

The equation (2.17) represents the general equations of ellipse. So the motion in general elliptical and the position of the vibrating particle at any instant depends on a, b and ϕ.

Case-I : Let $\phi = 0$ then the equation (2.17) reduces to $\frac{x}{a} - \frac{y}{b} = 0$ or, $y = \frac{b}{a}x$

This represents a straight line passing through the origin making an angle θ with positive direction of x -axis, such that $\tan\theta = \frac{b}{a}$ (figure 2.3a)

Case-II : If $\phi = x$ then from equation (2.17) reduces to $\frac{x}{a} + \frac{y}{b} = 0$ or, $y = \frac{-b}{a}x$

This is also represents a straight line passing through origin as shown is figure 23b.

Case-III: If $\phi = \pi/2$

From equation (2.17) we get

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, this represents an ellipse, axes of which coinciding with x and y axes (figure 2.3c).

Case-IV: If $\phi = \pi/2$ and a = b

Then from equation (2.17) we get

$x^2 + y^2 = a^2$, which represents a circle with radius 'a' the nature of the curve is shown in figure 2.3d.

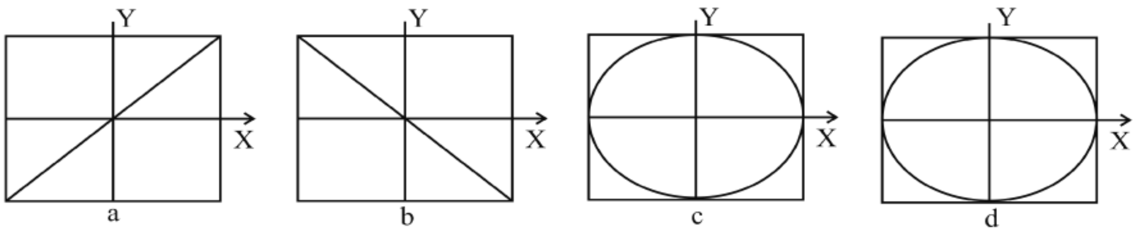


Fig. 2.3

2.6.2 Superposition of two mutually perpendicular harmonic oscillation having frequency ratio 1 : 2 .

Let us consider the case when the frequency of Y oscillation in twice the frequency of X oscillation.

Then two simple harmonic oscillations are given by

$x = a \cos \omega t$ ... (2.18)

and $y = b \cos (2\omega t + \phi)$ ... (2.19)

where a and b are their respective amplitudes and ϕ is the phase differenence between them.

Case-I : If $\phi = 0$, then $x = a \cos wt$ and $y = b \cos 2\omega t$

or, $y = b(2 \cos^2 \omega t - 1)$

$$\therefore \quad y = b\left(2\frac{x^2}{a^2}-1\right)\left[\because \cos\omega t = \frac{x}{a}\right]$$

The above equation is the equation of parabola, the nature of the curve is shown in figure. 2.4a.

Case-II : If $\phi = \frac{\pi}{2}$ then

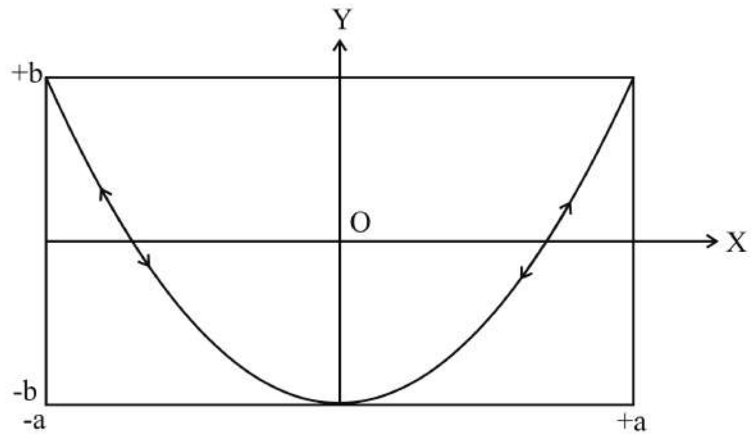


Fig. 2.4a

x = a cos ωt

$$\text{and } y = b \cos\left(2w\omega\frac{\pi}{2}\right)$$

$$= -\, b\sin 2\,\omega t$$

$$= -\,2b \sin \omega t \cos \omega t \ [\because \cos \omega t = x/a \text{ and } \sin \omega t = \sqrt{1 - x^2/a^2}\,]$$

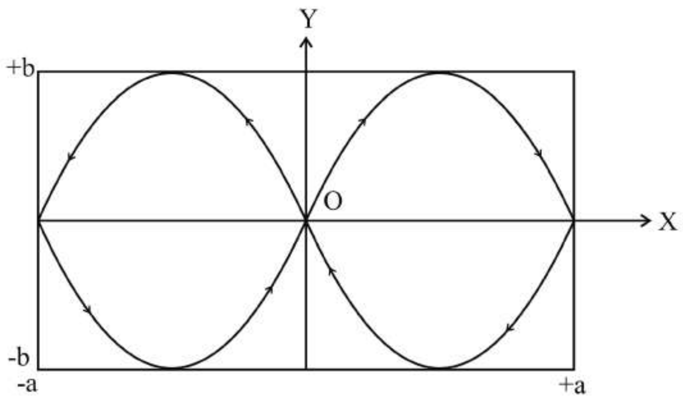


Fig. 2.4b

$$= -2b\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}$$

$$\therefore\ \frac{y^2}{b^2} = 4\frac{x^2}{a^2}\left(1-\frac{x^2}{a^2}\right)$$

The nature of the curve of the above equation is shown in figure. 2.4b

Case-III : when $\phi = \pi/2$

Then $x = a \cos \omega t$ and $y = b \cos (2\omega t + \pi)$

$= - b \cos 2\omega t$

$= - b (2 \cos^2 \omega t - 1)$

$\therefore\ y = b\left(1 - 2\frac{x^2}{a^2}\right)$. This is also a parabola nature of the curve is an shown in figure 2.4c.

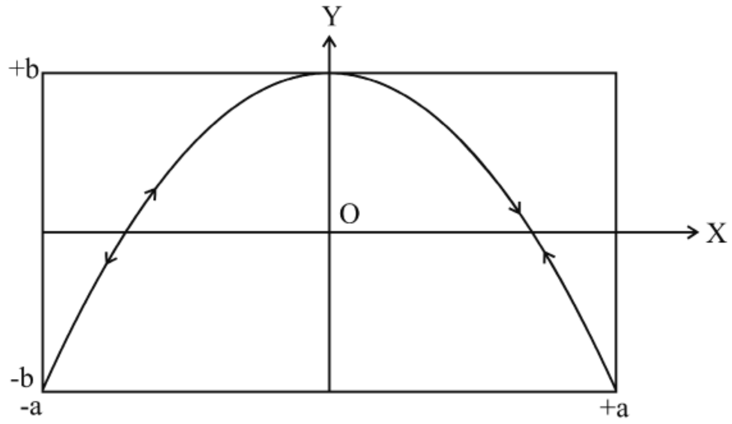


Fig. 2.4.c

2.6.3 Frequencies of any commensurate ratio:

In general, if the frequencies of two simple harmonic motions are in commensurate ratio ie. if $\frac{\omega_1}{\omega_2} = \frac{m}{n}$ where m and n are two integers and phase difference between the waves is $\phi = \frac{\pi}{2}$, then the resultants motions returns to their initial into after a time period

$T = \frac{m\omega_2}{2\pi} = \frac{n\omega_2}{\pi}$ and the number of loops in the path (fig. 2.3c, 24b) will be equal to the ratio at the frequencies. If the ratio of the frequencies be N, then the path of the particle will show N number of loops.

Exercise - 3

Show that the resultant of the two SHMs $x = \sin \omega t$ and $y = 2 \sin 2\omega t$ will be $y^2 = 16x^2 (1 - x^2)$

2.6.4 Lissajous figures

The figures or curves formed by the superposition of two SHMs at right angles to each other are known as Lissajous figures. The shape of these curves depend on the ratio of frequencies, amplitudues and the initial phase relationship of the components of SHMs.

The curves shown in figures 2.3 – 2.4 are the examples of Lissajous figures.

Demonstration of Lissajous figures :

The most appropriate method to demonstate the Liassajous figures is to use cathode ray oscilloscope (CRO).

The basic structure of CRO (as shown in figure 2.5) in an electron gun (G), the vertical (v_1, v_2) and horizontal (H_1, H_2) deflection plates, and a fluorescent screen (S).

A narrow beam of electrons from the electron gun is passed through the vertical plates and then horizontal plates which deflects the beam in vertical and horizontal directions respectively. The electrons beam finally impinge on the screen which produce vissible spot.

To display Lissajous figures on CRO screen, two SHM, are first converted into sinusoidal voltages by using two microphone and amplified by amplifiers, fed through the two deflecting chambers. Under the simultaneous action of the two voltages at right angles to each other, the spot traces out Lissajous figures on the CRO screen.

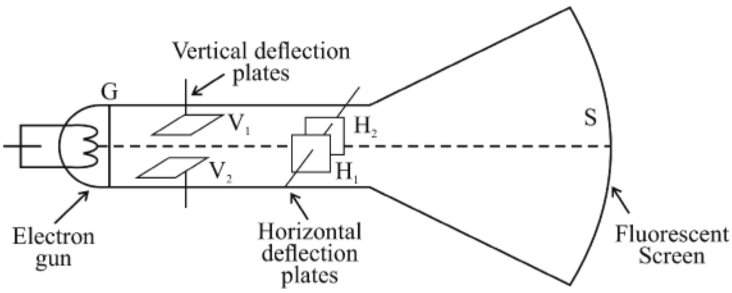


Fig. 2.5

4.7 Summary

The principle of superposition states that if we superpose two or more harmonic oscillation in a same straight line then it produces new type of oscillation, the resultant displacement is the vector sum of individual displacement at all subsequent times.

i.e., $x(t) = x_1(t) + x_2(t)$

when two or more colinear harmonic oscillation of same frequency.

$x_1 = a_1\cos(\omega t + \phi_1)$, $x_2 = a_2\cos(\omega t + \phi_2)$

$x_3 = a_3\cos(\omega t + \phi_3)$, ...etc are superposed, then after superposition the resultant displacement will be

$x = A\cos(\omega t + \delta)$

where $A^2 = \left(\sum_i a_i \cos\phi_i\right)^2 + \left(\sum_i a_i \cos\phi_i\right)^2$

and $\tan\delta = \dfrac{\sum_i a_i \cos\phi_i}{\sum_i a_i \cos\phi_i}$

If two colinear harmonic oscillations of different frequencies are superposed, then the resultant motion will not be simple harmonics, it produces beats

When two mutally perpendicular harmonic oscillations are superposed the resultant from traces out different curves. The general from of the curve is elliptical but for certain phases, it closes to a straight line, circle etc. These curves are known as Lissajous figures.

2.8 Questions and Problems

2.8.1 Two harmonic oscillations of frequencies ω having amplitude 2 cm and initial phases difference is $\frac{\pi}{2}$ are superposed. Calculate the amplitude and the phase of the resultant vibration.

2.8.2 A particle is subjected to two SHM in the same direction having equal amplitudes and frequencies. If the resultant amplitude is equal to the amplitude of the individual motion, what is the phase difference between them?

2.8.3 Equation of two SHMs are $x_1 = a\cos(\omega t + 30°)$ and $x_2 = a\cos(\omega t - 30°)$. Calculate the resultant equation of motion in vector method and complex number method when they are superposed.

2.8.4 In a CRO, the deflection of electrons by two mutually perpendicular linear harmonic osciallations of unequal amplitude of electric fields are given by

$x = 4\cos\omega t$

and $y = 3\cos(\omega t + \pi/6)$

what will be the resultant path of electrons ?

2.9 Solutions

Exercise - 1 :

Here $x_1 = 2\sin(\omega t + \pi/6)$

and $x_2 = 3\sin(\omega t + \pi/6)$

Thus, $a_1 = 2$, $a_2 = 3$, $\phi_1 = \pi/6$ and $\phi_2 = \pi/3$.

The resultant vibration is

$x = x_1 + x_2 = A \sin(\omega t + \delta)$

where, $A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi_1 - \phi_2)$

$= 2^2 + 3^2 + 2.2.3. \cos(\pi/6 - \pi/3)$

$$A = \sqrt{4 + 9 + 12\cos\left(-\frac{\pi}{6}\right)}$$

$\therefore$ A = 4.84

$$\text{Again, } \tan\delta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2} = \frac{2\sin\frac{\pi}{6} + 3\sin\frac{\pi}{3}}{2\cos\frac{\pi}{6} + 3\cos\frac{\pi}{3}} = 1.113$$

$$\therefore \delta = \tan^{-1} 1.113 = 48.1° = \frac{48.1 \times \pi}{180} \sim \frac{4\pi}{15}$$

$$\therefore \text{The phase} = \omega t + \delta = \omega t + \frac{4\pi}{15}$$

Exercise : 2

In the first case

Frequency of the tunning fork = 512 Hz

Beats/sec = $\Delta\vartheta = \vartheta_2 \sim \vartheta_1 = 2$

$\therefore$ Possible frequencies of note are 512 + 2 = 514 Hz

or, 512 – 2 = 510 Hz

Similarly, in second case, the possible frequencies of the note are 514 + 4 = 518 Hz

or, 514 – 4 = 510 Hz

Hence, the frequency of the note is 510 Hz

Exercise-3

Here, $x = \sin \omega t$

and $y = 2 \sin \omega t$

$\therefore$ y = 2. 2 sin ωt cos ωt

or, $\frac{y}{4} = x.\sqrt{1-x^2}$ [∵ sin ωt = x or, $\sin^2 \omega t = x^2$ or, $1 = \cos^2 \omega t = x^2$ ∴ cos ωt = $\sqrt{1-x^2}$]

$\therefore y^2 = 16x^2 (1 - x^2)$ Proved.

Solution of problems

2.8.1

Here $a_1 = a_2 = 2$cm

and $\phi_1 - \phi_2 = \frac{\pi}{2}$

∴ The amplitude of the resultant vibration is

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)}$$

$$= \sqrt{4. + 4 + 2.2.2.\cos\frac{\pi}{2}}$$

$$= \sqrt{8}$$

$$\therefore A = 2\sqrt{2} \text{ cm.}$$

Again, $\tan\delta = \frac{a_1\sin\phi_1 + a_2\sin\phi_2}{a_1\sin\phi_1 + a_2\cos\phi_2}$ [Let $\phi_1 = 0°$ and $\phi_2 = \pi/2$]

$$= \frac{2.\sin\frac{\pi}{2}}{2.\cos 0°} = 1$$

$$\therefore \delta = 45° = \frac{\pi}{4}$$

∴ The phase is $\left(\omega t + \frac{\pi}{4}\right)$

2.8.2 Here $a_1 = a_2 = A$

We know the amplitude of the resultant motion is $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2)}$

or, $A^2 = A^2 + A^2 + 2A^2\cos(\phi_1 - \phi_2) = 2A^2\left[1 + \cos(\phi_1 - \phi_2)\right]$ [$\phi_1 = 0°$ and ϕ_2 =where,

=phase difference between the two SHMs

or, $2\{1 + \cos(\phi_1 - \phi_2)\} = 1$

or, $1 + \cos(\phi_1 - \phi_2) = \frac{1}{2}$

or, $\cos(\phi_1 - \phi_2) = -\frac{1}{2}$

or, $(\phi_1 - \phi_2) = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$ or, $\frac{2\pi}{3}$

$\therefore$ Phase difference between the two SHMs is $(\phi_1 - \phi_2) = \frac{2\pi}{3}$

2.8.3 Vector method :

In figure $\vec{OA}$ is vector of length 'a' makes an angle $\omega t + 30°$ with x-axis.

$\therefore$ The projection $\vec{OA}$ along x-axis represents $x_1 = a\cos(wt + 30°)$

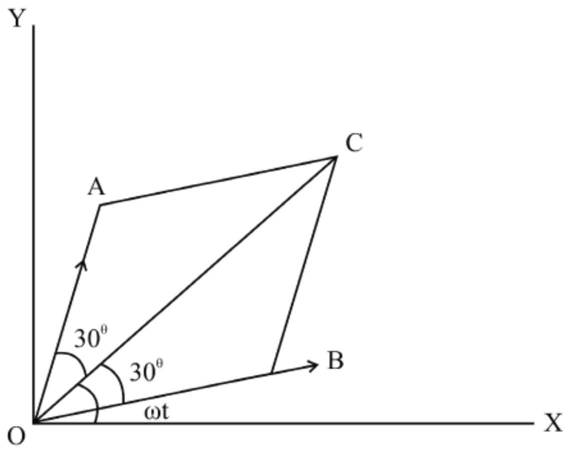
Similarly, $\vec{OB}$ is a vector of length 'a', makes an angle $wt - 30°$ with x-axis.

$\therefore$ $x_2 = a\cos(\omega t - 30°)$

$\therefore$ The resultant of $\vec{OA}$ and $\vec{OB}$ is

$A = \sqrt{a^2 + a^2 + 2.a.a.\cos 60°} = \sqrt{3a}$ [$\because$ Angle between $\vec{OA}$ and $\vec{OB}$ is 60°]

$\therefore$ The equation of motion of the resultant is $x = \sqrt{3a}\cos\omega t$ ·



Complex number method :

Here $x_1 = a\cos(\omega t + 30°)$

$= ae^{i(\omega t + 30°)}$ (Real Part only)

and $x_2 = a\cos(\omega t - 30°)$

$= ae^{i(\omega t - 30°)}$ (R.P only)

∴ The resultant

$x = x_1 + x_2$

$= a\left\{e^{i(\omega t+30°)} + e^{i(\omega t-30°)}\right\}$

$= ae^{i\omega t}\left(e^{i30°} + e^{-i30°}\right)$

$= ae^{i\omega t}\left(\cos 30° + (i\sin 30° + \cos 30° - i\sin 30°\right)$

$= ae^{i\omega t}\left(2\cos 30°\right)$

$= ae^{i\omega t}\left(2\cdot\sqrt{\frac{3}{2}}\right)$

$= \sqrt{3ae}^{i\omega t}$

$= \sqrt{3a}\cos\omega t$ (Real part only)

$\therefore\ x = \sqrt{3a}\cos\omega t$

2.8.4 Here x = 4 cos ωt and y = 3 cos (ωt + π/6)

we know the equation (2.17) is.....

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi$$

Here, a = 4, b = 3 and φ = π/6

$$\therefore\ \frac{x^2}{4^2} + \frac{y^2}{3^2} - \frac{2xy}{4.3}\cos\frac{\pi}{6} = \sin^2\frac{\pi}{6}$$

$$\text{or, } \frac{x^2}{16} + \frac{y^2}{9} - \frac{xy}{6}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4}$$

$$\therefore\ \frac{x^2}{16} + \frac{y^2}{9} - \frac{\sqrt{3}}{12}xy = \frac{1}{4}$$

∴ The resultant path is an ellipse.

Unit - 3 □ Damped Harmonic Motion

Structure

3.1 Objectives

● After studing this unit you will be able to estblish the differential equation for a damped harmonic oscillator and solve it in different damping conditions, such as weakly damped, critically damped and overdamped systems.

● Know the effect of damping on amplitude, energy and frequency of oscillation.

● find the relaxation time, logarithmic decrement and quality factor for a damped oscillator and able to draw the similarities between different natural system.

3.1 Introduction

In the previous chapter (unit-I) you learnt about the free simple harmonic motions. This is an ideal thing once such a system is set in motion it will continue to oscillate forever with a constant aplitude. But in real physical system, the amplitude of the oscillator gradually decreases with time and the oscillator eventually comes to rest. You must have observed that oscillations of a simple pendulum, torsional pendulum, a spring-mass system etc, the amplitude of vibration is gradually diminished and becomes imperceptible after some time. Hence, we may conclude that there is a force on the vibrating body and this may be due to viscosity of the medium or other frictional forces. This force is called the damping force. For small velocity, we may take the damping force is proportional to the instantaneous velocity of the body and in the opposite direction to the velocity.

In this unit, you will learn to establish and solve the equation of motion of a damped harmonic oscillator in different damping conditions.

3.2. Differential equation of a damped oscillator

Consider a particle of mass m is oscillating under the action of a restoring force proportional to the displacement (x) from its equilibrium position and a damping force proportional to the instantaneous velocity (dx/dt). The equation of motion of such as oscillator is

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt} \quad \text{......} \qquad \text{... ...} \qquad \text{...} \qquad (3.1)$$

where s is the restoring force per unit displacement and k is the damping force per unit velocity.

Equation (3.1) can be written as

$$\frac{d^2x}{dt^2} + 2b\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{...} \qquad \text{...} \qquad \text{... ...} \qquad \text{...} \qquad (3.2)$$

where $2b = \frac{k}{m}$ and $\omega = \sqrt{s/m}$, w is the angular frequency of vibration in absence of damping.

Equation (3.2) is knowen as differential equation of damped motion.

To solve this equation (3.2), let $x = Ae^{\alpha t}$ be the trial solution of equation (3.2), where A and α are constants.

Dirrerentiating x we get

$$\frac{dx}{dt} = A\alpha\, e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Putting these in equation (3.2) we get

$$(\alpha^2 + 2b\alpha + \omega^2)\, Ae^{\alpha t} = 0$$

$$\text{or, } (\alpha^2 + 2b\alpha + \omega^2)\, x = 0$$

$\therefore \quad \alpha^2 + 2b\alpha + \omega^2 = 0 \quad \because x \neq 0$

Now the roots of this equation are $\alpha_1 = -b + \sqrt{b^2 - \omega^2}$ and $\alpha_2 = -b - \sqrt{b^2 - \omega^2}$.

$\therefore$ The two possible solutions of equation (3.2) are

$x_1 = A_1 e^{\alpha_1 t}$ and $x_2 = A_2 e^{\alpha_2 t}$

Since the equation (3.2) is a linear homogeneous equation, so its general solution is given by the principle of superposition as

$$x = x_1 + x_2$$

$$= A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

$$= A_2 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + A_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}$$

$$\therefore x = e^{bt}\left(A_1 e^{\sqrt{b^2-\omega^2}t} + A_2 e^{-\sqrt{b^2-\omega^2}t}\right) \quad \dots \quad \dots \quad \dots \dots \quad \dots \quad (3.3)$$

Here A_1 and A_2 are two artitrary constants, which can be evaluate from initial conditions.

The nature of the soluti $\frac{x^2}{16} + \frac{y^2}{9} - \frac{\sqrt{3}}{12}xy = \frac{1}{4}$ on of equation (3.3) depends on the relative values of b and ω

i) if $b > \omega$, we say that the system is overdamped or heavy damping.

ii) if $b \simeq \omega$, we have a critically damped system. and

iii) if $b < \omega$, we have an under damped, the motion is damped oscillatory.

Now we shall consider three different cases.

3.2.1. Heavy damping b > ω :

When resistance to the motion is very strong, the system is said to be heavily damsped.

Now as $b > \omega$ the quantity $b^2 - \omega^2$ in equation (3.3) is positive definite. Let $\sqrt{b^2 - \omega^2} = b'$, equation (3.3) can be written as

$$x = e^{-bt}\left(A_1 e^{b't} + A_1 e^{-b't}\right) \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.4)$$

In equation (3.4) quantity b' is positive, so the motion is obviously non-oscillatory, aperiodic or dead beat type.

Now let at time t = 0, $x = x_0$,

$\therefore$ From equation (3.4) we get $x_0 = A_1 + A_2 \neq 0$

i.e. if the motion is started with initial displacement (x_0) but no initial velocity, the displacement (x) gradually falls of to zero with time due to presence of the term e^{-bt} from it initial value x_0 and finally the body returns to the equilibrium position without any oscillations. The motion is illustrated in figure - 3.1 (I)

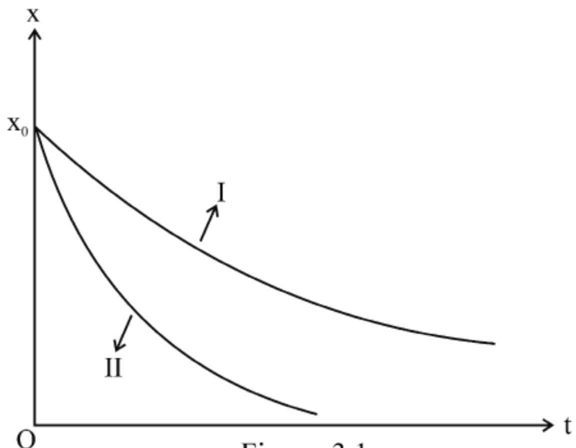


Figure 3.1

Again if at t = 0, $x = 0$

and velocity $\frac{dx}{dt} = \upsilon_0$, then from equation (3.4)

We get

$$A_1 + A_2 = 0 \text{ or } A_1 = -A_2$$

and $\frac{dx}{dt} = -be^{-bt}(A_1 e^{b't} + A_2 e^{-b't}) + e^{-bt}(A_1 b' e^{b't} - A_2 b' e^{-b't})$

$\because$ At t = 0 $\frac{dx}{dt} = \nu$

$$\therefore \nu_0 = -b(A_1 + A_2) + \therefore \nu_0 (A_1 - A_2)$$

$$= 2A_1 b' \qquad [\because A_1 = -A_2]$$

or, $A_1 = \frac{\nu_0}{2b'}$

$\therefore A_1 = \frac{\nu_0}{2\sqrt{b^2 - \omega^2}}$ and $A_2 \frac{\nu_0}{2\sqrt{b^2 - \omega^2}}$

Putting these values of A_1 and A_2 in equation (3.4) we get [$\therefore \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$]

$$\therefore x = \frac{\nu_0}{\sqrt{b^2 - \omega^2}} e^{-bt} \sinh(\sqrt{b^2 - \omega^2}\, t) \quad \ldots \quad \ldots \ldots \quad \ldots \quad (3.5)$$

Equation (3.5) discribes the behaviour of a heavily damped system which is disturbed from equilibrium by a sudden impulse at t = 0. The motion is illustrated in figure - 3.2.

When time is small the term e^{-bt} is very nearly equal to unity, so from equation (3.5) we see that the displacement depends on the factor sinh ($\sqrt{b^2 - \omega^2}\, t$) i.e. displacement increases with time as shown in figure 3.2 (a). But when the displacement reaches to a certain maximum value the term e^{-bt} dominates and then it decays exponentially with time, Figure - 3.2(b). The overall variation of displacement with time is shown in figure – 3.2 (c).

This type of motion is found in a pendulum immersed in a highly viscous liquid and also in a dead-beat galvanometer.

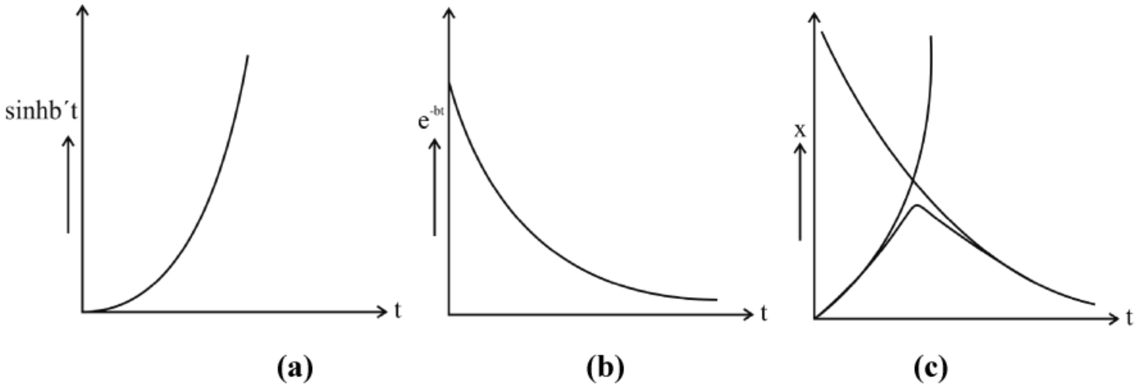


Figure 3.2

3.2.2. Critical damping, b ≃ ω:

When b = ω we see from equation (3.3) that

$$x = e^{-bt}(A_1 + A_2) \text{.........} (3.6) \quad [\because \sqrt{b^2 - \omega^2} = 0]$$

Since the constants A_1 and A_2 are multiplied by the same factor, so they are equivalent to one constant. Hence, the equation (3.6) does not satisfy the second order differential equation. Thus to get the solution for this condition, we take $b \to \omega$ ie $b - \omega \simeq \beta$, which is very small quantity. Using this condition, from equation (3.3) we get

$$x = e^{-bt}(A_1 e^{\beta t} + A_2 e^{-\beta t})$$
$$= e^{-bt}\{A_1(1 + \beta t + ...) + A_2(1 - \beta t + ...)\}$$
$$= e^{-bt}\{(A_1 + A_2) + (A_1 - A_2)\beta t\} \quad [\because \beta \text{ is very small higher order terms neglected}]$$

$$\therefore x = e^{-bt}(c_1 + c_2 t) \quad .... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (3.7)$$

where $c_1 = A_1 + A_2$ and $(A_1 - A_2)\beta = c_2$ are constants.

The motion represented by the equation (3.7) is also non-oscillatory. But here rate of decay is more faster than overdamped case. The motion is called critically damsped.

Now if at t = 0, $x = x_0$ and $\frac{dx}{dt} = 0$, then from equation (3.7) we get

$$x_0 = c_1$$

and $\frac{dx}{dt} = -be^{-bt}(c_1 + c_2t) + e^{-bt}c_2$

$$0 = -bc_1 + c_2 \qquad [\because \text{at } t = 0, \frac{dx}{dt} = 0]$$

$$\therefore c_2 = bc_1 = bx_0$$

Putting c_1 and c_2 in equation (3.7) we get

$$x = e^{-bt}(x_0 + bx_0t)$$

or $\quad x = e^{-bt}x_0(1 + \omega t)$ .... (3.8) $\quad [\because b \simeq \omega]$

From the above equation (3.8) we see that the displacement (x) decreases from its initial value x_0 with time. This is illustrated by the curve II in figure–3.1. Here the body returns quickly to the position of equilibrium.

Again if at t = 0, $x = 0$ and $\frac{dx}{dt} = v_0$ then from equation (3.7) we have

$$0 = c_1$$

and $\frac{dx}{dt} = -be^{-bt}(c_1 + c_2t) + e^{-bt}c_2$

or, $v_0 = c_2 \quad [\because \text{ at } t = 0, \frac{dx}{dt} = v_0, \text{ and } c_1 = 0]$

$\therefore$ Putting c_1 and c_2 in equation (3.7) we get

$$x = v_0 e^{-bt} \cdot t = v_0 t e^{-\omega t} \quad \ldots \quad (3.9)$$

and $\frac{dx}{dt} = v_0 e^{-\omega t}(1 - \omega t)$........... (3.10)

Now figure 3.3 represents the variation of displacement (x) against time (t) for $b \simeq \omega$ condition.

Here the oscillator is given a sudden impulse in equilibrium position, for small values of t, $e^{-\omega t}$ is very nearly equal to unity and the displacement increases almost linearly with time (from equation (3.9) $x = \nu_0 t$ as $e^{-\omega t} \simeq 1$) and becomes maximum when $\frac{dx}{dt} = 0$ i.e at $t = \frac{1}{\omega} = \frac{1}{b}$ (from equation 3.10), after which it decays to zero exponentially.

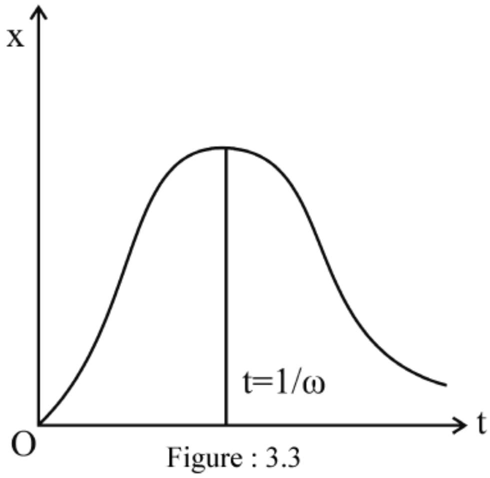


Figure : 3.3

In comparison equation (3.5) and (3.9) we see that the decay rate is much faster for $b \simeq \omega$ that for $b > \omega$.

In both the cases there are no oscillations as the displacement never becomes negative.

The motion is called the critically damped motion.

In a pointer type galvanometers where the pointer moves immediately to the correct position and stays there without oscillation, is example of this type critically damped motion.

3.2.3. Damped oscillation or light damping, $b < \omega$:

Here $b < \omega$, so $\sqrt{b^2 = \omega^2}$ is imaginary,

Let $\sqrt{b^2 = \omega^2} = i\sqrt{\omega^2 - b^2} = i\omega'$, where $\omega' = \sqrt{\omega^2 - b^2}$

From equation (3.3) we get

$$x = e^{-bt}(A, e^{i\omega' t} + A_2 e^{-i\omega' t})$$

or, $x = e^{-bt}\{A_1(\cos\omega' t + i \sin\omega' t) + A_2(\cos\omega' t - i \sin\omega' t)\}$

$$= e^{-bt}\{(A_1 + A_2)\cos\omega' t + i(A_1 - A_2)\sin\omega' t\}$$

$$= e^{-bt}(a_1 \cos\omega' t + a_2 \sin\omega' t)$$

where $a_1 = A_1 + A_2$ and $a_2 = i(A_1 - A_2)$

Putting $a_1 = A\cos\varphi$ and $a_2 = A\sin\varphi$ in the above equation we get

$$x = Ae^{-bt}\cos(\omega' t - \varphi) \quad .... (3.11)$$

Where A and φ are real constants, the values of which depend on the initial conditions.

Evaluation of A and φ –

Let at t = 0, $x = x_0$ and $\frac{dx}{dt} = \upsilon_0$, then from equation (3.11) get

$$x_0 = A\cos\varphi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3.12a)$$

Differentiating equation (3.11) we have

$$\frac{dx}{dt} = -Abe^{-bt}\cos(\omega' t - \varphi) - Ae^{-bt}\omega' \sin(\omega' t - \varphi)$$

or, $\upsilon_0 = -Ab\cos\varphi + A\omega' \sin\varphi$

$= -bx_0 + A\omega' \sin\varphi$

$$\text{or, } A\sin\varphi = \frac{\upsilon_0 + bx_0}{\omega'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3.12b)$$

From equations (3.12a) and (3.12b) we get

$$A = \sqrt{x^2 + \left(\frac{\upsilon_0 + bx_0}{\omega'}\right)^2} \text{ and}$$

$$\tan\varphi = \frac{\upsilon_0 + bx_0}{\omega' x_0}$$

Now if at t = 0, $x = x_0$ and $\frac{dx}{dt} = 0$ the

$$A = \sqrt{x_0^2 + \frac{b^2 x^2}{\omega'^2}} = x_0\sqrt{1 + \frac{b^2}{\omega^2 - b^2}} = x_0\sqrt{\frac{\omega^2}{\omega^2 - b^2}}$$

$$\therefore A = \frac{x_0\omega}{\sqrt{\omega^2 - b^2}} \text{ and } \tan\varphi = \frac{b}{\sqrt{\omega^2 - b^2}}$$

∴ Equation (3.11) can be written as

$$x = \frac{x_0\omega}{\sqrt{\omega^2 - b^2}} e^{-bt}\cos\left\{\sqrt{\omega^2 - b^2}\,t - \tan^{-1}\left(\frac{b}{\sqrt{\omega^2 - b^2}}\right)\right\} \ldots\ldots\ldots (3.13)$$

Equation (3.11) and (3.13) represent a damped oscillatory motion with amplitude Ae^{-bt} which goes on decreasing exponentially with time and a new frequencey $\omega' = \sqrt{\omega^2 - b^2}$ the variation of x with t, is shown in figure 3.4.

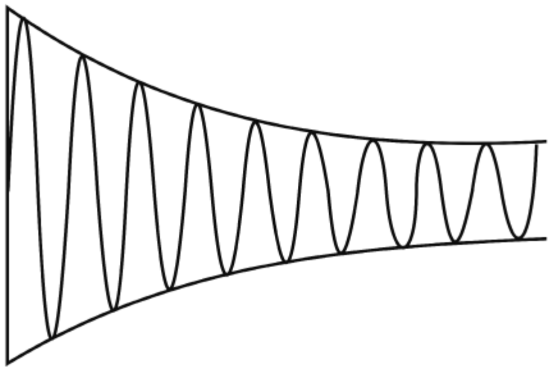


Figure : 3.4

Exercise–1

The amplitude of vibration of a damped oscillator decreases from 16 cm to 2 cm in 100 sec. If this oscillator performs 50 oscillations in this time. Calculate the periods with and without damping.

3.3. Energy of a damped oscillator

Let x be the instalments displacement of a pareticle of mass m at any time t, executing damped simple harmonic motions.

The potential energy of the particle is

$$E_p = \int_0^x sx\,dx = \frac{1}{2}sx^2 \; [\because \text{ s is the stiffness constant}]$$

And the kinetic energy at that instant is

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

$\therefore$ Total everty (E) = $E_p + E_k$

$$= \frac{1}{2}sx^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.14)$$

$[\because \omega = \sqrt{s/m}]$

Now using equation (3.11)

$$x = Ae^{-bt}\cos(\omega' t - \varphi)$$

$$\frac{dx}{dt} = bAe^{-bt}\cos(\omega't-\varphi) - A\omega'e^{-bt}\sin(\omega't-\varphi)$$

$$\therefore\ E = \frac{1}{2}m\omega^2 A^2e^{-bt}\cos^2(\omega't-\varphi) + \frac{1}{2}m\{-bAe^{-bt}\cos(\omega't-\varphi)$$

$$-A\omega'e^{-bt}\sin(\omega't-\varphi)\}^2$$

$$= \frac{1}{2}m\omega^2A^2e^{-2bt}\cos^2(\omega't-\varphi) + \frac{1}{2}mA^2e^{-2bt}\{b^2\cos^2(\omega't-\varphi)$$

$$+ \omega'^2\sin^2(\omega't-\varphi) + 2b\omega'\sin(\omega't-\varphi)\cos(\omega't-\varphi)\}$$

Since the damping is weak the sine and cosine functions vary more rapidly with time than e^{-2bt} in a period. So, the time averaged value of E can be obtained by taking the average over a period T of sine and consine functions.

$$\therefore \text{Time average energy, } \langle E\rangle = \frac{1}{T}\int_0^T E\,dt$$

$$\text{or, } E = \frac{1}{2T}\int_0^T m\omega^2A^2e^{-bt}\cos^2(\omega't-\varphi) + mA^2e^{-2bt}\{b^2\cos^2(\omega't-\varphi)$$

$$+ \omega'^2\sin^2(\omega't-\varphi) + 2b\omega'\sin(\omega't-\varphi)\cos(\omega't-\varphi)\}dt$$

$$\text{or, } E = \frac{1}{2T}mA^2e^{-2bt}\int_0^T[\omega^2\cos^2(\omega't-\varphi) + \{b^2\cos^2(\omega't-\varphi)$$

$$+ \omega'^2\sin^2(\omega't-\varphi) + 2b\omega'\sin(\omega't-\varphi)\cos(\omega't-\varphi)\}]dt.$$

$$= \frac{1}{2T}mA^2e^{-2bt}\left[\omega^2\cdot\frac{T}{2} + \{b^2\cdot\frac{T}{2} + \omega'^2\,\frac{T}{2} + 0\}\right]$$

$$= \frac{1}{4}mA^2e^{-2bt}(\omega^2 + b^2 + \omega'^2) \quad [\because \omega'^2 = \omega^2 - b^2]$$

$$= \frac{1}{4}mA^2e^{-2bt}(\omega^2 + b^2 + \omega^2 - b^2)$$

$$= \frac{1}{2}m\omega^2A^2e^{-2bt}$$

$= E_0 e^{-2bt}$ [Where $E_0 = \frac{1}{2} m\omega^2 A^2$]

$\therefore <E> = E_0 e^{-2bt}$........ (3.15)

From unit – 1 we can say $E_0 = \frac{1}{2} m\omega^2 A^2$ is the total energy of an undamped oscillator.

From equation (3.15) we see that the average energy of a weakly damped oscillator decays exponentially with time as in figure 3.5.

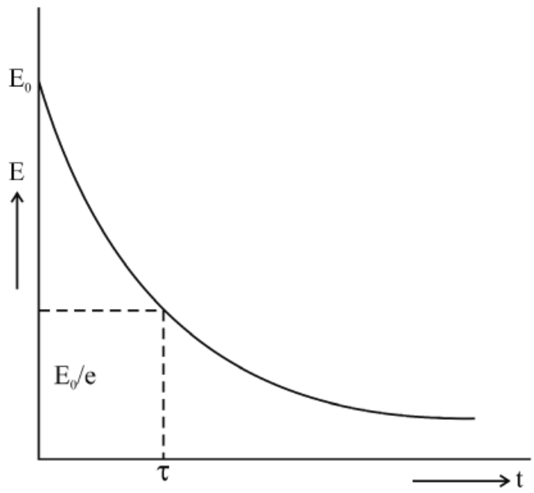


Figure : 3.5

Now if $<E> = \frac{1}{e} E_0$ i.e. the energy falls to $\frac{1}{e}$ of its initial value, then from equation (3.15) we get

$$\frac{1}{e} E_0 = E_0 e^{-2bt}$$

or, $e^{-1} = e^{-2bt}$

or, $2bt = 1$

$$\therefore \quad t = \frac{1}{2b} = \tau_1 \text{ (say)} \quad \ldots \quad (3.16)$$

This time (τ_1) is known as energy decay time constant.

3.4. Methods of characterising damped system

The characterise the damped motion we can relate b and ω in this section briefly.

3.4.1. Logarithmic decrement :

The displacement as given in equation (3.11) becomes maximum when cos $(\omega' t - 1) = \pm 1$

Let the oscillation starts from the mean position. Then after the time $t = \frac{T}{4}$ (where T is the time period) the particle goes to the extreme position ie say, $A_1 = A\,e^{-bT/4}$ and again it will be at the extreme position on the other side when $t = \frac{T}{4} + \frac{T}{2} = \frac{3T}{4}$ i.e.

$$A_2 = A\,e^{-3bT/4}$$

$$\text{Thus } \frac{A_1}{A_2} = e^{-\frac{bT}{4}+\frac{3bT}{4}} = e^{bT/2} = d \text{ (say)}$$

$$\text{or, } \log_e \frac{A_1}{A_2} = \frac{bT}{2} = \lambda \text{ (Say)}$$

Hence, the quantity d which is the ratio of two successive amplitudes (on the opposite sides of mean position) is called decrement and $\log_e d = \lambda$ is called the logarithrmic decrement of the damped oscillastion.

Therefore, we can define the logrithimic decrement as the natural logrithm of the ratio of two successive amplitudes of damed oscillations that are separated by the half of its time period.

$$\text{Now, if } A_1, A_2, A_3 \ldots\ldots = \frac{A_{n-1}}{A_n} = e^{bT/2} = e^{\lambda}$$

(here n is the number of swing)

$$\text{or, } \frac{A_1}{A_2}\cdot\frac{A_2}{A_3}\cdots\cdots\frac{A_{n-1}}{A_n} = e^{(n-1)\lambda}$$

$$\text{or, } \frac{A_1}{A_n} = e^{(n-1)\lambda}$$

$$\text{or, } \ln\frac{A_1}{A_n} = (n-1)\lambda$$

$$\therefore \lambda = \frac{1}{n-1}\ln\frac{A_1}{A_n} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.17)$$

Thus λ can be measured by noting A_1 and A_n and hence we can measure experimentally the damping co-efficient b.

Exercise–2 : A damped oscillator starting from rest has amplitude 4 cm after 101 swings. Its first amplitude was 40 cm and its time period is 2.3 sec. Calculate its logdecrement and relaxation time.

3.4.2. Decay constant or Relaxation time (τ) :

It is defined as the time in which the amplitude of a damped oscillatory motion reduces to $\frac{1}{e}$ of its initial value.

If τ be the relaxation time and A is the initial amplitude, then

$$A e^{-b\tau} = \frac{A}{e} \text{ or, } b\tau = 1 \therefore \tau = \frac{1}{b}$$

Thus the relaxation time varies inversely with damping co-efficient.

You have already known the energy decay time $\tau_1 = \frac{1}{2b}$ (equation 3.16). But here the relaxation time $\tau = \frac{1}{b}$. Hence $\tau = 2\tau_1$.

3.4.2. Quality factor or Q-factor

The average energy of a damped oscillator is

$$<E> = E_0 e^{-2bt}$$

$$\therefore \text{ Power dissipation (p)} = -\frac{d<E>}{dt} = 2bE_0 e^{-2bt} = 2b<E>$$

If $dt = T = \text{time} = 2\pi/\omega$, then we have the loss of energy in one period i.e. loss of energy per cycle is

$$-\frac{d<E>}{dt} = -\frac{(E_T)}{T} = 2b <E>$$

$$\text{or, } \frac{1}{2b}\cdot\frac{\omega}{2\pi} = -\frac{<E>}{<E_T>}$$

$$\text{or, } \frac{\omega}{2b} = \frac{m\omega}{k} = -\frac{<E>}{<E_T>} \qquad [\because 2b = \frac{k}{m}]$$

Hence we can define Q-factor as

$$\text{Quality factor (Q)} = \frac{\omega}{2b} = \frac{m\omega}{k} = \frac{\text{average energy of the oscillator}}{\text{average energy lost per cycle}}$$

Exercise – 3 : The quality factor of a turning fork of frequency 520 Hz is 1000. Calculate the time in which its energy becomes 20% of its initial value.

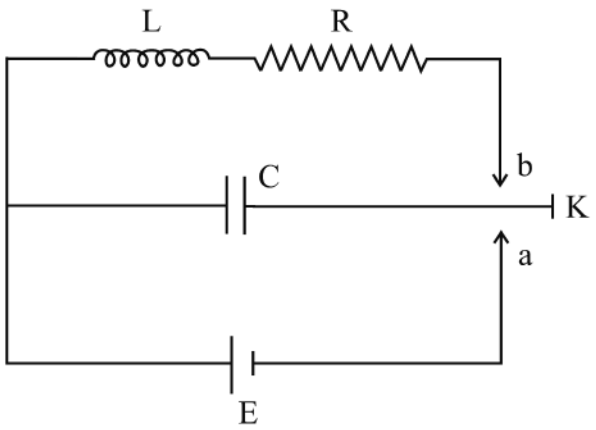


Figure : 3.6

3.5. Example of weakly damped system

You know that all harmonic oscillators in nature are not simple harmonic but they have some damping. We will discuss one of such motion here.

3.5.1. An LCR circuit

In Unit–I you have seen that an LC circuit is an ideal case, where the resistance of the inductor coil neglected. But here we consider the resistance of the circuit. Hence the circuit is an LCR circuit as shown in figure – 3.6.

Here the resistance R, an inductance L, and a capacitance c are connected in series. Now by pressing the key k downward, the capacitor is charged by a battery of emf E. If K is released, the battery is thrown out of the cvircuit and the capacitor begings to discharge through L and R giving a current i at any instant t.

Let at any instant t, the charge on the capacitor be q and current i, then the potential difference across the capacitor $(V_L) = q/c$, the potential difference across the resistor (V_R) = iR and the induced emf in the inductor $(V_L) = L\,di/dt$.

$$\therefore\ \frac{q}{c} + iR = -L\frac{di}{dt}$$

$$\text{or, } L\frac{di}{dt} + iR + \frac{q}{c} = 0$$

$$\text{or, } L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0 \qquad \left[\because i = \frac{dq}{dt}\right]$$

$$\text{or, } \frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega^2 q = 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3.18)$$

where $2b = \frac{R}{L}$ and $\omega^2 = \frac{1}{LC}$.

Equation (3.18) is of the same form as that of damped mechanical oscillator equation (3.2).

Therefore, the general solution of equation (3.18) is

$$q = e^{-bt}\left(A_1 e^{\sqrt{b^2-\omega^2}\,t} + A_2 e^{-\sqrt{b^2-\omega^2}\,t}\right) \dots \quad (3.19)$$

where A_1 and A_2 are arbitrary constants.

As before the motion of the chargew depends on the relative values of b and ω, three different cases may arise.

Case – I : Heavy damping $b > \omega$:

Here $b > \omega$ ie $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$

So, $\sqrt{b^2 - \omega^2}$ is real, thus the charge on the capacitor decreases exponentially. The motion is non-oscillatory or dead-beat.

Case–II : Critical damping $b \simeq \omega$:

The charge at any instant on the capacitor is given by

$$q = e^{-bt}(C_1 + C_2 t)$$

where C_1 and C_2 are constant.

Here also, the motion is non-oscillatory and quickly.

Case–III : Weak or light damping $b < \omega$:

Here $b < \omega$ ie $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$.

So, $\sqrt{b^2 - \omega^2}$ is imaginary and as before the solution of equation (3.18) is

$$q = Ae^{-bt}\cos(\omega' t - \varphi) \dots \quad (3.20)$$

where $\omega' = \sqrt{\omega^2 - b^2}$, A and φ are constants.

Here is before, $A = \frac{q_0\omega}{\sqrt{\omega^2 - b^2}}$ where q_0 is the initial charge on the capacitor

and $\tan\varphi = \frac{b}{\sqrt{\omega^2 - b^2}}$

$$\therefore A = \frac{q_0\left(\frac{1}{\sqrt{LC}}\right)}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \text{ and } \tan\varphi = \frac{R/2L}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

$$\text{Now } \omega' = \sqrt{\omega^2 - b^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$\therefore$ The time period of oscillation T is given by

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.21)$$

and frequency of oscillation

$$\eta = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (3.22)$$

If R is very small then frequency of oscillation

$$\eta = \frac{1}{2\pi\sqrt{LC}} \simeq \text{ as in ideal case. } \simeq$$

The quality factor,

$$Q = \frac{\omega'}{2b} = \frac{\omega' L}{R} \quad \frac{\omega L}{R} \quad [\because \omega' = \sqrt{\omega^2 - b^2} \quad \omega \text{ and } b < \omega]$$

The above equation shows that for a pure inductive circuit (R = 0), the quality factor will be infinite.

Exercise–4 : In an LCR circuit L = 1 m H and c = 2 μF. If R = 10 Ω, calculate the frequency of oscillation and the quality factor when the discharge is oscillatory.

3.6 Summary

- The differential equation of a damped harmonic oscillator is $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$

where $2b = \frac{k}{m}$ and $\omega^2 = S/m$

The solution of this equation for heavy damping is

$$x = e^{-bt}(A_1 e^{b't} + A_2 e^{-b't}) \text{ where } b' = \sqrt{b^2 - \omega^2}\,.$$

For critical damping the solution is

$$x = e^{-bt}(C_1 + C_2 t)$$

And for weak damping or light damping

$$x = Ae^{-bt}\cos(\omega' t - \varphi) \text{ where } \omega' = \sqrt{\omega^2 - b^2}$$

● The average energy of a weakly damped oscillator is

$$\langle E\rangle = E_0 e^{-2bt} \text{ where } E_0 = \frac{1}{2} m\omega^2 A^2.$$

● The logarithmic decrement is

$$\lambda = \frac{bT}{2} = \ln\frac{A_1}{A_2} = \frac{1}{n-1}\ln\frac{A_1}{A_n}$$

$$\text{Time period (T)} = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{k}{2m}\right)^2}}$$

● Relaxation time (τ) = $\frac{1}{b}$.

● Quality factor (Q) = $\frac{\omega}{2b} = \frac{m\omega}{k}$

● The differential equation describing flow of charge in LCR circuit is

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dA} + \omega^2 q = 0$$

where $2b = \frac{R}{L}$ and $\omega^2 = \frac{1}{LC}$

● In weakly damped circuit the motion of charge will be oscilllatory.

The frequency of oscillation (n) = $\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$

● For low resistance value (R is very small) Q = $\frac{\omega L}{R}$

3.7. Questions and Problems

3.7.1. A heavy mass 1 kg is suspended from a spring of stiffness cvonstant 25 Nm^{-1} and the damping factor $5kgs^{-1}$. Calculate the ratio of undamped frequency to damped frequency.

3.7.2. A mass 0.2 kg is suspended from a light spring of stiffness constant 20 Nm^{-1}. The average energy of oscillation of the system is found to decay to e^{-1} of its initial value is 100 sec. Calculate the damping constant (K) and Q-value of the oscillator.

3.7.3. Show that the fractional change in the natural frequenmcy of a damped simple harmonic oscillator is $\frac{1}{8Q^2}$, where Q is the quality factor of the oscillator.

3.7.4. A capacitor of capacity 1 μF, an inductance of 0.2 henry and a resistance of 800 Ω are connected in series. Is the circuit oscillatory?

2.7.5. The equation for displacement of a point on a damped oscillator is given by x = $5\,e^{-t/4}\sin\left(\frac{\pi}{2}\right)t$ metre. Find the velocity of the oscillating point at t = T, where T is the time period of oscillations.

3.8. Solutions

Exercise–1

Here, Initial amplitude (A_0) = 16 cm.

Final amplitude (A) = 2 cm.

Time (t) = 100 sec

Number of oscillations (N) = 50.

$\therefore$ Time period (T) = $\frac{T}{N}=\frac{100}{50}$ = 2 sec, (without damping)

To find time period with damping; we know

$\omega' = \omega^2 - b^2$

or, $\omega' = \sqrt{\omega^2 - b^2}$

or, $\dfrac{2\pi}{T'} = \sqrt{\left(\dfrac{2\pi}{T}\right)^2 - b^2}$

$$\therefore \quad T' = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{T}\right)^2 - b^2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Again we know

$A = A_0 e^{-bt}$

or, $2 = 16e^{-b.100}$

or, $8 = e^{100b}$

or, $\ln 8 = 100b$

$\therefore \quad b = \dfrac{\ln 8}{100} = 0.021$

$\therefore$ From equation (1) we get

$$T' = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{2}\right)^2 - 0.021^2}} = 1.999 \text{ sec.}$$

$\therefore$ Time period with damping is 1.999 sec and without damping = 2 sec.

Exercise–2

Here initial amplitude (A_1) = 40 cm. and after 101 swings i.e. n = 101, final amplitude A_n = 4 cm.

$\therefore$ Logdecremednt (λ) = $\dfrac{1}{n-1}\ln\dfrac{A_1}{A_n}$

$= \dfrac{1}{101-1}\ln\dfrac{40}{4}$

$= 0.023$

we know, relaxation time $\tau = \frac{T}{2\lambda}$

Here time period (T) = 2.3 sec

$$\therefore \tau = \frac{2.3}{2\times 0.023} = 50 \text{ sec.}$$

Exercise–3

Here Q = 1000 and frequency (n) = 520 Hz

we know $Q = \frac{\omega}{2b} = \frac{\omega\tau}{2} \quad [\because \tau = \frac{1}{b}]$

or, $Q = \frac{2\pi n\tau}{2} = \pi n\tau \quad [\therefore \omega = 2\pi n]$

$\therefore \quad \tau = \frac{Q}{\pi n} = \frac{1000}{\pi.520} = 0.612 \text{ sec}$

Again we know

$$E = E_0 e^{-2bt}, \text{ Here } \frac{E}{E_0} = \frac{20}{100} = \frac{1}{5}$$

$$\therefore \quad \frac{1}{5} = e^{-2t/\tau}$$

or, $5 = e^{2t/\tau}$

or, $\ln 5 = 2t/\tau$

or, $t = \frac{\tau \ln 5}{2} = \frac{0.612 \times \ln 5}{2} = 0.44$ sec.

Exercise–4

Here $L = 1 \text{ m4} = 10^{-3}$H

$C = 2\mu F = 2 \times 10^{-6}$F

and $R = 10\Omega$

We know the frequency of oscillation

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{10^{-3}\times 2\times 10^{-6}} - \frac{10^2}{4(10^{-3})^2}}$$

= 3.47 KHz.

Solutions

3.7.1. Here, stiffness constant (s) = 25 Nms^{-1}, mass (m) = 1 kg.

and damping factor (k) = 5 kgs^{-1},

we know $\omega'^2 = \omega^2 - b^2$

or $\frac{\omega'^2}{\omega^2} = 1 - \frac{b^2}{\omega^2}$

$$= 1 - \left(\frac{k/2m}{\sqrt{s/m}}\right)^2 \quad \because b = \frac{k}{2m} \text{ and } \omega = \sqrt{s/m}$$

$$= 1 - \frac{k^2 m}{4m^2.s}$$

$$= 1 - \frac{5^2}{4.1.25}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

or, $\frac{\omega'}{\omega} = \frac{\sqrt{3}}{2}$

or, $\frac{2\pi n'}{2\pi n} = \frac{\sqrt{3}}{2}$

or, $\frac{n}{n'} = 2/\sqrt{3}$ [$\because \omega = 2\pi n$] n is the frequency

$\therefore$ The ratio of undapmed to damped frequency is $2/\sqrt{3}$.

$= 2 : \sqrt{3}$

3.7.2. Here, mass (m) = 0.2 kg

Stiffness constant (s) = $20 Nm^{-1}$

energy $\langle E \rangle = \frac{1}{e} E_0$.

time (t) = 100 sec.

As we know $\langle E \rangle = E_0 e^{-2bt}$ where $2b = \frac{k}{m}$

$$\text{or, } \frac{1}{e} E_0 = E_0 e^{-2bt}$$

$$\text{or, } \frac{1}{e} = e^{-2bt}$$

$$\text{or, } 2bt = 1$$

$$\text{or, } 2b = {}^{1}\!/\!_{t} = \frac{1}{100} = 0.01$$

Again, Damping const (k) = 2b.m = $0.01 \times 0.2 = 2 \times 10^{-3}$ N.S. m^{-1}

$$\text{Now, Q - value} = \frac{\omega'}{2b} = \frac{1}{2b}\sqrt{\omega^2 - b^2}$$

$$= \frac{1}{2b}\sqrt{\frac{s}{m} - b^2}$$

$$= \frac{1}{0.01}\sqrt{\frac{20}{0.2} - \left(\frac{0.01}{2}\right)^2}$$

$$= 999.999$$

$$= 1000$$

3.7.3. we know $\omega' = \sqrt{\omega^2 - b^2}$

$$= \omega\left(1 - \frac{b^2}{\omega^2}\right)^{\frac{1}{2}} \quad \left[\therefore b = \frac{k}{2m} \text{ and } \omega^2 = {}^{s}\!/\!_{m}\right]$$

$$= \omega\left(1-\frac{k^2}{4m^2\omega^2}\right)^{\frac{1}{2}}$$

$$= \omega\left(1-\frac{k^2}{4\omega^2 m^2}\right)^{\frac{1}{2}}$$

$$= \omega\left(1-\frac{1}{2}\frac{k^2}{4\omega^2 m^2}\right)$$ [$\because \frac{k^2}{4\omega^2 m^2}$ is very small for small value of k]

or, $\frac{\omega'}{\omega} = 1 - \frac{k^2}{8m^2\omega^2}$

or, $1 - \frac{\omega'}{\omega} = \frac{k^2}{8m^2\omega^2} = \frac{1}{8}\left(\frac{k}{m\omega}\right)^2$

or, $\frac{\omega - \omega'}{\omega} = \frac{1}{8Q^2}$ $\quad\because$ Quality factor (Q) = $\frac{m\omega}{K}$

3.7.4. Here, capacity of capacitor (c) = 1 μ F = 10^{-6}F

inductance (L) = 0.2 henry

and resistance (R) = 800 Ω

Now, $b = \frac{R}{2L} = \frac{800}{2\times 0.2} = 2000$

and $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2\times 10^{-6}}}$ $(4.5 \times 10^{-4})^{-1}$

$\therefore \omega = 2222.22$

$\because b < \omega$, therefore the circuit is oscillatory.

3.7.5. Here $x = 5\,e^{-t/4} \sin\left(\frac{\pi}{2}\right)t$ .... (1)

This equation is similar to the equation

$x = ae^{-bt} \sin \omega t$

Comparing these two equations we get

$$\omega = \pi/2$$

$$\therefore \text{ Time period (T)} = \frac{2\pi}{\omega} = \frac{2\pi.2}{\pi} = 4 \text{ sec.}$$

Now differentiating equation (1) with respect to t,

$$\text{Velocity } (\nu) = \frac{dx}{dt} = 5\left(-\frac{1}{4}\right)e^{-t/4}\sin\left(\frac{\pi}{2}\right)t + 5\,e^{-t/4}.\frac{\pi}{2}.\cos\left(\frac{\pi}{2}\right)t$$

When t = T = 4 sec.

$$\nu = \frac{dx}{dt} = -\frac{5}{4}e^{-4/4}\sin\left(\frac{\pi}{2}\right).4 + 5e^{-4/4}.\frac{\pi}{2}.\cos\left(\frac{\pi}{2}\right).4$$

$$= -\frac{5}{4}e^{-4/4}\sin 2\pi + \frac{5\pi}{2}e^{-4/4}\cos 2\pi$$

$$= 0 + \frac{5\pi}{2}e^{-1} \quad [\because \sin 2\pi = 0, \cos 2\pi = 1]$$

$$= 2.89 \text{ m/s.}$$

Unit-4 ❐ Forced Vibrations and Resonance

Structure

4.0 Objectives

After studing this unit you will be able to–

- establish the differential equation of a system driven by periodic force and solve it.
- compute the energy of forced vibration, energy resonance and amplitude resonance.
- calculate the Q factor and sharpness of resonance.
- draw analogy between electrical and mechanical oscillation.

4.1 Introduction

In the previous unit we studied free oscillations of undamped system. In that case once the oscillations started it continue for ever with a constant amplitude and frequency. But in real physical system there are not constant due to damping force and the amplitude gradually decreses with time and the oscillator comes to rest. To maintain oscillations we have to apply energy to the system from an external source called a driver. Usually the frequencies of driver and the driven system may not same. At the initial stage of vibrations, if the frequency of the applied periodic force be not the same as the natural frequency of the vibrating body, the state is called unsteady state. But, when the body oscillates with the frequency of the applied force, the state is called steady state. However, if the frequency of the driving force is exactly equal to the natural frequency of the vibrating system, then the amplitude of the forced oscillations becomes very large and ther we say the resonance occurs. This type of osciliations are called forced oscillations.

In this unit, we shall discuss the motion of the system under the application of external periodic force.

4.2. Differential equation for weakly damped forced vibration.

Consider the motion of a mechanical oscillator of mass m under the action of a restoring force $-sx$, a damping force $-k\frac{dx}{dt}$ and an external periodic applied force $F = F_0 \cos\omega t$.

The equation of motion of such an oscillator is

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt} + F_0 \cos\omega t$$

where x is the displacement from the equilibrium position at any instant of time t, s is the stiffness constant. k is the damping constant, F_o is the amptitude and ω is the angular frequency of the applied force.

The above equation can be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f\cos\omega t \quad ...(4.1)$$

Here $2b = \frac{k}{m}$, $\omega_0 = \sqrt{\frac{s}{m}}$ and $f = \frac{F_0}{m}$.

Equation (4.1) is the differential equation of forced vibration.

4.3. Solution of the differential equation of forced vibration.

In the previous unit we see that in obsence of applied force, a weakly damped system ($b < \omega_0$) oscillates harmonically with an angular frequency $\omega_1 = \sqrt{\omega_0^2 - b^2}$. But when a driving force ($F_0 \cos \omega t$) of angular frequency ω is applied, it imposes its own frequency on the oscillator. Thus, we expect that the actual motion will be a result of super position of two oscillation. One of frequency ω_0 (of damped oscillator or driven system) and the other of frequency ω (of the driving force). Hence, for $\omega_0 \neq \omega$ the general solution of equation (4.1) can be written as–

$$x(t) = x_1(t) + x_2(t) \quad \ldots \quad (4.2)$$

where x_1, is a solution of complementary function of equation (4.1) i.e. when RHS is zero.

$$\therefore \quad \frac{d^2x_1}{dt^2} + 2b\frac{dx_1}{dt} + \omega_0^2 x_2 = f\cos\omega t \quad \ldots \quad (4.4)$$

you know the solution of equation (4.3) for weakly damped condition ($b < \omega_0$) is

$$x_1(t) = ae^{-bt}\cos(\omega_1 t + \varphi)$$

Here due to the presence of the factor e^{-bt} the complementary function decays exponentially and after some time it will disappear. It is known as transient solution. In this case, the system will oscillate with a frequency other than ω_0 and ω.

Hence, after some time the system will oscillate with the frequency of the driving force, then the solution is called steady solution.

Now, consider the solution of equation (4.4) is

$x_2(t) = A\cos(\omega t - \alpha)$ where amplitude (A) and phase angle (α) are unknown constants.

Now, $\frac{dx_2}{dt} = -A\omega\sin(\omega t - \alpha)$

and $\frac{d^2x_2}{dt^2} = -A\omega^2\cos(\omega t - \alpha)$

putting these in equation (4.4) we get

$$-A\omega^2\cos(\omega t - \alpha) - 2bA\omega\sin(\omega t - \alpha) + \omega_0^2 A\cos(\omega t - \alpha)$$
$$= f\cos\omega t$$
$$= f\cos\{(\omega t - \alpha) + \alpha\}$$

or, $(\omega_0^2 - \omega^2)A\cos(\omega t - \alpha) - 2bA\omega\sin(\omega t - \alpha)$
$$= f\cos(\omega t - \alpha)\cos\alpha - f\sin(\omega t - \alpha)\sin\alpha$$

Equating the co-efficients of $\cos(\omega t - \alpha)$ and $\sin(\omega t - \alpha)$ from both sides, we get

$$A(\omega_0^2 - \omega^2) = f\cos\alpha$$

and $2bA\omega = f\sin\alpha$

From above two equation we get

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

and $\tan\alpha = \dfrac{2b\omega}{\omega_0^2 - \omega^2}$ ... ... (4.6)

$\therefore$ The general solution of forced vibration is
$x = x_1 + x_2 = ae^{-bt} \cos(\omega_1 t + Q) +$

$$\frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} \cos\left(\omega t - \tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2}\right) \quad \ldots \quad \ldots \; \ldots\ldots. \quad (4.7)$$

Thus the first part of the solution (x_1) represents the natural damped motion, which will persists for longer time if the damping is small and after lapse of some time it will vanish as shown in figure 4.1a.

At the beginning, if ω_1 is close to w, during transient state of superposition, beats will be formed known as transient beats, as shown in figure 4.1b. But after some time the first term (x_1) will vanish and we can write $x = A \cos(\omega t - \alpha)$, which represents the sustained force vibration as in figure 4.1c.

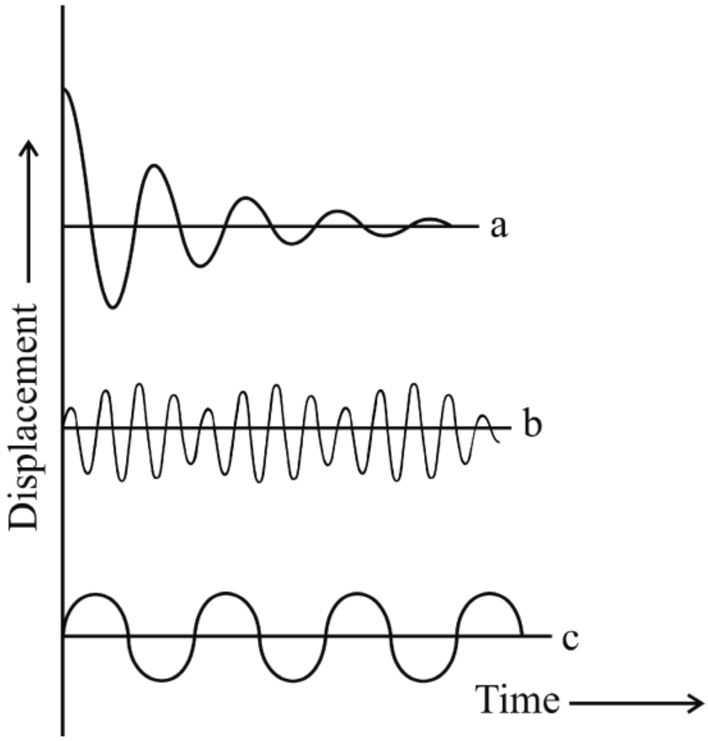


Figure 4.1

4.4. Energy of forced vibration and velocity or energy resonance.

The motion of a particle in the steady state under the action of a periodic force is given by

$x = A\cos(\omega t - \alpha)$

$\therefore$ velocity (20) = $-A\omega\sin(pt - \alpha)$

Hence the kinetic energy of the forced vibration at any instant is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \alpha)$$

Since the motion is a steady simple harmonic motion, so the maximum kinetic energy is the total energy of the system.

$$\therefore \text{ Total energy (E)} = \frac{1}{2}m\omega^2A^2$$

$$= \frac{1}{2}m\omega^2\frac{f^2}{(\omega_0^2 - \omega)^2 + 4b^2\omega^2}$$

$$= \frac{\frac{1}{2}mf^2}{\omega_0^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + 4b^2}$$

$$\therefore\ E = \frac{\frac{1}{2}mf^2}{\omega_0^2\Delta^2 + 4b^2} \quad \ldots\ldots\ldots \qquad (4.8)$$

Writing $\Delta = \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}$

Now, if $\omega_0 = \omega$, $\Delta = 0$, the energy of the system is maximum for any value of b.

Thus, when the frequency of the forcing system (driver) coincider with that of the natural frequency of the forced system (driven system), the energy of the forced system is maximum. Thus phonomenon is known as energy resonance or velocity resonance or simply resonance.

$\therefore$ The energy at resonance is $E_{max} = \dfrac{\frac{1}{2}mf^2}{4b^2}$ ... ...(4.9)

Figure 4.2 gives the variation of energy (E) with Δ for different values of b, the curves are symmetrical about Δ = 0 with its maximum value at Δ = 0 i.e. at $\omega = \omega_0$. It is also clean from the figure that smaller the value of b, greater is the value of energy for a particular value of Δ.

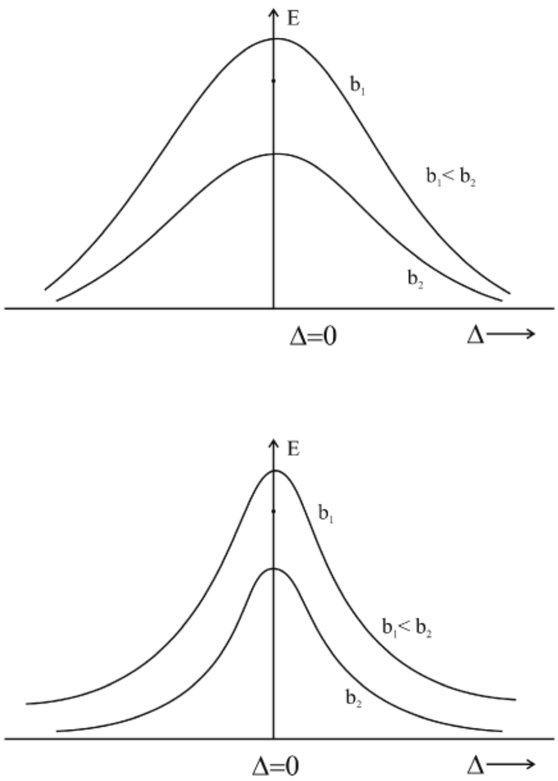


Figure 4.2

4.5. Amplitude resonance

From equation (4.5) we have the amplitude

$A = \dfrac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}}$ is maximum for a particular value of ω and we say there is amplitude resonance between the driver and driver system.

The amplitude A to be maximum when

$$\frac{dA}{d\omega}=0$$

$$\therefore \quad \frac{dA}{d\omega}=f\frac{-\frac{1}{2}\{2(\omega_0^2-\omega^2)\cdot 2\omega+4b^2\cdot 2\omega\}}{\{(\omega_0^2-\omega^2)^2+4b^2\omega^2\}^{\frac{3}{2}}}=0$$

or, $-4(\omega_0^2-\omega^2)\omega+8b^2\omega=0$

or, $(\omega_0^2-\omega^2)-2b^2=0$

or, $\omega^2=\omega_0^2-2b^2$ ... ... (4.10a)

or, $\omega=\{\omega_0^2(1-\frac{2b^2}{\omega_0^2})\}^{\frac{1}{2}}=\omega_0(1-\frac{1}{2}\cdot\frac{2b^2}{\omega_0^2})$ for small value of b.

$\therefore \quad \omega=\omega_0-\frac{b^2}{\omega_0}$ ... ...(4.10b)

Thus for amplitude resonance the angular frequency (ω) is slightly smaller than that at energy resonance.

The variation of the amplitude of oscillation of the driven system with frequency of the driver (ω) at different damping constants (b) is shown is figure 4.3. Here at ω = 0 for all values of b the amplitude is $\frac{f}{\omega_0}$ and as damping increases the maximum value of amplitude is shifted towards the lower value of ω. Again we see that as b decreases the amplitude increases for any value of ω and becomes infinite for b = 0 at $\omega = \omega_0$.

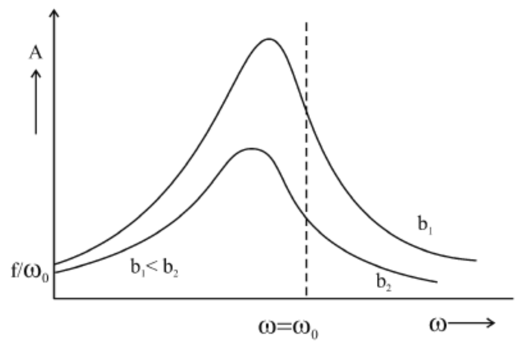


Figure 4.3

4.6. Phase of the forced vibrations

We have from equation (4.6) the phase angle of the steady motion is

$$\alpha=\tan^{-1}\frac{2b\omega}{\omega_0^2-\omega^2} \text{ and } \sin\alpha=\frac{2bA\omega}{f}$$

from this value of sin α, we see that α. is always positive, that means α lies between 0 and π.

Suppose the angular frequency (ω) of the applied force is increased gradually from 0 to ∞, then

(i) When $\omega = 0$, tan $\alpha = 0$. Hence $\alpha = 0$.

Thus there is no phase difference between driver and driven systems.

(ii) When $\omega < \omega_0$, tan α is positive, that means the phase difference lies between 0 and $\frac{\pi}{2}$.

(iii) When $\omega = \omega_0$, tan $\alpha = \infty$, hence $\alpha = \frac{\pi}{2}$, Thus at resonance the driven system lags behind the driver by an angle $\frac{\pi}{2}$.

(iv) When $\omega > \omega_0$ in this case tan α is negative, i.e. the angle α is in second quadrant or $\frac{\pi}{2} < \alpha < \pi$.

Again as $\omega \to \infty$, tan $\alpha \to 0$ in this case $\alpha \to \pi$.

Hence for all values of ω, α lies between 0 and π and at resonance it is $\frac{\pi}{2}$

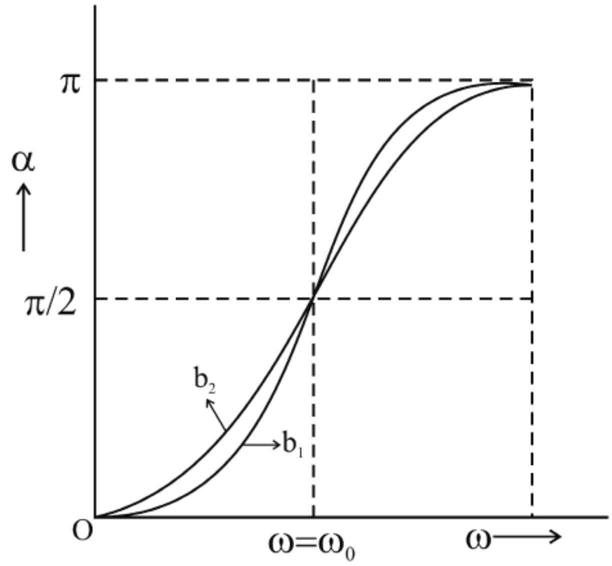


Figure : 4.4

Again $\alpha = \tan^{-1} \frac{2b\omega}{\omega_0^2 - \omega^2}$

or, $$\frac{d\alpha}{d\omega} = \frac{1}{1 + \frac{4b^2\omega^2}{(\omega_0^2 - \omega^2)^2}} \cdot \frac{2b(\omega_0^2 - \omega^2) + 2b\omega \cdot 2\omega}{(\omega_0^2 - \omega^2)^2}$$

$$= \frac{2b(\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}$$

At resonance $(\omega = \omega_0)$ (we get $\left(\frac{d\alpha}{d\omega}\right)_{\omega=\omega_0}$

$$= \frac{2b(\omega_0^2 + \omega_0^2)}{4b^2\omega_0^2} = \frac{1}{b}$$

Thus, smaller the values of b, the greater is the rate of change of phase angle (α) near resonance frequency. The vairation of α and ω is shown in figure 4.4.

4.7. Power absorbed in forced vibration

(a) Power supplied by the driving force:

When a damped simple hermonic oscileator is acted upon by an external periodic force $F_0 \cos \omega t$, the velocity of the forced system is given by

$$v = \frac{dx}{dt} = \frac{d}{dt}\{A\cos(\omega t - \alpha)\} \quad [\because x = A \cos (\omega t - \alpha)]$$

$\therefore \quad v = -A\omega\sin(\omega t - \alpha)$ ... ... (4.11)

$\therefore$ The instantaneous power supplied to the system is

P = Force $\times$ Velocity

$= F_0 \cos \omega t \{-A\omega \sin (\omega t - \alpha)\}$

$\therefore$ Average power is given by

$$P_{av} = -\frac{1}{T}\int_0^T F_0 A\omega \cos\omega t \sin(\omega t - \alpha)dt$$

$$= -\frac{F_0 A\omega}{T}\int_0^T (\cos\omega t \sin\omega t \cos\alpha - \cos^2\omega t \sin\alpha)dt$$

$$= -\frac{F_0 A\omega}{T}\left(-\frac{T}{2}\sin\alpha\right)$$

$$[\because \int_0^T \cos\omega t \sin\omega t dt = 0 \text{ and } \int_0^T \cos\omega t dt = \frac{T}{2}$$

or, $P_{av} = \frac{F_0 A\omega}{2} \sin\alpha$

$$= \frac{F_0 A\omega}{2} \cdot \frac{2bA\omega}{f} \quad [\text{From equation (4.6) } \sin\alpha = \frac{2bAw}{f}$$

$$\text{and } 2b = \frac{k}{m}]$$

$$= \frac{F_0 A\omega}{2} \cdot \frac{KA\omega}{mf} \quad [\because F_0 = mf]$$

$\therefore$ Average power supplied $(P_{av}) = \frac{1}{2} kA^2\omega^2$ ... ... (4.12)

(b) Power dissipated by the damped oscillator:

The power supplied by the driving force is wtilised in doing work against the damping force $k\frac{dx}{dt}$

$\therefore$ The rate of work done by the damping force

$$= k\frac{dx}{dt} \cdot \frac{dx}{dt} = k\left(\frac{dx}{dt}\right)^2$$

$$= kA^2\omega^2 \sin^2(\omega t - \alpha) \quad [\text{by equation 4.11}]$$

$\therefore$ Average power loss

= Average power loss

$$= \frac{1}{T}\int_0^T kA^2\omega^2 \sin^2(\omega t - \alpha)dt = \frac{1}{T}kA^2\omega^2 \frac{T}{2} = \frac{1}{2}kA^2\omega^2$$

... ... (4.13)

Thus from equations (4.12) and (4.13) we see that in the steady state of forced vibration, the average power supplied by the driving force is equal to that being dissipated by the damping force.

Exercise-1

A particle of mass 0.02 kg is subjected to a restoring force 1.0 N/m and an external simusoidal force of constant amplitude. Find the frequency of the driving force for which there will be energy resonance.

Exercise-2

Find the difference of the frequency from resonance for which amplitude of velocity of oscillation have half the value at resonance.

Exercise-3

Calculate the ratio of average kinetic energy to average potential energy of a vibrating system, when a periodic force $F = F_0 \cos\omega t$ is applied on the system.

4.8. Quality factor and sharpness of resonance

Here we study the response of the oscillator to the driving force, when the angular frequency of the driving force is varied slowly.

Now from equation (4.12) we have

Average power suppied $Pav = \frac{1}{2}k\omega^2A^2$

Putting the value of k and A we get

$$Pav = \frac{1}{2}\frac{2bm\omega^2f^2}{(\omega_0^2-\omega^2)+4b^2\omega^2}$$

$$= \frac{mb\omega^2f^2}{\omega^2\omega_0^2(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0})^2+4b^2\omega^2}$$

or, $P_{av} = \frac{mbf^2}{\omega_0^2\Delta^2+4b^2}$ $\quad\therefore\ \Delta = \frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}$

Now the maximum value of P_{av} will accur for the frequency $\omega = \omega_0$. i.e. power will be maximum at resonance.

$$\therefore \quad P_{max} = \frac{mbf^2}{4b^2}$$

$$\therefore \quad \frac{P_{av}}{P_{max}} = \frac{4b^2}{\omega_0^2\Delta^2 + 4b^2} \quad \dots \dots \dots \dots \dots \dots \dots \quad \dots(4.14)$$

At resonance ($\omega = \omega_0$), so, $\Delta = 0$ and $\frac{P_{av}}{P_{max}} = 1$. But below and above the resonance $\Delta \neq 0$ and $\frac{P_{av}}{P_{max}} < 1$. Quantitatively sharpness of resonance (S) is defined as the reciprocal of $|\Delta|$ at which $P_{av} = \frac{1}{2}P_{max}$. From figure 4.5 we see that there are two values of ω (ω_1 and ω_2 say) for which P_{av} is half of its resonance value. The frequencies w_1 and w_2 are called half power frequencies.

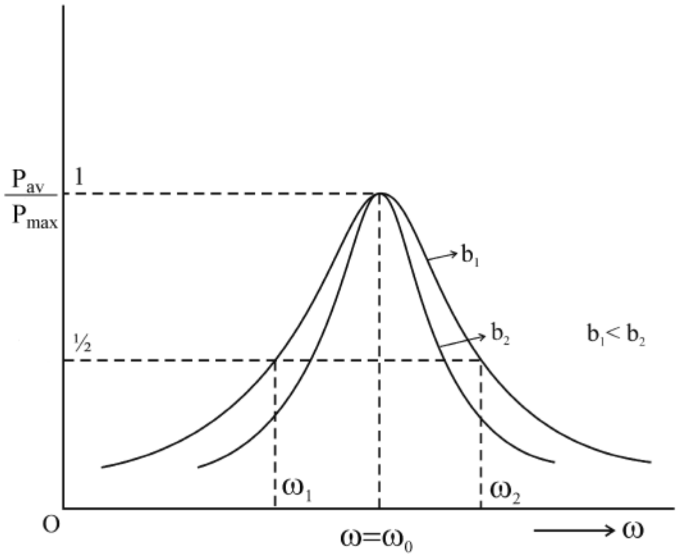


Figure 4.5

From equation (4.14) we have $\frac{4b^2}{\omega_0^2\Delta^2 + 4b^2} = \frac{1}{2}$

or, $8b^2 = \omega_0^2\Delta^2 + 4b^2$

or, $\omega_0^2\Delta^2 = 4b^2$

or, $\omega_0\Delta = \pm 2b$

$$\therefore \quad \left|\frac{1}{\Delta}\right| = \frac{\omega_0}{2b} \quad ...(4.15)$$

$$\text{or,} \quad \omega_0\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) = \pm 2b$$

$$\text{or,} \quad \frac{\omega_0^2}{\omega} - \omega = \pm 2b$$

$$\text{or,} \quad \omega_0^2 - \omega^2 = \pm 2b\omega$$

$$\therefore \quad \omega^2 \pm 2b\omega - \omega_0^2 = 0 \quad \ldots \ldots \quad (4.16)$$

Thus equation (4.16) can be written as

$$\omega^2 + 2b\omega - \omega_0^2 = 0$$

$$\text{and,} \quad \omega^2 - 2b\omega - \omega_0^2 = 0$$

Each of these equations have one positive root and one negative root. Since w is positive hence taking positive roots only, we have

$$\omega_1 = \sqrt{\omega_0^2 + b^2} - b$$

$$\text{and,} \quad \omega_2 = \sqrt{\omega_0^2 + b^2} + b$$

$$\therefore \quad \omega_2 - \omega_1 = 2b$$

The frequency interval $(\omega_2 - \omega_1)$ is called the half band width or simply baudwidth.

Now, sharpness of resonance (s) = $\frac{1}{\Delta} = \frac{\omega_0}{2b}$ [From equation (4.15)]

$$\text{or,} \quad S = \frac{\omega_0}{2b} = \frac{\omega_0 m}{k} = \frac{\omega_0}{\omega_2 - \omega_1} \quad \left[\because 2b = \frac{k}{m}\right]$$

$$\therefore \quad S = \frac{\omega_0}{2b} = \frac{\omega_0 m}{k} = \frac{\omega_0}{\omega_2 - \omega_1} = Q \quad \ldots \quad (4.17)$$

The quantity $\frac{\omega_0}{\omega_2 - \omega_1}$ is called the quality factor or Q-factor.

Thus Q is a measure of the sharpness or resonance. Smaller the damping smaller is the band width and higher the quality factors (It is also shown is figure 4.5).

4.9. An LCR circuit

Let a pure ohmic resistance R, a pure inductor L and a pure capacitor c connected in series with a sinusoidal source of e.m.f E = E_0 coswt as shown in figure 4.6.

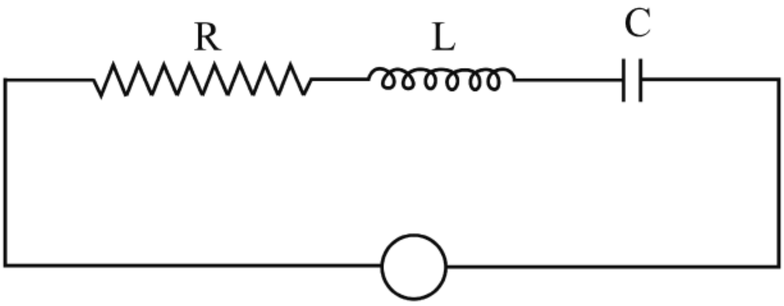


Figure 4.6

Let i be the current in the circuit at any instant of time and q be the charge on the capacitor plate at that instant.

The potential difference across the capacitor is $V_c = \frac{q}{C}$, the potential difference across the resistance, $V_R = iR$ and the potential difference across the indictor is $V_L = L\frac{di}{dt}$.

∵ The applied e.m.f is E = $E_0\cos\omega t$.

∴ We can write

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \cos\omega t$$

or, $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \cos\omega t \qquad \left[\because i = \frac{dq}{dt}\right]$$

or, $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E_0}{L}\cos\omega t$$

or, $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2 q = e_0 \cos\omega t \quad \ldots \quad (4.18)$$

Where $2b = \frac{R}{L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$ and $e_0 = \frac{E_0}{L}$

The equation (4.18) is exactly same as the mechanical equation of driven harmonic oscillator equation (4.1). Hence the steady state solution of this equation (4.18) can be written as

$$q = q_0 \cos(\omega t - \alpha)$$

Where $$q_0 = \frac{e_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}}$$

$$= \frac{e_0}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2}{L^2}\omega^2}}$$

$$= \frac{E_0}{L \cdot \frac{1}{L} \cdot \omega\sqrt{(\frac{1}{\omega c} - L\omega)^2 + R^2}}$$

$$\therefore \quad q_0 = \frac{E_0}{\omega\sqrt{(\omega L - \frac{1}{\omega c})^2 + R^2}} \quad \ldots \quad (4.19)$$

and $$\tan\alpha = \frac{2b\omega}{\omega_0^2 - \omega^2} = \frac{(R/L)\omega}{\frac{1}{Lc} - \omega^2} = \frac{R}{\frac{1}{\omega c} - \omega L}$$

$$\therefore \quad q = \frac{E_0}{\omega\sqrt{R^2 + (\omega L - \frac{1}{\omega c})^2}}\cos(\omega t - \alpha) \quad \ldots \quad (4.21)$$

And current is

$$i = \frac{dq}{dt} = \frac{-E_0\omega}{\omega\sqrt{R^2 + (\omega L - \frac{1}{\omega c})^2}}\sin(\omega t - \alpha)$$

$$\therefore \quad i = \frac{E_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \cos(\omega t - \varphi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots(4.22)$$

where $\varphi = \alpha - \frac{\pi}{2}$

$$\tan\varphi = \tan\left(\alpha - \frac{\pi}{2}\right) = -\cot\alpha = \frac{\omega L - \frac{1}{\omega C}}{R} \quad \text{[using equation 4.20]}$$

φ is the phase difference between applied emf ($E = E_0 \cos \omega t$) and current in the circuit.

Again we can write equation (4.22) as

$$i = i_0 \cos(\omega t - \varphi) \text{ where } i_0 = \frac{E_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.23)$$

i_0 is the amplitude of the current.

From equation (4.23) we see that the amptitude or peak value of current is a function of frequency (ω).

Now i_0 will be maximum when

$$\omega L - \frac{1}{\omega C} = 0 \qquad \text{or, } \omega L = \frac{1}{\omega C}$$

$$\text{or, } \omega^2 = \frac{1}{LC} \qquad \text{or, } \omega = \frac{1}{\sqrt{LC}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.24)$$

$\therefore$ The value of i_0 will be maximum when

$$\omega = \frac{1}{\sqrt{LC}} \qquad \text{or, } f = \frac{1}{2\pi\sqrt{LC}}$$

where f is the frequency of ascillation.

Thus the maximum value of i_0 is

$$i_{0\max} = \frac{E_0}{R}$$

When $\omega L = \frac{1}{\omega c}$ the phase difference between current and emf is $\varphi = 0$. ie is same phase.

This condition is called resonance condition.

For different values of R, the frequency variation of peak current is shown in figure 4.7. You see that lower the value of resistance, higher the peak value of current and sharpen is the resonance.

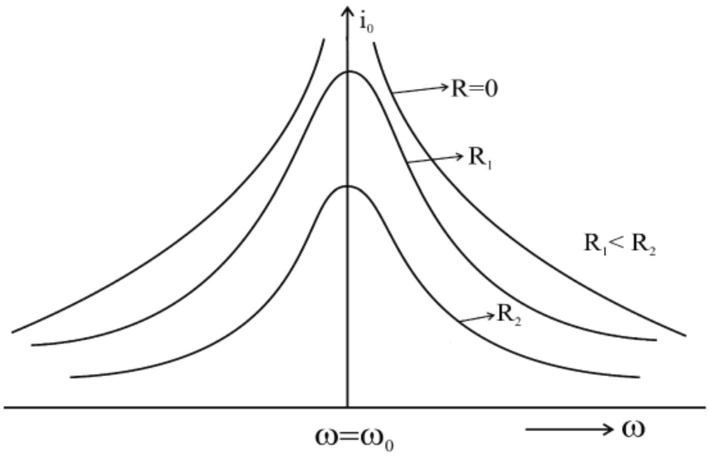


Figure 4.7

Now the power is an electric circuit is defined as the product of current and emf. For LCR cuircuit we can write

Power (P) $= Ei = E_0 i_0 \cos(\omega t - \varphi)\cos\omega t$

$= E_0 i_0 [\cos^2\omega t \cos\varphi + \sin\omega t \cos\omega t \sin\varphi]$

$\therefore$ Average power <P> $= \frac{i}{T}\int_0^T E_0 i_0$

$(\cos^2\omega t \cos\varphi + \sin\omega t \text{ coswt } \sin\varphi)dt$

$= \frac{E_0 i_0}{2}\cos\varphi$

$= E_{rms} i_{rms} \cos\varphi$ ... ... (4.25)

Where $i_{rms} = \frac{i_0}{\sqrt{2}}$ and $E_{rms} = \frac{E_0}{\sqrt{2}}$ are the root mean square values of current and emf respectively.

The term $\cos\varphi$ is called the power factor of the circuit.

Again we know the quantity factor (Q) is [from equation 4.17]

$$Q = \frac{\omega_0}{2b} = \frac{\omega_0}{R/L} = \frac{\omega_0 L}{R} \quad \ldots \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.26)$$

Where $\omega_0 = \frac{1}{\sqrt{Lc}}$.

$$\therefore \; Q = \frac{L}{R}\frac{1}{\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.27)$$

$\therefore$ The Q factor of the circuit depends on the values of L, C and R. The Q of a circuit determines the ability to select a narrow band of frequencies from wide range of frequencies. This property is used in radio and TV sets. This circuit is also called tunning circuit.

4.10. Summary

- The differential equation of forced vibration under the application of external periodic froce $F = F_0$ coswt to the oscillator is

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f\cos\omega t$$

where $2b = \frac{k}{m}$, $\omega_0 = \sqrt{\frac{s}{m}}$ and $f = \frac{F_0}{m}$

- The general solution of the differential equation is

$$x(t) = ae^{-bt}\cos(w_1 t + \varphi) + A\cos(wt - \alpha)$$

Where $A = \frac{f}{\sqrt{(\omega_0 - \omega)^2 + 4b^2\omega^2}}$ and $\alpha = \tan^{-1}\frac{2b\omega}{\omega_0^2 - \omega^2}$

are the steady state amplitude and phase angle respectively.

- Energy of the oscillator under forced vibration is

$$E = \frac{\frac{1}{2}mf^2}{\omega_0^2\Delta^2 + 4b^2} \text{ where } \Delta = \frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}$$

Energy is maximum when $\omega = \omega_0$, called energy resonance or simply resonance.

- Amplitude resonance ocurrs at $\omega = \sqrt{\omega_0^2 - 2b^2}$.
- The average power absorbed by a forced oscillator is

$$< P_{av} >= \frac{1}{2} kA^2\omega^2 .$$

- The quality factor of a forced oscillator is

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0 m}{k} = \frac{\omega_0}{2b} = S$$

where S is the sharpness of resonance.

$\omega_2 - \omega_1$ is called band width.

- The differential equation of LCR circuit is

$$\frac{d^2q}{dA^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{Lc}q = \frac{F_0}{L}\cos\omega t$$

In steady state the solution is

$$q = \frac{E_0}{L\sqrt{\left(\frac{1}{Lc} - \omega^2\right)^2 + \left(\frac{R\omega}{L}\right)^2}}\cos(\omega t - \alpha)$$

where $\alpha = \tan^{-1}\dfrac{R}{\frac{1}{\omega c} - \omega L}$

4.11. Questions and Problems

4.11.1. A body of mass 100 gm is suspended from a spring of spring constant(s) 100 Nm^{-1} and a resisting force (k) 5 NSm^{-1}. Set up the differential equation of motion and find the angular frequency for damped oscillations. Now a periodic force F = 5 cos 10 tN is applied. Calculate the amplitude of forced oscillations in the steady-state.

4.11.2. An oscillator of mass 1 gm is acted upon by a restoring force of 10^4N/m of displacement, a retarding force of 4NS/m and a driving force coswt N. Find the value of maximum possible amplitude.

4.11.3. A series LCR circuit with L = 0.05 H, C = 50 μF and R = 10π is corrected to an alternating supply at 200 volts and 50 Hz. Find the peak value of current and the power factor of the circuit.

4.11.4. A capacitor of capacitance 100 μF, a coil of resistance 50Ω and inductance 0.5H are connected is series with a 220v, 50Hz, AC source. Calculate the Q-factor of the circuit.

4.12. Solutions

Exercise—1

Here, mass (m) = 0.02 kg, Restoring force (5) = 1 N/m. We know at resonance frequency at during force is equal to the natural frequency of the vibrator ie.

$$\omega = \omega_0 = \sqrt{s/m} = \sqrt{1/.02} = 7.07 \text{rads}^{-1}$$

Exercise—2

Here displacement $x = A\cos(\omega t - \alpha)$

$\therefore$ Velocity (v) $= -A\omega \sin(\omega t - \alpha)$

$= v_0 \sin(\omega t - \alpha)$

$\therefore$ The amplitude of velocity

$$v_0 = A\omega = \frac{f.\omega}{\sqrt{(\omega^2 - \omega^2)^2 + 4b^2\omega^2}} \text{ for resonance}$$

$$\omega = \omega_0, \text{ or } v_{0\,max} = \frac{f\omega}{2b\omega} = \frac{f}{2b}$$

Now, for, $v_0 = \frac{v_{0max}}{2}$

$$\frac{f\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} = \frac{1}{2}\cdot\frac{f}{2b}$$

$$\text{or, } \left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2 = 16b^2\omega^2$$

$$\text{or, } \{(\omega_0 - \omega)(\omega_0 + \omega)\}^2 = 12b^2\omega^2$$

$$\text{or, } \{(\omega_0 - \omega)2\omega\}^2 = 12b^2\omega^2 \text{ for small damping } \omega \simeq \omega_0$$

or, $(\omega_0 - \omega)^2 = 3b^2$

$\therefore\ \omega_0 - \omega = \sqrt{3}b$

Exercise—3

We know $x = A\cos(\omega t - \alpha)$

$\therefore$ Kinetic energy (E) $= \frac{1}{2}mv^2 = \frac{1}{2}m\{-A\omega\sin(\omega t - \alpha)\}^2$

$$= \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \alpha)$$

$\therefore$ Average K. E. <E> $= \frac{1}{4}mA^2\omega^2$

And average poltential energy <P.E> $= \frac{1}{T}\int_0^T \frac{1}{2}sx^2 dt$

$$= \frac{s}{2T}\int_0^T A^2\cos^2(\omega t - \alpha)dt$$

$$= \frac{1}{2}\frac{sA^2}{T}\cdot\frac{T}{2}\left[\because \int_0^T \cos^2(\omega t - \alpha)dt = \frac{T}{2}\right]$$

$$= \frac{1}{4}sA^2$$

$$= \frac{1}{4}mA^2\omega_0^2\left[\because \omega_0^2 = \frac{s}{m}\right]$$

$$\therefore\ \frac{\text{Average kinetic energy}}{\text{Average potential energy}} = \frac{\frac{1}{4}mA^2\omega^2}{\frac{1}{4}mA^2\omega_0} = \frac{\omega^2}{\omega_0^2}$$

4.11.1. Here m = 100 gm = 0.1 kg, spring constant (s) = 100 Nm^{-1}, resisting force (k) = 5 NSm^{-1}, F_0 = 5N, ω = 10 rad/sec.

$\therefore$ Damping constant (2b) $= \frac{k}{m} = \frac{5}{0.1} = 50$

$\therefore$ b = 25.

$$\omega_0^2 = s/m = \frac{100}{0.1} = 1000$$

$\therefore$ The frequency for damped oscillations

$$(\omega_1) = \sqrt{\omega_0^2 - b^2} = \sqrt{1000^2 - 25^2} = 19.36 \text{ rad/s}$$

and the differential equation is $\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 1000x = 0$

We know the amplitude

$$A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} \quad \text{here } f = \frac{F_0}{m} = \frac{5}{0.1} = 50 \text{ m/s}^2$$

$$= \frac{50}{\sqrt{(1000 - 10^2)^2 + (50 \times 10)^2}}$$

$$= 0.05\text{m}$$

4.11.2. Here, m = 1 gm = 10^{-3} kg

$s = 10^4 \text{ N/m}$, $k = 4\text{NS/m}$ and $f = \frac{F_0}{m} = \frac{1}{10.3} \text{m/s}^2$

$$\therefore \omega_0^2 = \frac{s}{m} = \frac{10^4}{10^{-3}} = 10^7, \quad b = \frac{k}{2m} = \frac{4}{2 \times 10^{-3}} = 2 \times 10^3$$

we know the maximum amplitude

$$A_{max} = \frac{f}{2b\sqrt{\omega_0^2 - b^2}}$$

$\left[\because A = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}}\right]$ Now at amplitude resonance

(equation 4.10a) $\omega^2 = \omega_0^2 - 2b^2$,

$$\therefore A_{max} = \frac{f}{\sqrt{(\omega_0^2 - \omega_0^2 + 2b^2)^2 + 4b^2(\omega_0^2 - 2b^2)}}$$

$$= \frac{f}{\sqrt{4b^4 + 4b^2\omega_0^2 - 8b^4}}$$

$$= \frac{f}{\sqrt{4b^2 w_0^2 - 4b^4}}$$

$$= \frac{f}{2b\sqrt{w_0^2 - b^2}}\ ]$$

$$\therefore \ A_{max} = \frac{10^3}{2.2\times10^3\sqrt{10^7 - 4\times10^6}}$$

$$= 1.02\times10^{-4}\,m$$

4.11.3. Here L = 0.05H, C = 50μF = 50 × 10^{-6}F

R = 10π, w = 2πf = 25.50 = 314 rad/s

and E_0 = 200 vots

we know the peak value of current

$$i_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{200}{\sqrt{10^2 + \left(314\times0.05 - \frac{1}{314\times50\times10^{-6}}\right)^2}}$$

$$= \frac{200}{49.02} = 4.08 \text{ amp}$$

$\therefore$ $\mathbf{i_0}$ = 4.08 amp.

Again we know the power factor of the circuit

$$\cos\varphi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{10}{49.02} = 0.204$$

4.11.4. Here, $C = 100\mu F$, $R = 50\Omega$ and $L = 0.5H = 100 \times 10^{-6}F$

we know the Q factor (Q) $= \frac{1}{R}\sqrt{\frac{L}{C}}$ [using Equation 4.27]

$$= \frac{1}{50}\sqrt{\frac{0.5}{100 \times 10^{-6}}} = 1.414$$

$\therefore$ Q of the circuit = 1.414.

Unit : 5 □ Fourier's Theorem

Structure

5.1 Objectives

After studing this unit you will be able to know that any periodic functions can be expressed as a Fourier series.

Derive Fourier Series of a given periodic function by evaluating Fourier co-efficients.

Write a given function in terms of sine and cosine terms in Fourier Series.

5.1 Introduction

Any function f(t) is periodic function if it can be expressed as f(t) = f(t + T) when the time interval T is the period of the function. The functions sin ωt and cos ωt are periodic functions of period $T = \frac{2\pi}{\omega}$. The square wave, saw-toothwave etc are not sinusoidal but they are also periodic functions.

When several simple harmonic vibrations of frequencies integral multiple of its fundamental frequency combine, then they may produce some complex vibrations which are not simple harmonic Fourier's theorem is of great importance in the synthesis as well as analysis of these different types of periodic vibrations. The analysis of this complex note helps us to determine the quality of sound. Fourier's theorem has extensive applications in different branches of physics.

In this unit are will discuss the proces of solving some periodic vibrations with the help of Fourier's theorem.

5.2 Statement of Fourier's theorem

Fourier's theorem states that Any finite periodic motion can be expressed as the sum of a series of simple harmonic motions of commensurate periods.

Mathematically it can be written as

$f(t) = a_0 + A_1 \cos(\omega t + \phi_1) + A_2 \cos(2\omega t + \phi_2) + A_3 \cos(3\omega t + \phi_3) + ... + A_n \cos(n\omega t + \phi_n) + ...$ (5.1)

or, $f(t) = a_0 + A_1 \cos \omega t \cos \phi_1 + A_1 \cos 2\omega t \cos \phi_2 + ... - A_1 \sin \omega t \sin \phi_1 - A_2 \sin 2\omega t \sin \phi ......$

$= a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + ... + a_n \cos n\omega t + ... + b_1 \sin \omega t + b_2 \sin 2\omega t + ... + b_n \sin n\omega t + ...$

where, $a_1 = A_1 \cos \phi_1$, $a_2 = A_2 \cos \phi_2$ ....$a_n = A_n \cos \phi_n$

and $b_1 - A_1 \sin \phi_1$, $b_2 = -A_2 \sin \phi_2$ ..... $b_n = -A_n \sin \phi_n$

$$\therefore\ A_n = \sqrt{a_n^2 + b_n^2} \text{ and } \tan \phi_n = -\frac{b_n}{a_n}$$

∴ We can write

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (5.2)$$

This series is known as Fourier series,

a_0, a_1 ...... a_n and b_1, b_2....b_n etc are known as Fourier co-efficients or constants. The values of these constants can easily be determined

Limitations of Fourier's theorem :

To analyse any complex periodic functions using Fouriers theorem, the following conditions should be satisfied by the function.

(i) **Single Valued :** That means the displacement should be only one direction at any time, there may not be two or more values of displcements.

(ii) **Continuous :** The function should be continuous in nature, that means it should have no discontinuity.

(iii) **Finite :** The displacement during vibration should always remain finite, i.e, if should not infinite value at any instant.

In case of all mechanical vibrations all the above three conditions are satisfied.

5.2.1 Evaluation of Fourier's Co-efficients :

(i) Determination of a_0 :

To determine a_0 we integrate equation (5.2) with respect to t both side for one complete cycle i.e., t = 0 to t = T.

∴ we have,

$$\int_0^T f(t)dt = \int_0^T a_0 dt + \sum_{n=0}^{\infty}\left[\int_0^T (a_n \cos n\omega t + b_n \sin n\,\omega t)dt\right]$$

$$= a_0 T \left[\because \int_0^T \cos n\omega t\, dt = \int_0^T \sin n\,\omega t dt = 0\right]$$

$$\therefore\ a_0 = \frac{1}{T}\int_0^T f(t)dt \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

(ii) Determination of a_n :

To determine the coefficient a_n, i.e the coefficients of cosine terms, multiply equation (5.2) by $\cos k\omega t$ and integrate with respect to t from t = 0 to t = 7, k is a positive integer,

$$\text{Then } \int_0^T f(t)\cos k\omega t = \int_0^T (a_0 + a_1\cos\omega t + a_2\cos 2\omega t + \ldots\ldots$$

$$+ a_n \cos n\omega t + \ldots\ldots)\ \cos k\, nt\, dt + \int (b_1 \sin\omega t + b_2 \sin 2\omega t + \ldots\ldots + b_n \sin n\omega t + \ldots.)\cos knt\, dt\ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

Now, $\int_0^T \cos n\omega t \cos k\,nt\,dt =$

$$= \frac{1}{2}\left[\int_0^T \{\cos(n+k)\omega t + \cos(k-n)\omega t\} dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{(n+k)\omega}\sin(n+k)\omega t + \frac{1}{(k-n)\omega}\sin(k-n)\omega t\right]_0^T$$

$$= \frac{1}{2}\left[\frac{1}{(n+k)\omega}\sin(n+k)\frac{2\pi}{T}\cdot T + \frac{1}{(k-n)\omega}\sin(k-n)\frac{2\pi}{T}\cdot T\right]_0^T$$

= 0 when $k \neq n$ [... $\sin 2n\pi = 0$]

But when k = n,

$$\int_0^T \cos n\omega t \cos kntdt = \int_0^T \cos^2 n\omega tdt = \frac{T}{2}$$

Again, $\int_0^T \sin n\omega t \cos k\omega t\,dt$

$$= \frac{1}{2}\left[\int_0^T \{\sin(k+n)\omega t + \sin(k-n)\omega t\} dt\right]$$

= 0 for all values of k

∴ We can write equation (5.4) as

$$\int_0^T f(t)\cos n\omega t = a_n \frac{T}{2}$$

$$\therefore\ a_n = \frac{2}{7}\int_0^T f(t)\cos n\omega t\,dt \quad \text{.... ... ...} \quad (5.5)$$

(iii) Determination b_n :

In the same way as above,we can determine b_n by multiplying equation (5.2) with sin n wt and integrating w.r.t z between the limits 0 to T we get

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (5.6)$$

Thus the Fourier's co-efficients can be calculated by knowing the function f(t).

5.3 Application of Fourier's theorem :

Fourier's theorem is applied both for analysis and systhesis of the complex curves. To find the component vibrations from the resultant one, is known as analysis of the curve and to find the resultant vibration from the component vibration is known as synthesis.

5.3.1 Analysis of square-wave :

Let the displacement curve of a vibrating particle be given by y = k from

$t = 0$ to $t = \frac{T}{2}$ and y= 0 from

$t = \frac{T}{2}$ to T, where T is the

time period of vibration.

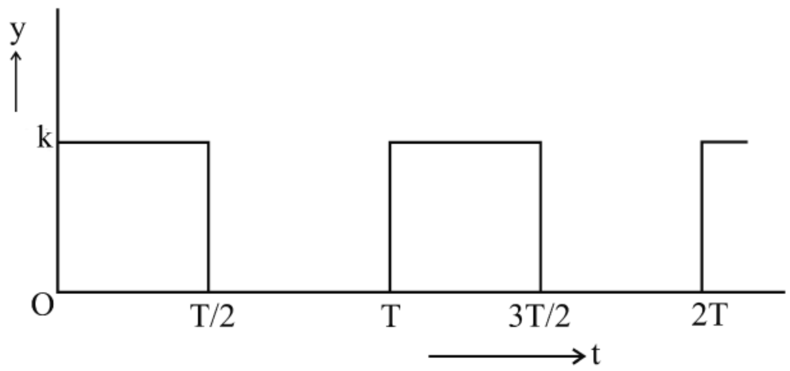


Figure : 5.1

Let us write

$$y = f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (5.7)$$

Now, $a_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\left[\int_0^{T/2} ydt + \int_{T/2}^{T} ydt\right]$

$$= \frac{1}{T}\int_0^{T/2} k\,dt + 0 \qquad [\because y = k \text{ for } t = 0 \text{ to } T/2]$$

$$= \frac{1}{T}\cdot k\frac{T}{2} = \frac{k}{2}$$

$$\therefore a_0 = \frac{k}{2}$$

Again, $a_n = \frac{2}{T}\int_0^T y\cos n\omega t\,dt = \frac{2k}{T}\int_0^{T/2}\cos n\omega t\,dt = \frac{2k}{T}\cdot\frac{1}{n\omega}[\sin n\,\omega t]_0^{T/2}$

or, $a_n = \frac{2k}{n\omega T}\left(\sin n\frac{2\pi}{T}\cdot\frac{T}{2} - 0\right)$

$$= \frac{2K}{n\omega T}\cdot 0 = 0$$

$$b_n = \frac{2}{T}\int_0^T y\sin n\omega t\,dt = \frac{2}{T}\left[\int_0^{T/2} k\sin n\,\omega t + \int_{T/2}^{T} 0.\sin n\omega t dt\right]$$

$$= \frac{2k}{Tn\omega}[-\cos n\,\omega t]_0^{T/2}$$

$$= \frac{2k}{Tn\omega}[1 - \cos n\pi] \qquad \left[\because \frac{\omega T}{2} = \frac{2\pi}{T}\cdot\frac{T}{2} = \pi\right]$$

$$= \frac{2k}{2kn}\left[1 - (-1)^n\right]$$

$= \frac{k}{n\pi}\cdot 2$ for n odd

= 0 for n even.

$\therefore\ b_n = \frac{2k}{n\pi}$ when n is odd.

$\therefore$ Only the odd terms of the sine series appear in the expression.

∴ Putting a_0, a_n, b_n in equation (5.7) we get

$$y = f(t) = \frac{k}{2} + \frac{2k}{A} \text{ (sin wt + } \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + ... \right) \quad ... \quad ... \quad ... \qquad (5.8)$$

when you that this series, you get a square are waveform.

5.3.2 Analysis of saw-tooth wave :

Let the displacement increase uniformly with l from y = 0 to y = k as t increases from 0 to T.

The resulting wave form is a saw-tooth wave form as shown in figure 5.2.

Here, from figure we can write

$$\frac{y}{t} = \frac{k}{T}$$

$$\therefore \; y = \frac{k}{T} t$$

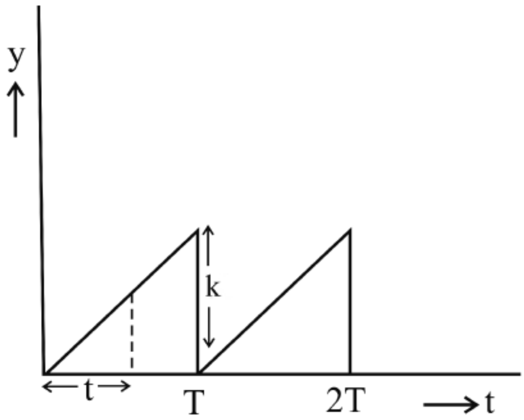


Figure : 5.2

$$\text{Hence, } a_0 = \frac{1}{T} \int_0^T y dt$$

$$= \frac{1}{T} \int_0^T \frac{k}{T} t dt$$

$$= \frac{k}{T^2} \cdot \frac{T^2}{2}$$

$$\therefore a_0 = \frac{k}{2}$$

$$\text{and } a_n = \frac{2}{T} \int_0^T \cos n\omega t \, dt = \frac{2k}{T^2} \int_0^T t \cos n\omega t \, dt$$

$$= \frac{2k}{T^2} \left[\frac{t}{n\omega} \sin s\omega t \right]_0^T - \frac{2k}{T^2} \int_0^T \frac{1}{n\omega} \sin n\, wt \, dt \qquad \text{[Integrating by parts]}$$

$$= 0 - \frac{2k}{T^2}\frac{1}{n^2\omega^2}\left[\cos n\,\omega t\right]_0^T$$

$$= 0$$

Now, $b_s = \frac{2}{T}\int_0^T y \sin n\,\omega t\, dt$

$$= \frac{2k}{T^2}\int_0^T t \sin n\,\omega t\, dt$$

$$= \frac{2k}{T^2}\left[-\frac{2}{n\omega}\cos n\,\omega t + \frac{1}{n^2\omega^2}\sin n\,\omega t\right]_0^T$$

$$= \frac{2k}{T^2}\left[-\frac{T}{n\omega}\cos 2\pi n + \frac{1}{n\omega^2}\sin 2\pi n\right]$$

$$= -\frac{2k}{T^2}\cdot\frac{T}{n\omega}\cdot 1$$

$$= -\frac{2k}{2\pi n}$$

$$\therefore\ b_s = -\frac{k}{n\pi}$$

$$\therefore\ y = f(t) = \frac{k}{2} - \frac{k}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\,\omega t + \frac{1}{3}\sin 3\omega t + ...\right) \quad (5.9)$$

Here all harmonies ar present.

5.4 Importance of Fourier's theorem

To analysis and synthesis of vibration, Fourier's theorem has important application. Using this theorem you can find out the harmomic components in a particular vibration. This theorem tells us which components of harmonies are to be taken to produce a vibration of given periods. This theorem enables in to study qualitatively the quality of a musical sound. It also helps us to design a rectifier circuit.

Exercise :1

Find the Fourier series for the function f(x) defined as f(x) = k, $-\pi < x < 0$

$= -k, 0 < x < \pi$

5.5 Summary

Fourier series is

$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$ where a_0, a_n and b_n are constant called Fouriers's coefficients.

Fourier coefficients are

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\text{con}\, n\,\omega t\, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\,\omega t\, dt$$

5.6 Questions and problems

5.6.1 An alternating e. m. t of sine form is applied to a half-wave rectifier. Obtain an expression for the rectified voltage in harmonic series.

5.6.2 Find the Fouriers' series for the function defined as

$f(x) = -1, -\pi < x < 0$

$= 1, 0 < x < \pi$

5.7 Solutions

Exercise-1 :

Here, $f(x) = k, -\pi < x < 0$

$= -k, 0 < x < \pi$

we know the fourier series as

$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$

$$\therefore \ a_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2k}\left[\int_{-\pi}^{0} k\ dx + \int_{0}^{\pi}(-k)dx\right]$$

$$= \frac{k}{2k}(0+\pi-\pi+0) = 0$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} k\cos nx\, dx - \int_{0}^{0} k\cos nx\, dx\right]$$

$$= \frac{k}{\pi n}\left[[\sin nx]_{\pi}^{0} - [\sin nx]_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{k}{\pi}\left[\int_{-\pi}^{0}\sin nx\, dx - \int_{0}^{\pi}\sin nx\, dx\right]$$

$$= \frac{k}{\pi n}\left[\{-\cos nx\}_{-\pi}^{0} + \{-\cos nx\}_{0}^{\pi}\right]$$

$$= \frac{k}{\pi n}\left[\{-\cos n\pi + 1\} + \{-\cos nx + 1\}\right]$$

$$= \frac{2k}{n\pi}(1-\cos n\pi) = 0$$

$$= \frac{4k}{n\pi} \text{ for n odd}$$

$$\therefore \ f(x) = \sum_{1,3,5,\ldots}^{\infty} \frac{4k}{n\pi}\sin n\pi = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

5.7.1 Here we consider the function for half wave rectifier as

$$y(t) = A\sin\omega t,\ 0 < t < \frac{T}{2}$$

$$= 0,\ \frac{T}{2} < t < T$$

$\therefore$ The Fourier series in

$$y(t) = a_0 + \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=0}^{\infty} b_n \sin n\omega t$$

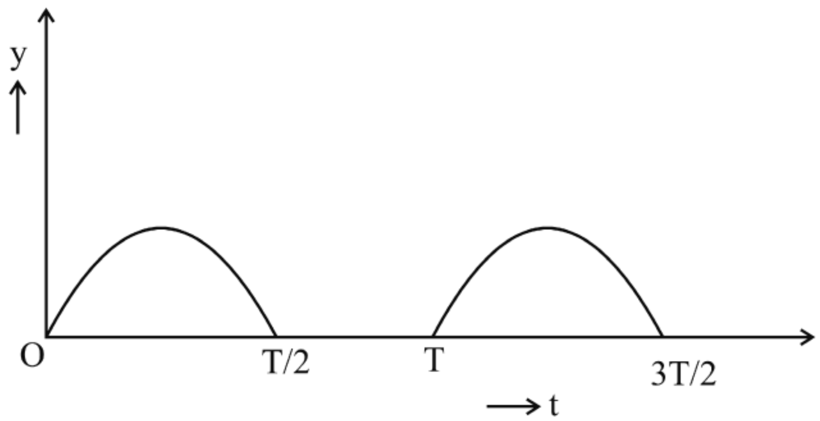


Figure : 5.7

$$\therefore a_0 = \frac{1}{T}\int_0^T y(t)\,dt$$

$$= \frac{A}{T}\int_0^{T/2} \sin\omega t\,dt$$

$$= \frac{A}{\omega T}\left[\cos\omega t\right]_0^{T/2}$$

$$= -\frac{A}{2\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{A}{\pi}$$

and, $\therefore a_n = \dfrac{2}{T}\displaystyle\int_0^{T/2} A\sin\omega t\cos n\omega t\,dt$

$$= \frac{2A}{2T}\int_0^{T/2}\left[\sin(\omega t + n\omega t) + \sin(\omega t - n\omega t)\right]dt$$

$$= \frac{A}{T}\left[\frac{-1}{(1+n)\omega}\cos(1+n)\omega t - \frac{1}{(1-n)\omega}\cos(1-n)\omega t\right]_0^{T/2}$$

$$= \frac{-A}{T\omega}\left[\left\{\frac{1}{1+n}\cos(1+n)\pi - \frac{1}{1+n}\right\} - \frac{1}{1-n}\cos(1-n)\pi + \frac{1}{1-n}\right]$$

Now, for n = 1 $a_1 = -\frac{A}{2\pi}\left[\left\{\frac{1}{2}\cos 2\pi - \frac{1}{2}\right\} - \frac{1}{1-n}\left\{\cos(1-n)\pi - 1\right\}\right]$

$$= -\frac{A}{2\pi}\left[0 + \frac{1}{1-n}\frac{-\sin^2\frac{(1-n)k}{2}}{2}\right] = 0 \text{ for } n = 1$$

$\therefore$ $a_1 = 0$, $a_2 = -\frac{A}{2\pi}\left\{\frac{1}{3}\cos 3\pi - \frac{1}{3} - \frac{1}{4}\cos\pi - 1\right\}$

$$= -\frac{A}{2\pi}\left\{\frac{1}{3}\cdot(-1) - \frac{1}{3} - 2\right\}$$

$$= +\frac{A}{2\pi}\left(\frac{2}{3} - 2\right)$$

$$= A\left(\frac{1}{3\pi} - \frac{1}{\pi}\right)$$

Similarly, $a_3 = 0$, $a_4 = A\left(\frac{1}{5\pi} - \frac{1}{3\pi}\right)$

$a_5 = 0$, $a_6 = A\left(\frac{1}{7\pi} - \frac{1}{5\pi}\right)$

$$\therefore a_n = A\left[\frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi}\right] \text{ when n is an even integer}$$

Now, $b_n = \frac{2A}{T}\int_0^{T/2} \sin\omega t \sin n\omega t\, dt$

$$= \frac{2A}{2T}\int_0^{T/2}\left[\cos(n-1)\omega t - \cos(n+1)\omega t\right]dt$$

$$= \frac{A}{T\omega}\left[\frac{\sin(n-1)\omega t}{n-1}\right]_0^{T/2} - \frac{A}{\omega T}\left[\frac{\sin(n+1)\omega t}{n+1}\right]_0^{T/2}$$

$$= \frac{A}{2\pi(n-1)}\left[\sin(n-1)\pi\right] - \frac{A}{2\pi(n+1)}\left[\sin(n+1)\pi\right]$$

= 0 except for n =1

$$\therefore b_1 = \frac{2A}{T}\int_0^{T/2}\sin^2\omega t\, dt = \frac{2A}{T}\cdot\frac{T}{4} = \frac{A}{2}$$

$$\therefore y = \frac{A}{\pi} - \frac{A}{\pi}\left\{\left(\frac{1}{3}-1\right)\cos 2\omega t + \left(\frac{1}{3}-\frac{1}{5}\right)\cos 4\omega t + \left(\frac{1}{7}-\frac{1}{5}\right)\cos 6\omega t + ...3 + \frac{A}{2}\sin\omega t\right\}$$

$$\therefore y = \frac{A}{\pi} - \frac{2A}{\pi}\left\{\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t + ...\right) + \frac{A}{2}\sin\omega t\right\}$$

5.7.2 Here $f(x) = -1, -\pi < x > 0$

$= 1, 0 < x < \pi$

Fouuier's series is

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=0}^{\infty} b_n \sin nx$$

Here, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f dx(x) = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_0^{\pi}(1)dx\right] = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] = 0$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{2}{nx}(1 - \cos n\pi)$$

= 0 for n even

$= \dfrac{4}{nx}$ for n odd.

$$\therefore\ f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)$$

Unit : 6 ☐ Wave Motion

Structure

6.0 Objectives

After studing this unit, you will be able to

- define wave motion and difference between longitudinal and transverse waves.
- establish the wave equation in elastic medium.
- derive expressions for energy and intensity of progressive wave.
- know the bel and decibel, phase velocity and group velocity.
- write two and three dimensional wave equations.

6.1 Introduction

When a particle of a material medium is slightly displaced from its equilibrium position and released, then it starts to vibrate about its equilibrium position by virtue of the elastic and inertial properties of the medium. Since the neighbouring particles are bound in each other by the force of cohesion. So this vibration is transmited to the neighbouring ones. Such a process at transfer of vibration from one particle to an other particle is known as a wave motion. Through wave motion, a disturbance and energy are carried out from one point to another point of the material medium, but he medium itself however is not physically transported. You have seen the water waves formed on the surface of water in pond, when a stone is dropped in the pond, the molecules of water move up and down at the same place, therefore a cork placed on the surface of the water moves up up and down at the same place as water moves across the surface of the pond. Thus each particle of the medium executes similar vibration about its mean position with same frequency and amplitude but not in same phase. The difference in phases increase as the distance of the particles increase from the source. Hence the wave motion, in general refer to the transfer of evergy from one point to another point of the medium.

There are two types of wave motions, transverse wave and longitudinal wave motions. In a transverse wave motion the particle of the medium vibrate at right angles about the mean position of the direction of propagation of the wave.

But in the longitudinal wave motion the particles of the medium vibrate about mean position in the same direction in which the wave is propagated.

There are source waves in which no material medium in requried, viz heat radiation, light waves, x-ray, r-ray, radio waves etc. These waves in general are called electromagnetic waves.

In this chapter we shall discuss the elastic waves in material medium, i.e., only sound wave.

6.2 Velocity of plane longitudinal waves in fluid (in an elastic medium)

Consider a cylinder of fluid of cross-section A perpendicular to the direction of propagationof the wave. Let x and $x + \delta x$ be the positions of two close sections A_1 and B_1 of the cylinder (figure 6.1) with respect to an arbitrary origin such that $A_1B_1 = x + \delta x - x = \delta x$.

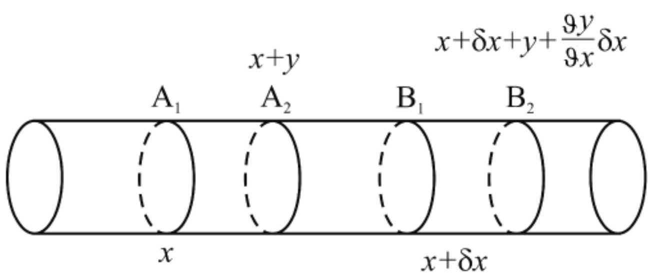


Figure : 6.1

Due to disturbance, let the particles on the plane A_1 be displaced to A_2 by y i.e., the co-ordinate of A_2 is $x + y$, and B_1 to B_2 such that the positon of B_2 is $x + \delta x + y + \frac{\partial y}{\partial x}\delta x$.

Thus the thicknes of the layer

$$A_2B_2 = \left(x + \delta x + y + \frac{\partial y}{\partial x}\delta x\right) - (x - y) = \delta x + \frac{\partial y}{\partial x}\delta x$$

∴ Initial volume between A_1 and B_1 is $V_i = A\delta x$ and final volume between A_2 and B_2 is $v_f = A\left(\delta x + \frac{\partial y}{\partial x}\delta x\right)$

∴ Increase in volume $(\Delta V) = V_f - V_i = A\frac{\partial y}{\partial x}\delta x$

Hence, the volume strain $= -\dfrac{A\frac{\partial y}{\partial x}\delta x}{A\delta x} = -\dfrac{\partial y}{\partial x}$

This negative sign indicates that the colum is compressed.This is due to the excess pressure over the atmospheric pressure.

Let P be the excess pressure on A_2 and that on B_2 be $P + \frac{\partial P}{\partial x}\delta x$

∴ The resulting force on the slice A_2 and B_2 along x axis is $A\left\{P - \left(P + \frac{\partial P}{\partial x}dx\right)\right\}$ = $-\frac{\partial P}{\partial x}\delta x$

From Newton's Law this force must be equal to mass of the layer ($\rho\delta x$) multiplied by the acceleration $\frac{\partial^2 y}{\partial t^2}$, where ρ is the equilibrium density of the fluid.

Thus $\rho\delta x\frac{\partial^2 y}{\partial t^2} = -\frac{\partial P}{\partial x}\delta x$

$$\text{or, } \rho\frac{\partial^2 y}{\partial t^2} = -\frac{\partial P}{\partial x} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.1)$$

Again from Hooke's law, we get

Bulk modules (κ) = $\dfrac{\text{volume stress}}{\text{volume strain}} = -\dfrac{P}{\partial y/\partial x}$

$$\therefore \; P = -\kappa \frac{\partial y}{\partial x} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

Now, from equation (6.1) and (6.2) we get,

$$\rho \frac{\partial^2 y}{\partial t^2} = -\frac{\partial}{\partial x}\left(-\kappa \frac{\partial y}{\partial x}\right)$$

$$\text{or, } \rho \frac{\partial^2 y}{\partial t^2} = +\kappa \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \; \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.3)$$

where $c = \sqrt{\frac{\kappa}{\rho}}$

Equation (6.3) is the differential eqution of progressive wave and c is the velocity of wave in fluid.

The solution of equation (6.3) is given by

$y = f_1(ct - x) + f_2\,(ct + x)$ where f_1 and f_2 are two arbitrary functions. $f_1\,(ct - x)$ represents the wave travelling in the positive direction of x axis and $f_2(ct + x)$ represents a wave travelling in the negative direction.

This treatment is applicable to all case, of plane progressive waves, but for fluid the appropriate modulus is adiabatic bulk modulus, young's modulus for long narrow solid bar and axial modulus for unlimited solid medium.

6.2.1 Acoustic pressure

Let a simple harmonic wave in a gas travelling in the positiev direction of x axis with a velocity c. The displacement at x is given by

$$y = a \sin \frac{2\pi}{\lambda}(ct - x)$$

and the excess pressure

$$\delta P = -\kappa \frac{\partial y}{\partial x}$$

$$= \frac{2\pi a\kappa}{\lambda}\cos\frac{2\pi}{\lambda}(ct - x)$$

∴ The maximum excess pressure

$$\delta P_{max} = \frac{2\pi a\kappa}{\lambda} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (6.4)$$

This is known as acoustic pressure.

Again acoustic pressure can be expressed as R. M. S acoustic pressure

$$\therefore \text{R. M. S acoustic pressure} = \left[\frac{1}{T}\int_0^T\left(\frac{2\pi a\kappa}{\lambda}\right)^2 \cos^2\frac{2\pi}{\lambda}(ct - x)dt\right]^{1/2}$$

$$= \frac{2\pi a\kappa}{\lambda}\left[\frac{1}{T}\cdot\frac{T}{2}\right]^{1/2}$$

$$= \sqrt{2}\,\frac{\pi a\kappa}{\lambda} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (6.5)$$

Exercise – 1

Show that the function $\Psi(x. t) = f(x - ct) + g(x + ct)$ satisfies the one dimensinoal wave equation.

6.3 Energy density and intensity of plane progressive wave

When sound wave pass through a medium the particles of the medium vibrate. So, these particles posses extra amount of energy which is partly kinetic and partly potential. The average enery per unit volume of the medium is called energy density.

Kinetic energy :

Let a plane harmonic longitudinal wave is moving through a medium along positive direction of x axis. The displacement of a particle at the position x at any time t may be written as

$$y(x, t) = a\cos(\omega t - kx) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (6.6)$$

where, ω = angular frequency, k = propagation constant and a = amplitude of the vibration.

Consider an infinitesimally thin layer of thickness δx and of unit cross-section, the mass of this layer is $\rho\delta x$, where ρ is the density of the medium.Then the kinetic energy of this layer is $E_K = \frac{1}{2}\rho\delta x\left(\frac{\partial y}{\partial t}\right)^2$

$$= \frac{1}{2}\rho\delta x\, a^2\omega^2 \sin^2(\omega t - kx) \left[\because \frac{\partial y}{\partial t} = -a\omega\sin(\omega t - kx)\right]$$

$\therefore$ The average kinetic energy over a complete cycle of period T is

$$< E_k >= \frac{1}{T}\int_0^T \frac{1}{2}a^2\omega^2\rho\delta x \sin^2(\omega t - kx)dt$$

$$= \frac{1}{T}\cdot\frac{1}{2}a^2\omega^2\rho\delta x\cdot\frac{T}{2}$$

$$= \frac{1}{4}a^2\omega^2\rho\delta x$$

Hence, the average kinetic energy per unit volume $< E_k >= \frac{1}{4}a^2\omega^2\rho$ ... ... (6.7)

Potential Energy ;

The force acting on the element of Thickness δx and of unit cross-section is given by

$$F = \rho\delta x \frac{\delta^2 y}{\delta t^2}$$

Now, $\frac{\delta^2 y}{\delta t^2} = -a\omega^2 \cos(\omega t - kx)$

$\therefore$ $F = - a\omega^2\rho\delta x \cos(\omega t - kx)$

$= - \rho\omega^2 \delta x\, y$ [Putting $y = a \cos (\omega t - kx)$]

The negative sign indicaets that the force is directed towards the equilibrium position i.e. opposite to the displacement.

Hence the work done to produce a small displacement dy is

$dE_P = \rho\omega^2\delta x\, ydy$

$\therefore$ The total workdone for the displacement from 0 to y. which is stored in the volume of the medium as potential energy of the wave is given by

$$E_P = \int_0^y \rho\omega^2 \delta x \, y dy = \rho\omega^2 \delta x \frac{y^2}{2}$$

$$= \frac{1}{2}\rho\omega^2 a^2 \delta x \cos^2(\omega t - kx)$$

∴ Average potential energy per unit volume over a complete period T is

$$\langle E_P \rangle = \frac{1}{2}\rho\omega^2 a^2 \frac{1}{T}\int_0^T \cos^2(\omega t - kx)dt$$

$$= \frac{1}{4}a^2\omega^2\rho \quad \ldots \quad \ldots \quad \ldots \quad (6.8)$$

Thus from equations (6.7) and (6.8), we see that the average kinetic energy per unit volume is equal to the average potential energy per unit volume.

So, the total energy per unit volume, ie., energy density

$E = \langle E_k \rangle + \langle E_P \rangle$

$$= \frac{1}{4}a^2\omega^2\rho + \frac{1}{4}a^2\omega^2\rho$$

$$\therefore\ E = \frac{1}{2}a^2\omega^2\rho \quad \ldots \quad \ldots \quad \ldots \quad (6.9)$$

Now, we know $\omega = 2\pi n$ when n is frequency of the wave

$$\therefore\ E = \frac{1}{2}a^2(2\pi n)^2\rho = 2\pi^2 a^2 n^2 \rho \quad \ldots \quad \ldots \quad \ldots \quad (6.10)$$

From equation (6.10) we see that the energy density is proportional to the density of the medium, square of the amplitude and square of the frequency of the wave.

Intensity of the wave :

The intensity of sound wave at a point in the energy flowing per unit area perpendicular to the direction of propatation per unit time, hence it is the energy contained in a volume of unit cross-sectional area and of length c.

∴ Intensity (I) = Energy density × c

$$= 2\pi^2 a^2 n^2 \rho c \quad \ldots \quad \ldots \quad \ldots \quad (6.11)$$

From equation (6.5) we have

R. M. S acoustic pressure, $P_{r.m.s} = \dfrac{\sqrt{2}\pi a\kappa}{\lambda}$

or, $P_{r.m.s} = \sqrt{2}\pi a\rho c^2 \cdot \dfrac{n}{c}$ $\quad [\therefore c = n\lambda \text{ or, } \dfrac{1}{\lambda} = \dfrac{n}{c} \text{ and } c = \sqrt{\dfrac{\kappa}{\rho}} \text{ or, } \kappa = \rho c^2]$

$$\therefore \frac{I}{P_{rms}^2} = \frac{2\pi^2 a^2 n^2 \rho c}{2\pi^2 a^2 \rho^2 c^2 n^2} = \frac{1}{\rho c}$$

$$\therefore I = \frac{P_{rms}^2}{\rho c} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.12)$$

Exercise-2

Compare the acoustic pressure for identical sound waves of equal intensity in air and water given for air $\rho_0 = 1.21 kg/m^3$. $c_0 = 343$ m/s and for water $P_\omega = 998 kg/m^3$, $c_\omega = 1480$ m/s.

Exercise-3

A plane harmonic wave has an RMS excess pressure of $20 N/m^2$. The frequency in 1 KHz and the phase velocity 350 m/s. Calculate the maximum value of particle displacement. [Given bulk modules (k) = $1.4 \times 10^5 N/m^2$]

6.3.1 The Bel and the Decibel

The sensation of loudness depends upon the intensity of sound but there no relation between the two. Loudness in the sensation which depends upon the ability of hearing of the listener but intensity is a purely physical quantity.

The relation between the intensity and the loudness is Weber-Fechner law. This law states that the change in Loudness dL is directly proportional to dI/I, where dI is a small change in intensity I.

$\therefore dL = K_0 \dfrac{dI}{I}$ where K_0 is constant Integrating the above equation between the limits L_1 to L_2 corresponding intensities I_1 to I_2 ,

we have $L_2 - L_1 = K_0 \ln \dfrac{I}{I_1} = 2.303 K_0 \dfrac{I_2}{I_1} = C \log_{10} \dfrac{I_2}{I_1}$

Where, C = 2.303 k_0a constant.

The quantity $\log_{10}\frac{I_2}{I_1}$, gives the intensity level, in unit of bel, of I_2 relative to I_1 which is taken as a standard intensity.

The threshold audibility, that is the lower limit of audibility, for arole of frequency 1KHz corresponds to an intensity 10^{-12} watt/m^2 and the corresponding acoustic pressure 2×10^{-5} N/m^2 is taken as the standard intensity (I_0).

\ The intensity level (IL) of sound of intensity I is defined as

$$IL = \log_{10}\frac{I}{I_0} \text{ bel}$$

$$\therefore \quad IL = \log_{10}\frac{I}{I_0} \text{dB} \quad \ldots \qquad (6.13)$$

where IL is expressed in decibel (dB).

1 bel = 10 dB

The modern sound deterctors, such as head phones, loud speakers etc responds to the changes in accoustic pressure rather than to the intensity.

So, it is more useful to express equation (6.13) in terms of sound pressure.

From equation (6.12) we have intesity $I = P_{rms}^2/\rho C$

$$\therefore \quad \frac{I}{I_0} = \left(\frac{P_{rms}}{P_0}\right)^2$$

So, we may define sound pressure level (SPL) as

$$SPL = 20\log_{10}\frac{P_{rms}}{P_0} \quad \ldots \qquad (6.14)$$

in dB, where P_0 is the effective pressure (RMS) at the standard sound.

6.3.2 Musical Scale

A musical scale is a collection of notes having certain relation to one another as regards the frequency of vibration. The note of lowest frequency of the scale is called key note or tonic. The physists prefer 256 Hz as a key note but th musicians concert, the pitch however uses a frequency of 264 Hz. The not having a frequency twice that of key note is called an octave. Between a note and its octave, the human ear can distinguish a number of notes of definite frequencies. By introducing a number of notes between keynote and the octave a musical scale is constructed.

6.4 Phase velocity and Group velocity

Phase velocity : When a monochromatic harmonic wave moves through a medium, the velocity with which the planes of constant phase move is called the phase velocity or wave velocity.

Let a plane progressive harmonic wave is propagating in the positive direction of x axis is expressed as

$$y = a \sin(\omega t - kx) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.15)$$

where y is the displacement of the particle at x at any time t, 'a' is the amplitude, ω is the angular frequency and $K\left(=\frac{2\pi}{\lambda}\right)$ is the wave vector of the wave.

For planes of constant phase we have $\omega t - kx$ = constant

$$\text{or, } \frac{d}{dt}(\omega t - kx) = 0$$

$$\text{or, } \omega - k\frac{dx}{dt} = 0$$

$$\therefore \frac{dx}{dt} = \frac{\omega}{k} = v_\rho \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.16)$$

where v_ρ is called the phase velocity.

Group Velocity :

In a dispersive medium a wave group may be formed by the superposition of an infinite numer of plane simple harmonic waves. The amplitudes and phases are such that the group has maximum of amplitude that falls off to zero near the maximum. The maxmimum amplitude of the wave group travels with a velocity different from that of the component waves. This velocity of maximum amplitude is called group velocity.

Let us consider the group to be formed by the superposition of two waves of equal amplitude, but slightly different angular frequencies ω and $\omega + \Delta\omega$, travelling with propagation constants κ and $\kappa + \Delta\kappa$ respectively.

$\therefore$ We can write the waves as

$y_1 = a\sin(\omega t - kx)$ and $y_2 = a\sin\{(\omega + \Delta\omega)t - (k + \Delta k)x\}$

$\therefore$ The resultant wave is

$y = y_1 + y_2$

$= a \sin(\omega t - kx) + a \sin\{(\omega + \Delta\omega)t - (k + \Delta k)x\}$

$$= 2a\sin\frac{1}{2}\{(\omega t - kx) + (\omega + \Delta\omega)t - (k + \Delta k)x\}$$

$$\cos\frac{1}{2}\{(\omega t - kx) - (\omega + \Delta\omega)t + (k + \Delta k)x\}$$

$$= 2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\sin\left(\frac{2\omega + \Delta\omega}{2}t - \frac{2k + \Delta k}{2}x\right)$$

$$\therefore\ y = 2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\sin(\omega t - kx) \quad \text{... (6.17)}$$

or, $y = A \sin \omega (\omega t - kx)$ $[\because \omega >> \Delta\omega$ and $k >> \Delta k]$

where $A = 2a\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$

Thus we see that the amplitude is also a wave.

Now the amplitude with be maximum ($A_{max} = 2a$)

when, $\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x = 0$

or, $\frac{\Delta\omega}{2}t = \frac{\Delta k}{2}x$

or, $\frac{\Delta\omega}{\Delta k} = \frac{x}{t}$

Now, $\underset{\Delta k \to 0}{Lt}\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} = \frac{x}{t} = v_g$

$$\therefore\ v_g = \frac{dw}{dk} \quad \text{... (6.17)}$$

here v_g is the velocity of maximum amplitude or group velocity.

Relation between group velocity and phase velocity

From equation (6.16) the phase velocity is $v_P = \frac{\omega}{k}$

or, $\omega = kv_p$

differenting we get

$$v_g + k\frac{dv_p}{dk}$$

But group velocity, $v_g = \frac{d\omega}{dk}$

$$\therefore v_g = v_g + k\frac{dv_p}{dk}$$

$$= v_p + k\frac{dv_p}{d\lambda}\frac{d\lambda}{dk}$$

$$= v_p - k\frac{2\pi}{k^2}\frac{dv_p}{d\lambda} \qquad \because \lambda = \frac{2\pi}{k} \text{ or, } \frac{d\lambda}{dk} = 2\pi\left(-\frac{1}{k^2}\right)$$

$$\therefore v_g = v_p - \lambda\frac{dv_p}{d\lambda} \qquad \text{... (6.18)}$$

Now, (i) it $\frac{dv_p}{d\lambda}$ is positive quantity then $v_g < v_p$.

This is normal dispersion.

(ii) If $\frac{dv_p}{d\lambda} = 0$, then $v_g = v_p$.This happens in a non dispersive medium.

(iii) If $\frac{dv_p}{d\lambda}$ is negative quantity then $v_g > g_p$. This is the case of anomalous dispersion.

6.5 Waves in two and three dimensions

When a plane progressive wave amoving along positive direction of x axis with velocity c, then the function be represented by $\psi = f(x - ct)$.

Here ψ is the disturbance associated with the wave, which may be the displacement of the particle or any characteristics of the wave, called the wave field parameter.

∴ The plane wave differential equation in one demension is

$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$ here $\psi = \psi(x, t)$ is. the waves are constrained to move along a line as in the case of vibration of string, where the particles vibrate is perpendicular direction.

But, suface waves or ripple on water caused by dropping a pebble into a quite pond are two dimensional (2D).

In such cases, the displacement is a function of x, y and t i.e $\psi = \psi(x, y, t)$ and the differential equagoin in 2D is

$$\frac{\partial^2\psi(x,y,z)}{\partial t^2} = c^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \quad \text{... (6.14)}$$

The solution of this eqution is $\psi = a\sin\left(\omega t - \vec{k}\,.\,\vec{r}\right)$ ... (6.20)

Where $\vec{k}\,.\,\vec{r} = \left(\hat{i}k_x + \hat{i}k_y\right)$ and $\vec{r} = \hat{i}x + \hat{j}y$

$\therefore\ \vec{\mathbf{k}}\cdot\vec{\mathbf{r}} = k_x x + k_y y$

Similarly, in three dimension $\psi = \psi(x, y, z, t)$ and the differential equation is

$$\frac{\partial^2\psi(x,y,z,t)}{\partial t^2} = c^2\left(\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right)$$

$$= c^2\nabla^2\psi \quad \text{... (6.21)}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$

is an opertor.

And the solution of equation (6.21)(is

$$\psi = a\sin\left(\omega t - \vec{k}\cdot\vec{r}\right) \quad \ldots (6.22)$$

where $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$, $\vec{k} = \hat{i}k_x + \hat{j}k_y + \hat{k}k_z$ and $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Plane wave and spherical wave :

When a disturbance is caused in a medium, the particles of the medium vibrate and the continuous locus of all particles vibrating in came phase at any instant of time is called the wave front. Thus a wavefront is a surface of constant phase.

Now in a homogeneous isotropic medium a point source of sound sends waves in all directions travelling with same speed so that they arrive simultaneously at the surface of a sphere with the spoint source as its centre. Hence, the wave front is a sphere and this wave is called spherical wave.

But is if the point source is at a large distance, then a small portion of the spherical wave front may considered to be a plane and is called plane wave front. Thus a plane wave is a wave is one dimensional medium.

Summary :

- Differential equation of Progressive wave in one dimension is $\frac{\partial^2\psi(x,t)}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$

  in 2D, $\frac{\partial^2\psi(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$

  in 3D, $\frac{\partial^2\psi(x,y,z,t)}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = c^2\nabla^2\psi$

- R. M. S acoustic Pressure $(P_{rms}) = \frac{\sqrt{2}\,\pi ak}{\lambda}$

- Energy density, $E = 2\pi^2 a^2 n^2 \rho$

- Intensity, $I = \frac{P^2 rms}{\rho c} = 2\pi^2 a^2 n^2 \rho c$
- Decible, Intensity lebel $= 10 \log_{10} \frac{I}{I_0}$ db
- Relation between group velocity and phase velocity

$$v_g = v_p - \lambda \frac{dv_p}{d\lambda}$$

6.7 Qustions and problems

6.7.1 An increase of pressure of 100 Kp_a causes a certain volume at water to decrease by $5 \times 10^{-3}\%$ of its original volume. Calculate the speed of sound in water.

6.7.1 Calculate the change in intensity level when the intensity of sound increases by 10^6 times its original value.

6.7.3 A microphone emits a 1KHz pure done having an intensity level of 70 dB. Calculate the actual intensity.

6.7.4 For gravity waves in a liquid the phase velocity v_p depends on the wave length λ according to the formula $v_p = g\lambda^{1/2}$, g being a constant. Show that the group velocity is half the phase velocity.

6.8. Solution

Exercise-1

Here $\psi = f(x - ct) + g(x + ct)$

Now, $\frac{\partial \psi}{\partial t} = -cf'(x - ct) + cg'(x + ct)$

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 f''(x - ct) + c^2 g''(x + ct)$$

$$= c^2 \left[f''(x - ct) + g''(x + ct) \right] \quad \text{... (3)}$$

Again, $\frac{\partial \psi}{\partial x} = f'(x - ct) + g'(x + ct)$

$$\frac{\partial^2\psi}{\partial x^2} = f''(x-ct) + g''(x+ct) \quad \text{... (2)}$$

$\therefore$ From equation (1) and (2) we get,

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$$

Thus ψ satisfies the differential wave equation.

Exercise-2.

we know intensity $I = \frac{P^2{}_{rms}}{\rho c}$

$\therefore$ For air, $I = \frac{P_o^2}{\rho_o c_o} = I = \frac{P_o^2}{1.21 \times 343}$ ... (1)

and for water, $I = \frac{P_\omega^2}{\rho_\omega C_\omega} = \frac{P_\omega^2}{998 \times 1480}$ ... (2)

$$\therefore \frac{P_o^2}{1.21 \times 343} = \frac{P_\omega^2}{998 \times 1480}$$

or, $\frac{P_\omega^2}{P_0^2} = \frac{998 \times 1480}{1.21 \times 343} = 3558.9$

$$\therefore \frac{P_\omega}{P_o} = 59.66$$

$\therefore P_\omega : P_o = 60 : 1$

Exercise-3

Here, P_{rms} = 20 N/m^2 , wave length $(\lambda) = \frac{v}{n}$

frequency (n) = 1kHz = 10^3 Hz

velocity (v) = 350 m/s

$$\therefore \lambda = \frac{v}{n} = \frac{250}{10^3} = 350 \times 10^{-3} \text{m}$$

and $k = 1.4 \times 10^5 \text{N/m}^2$

Let the displacement $y = a \sin \frac{2\pi}{\lambda}(ct - x)$

$\therefore$ Maximum displacement = a

Now, we know, $\rho_{rms} = \frac{\sqrt{2}\pi ak}{\lambda}$

$$\therefore \quad a = \frac{P_{rms}\lambda}{\sqrt{2}\pi k} = \frac{20 \times 350 \times 10^{-3}}{\sqrt{2}\pi \times 1.4 \times 10^{-3}} = 1.126 \times 10^{-5} \text{m.}$$

6.7.1 Here, Bulk modulus $k = \frac{P}{\frac{\partial v}{V}} = \frac{100 \times 10^3}{5 \times 10^{-5}} = 2 \times 10^9$ Pa

and density of water ρ = 1 gm/c.c = 10^3 kg/m^3.

$\therefore$ velocity of sound in water,

$$v = \sqrt{\frac{k}{\rho}} = \sqrt{\frac{2 \times 10^9}{10^3}} = 1414.21 \text{ m/s}$$

6.7.2 Here, initial intensity = I_0

Final intensity = I

$$\therefore \quad \frac{I}{I_0} = 10^6$$

$\therefore$ Increase in intensity level, L = $10 \log_{10} \frac{I}{I_0} = 10 \log_{10} 10^6$ = 60 dB.

6.7.3 Here, intensity level L = 70 dB we know the reference intensity $(I_0) = 10^{-12}$ ω/m²

$$\therefore \quad 70 = 10\log_{10}\frac{1}{10^{-12}} =$$

$$\text{or,} \quad \log_{10} I = \frac{70}{120} = 0.58$$

$$\therefore \quad I = 10^{0.58} = 3.83\ \omega/\omega^2$$

6.7.4 Here $v_p = g\lambda^{1/2}$

We have group velocity

$$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

$$= v_p - \lambda\frac{d}{d\lambda}\left(g\lambda^{1/2}\right)$$

$$= v_p - \lambda g\frac{1}{2}\lambda^{-1/2}$$

$$= v_p - g\cdot\frac{1}{2}\pi^{1/2}$$

$$= v_p - \frac{1}{2}v_p$$

$$= \frac{1}{2}v_p$$

$$\therefore \quad v_g = \frac{1}{2}v_p$$

Unit : 7 ❐ Vibration of Strings

Structure

7.0 Objectives

- After studing this unit you will be able to construct the differential equation of stretched string and solve it.
- define the Eigenfunctions and Eigenfrequencies.
- determine the energy of vibration of a string.
- learn the nature of vibration of string when it is plucked and when it is struck.

7.1 Introduction

In the previous units you have studied about the vibration of a system whose inertia and elasticity are concentrated in a region, while in wave motion they are distributed throughout the medium. But in stretched string. It acts as a vibrator and also a medium of wave propagation. Again in the world of music, vibration of string plays and important role. Instruments like guitar, Piano, Siutar, violin etc all are based on the transverse vibration of stretched strings. Thus the study of vibration of string is very important both in theoretical and practical points of view.

According to Lord Rayleigh an ideal string is defined as "a perfectly uniform and perfectly flexible filament of solid matter stretched between two fixed points.". A practical string is neither perfectly uniform nor perfectly flexible. The strings used in music only closely approximate. Real strings always have some amount of rigidity, whose effect decrease as the length to diamter ratio increases. The transverse vibration of an ideal string is controlled by its tension not by its rigidity. When a point of a stretched string is displaced from its rest position either by striking or plucking or bowing, the point vibrates and waves travel along the length of the string an either side. The reflected waves from the fixed supports superposed on the incident waves, producing stationary wave patterns. The string then produce a musical note of definite pitch and quality.

7.2. Differential equation and velocity of transverse waves along a stretched string.

Let a perfectly flexible string stretched by a tension T between two fixed supports lie at rest along x-axis.

Consider a small segment AB of length δx of the string when it is undisturbed. Let A_1B_1 represents the element when the string is displaced in XY plane and co-ordinates of A_1, B_1 be (x, y) and $(x + \delta x, y + \delta y)$ respectively. Again consider that the displacements are very small, so the tension remains same when the string vibrates.

Let A_1A_2 and B_1B_2 are two tangents at A_1 and B_1 making angles θ_1 and θ_2 with x-axis respectively, as shown in figure 7.1.

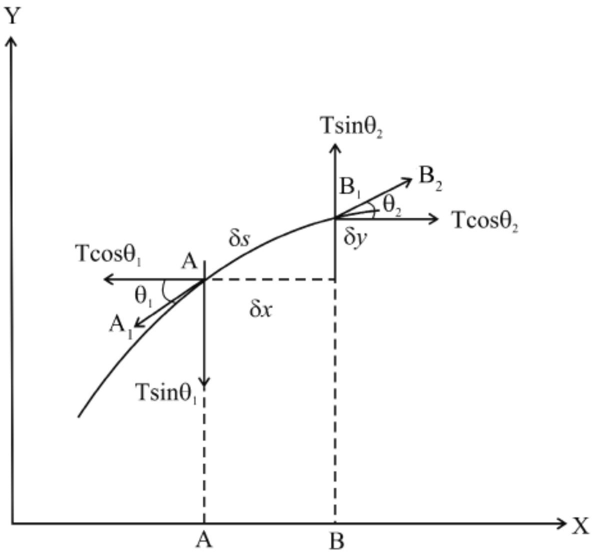


Figure : 7.1

Now the force on the element at A_1 and B_1 is T along A_1A_2 and B_1B_2.

Hence, the resultant force on the element along X-axis is

$$F_x = T\cos\theta_2 - T\cos\theta_1 = 0 \quad [\because \theta_1 \text{ and } \theta_2 \text{ are small } \cos\theta_1 = \cos\theta_2 = 1 \text{ and } \sin\theta = \tan\theta]$$

And the resultant force on the element along Y axis is

$$F_y = T\sin\theta_2 - T\sin\theta_1$$

$$= T(\tan\theta_2 - \tan\theta_1)$$

$$= T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

$$= T\left\{\frac{\partial}{\partial x}\left(y + \frac{\partial y}{\partial x}\delta x\right) - \frac{\partial y}{\partial x}\right\}$$

$$= T\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\delta x - \frac{\partial y}{\partial x}\right)$$

$$= T\,\frac{\partial^2 y}{\partial x^2}\delta x$$

If m is the mass per unit length of the string, the mass of the element A_1B_1 ($\delta l\, \hbar\, \delta x$) is $m\,\delta x$ and its acceleration is $\frac{\partial^2 y}{\partial t^2}$, then from Newton is law of motion, we can write

$$T\frac{\partial^2 y}{\partial x^2}\delta x = m\delta x\frac{\partial^2 y}{\partial t^2}$$

$$\therefore \frac{\partial y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \ldots \quad (7.1)$$

where $c = \sqrt{T/m}$ ... (7.2)

Equation (7.1) is the differential equation for transverse wave along the stretched string and c is the velocity of the wave.

The general solution of equation (7.1) is

$$y = f_1(ct - x) + f_2(ct + x) \quad \ldots \quad (7.3).$$

which represents two transverse waves travelling along the positive and negative difections of X-axis with velocity (c).

7.2.1. Solution of differential equation :

The general solution of the wave eqution (7.1) can be obtained by the method of separation of variables. Since y is function of x and t, so we can write,

$$Y = X(x)\,T(t) \quad \ldots \ldots \ldots \ldots \quad (7.4)$$

where X(x) is the function of x only and T(t) is function of t only.

Substituting for xy in equation (7.1) we get

$$X\frac{\partial^2 T(t)}{\partial t^2} = c^2 T\frac{\partial^2 X(x)}{\partial x^2}$$

or, $$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2} \quad \ldots \ldots \ldots \ldots \quad (7.5)$$

Since left hand side of equation (7.5) in function of t only and the right hand side is a function of x only, this is possible only when both sides is equal to a constant. The value of this constant must be negative, as otherwise y will not be periodic and go on increasing sor decreasing with x or t which would not obay the boundary conditions. Let the constant as $-\omega^2$ where $\omega^2 > 0$.

Therefore equation (7.5) gives

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \text{ or, } \frac{\partial^2 T}{\partial t^2} = -\omega^2 t = 0 \quad \ldots \ldots \ldots \ldots \quad (7.6)$$

and $$\frac{C^2}{X}\frac{\partial^2 X}{\partial x^2} = -\omega^2 \text{ or, } \frac{\partial X}{\partial x^2} + \frac{\omega^2}{c^2} = 0 \quad \ldots \ldots \ldots \ldots \quad (7.7)$$

Equtions (7.6) and (7.7) are standard equation of SHM. So the solution are

$$T = a_1 \cos\omega t + b_1 \sin\omega t \text{ and } X = a_2 \cos\frac{\omega x}{c} + b_2 \sin\frac{\omega x}{c}$$

where a_1, b_1, a_2 and b_2 are constants to be determined from the initial boundary conditions.

Thus the solution of the wave equation for the stretched string is $Y = XT = (a_2 \cos\frac{\omega x}{c}$

$$+ b_2 \sin\frac{\omega x}{c})(a_1 \cos\omega t + b_1 \sin\omega t) \quad \ldots \ldots \ldots \ldots \quad (7.8)$$

Now let $a_1 = A_1 \sin\varphi$ and $b_1 = A_1 \cos\varphi$ where $A_1 = \sqrt{a_1^2 + b_1^2}$ and $\tan\varphi = {}^{a_1}/_{b_1}$.

Then $a_1 \cos\omega t + b_1 \sin\omega t = A_1 \sin(\omega t + \varphi)$

Thus we can write equation (7.8) as

$$y = (a_2 \cos\frac{\omega x}{c} + b_2 \sin\frac{\omega x}{c})\, A_1 \sin(\omega t + \varphi)$$

$$\therefore \quad y = (A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c})\sin(\omega t + \varphi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.9)$$

where $A = A_1a_2$ and $B = A_1b_2$ are also constants.

7.2.2. Eigen functions and Eigen frequencies :

Suppose the stretched string of length ℓ, righdly fixed at $x = 0$ and at $x = \ell$. Thus $x = 0$ and $x = \ell$, $y = 0$ for all values of t.

$\therefore$ First condition gives (from equation 7.9)

$$0 = A\sin(\omega t + \varphi)$$

$$\therefore \quad A = 0 \quad [\because \sin(\omega t + \varphi) \text{ cannot be equal to zero}]$$

$\therefore$ The equation (7.9) becomes

$$y = B\sin\frac{\omega x}{c}\sin(\omega t + \varphi) \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.10)$$

Again from second condition we get

$$o = B\sin\frac{\omega\ell}{c}\sin(\omega t + \varphi)$$

which gives, $\sin\frac{\omega\ell}{c} = 0$

$$\text{or, } \frac{\omega\ell}{c} = S\pi \text{ where s = 1, 2, 3........ positive integers.}$$

$$\therefore \omega = \frac{s\pi c}{\ell} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.11)$$

$\therefore$ Equation (7.10) becomes

$$y = \sum_{s=1}^{\infty} B_s \sin\frac{s\pi x}{\ell}\sin\left(\frac{s\pi ct}{\ell} + \varphi_s\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.12)$$

We can rewrite the above equation as

$$y = \sum_{s=1}^{\infty}(a_s\cos\frac{s\pi ct}{\ell} + b_s\sin\frac{s\pi ct}{\ell})\sin\frac{s\pi x}{\ell} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.13)$$

Where $a_s = B_s\sin\varphi_s$ and $b_s = B_s\cos\varphi_s$ are new constants.

Equations (7.12) and (7.13) are the general solutions of the differential equation of the transverse vibration of the string in different forms and s represents the modes of vibrations.

Now from equation (7.11) we can write the s th mode of frequency of the vibrating string as

$$f_s = \frac{sc}{2\ell} = \frac{2}{2\ell}\sqrt{T/m} \quad \text{..... ... . (7.14)} \quad [\because \omega = 2\pi f]$$

For s = 1 the fundamental frequency is

$$f_1 = \frac{1}{2\ell}\sqrt{T/m},$$

for s = 2, $f_2 = \frac{2}{2\ell}\sqrt{T/m} = 2f_1$

for s = 3, $f_3 = \frac{3}{2\ell}\sqrt{T/m} = 3f_1$, and so on.

Thus the higher harmonics have frequencies which are integral multiple of f_1.

The frequencies f_s given by equation (7.14) or angular frequencies ω_s in equation (7.11) are called characteristics frequencies or eigen frequencies.

Suppose we have an operator $\hat{A}$ and functions Ψ_i such that $\hat{A}\Psi_i = a_i\Psi_i$ where a_i are constants.

Then Ψ_i are called eigen function and a_i eigen values of the operator $\hat{A}$.

Here the function sin $\frac{s\pi x}{\ell}$ is the eigen function of the operator $\frac{d^2}{dx^2}$.

From equation (7.7) we see that

$$\frac{d^2}{dx^2}\left(\sin\frac{s\pi x}{\ell}\right) = -\frac{\omega^2}{c^2}\sin\frac{s\pi x}{\ell}$$

$$= -\left(\frac{s\pi c}{\ell}\right)^2\frac{1}{c^2}\sin\frac{s\pi x}{\ell} \quad [\because \omega = \frac{s\pi c}{\ell}]$$

$$= -\left(\frac{s\pi}{\ell}\right)^2\sin\frac{s\pi x}{\ell}$$

$\therefore$ The eigen values of the operator $\frac{d^2}{dx^2}$ are $-\left(\frac{s\pi}{\ell}\right)^2$

and the corresponding eigen functions are sin $\frac{s\pi x}{\ell}$

Again the eigen functions of a stretched string are orthogonal because they satisfy orthogonality condition

$$\int_0^{\ell} \sin\frac{s\pi x}{\ell} \sin\frac{k\pi x}{\ell} dx = \delta_{sk} \text{ [where k is an integer]}$$

where $\delta_{sk} = 0$ where $s \neq k$

and $\delta_{sk} = \ell/2$ when s = k

7.3. Stationary waves in a vibrating string

From equation (7.12) we get

$$y = \sum_{s=1}^{\infty} B_s \sin\left(\frac{s\pi ct}{\ell} + \varphi_s\right) \sin\frac{s\pi x}{\ell}$$

or, $y_s = B_s \sin\left(\frac{s\pi ct}{\ell} + \varphi_s\right) \sin\frac{s\pi x}{\ell}$ taking sth mode only.

or, y_s

$$= \frac{B_s}{2}\left[\cos\left\{\left(\frac{s\pi ct}{\ell} + \varphi_s\right) - \frac{s\pi x}{\ell}\right\}\right] - \frac{B_s}{2}\left[\cos\left\{\left(\frac{s\pi ct}{\ell} + \varphi_s\right) + \frac{s\pi x}{\ell}\right\}\right]$$

This equation gives the displacement of the vibrating string at any distance *x* and at any time 1 in 5th mode. Which is the superposition of two waves propagating in two opposite directions.

The resultant amplitude at any point is $B_s \sin\frac{s\pi x}{\ell}$.

It varies with *x* and has maximum values (antinodes)

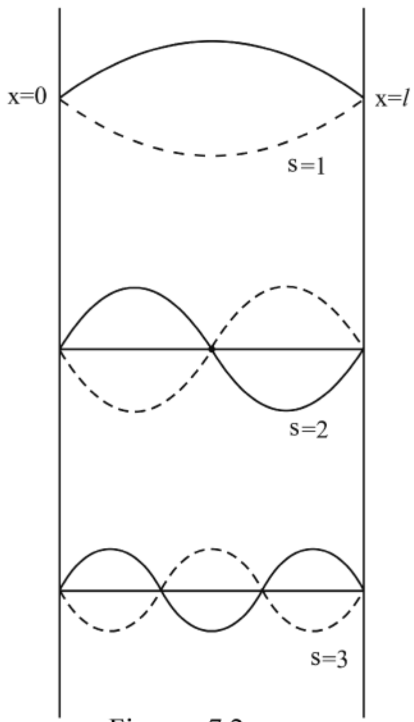


Figure : 7.2

when $\frac{s\pi x}{\ell} = (2n+1)\frac{\pi}{2}$ where n = 0, 1, 2, 3..........

or, $x = (2n+1)\frac{\ell}{2s}$

or, $x = \frac{\ell}{2s}, \frac{3\ell}{2s}, \frac{5\ell}{2s}$..........,

And has minimum values (nodes) when

$$\frac{s\pi x}{\ell} = n\pi \text{ where } n = 0, 1, 2, 3..........$$

or, $x = 0, \frac{\ell}{s}, \frac{2\ell}{s}$.......... ℓ

The stationary wave patterns for few modes are shown in figure - 7.2.

7.4. Energy of a vibrating string

We know from equation (7.12) the displacement y at any point x at any instant at time t for a vibrating stretched string rigidly fixed at both ends is

$$y(x, t) = \sum_{s=1}^{\infty} B_s \sin\frac{s\pi x}{\ell} \sin\left(\frac{s\pi ct}{\ell} + \varphi_s\right)$$

Now kinetic energy at the string is

$$E_k = \int_0^{\ell} \frac{1}{2}(m\,dx)\left(\frac{dy}{dt}\right)^2$$

$$\text{Here, } \frac{dy}{dt} = \sum_{s=1}^{\infty} B_s \frac{s\pi x}{\ell} \cos\left(\frac{s\pi ct}{\ell} + \varphi_s\right)$$

$$\therefore E_k = \frac{m}{2}\int_0^{\ell}\left\{\sum_{s=1}^{\infty}\frac{B_s s\pi c}{\ell}\sin\frac{s\pi x}{\ell}\cos\left(\frac{s\pi ct}{\ell} + \varphi_s\right)\right\}^2 dx$$

$$= \frac{m}{2}\int_0^{l}\frac{\pi^2 c^2}{l^2}[\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\left\{B_s B_n Sn \sin\frac{s\pi x}{\ell}\sin\frac{n\pi x}{\ell}\cos\left(\frac{s\pi ct}{\ell} + \varphi_s\right)\cos\left(\frac{n\pi ct}{\ell} + \varphi_n\right)\right\} dx$$

since $\int_0^{\ell} \sin\frac{s\pi x}{\ell} \sin\frac{n\pi x}{\ell} dx = 0$ for $n \neq s$

$$= \frac{1}{2} \text{ for } n = s$$

$\therefore$ We can write $E_k = \frac{\pi^2 c^2 m}{2\ell^2} \cdot \frac{\ell}{2} \sum_{s=1}^{\infty} B_s^2 s^2 \cos^2\left(\frac{s\pi ct}{\ell} + \varphi_s\right)$

$$\therefore E_k = \frac{m\pi^2 c^2}{4\ell} \sum_{s=1}^{\infty} s^2 B_s^2 \cos^2\left(\frac{s\pi ct}{\ell} + \varphi_s\right) \quad \ldots \quad (7.15)$$

To calculate the potential energy at the element δs in displaced position. The work done against tension when the element is stretched from δx to δs is $T(\delta s - \delta x)$, or the potential energy of the element is $T(\delta s - \delta x)$.

From fiture (7.1)

$$\delta s = \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$= \delta x \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{\frac{1}{2}}$$

$$= \delta x \left\{1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right\} \text{ neglecting higher order terms.}$$

$$\therefore T(\delta s - \delta x) = T\,[\delta x]\left\{1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \delta x\right\}$$

$$= \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \delta x$$

Hence the potential energy of the whole string is

$$E_p = \frac{T}{2}\int_0^{\ell}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$\text{Now}\left(\frac{\partial y}{\partial x}\right)^2 = \left\{\sum_{s=1}^{\infty} B_s \frac{s\pi}{\ell}\cos\frac{s\pi x}{\ell}\sin\left(\frac{s\pi ct}{\ell}+\varphi_s\right)\right\}^2$$

$$\therefore E_p = \frac{mc^2}{2}\int_0^{\ell}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\left\{\frac{ns\pi^2 B_s B_n}{\ell^2}\cos\frac{s\pi x}{\ell}\cos\frac{n\pi x}{\ell}\sin\left(\frac{s\pi ct}{\ell}+\varphi_2\right)\ \sin\left(\frac{n\pi ct}{\ell}+\varphi_n\right)\right\}dx$$

$$\text{Since}\int_0^{\ell}\cos\frac{s\pi x}{\ell}\cos\frac{n\pi x}{\ell}dx = 0 \text{ for } n \neq s$$

$$= \tfrac{1}{2} \text{ for } n = s$$

$$\therefore E_p = \frac{mc^2\pi^2}{2\ell^2}\cdot\frac{\ell}{2}\sum_{s=1}^{\infty}s^2 B_s^2 \sin^2\left(\frac{s\pi ct}{\ell}+\varphi_s\right)$$

$$= \frac{m\pi^2c^2}{4\ell}\sum_{s=1}^{\infty}s^2B_s^2\sin^2\left(\frac{s\pi ct}{\ell}+\varphi_s\right) \quad \ldots \quad (7.16)$$

∴ The total energy of the vibrating strisng is

$$E = E_k + E_p$$

$$= \frac{m\pi^2c^2}{4\ell}\sum_{s=1}^{\infty}s^2B_s^2\left\{\cos^2\left(\frac{s\pi ct}{\ell}+\varphi_s\right)+\sin^2\left(\frac{s\pi ct}{\ell}+\varphi_s\right)\right\}$$

$$= \sum_{s=1}^{\infty}\frac{M\ell}{4\ell^2}c^2s^2B^2\ c^2s^2B_s^2$$

$$= \sum_{s=1}^{\infty}\frac{M\pi^2}{4\ell^2}c^2s^2B^2 \quad \text{where } M = m\ell \text{ is the mass of the string}$$

$$\therefore E = \sum_{s=1}^{\infty} M\pi^2B_s^2\left(\frac{sc}{2\ell}\right)^2$$

$$\therefore\ E = M\pi^2 \sum_{s=1}^{\infty} B_s^2 f_s^2 ....(7.17)\ [\because\ f_s = \frac{sc}{2\ell} \text{ eqn. (7.14) }]$$

Hence, the energy of vibration of a string for a particular mode is proportional to the square of the frequency (f_s) and the square of the amplitude (B_s) of that mode of vibration.

Exercise–1

Consider two strings of same length and same material. Tension in the two strings are in ratio 1 : 4 and the ratio of radii are 1 : 2. Compare the frequencies of the fundamental modes of vibration.

Exercise–2

A heavy chain of length ℓ and mass per unit length m is suspended vartically from one end. A transverse wave is initiated at the upper most end. Show that the time taken by the wave to travel down to the lower end is $2\sqrt{\ell/g}$.

7.5. Plucked String

Let a string of length ℓ is fixed at two ends be plucked at a point P of distance h from the end A, which is taken as the origin (Figure - 7.3). And, let the vertical displacement of the string at P at time t = 0 is k, which is very small. Hence the two portions of the string form two sides of a trangle and then released. The string is said to be plucked and the vibration occurs due to plucking. This type of vibrations can be found in musical instrument like sitar, guitar, tanpura etc.

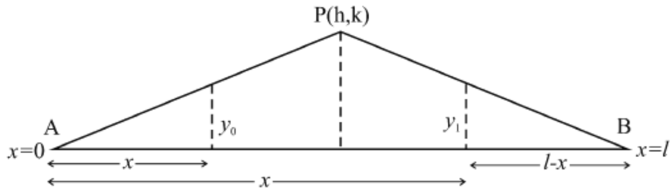


Figure : 7.3

If at t = 0, $y_0(x)$ be the vertical displacement of a point at a distance $x <$ h from A, then from figure – 7.3

We can write

$$\frac{y_0(x)}{x} = \frac{k}{h} \text{ or, } y_0 = \frac{k}{h}x \text{ when } 0 < x h \text{ and for } h < x < \ell$$

$$\frac{y_0(x)}{\ell - x} = \frac{k}{\ell - h} \text{ or, } y_0 = \frac{k}{\ell - h}(\ell - x)$$

Now, the general expression for the displacement of a stretched string at any point x and at any time t is (from equation 7.13).

$$y = \sum_{s=1}^{\infty}(a_s \cos\frac{s\pi ct}{\ell} + b_s \sin\frac{s\pi ct}{\ell})\sin\frac{s\pi x}{\ell} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7.18)$$

Now initially at t = 0 velocity of the string

$$\frac{\partial y}{\partial t} = 0$$

$$\therefore \frac{\partial y}{\partial t} = \sum_{s=1}^{\infty}(-a_s \frac{s\pi c}{\ell}\sin\frac{s\pi ct}{\ell} + b_s\frac{s\pi c}{\ell}\cos\frac{s\pi ct}{\ell})\sin\frac{s\pi x}{\ell}$$

$$\therefore \left.\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{s=1}^{\infty}\frac{b_s s\pi c}{\ell}\sin\frac{s\pi x}{\ell}$$

Since this is true for all values of x at t = 0

$$\therefore b_1 = b_2 = \ldots\ldots\ldots = b_3 = \ldots\ldots\ldots = 0$$

Hence we can write equation (7.18) as

$$y(x, t) = \sum_{s=1}^{\infty} a_s \cos\frac{s\pi ct}{\ell}\sin\frac{s\pi x}{\ell} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7.19)$$

Now at t = 0

$$y_0(x, 0) = \sum_{s=1}^{\infty} a_s \sin\frac{s\pi x}{\ell} \quad \text{i.e. y is function at } x \text{ only}$$

Multiplying both sides at the above equation by sin $\frac{m\pi x}{\ell}$ and integrating w.r.t x from x = 0 to $x = \ell$, we get

$$\int_0^{\ell} y_0 \sin\frac{m\pi x}{l}dx \quad = \sum_{s=1}^{\infty}\int_0^{\ell} a_s \sin\frac{\sin m\pi x}{\ell}dx$$

$$= 0 \text{ for } s \neq m$$

$$= \frac{1}{2}a_s \text{ for } s = m$$

$$\therefore a_s = \frac{2}{\ell}\int_0^{\ell} y_0 \sin\frac{s\pi x}{\ell}dx \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7.20)$$

Putting the values of y_0 we get

$$\int_0^{\ell} y_0 \sin\frac{s\pi x}{\ell}\,dx = \int_0^{h}\frac{k}{h}x\sin\frac{s\pi x}{\ell}\,dx + \int_0^{h}\frac{k(\ell - x)}{\ell - h}\sin\frac{s\pi x}{\ell}\,dx$$

$$= \frac{k}{h}\int_0^{h} x\sin\frac{s\pi x}{\ell}\,dx + \frac{k\ell}{\ell - h}\int_{\ell}^{h}\sin\frac{s\pi x}{\ell}\,dx - \frac{k}{\ell - h}\int_{\ell}^{h} x\sin\frac{s\pi x}{\ell}\,dx$$

$$= \frac{k}{h}\left[-\frac{\ell x}{s\pi}\cos\frac{s\pi x}{\ell} + \frac{\ell^2}{s^2\pi^2}\sin\frac{s\pi x}{\ell}\right]_0^h - \frac{k\ell}{\ell - h}\cdot\frac{\ell}{s\pi}\left[\cos\frac{s\pi x}{\ell}\right]_h^{\ell}$$

$$- \frac{k}{\ell - h}\left[-\frac{\ell x}{s\pi}\cos\frac{s\pi x}{\ell} + \frac{\ell^2}{s^2\pi^2}\sin\frac{s\pi x}{\ell}\right]_h^{\ell}$$

$$= \frac{k}{h}\left[-\frac{\ell h}{s\pi}\cos\frac{s\pi h}{\ell} + \frac{\ell^2}{s^2\pi^2}\sin\frac{s\pi h}{\ell}\right] - \frac{k\ell^2}{(\ell - h)s\pi}\left[\cos s\pi - \cos\frac{s\pi h}{\ell}\right]$$

$$- \frac{k}{\ell - h}\left[-\frac{\ell^2}{s\pi}\cos s\pi + \frac{\ell h}{s\pi}\cos\frac{s\pi h}{\ell} - \frac{\ell^2}{s^2\pi^2}\sin\frac{s\pi h}{\ell}\right]$$

$$= \left[-\frac{k\ell}{s\pi} + \frac{k\ell^2}{(\ell - h)s\pi} - \frac{k\ell h}{(\ell - h)s\pi}\right]\cos\frac{s\pi h}{\ell} + \left[\frac{k\ell^2}{hs^2\pi^2} + \frac{k\ell^2}{(\ell - h)s^2\pi^2}\right]\sin\frac{s\pi h}{\ell} +$$

$$\left[-\frac{k\ell^2}{(\ell - h)s\pi} + \frac{k\ell^2}{(\ell - h)s\pi}\right]\cos s\lambda$$

$$= \frac{-k\ell^2 + k\ell h + k\ell^2 - k\ell h}{(\ell - h)s\pi}\cos\frac{s\pi h}{\ell} + \frac{k\ell^2(\ell - h) + k\ell^2 h}{hs^2\pi^2(\ell - h)}\cdot\sin\frac{s\pi h}{\ell},$$

$$= \frac{k\ell^3}{s^2\pi^2 h(\ell - h)}\sin\frac{s\pi h}{\ell}.$$

$$\therefore\ a_s = \frac{2}{\ell}\int_0^{\ell} y_0 \sin\frac{s\pi x}{\ell}\,dx$$

$$\therefore\ a_s = \frac{2k\ell^2}{s^2\pi^2(\ell - h)h}\sin\frac{s\pi h}{\ell} \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (7.21)$$

Thus we can write equation (7.19) as

$$y = \frac{2k\ell^2}{\pi^2 h(\ell - h)}\sum_{s=1}^{\infty}\frac{\ell}{s^2}\sin\frac{s\pi h}{\ell}\sin\frac{s\pi x}{\ell}\cos\frac{s\pi ct}{\ell} \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots \quad (7.22)$$

It shows that the maximum amplitude of the sth harmonic is inversely proportional to s^2.

Again we see the sth harmonic will be absent if $\sin\frac{s\pi h}{\ell} = 0$.

i.e. when $\frac{s\pi h}{\ell} = n\pi$ where n is any integer.

or, $s = \frac{n\ell}{h}$.

When the string is plucked at the mid point

i.e. if $h = \ell/2$, $s = 2n$ putting $n = 1, 2, 3$...... we see that the 2nd, 4th, 6th and all even harmonics will be absent from the resultant vibration.

Simarlarly if $h = \ell/3$ then 3rd, 6th, 9th......etc., harmonics will be absent.

Hence, if the string be plucked at a point where nodes of certain harmonics fall, than those harmonics will be absent. This is known as Young's law or Young - Helmholtz law.

Effect of touching

If the plucked point is touched immediately after plucking, then all vibrations will be stopped, for those harmonics which have not anode at the point are stopped by damping, where as by Young's law those harmonics which have a mode at the point of plucking are not generated and hence absent from the begining.

7.6. Struck String

You know that in musical instruments like Santoor or in Piano, where the string is struck by a hammer and a sudden impulse is imported to a small part of the string struck and initial velocity at all points is zero, except the region struck. The waves move in opposite directions from the point of struck and get reflected from the fixed ends. Thus we may say that the initial conditions of a plucked string are 'Static', while those of struck string are dynamic.

The general expression for the displacement of a stretched string at any point x and at any time t is given from equation (7.13) as

$y(x, t) =$ .... (7.23)

Let the string is struck by a hammer over an infinitelly small region from $x = h$ to $x = h + \Delta x$ and let the instantaneous velocity imparted to the region at time t = 0 be υ_0

Hence at t = 0, y = 0 and velicity $\left(\frac{\partial y}{\partial t}\right) = 0$ for all values of x except the small region of striking Δx where it is υ_0.

$\because$ At t = 0, y = 0, then from equation (7.23) we get

$$\sum_{s=1}^{\infty} a_s \sin\frac{s\pi x}{\ell} = 0$$

$$\therefore\ a_s = 0\ \left[\because \sin\frac{s\pi x}{\ell} \neq 0\right]$$

$\therefore$ Equation (7.23) will be

$$y = \sum_{s=1}^{\infty} b_s \sin\frac{s\pi ct}{\ell}\sin\frac{s\pi x}{\ell} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (7.24)$$

Now $\frac{\partial y}{\partial t} = \sum_{s=1}^{\infty} b_s \frac{s\pi c}{\ell}\cos\frac{s\pi ct}{\ell}\sin\frac{s\pi x}{\ell}$

At, t = 0 we can write

$$\left.\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{s=1}^{\infty} b_s \frac{s\pi c}{\ell}\sin\frac{s\pi x}{\ell}$$

The constant b_s can be determined by the Fourier's theorem method i.e. multiplying both sides of the above equation by $\sin\frac{s\pi x}{\ell}$ and integrating from $x = 0$ to $x = \ell$ we get

$$\int_0^{\ell}\left(\frac{\partial y}{\partial t}\right)_{t=0}\sin\frac{s\pi x}{\ell}dx = \int_0^{\ell}\left[\sum_{s=1}^{\infty}\frac{b_s s\pi c}{\ell}\sin^2\frac{s\pi x}{\ell}dx\right]$$

since $\int_0^{\ell} \sin^2 \frac{s\pi x}{\ell} dx = \frac{1}{2}$

∴ We can write

$$\frac{s\pi c b_s}{\ell} \cdot \frac{\ell}{2} = \int_0^{\ell} \left(\frac{\partial y}{\partial t}\right)_{t=0} \sin \frac{s\pi x}{\ell} dx$$

or, $\frac{s\pi c b_s}{2} = \int_h^{h+\Delta x} \upsilon_0 \sin \frac{s\pi x}{\ell} dx$ $\left[\because \frac{\partial y}{\partial t}\right)_{t=0} = 0$ except the region h to h + Δx

$$= v_0 \frac{\ell}{s\pi}\left[-\cos\frac{s\pi x}{\ell}\right]_h^{h+\Delta x}$$

$$= \frac{\upsilon_0 \ell}{s\pi}\left[= \cos\frac{s\pi}{\ell} + (h + \Delta x + \cos\frac{s\pi h}{\ell}\right]$$

$$= \frac{\upsilon_0 \ell}{s\pi}\left[= \cos\frac{s\pi h}{\ell} - \cos\frac{s\pi \nabla x}{\ell} + \sin\frac{s\pi h}{\ell} \sin\frac{s\pi \Delta x}{\ell} + \cos\frac{s\pi h}{\ell}\right]$$

$$= \frac{\upsilon_o \ell}{s\pi} \cdot \frac{s\pi \Delta x}{\ell} \sin\frac{s\pi h}{\ell}$$ [∵ Δ x is very small

$\therefore \cos\frac{s\pi \Delta x}{\ell} = \ell$ and $\sin \frac{s\pi \Delta x}{\ell} = s\pi \Delta x/1$]

$$\therefore b_s = \frac{2\upsilon_0 \Delta x}{s\pi c} \sin\frac{s\pi h}{\ell}$$

$$\therefore y = \frac{2 v_0 \Delta x}{\pi c} \sum_{s=1}^{\infty} \frac{\ell}{s} \sin\frac{s\pi h}{\ell} \sin\frac{s\pi x}{\ell} \sin\frac{s\pi c t}{\ell} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7.24)$$

Thus the amplitude of the sth node of vibration is inversely proportional to s.

Here, due to presence of the term $\sin\frac{s\pi h}{\ell}$ in the expression of displacement (y) the 2nd, 4th, 6th.........etc. all even harmonics will be absent, if the string is stuck at the mid point and 3rd, 6th, 9th.......etc. harmonics will be absent, if the string is struck at h = $\ell/3$. Hence Young's law is also applicable here.

Effect of touching the string immediately after strucking is same as plucked string.

7.7. Summary

- Differential equation for transverse wave along the stretched string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

- Velocity of the wave $c = \sqrt{T/m}$
- Solution of the wave equation is

$$y = \sum_{s=1}^{\infty}\left(a_s \cos\frac{s\pi ct}{\ell} + b_s \sin\frac{s\pi ct}{\ell}\right)$$

- Frequency of a vibrating string for Sth mode

$$f_1 = \frac{sc}{2\ell} = \frac{s}{2\ell}\sqrt{T/m}$$

$f_1 = \frac{1}{2\ell}\sqrt{T/m}$ is the frequency of the fundamental.

- Energy of the vibrating string is

$$E = M\pi^2 \sum_{s=1}^{\infty} B_s^2 f_s^2.$$

- The general expression for displacement in plucked string is

$$y = \frac{2k\ell^2}{\pi^2 h(\ell - h)} \sum_{s=1}^{\infty} \frac{\ell}{s^2} \sin\frac{s\pi h}{\ell} \sin\frac{s\pi x}{\ell} \cos\frac{s\pi ct}{\ell}$$

and for struck string is

$$y = \frac{2v_0 \Delta x}{\pi c} \sum_{s=1}^{\infty} \frac{\ell}{s} \sin\frac{s\pi h}{\ell} \sin\frac{s\pi x}{\ell} \sin\frac{s\pi ct}{\ell}$$

7.8. Questions and problems

7.8.1. Find the ratio of intensity of the fundamentals at $\ell/2$ and $\ell/4$ when the string is struck at $\ell/3$.

7.8.2. A stretched string of length ℓ is plucked through K at a distance $\ell/3$ from one end. Find the maximum aplitude of vibration.

7.8.3. A 2m long wire having a linear mass density 0.0025 kg/m is stretched between two fixed supports such that stwo adjacent harmonic frequencies are 252 Hz and 336 Hz. Calculate a) the fundamental frequency and b) the tension of the wire.

7.9. Solutions

Exercise–1

We know the frequency of fundamental is

$$f = \frac{1}{2\ell}\sqrt{T/m} = \frac{1}{2\ell}\sqrt{\frac{T}{\pi r^2 \rho}}$$

Where T = Tension, r = radius, ρ = density of the wire.

Now the ratio of frequencies for two strings

$$\frac{f_1}{f_2} = \frac{1}{2\ell}\sqrt{\frac{T_1}{\pi r_1^2 \rho}} \times 2\ell\sqrt{\frac{\pi r_2^2 \rho}{T_2}} \quad [\text{ Since } \ell \text{ and } \rho \text{ are same}]$$

$$= \sqrt{\frac{T_1}{T_2} \cdot \frac{r_2^2}{r_1^2}}$$

Here $\frac{T_1}{T_2} = \frac{1}{4}$ and $\frac{r_1}{r_2} = \frac{1}{2}$

$$\therefore \frac{f_1}{f_2} = \sqrt{\frac{1}{4} \cdot \frac{2^2}{1}} = 1$$

$\therefore f_1 = f_2$ i.e. the frequencies are same.

Exercise–2

Let an element dx of the chain at a distance x from the upper most end.

Therefore, the tension T at this point is due to the weight at the lower part of the chain of length (ℓ – x)

$\therefore$ T = (ℓ – x)m.g

Now, the velocity at the transverse wave at this point is

$$c = \sqrt{T/m} = \sqrt{\frac{(\ell - x).m.g.}{m}}$$

or, $$c = \frac{dx}{dt} = \sqrt{(\ell - x)g}$$

or, $$dt = \frac{dx}{\sqrt{(\ell - x)g}}$$

$\therefore$ Time required to travel the wave from $x = 0$ to $x = \ell$ is

$$t = \int_0^{\ell} \frac{dx}{\sqrt{(\ell - x)g}} = \frac{1}{\sqrt{g}}\left[-2(\ell - x)^{\frac{1}{2}}\right]_0^{\ell}$$

$$= \frac{2}{\sqrt{g}}\left[-(\ell - \ell) + \ell\right]^{\frac{1}{2}} = 2\sqrt{\ell/g}$$

7.8.1. We know for struck string

$$y = \frac{2\upsilon_0 \Delta x}{\pi c}\sum_{s=1}^{\infty}\frac{1}{s}\sin\frac{1.\pi.\ell}{\ell}\sin\frac{1.\pi.\ell}{2\ell}$$

Here, h = $\ell/3$, x = $\ell/2$, s = 1 for first case

$$\text{Amplitude } (A_1) = \frac{2\upsilon_0 \Delta x}{\pi c}.\frac{1}{1}\sin\frac{1.\pi.\ell}{\ell}\sin\frac{1.\pi.\ell}{2\ell}$$

$$= \frac{2\upsilon_0 \Delta x}{\pi c}\sin \pi/3$$

For second case, h = $\ell/3$, x = $\ell/4$, s = 1

$$\therefore \text{Amplitude } (A_2) = \frac{2\upsilon_0 \Delta x}{\pi c}\sin\frac{1.\pi.\ell}{3\ell}\sin\frac{1.\pi.\ell}{4\ell}$$

$$= \frac{2\upsilon_0 \Delta x}{\pi c} \sin\frac{\pi}{3} . \sin\frac{\pi}{4}$$

$$\therefore \frac{A_1}{A_2} = \frac{1}{\sin \pi/4} = \frac{1}{1/\sqrt{2}} = \sqrt{2}$$

$\therefore$ The ratio of intensities is $\frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 = \left(\frac{\sqrt{2}}{1}\right)^2 = \frac{2}{1}$

7.8.2. We know the displacement for sth mode

$$y = \frac{2k\ell^2}{\pi^2 \mathbf{h}(\ell - h)} \frac{1}{s^2} \sin\frac{s\pi h}{\ell} \sin\frac{s\pi x}{\ell} \cos\frac{s\pi ct}{\ell}$$

Here, h = $\ell/3$ and for maximum amplitude $\sin\frac{s\pi x}{\ell}$ = 1, and s = 2.

$\therefore$ The maximum amplitude

$$A = \frac{2k\ell^2}{\pi^2 h(\ell - h)} . \frac{1}{S^2} \sin\frac{s\pi h}{\ell} .$$

$$= \frac{2k\ell^2 . 3}{\pi^2 \ell(\ell - \ell/3)} . \frac{1}{2^2} \sin\frac{2\pi\ell}{3\ell}$$

$$= \frac{6k\ell^2 . 3}{\pi^2 \ell . 2\ell} \frac{1}{4} \sin\frac{2.\pi}{3}$$

$$= \frac{9k}{\pi^2 4} \sin 120^0$$

$$= \frac{9k}{4\pi^2} . \frac{\sqrt{3}}{2}$$

$$\therefore A = \frac{9\sqrt{3}k}{8\pi^2}$$

7.8.3. We know, the length of the wire $\ell = s\lambda/2$

or, $\lambda = 2\ell/s$

$\therefore$ The frequency $f_s = \frac{c}{\lambda} = \frac{cs}{2\ell}$

Here $f_s = \frac{cs}{2\ell} = 252$

$\therefore \frac{c.s}{2.2} = 252$

or, $\frac{cs}{4} = 252$

and $\frac{c(s+1)}{4} = 336$

$\therefore \frac{s+1}{s} = \frac{336}{252}$ or, $1 - \frac{1}{s} = \frac{336}{252}$

or, $\frac{1}{s} = 1 - \frac{336}{252}$ $\quad \therefore s = 3$

$\therefore$ Velocity (c) $= \frac{252 \times 4}{s} = \frac{252 \times 4}{3} = 336$ m/s.

$\therefore$ Fundamental frequency of the wire

$$f_1 = \frac{c}{2\ell} = \frac{336}{2.2} = 84 \text{ Hz.}$$

$\therefore$ Velocity (c) $= \sqrt{T/m}$ or, $T = mc^2 = 0.0025 \times 336^2$

$\therefore$ Tension (T) = 282.24 N.

Unit : 8 ❐ Acoustics of Buildings

Structure

8.0 Objectives

After studying this unit you wil be able to

- define reveberation reverberation time.
- learn the term absorption co-efficient, live room, dead rom.
- complete the growth and decay of sound and hence the reverberation time i.e. Sabine formula and Erying's formula.
- desigin a good auditorium.

8.1 Introduction

The branch of physics that deals with the process of production, reception and propagation of sound is called acousties.

In this unit you will study. how to construct a conference room. concert hall, auditorium etc, from an acoustical point of view. The aim of this design is to reduce the noise and obtaining optimum listening condition. Sabine from his experimental studies found that the acoustical properties of a room depends on the reverberation chracteristics.

8.2 Reverberation

It is observed that when a source is sounding continuously of constant intensity in a closed room, the sound waves spread over in all directions. The waves receive by the listenters are (i) direct waves and (ii) reflected waves due to multiple reflections from the walls, ceiling and floor of the hall. The sound energy in the room increases to a certain maximum value, when the steady state is reached between the rate of energy emitted from the source and that of absorbed by the walls and other materials in the room. Now if the source of sound in cut off, the listeners get the impression of prolongation of persistence of sound for some time after the original source of sound has ceased, because the sound dies away slowly due to absorption within the hall. This prolongation or persistence of sound due to repeated reflection is called reverberation.

The time taken for the sound to fall below hte minimum audibility after the source of sound is cut off is called reverberation time. The reverberation time depends on the size of the room or auditorium, the nature of the reflecting material on the wall, the ceiling and the area of the flecting surfaces.

Accourding to W. C sabine, the reverbertion time T is the time required for sound to fall from the initial intensity to one-millionth of its initial value, i.e., to fall by 60dB in loudness.

So the relation is

$$\frac{I}{I_0} = e^{-KT} = 10^{-6}$$

$$\text{or, } \frac{I_0}{I} = e^{-KT} = 10^{6}$$

$$\therefore\ T = \frac{1}{K}\ln\frac{I_0}{I} = \frac{1}{K}\ln 10^6 = \frac{2.303}{K}\log_{10} 10^6$$

$$= \frac{2.303 \times 6}{K} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8.1)$$

where I_0 is the initial intensity I is the intensity at time T and K is a constant.

8.3 Absorption co-efficient

The acoustic absorption co-efficient for the material of a surface is defined as the ratio of the sound energy absorbed by the surface of the material to the incident energy.

If a_1, a_2, a_3 ...... etc are the absorption co-efficient at each reflection of the surface having areas ds_1,ds_2,ds_3..... etc respectively in the room, then the average value of absorption co-efficient $\bar{a}$ is

$$\bar{a} = \frac{a_1 ds_1 + a_2 ds_2 + a_3 ds_3 + \dots\dots\dots}{ds_1 + ds_2 + ds_3 + \dots\dots\dots\dots} = \frac{\sum_i a_i ds_i}{\sum_i ds_i}$$

$$\therefore \quad \frac{\sum_i a_i ds_i}{S} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (3.2)$$

where S is the total area of all surface.

The standard unit of absorption co-efficient is Sabine, which is the amount of sound energy absorbed by one square-foot of a perfectly absorbing surface i.e., an open window.

Live room and dead room

If the loudness in a room increase the reverberation, then the room is called a live room. In this case the absorption co-efficient is less than 0.4. But when the absorption co-efficient is more than 0 .4,then the reverberation is very small and the rom is called dead room. Again if the absorption co-efficient is unity then there will be no reverberation the room is called a perfectly dead room.

8.4 Reverberation formula from live room

Sabine developed the reverberation formula to expression the growth and decay of sound in an auditorium on the following assumptions :

(i) the emission of sound energy by the source in all directions are at constant rate.

(ii) there is no interference of sound waves.

(iii) sound energy is uniformly distributed in the enclosure and absorption co-efficients are independent of the intensity of the incident sound.

Let u be the energy density of the sound field in the enclosure of any instant of time t. Consider an elementary volume dV at a distance r from an elementary area ds taken on the wall, r makes an angle θ with the normal to ds as in figure 8.1

The energy contained in dV is udV. Since the energy distribution is uniform in all directions, then the amount of energy reaching ds from dV will be

$dE = udv\frac{d\omega}{4\pi}$ where dω is the solid angle made by ds at the volume elements

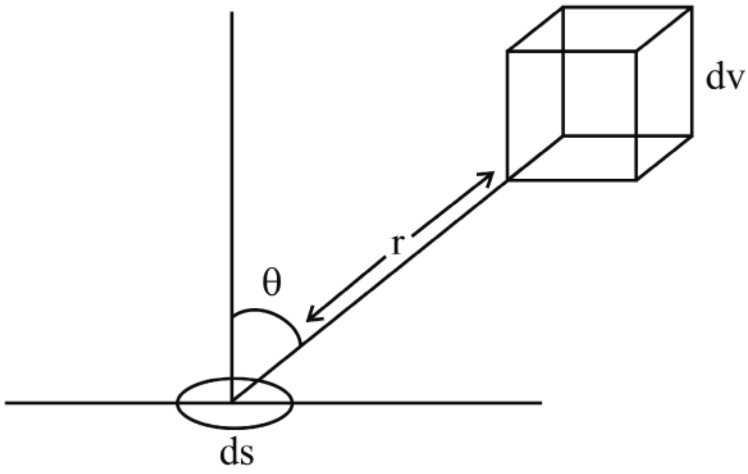


Figure : 8.1

$$\therefore \; dE = \frac{u}{4\pi} r^2 \sin\theta \, dr d\theta d\phi \cdot \frac{ds\cos\theta}{r^2}$$

Here $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

in spherical co-ordinate and $d\omega = \dfrac{ds\cos\theta}{r^2}$

$$\therefore \; dE = \frac{uds}{4\pi} \sin\theta\cos\theta \, dr \, d\theta \, d\phi \quad \text{... ... ...} \quad (8.3)$$

So, the energy passing per second through ds from the front side will be that contained in the hemisphere at radius c, where c is the velocity of sound.

Then $E = \dfrac{uds}{4\pi} \displaystyle\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \int_0^{C} dr \int_0^{2\pi} d\phi$

$$= \frac{uds}{4\pi} \cdot \frac{1}{2} - c.2\pi = \frac{uc}{4} ds$$

$\therefore$ The intensity of sound $I = \dfrac{E}{ds} = \dfrac{uc}{4}$

If the areas of different materials are ds_1, ds_2, ds_3...... etc. with absorption co-efficients a_1,a_2,a_3....... etc of the walls of the enclosure,

Then a = absorptive power = $\displaystyle\sum_{i=1}^{n} a_i ds_i$

Hence, the total rate of absorption of energy by the walls is

$$E_a = \frac{uc}{4}\sum_{i=1}^{n} a_i ds_i = \frac{auc}{4}$$

Now, if energy produced per second by the source be P, then according to the conservation principle, the rate of increase in energy of the whole enclosure of volume v will be

$$V\frac{du}{dt} = P - \frac{auc}{4} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8.4)$$

$$\text{or,}\quad \frac{vdu}{P - \frac{auc}{4}} = dt \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8.5)$$

Now, at t = 0, u = 0 and let after time

u = E, then integrating equation (8.5) we get

$$\int_0^t dt = v\int_0^E \frac{du}{P - \frac{auc}{4}}$$

$$\text{or,}\quad t = -\frac{4v}{ac}\left[\ell n\left(P - \frac{auc}{4}\right)\right]_0^E$$

$$= -\frac{4v}{ac}\left[\ell n\left(P - \frac{aEc}{4}\right) - \ell n\, P\right]$$

$$= -\frac{4v}{ac}\ell n\frac{P - aE\frac{c}{4}}{P}$$

$$\text{or,}\quad \frac{P}{P} - \frac{aEc}{4P} = e^{-cat/4v}$$

$$\text{or,}\quad \frac{aEc}{4P} = 1 - e^{-cat/4v}$$

$$\text{or,}\quad E = \frac{4P}{ac}\left(1 - e^{-cat/4v}\right)$$

$$\text{at } t = \infty,\ E = E_{max} = \frac{4P}{ac}$$

$$\therefore\ E = E_{max}\left(1 - e^{-cat/4v}\right) \quad \dots \quad (8.6)$$

Now, $I = \frac{4c}{4}$, so at time t, u = E and at t = ∞, E= E_{max}

$\therefore\ I = \frac{Ec}{4}$, $I_{max} = \frac{c\,E_{max}}{4}$, here I = intensity at time t.

∴ From equation (8.6) we get

$$I = I_{max}\left(1 - e^{-cat/uv}\right) \quad \dots \quad (8.7)$$

The I -t graph has been shown in the figure (8.2)

Decay of intensity :

When the intensity or energy density in the room attains its maximum value, the source of sound in cut off, then the energy density decays with time.

Now, to determine how energy density decay, consider the equation(8.4) by putting P = 0

We get,

$$v\frac{du}{dt} = \frac{auc}{4}$$

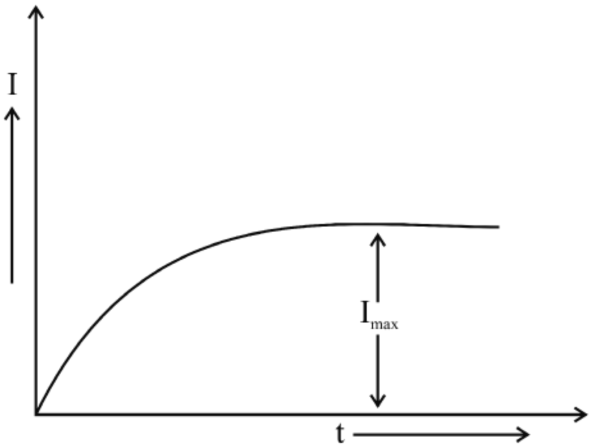


Figure : 8.2

$$\text{or,}\ dt = -\frac{4v}{ac}\frac{du}{u} \quad \dots \quad (8.8)$$

Now, at t = 0, u = E_{max} and let after time t, u = E, then integrating equation (8.8) between these limits.

$$\text{We get } \int_0^t dt = -\frac{4v}{ac}\int_{E_{max}}^{E}\frac{du}{u}$$

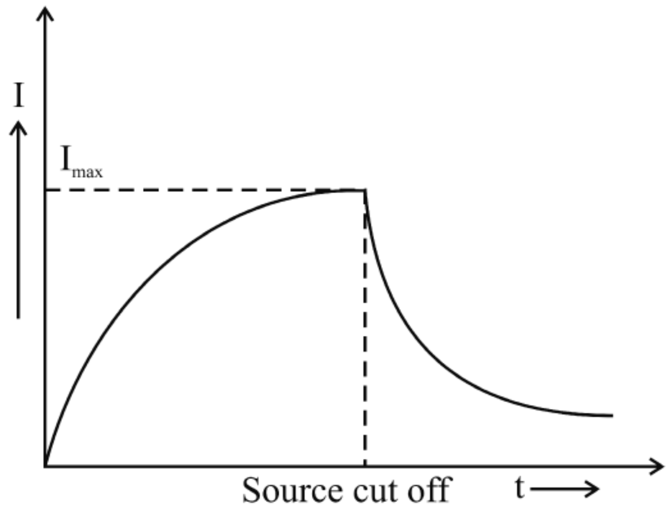


Figure : 8.3

$$\text{or, } t = -\frac{4v}{ac} \ln\frac{E}{E_{max}}$$

$$\text{or, } \frac{E}{E_{max}} = e^{-\frac{cat}{4v}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8.9)$$

$$\text{And } I = I_{max} e^{-cat/4v} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8.10)$$

The I – t graph has been shown in figure 8.3.

8.5 Reverberation time

From figure-8.3 we see that the intensity of sound in the room decays exponentially, Now we know the reverberation time is t = T, when $I = 10^{-6} I_0 = 10^{-6} I_{max}$ i.e., when the intensity is 10^{-6} the times of maximum intensity.

From equation (8.10) we get

$$\frac{I}{I_{max}} = e^{-cat/4v}$$

$$\text{or, } 10^{-6} = e^{-eaT/4v}$$

$$\text{or, } \ln\left(10^{6}\right) = \frac{caT}{4v}$$

or, $\frac{6\times2.303\times4v}{ca}=T$

$\therefore T=\frac{KV}{a}$ ... (8.11) where $K=\frac{6\times4\times2.303}{C}$

Equation (8.11) agrees with the sabine's empirical formula for reverberation time.

8.6 Eyring's formula for live room

From equation (8.11) we see that for a = 1, T is finite but not equal to zero, because for a perfectly dead room (a = 1) absorption is complete, so T must be equal to zero.

According to Eyring, every time the sound strikes the wall, a fraction of $\bar{a}$ of its intensity is absorbed and $(1-\bar{a})$ fraction is reflected. So, intensity of sound in the room after successive flection becomes.

$I_0(1-\bar{a}),\ I_0(1-\bar{a})^2 ......\ I_0(1-\bar{a})^n$, when I_0 is the initial intensity and $\bar{a}$ is the mean absorption. co-efficeient. So, after the source is cut all, the intensity after nth reflection becomes $I_0(1-\bar{a})^n$ and if after this it attains threshold audibility, then we get,

$\frac{I}{I_0}=10^{-6}$ where $I=I_0(1-\bar{a})^n$

or, $(1-\bar{a})^n=10^{-6}$

or, n $\log_e(1-\bar{a})=-6\times2.303$

$\therefore n=\frac{-6\times2.303}{\ln(1-\bar{a})}$

Now, if λ be the mean free path of sound in the room, i.e., the average distance traversed between two succesive reflections on the walls,

we get, $\lambda=\frac{V}{\frac{S}{4}}=\frac{4V}{S}$ where s = total surface area.

Again $n\lambda = cT$, where T = reverberation time

$\therefore\ n=\frac{cT}{\lambda}=-\frac{2.303\times6}{\ln(1-\bar{a})}$

or, $c.T.\frac{s}{4V} = -\frac{2.303\times 6}{\ln(1-\bar{a})}$

or, $T = -\frac{2.303\times 6\times 4V}{cs\ln(1-\bar{a})} = -\frac{KV}{\sin(1-\bar{a})}$ when $K = \frac{2.303\times 6\times 4}{c}$

$$\therefore \quad T = -\frac{kV}{\sin(1-\bar{a})} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (8.12)$$

This is known as Eryring's formula

Now,

$\ln(1-\bar{a}) = -\bar{a} = \frac{\bar{a}^2}{2} - \frac{\bar{a}^3}{2}\ldots\ldots\ldots \approx \bar{a}$ for $\bar{a}$ is small.

$T = \frac{KV}{S\bar{a}} = \frac{KV}{a}$ which is Sabine's formula.

But when $\bar{a} = 1$ then $\ln(1-\bar{a}) = -\infty$

$\therefore$ From equation (8.12) T = 0

Let

[$x = \ln(1-\bar{a})$

or, $1-\bar{a} = e^x$

when $\bar{a} = 1$

then $e^x = 0$

when $x = -\infty$

$e^{-\infty} = 0$]

Thus we can say Sabines formula is only for live room $(\bar{a} > 0.4)$, but Eyring's formula is valid for live room and also for dead room.

Exercise-I

The volume of a room is 500 m^3. The wall area of the room is 220 m^2 , the floor area is 120m^2 and the ceiling area is 120 m^2. The average absorption co-efficient for the walls is 0.03, ceiling is 0.08 and floor is 0.06.

Find the averge reverberation time.

Given velocity of sound 350 m/s.

Exercise-2

Find the reverberation time of an auditorium of volume 3000 m^3 and a total sound absorption of 70 metric Sabine. What is the additional sound absorption required for an optium reverberation time 4.2 see ? Give velocity of sound is 350 m/s.

8.7 Design of a good auditorium

The following are the requirement of a good auditorium

(i) Adequate loudness

(ii) Suitable reverberation time.

(iii) Uniform distribution of sound i.e., absence of echoes and focussing of sound.

(iv) Resonance must be as small as possible.

(v) Interference effect must be absent.

8.8 Summary

The reverberation time is defined as the time required for sound to fall from initial intensity to 10^{-6} times of its initial value or to fall by 60 dB in loudness.

Energy of growth of sound in a hall is written as

$$E = E_{max}\left(1 - e^{-cat/4v}\right)$$

And intensity $I = I_{may}\left(1 - e^{-cat/4v}\right)$

For decay of sound

$E = E_{max}e^{-cat/4v}$ and $I = I_{max}e^{-cat/4v}$

Reverberation time in live room

$T = \frac{Kv}{a}$, where $K = \frac{24 \times 2.303}{c}$

Erying's formula for reverberation time

$$T = -\frac{kV}{s\ln(1-\bar{a})}$$

8.9 Quesions and Problems

8.9.1 A hall of volume 3300 m^3 is found to have a reverberaton time of 3.3 sec. The second absorbing surface was an area of 750 m^3 calculate the average absorption co-efficient. Given velocity of sound = 350 m/s.

8.9.2 The reverberation time in a hall measuing 16.0 × 14.0 × 20.0 m^3 is 1.6 sec, when it is empty. What will be the reverberation time in the hall when an audience of 250 person is present?

Assume that sound absorption by each person is 4 metric Sabine.

8.10 Solution

Exercise-I

The average sound absorption co-efficient

$$\bar{a} = \frac{\sum_i a_i ds_i}{\sum_i ds_i} = \frac{a_1 ds_1 + a_2 ds_2 + a_3 ds_3}{ds_1 + ds_2 + ds_3}$$

Here, $a_1 = 0.03$, $ds_1 = 220\ m^2$

$a_2 = 0.03$, $ds_2 = 120\ m^2$

and $a_3 = 0.03$, $ds_3 = 120\ m^2$

$$\therefore\quad \bar{a} = \frac{0.03\times 220 + 0.08\times 120 + 0.06\times 120}{220+120+120}$$

$$= \frac{224}{460}$$

$$\therefore\quad a = \bar{a}\times 460 = 23.4$$

Now, reverberation lime

$$T = \frac{KV}{a} = \frac{0.158\times 500}{23.4} = 3.38\ \text{sec.}$$

Here, $K = \dfrac{24 \times 2.303}{c}$ c = 250 m/s

$= \dfrac{24 \times 2.303}{350} = 0.158$

$a = \sum_i \bar{a}_i ds_i$

v = 500 m^3

Exercise-2

We know reverberation time

$T = \dfrac{0.158v}{a}$ Here, in S. I. Unit K = 0.158 for c = 350 m/s.

V = 3000 m^3,a = 70 metric - sabine

$\therefore\ T = \dfrac{0.158 \times 3000}{70} = 6.77$ sec

Let, a′ is the total absorption for reverberation time T′ = 4.2 sec

∴ T′ = 4.2 sec

$\therefore\ T' = \dfrac{0.158v}{a^1}$

or, $a' = \dfrac{0.158 \times 3000}{4.2} = 112.86$ metric Sabine

∴ Additional sound absorption required = 112.86 – 70 = 42. 86 metric Sabine

8.91 Here, v = 3300 m^3,T = 3.3 sec, s = 750 m^2

Let the average absorption co-efficient as $\bar{a}$

We know, $T = \dfrac{Kv}{a} = \dfrac{Kv}{\bar{a}s}$, $K = \dfrac{24 \times 2.303}{c} = 0.159$ c = 350 m/s

$\therefore T = \dfrac{0.158 \times 3300}{750\bar{a}}$

$\therefore \quad \bar{a} = \dfrac{0.158 \times 3300}{750 \times 3.3} = 0.21$

8.9.10 Here v = 16 × 14 × 20 = 4480 m^3, T = 1.6 sec

we know, $T = \dfrac{0.158v}{a}$

$\therefore$ Total absorption for empty hall is

$a = \dfrac{0.158v}{T} = \dfrac{0.158 \times 4480}{1.6}$ =442.4 metric Sabine

Now, total absorption when the audience is present

$a' = (442.4+250 \times 4) = 1442.4$ metric Sabine

$\therefore$ The new reverberation time $T' = \dfrac{0.158 \times 4480}{1442.4} = 0.49$ sec

Unit : 9 □ Wave Optics

Structure

9.0 Objectives

- After studying this unit you willbe able to
- know what is light ?
- define the wave front and propagation of wave front using Huygen's princple
- explain laws of reflection and laws of refraction from the wave theory of light.
- Compute the lens formula from wave theory.

9.1 Introduction

In the previous units, you have gathered some knowledge about the wave nature of sound. Sound wave can be classified as transverse or longitudinal depending upon the direction of vibration of particles relative to the direction of propagation of the wave. In fact, we can classify waves in many ways. As for example, we have mechanical and non-mechanical waves depending on whether a wave needs a medium for propagation or not. Sound wave and water waves are mechanical waves areas as light waves are not. You have also some idea about reflection and refraction of light, which was explained by assuming that light is propagating in a straight line. But wave nature at light canalso explain these phenomena. In this unit we shall discuss how wave nature of light can explain the different physical phenomena of light.

9.2 Nature of light

The wave theory of light proposed by Hungers was very successfuly to explain some aptical phenomena. But to explain the propagation of light through vacuum it had to assume the existence of an all pervating ether medium. If E be the modules of elasticity and ρ, the density of such medkium, then the velocity of waves travelling through the medium is $v = \sqrt{\frac{E}{\rho}}$. Roman calculated the velocity of light (2×10^8 m/s) during eclipse of planet Jupiter due to its satellite (10). To account this high value of velocity of light, ether must possess high elasticity and lowd ensity. But these two properties are contrasy to each other. Experiments had been carried out to detect the presence of ether medium, but failed from the theory of relativity, we know that ether does not exists.

To overcome this difficulty a completes new concept regarding the nature of light wave was proposed by clerk Maxwell. According to him, light is a transverse electromagnetic wave. A changing magnetic field produces a changing electric field and vice vera.

That means when either magnetic field or electric fields changes with time, the other field is induced in space. This leads togenerate electro magnetic disturbance and no material medium is required to propagate this disturbance i.e., it can propagate in free space. These disturbance have the properties of a wave and are called electromagnetic waves. The velocity of these waves in free space is $C = \frac{1}{\sqrt{\mu_0 \in_0}}$ where

$\mu_0 = 4\pi \times 10^{-7}$ henry/m and $E_0 = 8.854 \times 10^{-12}$ farad/m. Therefore

$\therefore c = 3 \times 10^8$ m/s

The existance of electric magnetic waves was experimentally demonstrated and verified by Heritz. But this theory could not explain the photoelectric effect. Photoelectric effect was explained by Einstaein with the photon theory of light.

Thus we may conclude that there are some phenomena which can be explained by wave theory of light and some can be expaine on the basis of photon or particle nature of light. So, the nature of light is such that is possess the dual character i.e. sometimes it behaves as wave and sometimes as particle. Hence, we can say that the wave and particle nature of light are complementary to each other.

9.3 Wave Fron

When a disturbance is caused in a medium, the particles of the medium vibrate and the continuous locus of all particles vibrating in same place at any instant of time is called wave front.

You have seen that, when a piece of stone be thrown in a pond of still water, ripple are generated all around that point. The ripple consist of concetric circular troughs and crests, on which the particle of water vibrates in same phase. The wavefront in this case is circular.

Thus a wavefront is a surface of constant phase.

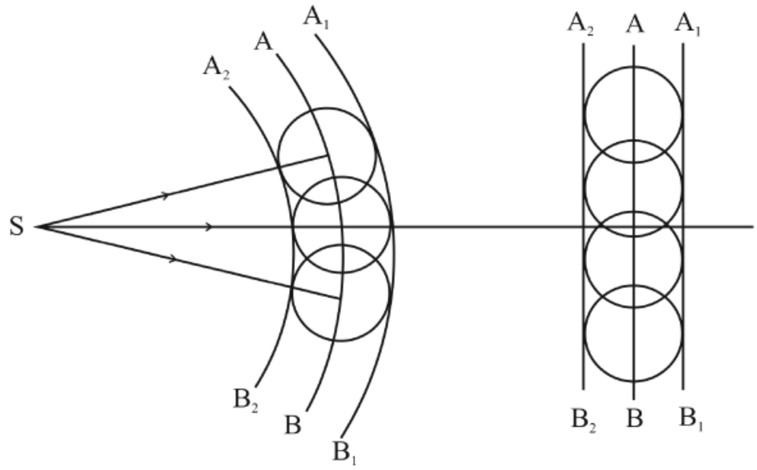


Figure : 9.1

Now in a homogeneous istropic medium a point source of light sens waves in all directions travelling with some speed, so that they al arrive simultaneously at she surface of a sphere wil the point source at its centre. Hence, the wavefront in a sphere. But if the point source is at a large distance, then a small portion of the spherical wavefront may be considered to be a plane as shown in figure 9.1. In that case, it is a plane wavefront. For a line source the wavefront will be cylindrical.

9.4 Huygens principle and propagation of wavefront

Let S is a point source of light from which light waves are propagated in all directions At an instant of time t′, all the particles in the medium will be on the spherical wave surface AB are vibrating in the same phase. AB is thus the portion, which have been drawn with S as centre and radius ct′, where c is the velocity of light. The surface AB is called the primary wavefront. To obtain the position of the new wavefront after time t, we can apply Hunger's Principle.

According to Hunger's principle, "every points on the primary wave front is the source of a new disturbance. It sends small secondary wavelets in all directions. These wavelets are spherical and move forward in a homogeneous medium with same velocity. At any instant the position and shape of the new wavefront can be obtained by drawing a surface enveloing the secondary wavelets."

To find the position of he new wavefront after t seconds, take a number of points on AB, with each point as centre and radius ct, sphere are drawn in turn. These spheres represent the secondary waves starting from these points A surface.

A_1 B_1 touching all these spheres in the forward direction is the new wave front as shownin figure 9.1. The construction can be repeated with A_1 B_1 to get the next wave front and so on.

In figure 9.1 we see that A_2B_2 is also on enveloping surface to the secondary wavelets. That is the wavefront moving towards the source. This is however, not an experimentally observed fact. Hence the wavefront does not propagate backward.

9.4.1 Laws of reflection from the wave theory :

Let PQ be a plane reflecting surface and AB the plane wavefront incident on it. When the wavefront touches the reflector A at an angle i with it, i is the angle of incidence, because it is the angle between the normal and the incident ray at A shown in figure 9.2. The point A becomes a surface of secondary wavelets which lend to spread out in the surrounding space by the time the disturbance from B reaches the the reflector at C, the secondary waves starting from A will have a radius BC. Draw a sphere with A as centre and radius equal to BC. A plane through C, touching the sphere at D, represents the reflected wave front.

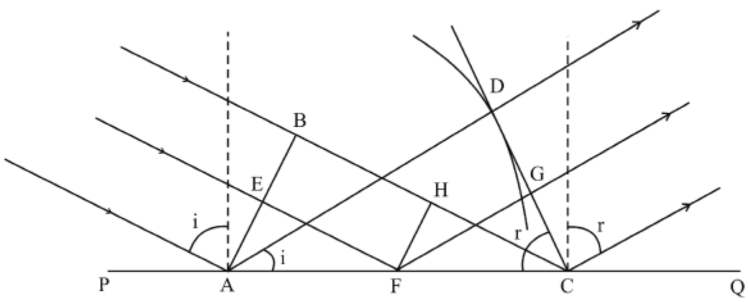


Figure : 9.2

Now, the disturbance from any point E on the incident wave front AB, after reflection from the surface PQ at F, msut reach the reflected wave front CD in the same time in which the disturbance from A reaches at D. If FG is drawn perpendicular to CD, then EF + FG must be equal to AD or BC. Now we have to prove that CD is the reflected wavefront.

To prove this we draw FH perpendicualr to BC and consider the triangles ΔABC and ΔADC.

Since, AD = BC, ∠ABC = ∠ADC and AC is common.

Therefore, ΔABC and ∴ ΔADC are congruent.

∴ ∠BAC = ∠DCA = r ... ... (9.1)

Again HF parallel to AB and ∠HFC = ∠BAC = ∠DCA = ∠GCF,

Again ∠CGF = ∠CHF = right angle

and ∠GCF = ∠HFC, FC is common.

∴ ΔFHC and ΔFGC are congruent

∴ FG = HC

or, EF + FG = BH + HC = BC

Hence, CGD is he reflected wave front

Now, ∠BAC = ∠DCA

But, i = ∠BAC and r = ∠DCA

∴ i = r

It is also clear from the construction the incident ray ; the reflected ray and the normal, all lie in the same plane.

Hence, the laws of reflection are verified from the wave theory.

9.4.2 Laws of refraction from the wave theory

Let a plane wavefront AB incident on a plane refracting surface PQ that separates the two media a and b. Let c_1 and c_2 be the velocities of the waves in the two media a and b respectively ($c_1 > c_2$) as shown in figure 9.3

By the times the disturbance from the point B reaches C at th surface of separation, the secondary waves from A have acquired a radius AD in the medium b.

Therefore, we have

$$\frac{AD}{C_2} = \frac{BC}{C_1} \quad \ldots \quad (9.2)$$

Draw a sphere with A as centre and AD as radius A tangent plane DC passing through C and touching the sphere will give the the refracted wavefront.

To prove that CD is the true refracted wavefront, the disturbance fromany point E on the incident wavefront after refration at F on the surface of seperation must reach the refracted wave front CD in the same time when the disturbance from A reaches at D.

Hence, $\frac{EF}{C_1} + \frac{FG}{C_2}$ must be equal to $\frac{AD}{C_2}$ or $\frac{BC}{C_1}$

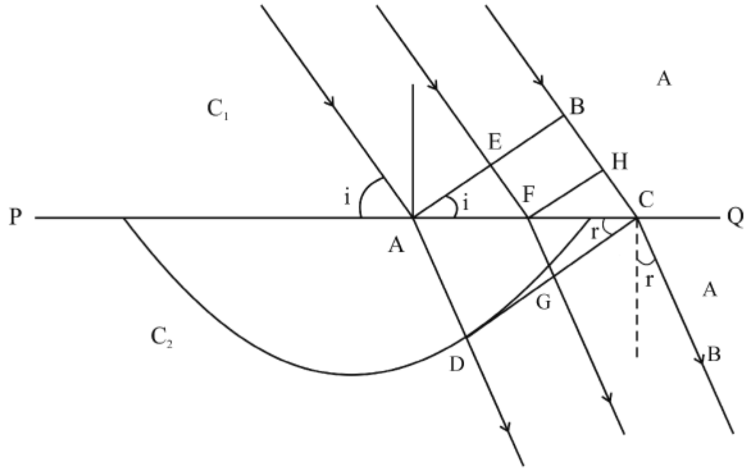


Figure : 9.3

Now, FG is perpendicular to CD and if we draw FH perpendicular to BC, then

BC = BH + HC

or, $\frac{BC}{C_1} = \frac{BH}{C_1} + \frac{HC}{C_1} = \frac{EF}{C_1} + \frac{FG}{C_2}$ ... ... (9.3)

[BH = EF]

$\therefore \frac{HC}{C_1}$ must be equal to $\frac{FG}{C_2}$

From figure 9.3, triangles ΔABC and ΔFHC are similar triangles

$\therefore \frac{FC}{AC} = \frac{HC}{BC}$ ... ... (9.4)

and also triangles ΔADC and ΔFGC are similar

$\therefore \frac{FC}{AC} = \frac{FG}{AD}$ ... ... (7.5)

From equations (9.4) and (9.5) we get

$\frac{HC}{BC} = \frac{FG}{AD}$ or, $\frac{AD}{BC} = \frac{FG}{HC}$

But from relation (9.2) we have

$$\frac{AD}{BC}=\frac{C_2}{C_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (9.6)$$

$$\text{or, } \frac{FG}{HC}=\frac{C_2}{C_1}$$

$$\text{or, } \frac{HC}{C_1}=\frac{FQ}{C_2} \text{ as required}$$

Thus it is proved that CD is the true refracted wave front

Again $\angle BAC = i$ and $\angle ACD = r$ angle of incidence and refracted angle respectively.

$$\therefore \frac{\sin i}{\sin r}=\frac{BC}{AC}\times\frac{AC}{AD}=\frac{BC}{AD}=\frac{C_1}{C_2}=\text{constant} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (9.7)$$

This is snells law and the constant is called the refractive index ${}_a\mu_b$, of the medium b with respect to the medium a.

It is also clear from figure 9.3 that the incident ray, the refracted rayand the normal at the point of incidence, all lie in the same plane.

Hence, the laws of refraction proved using the wave theory.

9.4.3 Refraction of a spherical wave through in this lens.

Let A and B are the poles of the two spherical surface of a thus conver lens of rail of curvature R_1 and R_2 respectively. Consider A_1AA_2 be the incident wavefront which diverges from the object point P. After imergence from the lens the wavefront becomes curved in apposite direction and goint to converage at Q, which is the image point of the object at P.

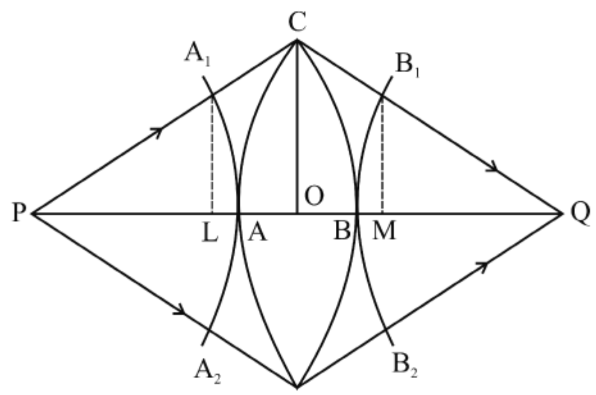


Figure : 9.4

Now, $PA = PA_1$ and $QB = QB_1$

Therefore, the optical path $A_1\ CB_1$ must be equal to the optical path AB, by Fermat's principles, the optical path between P and Q. via any path in same. If μ_1 and μ_2 are the refractive index of the surrounding medium and that of the material of the lens respectively then,

$\mu_1 (A_1CB_1) = \mu_2 AB$

or, $\mu_1 (A_1C + CB_1) = \mu_2 (AO + OB)$

For small aperture of the lens

$A_1C = LO = LA + AO$

and $B_1 = OM = OB + BM$

$\therefore \mu_1\{(LA + AO) + (OB + BM)\} = \mu_2 (AO + OB)$

or, $\mu_1 (LA + BM) + \mu_1 (AO + OB) = \mu_1 (AO + OB)$

or, $\mu_1 (LA + BM) = (\mu_2 - \mu_1)(AO + OB)$

$\simeq$

or, $LA + BM = \left(\frac{\mu_2}{\mu_1} - 1\right) (AO + OB)$ ... ... (9.7)

From geometry of the circles we have

$$LA = \frac{A_1L^2}{2AP} \quad \frac{A_1L^2}{2OP} = \frac{A_1L^2}{2u}$$

$$BM = \frac{B_1M^2}{2BQ} = \frac{B_1M^2}{2OQ} = \frac{B_1M^2}{2v}$$

$$AO = \frac{OC^2}{2R_1^2} \text{ and } OB = \frac{DC^2}{2R_2}$$

where OP = u, the object distance

OQ = v, the image distance

Putting these in equation (9.7) we.get

$$\frac{A_1L^2}{2u} + \frac{B_1M^2}{2v} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{OC^2}{2R_1} + \frac{OC^2}{2R_2}\right)$$

or, $\frac{1}{v} - \frac{1}{u} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ ... ... (9.8)

Since the lens is very thin, So $A_1L = B_1M = OC$ and according to sign convention v and R_1 are positive, while u and R_2 are negative.

When n = ∝ (ie object at infinity), v = f (ie. the image is at the second focus)

$$\therefore \frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} \Big/ \frac{1}{R_2}\right) \quad \text{(9.9)}$$

From equation (9.8) and (9.9) we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \text{(9.10)}$$

the usual lens formula

Again if the surrounding medium is air, then $\mu_1 = 1$ and let $\mu_2 = \mu$, l then from equation (9.9) we get

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \text{(9.10)}$$

This formula is known as lens-making formula.

Exercise-1

A convex lens of focal length 24 cm ($\mu = 1.5$) is totally immersed in water ($\mu = 1.33$). Find its focal length in water.

9.5 Summary

Velocity of light in free space is $C = \frac{1}{\sqrt{\mu_0 c_0}} = 3 \times 10^8 \, m/s$

A wavefront is a surface of constant phase

Usual lens formula $\frac{1}{v} = \frac{1}{v} = \frac{1}{f}$

Lens make is formula $\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

9.6 Question and Problems

9.6.1 The velocity of light in air is 3×10^3 m/s. Find the wavelength of sodium light. (wave length in air λ_a= 5893Å) in water of refractive index 1.33 and also find the velocity of light in water.

9.7 Solution

Exercise -1

Here ${}_a\mu_g = 1.5$ and ${}_a\mu_w = 1.33$

we know $\frac{1}{f} = \left(\frac{\mu_2}{\mu_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

When the lens in air, $\mu_2 = {}_a\mu_g = 1.5$

μ_1 = refrative index of air = 1

f = 24 cm

$$\therefore \frac{1}{24} = (1.5-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (1)$$

When the lens in water $\mu_1 = a\mu_w = 1.33$

$\mu_2 = a\,\mu_g = 1.5$

$$\therefore \frac{1}{f} = \left(\frac{1.5}{1.33} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.125-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad \ldots \ldots \ldots \ldots \quad (2)$$

Dividing equation (1) by (2) we get,

$$\frac{f}{2.4} = \frac{0.5}{0.125}$$

$\therefore$ f = 96 cm

9.6.1 Here c_1 = velocity of light in air = 3×10^8 m/s

c_2 = velocity of light in water of refractive index, $\mu = 1.33$

$$\therefore \mu = \frac{c_1}{c_2},\ c_2 = \frac{3\times10^8}{1.33} = 2.26\times10^8\,\text{m/s}$$

If λ_1 = 5893 Å for sodium light in air and λ_2 that in water, then $c_1 = \upsilon\lambda_1$ and $c_2 = \upsilon\lambda_2$, υ = frequency of light.

$$\therefore \frac{\lambda_1}{\lambda_2} = \frac{c_1}{c_2} = \mu$$

$$\therefore \lambda_2 = \frac{\lambda_1}{\lambda_2} = \frac{5893}{1.33} = 4431\text{Å}$$

Unit : 10 ❐ Interference of Light

Structure

10.0 Objectives

After studing this unit, you will be able to

- Use the principle of superposition of light waves to interprete constructive and destructive interference.
- know the coherent and incoherent sources of light.
- know the origin of the interference pattern produced by youngs double slit, biprism

and Lloyd's mirror.

- Compute the intensity of light distributed in the interference pattern.
- Compute the fringse-width in terms of wavelength of light.
- describe the various arrangements for producing interference by division of wave front and by division of amplitude.

10.1 Introduction

You are familiar with some optical phenomena like reflection, refraction, dispersion etc. in geometrical optics. All these phenomena are explained with rectilinear properties of light. But some phenomena as we observe in nature like, the bright colours are seen in an oil slick floating on water or in a sunlit soap bubble are caused by interference, which cannot be explained by rectilinear properties of light. Similarly we cannot explain the diffraction, polarization of light with rectilinear properties of light. To explain these we have to consider the wave nature of light.

In this unit, we shall discuss the interference phenomena with some experimental observations.

10.2 Principle of Superposition

You have already learnt about the principle of superposition of waves in the previous section (unit-2). According to this principle in any medium, when two or more disturbances acting at a point simultaneously the resultant disturbance at that point is the vector sum of the individual disturbances, provided the disturbances are small.

Suppose y_1 is the displacement of a particle at a given point at any instant and y_2 is that due to other wave. When these two waves simultaneously arrive at a point, the resultant displacement y of the point is given by the principle of superposition as

$$y = y_1 \pm y_2$$

Positive sign is to be taken when the displacements are in the same direction and the negative, when they are in opposite directions.

In case of light waves, when monochromatic waves of light from two sources proceed almost in the same direction and superpose at a point, then the intensity of light at that point will be maximum or minimum according as the waves meet at that point in same phase or in opposite phase. This phenomenon is known as interference of light. This phenomena requires for its explanation the wave nature of light.

10.3 Young's double slit experiment

Young gave the first demonstration of the interference of light waves. Here, the monochromatic light is allowed to pass through a narrow slit s and then fall on the two identical narrow closely spaced. slits s_1 and s_2 as shown in figure–10.1. The cyndrical waves emerging from the slits s_1 and s_2 overlap.

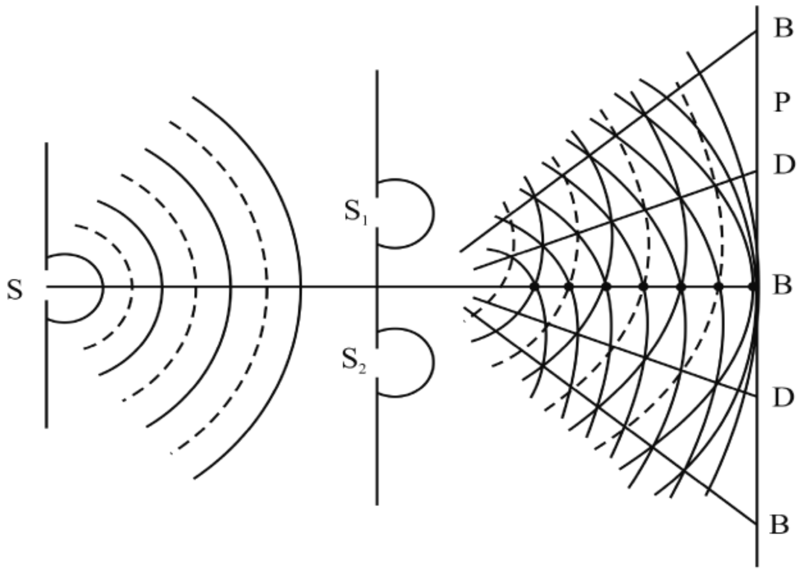


Figure : 10.1

Since the slits are equaidistant from S, so the phase of the wave at S_1 will be same as that at S_2. Hence, sources S_1 and S_2 act as secondary coherent sources of light. The waves leaving from S_1 and S_2 interfere and produce alternate bright and dark bands on the screen at P.

Young's experiment is known as double slit experiment of interference by the division of wavefront.

The wave front from S is divided at S_1 and S_2 .

In the original experiment Young used sunlight as source of light and found coloured bands on the screen.

10.3.1. Analytical treatment of interference.

Let s be a narrow stit is illuminated with a monochromatic light of wavelength. S_1 and S_2 are two similar parallel slits, very close to each other and equidistant from S (figure–10.1)

According to Hygyen's principal cylindrical wavelets spread out from the slit S arriving at S_1 and S_2 in same time as $SS_1=SS_2$ The secondary wavelets start from S_1 and S_2 with equal phase diverge towards the screen (P). S_1 and S_2 may be considered as coherent sources.

Let the light waves at S_1 be represented by

$$y = a_1 e^{i\omega t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad .(10.1)$$

where a_1 is the amplitude and ω is the angular frequencey of the wave.

In travelling from S_1 to any point ρ on the screen the phase changes by $\frac{2\pi}{\lambda} S_1\rho = \frac{2\pi}{\lambda} x_1 = Kx_1$ here $S_1\rho = x_1$ and $K - \frac{2\pi}{\lambda}$, propagation constant [figure10.2]

Hence at time t the complex disturbance at P due to light from s_1 will be

$$y_1 = a_1 e .... \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad .(10.2)$$

Similarly, the disturbance at P due to light from s_1 may be written as

$$y_2 = a_2 e^{i(\omega t - kx_2)} ...... \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad .(10.3)$$

where a_2 is the amplitude and $s_2 p = x_2$

∴ According to principle of superposition the resultant disturbance at P, is given by

$$y = y_1 + y_2 = a_1 e^{i(\omega t = kx_1)} + a_2 e^{i(\omega t - kx_2)}$$

$$= e^{i\omega t}\left(a_1 e^{-kx_1} + a_2 e^{-ikx_2}\right)$$

$$e^{i\omega t}\left\{a_1\left(\cos kx_1 - i\sin kx_1\right) + a_2\left(\cos kx_2 - i\sin kx_2\right)\right\}$$

$$= e^{i\omega t}\left\{\left(a_1 \cos kx_1 \_ a_2 \cos kx_2\right) - i\left(\sin kx_1 + \sin kx_2\right)\right\}$$

$= e^{i\omega t}(a - ib)$

$= e^{i\omega t}\sqrt{a^2 + b^2}e^{-i\varphi}$

$= \sqrt{a^2 + b^2}e^{i(\omega t - \varphi)}$

$\therefore\ y = Ae^{i(\omega t - \varphi)}$ ... ... (10.4)

Where $a = a_1 \cos kx_1 + a_2 \cos kx_2$

$b = a_1 \sin kx_1 + a_2 \sin kx_2$

and $\tan\varphi = \frac{b}{a} = \frac{a_1 \sin kx_1 + a_2 \sin kx_2}{a_1 \cos kx_1 + a_2 \cos kx_2}$

Now, $A^2 = a^2 + b^2 = (a_1 \cos kx_1 + a_2 \cos kx_2)^2 + (a_1 \sin kx_1 + a_2 \sin kx_2)^2$

$= a_1^2 + a_2^2 + 2a_1a_2 \cos k(x_2 - x_1)$

$\therefore$ The amptitude of the resultant disturbance is $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\delta}$ ....(10.5)

Where $\delta = k(x_2 - x_1) = \frac{2\pi}{\lambda}(x_2 - x_1)$ ......(10.6) is the phase diference between the two waves at ρ due to path difference $x_2 - x_1$.

Since, intensity at ρ is proportional to A^2

$\therefore$ Intensity at ρ is

$I = a_1^2 + a_2^2 + 2a_1a_2 \cos d$ .....(10.7) [taking proportionatity constant = 1]

(i) Condition for constructive interference

Now it $\delta = 2n\pi$ where n = 0, 1, 2 ......

or, $x_2 - x_1 = s_1\rho - s_1\rho$

$= 2n\frac{\lambda}{2}$ (using equatioin 10.6)

Then the intensity will be maximum

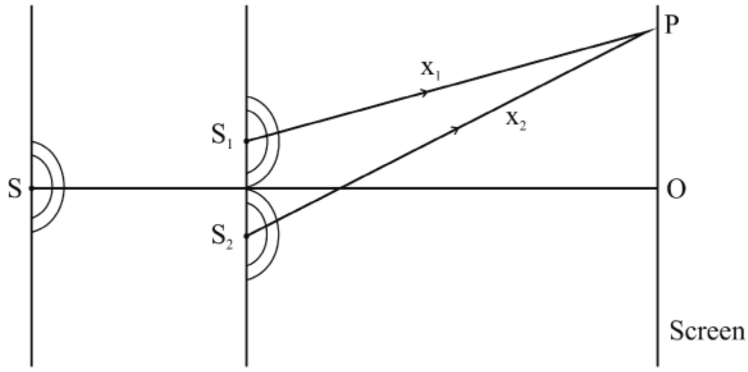


Figure : 10.2

$$I_{max} = a_1^2 + a_2^2 + 2a_1a_2 \quad [\because \cos\delta = \cos 2n\pi = 1]$$

$$I_{max} = (a_1 + a_2)^2 \quad .... \quad ... \quad ... \quad ... \quad ... \quad ... \qquad (10.7)$$

Thus when the path difference $(x_2 - x_1)$ of the point ρ from the two sources $(s_1 \text{ and } s_2)$ is the even multiple of $\frac{\lambda}{2}$ the intensity of light at that point becomes ,,,, maximum and we get a bright band there. This is knows as constructive interference. When n = 0, we get the central bright band at the point O, for which the path difference is zero i.e. $x_1 = x_2$ and for n = 1 we get next bright band called 1st order bright band, for n = 2 second order and so on.

(ii) Condition for destructive interference

The intensity (I) will be minimum when cos hδ = –1, that is when $\delta = (2n+1)\pi$

or, $x_2 - x_1 = s_2p - s_1p = (2n+1)\frac{\lambda}{2}$

where n = 0, 1, 2, ...

From equation (10.6) we can write the minimum intensity

$$I_{min} = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 = a_2)^2 \quad ... \quad ... \quad ... \quad ... \quad ... \qquad (10.8)$$

Therefore, when the path difference of the point P from s_1 and s_2 is odd multiple of $\frac{\lambda}{2}$, the intensity of light at that point becomes minimum and we get a nearly dark band there. This is called destructive interference.

Now if amplitude of the two waves are equal i.e. $a_1 = a_2 = a$, then $I_{max} = (a_1 + a_2)^2 = 4a^2$ and $I_{min} = 0$

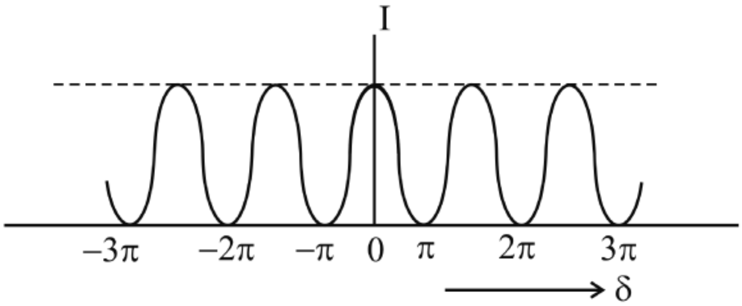


Figure : 10.3

The intensity distribution curve as shown in figure–10.3. This is known as the intensity pattern. The alternate dark and bright regions are called interference fringes.

10.3.2 Fringe width and shape of fringes in young's double slit.

Let two coherent sources s_1 and s_2 are separated by a distance d which are sending monochromatic light of wave length λ to produce interference fringes on the screen as shown in figure–10.4. The screen is placed parallel to the slits s_1 and s_2.

Let the distance of separation between the screen and sources is D. The point O on the screen is equidistant from s_1 and s_2. Hence the waves from s_1 and s_2 will arrive at O, at the same time. If the initial phase difference between the two coherent sources is zero then the waves will meet at O from s_1 and s_2 is same phase and will produce a bright band at O, called central bright band. Let the n th bright band be formed at P, distance x_n for O.

Then from figure–10.4

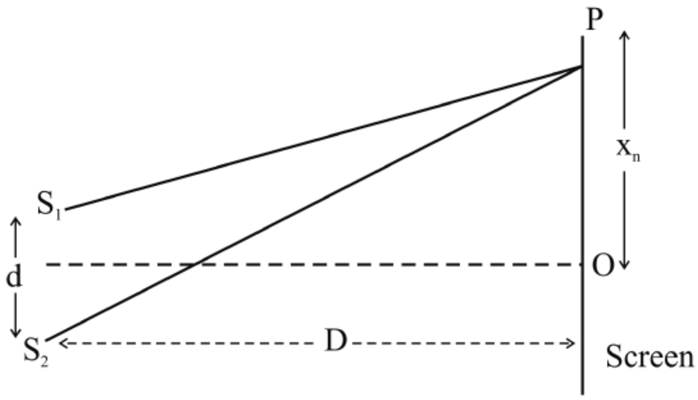


Figure : 10.4

$$S_2p^2 = D^2 + \left(x_n + \frac{d}{2}\right)^2 = D^2\left\{1 + \left(\frac{x_n + \frac{d}{2}}{D}\right)^2\right\}$$

or, $$S_2p = D\left\{1 + \left(\frac{x_n + \frac{d}{2}}{D}\right)^2\right\}^{\frac{1}{2}} = D\left\{1 + \frac{1}{2}\left(\frac{x_n + \frac{d}{2}}{D}\right)\right\} \quad \ldots \quad \ldots\ldots \quad \ldots\ldots \quad (10.9)$$

[$\because$ D>>d we have neglected the higher order terms]

similarly $$S_1p = D\left\{1 + \frac{1}{2}\left(\frac{x_n - \frac{d}{2}}{D}\right)^2\right\} \quad \ldots\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

... ... (10.10)

$$S_2p - S_1p = \text{path difference} = \Delta = \frac{x_n d}{D} \quad \ldots.(\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad 10.11)$$

Fringe width

For the n the bright fringe at p we have

$\Delta = 2n\frac{\lambda}{2}$ where n = 0, 1, 2, ...

or, $\frac{x_n d}{D} = 2n\frac{\lambda}{2} \qquad \therefore\ x_n = n\frac{\lambda D}{d}$ ......(1.12)

Similarly the distance of (n+1) the bright fringe from O will be

$$x_{n+1} = (n+1)\frac{\lambda D}{d} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.13)$$

$\therefore$ The distance betweent to consecutive bright bands is

$$x_n = (2n+1)\frac{\lambda}{2}\frac{D}{d} \quad \ldots. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.14)$$

Again if ρ be the position of nth dark fringe then, $x_n = (2n+1)\frac{\lambda}{2}\frac{D}{d}$ and $x_{n+1} = (2n+3)\frac{\lambda}{2}\frac{D}{d}$

Spacing between nth and (n+1)th dark fringe is

$$\beta = x_{n+1} - x_n = \frac{\lambda D}{d} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.15)$$

Hence, the specing between any two consecutive bright or dark fringes in equal and is given by $\beta = \frac{\lambda D}{d}$. This distance between two consecutive bright or dark fringe β is known as fringe width. Now, using this expression, you can experimentally measure the wave length of monochromatic light by measuring β, D and d.

Exercise–1

Calculate the intensity of maxima when two light waves $y_1 = 3 \sin(\omega t - k_1 x_1)$ and $y_2 = \sin(\omega t - k_2 x_2)$ interfere.

Shape of the fringes

Let s_1 and s_2 are two sources (slits) of monochromatic light Let O be the mid point between the slits as the origin (0,0) of a coordinate system. Consider the x-axis be along OX and Y-axis perpendicular to the plane containing the slits. If P(x,y) be any point, then are can write.

$$S_1P^2 = y^2 + \left(x - \frac{d}{2}\right)^2$$

and $S_2P^2 = y^2 + \left(x + \frac{d}{2}\right)^2$ where d is the distance between two slits.

The path difference

$$\Delta = S_2P - S_1p$$

or, $\Delta + S_1P = S_2P$

or, $\Delta+\left\{y^2+\left(x-\frac{d}{2}\right)^2\right\}^{\frac{1}{2}}=\left\{y^2+\left(x+\frac{d}{2}\right)\right\}^{\frac{1}{2}}$

squaring both sides and rearranging we get

$$2\Delta\left\{y^2+\left(s-\frac{d}{2}\right)^2\right\}^{\frac{1}{2}}=2xd-\Delta^2$$

Again squaring and rearranging,

$$x\frac{x^2}{\frac{\Delta^2}{4}}-\frac{y^2}{\left(d^2-\Delta^2\right)/4}=1$$

........(10.16)

This equatioin (10.16) represents a hyperbola with eccentricity (e) equal to

$$e=\left(\frac{\Delta^2}{4}+\frac{d^2-\Delta^2}{4}\right)^{\frac{1}{2}}/\left(\frac{\Delta}{2}\right)=\frac{d}{\Delta}$$

$$\left[\because e=\left(a^2+b^2\right)^{\frac{1}{2}}/a\right]$$

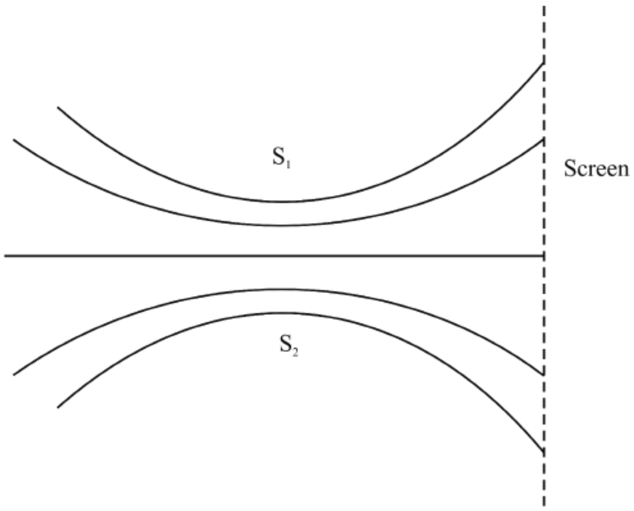


Figure : 10.5

Thus the loci of points of constant path difference Δ in xy place are hyperbola with s_1 and s_2 as foci on y axis.

In optical experiments the path difference Δ is very small $\left(\sim 10^{-8}\,cm\right)$ and $d\sim 10^{-2}\,cm$. Therefore e is very large and the hyperbola become practically straight lines, given by

$$y=\pm\left(\frac{d^2-\Delta^2}{\Delta^2}\right)^{\frac{1}{2}}x$$

If instead of slits we use two coherent point sources s_1 abd s_2 then in 3-dimensional space the loci of bright fringes of different orders will be a system of confocal hyperboloids with s_1, s_2 as foci (figure–10.5). If the screen be placed parallel to the time joining s_1

and s_2 then short straight time fringes are obtained. Again if the screen is placed perpendicualr to the lime joining s_1, s_2 we shall observe a number of atternately bright and dark concentric circles with their common centre on the intersection of line s_1, s_2 with the screen.

These fringes are called non-localised fringes, because they can be obtained on a screen from any where.

10.3.3 White light fringes and colour effect.

You know that the distance of the nth bright fringe from the central fringe is

$$x_n = n\frac{\lambda D}{d}.$$

where d = distance between two slits and

D = distance between the slits and the screen.

Here d and D are constant. Thus when slits are illuminated by white light source, then for n = 0,

x_n =0, the central bright band will be white, because light of all wave lengths in white light will coinsideat this point.

For higher orders n > O, x_n will be greater for light of longer wave length and less for shorter one.

As wavelength of red light λ_r is longer than λ_v for violet light, hence all bright bands will be coloured except central one, in which red will be in the outermost positiong and violet will be in innermost position.

10.3.4 Conservation of energy

When two light waves interfere, at first sight it appears that conservation of energy, is violated, for the energy at the minimum points is lost. According to the law of conservation of energy, the energy cannot be destroyed, but it is transferred from points of minimum intensity to points of maximum intensity.

In absence of interference phenomena due to two waves of amplitudes a_1 and a_2 the intensity is

$$I = I_1 + I_2 = a_1^2 + a_2^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

When the two light waves from coherent sources interfere the maximum and minimum intensity of the resultant disturbance after interference are obtained from equations (10.7) and (10.8)

$I_{max} = (a_1 + a_2)^2$ and $I_{max} = (a_1 - a_2)^2$

$\therefore$ Average intensity is $I_{av} = \dfrac{I_{max} + I_{min}}{2}$

$$= \frac{(a_1 + a_2)^2 + (a_1 - a_2)^2}{2}$$

$= a_1^2 + a_2^2$ ......(ii)

So, the conservation of energy is proved.

10.4 Conditioins for interference

In order to observe a distinct and well defined interference pattern the following conditions must be satisfied.

(i) The two interfering waves of light must be coherent.

If the waves are coherent, then they maintain a constant phase difference over time and space. Hence a stationary interference pattern will observe.

(ii) The two interfering waves must have same or nearly same frequency or wavelength. Their amplitudes must also be equal or nearly equarl, otherwise the intensity variations of dark and bright fringes cannot be recognised.

(iii) If the interfering waves are polarised, they must be in the same state of polarization.

(iv) Path difference must not be large.

10.5 Types of interference

On the basis of production of coheerent sources, the interference may be classified into two classes division of wavefront and division of amplitude.

(i) Division of wavefront :

In this class of interference, the incident wave front is divided into two parts by reflection, refraction etc. to produced two coherent interfering beams. In order to maintain coherence it is essential to use narrow sources in these cases.

Examples of this types are the fringes formed by biprism, Lloyd's mirror etc.

(ii) Division of amplitude

Here the amplitude of the incident bean is divided into two or more parts by partical reflection or refraction. to produce two or more coherent inter fring beams. A broad source is required here to produce interference patter. Examples of this class are the fringes fromed by the thin film, Newton's rings, Michelson's inter ferometer etc.

10.6 Interference by division of wave front

10.6.1 Fresnel's biprism

Fresnel's produced two coherent sources by division of wavefront using a biprism. The biprism consists of two prism of very small refracting angles joined base to base in practice, it is constructed from a thin glass plate by proper grinding and polishing. The obtuse angle of the prism is about 179° and two side angles are about 30′ each (figure–10.6).

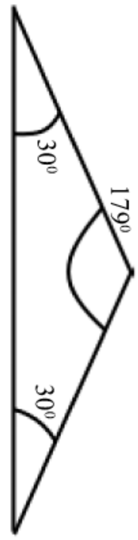


Figure : 10.6

Experimental set up:

The experimental arrangement for obtaining interference fringes using biprism is shown in figure 10.7.

Light from a narrow slit s, illuminated by a monochromatic light. The light from s incident symmetrically on the birpism ABC, placed at a small distance from the slit s, with its refracting edge parallel to the length of the slit, the incident wavefront is divided into two parts and suffer separate retractions from the upper and lower parts of the

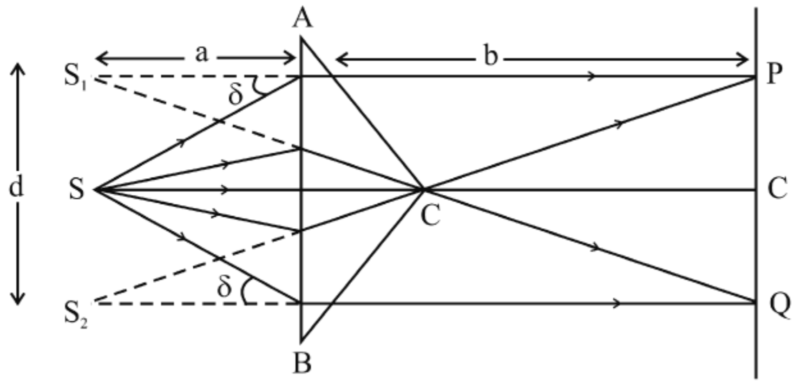


Figure : 10.7

biprism. The two refracted wavefronts appears to diverge from two vistual sources s_1 and s_2. Thus s_1 and s_2 can be considered as two coherent sources. The emergent wavefronts meet at small angles and produce interference pattern on the screen in the overlapping region PQ or may be seen through an eyepiece. The fringes are not localised, so the screen may be placed any where within a suitable distance. The typical fringe pattern is shown if figure 10.8.

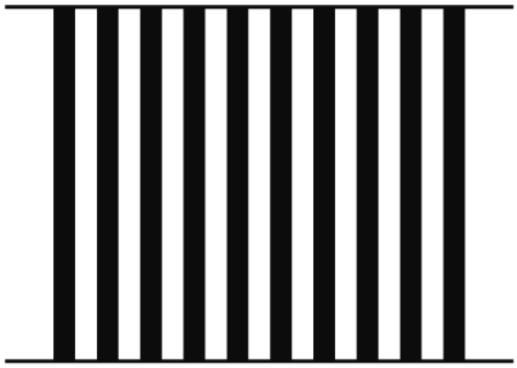


Figure : 10.8

10.6.2 Thory of biprism

Theory of the interference fringe formation of biprism is same as that of Young's double slit, as described in article–10.3

The unknown wavelength λ of monochromatic light can be determined by using the formula ... . (10.14 or 10.15)

$$\beta = \frac{\lambda D}{d}$$

$$\text{or, } \lambda = \beta \frac{d}{D} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad .(10.17)$$

Where β = fringe width, d = distance between the two virtual sources s_1 and s_2, D = (a+b), the distance of the slit s from the eyepiece (figure–10.7)

10.6.3 Experimental procedure :

A narrow slit s, the biprism and a micrometer eyepice are mounted on the uprights of an optical bench all are adjusted properly to obtain the interference fringes. The slit is illuminated by a monochroniatic light.

Measurement of β :

When the fringes are observed in the field of view of the eyepiece, the cross-wire of the eyepiece is mde to coinside with the centre of one of the bright fringes. Now the fringe width β is measured by setting the cross-wire at successive bright fringes with the help of micrometer screw fitted with the eyepiece.

Measurement of D :

The distance D, between the slit and eyepiece can be measured directly from the optical bench scale.

Measurement of d:

A convex lens of suitable focal length (f) is placed in between the biprism and eyepiece such that D > 4f. Under this condition, there are two conjugate positions of the convex lens for which sharp real images of s_1 and s_2 are obtained in the field of view of the eyepiece. If d_1 and d_2 are the distances between real images in the above two postions of the lens, then the magnification at one position will be inverse at the other positions.

$$\therefore \quad \frac{d_1}{d} = \frac{d}{d_2}$$

$$\text{or,} \quad d = \sqrt{d_1 d_2}$$

Hence measuring d_1, d_2 the distance d can be obtained.

Now to avoid the index error between the slit stand and eyepiece stand, we measure fringe width β_1 and β_2 at two different positios D_1 and D_2. Then by using equations (10.17) λ is given by

$$\lambda = d\frac{\beta_2 - \beta_1}{D_2 - D_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10.18)$$

knowing all these parameters λ can easily be determined.

Exercise–2

In a biprism experiment, the fringe width is 10^{-3}m for a wave length of 5893Å. If $\frac{b}{a} = 20$ where b is the distance between the biprism and screen, and x is the distance between slit and biprism, Calculate the refracting angle (α) of the biprism. m=1.5

Neccessity of narrow sources

A broad source can be considered as a large number of narrow virtual coherent sources. Thus each pair of conjugate points on the virtual sources produce interference fringes, which overlaps and a general illuminatioin will be observed.

10.6.4 Measurement of acute angle of biprism:

From figure–10.7, we see that the deviation ofthe rays SA and SC after passing through the biprism is given by $\angle SAS_1 = \angle SCS_2 = \delta$

Again, the deviation produced by the path of the ray by a thin prism $= (\mu - 1)\alpha$, where α is the acute angle of the prism and μ is the refractive index of the material of the prism.

From figure–10.7, again we have

$$\delta = \frac{d/2}{a} = \frac{d}{2a} \quad [\because \delta \text{ is very small}]$$

Here d is the distance between two virtual sources s_1 and s_2, a is the distance of biprism from slits.

$\therefore$ We can write

$$(\mu - 1)\alpha = \frac{d}{2a}$$

$$\therefore \; d = 2a(\mu - 1)\alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad .(10.19)$$

Hence, measuring d, a and knowing μ, the acute angle α can be determined.

Exercise–3

Calculate the separation between two coherent sources fformed by a biprism whose acute angle is 2° The distance between slite and biprism is 20 cm. $(\mu = 1.5)$

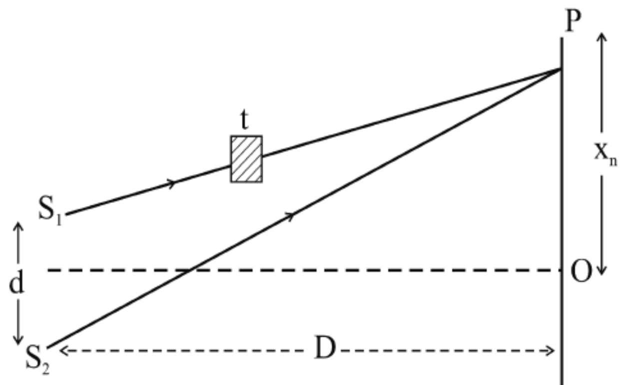


Figure : 10.9

10.6.5 Measurement of thickners of a transparent thin film

The biprism experiment can be used to measure the thickness of a thin sheet of transparent material such as mica, glass etc.

Suppose s_1 and s_2 are the two virtual coherent sources of monochromatic light which process interference fringes on the screen. Here the position of central fringe is at), as shown in figure 10.9 Hence the optical path $S_1O=S_2O$.

Let a thin sheet of transparent material of thickness t and refractive index μ be placed in the one of the paths (say S_1P) of the interfering rays. The optical path lengths S_1O and S_2O are now not equal and the central bright fring shifted to the position P from O.

Thus the optical path S_1P is again equal to the optical path S_2P.

$$\therefore\ S_1P+(\mu-1)t=S_2P$$

or, $$S_2P-S_1P=(\mu-1)t \quad \ldots \quad (10.20)$$

If P is the position originally occupied by nth order bright fringe, then

$$S_2P-S_1P=n\lambda \quad \ldots \quad (10.21)$$

∴ From equation (10.20) and equation (10.21) we get

$$(\mu-1)t=n\lambda \quad \ldots \quad (10.22)$$

The lateral shift of the central fringe of zero optical path difference is given as

$$x_n=OP=n\beta$$

or, $$n=\frac{x_n}{\beta} \quad \ldots \quad (10.23)$$

where fringe width $\beta=\dfrac{\lambda D}{d}$

∴ From equations (10.22) and (10.23) we get

$$(\mu-1)t=\frac{x_n}{\beta}\lambda$$

$$\therefore\ t=\frac{x_n\lambda}{(\mu-1)\beta}=\frac{x_n}{\mu-1}\cdot\frac{d}{D} \quad \ldots \quad (10.24)$$

This equation can be used to find the thickness of the transparent thin film(t)

Exercise–4

In a biprism experiment fringes were first observed with sodium light of wave length 589 nm and fringe width 0.347 mm. Sodium light was then replaced with white light and central fringe was located. On introducing a thin sheet of glass in the path of one of the beam, the central fringe was shifted by 2.143 mm. Calculate the thickness of the glass sheet of refraction index 1.542.

10.7 Change of phase on reflection : Stoke's treatment

Stoke's law states that when a light wave is reflected at the surface of a medum, which is optically derser than the medium through which the wave istravelling, a change of phase equal to π or a path difference of is introduced

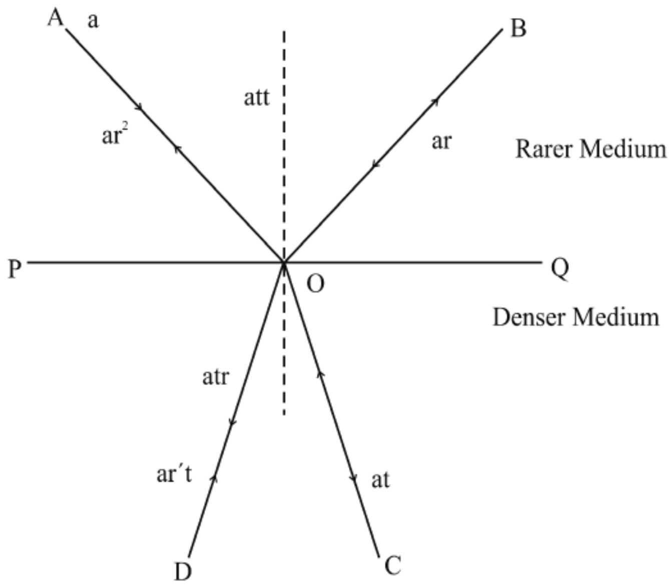


Figure : 10.10

Let PQ be the surface separating the denser medium from the rarer medium above it.

Let a wave AO of amplitude 'a' incident on the surface of the denser medium from the rarer medium is partly reflected along OB and partly refracted into the denser medium along OC, as shown in figure 10.10.

If r and t are the reflection co-efficient (fraction of the amplitude of the incident light which is reflected) and transmission co-efficient (fraction of the amplitude of) respectively, then the amplitude of the reflected wave OB = ar and that of refracted wave OC = at.

Now consider that the direction of reflected and refracted waves are reversed. So, on reversing the reflected wave BO, we get the amplitude ar^2 along OA and refracted amplitude art along OD. And on reversing the refracted wave CO, we get the reflected wave OD of amplitude ar't and refracted wave OA of amplitude at t where r' is the co-efficient of reflection at the surface of rarer medium and t' is the transmission co-efficient for the wave passing from denser to rarer medium.

Thus the two amplitudes along OA will combine together to produce the original amplitude, only if the total amplitude laong OD is zero

$at' + ar^2 = a$ and $art + ar't = 0$

or, $r = -r'$ ... ... (10.25)

The negative sign implies that when one wave has a positive displacement, the other has negative displacement. Which is equavalent to a phase change of π or a path difference of $\frac{\lambda}{2}$.

Hence a phase change of π will be introduced when the reflection takes place from the surface of a denser medium.

But when reflection takes place at the surface of a rarer medium, no change in phase or path difference takes place.

10.8 Lloyd's mirror

The Lloyd's mirror (MN) is a plane mirror polished on the front surface or a piece of a black glass plate, so that no reflection can take place from the back of the mirror.

S_1 is a narrow slit, illuminated by a source of monochromatic light placed parallel to the surface of the mirror. Light from S_1 is partly incident at a grazing angle on the surface of the mirror MN, while the rest reaches the screen directly.

The reflected light appears to diverge from S_2 as shown in figure-10.11, which is the virtual image of the source S_1. Thus S_1 and S_2 act as two coherent sources and the interference f

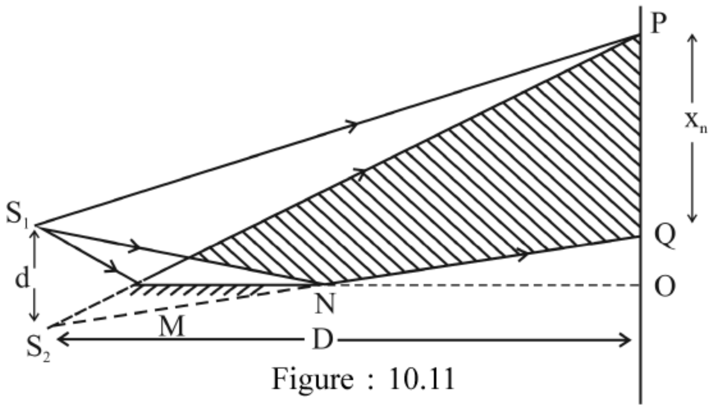


Figure : 10.11

The central point O on the screen for which $S_1O=S_2O$ receives only the direct light and this central fringe of zero path difference is not usually seen. To observe the central fringe, it is necessary to move the screen in contact with the end N of the mirror. It can also be seen by introducing a thin sheet of mica or glass in the path of direct ray. Which causes the entire pattern shifted upward. It is observed that the central fringe is black instead of bright, because the light reflected from the mirror (i.e. denser medium) suffers a phase change of π and the path difference between the rays at zero order is not zero but $\frac{\lambda}{2}$.

Fringe width

Now the path difference of the point P on the screen is

$\Delta = S_2P - S_1P$. If d is the distance between S_1 and S_2. D is the distance between the source and screen, then as young's double slit experiment (equatioin 10.11) we have.

$$\Delta = S_2P - S_1P = x_n \frac{d}{D}$$

Here for nth order dark fringe at P we have

$$x_n \frac{d}{D} = 2n\frac{\lambda}{2} \text{ for } n = 0, 1, 2, ...$$

$$\text{or, } x_n - n\lambda \frac{D}{d} \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \qquad (10.26)$$

$$\text{and for (n+1)th order dark fringe, } x_{n+1} = (n+1)\lambda \frac{D}{d} \quad ... \quad ... \qquad (10.27)$$

$\therefore$ The fringe width

$$\beta = x_{n+1} - x_n = (n+1)\frac{\lambda D}{d} - n\lambda \frac{D}{d}$$

$$\therefore \ \beta = \lambda \frac{D}{d}$$

Again for nth order bright fringe at p we have

$$x_n \frac{d}{D} = (2n+1)^{\frac{\lambda}{2}} \text{ for } n = 0, 1, 2....$$

$$\text{or, } x_n = (2n+1)\frac{D}{d}.\frac{\lambda}{2}$$

and for (n+1)the bright fringe we have

$$x_{n+1} = \{2(n+1)+1\}\frac{D}{d}\frac{\lambda}{2}$$

∴ The fringe width for bright fringes

$$\beta = x_{n+1} - x_n = \{2(n+1)+1\}\frac{D}{d}\frac{\lambda}{2} - (2n+1)\frac{D}{d}\frac{\lambda}{2}$$

$$\therefore \quad \beta = \frac{\lambda D}{d}$$

This shows that the fringes are equally spaced.

Exercise–5

In Lloyd's mirror experiment, the slit is at a distance 3 mm from the mirror. The screen is kept at a distance of 1.5 m from the source slit. Calculate the fringe width. Wavelenth of light is 5890Å. used.

10.9 Comparison of biprism and Lloyd's mirror fringes.

(i) In fresnel's bi-prism the conerent sources are composed of two virtual sources produced by refraction through the prism, while in Lloyd's mirror the coherent sources are of one real and the other is virtual produced by reflection at a plane mirror.

(ii) In biprism experiment fringes are formed or both sides of the central fringe, white in Lloyd's mirror method less than half of the fringes are seenn on one side of the central line.

(iii) In biprism the central fringe is bright while in Lloyd's mirror it is dark.

(iv) In biprism experiment, the distance between two corresponding points of the sources (d) is constant, so the fringe width is same for all parts of the source. But in Lloyd's mirror, the above distance is different fro different pairs of coherent point sources, causing unequal fringe width.

10.10 Interference by division of amplitude

10.10.1 Interference in thin wedge shaped film

Consider a thin wedge shaped filmis bounded by two plane surfaces AB and CD inclined at an angle α.

Let a ray PQ of monochromatic light of wavelength λ is incident at an angle i on the film. This ray is partly reflected from the front surface AB along QO and partly refracted along QR at an angle r. At R, it is again partly refracted along RS inside the medium and partly refracted out of the medium along RM. Smilar reflections and refractions occur at S, T etc. as showsn in figure–10.12. Since the rays QO and so, suffer only one reflection each, so they have almost equal intensities. But due to multiple reflections of the rest of the rays the intensities are ignorable.

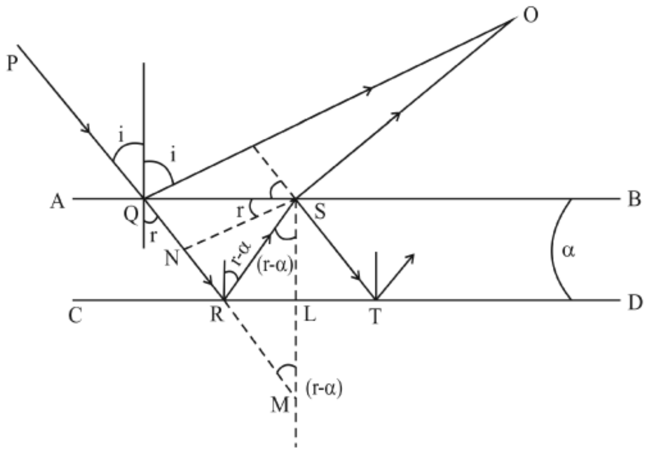


Figure : 10.12

As the two rays QO and so are derived from the same incident ray PQ and hence they are coherent. They combine to produce interference pattern.

Calculation of path difference.

To calculate the path difference between these reflected beams QO and SO, draw SN⊥QO, SN⊥QR and SLM⊥CD. produce QR and SL to meet at a point M.

We can write from geometry

RS = RM and SL = LM = d, the thickness of the film at S.

Hence the path difference between QO and SO is

$$\Delta = \mu\left(QN' + N'R + RS\right) - QN \quad \text{.........} \quad \ldots \quad \ldots \quad \ldots \qquad (10.28)$$

Where μ is the refractive index of the film by Snell's law we get,

$$\mu = \frac{\sin i}{\sin r} = \frac{QN/QS}{QN'/QS}$$

or, $QN = \mu QN'$

Therefore, from equation (10.28) we get

$$\Delta = \mu\left(N'R + RS\right) + \mu QN' - QN$$

$$= \mu\left(N'R + RS\right) + \mu QN' - \mu QN' \qquad \left[\because QN = \mu QN'\right]$$

$= \mu(N'R + RM) \quad [\because RS = RM]$

$= \mu N'M$

$= \mu SM\cos(r - d) \quad [\because N'M = SM\cos(r - d)]$

(from fig. 10.12)

$\therefore \Delta = 2d\mu\cos(r - d)$ .... (10.29)

Here the light (PQ) is reflected from the surface of the denser medium, which causes a phase change of π.

So, there is an extra path difference of $\frac{\pm 1}{2}$ is introduced for reflection from the surface of the denser medium (ie at Q)

Hence the effective path difference between the two rays is

$\Delta = 2d\mu\cos(r - d) \pm \frac{\lambda}{2}$ ... ... (10.30)

Conditions for maxima and minima

For maxima of brighness at O (i.e. for constructive interference), the condition is

$\Delta = 2d\mu\cos(r - d) \pm \frac{\lambda}{2} =$ even multiple of $\frac{1}{2}$

$= 2n\frac{\lambda}{2}$ where n = 0,1, 2...

or, $2d\mu\cos(r - d) = (2n \pm 1)\frac{\lambda}{2}$ ... ...(10.31)

Similarly condition for minima of brightness (ie for destructive interference)

$2d\mu\cos(r - \alpha) = 2n\frac{\lambda}{2}$ ... ... (10.32)

If the surfaces of the film are parallel, then $\alpha = 0$ in that case the path difference is

$\Delta = 2d\mu\cos r \pm \frac{\lambda}{2}$ ......(10.33)

Fringe width.

If x_n be the distance of nth order bright fringe from the thin edge of the wedge shaped thin film of thickness d, then from figure–10.13, we get

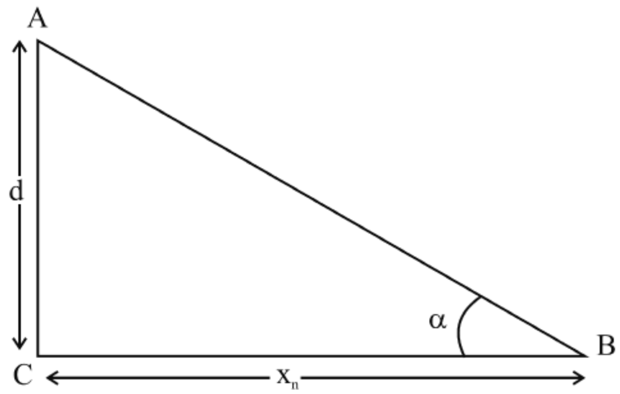


Figure : 10.13

$$\tan\alpha = \frac{d}{x_n}$$

or $d = x_n \tan\alpha$ .... (10.34)

Now, for normal incidence, r=0, then from equation (10.31), we have

$$2d\mu\cos\alpha = (2n+1)\frac{\lambda}{2}$$

or, $2d\mu\cos\alpha = (2n+1)\frac{\lambda}{2}$

or, $x_n = (2n+1)\frac{\lambda}{4\mu\sin\alpha}$

For (n+1)th bright fringe we have

$$x_{n+1} = \{2(n+1)+1\}\frac{\lambda}{4\mu\sin\alpha}$$

∴ Fringe width $\beta = x_{n+1} - x_n$

$$= [\{2(n+1)+1\}-(2n+1)]\frac{\lambda}{4\mu\sin\alpha}$$

$$= \frac{\lambda}{2\mu\sin\alpha}$$

If α is very small, then

$$\beta = \frac{\lambda}{2\mu\alpha} \quad \text{... ... ...} \quad (10.35)$$

Similarly for dark fringe the fringe width is

$$\beta = \frac{\lambda}{2\mu\alpha}$$

Thus the fringes are equispaced.

Exercise–6

A thin film of air in the form of wedge is formed between two glass plates. Find the angle of the wedge if the interference fringes are 1 mm apart when viewed normally with light of wavelength 589nm.

10.10.2 Fringes of equal width and fringes of equal inclination

In thin film interference fringes are produced due to path difference $\Delta = 2d\mu\cos r, (\alpha = 0)$ between the overlapping rays. For a given film, we see that the pathdifference arise (i) due to change in thickness (d) and (ii) due to change in angle of refraction (r) i.e. on angle of incidence (i).

Thus when a parallel beam of monochromatic light falls on a film of varying thickness μ, λ, r are held constant, maxima or minima represents the locus of points where the thickness is constant or equal width. This kind of fringes are observed in Newton's ring experiment.

Again if the thickness of the film (d) is constant i.e. if the film surfaces are plane parallel (i.e. $\alpha = 0$), and μ, λ are also held constants then different order number (n) of maxima or minima will be determined by the values of angle of refraction (r) and hence the angle of incidence (i).

Thus the fringes of a given order corresponds to the locus of points of equal r (or i). Such fringes are known as fringes of equal inclination. They are also called Haidinger's Fringes Enings of equal thickness are called Fizeaw Fringes.

Fringes with white light

When a parallel beam of white light falls on a thin wedge shapped film colour fringes are obtained, becouse the path difference $2\mu d \cos r$ depends upon μ, d and r. As $\lambda_r > \lambda_v$, the first order bright fringe of violet colour light ocurs at smaller d of the film and the corresponding red colour fringe will be at larger value of d. Thus we shall see the differently coloured fringes at different thickness. But if the thin edge of the film is zero i.e.d = 0 then at that point the fringe will be dark due to extra path difference of $\frac{\lambda}{2}$, as discuss earlier.

10.10.3 Necessity of broad source

In the case of interference in thin films, the narrow source limits the visibilty of the film.

To explain the situation, consider a ray SA starting from a narrow source s (figure–

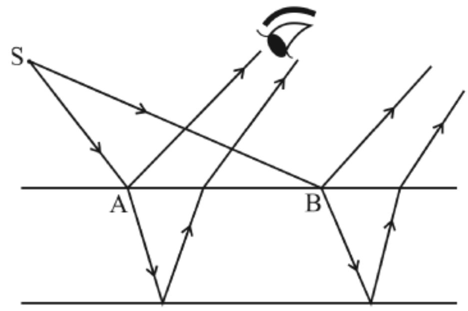


Figure : 10.14

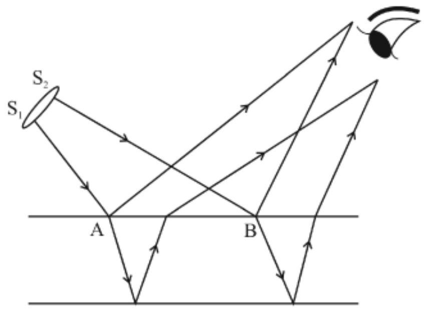


Figure : 10.15

10.14), after suffering reflection from the front surface and then reflection from the back surface of the film enters the eye, where as the ray SB incident incident at a different angle after reflection and internal reflection will not reach the eye. Thus a limited portion of the film is visible.

In an extended source the rays S_1A, S_2B after reflection and internal reflections will reach the eye from a large portion of the film as shown in figure 10.15. Thus the entire firm can be seen simultaneously.

10.11 Newton's rings

Newton's ring experiment is an example of interference formed by thin film. The convex surface of a long focal length plano-convex lens (L) is placed on a plane glass plate (G) A very thin air film of varying thickness is formed between the lower surface of the lens and the upper surface of the glass plate. The thickness of the air silm at the point of contact (O) is zero and increases towards the periphery of the lens as shown is figure 10.16a.

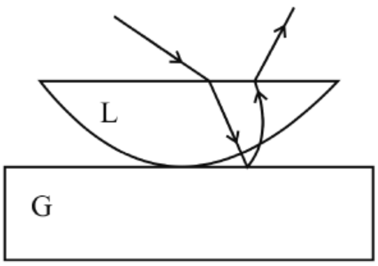


Figure : 10.16 (a)

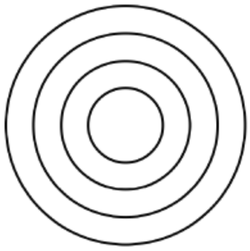


Figure : 10.16 (b)

If monochromalic light be allowed to fall normally on the film and observed by means of a low powr travelling microscope, the interference fringes in the form of concentric circular rings are found. These rings are known as newton's rings. These fringes are the loci of points of equal thickness and are localized in the air film.

Typical Newton's rings are shown infigure (10.16b)

Experimental arrangement

The experimental arrangement for observing Newton's rings is shown in figure– **16.11. The light from an extended.**

monochromatic source S is made paralled by a lens L_1. These parallel rays are reflected by a glass plate p held at 45° with the horizoltal. The light reflected from the plate P falls normally on air film enclosed between the glass plate (a) and the plano convex lens(L) as shown in figure– 10.17. Interference occurs between the rays reflected fromthe upper and lower surfaces of the air film. The concentric circular alternate dark and bright rings localized in the film are viewed by a low power travelling microscope (M) focussed on the film.

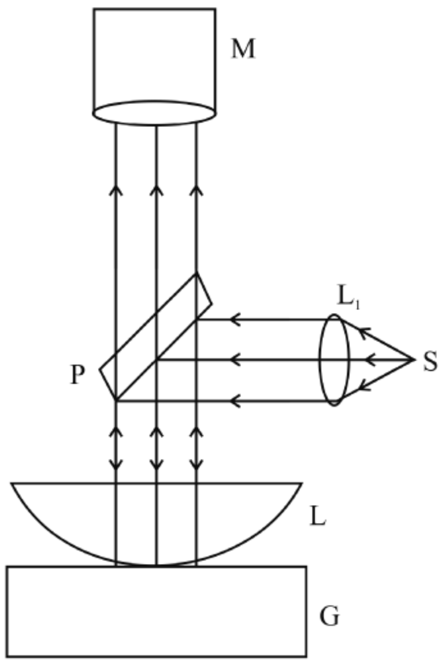


Figure : 10.17

Formation of fringes.

The path difference between two successive reflected rays QS_1R_1 reflected from the upper surface of the air film at Q and NS_2R_2 reflected from the lower surface of the film at N as shown in figure (10.16a) will be

$\Delta = 2d\mu\cos r \pm \frac{\lambda}{2}$ where d = thickness of the film at N, μ = refrative index of the film and r = inclination of the ray. Here the extra path difference $\frac{\lambda}{2}$ is introduced due to phase change of π on reflection at the lower surface of the film at G.

Now for normal incidence, r = 0, so the path difference is

$\Delta = 2d\mu \pm \frac{\lambda}{2}$ which is the condition for minimum intensity. Thus the central point (O) id dark.

For constructive interference i.e. for bright fringes, $\Delta = 2d\mu \pm \frac{\lambda}{2}$ = even multiple of $\frac{\lambda}{2}$

or, $2d\mu$ = odd multiple of $\frac{\lambda}{2}$

$\therefore\ 2d\mu = (2n-1)\frac{\lambda}{2}$ .. ..(10.38)

when n = 1, 2, 3 .....

And for destructive interference, i.e. for dark fringes $\Delta = 2d\mu \pm \frac{\lambda}{2}$ = odd multiple of $\frac{\lambda}{2}$

or, 2dm = even multiple of $\frac{\lambda}{2}$

$\therefore\ 2d\mu = 2n\frac{\lambda}{2} = n\lambda$ .. ..(10.39)

From equations (10.38) and (10.39) it is clear that, the bright or dark fringe of any particular order (n) occurs for a constant value of thickness (d) of the film. In the fil d is constant along a circle with centre at the point of contact (O). Hence the fringes will be concentric circlesfor different values of thickness (d).

Diameters of the rings.

Let r_n be the radius of the nth Newton's ring corresponding to the point P where the thickness of the film is d, as shown in figure 10.17

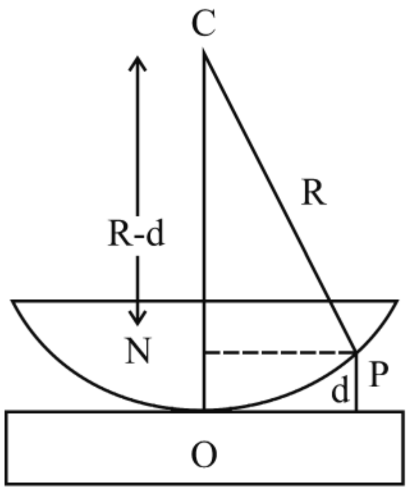


Figure : 10.17

Then from geometry.

$CP^2 = CN^2 + NP^2$

$R^2 = (R - d)^2 + r_n^2$

where R = radius of curvature of the plano convex lens $CP, r_n = NP$

and CN = CO–NO = R–d.

$R^2 = R^2 - 2Rd + r_n^2$ [∴ d is very small, d^2– neglected]

or, 2Rd = r_n^2

$r_n^2 = 2Rd$ ... ... (10.40)

We know the condition (from equation 10.38) for nth bright ring

$2d\mu = (2n-1)\frac{\lambda}{2}$

or, $\frac{r_n^2\mu}{R} = (2n-1)\frac{\lambda}{2}$

$$r_n^2 = (2n-1)\frac{\lambda R}{2\mu} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (10.41)$$

∴ Diameter of nth bright ring is

$$D_n = 2r_n = 2\sqrt{2n-1}\sqrt{\frac{\lambda R}{2\mu}}$$

$$= \sqrt{2n-1}.\sqrt{2\lambda R/\mu}$$

$$D_n = \sqrt{2n-1}\sqrt{2\lambda R/\mu} \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (10.42)$$

Since μ, λ and R are constant,

therefore $D_n \alpha \sqrt{2n-1}$

Hence the diameters of bright rings are proportional to the square roots of odd natural numbers

i.e. $D_1 : D_2 : D_3 ..... = \sqrt{1} : \sqrt{3} : \sqrt{5} ....,$

Similarly for nth dark ring we can write

$r_n^2 = \frac{n\lambda R}{\mu}$ using equations (10.39) and (10.40)

or, $r_n^2 = \frac{n\lambda R}{\mu}$

or, $r_n = \sqrt{\frac{n\lambda R}{\mu}}$, the diameter of nth darking is $D_n = \sqrt{\frac{4n\lambda R}{\mu}}$ ... ... (10.43)

Here also D_n is proportional to $\sqrt{n}$.

Now the difference in diameters of nth and (n+1) th order dark rings in

$$D_{n+1} - D_n = \sqrt{\frac{4\pi R}{\mu}}\left(\sqrt{n+1} - \sqrt{n}\right)$$

This shows that as the order number (n) increases this difference decreases, which means that the rings are gradually become narrower as their radii increase. Thus as we move outward from the centre, we see that the rings are more crowded.

Fringe width

From equation (10.42) we can write

$$D_{n+1}^2 - D_n^2 = \left[\left\{2(n+1)-1\right\}-(2n-1)\right]\frac{2\lambda R}{\mu}$$

$= \dfrac{4\lambda R}{\mu}$ where D_{n+1} and D_n are the diameters of two successive bright rings.

$$D_{n+1} - D_n = \frac{4\lambda R}{\mu(D_{n+1}+D_n)} \quad \text{writing } D_{n+1}+D_n \approx 2D_n$$

Fringe width, $\beta = \frac{1}{2}(D_{n+1}-D_n) = \frac{1}{2}\cdot\frac{4\lambda R}{\mu . 2D_n} = \frac{\lambda R}{\mu D_n}$ ... .(10.44)

Hence fringe width β decreases as the diameter of the ring (D_n) increase.

Newton's rings with white light :

The diameter of a ring depends upon the wavelength of light used (equation 10.42)

Thus if white light is used instead of monochromitic light a few coloured rings will be observed and beyond this there will be generall illumination due to overlapping of different rings, but the central spot will be dark.

10.12 Applications of Newton's rings

1) Determination of wavelength

Wavelength of monochromatic light can be measured with the help of newtons rings. The diameters of nth newton's ring for an air film are obtained from equations (10.42) and (10.43) are $D_n^2 = (2n-1)\left(\frac{2\lambda R}{\mu_{air}}\right) = (2n-1)(2\pi R)$ $[\because \mu_{air} = 1]$

or, $D_n^2 = 2(2n-1)\lambda R$ for brightring

and $D_n^2 = 4n\lambda R$ for darkring.

Now if D_n and D_{n+s} are the diameters of the nth and (n+s)th rings (bright or dark) then, from the above equation, we get

$$D_{n+s}^2 - D_n^2 = 4(N+S)\lambda R - 4n\lambda R$$

$$= 4R\lambda S$$

$$\lambda = \frac{D_{n+s}^2 - D_n^2}{4SR} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots. \ (10.45)$$

Hence, the wavelength λ of the monochromatic light can be obtained by measuring D_{n-s} , D_n, and by counting s. The radius R can be measured by a spherometer.

2. Determination of refractive index of a liquid:

To measure he refractive index of a liquid by newton's rings experiment. At first measure the diameter D_{n-s} and D_n i.e. the diameter of n+s and nth bright ordark rings with air film. After this, the diameters of these rings and measured again by forming the liquid film by pouring the experimental liquid between the lens and the glass plate within a container, without disturbing the arrangement.

For air film $\left(D_{n+s}^2 - D_n^2\right)_{air} = 4s\lambda R$

and for liquid $\left(D_{n+s}^2 - D_n^2\right)_{liquid} = \dfrac{4s\lambda R}{\mu}$

where μ is the refractive index of the experimental liquid.

$$\mu = \frac{\left(D_{n+s}^2 - D_n^2\right)_{air}}{\left(D_{n+s}^2 - D_n^2\right)_{liquid}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad .(10.46)$$

So, μ can be determined, if D_{n+s}, D_n for air and for liquid are known.

Exercise-7

Newton's ring experiment, the diameter of 10th dark ring is 0.5 cm. Find the radius of curvature of the lens. Given $\lambda = 5.9 \times 10^{-7}$ m

10.13 Michelson interferometer and its application

Michelsoninter fermeter:

The interference fringes of different shapes are produced in this interferometer by the method of division of amplitude.

Constructions :

The main optical parts of michelson interferometer consiists of two highly polished from silvered plane glass mirrors M_1 and M_2 placed at right angles to each other. G_1 and G_2 are two plane paralled glass plates of same material and thickness. The plates are held parallel to each other and inclined at 45° to the mirror M_2. The plate G1 is half silvered at the back, so that the incident beam is divided into reflected and ftransmitted beam of equal amplitudes. The mirror M_1 is mounted on a carriage to move it back and forth exactly parallel to itself with the help of a micromater screw fitted with a graduated drum to take readin of displacements of the order of 10^{-5} cm. The mirrors M_1 and M_2 can be tilted about both horizontal and vertical axes with the levelling screw at their back to make them exactly perpendicular to each other. The interference fringes are obserbed in the field of view of the telescope T as shown in figure–10.18.

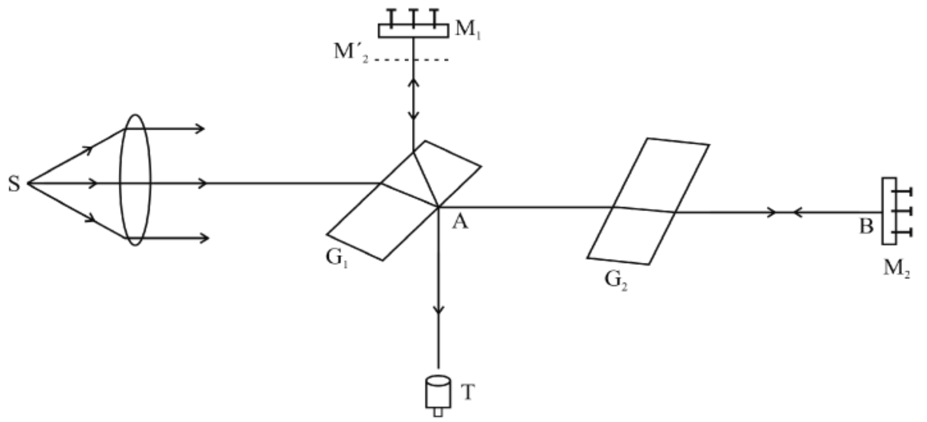


Figure : 10.18

Working principle

Monochromatic light from an extended source S is rendered parallel by a lens L and is made to fall on the glass plate G_1. It gets divided into two parts of equal amplitudes by partial reflection and transmission at the back surface of G_1. The reflected wave proceeds towards M_1 along AC and transmitted wave proceeds to M_2 along AB. Both the rays AC and AB fall normally on M_1 and M_2 respectively and retrace their paths after reflection and finally enter into the telescope T along AT.

Thus two beams along AT are produced from a single source by the division of the amplitude. These two beams produced interference under suitable conditions.

From figure 10.18 it is clear that the ray AC passes through G_1 thrice,but AB traverses the glass plate G_1 once only. To compensate for this an exactly similarly glass plate G_2 is introduced. Hence G_2 is called a compensator plate.

Form of fringes

If you look through the telescopeT twowards the mirror M_1, you will see the mirror M_1 and a virtual image M_2^1 of M_2 formed by reflection from the glass plate G_1. Thus we may consider that the two interfering beams received by the telescope are coming after reflection from M_1 M_2^1 (figure–10.19)

Depending on the path difference and angle between the mirrors M_1 and virtual mirror M_2 the fringes of different shapes such as circle, straight line etc may be formed.

Formation of circular fringes:

Circular fringes with monochromatic light is produced in Michelson interferometer, when mirrors M_1 and M_2 are exactly perpendicular to each other. The production of these fringes can be under stood from figure–10.19.

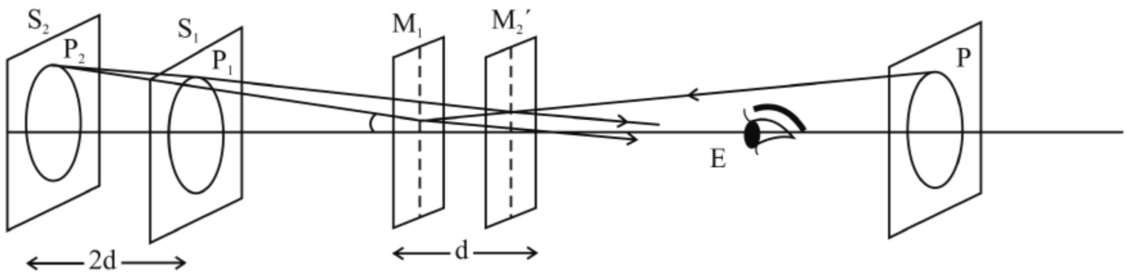


Figure : 10.19

Here M_1 and the image M_2^1 of M_2 are exactly parallel to each other. The real extended source is also replaced by its virtual image S formed by reflection in glass plate G_1 behind the eye E. This source S^1 produces two virtual images S_1 and S_2 after reflection from M_1 and M_2 respectively. These two virtual images S` and S_2 behave as virtual coherent sources and consequently the phases of the corresponding points in them are exactly the same at all instant. Thus the points P_1 and P_2 on S_1 and S_2 are the two virtual image points of the point P in S^1 formed by the reflections in M_1 and M_2^1 respectively are always exactly in phase.

If d is the distance between M_1 and M_2^1, then the distance of separation between S_1 and S_2 will be 2d. Let θ is the angle which the reflected beams make with the normal, then the path difference between the two rays entering to the eye, from the corresponding points P_1 and P_2 of the two virtual sources is 2d cosθ.

If 2d cos = nλ (where n = 0, 1, 2,....) the point P appears bright ie maximum and

if $2d\cos\theta = (2n+1)\frac{\lambda}{2}$ p, appears dark i.e minimum.

Thus for a given value of n, d and λ the angle θ will be constant, hence locus of the point P is a circle about the foot of the perpendicular from the eye to the mirrors as the centre.

So, a series of concentric alternate bright or dark circular fringes are osbserved. These fringes are called fringes of equal inclinations and are fromed at infinity. These fringes are non-localized.

Formation of localized fringes:

When two mirrors M_1 and the image M_2' of M_2 are not exactly parallel, then they enclose an wedge shaped film of air between them. Hence a ray PQ from the extended source S after reflection from M_2 and M_1 will give two reflected rays QE_1 and RE_2 which on producing backward will meet at a point O, where the interference fringes will be formed (figure10.20) To observed these fringes the eye must be focussed near to the mirror. Thus these fringes are localized fringes.

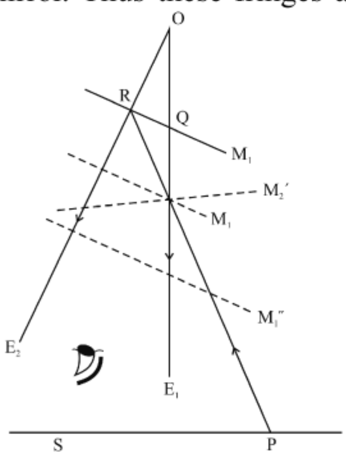
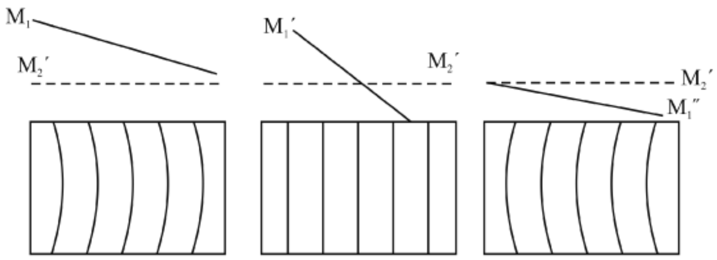


Figure : 10.20



(i) (i) (i)

Figure : 10.21

For a certain value of d, the light is incident on the film at various angles. So observe fringes are curved and are always convex towards the thin edge of the wedge as shown in figure {(i) and (iii) of 10.21}. But if the path difference 2d cosθ betwen the mirrors M_1 and M_2^1 approach to zero, the fringes becomes straight till the mirror M_1 interesects M_2^1 as shown in figure–10.21 (ii).

Localized white light fringes :

When a white light source is used a few coloured fringes with a central dark fringe can be observed. In observing these fringes, the mirrors M_1 and M_2^1 are adjusted for localised straight fringes. The position if often troublesome to find with white light. The position can be located with monochromatic light when the fringes are straight. The using white light a few coloured fringes (8–10) are observed on either side of a central dark fringe.

Applications of Michlson interferometer :

(i) Determination of wavelength of monochromatic light.

To measure the wavelength of monochromatic light the interferometer is adjusted $M_1 \parallel M_2$ to obtain circular fringes in the fild at view of the telescope. If the mirror M_1 is moved forward orbackward, the circular fringes appear or disapper at the centre. Now as the mirror is moved through a known distance the number of fringes disappearing of the centre is counted. Let D_1 is the initial thickness of the air film between the mirror M_1 and the image M_2^1 of M_2 corresponding to the n_1th order of bright fringe and d_2 for the (n+s)the order fringe.

Then $2d_1 = n\lambda$ and $2d_2 = (n+s)\lambda$

or, $2(d_2-d_1) = s\lambda$

$$\therefore \lambda = \frac{2(d_2-d_1)}{s}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (10.47)$$

Hence measuring d_1, d_2 and counting s, λ can be determined accurately.

2. Determination of the refractive index of a material :

To determine the refractive index of a material the interferometer is adjusted for localized white light fringes and a cross wire made to coincides with the central dark fringe. In this position the optical path difference between two interfering rays are equal (ie AB = AC). Now a thin film of thickness t and refractive index μ is introduced in the one of the path of the interfering rays, when an extrapath of $(\mu-\ell)t$ is introduced in the side of the film. Due to this extra path, the central fringe will be displaced from the wire. The mirror M_1 is then to be moved by a distance d(say) until the central fringe coinsides again with the wire. Hence we can write.

$2d\ (\mu-1)t$ or, $d = (\mu-1)t$

$$\therefore\ t = \frac{d}{m-1} \quad \ldots\ldots(10.48)$$

Thus knowing t and d, μ can be determined. Again knowing μ and d, t can be tdetermined also from this experiment.

Exercise–8

A Michelson interferometer is set for white light straight fringes. Shen a thin mica sheet of thickness 0.005 canmriginal position, the movable mirror is moved by 0.0025 cm. Calculate the refractive index of mica.

10.14 Summary

- The resultant amplitude of the waves after superposition in youngs double slit experiment is

  $A = \left(a_1^2 + a_2^2 + 2a_1a_2\cos\delta\right)$

- Where $\delta = \frac{2\pi}{\lambda}\left(x_2 - x_1\right), x_2 - x_1$ = path difference between the two waves and δ is the correspondingphase difference
- Condition for constructive interference is path difference = even multiple of $\frac{\lambda}{2} = 2n\frac{\lambda}{2}$ and for distructive interference

  path difference = odd multiple of $\frac{\lambda}{2} = \left(2n+1\right)\frac{\lambda}{2}$

- Fringe width $\beta = \frac{\lambda D}{d}$.$D =$ distance between slit and screen d = distance between two slits S_1 and S_2.
- Acute angle of the biprism is $\alpha = \frac{d}{2a\left(\mu - 1\right)}$ a = distance between the position of the slit and biprism.
- Conditions for bright and dark fringes in Loyd's mirror are odd multiple of $\frac{\lambda}{2}$ and even multiple of $\frac{\lambda}{2}$ respectively.
- The effective path difference between two interfering rays in a thin film is $\Delta = 2d\mu\cos\left(r - \alpha\right) \pm \frac{\lambda}{2}$
- Fringe width $\beta = \frac{\lambda}{2\mu\alpha}$ for thin film. α = angle between the upper and lower surface of the film.

- Wave length $\lambda = \frac{D_{n+s}^2 - D_n^2}{4SR}$ can be determined in Newton's ring experiment.
- Wave length $\lambda = \frac{2(d_2 - d_1)}{s}$ and the thick of a thin film $t = \frac{d}{\mu - 1}$, can be determined by michelsom interfero meter.

10.15 Questions and Problems

10.15.1 A monochromatic light of wavelength 5100Å from a narrow slit is incident on a double slit. If the overall sparation of 10 fringes on the screen 200 cm away is 3 cm find the distance of separation between the slits

10.15.2 Interference fringes are observed with a biprism of retracting angle 1° and refractive index 1.5 on a screen 100 cm away from it. If the distance tetween the source and birism is 10 cm. Calculate the fringe width. Given λ = 5890Å.

10.15.3 In a Lioyd's mirror experiment, calculate the ration of the intensities of the inderference minima and maxima if the mirror reflects only75% of the light incident on it.

10.15.4 Fringes are produced with monochromatic light of wavelength 689nm.. A thin film of glassof μ = 1.52 is placed normally in the path of one of the interfering beams. The central fringe is found to be shifted to a position occupaid by the 5th bright band from the centre. calculate the thickness of the glass film.

10.15.5 In Newton's ring experiment, the diameter nth dark ring is 8 mm and diameter of (n+5)th dark ring is 12 mm. If the radius of curvature of the lens is 8 m, find the wavelength of light used.

10.5.6 In Michelson's interferometer, 100 fringes cross the field of view when the movable mirror is displaced through 0.02948mm. Calculate the wave length of monochromatic light used.

10.16 Solution

Exercise–1

We know the maximum intensity

$I_{max} = (a_1 + a_2)^2$ there a_1 = 3, a_2 = 4

$= (3+4)^2$

=49

Exercise–2

$\beta = 10^{-3}\text{m}, \frac{b}{a} = 20, \lambda = 5893 \times 10^{-10}\,\mathbf{m}$

μ = 1.5 and α = ?

we know the fringe width

$$\beta = \frac{\lambda D}{\alpha} = \frac{\lambda(a+b)}{2a(\mu-1)\alpha} \qquad [\because d = 2a(\mu-1)\alpha \text{ equation } 10.19]$$

$$= \frac{\lambda}{2(\mu-1)\alpha}\left(1+\frac{b}{a}\right)$$

$$\therefore \quad \alpha = \frac{\lambda}{2(\mu-1)\beta}\left(1+\frac{b}{a}\right)$$

$$= \frac{5893 \times 10^{-10}}{2(1.5-1) \times 10^{-3}}(1+20)$$

= 0.0124

$$= 0.0124 \times \frac{180}{1}$$

= 0.711°

Exercise–3

Here, a = 20 cm= 0.2m, μ = 1.5

$$\alpha = \frac{\pi}{180}.2 = \frac{\pi}{90}\,\text{radian}$$

The distance of separation between two slits is $d = 2a(\mu-1)\alpha$

$$2 \times .20(1.5-1)\frac{\pi}{90}$$

=0.00698m.

Exercise–4

Here λ = 589 nm = 589×10^{-9}m, μ= 1.542 and distance x_n = 2.143 mm, = 0.347 mm

If n is the number of fringes contained in x_n, then

$$n = \frac{x_n}{\beta} = \frac{2.143}{0.347} = 6.175$$

Now $(\mu - 1)t = n\lambda$

$$t = \frac{n\lambda}{\mu - 1} = \frac{6.175 \times 589 \times 10^{-9}}{1.5 - 1}$$

$\therefore$ t = 7.27×10^{-6}m.

Exercise–5

Here,

Slit is at adistance from the mirror, thus the distance between two coherent source

d = 2×3 mm = 6×10^{-3}m.

λ = 5890Å = 5890×10^{-10}m.

D = 1.5m.

Fringe width $\beta = \frac{\lambda D}{d} = \frac{5890 \times 10^{-10} \times 1.5}{6 \times 10^{-3}} = 1.47 \times 10^{-4}\,\mathbf{m}$

Exercise–6

Here, $\mu_{air} = 1, \beta = 1mm = 10^{-3}\,m$

and $\lambda = 589nm = 589 \times 10^{-9}\,m.$

we know, $\beta = \frac{\lambda}{2\mu\alpha}$

$$\therefore \quad \alpha = \frac{\lambda}{2\mu_{air}\beta} = \frac{589 \times 10^{-9}}{2 \times 10^{-3}} = 2.945 \times 10^{-4}\,\text{rad}$$

$$\alpha = 2.945 \times 10^{-4} \times \frac{180°}{\lambda}$$

= 0.01688°.

Exercise–7

n = 10, D_n = 0.5 cm = 0.5×10^{-2}m, λ = 5.9×10^{-7}m

we know $D_n^2 = 4n\lambda R$

$$R = \frac{D_n^2}{4n\lambda} = \frac{(0.5)^2\times10^{-4}}{4\times10\times5.9\times10^{-7}} = 1.059\text{m}$$

Exercise–8

Here t = 0.005 cm

d = 0.0025 cm

we know $t = \frac{d}{\mu-1}$

or, $\mu - 1 = \frac{d}{t}$

$$\therefore\ \mu = 1 + \frac{d}{t} = 1 + \frac{0.0025}{0.005} = 1.5$$

10.15.1

Here, λ = 5100Å = 5100×10^{-8}cm.

Overall separation of 10 fringes is 3 cm

$\therefore\ \beta = \frac{3}{10}\text{cm}$, D = 200 cm

we know, $\beta = \frac{\lambda D}{d}$

$$\therefore\ d = \frac{\lambda D}{\beta} = \frac{5100\times10^{-8}\times200}{3/10} = 0.034\,\text{cm}$$

10.15.2 Here, $\alpha = 1° = \frac{\pi}{180}\text{rad}.\ \mu = 1.5,$

b = 100 cm, a = 10 cm, λ = 5890Å = 5890×10^{-8}cm

D = a+b = 900+10 = 110 cm

we know $\beta = \frac{\lambda D}{d}$ and d = 2a (μ–1)α

$$\therefore \ \beta = \frac{\lambda D}{2a(\mu-1)\alpha} = \frac{5890\times10^{-8}\times110\times180}{2.10(1.5-1)\pi} = 0.037 \text{ cm}$$

10.15.3 Here, reflectivity (R) = 75% = $\frac{75}{100} = \frac{3}{4}$

The amplitude reflection co-efficient, r $\sqrt{R} = \sqrt{\frac{3}{4}}$

so, if a_1 be the amplitude of direct beam, then the amplitude of rreflected beam is

$$a_2 = a_1 r = \sqrt{\frac{3}{4}}a_1$$

$$\frac{I_{min}}{I_{max}} = \frac{(a_1-a_2)^2}{(a_1+a_2)^2} = \left(\frac{a_1 - a_1\sqrt{\frac{3}{4}}}{a_1 + a_1\sqrt{\frac{3}{4}}}\right)^2$$

$$= \left(\frac{1-\sqrt{\frac{3}{4}}}{1+\sqrt{\frac{3}{4}}}\right)^2 = \left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^2 = \left[\frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})}\right]^2$$

$$= \left[\frac{(2-\sqrt{3})^2}{2^2-3}\right]^2 = \frac{5.15\times10^{-3}}{1} = 5.15\times10^{-3}$$

10.15.4 Here λ = 689 nm = 689×10–9 m, μ = 1.52

shift of the central fringe, $x_n = 5\beta$

we know, $t = \frac{x_n \lambda}{(\mu - 1)\beta}$ [using equation–10.24]

$$= \frac{5\beta \times 689 \times 10^{-9}}{(1 - 1.52)\beta}$$

$\therefore$ t = 6.625×10^{-6}m.

= 0.006625 mm

10.15.5

Here $D_n = 8\text{mm} = 8 \times 10^{-3}\text{m}$.

$D_{n+5} = 12\text{mm} = 12 \times 10^{-3}\text{m}$

R = 8m., s = 5

we know, $\lambda = \frac{D_n^2 + s - D_n^2}{4SR} = \frac{D_{n+5}^2 - D_n^2}{4.SR}$

$$= \frac{(12 \times 10^{-3})^2 - (8 \times 10^{-3})^2}{4.5.8}$$

$\lambda = 5 \times 10^{-7}\text{m}$ = 500 nm.

10.15.6

Here, n = 100, d = 0.02948mm = 0.2948×10^{-3}m

we know 2d = nλ

$\therefore \lambda = \frac{2d}{n} = \frac{2 \times 0.02948 \times 10^{-3}}{100} = 5.896 \times 10^{-7}\text{m}$

= 5896×10^{-10}m.

$\therefore$ λ = 5896Å

Unit : 11 ❒ Diffraction of Light

Structure

11.1 Objectives

After studying this unit, you will be ablet

- define diffraction
- discuss the concept of fresnel half period zone, zone plate and their applications.
- discuss the diffraction patterns due to straight edge, thinwire, narrow rectangular aperture of Fresnel class of diffraction
- explain Fraunhoffer's class of diffraction due to single slit, double slits and multiple slits i.e plane diffraction grating.

1.0 Introduction

In school level physics you have learnt that light is propagating in a straight line path. If an obstacle held in the path of light rays they from a sharp geometrical shadow.

But if you examine the shadow carefully, you will see that it is not so as you expected. The departure from the expected result confirms the light is not travelling in a straight line, it bends round the corners. Such an efect of bending at light round a sharedge of the opaque obstacle is called diffraction effect. This phenomenon was first observed by Grimaldi. The effect of bending is negligible small if the dimensions of the obstacle or aperture and large compared to the Navelenght of ligh. If the size of the aperture or obstacle is comparable to the Navelength of light, the bending becomes more pronounced, all waves like sound, light, x-ray etc. exhibit diffraction under suitable conditions.

Fresnel gave a satisfactory explanation of the diffraction of light, consideringHuygen's principle i.e. the principle of secondary wavelets and based on the superposition of waves. Diffraction phenomenon (diffraction fringes) occurs due to multual interference (diffraction fringes) ocurs due to mutual interference of secondary waves from different points of a particular wave front not blocked by the obstactle. Fraunhoffer also gave an explanation of diffraction considering the source and screen are placed at infinite distance from the obstacle. But Fresnel assumed both source and screen are placed at nearer points.

All optical instruments are not free from diffraction effects as a limited portion of the incident wave fron is used. Hence, diffraction effects are of great importance to understand the optical devices.

In this unit, we shall discuss both Fresnel and Fraunhoffer's class of diffractions.

11.2 Diffraction

11.2 Definition of Diffraction :

The bending of light round the edges of an obstable or apertures of sizes comparable with the wavelenght of light and spreading of light into the geometrical shadow of the object is called diffraction.

11.2.1 Difference between interference pattern and diffraction pattern

Interference	Diffraction
1. Interference in the result of superposition of light waves coming from two different wave fronts originating from the same source.	1. Diffraction pattern is the result of superposition of light waves coming from different parts of the same wave front.
2. Interference frinages may or may not be of equal width.	2. Fringes are not of the equal width
3. Minimum intensity points are perfectly dark.	3. Minimum intensity points are not pertectly dark.
4. All bright band are of uniform intensity.	4. Bright bands are of different intensity.

11.2.2 Classes of diffraction:

There are two classes of diffraction phenomenon known as Fresnel's diffraction and Fraunhoffer diffraction.

(i) Fresnel's diffraction : In this type of diffraction, the source of light or screen or both are at finite distances from the obstacle or parture. Hence, the incidident wave front is spherical or cylindrical.

(ii) Fraunhoffer diffraction : In this type of diffraction, the source of light and the screen are effectively at infinite distances from the obstacle or apperture. This may be made by using two convex lenses, to make the light source parallel before it falls on the aperture and the other to focus the light after diffraction on the screen. Thus the wave front is plane.

11.3 Fresnel's half period zones of a plane wave front

Fresnel explained the phenomenon of diffraction of light onthe basis of the multual interference of the secondary waves of wavelets from the different points of a wavefornt.

Let ABCD isa plane wave front of monochromatic light of wavelength λ is advancing twowards the right and P is a an external point at which the resultant intensity to be found out due to the wave from ABCD. Fresnel divided the wave front into a number of half period elements or zones called Fresnel's half period zone and find the effect of all the zones at the point P.

From P draw a perpendicular on ABCD at the point O. the foot of the perpendicular is called the pole of the wave with respect to P. Let PO = b and with P as centre and radii $b+\frac{\lambda}{2}, b+\frac{2\lambda}{2}, b+\frac{3\lambda}{2}$...... etc. Spheres are drawn, the sections of which made by the wavefront are concentric circles M_1, M_2, M_3 ..... M_n etc. with centre at O. The area enclosed bythe first circle (M_1) is called first half period zone. The annutar space between the first and (M_1) and the second circle (M_2) is called the second half period zone and so on, as shown in figure : 11.1.

Areas of zones:

The area of the nth zone i.e. the area between the circles M_n and M_{n-1} is $A_n = \pi\left[IM_n^2 - OM_{n-1}^2\right]$

or $A_m = \pi\left[\left(M_nP^2 - b^2\right) - \left(M_{n-1}P^2 - b^2\right)\right]$

$$= \pi\left[\left\{\left(b+\frac{n\lambda}{2}\right)^2 - b^2\right\} - \left\{\left(b+\overline{n-1}\frac{\lambda}{2}\right) - b^2\right\}\right]$$

$$= \pi\left[\left(n\lambda b + \frac{n^2\lambda^2}{4}\right) - \left\{(n-1)^2\frac{\lambda^2}{4} + b(n-1)\lambda\right\}\right]$$

$$= \pi\left[nb\lambda + \frac{n^2\lambda^2}{4} - \left(\frac{n^2\lambda^2}{4} - \frac{n\lambda^2}{2} + nb\lambda - b\lambda\right)\right]$$

$$= \pi\left[b\lambda + \frac{\lambda^2}{4}(2n-1)\right] \simeq \pi b\lambda \quad ....(11,1)$$

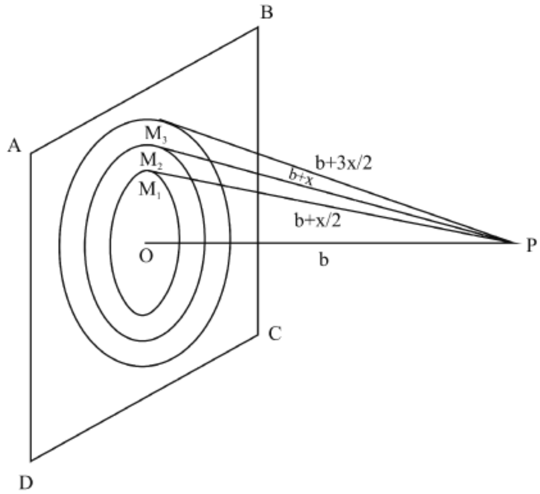


Figure : 11.1

since $b >> \lambda$ so λ^2 is neglected.

Thus the area of all zones are appoximately equal. But actually the area of the zone increases with increase in order number n.

Resultant amplitude due to all wave fronts.

Since each zone differs from its neighbour by a path difference of $\frac{\lambda}{2}$, hence the phase difference of the alternate zones will be λ. Thus if the resultant amplitude of wavelets from first half period zone is positive, that from second half period zone will be negative, that from third zone positive and so on

Let d_1, d_2, d_3 ..... etc half period zones respectively then,

Resultant amplitude, D= d1 – d2 + d3 –d4 _ ..... +(–1) n–1 an.... (11.2)

It can be written as,

$$D = \frac{d_1}{2} + \left(\frac{d_1 + d_3}{2} - d_2\right) + \left(\frac{d_3 + d_5}{2} - d_4\right) + ... + \frac{d_n}{2} \text{ (if n is odd)}$$

$$= \frac{d_1}{2} + \left(\frac{d_1 + d_3}{2} - d_2\right) + \left(\frac{d_3 + d_5}{2} - d_4\right) + ... + \frac{d_{n-1}}{2} - d_n \text{ (if n is even)}$$

As d_1 , d_2, d_3 .... are in descending order of magnitudes $\frac{d_1 + d_3}{2} - d_2$ and so on. Hence, all the terms of the above equation within bracket will be zero and we get,

$$D = \frac{d_1}{2} + \frac{d_n}{2} \text{ (When n is odd)}$$

and $D = \frac{d_1}{2} + \frac{d_{n-1}}{2} - d_n$ (When n is even)

If n is sufficiently large, the effect due to the nth zone becomes negligible and resultant amplitude due to the whole wave is

$$D = \frac{d_1}{2}$$

(for both odd or even n)

Thus the resultant amplitude at p due to the whole wave front is equal to half the amplitude to the secondary waves from the first half period zone.

As intensity is proportional to the amplitude

Intensity, $I = \frac{d_1^2}{4}$

Hence the intensity at P due to the wavelets from all zones is equal to one fourth of the intensity due to the waves from the first half period zone.

11.3.1 Rectilinear propagation of light.

We have seen in the previous article that the resultant disturbance at P is equal to the half of the disdurbance due to first half period zone only. The amplitude is also decreases rapidly as the order of the zone increases. Thus if small obstacle is placed at O it blocks a considerable number of half period zones, by which the light from the source is practically cut off i.e. no light will be received at P. In other words, light travels in straight line.

11.4. Zone plate.

zone placte is a transparent plate on which concentric circles of radii proportional to the square roots of natural numbers is $\sqrt{1}:\sqrt{2}:\sqrt{3}$........ etc. are drawn. The alternate even or odd annular spaces betweeen the circles are blocked such a plate behaves like a convex lens and produces image of a light source on a screan placed at a suitable distance.

If odd zones are transparent and even zones are blocked, then it is called a positive zone plate as shown in figure 11.2a. If even zones are transparent and odd zones are blocked, then it is called negative zone plate as in figure 11.2b.

Therory of zone plate.

Let O be a point source of monochromatic light emmitting spherical waves of wavelength λ whose effect at the point I on the screen is required.

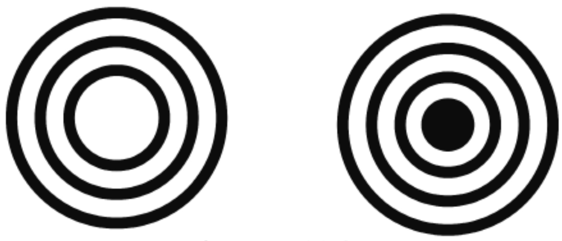


Figure : 11.2

Let an imaginary plane perpendicular to the plane of the paper through the point P of a transparent medium. Draw a perpendicular OP to the plane and produce it to I as shown in figure 11.3. Divide this plane into zones bounded by circle zones bounded by circle having centres at P and radii $PM_1 = r_1$

$PM_2 = r_2, PM_3 = r_3$

$PM_n = r_n$ Such that

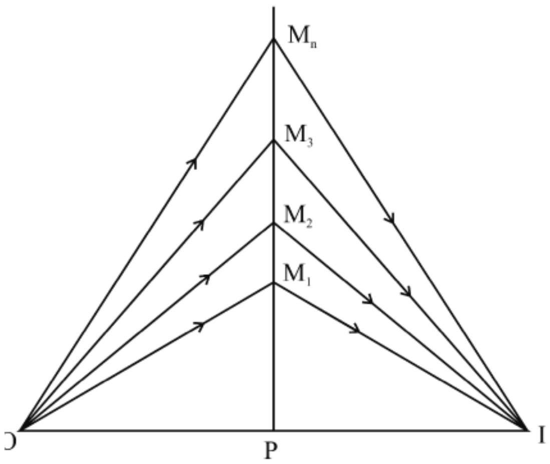


Figure : 11.3

$$OM_1 + IM_1 = OP + IP + \frac{\lambda}{2}$$

$$OM_2 + IM_2 = OP + IP + \frac{2\lambda}{2}$$

$$OM_n + IM_n = OP + IP + \frac{n\lambda}{2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (11.3)$$

The annular rings thus formed are half period zones for the image I, because the path difference between the corresponding points of two consecutive zones is $\frac{\lambda}{2.}$

Let OP = u and IP = v, then we have from (figure-11.3)

$$OM_n = \left(OP^2 + PM_n\right)^{\frac{1}{2}} = \left(u^2 + r_n^2\right)^{\frac{1}{2}}$$

$$= u\left(1 + \frac{r_n^2}{u^2}\right)^{\frac{1}{2}}$$

$$= u + \frac{r_n^2}{2u}\left[\because u >> r_n\right]$$

Similarly $IM_n = \left(v^2 + r_n^2\right)^{\frac{1}{2}} = v + \frac{r_n^2}{2v}\left[\therefore v >> r_n\right]$

Putting these values in equation (11.3) we get

$$u + \frac{r_n^2}{2u} + v + \frac{r_n^2}{2v} = u + v + \frac{n\lambda}{2}$$

or, $r_n^2\left(\frac{1}{4} + \frac{1}{v}\right) = n\lambda$

$$\therefore \quad \frac{1}{v} + \frac{1}{u} = \frac{n\lambda}{r_n^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (11.4)$$

Applying sign convention, we have from equation (11.4)

$$\frac{1}{v} - \frac{1}{u} - \frac{n\lambda}{r_n^2} = \frac{1}{f_n} \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (11.5)$$

Where $f_n = \frac{r_n^2}{n\lambda}$, Equation (11.5) is similar to convex lens fromula,

Thus zone plate may behave as a conven lens of focal length.

$$f_n = \frac{r_n^2}{n\lambda} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11.6)$$

Again from equation (11.4) we have

$$r_n^2 = \frac{n\lambda nv}{u+v}$$

Since u, v, λ are constants.

$r_n \alpha \sqrt{n}$, Thus the radii of the half period zones are proportional to the square roots of natural numbers.

The are of the nn zone is given by

$$\pi\left(r_n^2 - r_{n-1}^2\right) = \pi\left\{\frac{n\lambda uv}{u+v} - \frac{(n-1)\lambda uv}{u+v}\right\}$$

$$= \frac{\pi uv}{u+v}\lambda = \text{constant.}$$

This is independent of n, hence for a given object (u) and image (v) distances, the areas of all zones are same. Again, the are increases as u or u increases i.e. as the plate oves away from the object or image.

Again as all the zones are of equal area and hence the magnitude of the amplitudes $d_1, d_2 \ldots d_n$ at I due to secondary wavelets from the various zones, diminish only slightly with the order of the zone Hence the resultant amplitude at I

$$D = d_1 - d_2 + d_3 - d_4 + \ldots..$$

Now if the alternate zones say even are blocked, then the resultant amplitude at I will be

$$D = d_1 + d_3 + d_5 + \ldots..$$

Again if we block odd number of zones then we have the resultant amplitude at I;

$$D = d_2 + d_4 + d_6 + \ldots\ldots.$$

Thus we see that in both cases the resultant amplitude at I is many times greater than that due to the wavelets from all zones. Thus I will be the point of maximum intensity, i.e. the light from O will be focussed at I. Under this condition I is said to be the image of the object O.

11.4.1 Difference between convex lens and zone plate

(i) For a particular position of object, a lens produces only one image, where as a zone plate produces a number of images of diminishing intenity, because for a given λ the lens has only one focus but the zone plate has multiple focal lengths (n-dependent) $f_n = \frac{r_n^2}{n\lambda}$.

(ii) light from the consecutive clear zones of the zone plate arrives at the image point I after one complete period of the wave. Buit in a lens the rays reach the image point in same phase.

(iii) The focal length of a lens is given by the relation $\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ Where as that of the zone plate is $\frac{1}{f} = \frac{n\lambda}{r_n^2}$

(iv) For a zone plate focal length decreases as λ increases. Therefore, the focal length for red light is less than for violet $(\text{i.e.} f_r < f_v)$ for zone plate, but for convex lens $f_r > f_v$.

(v) The image is formed in a zone plate by diffraction, but for convex lens image formed by refraction, and image of convex lens is more intense than that of zone plate.

Exercise–1

What is the radius of the first half period zone in a zone plate behaving like a convex lens of focal length 60 cm for a light of wavelength 6000Å.

11.5. Fresnel's diffraction at a straight edge.

Let S be a narrow slit illuminated by a source of monochromatic light of wave length λ. The length of the slit is perpendicular to the plane of the paper. AB is a straight edge (e.g. edge of a razor blade) and the length of the edge is parallel to the length of the slit MN is the incident cylindrical wave front. O is a point on the screen XY. Join SAO is perpendicualar to the screen. The screen is perpendicular to the plane of the paper. According to geomatrical optics below the point O is the geometrical shadow and above is the illuminated portion. But in fact, it is observed that the dark and bright bands of unequal width above O and the intensity below O falls off repidly and becomes zero at a small distance from O, all are shown in figure (11.4a and b).

To study the intensity at any point Q at a distance x above O on the screen, join QS which interesects wave fron MN at P. Thus P is the pole of the wave front with respect

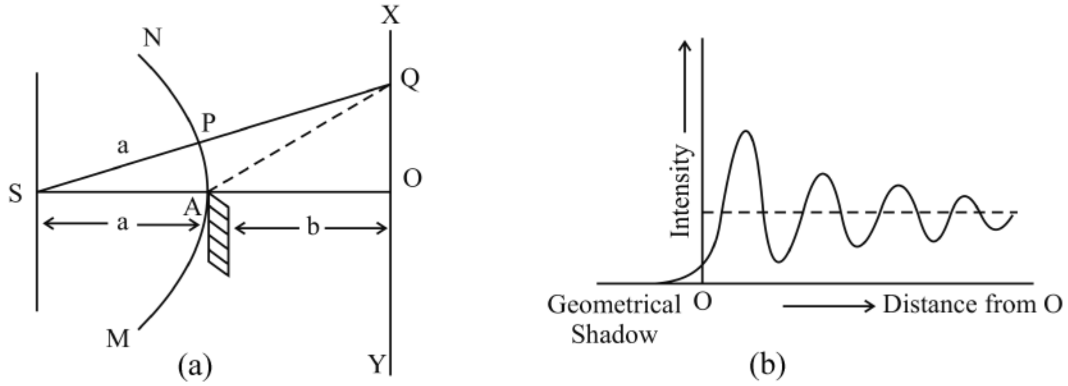


Figure : 11.4

to the point Q. With P as pole construct Fresnel's half period elements. The effect at Q depends on the number of half period elements contained in PA and the upper half of the wave PN.

The effect at O is due to the upper half (AN) of the wave front only. Hence, the displacement at O is half of the displacement that would have been there if the entire wave front was effective. The intensity at O, is therefore one fourth of that at a point for removed from O, where the whole wave front is effective.

As we move in the geometrical shadow, the pole P moves from A towards M and 1st, 2nd , 3rd. etc, half periods are intercepted and the intensity falls off gradually. Again it we move from O towards X, 1st, 2nd 3rd....etc. half period elements are exposed. The illumination at Q is due to the complete half of the wave surface PN and the resultant of the number of half period elements contained in PA.

The amplitude at Q will be maximum or minimum depending on the number of odd or even number of half period elements contains in AP. [As we know, $D_O = \frac{d_1}{2} + \frac{d_n}{2}$ odd number of elementas and $D_e = \frac{d_1}{2} + \frac{d_{n-1}}{2} - d_n$ for even number of elements, i.e. $D_O > D_e$.] The intensity at a maximum goes on decreasing and at a minimum goes on increasing, till finally at a large distance from O, we get a uniform illumination as shows in figure (11.4b)

The number of half period elements in AP (figure-11.4a) depends upon the path difference AQ –PQ.

Let SA = a and AO = b, then

$$AQ = \left(b^2 + x^2\right)^{\frac{1}{2}} = b\left(1 + \frac{x^2}{b^2}\right)^{\frac{1}{2}} = b + \frac{x^2}{2b} \left[\because b \simeq x\right]$$

Similarly

$$SQ = \left\{(a+b)^2 + x^2\right\} = a + b + \frac{x^2}{2(a+b)}$$

$$PQ = SQ - SP = a + b + \frac{x^2}{2(a+b)} - a \quad [\because SP = a]$$

$$= b + \frac{x^2}{2(a+b)}$$

Path difference, $AQ - PQ = b + \frac{x^2}{2b} - b - \frac{x^2}{2(a+b)}$

$$= \frac{ax^2}{2b(a+b)}$$

Now for maximum brightness at Q.

If $AQ - PQ = \frac{ax_n^2}{2b(a+b)} = (2n+1)\frac{\lambda}{2}$ [Where n = 0, 1, 2 .....]

$$\therefore \quad x_n = \sqrt{\frac{b(a+b)}{a}(2n+1)\lambda} \quad ....(11.7)$$

And similarly for minimum at Q

$$x_n = \sqrt{\frac{b(a+b)}{a}2n\lambda}$$

From equation (11.7) or, (11.8 we can write

$$x_{n+1}^2 - x^2 n = 2k \quad \text{where } k = \frac{b(a+b)\lambda}{a}$$

The fringe width

$$\beta = x_{n+1} - x_n = \frac{2k}{x_{n+1} + x_n} \simeq \frac{k}{x_n} \quad ....(11.9)$$

From the equation (11.9) we see that, as n increases x_n increases and β decreases.

Thus the spacing of dark or bright fringes decreases with the increase of order number (n).

Determination of wavelength

By measuring the distance between the position of first maximum intensity fringe and the one (say nth) most distant clearly visible bright fringe, by means of a micrometer eye-piece, we can measure λ by using the relation (11.7) Hence

$$x_1 = \sqrt{\frac{b(a+b)}{a}\lambda} \quad [\because n = 0]$$

For n the maxima

$$x_n = \sqrt{\frac{b(a+)}{a}(2n+1)\lambda}$$

$$x_n - x_1 = \sqrt{\frac{b(a+b)}{a}}\left(\sqrt{2n+1} - 1\right)$$

The distance b from the straight edge to the eye piece and a between the slit and the straight edge are measured. Hence knowing $x_n - x_1$, we can determine the wavelength of light.

11.6 Fresnel's diffraction by a turn wire

Let a narrow slit S is illuminated by monochromatic light of wavelength λ. A fine wire AB is placed parallel to the slit and is perpendicular to the plane of the paper. The screen XY is also perpendicular to the paper. Let waves from S are intercepted by ABA to produce the geometrical shadow PQ on the screen XY as Shown in figure–11.5.

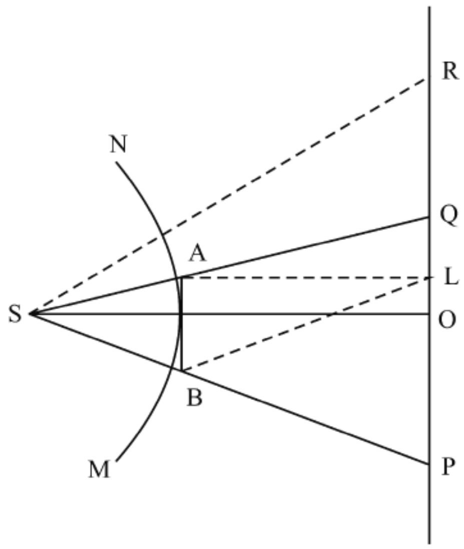


Figure : 11.5

We consider a point R outside the geometrical shadow on the screen. Join SR which interesets the wave fron MN at C. Thus C is the pole of the wavefront with respect to R. Thus C is the pole of the wave front with respect to R. The intensity at R due to the wave front above C is same for all points and effect due to the wave front BM is negligible. The intensity at R will be maximum or minimum depends on the number of half period elements contains in CA is odd or even. Thus the diffraction bands of gradually decreasing intensity will be observed in the illuminated protion of the screen.

Now within the geometical shadow if we consider a point L, the inter ference bands of equal width will be observed in this region due to the fact that the points A and B of the incident wave front are similar to the two coherent sources. Intensity at the point L will be maximum or minimum depends on the path difference BL–AL is equal to even multiple of $\frac{\lambda}{2}$ or odd multiple of $\frac{\lambda}{2}$

The fringe width $\beta = \frac{\lambda D}{d}$ where D = the distance between the wire and screen and d = the distance between two coherent sources (AB) ie the diameter of the wire.

$\therefore \ \beta = \frac{\lambda D}{2r}$ .... (11.10) where r is the radius of the wire.

The intensity distribution due to a narrow wire is shown in figure 11.6a

Here the centre of the geometrical shadow (O) is bright, because the path difference between AO and BD is zero.

If the wire is very thick, then from equation (11.10), we seen that as diameter increases the fringes width decreases and finally when the thickness of the wire becomes large engough, the interference fringes disappear. But outside of the geometrical shadow the diffraction bands are visible as shows in figure–11.6b

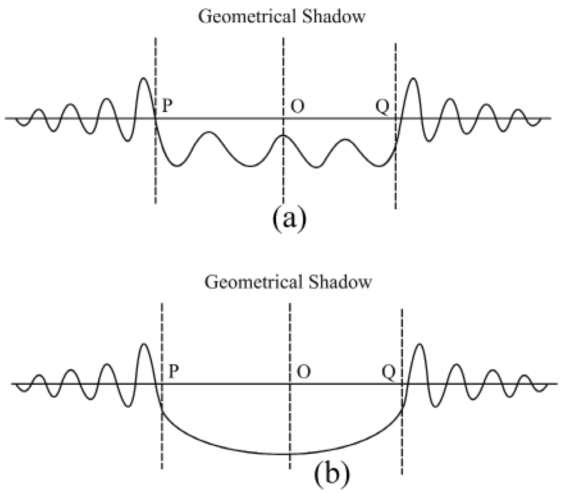


Figure : 11.6

11.7 Fresnel's diffraction due to a narrow rectangular a perative

Let a narrow rectangular aperture of width AB is placed between a narrow slit S illuminated by monochromatic light of wavelength λ and a screen XY, both are parpendicular to the plane of the paper. MN is the section of the cylindrical wave front from S incident on AB. On the screen, the region PQ is the illuminated protion, above Q and below P is the region of geometrical shadow, as shown in figure-11.7

The intensity at the piont C, the central region of PQ depends on the number of half period elements on the exposed wave front between A and B, C as a pole with respect to. O.

The intensity will be maximum or minimum if the number of half period elements are odd or even. respectively.

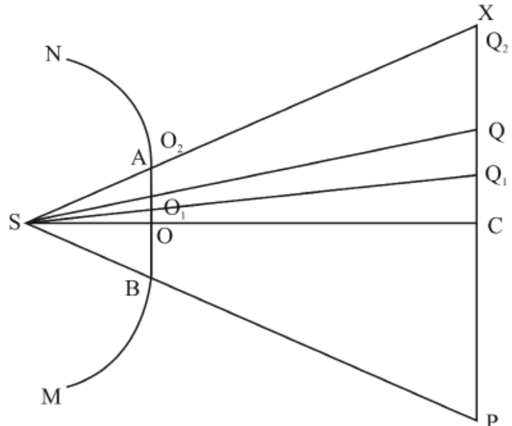


Figure : 11.7

If we consider another point Q_1 in the illuminated portion PQ ofthe screen, the intensity at Q_1 will be maximum or minimum according as odd or even number of half period elements are remaining in each half of the exposed above (O_1A) and below (O_1B) the new pole O_1.

Now we consider a point Q_2 in the region of geometricl shadow. Q_2 is the pole of the wave front with respect to the point Q_2. The intensity at Q_2 will depend on the number of half period elements exposed by the slit AB. The upper half of the wave front above O_2 is obstructed by the obstacle and the half period elements between O_2 A are also cut off by the obstacle.

Hence, O_2 gets light only for a few half period elements contained in the part AB of the lower half of the wave front. Thus when the path difference.

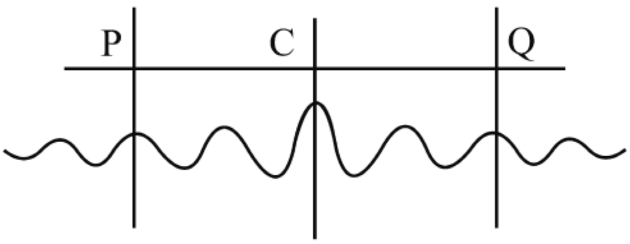


Figure : 11.8

$BQ_2 - AQ_2 = (2n+1)^{\frac{\lambda}{2}}$, Q_2 will be maximum [where n =0, 1, 2, ...]

and $BQ_2 - AQ_2 = 2n\frac{\lambda}{2}, Q_2$ will be minimum [where n = 1, 2, 3.....]

The conditions are similar to the interference due to two coherent sources. But if Q_2 is for away from Q the diffraction pattern will be indistinguishable. The intensity distribution due to narrow aperture is shown in figure. 11.8.

Exercise–2

A narrow slit is illuminated by a light of wave length 5890Å is located at a distance of 0.1m, from straight edge. If the measurements are made at a distance of 0.5 m from the edge, calculate the distance between the first and second dark band.

Fraunhoffer diffraction

11.8 Fraunhoffer diffraction in a single slit.

Let a narrow slit S_1S_2 of width 'a' placed perpendicular to the plane of the paper is illuminated by a parallel beam of monochromatic light of wavelength λ (i.e a plane wave front). Accordingto Muygens principle each point of the wave front on the slit plane maybe considered as a source of secondary wavelets. The secondary wavelets travelling is a direction parallel to OC comes to focus by a conven lens L on the screen at C. The wavelets travelling at an angle Q with the normal are brought to focus tat P (figure-11.9)

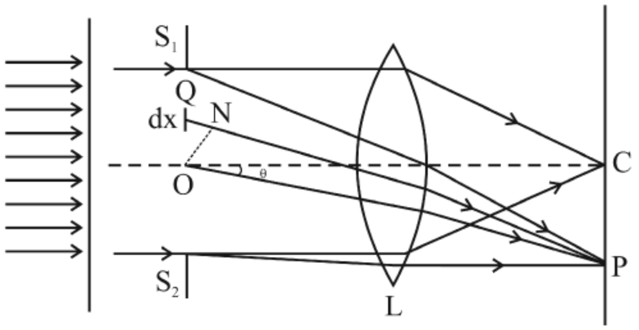


Figure : 11.9

To find the resultant intensity of light at P, let the complex distrurbance at any instant due to secondary waves fron the mid piunt o of the slit is represented by AE iwt where A is the amplitture and ω is the augular frequencey of the wave.

Now draw ON perpendicular to the direction of the diffracted rays from O.

Therefore, the path difference between the waves at P coming from O and from a point Q at a distance x from O is

QN = x sinθ

and the phase difference due to this path difference QN is $\frac{2\pi}{\lambda}QN = \frac{2\pi}{\lambda}x\sin\theta$ [where

$\frac{2\pi}{\lambda}\sin\theta$ ... ... (11.11)

Hence the disturbance at p due to secondary waves from Q for diffracting element dx can be written as dy = CAdx e i(ωt-kx) where c is the proportinality constant, as we consider amplitude is proportional to dx.

The resultant disturbance at p due do the waves coming from whole slit S_1S_2 is

$$y = \int_{\frac{-a}{2}}^{\frac{a}{2}} cAe^{i(\omega t - kx)} dx$$

$$= cAe^{i\omega t}\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{okx}dx = cAe^{i\omega t}\left[\frac{e^{okx(\;)}}{-ik}\right]_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= cAe^{i\omega t}\,\frac{e^{-ika/2} - e^{ika/2}}{-ik} - cAe^{i\omega t}\,\frac{\left(\cos\frac{ka}{2} - \sin\frac{ka}{2}\right) - \left(\cos\frac{ka}{2} + \sin\frac{ka}{2}\right)}{-ik}$$

$$= cAe^{i\omega t}\,\frac{2\sin\frac{ka}{2}}{k} \qquad \left[\because e^{\pm i\theta} = \cos\theta \pm i\sin\theta\right]$$

$$= cAe^{i\omega t}a\,\frac{\sin\frac{ka}{2}}{\frac{ka}{2}}$$

$$= cAae^{i\omega t}\,\frac{\sin\alpha}{\alpha} \quad \text{where } \alpha = \frac{ka}{2} = \frac{\pi a}{\lambda}\sin\theta \qquad \ldots \qquad \ldots \qquad 11.12$$

∴ The resultant intensity I at P is obtained by multiplying Y by its complex conjugate Y.

$$I = y.y^*$$

$$= (cAa)^2\,\frac{\sin^2\alpha}{\alpha^2}$$

$$\therefore \quad I = I_0\,\frac{\sin^2\alpha}{\alpha^2} \quad (11.13) \text{ where } I_0 = (cAa)^2$$

Thus from equations (11.12) and 11.13) we see that the resultant intemnsity at P depends on α , and α depends on θ, i.e. intensity depends on the angle of diffraction (θ).

Maxima and Minima–

The intensity (I) will be maximum or minimum at a point, when

$$\frac{dI}{d\alpha} = 0$$

$$\text{or, } \frac{d}{d\alpha}\left[I_0\left(\frac{\sin\alpha}{\alpha}\right)^2\right] = 0$$

or, $2\frac{\sin\alpha}{\alpha}\left[\frac{\alpha\text{cois}\alpha-\sin\alpha}{\alpha^2}\right]=0$

or, $\sin\alpha(\alpha\cos\alpha-\sin\alpha)=0$

Hence, either $\sin\alpha=0$ ... ... (11.14)

or, $\alpha\cos-\sin\alpha=0$

or, $\alpha=\tan\alpha$ ... ... (11.15)

But, it is found that $\frac{d^2I}{d\alpha^2}$ is positive for $\sin\alpha=0$ and negative for $\alpha=\tan\alpha$.

Hence for minima

$\alpha=n\pi$

or, $\alpha=n\pi$ where n =± 1± 2 ± 3....

∴ From equation (11.12) we get

$\frac{\pi}{\lambda}a\sin\theta=n\pi$

a sinθ = nλ ... ... (11.16)

Thus we get, first, second....etc. minima for n = ± 1, ± 2 ....etc/ Here n = O is excluded because for this α = O and since $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1$, that gives a maximum $(I=I_0)$, called central or principal maximum.

For maxima, we have from equation (11.15)

α = tanα, which is transcendental equation and can be solved graphically by plotting the curves y = α and y = tanα and finding the intersections as shown in figure-11.10. The figure shows that for α = 0 and other values of α which will give maxima are less but gradually approaching towards

$\pm\frac{3\lambda}{2},\pm\frac{5\lambda}{2},\ldots$.. etc. For α = 0 gives the central maxima and for other values give secondary maxima.

The intensity of principal maxima is

$I=I_0$ for a = 0

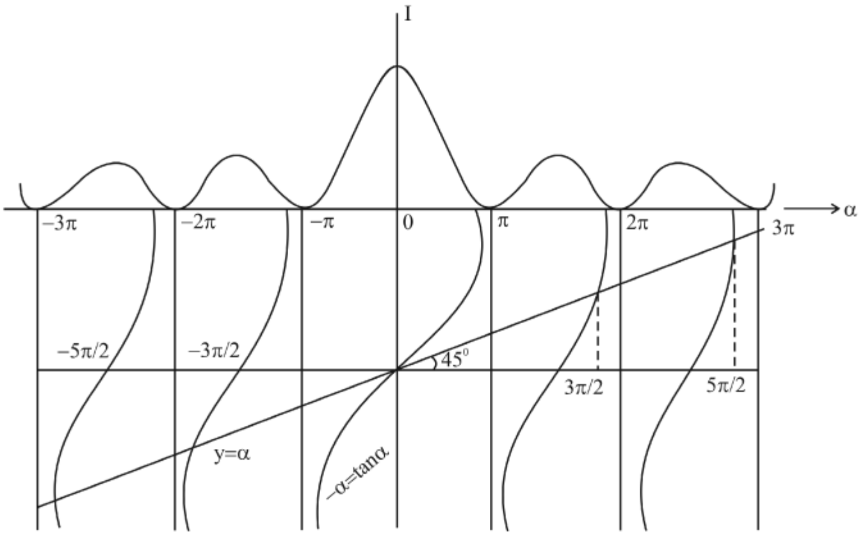


Figure : 11.10

And intensities I, I_2, I_3...... etc of the first, second third.....etc. Secondary maximia are

$$I_1 \simeq I_0 \frac{\sin^2 \frac{3\pi}{2}}{\left(\frac{3\pi}{2}\right)^2} = \frac{4}{9\pi^2} I_0$$

$$I_1 \simeq I_0 \frac{\sin^2 \frac{5\pi}{2}}{\left(\frac{5\pi}{2}\right)} = \frac{4}{25\pi^2} I_0 \quad \text{... etc.}$$

Thus the intensity of secondary maxima falls off rapidly. hence we se that secondary maxima of decreasing intensity occur on either side of central maxima. The intensity distribution in the diffractioin pattern due to a single slit is shown in figure-11.10.

Width of central maxima.

Let the angle of diffractioin θ for which the first minimum on either side of central maxima occurs. Then

$$a \sin\theta = \pm\lambda \qquad \because n \pm 1$$

or, $a\theta = \pm\lambda$ $\qquad \because q$ is very small .

or, $\theta = \pm \frac{\lambda}{a}$

∴ The angular width of the central maximum is $2\theta = \frac{2\lambda}{2}$ which is inversely proportional to the width (a) of the slit.

White light effect

If white light is used, then the central maxima becomes white, while the other maxima willbe concoured. Since the condition for minima is $a \sin\theta = n^1$

or, $\theta = \frac{\lambda}{a}$ for first order dark band.

so, as $\lambda_r > \lambda_v, \theta_r > \theta_v$. Thus the red maxima being farther apart than blue.

Exercisr-3

A screen is placed at a distance of 90 cm from a narrow slit. The slit is illuminated by a parallel bean of light of wavelength 6000Å. Calculate the width of the slit if the first minimum is at a distance of 1mm on either side of the central maximum.

11.9 Fraunhoffer diffraction in a double slit

Let a parallel beam of monochromatic light (i.e. plane wave front) of wavelength λ be incident normally on two slits, each of width 'a' and separated by an opaque space [b]. The distance between any pair of corresponding points of the two slits is d = (a+b), called double slit constant. According to Huygen's principle each point of the wave front on the plane of the slits may be considered as a source of secondary wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at c. The wavelets travelling at an angle θ with the normal are brought to focus at P, as shown in figure–11.11

To find the resultant intensity of light at P, let the complex disturbace at any instant due to secondary waves from the mid point O of the first slit is represented by $Ae^{i\omega t}$, where A is the amptitude and ω is the angular frequency of the wave.

Draw a perpendicular ON to the direction of the diffracted rays from O.

Therefore, the path difference between the waves at P coming from O and from a point Q at a distance x from O is

QN = $x \sin\theta$

and the phase difference due to this path difference QN is $\frac{2\pi}{\lambda}QN = \frac{2\pi}{\lambda}x\sin\theta = kx$ (11.17) where $k = \frac{2\pi}{\lambda}\sin\theta$

Hence, the disturbance at P due to secondary waves from Q for diffracting element dx can be written as dy = $cAdxe^{i(\omega t-kx)}$ where c is the proportionality constant, as we consider that the amplitude is propertional to dx.

The resultant disturbance at P due to both slits is

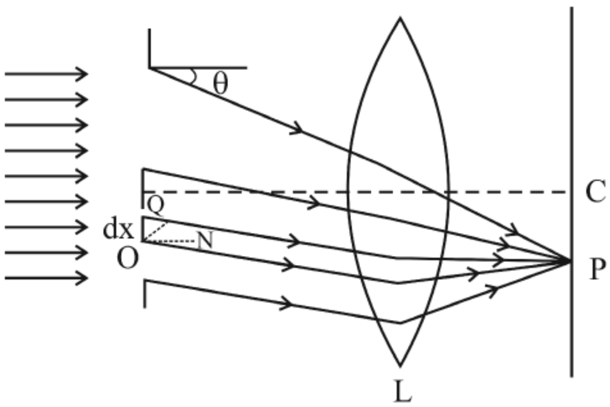


Figure : 11.11

$$y = \int_{-\frac{a}{2}}^{+\frac{a}{2}} cAe^{i(\omega t-kx))}dx + \int_{\frac{d-a}{2}}^{\frac{d+a}{2}} cAe^{i(\omega t-kx)}dx$$

$$= cAe^{i\omega t}\left[\left\{\frac{e^{-kx}}{-ik}\right\}_{-\frac{a}{2}}^{\frac{a}{2}} + \left\{\frac{e^{-ikx}}{ik}\right\}_{\frac{d-a}{2}}^{\frac{d+a}{2}}\right]$$

$$= cAe^{i\omega t}\left[\frac{e^{\frac{-1ka}{2}} - e^{\frac{ika}{2}}}{-ik} + \frac{e^{-k\left(\alpha+\frac{9}{2}\right)} - e^{-ik\left(d-\frac{a}{2}\right)}}{-ik}\right]$$

$$= cAe^{-\omega t}\left[\frac{2i\sin\frac{ak}{2}}{ik} + e^{-ikd}\,\frac{2i\sin\frac{ak}{2}}{ik}\right] \quad \left[\because \frac{e^{-i\theta} - e^{i\theta}}{2} = -\,i\sin\theta\right]$$

$$= cAae^{i\omega t}\frac{\sin\frac{ak}{2}}{\frac{ak}{2}}\left(1+e^{ikd}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \; 11.18)$$

∴ The resultant intensity I at P is obtained by multiplying y by its complex consjugate y*

∴ I = yy*

$$I = (CAa)^2\frac{\sin^2\frac{ka}{2}}{\left(\frac{ak}{2}\right)^2}\left(1+e^{-ikd}\right)\left(1+e^{ikd}\right)$$

$$= (CAa)^2\frac{\sin^2\frac{ka}{2}}{\left(\frac{ka}{2}\right)^2}\left(2+e^{-ikd}+e^{+ikd}\right)$$

$$= (CAa)^2\frac{\sin^2\frac{ka}{2}}{\left(\frac{ka}{2}\right)}\left(2+2\cos kd\right) \qquad \left[\because e^{i\theta}+e^{-i\theta}=2\cos\theta\right]$$

$$= 2(CAa)^2\frac{\sin^2\frac{ka}{2}}{\left(\frac{ka}{2}\right)^2} = 2\cos^2\frac{kd}{2} \qquad \left[\because 1+\cos 2\theta = 2\cos^2\theta\right]$$

$$I = 4I_0\frac{\sin^2\alpha}{\alpha^2}\cos^2\beta \quad . \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \; (11.19)$$

where $I_0 = (CAa)^2, \alpha = \frac{ka}{2} = \frac{\pi}{\lambda}a\sin\theta$ ... ... (11.20)

and $\beta = \frac{kd}{2} = \frac{\pi}{\lambda}(a+b)\sin\theta$ ... ... (11.21)

The resultant intensity depends on two factors

(i) $I_1 = 4I_0\frac{\sin^2\alpha}{\alpha^2}$ which gives the diffraction patterns due a single slit and (ii) $I_2 = \cos^2\beta$, which gives the intereference pattern of diffracted light waves from the two slits.

Nature of diffraction pattern

The intensity of central maximum $4I_0$ when $\alpha = 0$ and $\beta = 0$ i.e when $\theta = 0$., because

$\underset{\alpha \to 0}{\text{Lt}} \frac{\sin \alpha}{\alpha} = 1$ and at $\beta = 0$; $\cos \beta = 1$

The position of minima due to this factor is given when $\alpha = n\pi$ or, $a \sin \theta - n\lambda$ where $n = \pm 1, \pm 2....$ minima and for secondary maxima

$\alpha \approx \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \pm \frac{7\pi}{2}$ .....

Interference pattern: Considering the factor $I_2 = \cos^2 \beta$ for intensity variation, we find that the intensity will be maximum, if $\cos^2 \beta = 1$ i.e. $b = mp$ where $m = \pm 0, \pm 1, \pm 2....$

From equation (11.21) we get,

$$\beta = \frac{\pi}{\lambda}(a + b)\sin\theta = m\pi$$

$$\therefore (a + b)\sin\theta = m\lambda \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ... \quad (11.22)$$

The positions of interference minima due to $I_2 = \cos^2 \beta$ is given by

$\cos\beta = 0$ i.e. $\beta = (2S + 1)\frac{\pi}{2}$ where s = 0, 1, 2 .....

$$\because (a + b)\sin q = (2s + 1)\frac{1}{2} \quad (11.23)$$

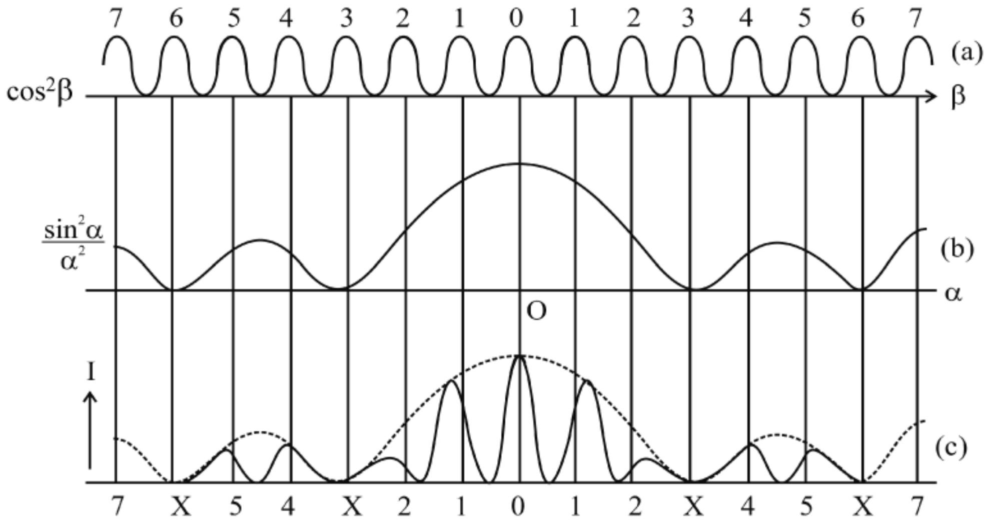


Figure : 11.12

In figure (11.12a) $\cos^2\beta$ is plotted against , in figure (11.12b) $\frac{\sin^2\alpha}{\alpha^2}$ is plotted against α and and figure (11.12c) shows the intensity distribution (I) in the resultant diffraction pattern due to double slit.

Missing orders in double slit pattern

Let a be the width of each slit and b is the width of the opaque space.

Then we know the condition of interference maxima is $(a+b)\sin\theta = m\lambda$

and the condition for diffraction minima

$a\sin\theta = n\lambda$

Where m and n are integers.

If the value of a and b are such that both the equations are satisfied for the same value of θ, then a certain interference maxima will overlap the diffraction minima and hence that order will be missing.

Hence we get from the above two equations.

$$\frac{a+b}{a} = \frac{m}{n} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (11.24)$$

If a = b, then

$$\frac{m}{n} = \frac{2a}{a} = 2 \qquad \text{or, } m = 2n$$

If n = 1, 2, 3 .... etc. then m = 2, 4, 6 .... etc.

This shows that the 2, 4, 6 ..... etc. orders of interference maxima will be missing in the diffraction pattern. Hence, there will be three interference maxima in the central diffraction maxima (corresponding $m = 0, \pm 1$)

Similarly if b = 2a

$$\frac{a+2a}{a} = \frac{m}{n} \text{ or, } m = 3n$$

Thus if n = 1, 2, 3...., m = 3, 6, 9 .... This means 3rd, 6th, 9th etc. orders of interference maxima will be missing in the diffraction pattern and there will be five intereference maxima in the central diffraction maxima (corresponding $m = 0, \pm 1, \pm 2$). The position of the third interference maximum corresponds to the first diffraction. minimum (figure-11.12)

Exercise-4

Find the missing orders for a double slit Fraunhoffer pattern if the width of each slit is 0.15 mm and they are separated by a distance of 0.60 mm.

11.10 Multiple slits: Plane diffraction grating.

A plane diffraction grating consists of a large number (N) of identical parallel and equidistant shlits of same width. It is constructed by ruling equidistant parallel lines on a glass plate by a fine diamond point. Each ruled line acts as an opaque space and the transparent portion between two consecutive ruled lines acts as a slit. The number of rulings on a plane diffraction grating is of the order of 10,000 per cm.

If 'a' be the width of the clear space and 'b' be the width of a ruled line, then the distance (a+b) is called the grating element or grating constant. The points in two consecutive spaces separated by a distance (a+b) are called corresponding points.

Since the process of making a large number of exactly parallel and equidistant lines on a glass plate by a diamond point is very difficult and hence too costly. So, for practional purposes, replicas of the original grating are prepared. For this, on the original grating surface a thin layer of properly diluted collodion solution is poured and the solution is allowed to harden into a tought film, which can be easily separated from the master grating under water. The film is then fixed between two glass plates. This serves as a plane diffraction replica grating.

Theory: Let a plane diffration grating of N slits each of width 'a' and equal opaque space between two successive slits is 'b' consider a parallel beam of monochromatic light of wavelength λ incident normally on the grating surface. According to Huygen's principle each point of the incident plane wave front on the slits may be considered as a source of secondary wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at p, as shown in figure – 11.13.

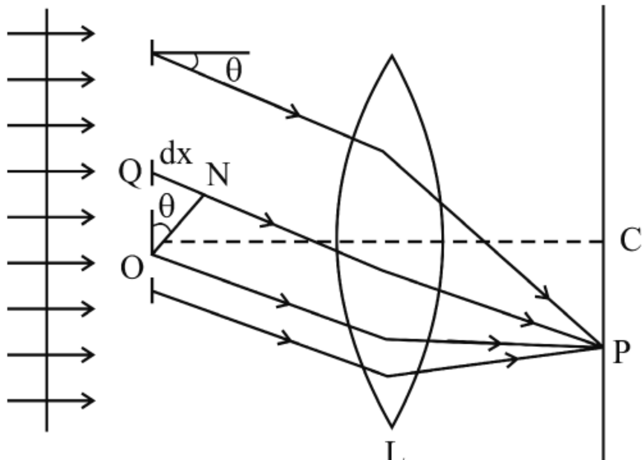


Figure : 11.13

To find the resultant intensity at light at P, let the complex disturbance at any instant due to secondary waves from the point O (the centre of the first slit) is represented by $Ae^{i\omega t}$, where A is the amplitude and ω is the angular frequency of the wave.

Now draw on perpendicular to the direction of the diffracted rays from O.

Therefore, the path difference betwen the waves at P coming from O and from a point Q a distance x from O is

QN = x sin θ and the phase difference due to this path difference QN is

$\frac{2\pi}{\lambda}QN = \frac{2\pi}{\lambda}x\sin\theta = Kx$ ..... (11.25) where $k = \frac{2\pi}{\lambda}\sin\theta$.

Hence, the disturbance at P due to secondary waves from Q for diffracting element dx can be sritten as,

$dy = CAcxe^{i(\omega t - kn)}$. Where c is the proportionality constant, as we consider amplitude is proportional to dx.

∴ The resultant complex disturbance at P due to the all slits is

$$y = \left[\int_{\frac{-a}{2}}^{\frac{+a}{2}} e^{i(\omega t - kx)}dx + \int_{d-\frac{a}{2}}^{d+\frac{a}{2}} e^{i(\omega t - kx)}dx + ... + \int_{(N-1)d-\frac{a}{2}}^{(N-1)d+\frac{a}{2}} e^{i(\omega t - kn)}dx\right]CA$$

Where the grating element d = a + b

$$\therefore y = CAe^{i\omega t}\left[\left\{\frac{e^{-ikx}}{-ik}\right\}_{\frac{-a}{2}}^{\frac{a}{2}} + \left\{\frac{e^{-ikx}}{-ik}\right\}_{d-\frac{a}{2}}^{d+\frac{a}{2}} + ... + \left\{\frac{e^{-kx}}{-ik}\right\}_{(N-1)d\frac{-a}{2}}^{(N-1)d+\frac{a}{2}}\right]$$

$$= CAe^{i\omega t}\frac{\sin\frac{ak}{2}}{\frac{ak}{2}}\left[1 + e^{-ikd} + ... + e^{-i(N-1)d}\right]$$

$$= CAae^{i\omega t}\frac{\sin\frac{ak}{2}}{\frac{ak}{2}}.\frac{e^{-iNkd} - 1}{e^{-ikd} - 1} \quad \left[\because 1 + r + r^2 + ...r^{n-1} = \frac{r^n - 1}{r - 1}\right]$$

The resultant intensity (I) at P is obtained by multiphying y by its complex conjugate y*

$\therefore$ I = yy*

$$=(CAa)^2 \frac{\text{Sin}^2 \frac{ak}{2}}{\left(\frac{ak}{2}\right)^2} \frac{e^{-iNkd}-1}{e^{-ikd}-1} \cdot \frac{e^{iNkd}-1}{e^{ikd}-1}$$

$$=(CAa)^2 \frac{\sin^2 \frac{ak}{2}}{\left(\frac{ak}{2}\right)^2} \cdot \frac{2-e^{iNkd}-e^{-iNkd}}{2-e^{ikd}-e^{-ikd}}$$

$$=(CAa)^2 \frac{\text{Sin}^2 \frac{ak}{2}}{\left(\frac{ak}{2}\right)^2} \frac{\sin^2 NKd}{\sin^2 kd} \qquad \left[\because e^{i\theta}+e^{-i\theta}=2\cos\theta\right]$$

$$=(CAa)^2 \frac{\text{Sin}^2 \frac{ak}{2}}{\left(\frac{ak}{2}\right)^2} \frac{\sin^2 NKd}{\sin^2 kd} \qquad \left[\because 1-\cos 2\theta = 2\sin^2\theta\right]$$

$$I = I_0 \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2 N\beta}{\sin^2\beta} \quad \ldots \quad (11.26)$$

$I_0 = (CAa)^2$, $\alpha = \frac{ak}{2} = \frac{\pi}{\lambda} a\sin\theta$ and $\beta = \frac{\pi}{\lambda} d\sin\theta$

$$\beta = \frac{\pi}{\lambda}(a+b)\sin\theta \quad \ldots \quad (11.27)$$

$\therefore$ The resultant intensity depends on two factors (i) $I_1 = I_0 \frac{\sin^2\alpha}{\alpha^2}$ which gives the diffraction pattern due a single slit and (ii) $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$, which gives the interference pattern of diftracted light waves from the N slits.

Principal maxima:

Let us confine our observation to the neighbourhood of the central part of the pattern., where the variation of $I_1 = \frac{\sin^2 \alpha}{\alpha^2}$ is very small and the condition for maxima solely depends on the factor $I_2 = \frac{\sin^2 B\beta}{\sin^2 \beta}$ Now, $\frac{\sin^2 B\beta}{\sin^2 \beta}$ will be maximum when $\beta = n\pi$, where $n = \pm 0, \pm 1 \pm 2$ .... etc.

Now; $\beta = \frac{kd}{2} = \frac{\pi}{\lambda}(a+b)\sin\theta = n\pi$ [using equ. (11.27)]

or, $(a+b)\sin\theta = n\lambda$ .... (11.28)

These are known as principal maxima.

For n = 0, we get zero order spectrum and for n = ± 1, ± 2 ... etc. We get first order, second order .... etc. principal maxima. The sign indicates that there are two principal maxima of the same order lying on either side of zero-order maxima.

Again, for $\beta = b\pi, I_2 = \frac{o}{o}$ which is indeterminate.

To find the value of this limit, we get the maximum value of I_2, When

$$\underset{\beta\to n\pi}{Lt} \frac{\sin N\beta}{\sin\beta} = \underset{\beta\to b\pi}{Lt} \frac{N\cos N\beta}{\cos\beta} = N \qquad \text{[using L Hospital's rute]}$$

From equation (11.26) we get

$$I = I_0 \frac{\sin^2\alpha}{\alpha^2}.N^2 \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (11.29)$$

Thus the resultant intensity is proportional to N^2, i.e the principal maxima increases with number of slits (N) increases, but due to the factor $\frac{\sin^2\alpha}{\alpha^2}$, whose value decreases with the increase of the angle of diffration (θ). Hence, the intensity of principal maxima decreases with increase in order number of fringes.

Conditions for secondary minima and maxima:

The factor $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ depends on β and for maxima or minima $\frac{dI_2}{d\beta} = 0$

$$\frac{dI_2}{d\beta} = \frac{\sin^2(2N\sin N\beta\cos N\beta) - 2\sin^2 N\beta\sin\beta\cos\beta}{\sin^4\beta}$$

$$= \frac{2N\sin N\beta\cos N\beta}{\sin^2\beta} - \frac{2\sin^2 N\beta}{\sin^2\beta}.\frac{\cos\beta}{\sin\beta}$$

$$= \frac{2\sin^2 N\beta}{\sin^2\beta}(N\cot N\beta - \cot\beta) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11.30)$$

For maxima or, minima $\frac{dI_2}{d\beta} = 0$

or, $\frac{2\sin^2 N\beta}{\sin^2\beta}(N\cot N\beta - \cot\beta) = 0$

Hence, either (i) $\frac{\sin^2 N\beta}{\sin^2\beta} = 0,$

or, (ii) $N\cot N\beta - \cot\beta = 0$

or, NCot β + N=Cotβ

Secondary minima

When sin $N\beta = 0$ but $\sin\beta \neq o$, the factor $\frac{\sin N\beta}{\sin\beta}$ becomes zero and hence intensity is minimum.

Thus for minimum

SinNβ = 0

or, $N\beta = \pm m\pi$

or, $N\frac{\pi}{\lambda}(a+b)\sin\theta = \pm m\pi \quad \left[\because \beta = \frac{\pi}{\lambda}(a+b)\sin\theta\right]$

or, $(a+b)\sin\theta = \pm\frac{m}{N}\lambda$ ... ...(11.31)

Where m has all the integral values except O,N, 2N ....nN as for these values of m, sinβ becomes zero and we get principal maxima. Hence there will be (N–1) minima between twoconsecuted principal maxima.

Secondary maxima

From the condition N cot Nβ= cotβ , $\frac{dI_2}{d\beta} = 0$

It can also be shown that in this condition $\frac{d^2I_2}{d\beta^2}$ is negative. Thus for those values of β other than $\beta = b\pi$ satisfy the condition $N\cot N\beta = \cot\beta$ give rise to another set of maxima, called the secondary maxima. Its intensity is much less than the intensity of principal maxima.

Now, to find the intensity of secondary maxima, we have

$$N\cot + N\beta = \cot\beta$$

or, $$N^2\frac{\cos^2 N\beta}{\sin^2 N\beta} = \frac{\cos^2\beta}{\sin^2\beta}$$

or, $$\frac{N^2\left(1-\sin^2 N\beta\right)}{\sin^2 N\beta} = \frac{1-\sin^2\beta}{\sin^2\beta}$$

or, $$\frac{N^2\left(1-\sin^2 N\beta\right)}{1-\sin^2\beta} = \frac{N^2\sin^2\beta}{N^2\sin^2\beta} = \frac{N^2}{1+\left(N^2-1\right)\sin^2\beta}\left[\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right]$$

$$\therefore \frac{\text{Intensity of secondary maxima (Ism)}}{\text{Intensity of principal maxima (Ipm)}} = \frac{N^2}{1+\left(N^2\right)\sin^2\beta}\cdot\frac{1}{N^2}$$

$$= \frac{1}{1+\left(N^2-1\right)\sin^2\beta}$$

It is clear that as N increases the intensity of secondary maxima relative to the principal maxima decreases and becomes negligible when N is very large.

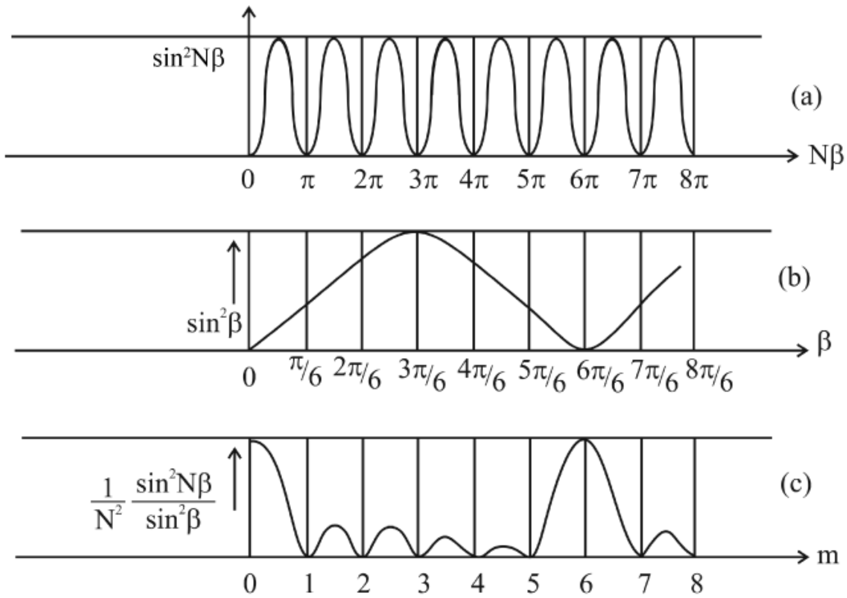


Figure : 11.14

To understant the variation of intensity of secondary maxima we have plotted $\sin^2 N\beta$ againnst β in figure (11.14a) and the product of ratio and and $\left(\frac{1}{N^2}\right)$ against m, foir N = 6. From figure it is evident that the secondary maximal falls off between two successive principal maxima and are unequal spaced. Due to back of symmetry the peaks of the secondary maxima shifted. towards the adjacent principal maxima.

Absent spectra:

The condition for the principal maxima in a grating spectra is

$(a+b) \sin\theta = n\lambda$ where θ = angle of diffration

m = order number.

a + b = grating constant.

The condition for 5th diffraction minima in the same direction is

$a \sin\theta = s\lambda$.

Now, if these two conditions satisfy simutaneously, then nth order principal maxima will be absent.

Therefore, the condition for absent spectra is

$$\frac{a+b}{a} = \frac{n}{s} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \; (11.32)$$

If b = a, then n = 2s ie 2, 4, 6, ...etc. (s = 1, 2, 3...) order of principal maxima will be absent, corresponding to 1, 2, 3 etc. diffraction minima.

Overlapping of spectral lines

We know the conditin for principal maxima is

$(a+b)\sin\theta = n\lambda$

Now, for a given grating d = a + b = Const., so for constant θ, $n\lambda$ is constant. Thus if the incident light has a large range of wavelengths, then the lines of shorter wavelength and higher order overlap on the lines of longer wavelength lower order.

For example, 3rd order of light of wavelength.

λ = 700 nm, 4th oirder of light λ = 525 nm and 5th order of light wavelength λ = 420 nm wll be formed in the same direction, i.e. they all overlap, because

$n\lambda = 3\times700 = 4\times525 = 5\times420$ = constant.

Exercise–5

How many orders would be visible, if the wavelength of incident light is 5460Å and the number of lines inthe grating is 6000 lines/cm?

11.11 Summary

- When the distance between the source of light and the screen or both from the diffracting operature /obstacle is finite, the diffraction pattern of this type is Fresnel class of diffraction.
- In case of Fraunhoffer class both source and screen are at infinite distance from the slit.
- The area of each fresnel half-period zone is nearly equal to so $\pi b\lambda$.
- Phase change of the alternate zones is π.
- A zone plate is an optical device in which alternate half period zones are blackended. It is equaivalent to a convex lens.
- The focal length of zone plate, $f_n = \frac{r_n^2}{v\lambda}$.
- Position of nth bright and dark band due to straight edge are $x_n = \sqrt{\frac{b(a+b)}{a}(2n+1)\lambda}$

and $x_n = \sqrt{\frac{b(a+b)}{a} 2n\lambda}$ respectively.

- condition for minima of single slit diffraction pattern is a sinθ = nλ.
- $\frac{a+b}{a} = \frac{m}{n}$ is the condition of missing order of interference maxima in the diffraction pattern of double slit.
- $(a+b)\sin\theta = n\lambda$ is the condition of principal maxima of grating diffraction pattern.
- Expressions for intensity for single slit $I = I_0 \frac{\sin^2\alpha}{\alpha^2}$ where $\alpha = \frac{\pi a}{\lambda}\sin\theta$, for double slit $I = 4I_0 \frac{\sin^2\alpha}{\alpha^2}\cos^2\beta$ where $\alpha = \frac{\pi a}{\lambda}\sin\theta$ and $\beta = \frac{\pi}{\lambda}(a+b)\sin\theta$ and for grating
- $I = I_0 \frac{\sin^2\alpha}{\alpha^2} \cdot \frac{\sin^2 N\beta}{\sin^2\beta}$. Where $\alpha = \frac{\pi a}{\lambda}\sin\theta$ and $\beta = \frac{\pi}{\lambda}(a+b)\sin\theta$.

11.12 Questions and problems

11.12.1 An object is placed at 20 cm from a zone plate and the brighest image is situated at 20cm from the zone plate, the wavelength $\lambda = 6000Å$. Find the number of fresnel's zones in a radius of 3 cm of the plate.

11.12.2 Find the angular width of the central bright fringe in the Fraunhoffer diffraction pattern of a single slit of width 0.24mm. Wavelength of light used is 5890Å.

11.12.3 Fraunhoffer double slit diffraction pattern is observed in the focal plane of a lens of focal length 0.5m. The wavelength of incident light is 600 nm. The didtance between twomaxima adjacement to the macimum of zero or der is 5 mm and the 4th order maximum is missing. Find the width of each slit and the distance between their centres.

11.12.4 Light is incident normally on a grating of toal ruled width 0.005m with 2500 lines in all. Calculate the angular separation of the two sodium lines in the first order spectrum.

11.13. Solutions

Exercise–1 Here, f = 60 cm, n = 1 and λ= 6000Å = 6000×10^{-8} cm.

we know $f = \frac{r_n^2}{n\lambda}$

or, $r_1^2 = f.1.\lambda = 60 \times 1 \times 6000 \times 10^{-8} = 3 \times 10^{-4}$

$\therefore\ r_1 = 6 \times 10^{-2} = 0.06\text{cm}$

Exercise–2:

Here, λ = 5890Å = 5890×10^{-10}m, a = 0.1m, b = 0.5m

we know for dark band $x_n = \sqrt{\frac{b(a+b)}{a} 2n\lambda}$

$$x_2 - x_1 = \sqrt{\frac{b(a+b)2\lambda}{a}}\left(\sqrt{2}-1\right) = \sqrt{\frac{0.5(0.1+0.5)\times 2\times 5890\times 10^{-10}}{0.1}}\left(\sqrt{2}-1\right)$$

= 7.7×10^{-4}m.

Exercise–3

Here, Distance between slit and sereen (D) = 90 cm

λ = 6000Å = 6000×10^{-8} cm, n = 1, x_1 = 0.1 cm.

we know the condition for minima is

a $\sin\theta = n\lambda$ θ is very small

or, $\tan\theta = \frac{n\lambda}{a}$ $\quad\therefore\ \sin\theta = \theta - \tan\theta$

or, $\frac{x_n}{D} = \frac{n\lambda}{a}$

$$a = \frac{n\lambda D}{x_n} = \frac{1 \times 6000 \times 10^{-8} \times 90}{0.1} = 54 \times 10^{-3} = 0.054$$

Exercise-4

Here a = 0.15 mm, b = 0.60mm.

we know $\frac{a+b}{a} = \frac{m}{n}$

or, $\frac{m}{n}=\frac{0.15+0.6}{0.5}=5$

or, m = 5n

when n = 1, 2, 3.....etc then m = 5, 10, 15 .... etc.

∴ The 5th, 10th, 15th ...etc orders of interference maxima will be missing in the diffraction pattern.

Exercise-5

Here, λ = 5460Å = 5460×10^{-8} cm. and number of lines per cm is

$$(m)=\frac{1}{a+b}=6000 \text{ lines/cm}$$

For principal maxima, we know

$(a+b)\sin\theta = n\lambda$

or, $\sin\theta=\frac{1}{a+b}n\lambda = mn\lambda$

Here, θ = 90° (Maximum)

$mn\lambda = \sin 90° = 1$

or, $n=\frac{1}{m\lambda}=\frac{1}{6000\times5460\times10^{-8}}=3.04$

∴ n=3, i.e. 3 orders will be visible only.

11.12.1

Here u = 20 cm, v =20 cm, λ= 6000Å = 6000×10^{-8} cm radius of nth zone (r_n) = 3 cm.

we know

$$\frac{1}{v}+\frac{1}{u}=\frac{n\lambda}{r_n^2}=\frac{1}{f}$$

or, $\frac{1}{20}+\frac{1}{20}=\frac{1}{f}$ ∴ f = 10 cm

Now, $\frac{n\lambda}{r_n^2} = \frac{1}{10}$

or, $n = \frac{r_n^2}{10\lambda} = \frac{3^2}{10 \times 6000 \times 10^{-8}} = 15,000$

The number of Fresnel's zone is 15,000

11.12.2

Here, a = 0.24 mm=0.024 cm. λ= 5890Å = 5890×10^{-8} cm.

we know for single slit minima.

a $\sin\theta = n\lambda$ for n = 1

$$\sin\theta = \frac{\lambda}{a} = \frac{5890 \times 10^{-8}}{0.024} = 2.45 \times 10^{-3}$$

$\therefore$ θ = 0.140°

Angular with of the central maxima (2θ) = 0.28°

11.12.3

Here, f = 0.5 m, $\lambda = 600 \times 10^{-9}$m.

The distance between two maxima (x) = 5 mm = 5×10^{-3}m.

Adjacent to maximum of zero order

For missing of 4th order maximum, we have

$$\frac{a+b}{a} = 4$$

or, b = 3a.

Now, for 1st order maximum

(a+b) $\sin\theta = \lambda$

or, $\theta = \frac{\lambda}{a+b} = \frac{\lambda}{4a}$ [$\because$ θ is very small]

Now, $2\theta = \frac{x}{f}$

or, $x = 2\theta f = \frac{2f\lambda}{4a}$

$$\therefore\ a = \frac{f\lambda}{2x} = \frac{0.5\times600\times10^{-9}}{2\times5\times10^{-3}} = 3\times10^{-5} = 0.03\text{mm}$$

$\therefore$ b = 3a = 0.009 mm.

11.12.4 Here, width of ruling = 0.005m.

Total Number of slits = 2500

$$\therefore\ \text{Number of rulings per metre } = \frac{2500}{0.005} = 5\times10^{5}\text{No/m}$$

$$\therefore\ \text{Grating element } (a+b) = \frac{1}{5\times10^{5}} = 2\times10^{-6}\text{m.}$$

For sodium light, $\lambda_1 = 5890\times10^{-10}$m

and $\lambda_2 = 5896\times10^{-10}$m

Now, We know for principal maxima

$(a+b)\sin\theta = n\lambda$ for n =1

$$\text{or, } \sin\theta_1 = \frac{\lambda_1}{a+b} = \frac{5890\times10^{-10}}{2\times10^{-6}} = 0.2945$$

or, $\theta_1 = 17°7'$

Similarly

$$\sin\theta_2 = \frac{\lambda_2}{a+b} = \frac{5896\times10^{-10}}{2\times10^{-6}} = 0.2948$$

or, $\theta_2 = 17°8'$

Angular separation $\theta_2 - \theta_1 = 17°8' - 17°7' = 1'$.

Unit : 12 Polarization

Structure

12.0 Objectives

After studying this unit you will be able to

- define polarization of light, different types of polarization.

- know the process of producing plane polarized light.
- explain laws of Brewster's, Malus.
- construct Nicol prism, polaroids, reterdation plates.
- explain dischroism, optical activity, optical rotation.
- draw the wave surfaces within the uniaxial crystal using Huygen's theory of double refraction.
- determine experimentally specific rotation of optically active solutions, using polarimeter.

12.1 Introduction

In the presions units you have learnt about interfernce and diffraction phenomena of light which proved that light is a wave. But, whether the light waves are longitudinal or transverse or whether the vibrations are linear or circular cannot be deduced from the above two phenomena. Polarization is the only phenomenon which explains the light must be a transverse wave. Again, you know that the light is an electronagnetic wave consisting of vibrating electric and magnetic field vectors at right angles to each other and also perpendicular to the direction of propagation. The electric field vector is responsible for the sensation of vision, so it is called light vector. Ordinary or unpolarized light consists of large number of waves, the light vectors of the component waves will remain in any plane about and at right angles to the direction of propgagation, because such light is emitted by an atom or a group of atoms of the source vibrating independently. Such an ordinary light beam with electric vectors arranged symmetrically about the direction of propagation is called unpolarized light. If by some means, these vibrations in an unpolarized light are so cut off that only vibration of a constant mode and direction remain, the light so obtained is called polarized light, the phenomenon is called polarization, as shown in figure–12.1.

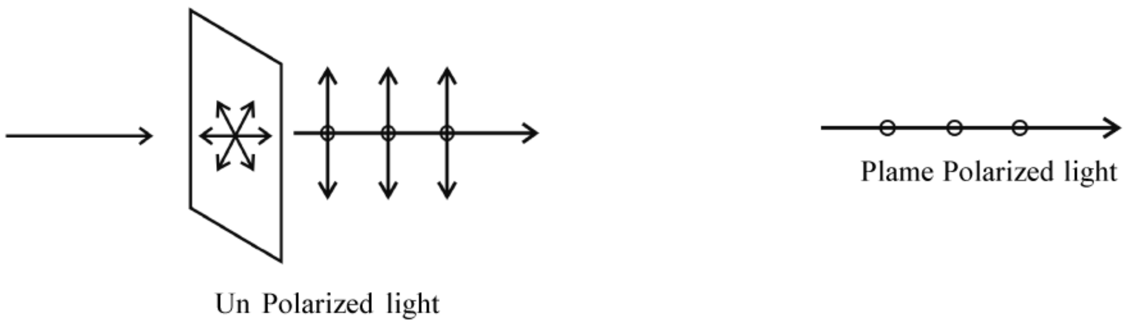


Figure : 12.1

When two plane polarized light waves superposed under suitable conditions, then the resultant tight vectors rotate in a plance perpendicular to the direction of propagation and if the magnitude of the resultant hight vector remains constant, then the tip of light vector appears to trace a circle at a fixed space.

Such a light is called circularly polarized light. If the rotation, when looking towards the incomeing light is clockwise then the light is called right circularly polarized light. If the light vector rotates anticlockwise, then it is called left circularly polarized light.

On the other hand, it the magnitude of the resultant light vector varies periodically between a amaximum and minimum value, then the tip of the light vector appears to trace on a eliptic path, such light is called elliptically polarized light.

A mixture of polarized and unpolarized lights is known as partially polarized light.

In this unit we shall discuss about the production, detections and other phenomena of polarized light.

12.2 Methods of producing plane polarized light.

1. Polarization by reflection
2. Polarization by double refraction
3. Polarization by dichroism.

12.3 Polarization by reflection and Brewster's law

Malus observed that when a beam of ordinary light is incident on the surface of a transparent medium (like glass plate, dielectric surface, water etc. the reflected beam is partially plane polarized. The degree of polarization depends on the angle of incidence.

Sir David Brewster performed a series of experiments on the polarization of light by reftections at a number of different media. it is found that, for a particular angle of incidence when the reflected and refracted rays are perpendicular to each other then the reflected ray is completely plane polarized.

This angle of incidence at which the reflected and refracted rays becomes mulually perpendicular and the reflected ray is completely polarized is known as angle of polarization or Brewster's angle.

Brewster's proved that the tangent of the angle (θ_B) at which polarization is obtained by reflection is numerically equal to the refractive index of the meduium

If μ is the refractive index of the medium then $\mu = \tan\theta_B$ ... ... (12,1)

This is known as Brewster's law.

Proof:

Consider a ray of unpolarized light from the source S incidents on a medium of refractive index μ. OA is the reflected polarized light with vibrations perpendicular to the plane of incidence and OB is the refracted light, which is partially polarized, as shows n in figure 12.2.

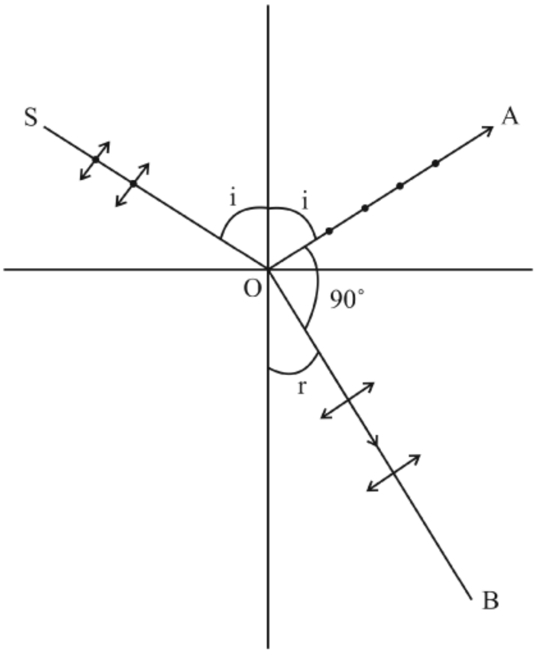


Figure : 12.2

According to Snell's law $\mu = \frac{\sin i}{\sin r}$, where μ is the refractive index of the medium. Here, angle of incidence $i = \theta_B$ (Brewster's angle) and r is the angle of refraction.

$\therefore \ \theta_B + r + 90° = 180°$

or, r = $90° - \theta_B$

Hence $\mu = \frac{\sin \theta_B}{\sin(90° - \theta_B)} = \frac{\sin \theta_B}{\cos \theta_B} = \tan \theta_B$

$\therefore \ \mu = \tan \theta_{B,} \ \theta_B$ is also called polarizing angle. This is known as Brewster's law.

12.4 Double refraction

In the isotropic refracting medium (like glass, water, air, etc.) the refractive index is same in all directions. Thus when a light is incident on an isotropic medium, it refracts as a single ray in all directions. Because the atoms in a crystal of isotropic material are arranged in a regular periodic matter. If the arrangement of atoms within a crystal are differ in different directions then the physical properties very with directions. This type of crystals are called anisotropic crystal. Calcite, quartz, tourmaline are the examples of anisotropic materials.

Bartholinus discovered that when a ray of unpolarized light is incident on a crystal like calcite, quartz etc, it is refracted into two rays. one of these rays obeys the laws of refraction of light, is called ordinary ray(E-ray)

Both of these O-ray and E-ray are plane polarized, whose vibration are along and perpendicular to the principal section, i.e. they are mutually perpendicular to each others shown in figure-12.3

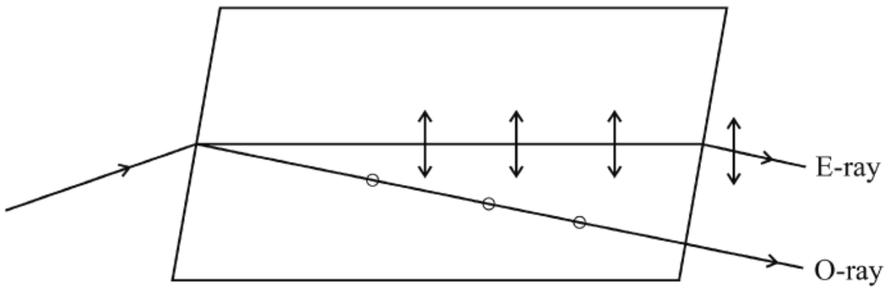


Figure :

This phenomanon of refraction where a single incident ray is refrated into two rays is called double refraction or birefringence and the crystals that exhibits the phenomena are called doubly refracting crystal or birefringent.

12.5 Geometry of calcite crystal, optic axis and principal section.

Calcite ($CaCO_3$) is a colourless, transparent crystal also known as iceland spar, found is nature. it is a rhombohedral crystal containing six parallelogram foces. The two opposite angles of each parallelogram are at 101°55′ and 78°5′. The two diagonally opposite corners A and C (as in figure-12.4) Where three obtuse angles of 101°55′ meet, are called blunt corners of the crystal. The rest of the six corners there is one obtuse angle and two acute angles.

Optic axis:

If a straight line by drawn through any one of the blunt corners (A or C) and that bisects these blunt convers is known as optic axis.

In fact any line parallel to this line is also an optic axis. So, the optic axis is a direction but not a particular straight line.

Optic axis of a crystal is also defined as a direction along which if a ray pass then, there will be no double refraction of the incident ray, both the ordinary and oxtra-ordinary rays travel with same speed along this direction.

Principal section :

Principal section is a plane containing the optic axis and perpendicular to the opposite refracting faces of the crystal. Since there are infinite number of lines parallel to the direction of optic axis, so there are infinite number of principal sections. ACC_1A_1 is one of such principal section, as shown in figure-12.4.

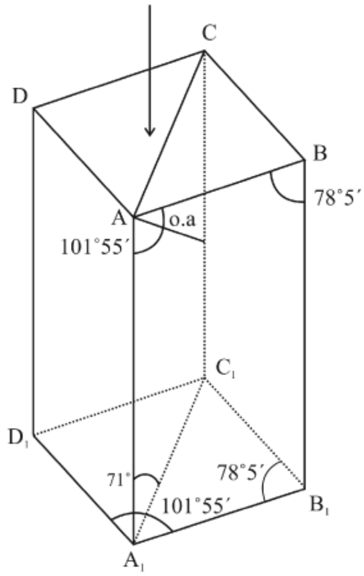


Figure : 12.4

12.6 Nicol prism

Nicol prism is an optical device made from a calcite crystal and is used to produce and analyse the polarized light.

Principle

When an unpolarized light is incident on a unixial crystal (Calcite), it splits up into the ordinary ray (O-ray) and extra ordinary ray (E-ray). Both are plane polarized. One of these rays is cutoff by the process of total internal reflection and the other is transmitted as a plane polarized light.

This prism was designed by Willium Nicol and is known as Nicol Prism after his name.

Construction:

It is constructed from a calcite crystal whose length is three times of its breadth. The end faces AC and A_1C_1 of the crystal are cut down to reduce the angles of pricipal section to 112° and 68° inplace of 109° and 71° respectively. The crystal is then cut into two pieces along the plane A′C′ (as shown infigue 12.5), perpendicular to both the principal section and the two end faces of the crystal. Two cut surfaces are ground, polished optically flat and then cemented together with canada balsam, whose refractive index $(\mu_{cB} = 1.55)$ is greaterthan the refractive index of E-ray $(\mu_E = 1.486)$, but less than the prism are blackened to absorb totally reflected rays.

Action :

(i) Nicol prism as polarizer

When an unpolarized light falls on a nicol prism, the ray splits up into two refracted rays. ordinary ray (O-ray) and extra ordinary ray (E-ray), they travel through the crystal. Both the rays are plane polarized, of which the plane of vibration of O-ray is perpendicualar to the principal section and that of E-Ray is in the plane of the principal section of the crystal. When these two rays are incident on canada balsman layer O-ray suffers total internal reflection, as it is passes from denser to rarer medium $(\mu_o > \mu_{CB})$ and the geometry of the Nicol prism is such that the angle of incidence of O-ray is greater than the critical angle. Finally it is absorbed by the black laryer onthe sides of the prism. But when the E-ray incidets on canada balsam layer it traversing from rarer to denser medium $(\mu_{CB} > M_E)$ and is transmitted through the canada balsam layer, Finally emerges through the Nicol as a plane polarized light. Whose vibrations are parallel to the principal section (Figure-12.15).

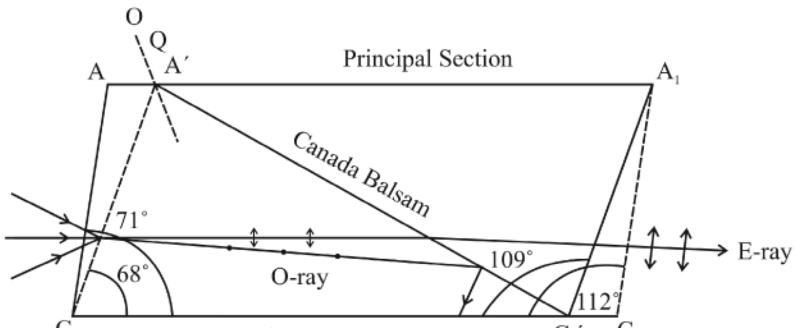


Figure : 12.5

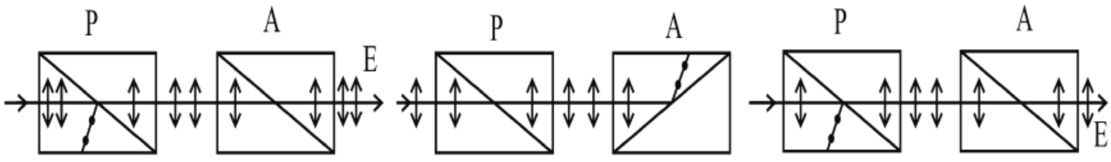
(ii) Nicol prism as analyzer :

When two Nicol prisms P and A are placed adjacent to each other (Figure–12.6), one of them (P) acts as a polarizer and the other (A) as analyser.

When an unpolarized light is incident on the Nicol prism P, the emergent ray will be plane poolarized (E-ray). Now if this ray incidents on the second Nicol (A) whose principal section is parallel to that of P, then as long as the second Nicol remains parallel to the principal section of the first nicol (P), E-ray will be tansmitted through the second Nicol (A). The intensity of the transmitted light will remain maximum, as shown in figure-12.6a.

Now if the second prison (A) is rotated, the intensity of the emitted E-ray from the second Nicol decreases gradually and ultimately becomes zero, when the principal section of Nicol A is exactly perpendicular to that of the Nicol P (figure 12.6b). In this position the E-ray behaves as O-ray inside the prism A and is totally reflected by the canada balsam layer and two Nicol prisms P and A are said to be crossed.

If A is further rolated throlugh another 90°, the intensity of the emergent light from A will again be maximum i.e., the principal sections of two prisom are in parallel position again (Figure-12.6c).



(a) Figure : 12.6 (b) (c)

Thus the prism P produces plane polarized light and the prism A detects it. Hence Nicol prism P is called a polarizer and the nicol prism A is called an analyzer.

12.7 Malus's Law

Malus obtained a relation between the intensity of light transmitted by the analyzer with the angle between the planes of polarizer and analyzer. This relation is known as Malus's law.

It states that the intensity of plane polarized light transmitted through an analyzer is proportional to the square of cosine of the angle between the planes of transmission of the analyzer and the plane of polarizer.

Proof:

Let a plane polarized light of amplitude A is incident on an analyzer and the angle between the planes of transmission of the analyzer and that of polarizer is θ (figure-12.7)

Now the amplitude A of the plane polarized light emerging from the polarizer may be resolved into two components, A $\cos\theta$ and A $\sin\theta$, along and perpendicular to the plane of transmission of the alalyzer respetively. The component A $\sin\theta$ will be reflected from the analyzer, but the parallel component A $\cos\theta$ will be transmitted through the analyser.

Therefore, the intensity of the transmitted light from the analyzer is,

$I = A^2 \cos^2\theta = I_0 \cos^2\theta$

$\therefore\ I \alpha \cos^2\theta$ .... (12.2)

where $I_0 = A^2$, the intensity of the polarized light incident on the analyzer and is a constant quantity.

Thus the intensity of transmitted light is proportional to the square of the cosine of the angle between the planes of transmission of polarizer and analyzer.

This is Malus's Law.

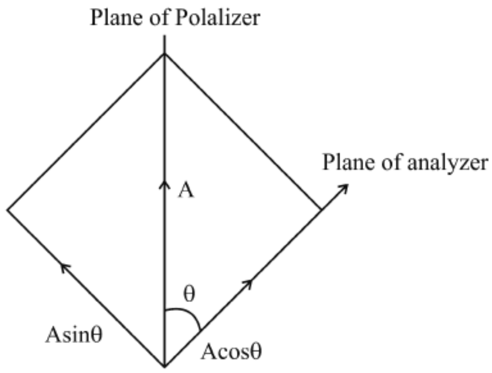


Figure : 12.7

2.8 Dichroism and polaroids

The phenomenon by this a doubly refracting crystal (e.g. calcite, tourmaline) absorbs one of the doubly refrecting rays (E. rays or O-ray) strongly and the other passes through the crystal with a minimum loss is called dichroism and the crystal with a minimum loss is called dichroism and the crystals that show this property are called dicroic crystals.

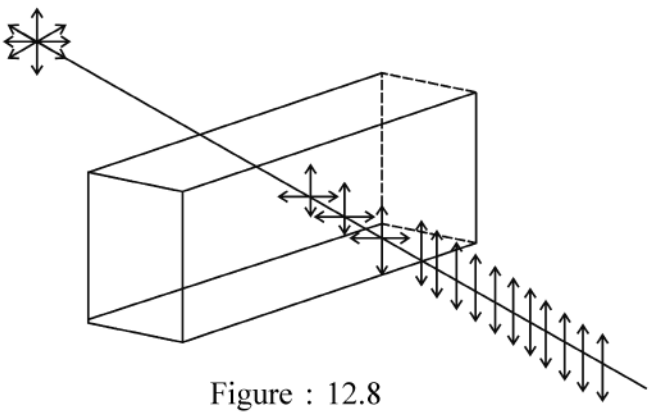


Figure : 12.8

Thus when an unpolarized light passes through this crystal it will produce plane polarized light, as shown in figure 12.8.

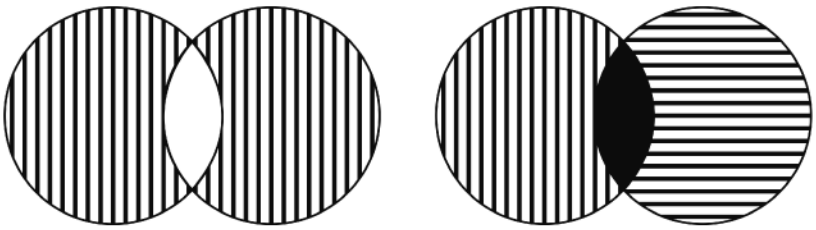
A thin piece of four maline crystal can be used as a polarizer by cutting its faces parallel to optic axis. However, its use as polarizer is limited, because the polarized light is coloured due to unequal obsorption of light of various wavelengths.

Polaroids:

A polaroid is a thin transparent film which can produce and analyze the plane polarized light. Because of certain advantages instead of Nicol prism polaroids are used to producing and analyzing plane polarized light in Laboratory.

The polaroid is prepared from suspension of small herapathite (Iodo-sulphate of quinine) crystals in nitro cellulose. In this way, a large fine sheet is produced which contains million of tiny crystals with their optic axes all parallel. This is mounted between two thin sheets of glass to give more stability. The construction of polaroid is based on dichroism.

Another type, called H-polaroid, is formed by stretching polyvinyl alcohol film so as to orient the complex molecules in the direction of stress, which makes it doubly refracting and when saturated with iodine it exhibits dichroism.



(a) Figure : 12.9 (b)

When two polaroids are not crossed, the transmitted beam is plane polarized (figure 12.9a) and when they are crossed (figure-12.9b), there is no light prassing through.

Polaroids are widely use in everyday life e.g. in sunglasses, head lights and wind screens of a car, windows of railway trains, aeroplanes, in stereoscopic motion pictures etc.

It is also used as polarizer and analyzer.

Exercise--1

A ray of light is incident on a glass plate of refractive index 1.5 at the polarizing angle. Find the polariziang angle and the angle of refraction.

12.9. Huygen's theory of double refraction.

The phenomenon of double refractions in a uniaxial crystal was explained by Huygen with the help of his theory of secondary wavelets. According to him, when an umpolarized light is indicent on a doubly refracting crystal, two wave fronts. are produced, one for the ordinary ray and other for extra ordinary ray. The wave front of o-ray will be spherical, as it obeys Snell's law, so the velocity of O-ray in all directions is same. But the wavefront of E-ray will be an ellipsoid, as E-ray propagates with different speed in different directions and does not obey snell's law of refraction in uniaxial crystal.

Since the optical properties of uniaxial crystals are perfectly symmetrical about optic axis, hence along optic axis O-ray and E-ray travel with same speed and no double refraction occurs along optic axis. The nature of wave surfaces is shown in figure along optic axis. The nature of wave surfaces is showning figure–12.10.

The crystals like calcite in which E-ray travels faster than O-ray $(\mu_0 > \mu_E)$ in the direction perpendicular to the optic axis are called negative crystals. In this case ellipsoid lies outside the sphere. Crystals like quartz O-ray travels faster than E-ray $(\mu_E > \mu_0)$ in the direction perpendicular to the optic axis are called positive crystals. Here sphere is outside the ellipoid.

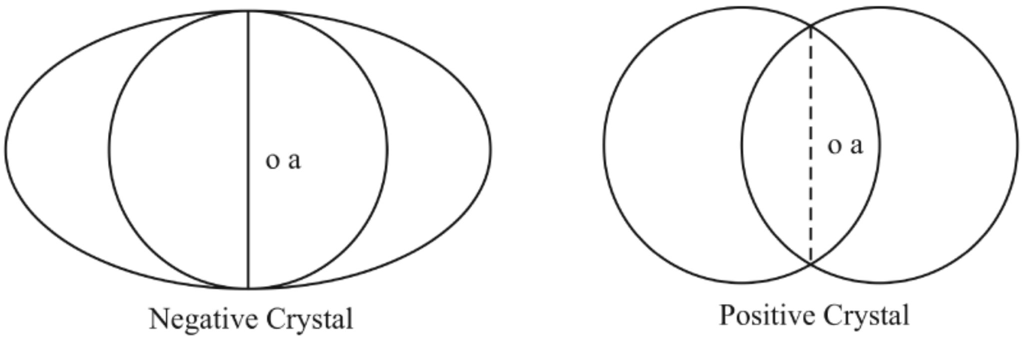


Figure : 12.10

12.9.1 Huygen's construction of surfaces in uniaxial crystal.

For constraction of wave surfaces on the plane of incidence in a unixial crystal, we take only the examples of a negative crystals.

Case I– Optic axis inclined to the upper face and lying in the plane of the incidence.

Consider a plane wave front AB is incident obliquely on the upper surface of a unixial negative crystal, cut is such a way that the optic axis (Ax) lies in the plane of the incidence but is inclined to the upper face at an angle as shown in figure–12.11.

The point A of the wavefront AB, where it strikes the crystal surface first becomes the centre of secondary wavelets of both ordinary (O) and extraordinary (E) waves. During the time t the disturbance from B reaches C, the ordinary and extra ordinary wavefront from A will move to the positions D and F respectively.

Thus

$$t = \frac{BC}{C} = \frac{AD}{v_0} = \frac{AF}{v_E}$$

$$\therefore \; AD = \frac{BC}{\frac{C}{v_0}} = \frac{BC}{\mu_0} \quad .... \quad ... \quad ... \quad ... \quad ... \qquad (12.3)$$

$$AF = \frac{BC}{\left(\frac{C}{v_E}\right)} = \frac{BC}{\mu_E} \quad .... \quad ... \quad ... \quad ... \quad ... \qquad (12.4)$$

Where C is the velocity of light in air, v_0 and v_E are the velocitesof O-ray and E-ray in the crystal respectively, m_0 and m_E are the respective refractive indices.

Now dra a circle with A as centre and AD as radius, the circle will cut the optic axis at X. The circle represents the position of ordinary wave surface. Again for negative crystal $\mu_0 > \mu_E$ but along optic axis $\mu_0 > \mu_E$, which is the maximum value of μ_E. Thus AX is the seniminor axis of the ellipse. The seminajor axis is given by $\frac{BC}{(\mu_E)}$. Draw an ellipse with the given semi-major and minor axes touching the circle at X, it gives the position of extra ordinary wave surface.

The tangents CD and CF are drawn from C of the ordinary and extra ordinary wave surfaces represent the ordinary and extra ordinary wave fronts. The straight lines AD and AF are the directions of O-ray and E-ray inside the crystal respectively.

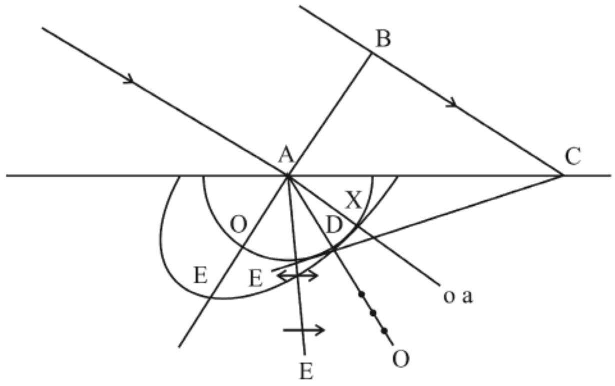


Figure : 12.11

Case-II optic axis parallel to upper face and lying in the plane of incidence.

Depending upon the angle of inciedence here two different cases arise.

(i) Oblique incidence

As the optic axis is parallel to the upper face of the crystal and lies inthe plane of incidence, the positions of O and E wave fronts can be drawn by similar method of constructiin as in case-I

The sphere and ellipse will touch at X along AX, the ordinary and extra ordinary wavefronts CD and CF corresponding tothe incident wave front AB are drawn, as shown in figure 12.12. It is also clear that the E-ray and O-ray travel different directions with different velocities.

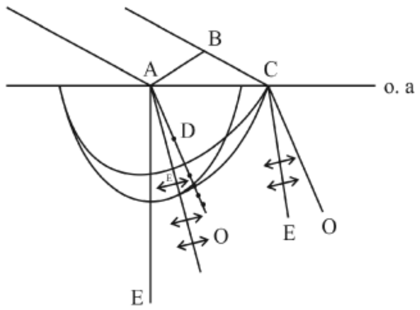


Figure : 12.12

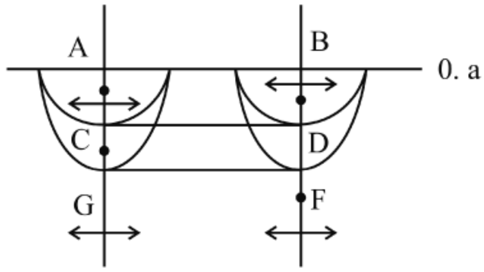


Figure : 12.13

(ii) Normal incidence:

For normal incidence the construction is as shown in figure -12.13. CD and GF are the wave fronts of ordinary and extra ordinary wave surfaces, corresponding to indident wave front AB. The O and E wave fronts are parallel toeach other and they travel in the same directiin with different velocities. Hence there will be a path difference between O-ray and E-ray the fact is used inthe constructiin of the quarter and half wave plates.

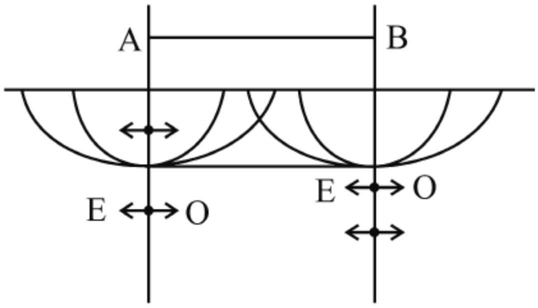


Figure : 12.14

Case III. Optic axis perpendicular to the upper face.

Here the optic axis is perpendicular to the face of the crystal, the wavefronts of O and E waves can be drawn by simuilar method of construction as in case-I The two wavefronts touch each other at x. The O-ray and E-ray travel with different velocities in different directions. Since the incidence is normal there is no double refraction, as the O and the E rays travel with the same velocity inthe direction of optic axis. The two rays (O and E) coinside in the crystal (oo or EE) as illustrated in figure-12.14.

12.10 Super position of two plane polarized waves with vibrations at right angles.

Consider a plane polarized monochromatic light be indident normally on a calcite crystal cut with faces parallel to the optic axis (figure-12.14).

Suppose the electric vector makes an angle withthe optic axis, on entering the crystal at P the amplitude A of the incident light breaks into two componounts O-wave of amplitude $b = A\sin\theta$ with vibration perpendicular to optic axis and E-wave of amplitude $a = A\cos\theta$ with vibrations along optic axis.

Now the O and E-rays travel throught the crystal with different velocities $(v_E > v_0)$ and therefore, on energence from the crystal the path difference between the rays would be $(\mu_0 - \mu_e)$, where μ_0 and μ_E are the refractive indices of O and E-rays respectively, is the thickness of the crystal.

∴ Phase difference between the rays is

$$\phi = \frac{2\pi}{\lambda}(\mu_0 - \mu_E)t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12.5)$$

So the equations of energent rays may be written as

$X = A \sin\theta \cos\omega t = a \text{ con}\omega t$ ... (12.6)

and $Y = A \cos\theta \cos(\omega t+\varphi) = b \cos(\omega t+\varphi)$ ... ... (12.7)

Where ω = the angular frequence of vibrations

$a = \sin\theta$ and b $A \cos\theta$.

Now in the same way as we deduced euqations(2.17) in the article 2.61. we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\varphi = \sin^2\varphi$ ... ... (12.8)

This is a general equation of the ellipse.

Again proceeding in the same way as in article -2.6.1, we can conclude that–

In polarized light, the nature of the resultant emerging light from the crystal are

(i) plane or linearly polarized

(ii) circularly polarized and

(iii) elliptically polarized for different values of φ

12.11. Retardation plates.

An optical device, which makes a finite path difference between O-ray and E-ray by retarding the motion of one of these two rays is known as retardation plate. There are two different types of retardation plates.

(i) Quarter wave plate :

A plate of doubly refracting material cut with its optic axis parallel to the refracting surface. Thickness of the crystal is so adjusted that it produces a path difference of $\frac{\lambda}{4}$

or phase difference $\frac{\lambda}{2}$ between the O-ray and E-rays, then the plate is called quarter wave plate or $\frac{\lambda}{4}$ plate.

For a negative crystal (e.g. calcite) the velocity of E-ray is greater than the velocity of O-Ray, thence $\mu_0 > \mu_e$

If t is the thickness of the crystal plate, the the path difference between the two rays (O-ray and E-ray)

will be

$(\mu_0 - \mu_e)t$

Hence for negative crystal.

$$(\mu_0 - \mu_e)t \; (\mu_0 - \mu_e)t = \frac{\lambda}{4} \; ..... \; (12.9), \quad \therefore \quad t = \frac{\lambda}{4(\mu_0 > \mu_e)}$$

For positive crystal (e.g. quartz) $\mu_e > \mu_0$, so that

$$(\mu_e - \mu_0)t = \frac{\lambda}{4} \; ......(12.10), \quad t = \frac{\lambda}{4(\mu_e - \mu_0)}$$

Quarter wave plate is used to produce circularly and elliptically polarized light by placing them in the path of a plane polarized light.

(ii) Half wave plate

A plate of doubly refracting material cut with its optic parallel to the refracting surface.

Thickness of the crystal is so adjusted that it produces a path difference of $\frac{\lambda}{2}$ or phase difference π between O-ray and E-rays, then the plate is called half wave plate or $\frac{\lambda}{2}$ plate.

For a negative crystal (e.g. calcite) $\mu_0 > \mu_e$, if t is the thickness of the crystal plate then by definition we get

$$(\mu_0 - \mu_e)t = \frac{\lambda}{2} \qquad \therefore \; t = \frac{\lambda}{2(\mu_0 - \mu_e)} \quad ... \quad ... \quad ... \quad ... \quad ... \; (12.11)$$

And for a positive crystal (e.g. quartz) $\mu_e > \mu_0$

Therefore $t = \dfrac{\lambda}{2(\mu_e > \mu_0)}$ ... ... (12.12)

Exercise -2

Calculate the thickness of a quartz half wave plate for the line 6563Å for which the extraordinary and ordinary refractive indices $\mu_E = 1.55085$ and $\mu_0 = 1.54184$.

Exercise–3

Calculate the thickness of quarter wave plate of quartz $\lambda = 5.8 \times 10^{-17}$ um, $\mu_E = 1.553$ and $\mu_0 = 1.544$

12.12. optical activity

It is observed that, when plane polarized light passes through certain substances like quartz, aquas solution of sugar etc. along the optical axis, they rotate the plane of poarization and the direction of propagation through a certain angle. This phenomenon is known as optical activity or rotatory polarization. In case of crystals this ability of rotation arises diquids and the solutions the optical activity is due to certain structural asymmetry of the molecules themselves.

The angle of optical rotation depends on the thickness of the crystal, density of the material or concentration of solutions, wavelength of the light used and temperature of the crystal.

The substances which can rotate the direction of plane of polarization of the incident plane polarized light are called optically active substances.

There are two types of optically active substances. The substances which rotate the plane of polarization clockwise i.e. towards right of the direction of propagation of light are called dextro rotatory or right handed (e.g. quartz), while those rotate the plane of polarization anticlockwise are called laevo rotatory or left handed. (e.g. fruit sugar).

12.13 Biot's laws of optical activity and specific rotation

Biot conducted a services of experiments on optical rotation with various optically active substances and formulated the following laws—

(1) The angle of rotation (θ) of the plane of polarization for a given wavelength and temperature is directly proportional to the length (l) of the optically active substance i.e. $\theta \; \alpha \; \lambda$.

(2) The angle of rotation (θ) is directly proportional to the concentratio (c) of the solution i.e. θ α c.

(3) The rotation (θ) produced by a mxiture of optically active substances is equal to the algebraic sum of individual rotations e.e. $\theta = \sum_{n=1}^{n} \theta_n$, here clockwise and anticlockwise rotations are taken with opposite signs.

(4) The angle of rotation (θ) is inversely proportional (approximately) to the square of the wavelength (λ) of light. For quartz $\theta = A + B/\lambda^2$ where A and B are constants.

(5) The angle of rotation is also depends on temperature of the active substance.

Specitic Rotation:

In case of solution, combining the first and second law a relation is obtained as

θ = SlC, where s is a constant and is called specific rotation or rotatory power of the solution.

Thus the specific rotation of a solution at a given temperature and for a given wavelength of light is defined as the rotation in degrees by one decimetre length of the solution containing 1gm of optically active material per c.c. of solution.

Therefore specific rotation S_λ^T at a given temperature and for a given wavelength is given by

$S_\lambda^T = \frac{\theta}{lc}$ .....(12.13) where l is length of the solution in decimetre, c is the concentration of the solution in gm/c.c and θ is the angle of rotation of plane of polarization. But if λ is expressed in cm. then

$$S_\lambda^T = \frac{10\theta}{lc} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (12.14)$$

Exercise–4

A tube 20 cm long filled with a solution of came sugar placed in the path of a polarized light, given an optical rotation of 12°. Find the strenght of solution if the specific rotation of cane sugar is 66°.

12.14 Fresnel's Explanation of optical rotation

According to Frenel, a plane polarized light can be assumed to be the superposition of two equal but opposite circularly polarized light. To explain the optical rotation base on above principle, Fresnel assumed the followings.

(i) A plane polarized light incident parallel to the optic axis of the optically active crystal, is split up into two circularly polarized waves, one of which is right handed and the other is left-handed.

(ii) In an optically inactive crystals both the circularly polarized light travel in same speed, but in active substances they travel with different speeds. If the crystal is dextro rotatory the right-handed circularly polarized light travels faster, but in laevo rotaroy substance the anticlockwise travesl faster.

(iii) On emergent the circular vibrations will socmbine to form a plane polarized light, but the plane of polarization rotates clockwise by a crtain amount $(\delta/2)$ with respect to the incident plane, (as in figure-12.15b)

This can be explained by assuming that the incident their optical vibration at z = 0 be represented by

$$x = a\cos\omega t \quad \ldots \quad (12.15)$$

which, on entering the crystal, splits up into two equal and oppisite (L and R) circular motions (figure-12.15a) OL and OR, OA is the resultant of these two vectors.

Thus x = a cosωt(equation 12.15) can be resolved into two circular motions:

$$\left.\begin{aligned} x_1 &= \frac{1}{2}a\cos\omega t \\ y_1 &= \frac{1}{2}a\sin\omega t \end{aligned}\right\} \text{left circular motion and}$$

$$\left.\begin{aligned} x_2 &= \frac{1}{2}a\cos\omega t \\ y_2 &= \frac{1}{2}a\sin\omega t \end{aligned}\right\} \text{right circular motion and}$$

These circular components will travel through the crystal with different speeds and as a result on emergence, some phase difference will be introduced between them. Assuming the clockwise component moves faster and the phase difference introduced due to faster movement of clockwise component is δ. Then the energent circular components are:

$$\left.\begin{aligned} x_1 &= \frac{1}{2}a\cos\omega t \\ y_1 &= \frac{1}{2}a\sin\omega t \end{aligned}\right\} \text{ and } \left.\begin{aligned} x_2 &= \frac{1}{2}a\cos(\omega t+\delta) \\ y_2 &= \frac{1}{2}a\sin(\omega t+\delta) \end{aligned}\right\}$$

On emergence OL, OR and OA′ are the left handed, right-handed and the resultant of them respectively, as shown in figure (12.15b)

The resultant vibrations along the x and y axes are

$$x = x_1 + x_2 = \frac{1}{2}a\{(\cos\omega t + \cos(\omega t+\delta)\}$$

$$= a\cos\left(\omega t + \frac{\delta}{2}\right)\cos\frac{\delta}{2}$$

and $y = y_1 + y_2 = \frac{1}{2}a\{\sin\omega t - \sin(\omega t+\delta)\}$

$$= -a\cos\left(\omega t + \frac{\delta}{2}\right)\sin\frac{\delta}{2}$$

Now dividing above two equations, we get

$$\frac{y}{x} = -\tan\frac{\delta}{2} \text{ or, } y = -\tan\frac{\delta}{2} \quad \ldots\ldots(12.16)$$

$$y = x\tan\left(-\frac{\delta}{2}\right) \quad \ldots\ldots(12.17)$$

Equation (12.17) represents a straight lime, So the emargent light is plane polarized, whose direction of vibration makes and angle

$\left(\frac{-\delta}{2}\right)$ with x axis i.e relative to the incident viractions. Now it μ_L and μ_R are the retractive indices for left handed and right handed circular components respectively within the crystal of thickness 't' then the path difference between the two circular vibrations is $\Delta = (\mu_L - \mu_R)\ell$ and hence the phase difference is $\theta = \frac{\delta}{2} = \frac{\pi l}{\lambda}(\mu_L - \mu_R)$.

Thus the angle of rotation of plane of polarization is $\theta = \frac{\delta}{2} = \frac{\pi\ell}{\lambda}(\mu_L - \mu_R)$ ....(12.18)

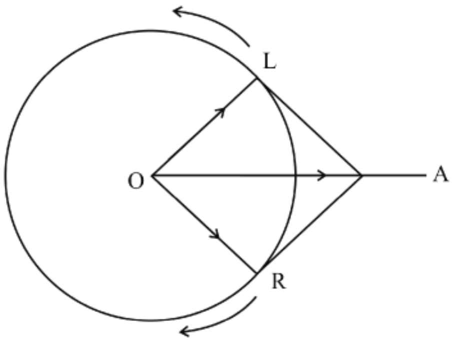


Figure : 12.15(a)

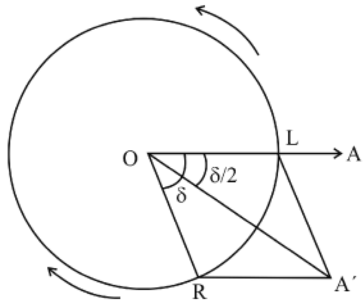


Figure : 12.15(b)

Again we can write

$$\delta = \frac{2\pi}{\lambda}\left(\mu_L - \mu_R\right)\ell$$

$$= \frac{2\pi}{\lambda}\left(\frac{C}{v_L} - \frac{C}{v_R}\right)\ell \quad \left[\because \mu = \frac{C}{v}\right]$$

$$= \frac{2\pi C\ell}{\lambda}\left(\frac{\ell}{v_L} - \frac{\ell}{v_R}\right)$$

$$= \frac{2\pi C\ell}{\lambda}\left(\frac{\ell}{\gamma\lambda_L} - \frac{\ell}{\gamma\lambda_R}\right) \qquad \begin{bmatrix} v_L & = & \upsilon\lambda_L \\ v_R & = & \upsilon\lambda_R \\ C & = & \upsilon\lambda \end{bmatrix}$$

$$\frac{2\pi C\ell}{\upsilon\lambda}\left(\frac{\ell}{\lambda_L} - \frac{\ell}{\lambda_R}\right)$$

$$= 2\pi\ell\left(\frac{\lambda_R - \lambda_L}{\lambda_L\lambda_R}\right)$$

$$\therefore\ \delta = 2\pi\ell\frac{\Delta\lambda}{\lambda^2} \quad [\text{Taking } \lambda_L = \lambda \text{ and } \lambda_R = \lambda + \Delta\lambda]$$

From this equation we see that the angle of rotation varies as thickness (ℓ) and inversely Proportional to λ^2

Exercise–5

The rotation of the plane of polarization in a certain substance is 10° per cm. Calculate the difference between the retractive indices for right and left circularly polarized beamsin the substance Given $\ell = 589.6\text{nm}$.

12.15 Polarimeter:

Polarimeter is an instrument, which measures the optical rotation of the plane of polarizatioin of a plane polarized light produced by an optically active substance. Different types of polarimeters have been designed to measure accurately the angle of rotation.

A. Laurent's half shade polarimeter:

Monochromatic light from a narrow slit S is made parallel by a convex lens L incident on the polarizer Nicol (P), which converts unpolarized light to polarized light, This polarized light incidents normally on the half shade plate (H). After passing from H the light enters the tube (T) containing the active solution whose specific rotation is to be determined. The light transmitted from T passes through the analyser Nicol(A). The emerging light is viewed through the telescope (E). The nalyser (A) Nicol can be rotated about the axis of the light i.e. the axis of the tube and its rotation can be measured in degrees by a circular scale with the help of vernier as shown in figure–12.16

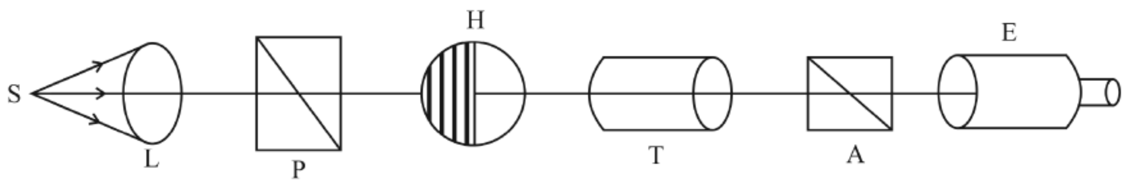


Figure : 12.16

Working:

The polarimeter tube T is first filled with distilled water. The analyzer A rotate slowly till the field of view is totally dark or bright. The position of the analyzer is recorded from the circular scale and the vernier scale.

Next, the tube T is filled with experimental solution, replacing water. As the solution rotates the plane of vibration, So the analyzer (A) is to be rotated in clockwise or anticlockwise to obtain equally dark or bright field of view. This position of analyzer is recorded from the scale. The difference between the two positions of the analyzer gives the angle of rotation (θ).

Determination of specific rotation :

knowing for a known concentration in gm/c.c. and length of the tube ℓ, the specific rolatin s is obtained from the relation

$S=\frac{\theta}{\ell c}$., here ℓ in decemeter.

Action of half shade plate:

The half-shade plate is a semi-circular plates ADB of glass and ACB of quartz cenmented together diametrically along AB. The optic axis of the quartz plate is parallel to the line of separation AOB. The thickness of the quartz plate is such that it introduces a phase difference of π between the O and E vibration. The thicknes of the glass plate is such that it absorbs the same amount of light as is done by the quartz plate.

Let the light after passing through the polarizer (P) is incident normally on the half-shade plate with vibrations along OP. On passing through the glass the vibrations will remain along OP, but it splite up into E and O components when passing through the quartz plate portion. The vibrations of O-component are along OD (perpendicular to optic axis) and those of E-components along OA (parallel to optic axis).

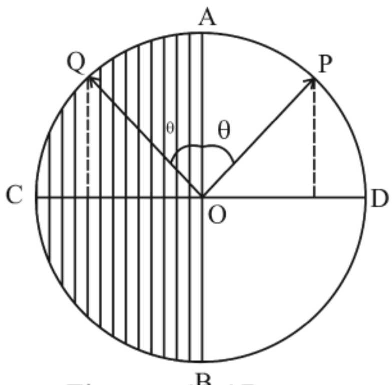


Figure : 12.17

On passing through the quartz plate a phase difference of π is introduced between these two vibrations. So, the O-vibrations will be along OC instead of OD on emergence and E-vibrations remains along OA.

The resultant vibration from the quartz plate will be along OS after emergence, such that

$\angle POA = \angle QOA = \theta$

Now if the principal plane of the analyzing Nicol is parallel to OP, the plane polarized light through glass half will pass and tence it will appear brighter that the quartz half from which light is partly obstructed. Similary if the plane of Nicol is parallel to OQ the quartz half will be brighter than glass half.

Again when the plane of Nicol is parallel to AOB, the two halves will be euqally bright, because OP and OQ are inclined equally to its principal plane and hence two components are equally bright.

Now if the principal plane of analyzer is at right angle to AOB ie parallel to COD again the components OP and OQ are equal, but as the intensity of the components are small, so the two haves are said to be equally dark.

The eye can easily detect a small change when the two haves are equally dark. The reading are thus taken in this position.

Bi quartz polarimeter:

Half shade plate can be used for a particular wavelength. So it cannot be used for any other wavelength of light. This disadvantage can be over come by using a biquartz plate, because it can be used even with white light.

Bi-quartz polarimeter has almost the same arrangement as that of Laurent's half shade polarimeter except a bi-quartz plate is used instead of half-shade plate and white light is used in place of monochromatic light.

Action of bi-quartz plate

Bi-quartz plate consists of two semi-circular quartz plates ACB and ADB, one of which is lelf handed (L) and the other is right handed (R) quartz cut with their optic axes perpendicualr to their refracting surfaces. The two plates are then cemented toggether along AB to form a circular plate.

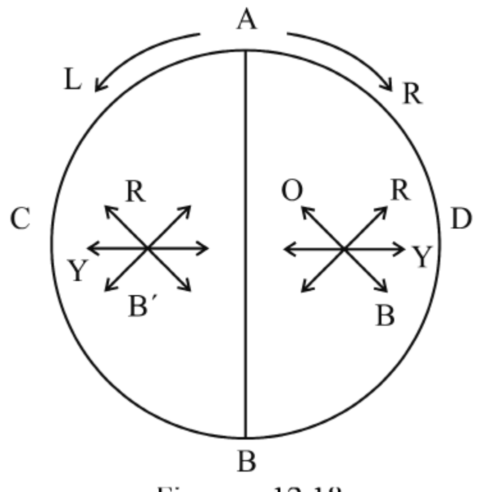


Figure : 12.18

When a plane polarized white light is incident normally on the plate each half of this plate will rotate each colour equally in opposite directions. The thickness of the plates are equal and they are so adjusted that only the yellow light will have 90° rotation of the plane of polarization.

If AOB is the direction of incident vibration, then after passage through the bi-quartz, the vibration of yellow light (γ) are along perpendicular to AOB ie along COD and the vibrations of the other colours, red (R) and blue (B) are along different directions as shown in figure–12.18. Thus if the principal section of analyzer Nicol is parallel to AOB, the yellow light will be quahead in both halves of the field, while other colours will be in the same proportion in each half. Thus two halves will then appear equally illuminated with a reddish violet tint, This is known as the tint of passage. This tint position is so sensitive that a small rotation of analyzing Nicol from this position will make one half blue and other half read.

Determinatioin of Specific rotation:

In order to measure the specific rotation, the position of the analyzer should be adjusted that the both halves of the field of view get the tint of passage with active solution and with distilled water in the tube. The difference between the readings of these two positions of the analyzer gives the optical rotation 'θ' of the optically active substance.

Hence knowing C and ℓ the specific rotation can be found using the equation $S_\lambda^T = \frac{\theta}{lc}$, when ℓ is the length of the tube taken in decimeter and C the concentrations in gm/cc.

12.16 Summary

- Polarization–vibrations of light are in particular plane and is perpendicular to the direction of propagation.
- Circularly polarized light–the magnitude of light vectors are same but the orientations change continlously
- Elliptically polarized light–the magnitude and orientations of light vector change continuously.
- Brewster's law, $\mu - \tan\theta_B$.
- Malus's law $\mu_E > \mu_0$
- Positive crystal $\mu_E > \mu_0$, E-ray refractive index μ_E
- O-ray refrative index μ_0
- Negative crystal $\mu_E < \mu_0$
- Quarter wave plate thickness $t = \frac{\lambda}{4(\mu_0 - \mu_e)}$
- for negative crystal and $t = \frac{\lambda}{4(\mu_e - \mu_0)}$ for positions crystal
- Similarly for half wave plate $t = \frac{\lambda}{2(\mu_0 \pm \mu_E)}$
- Specific rotations $S_\lambda^T = \frac{\theta}{lc}$

- The angle of rotation of the plane of polarization

$$\theta = \frac{\delta}{2} = \frac{\pi \ell}{\lambda}(\mu_L - \mu_R)$$

12.17. Questions and problems:

12.17.1 The critical angle of glass with respect to air is 41.81°. What are the angle of refraction for light and polarizaing angle?

12.17.2 A polarizer and an analyzer are oriented, So that maximum amount of light is transmitted. To what fractioin of its maximum value is the intensity of the transmitted light reduced when the analyzer is rotated through 30°?

12.17.3 Determine the wavelength of light used when quarter wave plate of thickness 1.7×10^{-5} m is used for delection Given $\mu_0 = 1.65$ and $\mu_E = 1.64$

12.17.4 Calculate the thickness of half wave plate for sodium light $(\lambda = 5893Å)$, given $\mu_0 = 1.65$ and ratio of velocity of O-ray and E-ray is 1.007.

12.17.5 A certain length of 6% solution rotates the plane of polarization by 22°. How much length of 12% solution of the same substance will cause a rotation of 30°?

12.18 Solution

Exercise–1

Here $\mu = 1.5$

We know the Brewster's law as

$$\mu = \tan\theta_B$$

$$\therefore \theta_B = \tan^{-1}\mu = \tan^{-1}1.5 = 56.3°$$

Again we know $\theta_B + r = 90°$

$$\therefore \; r = 90° - 56.3° = 33.7°$$

$\therefore$ Polarizing angle (θ_B) = 56.3° and angle of refraction (r) = 33.7°

Exercise–2

Here, $\lambda = 6563Å = 6563\times10^{-8}\text{cm}$

$\mu_E = 1.55085$ and $\mu_0 = 1.54184$

we know for half wave plate.

thickness $t=\dfrac{\lambda}{2(\mu_E-\mu_0)}=\dfrac{6563\times10^{-8}}{2(1.55085-1.54184)}$

$\therefore$ t = 3.64×10^{-3} cm.

Exercise–3

Here, $\lambda=5.8\times10^{-7}\text{cm}$; $\mu_E=1.553$ and

$\mu_0=1.544$

$\therefore$ Thickness $t=\dfrac{\lambda}{4(\mu_E-\mu_0)}=\dfrac{5.8\times10^{-7}}{4(1.553-1.544)}$

t = 1.61×10^{-5}m.

Exercise-4

Here, ℓ = 20 cm, θ = 12° and s = 66°

we know $S=\dfrac{10\theta}{lc}=\dfrac{10\times12}{20\times C}=\dfrac{6}{c}$

$\therefore\ C=\dfrac{6}{s}=\dfrac{6}{66}=\dfrac{1}{11}=0.099$

$\therefore$ C = 9.9%

Exercise–5

Here $\dfrac{\theta°}{L}=10°$ per cem and λ = 589.6nm

$=\dfrac{10.\pi}{180}\text{rad/cm}$ = 589.6×10^{-7} cm

we know angle of rotations

$$\theta=\frac{\pi\ell}{\lambda}(\mu_R-\mu_L)$$

$$\mu_R-\mu_R=\frac{\theta\lambda}{l\pi}=\frac{10.\pi}{180}.\frac{589.6\times10^{-7}}{\lambda}=3.27\times10^{-6}$$

12.17.1

Here $\theta_C = 41.81°$

we know $\mu = \dfrac{\sin i}{\sin r} = \dfrac{\sin 90°}{\sin\theta_C} = \dfrac{\ell}{\sin\theta_C} = \dfrac{\ell}{\sin\theta_C} = \dfrac{\ell}{\sin 41.81} = 1.5$

[$\because \theta_C$ is the critical angle]

Again from brewster's law

m= $\tan q_B$, q_B is the polarizing angle

$\theta_B - \tan^{-1}\mu = \tan^{-1}1.5 = 56.3°$

Again, $\theta_B + r = 90°, -r = 90° - \theta_B = 90° - 56.3° = 33.7°$

The angle of refraction r = 33.7° and polarizing angle $\theta_B = 56.3°$.

12.7.17.2

Let I_0 is the maximum intensity of the transmitted light.

From Malus's law, we have

$I = I_0\cos^2\theta = I_0\cos^2 30°$ Here $\theta = 30°$

$= I_0 \times 0.75$

$\therefore \dfrac{I}{I_0} = 0.75$

12.17.3 Here, t = 1.7×10^{-5}cm, $\mu_0 = 1.65$ and $\mu_E = 1.64$

we know $t = \dfrac{\lambda}{4(\mu_0 - \mu_E)}$

or $\lambda = 4t(\mu_0 - \mu_E) = 4\times1.7\times10^{-5}(1.65 - 1.64)$

$\therefore \lambda = 6.8\times10^{-7} = 6800\times10^{-10}m = 6800Å$

12.17.4 Here λ=5893Å=5893×10^{-8}cm μ_0= 1.54 and $\dfrac{v_0}{v_E} = 1.007$

we know $\dfrac{\mu_E}{\mu_0} = \dfrac{v_0}{v_E} = 1.007 \quad \therefore \quad \mu_E = 1.007\mu_0 = 1.007\times1.54 = 1.55$

Thickness of half wave platies $t = \frac{\lambda}{2(\mu_E - \mu_0)} = \frac{5893 \times 10^{-8}}{2(1.55 - 1.54)} = 2.95 \times 10^{-3}$

$\therefore$ $t - 2.95 \times 10^{-3}$ cm.

12.17.5

Here $C_1 = 6\%, \theta_1 = 22°$ and $C_2 = 12\%, \theta_2 = 30°$

we know $S_\lambda^T = \frac{\theta}{\ell c} = \frac{\theta_1}{\ell_1 c_1} = \frac{\theta_2}{\ell_2 c_2}$

or, $\frac{22}{\ell_1 \times 6} = \frac{30}{\ell_2 \times 12}$

or, $\frac{\ell_2}{\ell_1} = \frac{30 \times 6}{22 \times 12} = 0.68$

$\therefore$ $\ell_2 = 0.68l.$

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Notes