QP Code: 22/PT/13/IA

POST-GRADUATE COURSE
Term End Examination — June, 2022/December, 2022
MATHEMATICS
Paper-1A : ABSTRACT ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Let G be a non-commutative group of order 343. Find the order of $Z(G)$.

b) Let $f: G \to G^{/}$ be a group homomorphism. Prove that $\ker f = \{e_G\}$ iff f is a monomorphism, where e_G denotes the identity element of G.

c) Let G be a group in which $(ab)^2 = a^2b^2$ for all $a, b \in G$. Show that $H = \{g^2 \mid g \in G\}$ is normal subgroup of G.

d) If G is a group and $G^{/}$ is its derived subgroup, then prove that $G^{/}$ is a normal subgroup of G.

e) Show that the integral domain Z of all integers is a Euclidean domain with valuation υ defined by $\upsilon(a) = |a|$ for all $a \in Z \setminus \{0\}$.

f) Find a splitting field of the polynomial $(x^3 - 2)$ over Q, the field of rational numbers.

g) If U is an ideal of a ring R and $I \in U$, then prove that $U = R$.

2. a) Let ϕ be an epimorphism from a group G onto a group H. Prove that $G/\ker\phi \cong H$. 6

   b) Prove that in a group G, the relation of conjugacy is an equivalence relation. 4

3. a) Prove that every group is isomorphic to a permutation group. 5

   b) Prove that every infinite cyclic group is isomorphic to the additive group of integers Z. 5

4. a) Prove that the set of atomorphisms of a group G forms a group under the function composition operation. 5

   b) If $O(G) = p^n$, where G is a group and p is a prime number, then prove that the centre $Z(G)$ will contain more than 1 element. 5

5. a) Let R be commutative ring with identity. Prove that an ideal I of R is maximal iff the quotient ring R/I is a field. 6

   b) Prove that a commutative ring with unity is without proper ideals if and only if it is a field. 4

6. a) Define Euclidean domain. Prove that every Euclidean domain is a Principal ideal domain. 6

   b) Prove that $x^2 + 1$ is irreducible over the ring of integers mod 7. 4

7. a) Define extension of a field F with example. When is an element $\alpha \in G$, an extension of F, said to be algebraic over F? 2

   b) If the field G is an extension of the field F of finite degree n, then prove that $c \in G$ is a root of a polynomial of degree at most n with coefficients in F. 3

   c) Let $f(x)$ be a non-constant polynomial over a field K. Prove that there exists a splitting field of $f(x)$ over K. 5