POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-2A : REAL ANALYSIS & METRIC SPACES

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Let $A = \left\{ x + \frac{1}{x} : x > 0 \right\}$. Find inf A. Is A bounded above ?

   b) Let $x \in \mathbb{R}$ and $C_x = \{ r \in \mathbb{Q} : r < x \}$. Is the map $x \to C_x$ of $\mathbb{R}$ into the power set $P(\mathbb{R})$ injective ?

   c) The set of all rational numbers is measurable. — Justify.

   d) Let, $f(x) = \frac{1}{3} + \cos x$ for all x in $[0, 2\pi]$. Find f^+ and f^-.

   e) Show that $\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt{ln\, n}}$ is not a Fourier series.

   f) State *true* or *false* :
   The Euclidean n-space $\mathbb{R}^n$ is separable. — Justify.

   g) State *true* or *false* :
   The subspace $\mathbb{Z}$ of $\mathbb{R}$ is a complete metric space. — Justify.

   h) If A is a countable connected subset of $\mathbb{R}$ then which of the followings are *true* ?
   (i) A must be an empty set
   (ii) A must be a singleton set
   (iii) A is a finite set
   (iv) A is an interval.

2. a) Prove that every open set of reals is a countable union of disjoint open intervals in $\mathbb{R}$. 5

   b) State *true* or *false* :
   If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation, then f is a bounded function. Justify. 2

   c) Let $\{ E_k \}$ be a decreasing sequence of measurable sets with E_1 bounded and $E = \bigcap_{k=1}^{\infty} E_k$. Show that E is measurable and $m(E) = \lim_{n \to \infty} m(E_n)$. 3

3. a) Let $\{f_n : E \to \mathbb{R}\}_{n \in \mathbb{N}}$ be a sequence of measurable functions and $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in E$. Show that f is measurable. 5

   b) If $f(x) = \dfrac{1}{x^P}$ in $0 < x \leq 1$; for $P < 1$ then show that $L - \int_0^1 f \, dx = \dfrac{1}{P+1}$. 5

4. a) Let, $f(x) = \begin{cases} x - \pi, & x \in (-\pi, 0) \\ \pi - x, & x \in [0, \pi) \end{cases}$. Find the Fourier series for f. Then show that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$. 6

   b) Show by definition that R-S integral $\int_0^2 x^2 \, d[x] = 5$, where $[\ x\ ]$ denotes the greatest integer not greater than x. 3

   c) State Lebesgue Dominated Convergence Theorem. 1

5. a) If (X, d) is a metric space, then show that $\left(X, \dfrac{d}{d+1}\right)$ is also a metric space. 4

   b) Give an example of subsets A, B of reals with usual metric such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. 2

   c) Let (X, d) be a complete metric space and $Y \subseteq X$. Then show that (Y, d_Y) is complete if and only if Y is closed in (X, d). 4

6. a) Prove that every complete metric space is of second category. 6

   b) Examine if $f(x) = x^2$ is a uniformly continuous function over $\mathbb{R}$ with usual metric. 4

7. a) Show that continuous image of a compact metric space is compact. 5

   b) If $\{G_n\}$ is a sequence of connected sets in a metric space with $G_n \cap G_{n+1} \neq \phi$ for all n, show that $\bigcup_{n=1}^{\infty} G_n$ is connected. 3

   c) State *true* of *false* :
   Continuous image of locally connected space may not be locally connected. — Justify. 2