QP Code: 22/PT/13/IIB

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-2B : COMPLEX ANALYSIS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Symbols have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Show that the function $u(x,y)=\frac{1}{2}(x^2+y^2)$ is harmonic.

   b) Find the radius of convergence of the power series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} z^n$.

   c) Show that the point at infinity is a simple zero of $f(z)=\frac{z^2-2}{z^3-3z+4}$.

   d) Examine the singularity of the function $f(z)=\sin\frac{1}{z}$ at $z = 0$.

   e) Find the residue of $f(z)=\cot z$ at $z = 0$.

   f) Evaluate $\int_{|z|=2} \frac{dz}{e^z (z-1)^2}$.

   g) Determine all bilinear transformations which have fixed points $-i$ and i.

2. a) Show that the function $f(z)=u(x,y)+iv(x,y)$ is differentiable in a domain D if u_x, u_y, v_x, v_y exist, are continuous and satisfy Cauchy-Riemann equations in D.

   b) Let $f(z)$ be a bilinear transformation such that $f(i)=0$, $f(0)=-1$ and $f(-1)=\infty$. Show that $\omega=f(z)$ transforms

   (i) the real axis $\operatorname{Im} z=0$ on $|w|=1$

   (ii) the upper half plane $\operatorname{Im} z>0$ on $|w|<1$

   (iii) the lower half plane $\operatorname{Im} z<0$ on $|w|>1$. 5 + 5

3. a) State and prove Cauchy integral formula for first derivative.

   b) Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

   (i) $1 < |z| < 2$

   (ii) $|z| > 2$

   (iii) $|z-1| > 1$. 5 + 5

4. a) State and prove Riemann's theorem on removable singularity for an analytic function.

   b) Evaluate $\int_C \dfrac{\cos z}{z^3}\,dz$, where C is a positively oriented closed curve around the origin.

   c) Locate and name all the singularities of the function

   $f(z)=\dfrac{z^2-3z}{z^2+2z+2}$. 5 + 3 + 2

5. a) Given a rectifiable curve L, suppose that the series $f(z)=\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on L and every term of $\{f_n(z)\}$ is continuous on L. Prove that

   $$\int_L f(z)dz = \sum_{n=1}^{\infty} \int_L f_n(z)dz.$$

   b) Show that the zeros of an analytic function are isolated points.

   c) Evaluate $\displaystyle\int_{|z|=2} \frac{e^z}{z(z-1)^2}\,dz$ using Cauchy's residue theorem.

   4 + 3 + 3

6. a) State and prove Rouche's theorem.

   b) Show by the method of contour integration

   $$\int_0^{\infty} \frac{\sin mx}{x}\,dx = \frac{\pi}{2}.$$ 5 + 5

7. a) Prove that if a bilinear transformation $w=f(z)$ has two fixed points p and q then it can be expressed as

   $$\frac{w-p}{w-q} = k\left(\frac{z-p}{z-q}\right)$$

   where k is a constant.

   b) If $z=\alpha$ is a pole of $f(z)$ then show that $|f(z)| \to \infty$ as $z \to \alpha$.

   c) For what values of z does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}$ converge and find its sum. 4 + 3 + 3