QP Code: 22/PT/13/IIIA

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-3A : ORDINARY DIFFERENTIAL EQUATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Symbols / notations have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Find the singular solution of the differential equation satisfied by the family of curves $c^2x^2 - 2yc + 4 = 0$ where c is a parameter.

b) Are the solutions e^x, e^{-x} and e^{2x} of the differential equation
$y''' - 2y'' - y' + 2y = 0$,
linearly dependent ?

c) If S is defined by the rectangle $|x| \leq a$, $|y| \leq b$, show that $f(x, y) = x^2 + y^2$, satisfies the Lipschitz condition. Find the Lipschitz constant.

d) Solve the equation $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ with the initial condition $y(0) = 0$ by Picard's method to obtain y for $x = 0 \cdot 15$ correct to 3 decimal places.

e) Locate the critical point and find its nature for the system
$\dot{x} = x + y,\ \dot{y} = x - y + 1$.

f) Let y_1 and y_2 be two solutions of the problem

$$\left.\begin{array}{l} \frac{d^2y}{dt^2} + a\frac{dy}{dt} + by(t) = 0, t \in \mathbb{R} \\ y(0) = 0 \end{array}\right\}$$

where a and b are constants. Find the Wronskian of y_1 and y_2.

g) Prove the Rodrigues formula

$$L_n(z)=e^z.\frac{d^n}{dz^n}(z^n \cdot e^{-z})$$

where $L_n(z)$ is the Laguerre polynomial of order n.

2. a) Given that $y=(t+\frac{1}{t})$ is a solution of $t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-y=0$. Solve the equation by reducing the order of the equation. 5

b) If the n vector functions $\vec{\phi}_1, \vec{\phi}_2, \ldots\ldots\vec{\phi}_n$ are the n solutions of the homogeneous linear vector differential equation $\frac{d\vec{y}}{dx}=A(x)\vec{y}$ and the Wronskian $W(\vec{\phi}_1, \vec{\phi}_2, \ldots\ldots\vec{\phi}_n)=0$ for some $x_0 \in [a,b]$ then prove that $\vec{\phi}_1, \vec{\phi}_2, \ldots\ldots\vec{\phi}_n$ are linearly dependent on [a, b]. 5

3. a) Use Picard's method to compute approximately the value of y when x = 0·1 from the initial value problem $\frac{dy}{dx} = x+y$ where $y(0)=1$. Check the result with the exact value. 5

b) State Picard's existence and uniqueness theorem for IVP of differential equation.

Prove that for the IVP, $\frac{dy}{dt}=y^2+\cos^2 t,\ y(0)=0$, the interval of existence of the solution is [0, ½], given that R is the rectangle containing the origin : $R=\{(x, y): 0\le x\le a,\ |y|\le b,\ a>\frac{1}{2}\}$. 5

4. a) Obtain the Green's function and hence find the solution of the following BVP :

$$\frac{d^2u}{dx^2} = f(x),\ 0 \le x\le 1$$

subject to $u(0)=\alpha,\ u'(1)=\beta$, (α, β are constants). 5

b) Find the general solution of

$$(x^2+1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=6(x^2+1)^2$$ given that $y = x$ and $y=x^2-1$ are two independent solutions of the corresponding homogeneous equation. 5

5. a) Determine the nature of the critical point for the system $\dot{x} = \sin y,\ \dot{y} = \cos x$ and find the equation of the phase path. 5

b) Find the nature and the stability property of the critical point of the system $\dot{x} = -ax + y,\ \dot{y} = -x - ay$ for $a < 0$ and $a > 0$. 5

6. a) Find the general solution of the homogeneous linear system

$$\frac{dx}{dt} = Ax$$

where $A = \begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix}$ and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$ 5

b) Prove that

$$J_{2n}(z) = (-1)^n \cdot \frac{2}{\pi} \int_0^{\pi/2} \cos 2n\phi . \cos(z . \sin\phi)\, d\phi .$$

where $J_{2n}(z)$ is the Bessel function of first kind and of order n. 5

7. a) If $P_n(x)$ denotes the Legendre Polynomial of degree n, prove that

$$\int_{-1}^{1} P_m(x) . P_n(x)\, dx = \begin{cases} 0. & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n \end{cases}$$ 5

b) Prove that if $m < n$,

$$\frac{d^m}{dt^m}\{H_n(t)\} = \frac{2^m \lfloor n}{\lfloor (n-m)} H_{n-m}(t)$$

where $H_n(t)$ is Hermite's polynomial of degree n. 5