POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-3B : PARTIAL DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTION

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$
   a) Define Pfaffian differential equation in two independent variables. Prove that it always possesses an integrating factor.
   b) If the Dirichlet problem for a bounded region has a solution, then prove that it is unique.
   c) Form a partial differential equation by eliminating arbitrary functions *f* and *g* from $z = f(x^2 - y) + g(x^2 + y)$.
   d) Find the complete integral of $(p^2 + q^2)x = pz$.
   e) Find the value of $u(1, 1)$ for the initial value problem $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(0, y) = 4e^{-2y}$.
   f) Solve $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$.
   g) Obtain all the points (x, y) at which the equation
   $$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} - xy\frac{\partial u}{\partial x} = 0$$ is
   (i) hyperbolic (ii) parabolic (iii) elliptic.

2. a) What do you mean by compatibility of two first order partial differential equations ? Derive a necessary and sufficient condition for the compatibility of the two first order partial differential equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$. 5
   b) Show that the equations $xp - yq = x$, $x^2 p + q = xz$ are compatible and find their solution. 5

3. a) Prove that a necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation. 6
   b) Obtain the solution of the equation $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$ which passes through the *x*-axis. 4

4. a) Reduce the equation :

$$y^2 z_{xx} - 2xy\, z_{xy} + x^2 z_{yy} = \frac{y^2}{x} z_x + \frac{x^2}{y} z_y$$ to its canonical form and hence solve it. 7

b) Show that if f and g are arbitrary functions of a single variable, then $u = f(x - vt + iay) + g(x - vt - iay)$ is a solution of the equation

$$u_{xx} + y_{yy} = \frac{1}{c^2} u_{tt} \text{ where } a^2 = 1 - \frac{v^2}{c^2}.$$ 3

5. a) Show that the general solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$ is given by $u(x, t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\theta)\, d\theta$, where initial deflection is $f(x)$ and initial velocity is $g(x)$. 5

b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $u = u_0 \sin^3\left(\frac{\pi x}{l}\right), 0 \leq x \leq l$ and then released. Find the displacement of any point x of the string at any time $t > 0$. 5

6. a) Discuss the Riemann method in solving general linear hyperbolic partial differential equation of second order. 5

b) Prove that for the equation

$\frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0$, the Green's function is

$$w(z, y, \xi, \eta) = \frac{(x+y)\{2xy + (\xi - \eta)(x - y) + 2\xi\eta\}}{(\xi+\eta)^3}$$

Hence find the solution of the differential equation which satisfies the condition

$$z = 0, \frac{\partial z}{\partial x} = 3x^2 \text{ on } y = x.$$ 5

7. Prove that the solution of the Laplace equation $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, within the region of the plane bounded by $r = a$, $r = b$, $\theta = 0$, $\theta = \frac{\pi}{2}$, with its value along the boundary $r = a$ as $\theta\left(\frac{\pi}{2} - \theta\right)$ and along the other boundary as zero is given by

$$u(r, \theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{r}{b}\right)^{4n-2} - \left(\frac{b}{r}\right)^{4n-2}}{\left(\frac{a}{b}\right)^{4n-2} - \left(\frac{b}{a}\right)^{4n-2}} \cdot \frac{\sin(4n-2)\theta}{(2n-1)^3}$$ 10