# POST-GRADUATE COURSE

## Term End Examination — June, 2022/December, 2022

## MATHEMATICS

### Paper-6B : FUNCTIONAL ANALYSIS

Time : 2 hours ] [ Full Marks : 50

Weightage of Marks : 80%

**Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.**

**Use of scientific calculator is strictly prohibited.**

*( Notations and symbols have their usual meanings )*

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Show that Ker $(f)$ is closed for any continuous linear functional $f$ on a normed linear space $X$.

   b) Prove or disprove : In a normed linear space the norm function is uniformly continuous.

   c) Let $T: X \to Y$ be a linear operator. If $T^{-1}: Y \to X$ exists then show that $T^{-1}$ is a linear operator on $Y$.

   d) Let $A: H \to H$ be a self-adjoint operator. Show that eigen vectors corresponding to distinct eigen values of $A$ are orthogonal.

   e) Let $C[0, 1]$ be the set of all complex-valued continuous functions on $[0, 1]$. For each $x \in C[0, 1]$, define $\|x\| = \int_0^1 |x(t)|\, dt$. Is it a norm on $C[0, 1]$ ? Justify.

   f) Show that $l_p$ cannot be an inner product space if $p \neq 2$.

   g) Show that the orthogonal complement of any subset of an inner product space $X$ is a closed linear subspace of $X$.

2. a) Let $T: X \to Y$ be a linear operator where $X$ and $Y$ are normed linear spaces over the same scalar field. Prove that $T$ is continuous if and only if $T$ is bounded.

   b) State and prove Riesz Representation theorem in a Hilbert space.

   4 + 6

3. a) Let $H$ be a Hilbert space and $A: H \to H$ be a bounded linear operator. Prove that there is a unique bounded linear operator $B: H \to H$ such that for all $x, y \in H, (Ax, y) = (x, By)$.

   b) State and prove closed graph theorem. 5 + 5

4. a) If $X$ be a normed linear space such that the unit sphere $\{x \in X : \| x \| = 1\}$ is compact, then prove that $X$ is finite dimensional.

   b) Let $T : l_2 \to l_1$ be defined by

   $T(x_1, x_2, x_3, \ldots) = (0, 0, x_1, x_2, x_3, \ldots)$ for all $(x_1, x_2, x_3, \ldots) \in l_2$.

   Examine whether $T$ is a self-adjoint bounded linear operator on $l_2$. 5 + 5

5. a) Let $A : H \to H$ be a bounded linear operator. Show that the following statements are equivalent :

   (i) $A^* A = I$, the identity operator,

   (ii) $(Ax, Ay) = (x, y)$ for all $x, y \in H$,

   (iii) $\| Ax \| = \| x \|$ for all $x \in H$.

   b) Let $x_0$ be a non-zero vector in a normed linear space $X$. Show that there is a bounded linear functional $f$ defined on $X$ such that $\| f \| = 1$ and $f(x_0) = \| x_0 \|$. 5 + 5

6. Let $H$ be a Hilbert space and $(e_n)$ be an orthonormal sequence in $H$. Prove that the following conditions are equivalent :

   (i) $(e_n)$ is complete

   (ii) $x \perp e_n$ for $n$ = 1, 2, 3, ... implies that $x = 0$

   (iii) For every $x \in H$, $x = \sum_{n=1}^{\infty} (x, e_n) e_n$

   (iv) For every $x \in H$, $\| x \|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$. 10

7. a) Prove that every finite dimensional normed linear space is a Banach space.

   b) Prove that every closed convex subset of a Hilbert space $H$ has a unique member of smallest norm.

   c) Show that the norm in a linear space $X$ is a sublinear functional on $X$. 4 + 4 + 2