QP Code: 22/PT/13/VIIA

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-7A : DIFFERENTIAL EQUATIONS, INTEGRAL TRANSFORMATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Notations and symbols have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) If $F\{f(x)\} = \overline{f}(K)$, then find $F\{f(x-a)\}$, K being the Fourier transform parameter.

   b) Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$.

   c) Using Fourier inversion theorem, show that
   $$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, \mathrm{d}x = \pi .$$

   d) Using initial value theorem of Laplace transform, find $f(0)$ when $F(p) = \dfrac{1}{p(p^2 + a^2)}$, p being the Laplace transform parameter.

   e) Find $L^{-1}\left\{ \tan^{-1}\left(\dfrac{1}{p}\right)\right\}$, p being the Laplace transform parameter.

   f) If a function $\dfrac{f(t)}{t}$ satisfies the existence conditions of its Laplace transform and $L\{f(t)\} = F(p)$, which exists in the domain $\mathrm{Re}(p) > a$, then show that
   $$L\left\{\frac{f(t)}{t}\right\} = \int_{p}^{\infty} F(q)\,\mathrm{d}q .$$

   g) Find the Hankel transform of e^{-ax}, taking $xJ_0(px)$ as the Kernel of the transform.

2. Find the Fourier transform of $\frac{a}{x^2+a^2}$, $a>0$. 10

3. a) Deduce the Parseval's relation for Fourier transforms.

   b) Find Laplace inversion of $\cot^{-1} p$. 7 + 3

4. Find the solution of the following problem of free vibration of a stretched string of infinite length.

   (i) $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \ -\infty < x < \infty$

   (ii) $u(x,0) = f(x)$

   (iii) $u_t(x,0) = g(x)$

   (iv) $u, u_x \to 0$ as $|x| \to \infty$. 10

5. a) By the use of Laplace transform find the solution of the equation

   $t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = \sin t$

   satisfying the initial condition $y(0)=1$.

   b) Find $L\left\{\frac{\sin at}{t}\right\}$. 8 + 2

6. a) State and prove the convolution theorem for Laplace transform.

   b) By the use of Laplace transform, find the solution of the equation

   $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \ x>0, \ t>0,$

   with the initial and boundary conditions

   $u(x, 0) = 0$, for $x>0$

   $u(0, t) = 0$, for $t>0$. 6 + 4

7. a) Find $L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\}$.

   b) Solve $y'' + y = t, \ y(0)=1, y'(0) = -2$, by Laplace transform.

   c) Write down the Hankel transform of order γ of $\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\gamma^2}{r^2}f$.

   What will happen where $\gamma = 0$? 2 + 4 + 4