POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Special Paper : Pure Mathematics

Paper - 10A(i) : Advanced Differential Geometry

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

(Notations have their usual meanings.)

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Test whether f is a homeomorphism or not.

b) Define a differentiable curve on a manifold.

c) Evaluate $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$.

d) Compute $(du^1 + du^2) \wedge (du^1 + du^2 + du^3)$.

e) Is $R_a R_b = R_{ab}$?

f) Define an exact form.

g) Show that $R(X, Y,)Z = -R(Y, X,)Z$.

2. a) Find $X_p f$ where $f = (x^1)^2 + x^1 x^2$. 3

b) Find the integral curve for the following vector field $X = e^{-x^1} \frac{\partial}{\partial x^1}$ on $\mathbb{R}$. 4

c) What is the integral curve for a null vector ? 3

3. a) Define a pull back map on a differentiable manifold. 3

b) If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N where $(x^1, \ldots, x^n)$ is local coordinate system in a neighbourhood of a point p of M and $(y^1, \ldots, y^m)$ is the local coordinate system in a neighbourhood $f(p)$ of N, prove that

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m}\left(\frac{\partial f^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)}$$

where $f^k = v^k . f$, $i = 1, \ldots, n$, $k = 1, \ldots, m$. 7

4. a) Define a 1-parameter group of transformations on a manifold. 4

b) Let X, Y generate ϕ_t and ψ_s respectively as its local 1-parameter group of transformations. Prove that $\phi_t . \psi_s = \psi_s . \phi_t$ iff $[X, Y] = \theta$. 3 + 3

5. a) Define Exterior differentiation on manifold. Find $\mathrm{d}(x^2 y \,\mathrm{d}y)$. 3 + 2

b) Show that a set of 1-forms $\{\omega_1, \omega_2, \ldots, \omega_k\}$ is linearly dependent if and only if $\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k = 0$. 5

6. a) Show that
$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$. 4

b) If $(x^1, \ldots, x^n)$ is a local coordinate system of a point of a manifold and

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^h_{ijk}\frac{\partial}{\partial x^h}$$

show that

$$R^k_{ijm} = \frac{\partial}{\partial x^i}\Gamma^k_{jm} - \frac{\partial}{\partial x^j}\Gamma^k_{im} + \Gamma^t_{jm}\Gamma^k_{tj} - \Gamma^t_{im}\Gamma^k_{tj}.$$ 6

7. a) Define Riemannian manifold. 4

b) Let ∇ be a metric connection of a Riemannian manifold (M, g) and $\tilde{\nabla}$ be another linear connection given by

$$\tilde{\nabla}_X y = \tilde{\nabla}_X y + T(X,Y)$$

where T is the torsion tensor of M. Show that the following conditions are equivalent

i) $\tilde{\nabla} g = 0$

ii) $g(T(X,Y,)Z) + g(Y, T(X,Z)) = 0$ 6

POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Special Paper : Applied Mathematics
Paper - 10A(ii) : Fluid Mechanics

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Show that for a two-dimensional motion in a homogeneous incompressible fluid there exists a function ψ which is constant along a streamline.

b) Obtain stream function and velocity potential for a fluid motion where complex potential is given by $\omega = \mu / z$. Obtain the velocity components at a point.

c) Explain an Axisymmetric motion in a liquid. Express velocity components in terms of Stokes' stream function for an axisymmetric motion.

d) Define source, sink and doublets in three dimensions in a hydrodynamical system and obtain velocity potential for these systems.

e) Define strength of a vortex filament and show that the strength is constant along the filament for all time in a homogeneous incompressible liquid under conservative force field.

f) Obtain velocity potential for a stationary wave in deep water.

g) State Milne-Thomson's Circle theorem.

2. a) Define hydrodynamical image system. Obtain velocity potential and stream function when a source of strength m is placed in front of a circular cylinder and then explain image of a source in front of the circular cylinder. 5

b) Discuss the irrotational motion of an ideal liquid past a fixed sphere in a uniform stream. Obtain the equation of streamlines in the liquid outside the sphere. 5

3. a) If $u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z = \mathrm{d}\theta + \lambda \mathrm{d}\chi$, where θ, λ, χ are functions of *x, y, z, t* prove that the vortex lines at any time are the lines of intersection of the surfaces λ = constant and χ = constant. 5

b) What is the Karman vortex street ? When the upper row of vortices of strength k are placed at points $z = ma + \frac{1}{2}ib$ and lower row of vortices of strength $-k$ are at $z = \left(m + \frac{1}{2}\right)a - \frac{1}{2}ib$, $m = 0, \pm 1, \pm 2 \ldots$, show that the lower row advances with velocity $V = \frac{k}{2a}\tanh\frac{\pi b}{a}$. 5

4. a) Consider the propagation of a simple harmonic progressive wave $\eta = a\sin(mx - nt)$ at the surface of water of uniform depth h. Show that the velocity of propagation *'C'* is given by $C^2 = \frac{g}{m}\tanh mh$. 5

b) Show that the total energy of a simple harmonic progressive wave per wavelength λ is $\frac{1}{2}\zeta g a^2 \lambda$, a being the amplitude of the wave where ζ is the density of water. 5

5. a) State and prove the Blasius theorem for a two-dimensional irrotational motion of an incompressible homogeneous liquid. 5

b) A circular cylinder is placed in uniform stream, find the force acting on the cylinder. 5

6. a) Show that, if a viscous liquid flows steadily under a constant pressure gradient P in a cylinder whose generators are parallel to the axis of Z, the velocity component ω parallel to the axis satisfies the equation $\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = -\frac{P}{\mu}$ where $\frac{\mathrm{d}p}{\mathrm{d}z} = -P$. 5

b) Consider a steady flow of an incompressible viscous fluid through a rectangular tube of uniform cross-section. Find the volume rate of flow at any cross-section. 5

7. a) Obtain the vorticity equation for a viscous incompressible fluid. 5

b) Find the Blasius solution for the two-dimensional flow over a flat plate. 5