POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 1A : Abstract Algebra

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Let G be a finite group and H be a unique subgroup of G of order n. Prove that H is a normal subgroup of G.

   b) Let G be a group and H be a subgroup of $Z(G)$, where $Z(G)$ is the centre of G. Prove that H is normal in G.

   c) Show that a cyclic group of order 8 is homomorphic to a cyclic group of order 4.

   d) Let U be an ideal of a ring R and V be an ideal of U considering U as a ring. Is V an ideal of R ? Justify with an example.

   e) Prove that in an integral domain every prime element is irreducible.

   f) Prove that $\dfrac{R[x]}{<x^2+1>}$ is a field, where R denotes the field of real numbers and $<x^2+1>$ is the ideal generated by x^2+1.

   g) Let S be the splitting field of the polynomial x^3-1 over Q. Find $[S:Q]$.

   h) Prove that every finite field extension is an algebraic extension.

2. a) Let G be a group of order p^2 where p is a prime. Prove that G is commutative. 5

   b) Let G be a group such that $\dfrac{G}{Z(G)}$ is cyclic. Prove that G is commutative. 5

3. a) State and prove 'Fundamental theorem of homomorphism' on groups. 5

   b) Prove that $I_{nn}(G)$, the set of all inner automorphisms of a group G, is a normal subgroup of $\mathrm{Aut}(G)$, the group of all automorphisms of G. 5

4. a) Let G be a group and G' be its derived group. Prove that G' is a normal subgroup of G and G/G' is commutative. 5

   b) Let F be a finite field. Prove that the mapping $\phi : F \to F$, defined by $\phi(x) = x^p$, $\forall x \in F$ is a monomorphism. 5

5. a) Let R be a commutative ring with 1. Prove that an ideal I of R is a maximal ideal of R iff the quotient ring R/I is a field. 5

   b) Find all maximal ideals of the ring of integers Z. 5

6. a) Define a Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. 5

   b) Prove that an ideal $< p(x) > \neq (0)$ of $F[x]$ where F is a field, is a maximal ideal iff $p(x)$ is irreducible in $F[x]$. 5

7. a) Let K be a field and $f(x)$ be a non-constant polynomial over K. Prove that there exists a field extension F/K such that F contains a root of $f(x)$. 5

   b) Prove that the polynomials $x^2 - 2x - 1$ and $x^2 - 2$ have the same splitting field over Q. 5