POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 1B : Linear Algebra

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Find a basis of the vector space of all real polynomials of degree less than or equal to 3 and hence find its dimension.

b) Let A be an $m \times n$ matrix over any field F. Define row rank and column rank of A and write (no proof is required) the relation between them.

c) Suppose V_1 and V_2 are two subspaces of a vector space V. Prove that the subspace $V_1 + V_2$ is the smallest subspace of V containing $V_1 \cup V_2$.

d) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ cannot be one-one. Justify.

e) Find the quadratic form on $\mathbb{R}^3$ determined by the symmetric matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This quadratic form is positive definite. Justify.

f) Find the minimal polynomials and characteristic polynomials of the following matrices :

$$A = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

g) Every orthonormal set in an inner product space is linearly independent. Justify.

h) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by
$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.
Find the matrix of T with respect to the standard ordered basis of $\mathbb{R}^3$.

2. a) Let $T : V \to W$ be a linear transformation. Define nullity and rank of T. Prove that if dim V is finite then rank T is finite. (1 + 1) + 2

b) Let V_1 and V_2 be two subspaces of a vector space V. Define ordinary sum $V_1 + V_2$ and direct sum $V_1 \oplus V_2$. Suppose $V = \mathbb{R}^3$ and $V_1 =$ the subspace of V spanned by $\{(1,2,2),(2,-2,1)\}$. Find a subspace V_2 of V such that $V = V_1 \oplus V_2$. (1 + 1) + 4

3. a) Let V and W be two finite dimensional vector spaces over the same field such that dim V = dim W and $T : V \to W$ be a linear transformation. Prove that T is one-one if and only if T is onto. Give an example to illustrate that the result is not true for dim $V \neq$ dim W. 3 + 2

   b) Suppose $T : V \to W$ is a linear transformation. Prove that T is onto if and only if T maps spanning set to spanning set. 5

4. a) Find a linear operator T on $\mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$. What are the eigenvalues of such an operator ? Does there exist more than one such operator ? 3 + 1 + 1

   b) Prove that two eigenvectors corresponding to two distinct eigenvalues of a square matrix (or of a linear operator defined on a finite dimensional vector space) are linearly independent. 3

   c) Consider the subspaces $U = \{ (a, b, 0) : a, b \in \mathbb{R} \}$ and $V = \{ (0, b, c) : b, c \in \mathbb{R} \}$ of the vector space $\mathbb{R}^3$ and find dim $(U \cap V)$. 2

5. a) For $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$ in $\mathbb{R}^2$, define $< \alpha, \beta > = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$. Verify that this defines an inner product on $\mathbb{R}^2$. 5

   b) Apply Gram-Schmidt process to find an orthonormal basis for the inner product space $\mathbb{R}^3$ with standard inner product that contains $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right)$. 5

6. a) Find the quadratic form $q(x, y, z)$ corresponding to the real symmetric matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$. Find an orthogonal matrix P such that $P^t AP$ is a diagonal matrix and hence determine the rank, signature and sign of $q(x, y, z)$. 1 + 5 + 2

   b) Prove that all the eigenvalues of a nilpotent matrix are zero. 2

7. a) Suppose V is a finite dimensional vector space over a field F and $T : V \to V$ is a linear transformation. When is T said to be diagonalizable ?

   Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $T(x, y) = (x + y, y)$, is not diagonalizable. 1 + 3

   b) What do you mean by an elementary Jordan matrix of order 3 corresponding to a scalar λ ?

   Find the possible Jordan forms of a 4×4 real matrix with characteristic polynomial $(x - 2)^3 (x + 5)$. 1 + 5