POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 2A : Real Analysis & Metric Spaces

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : 2 × 5 = 10

a) Let $f : [\,0,\,1\,] \to \mathbb{R}$, $\mathbb{R}$ being the set of real numbers, is defined as follows :

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Examine whether the function f is of bounded variation on [0 , 1].

b) Show that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is a Fourier series corresponding to a periodic function of period 2π in $[-\pi, \pi]$.

c) If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in a metric space (X, d), show that real sequence $\{d(x_n, y_n)\}$ is convergent.

d) Examine whether the set Q of all rationals in $\mathbb{R}$ with usual metric is a disconnected set.

e) Prove or disprove : A homeomorphic image of a complete metric space is a complete metric space.

f) Show that the Riemann-Stieltjes integral $\int_0^4 x \, d([x] - x)$ exists and $\int_0^4 x \, d([x] - x) = 2$ where $[\,x\,]$ denotes the greatest integer not exceeding x.

g) Show that every function $f : E \to \mathbb{R}$ is Lebesgue measurable if E is Lebesgue measurable and $m(E) = 0$ where 'm' denotes the Lebesgue measure.

h) Define the outer measure m^* on $P(\mathbb{R})$, the power set of $\mathbb{R}$. If $m^*(A) = 0$ where $A \subset \mathbb{R}$, show that A is measurable and $m(A) = 0$.

2. a) If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation on [a, b], show that f can be expressed as the difference of two monotonic increasing functions on [a, b]. Is the representation of f unique ? Justify. 4 + 2

b) If $f : [a, b] \to \mathbb{R}$ is continuous function and $g : [a, b] \to \mathbb{R}$ is a function of bounded variation on [a, b], show that

$$\left| \int_a^b f \, dg \right| \le M V_a^b(g)$$

where $M = \sup_{a \le x \le b} |f(x)|$ and $V_a^b(g)$ denotes the total variation of g on [a, b]. 4

3. a) If a bounded set $E = \bigcup_{k=1}^{\infty} E_k$, show that

$$m^*(E) \le \sum_{k=1}^{\infty} m^*(E_k).$$ 5

b) When is a function $f : E \to \mathbb{R}$ where E is a Lebesgue measurable set, said to be Lebesgue measurable ? Show that a bounded set E in $\mathbb{R}$ is measurable if and only if the characteristic function χ_E of the set E is measurable. 1 + 4

4. a) Let $\{f_n\}$ be a sequence of real-valued measurable functions each defined on a measurable set E. If $f(x) = \underset{n \to \infty}{Lt} f_n(x)$ for each $x \in E$, show that f is also a measurable function. 5

b) If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, show that f is Lebesgue integrable on $[a, b]$ and $(R)\int_a^b f = (L)\int_a^b f$. 5

5. a) If f and g are two bounded measurable functions each defined on a measurable set E of finite measure such that $f = g$ *a.e.* on E, show that $(L)\int_E f = (L)\int_E g$. Is the converse of the result true ? Justify. 3 + 2

b) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions each defined on a measurable set E of finite measure such that $|f_n(x)| \le K$ for all n and for all $x \in E$ where K is a positive constant. If $f(x) = \underset{n \to \infty}{Lt} f_n(x)$ for each $x \in E$, show that f is Lebesgue integrable on E and $\underset{n \to \infty}{Lt} (L)\int_E f_n = (L)\int_E f$. 5

6. a) When is a family of subsets in metric space said to have finite intersection property (FIP) ? Prove that a metric space (X, d) is compact if and only if each family of closed subsets of X with FIP has non-empty intersection. 1 + 5

b) Let $f, g : (X, d) \to (Y, \rho)$ be continuous functions where (X, d) and (Y, ρ) are metric spaces. Show that the set $\{x \in X : f(x) \ne g(x)\}$ is an open set in X. 4

7. a) Let (X, d) be a complete metric space. If $T : X \to X$ is a contraction mapping, show that T has a unique fixed point. 5

b) If $f : (X, d) \to (Y, \rho)$ is a uniformly continuous function where (X, d) and (Y, ρ) are metric spaces, show that f transforms a Cauchy sequence in X into a Cauchy sequence in Y. Is the result true for continuous function ? Justify. 3 + 2