POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 2B : Complex Analysis

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : 2 × 5 = 10

a) Show that $f(z) = |z|^2$ is differentiable nowhere except at $z = 0$.

b) What kind of singularity has the function $\sin\frac{1}{z-1}$?

c) Find the bilinear transformation that maps the points ∞, i, 0 into the points 0, i, ∞ respectively.

d) State Laurent's theorem.

e) Find the residue of the function $f(z) = \frac{z^2}{z^2 + a^2}$ at $z = ia$.

f) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{2^n + 1}$.

g) For what values of z does the series $\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}$ converges ? Find its sum.

h) Evaluate $\oint_{|z|=2} \frac{z}{z^2 - 1} dz$.

2. a) Define uniform continuity of a function $f(z)$ in a region R. Show that $f(z) = z^2$ is uniformly continuous in $|z| < 1$, while $g(z) = \frac{1}{z}$ is not uniformly continuous in that region. 1 + 4

b) Let $f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$, $z \neq 0$

$= 0$, $z = 0$.

Show that though *C-R* equations are satisfied at (0,0), $f'(0)$ does not exist. 5

3. a) State and prove Morera's theorem. 6

b) Evaluate $\int_C \frac{e^z}{(z+1)^2} dz$ where C is the circle $|z - 1| = 3$. 4

4. a) State Cauchy-Hadamard theorem for a power series. Prove that the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence. 1 + 3

b) Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent's series valid for (i) $1 < |z| < 3$, (ii) $|z| > 3$, (iii) $0 < |z+1| < 2$. 6

5. a) State and prove Riemann's theorem on removable singularity for an analytic function. 5

b) Find all the singularities of $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$. 3

c) Show that the point at infinity is a simple zero of $f(z) = \dfrac{z^2 - 2}{z^3 + 3z + 2}$. 2

6. a) State and prove Rouche's theorem. 5

b) Using Cauchy's residue theorem evaluate $\oint_{|z|=2} \dfrac{e^z}{z(z-1)^2} dz$. 3

c) If $f(z) = \dfrac{(z^2+1)^2}{(z^2+2z+3)^3}$, evaluate $\dfrac{1}{2\pi i} \displaystyle\int_{|z|=2} \dfrac{f'(z)}{f(z)} dz$. 2

7. a) Prove that if a bilinear transformation has two fixed points p and q then it can be expressed as

$$\frac{\omega - p}{\omega - q} = K\left(\frac{z-p}{z-q}\right)$$

where K is a constant. 4

b) Evaluate $\displaystyle\int_0^{\infty} \frac{\sin mx}{x} dx$. 6