POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 3A : Ordinary Differential Equations And Special Functions

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle.

b) If S is defined by the rectangle $|x| \leq a$, $|y| \leq b$, show that the function $f(x, y) = x \sin y + y \cos x$ satisfies the Lipschitz condition and hence find the Lipschitz constant.

c) Show that the solutions e^x, e^{-x} and e^{2x} of $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$ are linearly independent.

d) Consider the linear system $\frac{dx}{dt} = 3x + 4y$,

$$\frac{dy}{dt} = 2x + y.$$

Show that $\begin{matrix} x = 2e^{5t} \\ y = e^{5t} \end{matrix}$ and $\begin{matrix} x = e^{-t} \\ y = e^{-t} \end{matrix}$ are solutions of this system.

e) Find the nature and stability property of the critical point of the system $\dot{x} = -ax + y$, $\dot{y} = -x - ay$ for $a < 0$ and $a > 0$.

f) Prove that $H_n'(z) = 2nH_{n-1}(z)$, $n \geq 1$.

g) Prove that $P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n}(z^2 - 1)^n$.

h) Solve the equation $x^2 u'' + 2xu' = x^2$, $0 \leq x \leq 1$ with the boundary condition $u(0)$ is finite and $u(1) + u'(1) = 0$ by using Green's function method.

2. a) Find the general solution of

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2 + 1)^2,$$

given that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of the corresponding homogeneous equation. 5

b) Find the adjoint differential expression $L^*[v]$ of the differential expression $L[u]$ defined by

$$L[u] = a_0 u^{(n)} + a_1 u^{(n-1)} + \ldots + a_r u^{(n-r)} + \ldots + a_n u.$$

Establish the Lagrange's identity. 5

3. a) Let $u_1, u_2, \ldots, u_n$ be n solutions of the differential equation

$$a_0(t)x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \ldots + a_n(t)x_n(t) = 0$$

where $a_i : I \to R\,(i = 0,1,\ldots,n)$ are continuous and $a_0(t) \neq 0$ for any $t \in I$.

Prove that if the Wronskian of the solutions $u_1, \ldots, u_n$ vanishes at any point of (a, b), these n solutions are linearly dependent. 5

b) Find the general solution of

$$(x^2 + 2x)\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + 2y = (x+2)^2,$$

given that $y = x+1$ and $y = x^2$ are linearly independent solutions of the corresponding homogeneous equation. 5

4. a) If the vector functions $\phi_1, \phi_2, \ldots, \phi_n$ be the n solutions of the homogenous linear vector differential equation $\frac{d\overline{X}}{dt} = A(t)\overline{X}$ on [a, b], then the n solutions are linearly independent on [a, b] iff $W(\phi_1, \ldots, \phi_n)(t) \neq 0$ for all t in [a, b]. 5

b) Find the general solution of

$$\frac{d\overline{X}}{dt} = \begin{pmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{pmatrix}\overline{X}, \text{ where } \overline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

5

5. a) Let $f(x,y)$ be a continuous function defined over a rectangle $D = \{(x,y) : |x - x_0| \leq a, |y - y_0| \leq b\}$ where a, b are some positive real numbers. Let $\frac{\partial f}{\partial y}$ be defined and continues on D and $\left|\frac{\partial f}{\partial y}\right| \leq k$ for each $(x,y) \in D$ for some $k > 0$. Then prove that f satisfies a Lipschitz condition on D with Lipschitz constant k. 5

b) Consider the initial-value problem :

$\frac{dy}{dx} = y^2$, $y(0) = 2$. Let R be the rectangle $R : \{(x,y) : |x| \leq a, |y-2| \leq b, a > 0, b > 0\}$.

Find the largest interval of existence of its solutions. 5

6. a) State and prove Sturm separation theorem. 5

b) Find the Green's function for the equation $\frac{d^2u}{dx^2} = f(x), 0 \leq x \leq 1$ subject to the boundary condition $u(0) = u'(0)$ and $u(1) = -u'(1)$. Hence find the solution of the problem. 5

7. a) Show that $H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}(e^{-z^2})$. 5

b) Prove that

(i) $z J_n(z) = z J_{n-1}(z) - n J_n(z)$

(ii) $z J_n'(z) = -z J_{n+1}(z) + n J_n(z)$. 5