POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 6A : General Topology

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$, obtain limit points of *A*, if any, of *A* if *A* is endowed with (i) usual topology of reals, (ii) discrete topology, (iii) cofinite topology.

b) In a topological space (X, τ) if $A \subset X$, show that $Bdr(A) = \phi$ if and only if *A* is clopen (both closed and open).

c) Give an example of a mapping $f : (X, \tau) \to (Y, \tau')$ where (X, τ), (Y, τ') are suitable topological spaces so that *f* is both open and closed but not continuous.

d) Show that a finite topological space that is T_1 has discrete topology.

e) Prove that in a topological space (X, τ) the set *A* consisting of the elements of a convergent sequence along with its limit point is compact.

f) Prove that the set of all irrational numbers with the usual topology induced from the usual topology of reals is totally disconnected.

g) Construct a uniformly υ for the space *R* of reals which induces the usual topology of *R*.

2. a) State the Kuratowski closure axioms. Prove that the involved operator c (say): $\wp(X) \to \wp(X)$ generates a topology τ (say) on X such that $c(A) =$ closure of A in (X, τ), for all $A \subseteq X$, where X denotes the underlying set. 4

b) In a topological space (X, τ) if G is an open set and $A \subset X$ then show that $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$. 3

c) Prove that the real number space *R* with the lower limit topology is separable but is not second countable. 3

3. a) Let $f : (X, \tau) \to (Y, \tau')$. Prove that following statements are equivalent :

i) *f* is continuous.

ii) For any closed set *F* in *Y*, $f^{-1}(F)$ is closed in *X*.

iii) $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.

iv) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subject B of Y. 6

b) Define a net. Prove that in a topological space (X, τ), a point $u \in X$ is a limit point of $A \subset X$ if and only if there is a net in $A \setminus \{u\}$ such that the net converges to u. 4

4. a) Prove that a topological space (X, τ) is T_2 if and only if every net in X converges to at most one point in X. 3

b) State and prove Tietze Extension theorem. 7

5. a) Define compactness. Is compactness a hereditary property ? Answer with reasons. 2

b) Prove that continuous image of a compact space is compact. 3

c) Define the one-point compactification (X_u, τ_u) of a non-compact locally compact T_2 space (X, τ). Then show that (X_u, τ_u) is a compact T_2 space. 5

6. a) In a topological space (X, τ) if A is connected and $A \subset B \subset \overline{A}$ then prove that B is connected. 4

b) Prove that the product space $(X \times Y, \tau \times \tau')$ is connected if and only if (X, τ) and (Y, τ') are connected. 5

c) Prove that every component of a topological space is closed. 1

7. a) Using connectedness, prove that any continuous function $f : [0,1] \to [0,1]$ has a fixed point u *i.e.* $f(u) = u$. 5

b) Define a uniformity on a non-void set X. Let (X, υ) be a uniform space. Define $\tau = \{G \subset X \ : \text{ for each } x \in G, \text{ there is a member } U \in \upsilon \text{ such that } U(x) \subset G\}$. Prove that τ is a topology on X. 5