POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 6B : Functional Analysis

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Notations and symbols have their usual meanings.
Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

   a) If X be a non-trivial normed linear space, then prove that the conjugate space of X is also non-trivial.

   b) Let $X = \mathbb{R}^2$. Define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ by
   $\|(x,y)\|_1 = \max\{|x|,|y|\}$, $(x,y) \in \mathbb{R}^2$
   and $\|(x,y)\|_2 = |x| + |y|$, $(x,y) \in \mathbb{R}^2$.
   Prove that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

   c) Is $C[a,b]$ with respect to sub-norm compact ? Justify your answer.

   d) For what values of $p (1 \le p < \infty)$, l_p is an inner product space ? Justify your answer.

   e) If M is a subspace in a normed linear space X, then prove that the closure of M, $\overline{M}$ is also a subspace of X.

   f) If T is a self-adjoint operator in a Hilbert space H, and S is any bounded linear operator in H, show that S^*TS is self-adjoint.

   g) If x and y are two elements in a real Hilbert space such that $\|x\| = \|y\|$. Prove that $< x+y, x-y > = 0$.

2. a) Prove that l_2 is self dual. 5

   b) Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space H. For $x \in H$, define $y = \sum_{k=1}^{\infty} < x, e_k > e_k$; prove that $(x - y) \perp e_k \; (k = 1, 2, \ldots)$. 5

3. a) Let X and Y are Banach spaces over the same field of scalars, and let $T : X \to X$ be a closed linear operator. Prove that (i) If C is compact in X then $T(C)$ is closed in Y and (ii) If K is compact in Y then $T^{-1}(K)$ is closed in X. 3 + 4

   b) Show that a sublinear functional p in a linear space X satisfies (i) $p(\underline{0}) = 0$ and (ii) $p(-x) \ge -p(x)$ for $x \in X$. 1 + 2

4. a) Verify that the closed unit ball in the sequence space l_2 is bounded without being totally bounded. 2 + 3

b) In a normed linear space X, verify that for a fixed member $a \in X$, the function $f : X \to X$ given by $f(x) = x + a$; $x \in X$ is a homeomorphism. Hence deduce that the translate of an open set in X in an open set.

3 + 2

5. a) Show that the space of all real polynomials of degree $\leq n$ in the closed interval $[a, b]$ is isomorphic to the Euclidean space $\mathbb{R}^{n+1}$. 5

b) Let X and Y be two normed linear spaces over the same field of scalars and let $F, G : X \to Y$ be two bounded linear operators such that F and G agree over a dense set in X, show that $F \equiv G$. 5

6. a) Let H be a Hilbert space and $A : H \to H$ be a bounded linear operator. Prove that there is a unique bounded linear operator $B : H \to H$ such that for all $x, y \in H$

$< Ax, y > = < x, By >$. 5

b) Let $L_2[0, 2\pi]$ be the real Hilbert space of all square integrable functions f over $[0, 2\pi]$ with inner product function $< f, g > = \int_0^{2\pi} f\, g \, dt$, $f, g \in L_2[0, 2\pi]$. Show that $l_0(t) = \frac{1}{\sqrt{2\pi}}$, $ln(t) = \frac{\cos nt}{\sqrt{\pi}}, n = 1, 2, \ldots$, where $0 \leq t \leq 2\pi$ forms an orthonormal sequence $L_2[0, 2\pi]$. 5

7. a) State and prove the closed graph theorem. 1 + 7

b) Show that the norm in a linear space X is a sublinear functional over X. 2