POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 7A : Differential Equations and Integral Transformations

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

(Notations have usual meanings.)

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Express the Fourier transform of the derivative of a function in terms of the Fourier transform of the function.

b) Write down the Hankel transform of order γ of $\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\gamma^2}{r^2} f$. What will happen when $\gamma = 0$?

c) If $F_c(k)$ and $\overline{G}_c(k)$ are the Fourier cosine transforms of the functions $f(x)$ and $\overline{g}(x)$ respectively with parameter k, $\overline{g}(x)$ being the complex conjugate of $g(x)$, then show that $\int_0^\infty F_c(k)\overline{G}_c(k)dk = \int_0^\infty f(x)g(x)dt$.

d) State the conditions for existence of Laplace transform of a function $f(t)$.

e) Deduce the initial value theorem for Laplace transform.

f) Assuming $L[\mathfrak{I}_0(t)] = \frac{1}{\sqrt{\phi^2+1}}$ and $L[\sin t]$, use convolution theorem to show that $\sin t = \int_0^t \mathfrak{I}_0(\tau)\mathfrak{I}_0(t-\tau)d\tau$.

g) Find $H_0\left(\frac{\sin r}{r^2}\right)$.

2. Deduce Parseval's relation of Fourier transforms. 10

3. Find the Fourier transform of $\frac{1}{x^2+1}$. 10

4. Solve the following problem of two-dimensional flow of a perfect fluid in a half-space where the fluid is introduced with uniform velocity U through a slit on the boundary :

i) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, -\infty < x < \infty, y \geq 0$

ii) $\frac{\partial\phi}{\partial y} = \begin{cases} -U \text{ for } |x| \leq a, y = 0 \\ 0 \text{ for } |x| > a, y = 0 \end{cases}$

iii) $\phi(x, y) \to 0$ as $y \to \infty$. 10

5. Let the function $F(p)$ be given function of the complex variable p in the domain $\mathrm{Re}(p) > a$ and is the Laplace transform of a function $f(t)$ of real variable t such that (i) $f(t) = 0$ for $t < 0$, (ii) in any finite interval of t, the function $f(t)$ is piecewise continuous and (iii) $f(t)$ is of exponential order $O(e^{at})$ as $t \to \infty$. Then show that

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,\mathrm{d}p\text{, where } \gamma > a.$$ 10

6. By the use of Laplace transform, find the solution of the equation

$$\frac{\partial u}{\partial t} = \lambda\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right),\ 0 \le r \le a,\ t > 0$$

satisfying the initial condition $u(r,0) = 0$, $0 \le r \le a$ and the boundary condition $u(a,t) = U$, $t > 0$, U being constant. 10

7. Applying Hankel transform, find the solution of the equation $\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$ with initial conditions $u(r,0) = \frac{1}{\sqrt{1+r^2}}$ and $\frac{\partial u}{\partial t}(r,0) = 0$, given that $\int_0^\infty e^{-\xi z}, \mathfrak{J}_0(\xi r)\,\mathrm{d}\xi = \frac{1}{\sqrt{z^2+r^2}}$. 10