POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Paper - 7B : Integral Equations And Generalised Functions

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : 2 × 5 = 10

a) Reduce the following initial value problem :

$\phi''(x) = \phi(x), \quad 0 \le x \le 1$

$\phi(0) = 0, \quad \phi'(0) = 1$

to an integral equation.

b) Derive the boundary value problem corresponding to the integral equation

$$\phi(x) - \lambda \int_0^1 k(x,t)\phi(t)\,dt = f(x), \; 0 \le x \le 1$$

where $k(x,t) = \begin{cases} x(1-t), x \le t \\ t(1-x), x > t \end{cases}$

c) Find the Neumann series corresponding to the integral equation $\phi(x) = e^x + \lambda \int_0^1 \phi(t)\,dt$.

d) Find the first and second iterated kernel of the integral equation

$$\phi(x) - \lambda \int_0^{\pi/2} \sin(x-t)\phi(t)\,dt = f(x), \; 0 \le x \le \frac{\pi}{2}.$$

e) For what values of λ, the integral equation

$$\phi(x) = \lambda \int_0^{2\pi} (x^2 t + x t^2)\phi(t)\,dt + f(x)$$

does not possess unique solution ?

f) Show that the integral equation $\phi(x) = \frac{2}{e^2 - 1}\int_0^1 e^{x+t}\phi(t)\,dt$ has only trivial solution.

g) Show that $\lambda_1 = 2/\pi$ is an eigenvalue of the integral equation $\phi(x) = \lambda_1 \int_0^{\pi} \cos(x+t)\phi(t)\,dt$.

Find the corresponding eigenfunction.

2. a) Convert the following boundary value problem into an integral equation :

$y''(x) + \lambda y(x) = f(x), \; 0 \le x \le 1$

$y(0) = c_1, \; y(1) + y'(1) = c_2$

where c_1 and c_2 are known constants.

b) Convert the following initial value problem to an integral equation :

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0\,,\ 0 \le x \le 1\,,$$

$$y(0) = 1\,,\ y'(0) = 0\,.$$ 5 + 5

3. a) Prove that the integral equation

$$\phi(x) = \lambda \int_0^1 [t\sqrt{x} - x\sqrt{t}]\phi(t)\,dt$$

does not have real eigenvalues and eigenfunctions.

b) Solve the following integral equation by determining its Neumann series

$$\phi(x) - \lambda \int_0^1 (1 - 3xt)\phi(t)\,dt = 1\,.$$ 5 + 5

4. a) Solve the following integral equation with degenerate kernel :

$$\phi(x) = f(x) + \lambda \int_0^1 xt\,\phi(t)\,dt\,,\ 0 < x < 1\,.$$

b) Use Laplace transform method to solve

$$\phi(x) = x^2 + \int_0^x \sin(x - t)\phi(t)\,dt\,,\ 0 \le x < 1\,.$$

5 + 5

5. a) Use Hilbert-Schmidt theorem to solve

$$\phi(x) - \lambda \int_0^1 k(x,t)\phi(t)\,dt = f(x)\,,\ 0 \le x \le 1\,,$$

where $k(x,t) = \begin{cases} x(1-t), x \le t \\ t(1-x), x > t \end{cases}$, $\lambda \ne n^2 r^2$.

b) Find the eigenvalues and eigenvectors of the following integral equation :

$$\phi(x) = \lambda \int_1^2 \left(xt + \frac{1}{xt}\right)\phi(t)\,dt\,.$$ 5 + 5

6. Analyse and solve the integral equation

$$\phi(x) - \lambda \int_0^{\pi} \sin(x + t)\phi(t)\,dt = f(x)\,,\ 0 \le x \le \pi\,.$$

Does the solution exist for $\lambda = \pm 2/\pi$? Justify your answer. 10

7. a) Prove that eigenfunctions of a symmetric kernel corresponding to different eigenvalues, are orthogonal.

b) Find the resolvent kernel to solve the following Volterra integral equation :

$$\phi(x) = f(x) + \int_0^x e^{x-t}\phi(t)\,dt\,,\ 0 \le x \le 1\,.$$ 5 + 5