POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Special Paper : Pure Mathematics
Paper - 9B(i) : Topological Group

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

In the following, G stands for a topological group and e for the identity element of G, if not stated otherwise.

Answer Question No. 1 and any *four* from the rest.

1. Prove any *five* of the following : $2 \times 5 = 10$

   a) In G, T_1 and T_2 axioms are equivalent.

   b) If $A \subseteq G$ be a closed set, then so also is xAx^{-1} for $x \in G$.

   c) $X = [\,0,1\,)$ is a topological space with topology $\tau = \{\phi, [\,0, \alpha), 0 < \alpha < 1\,\}$; it is not a T_1 space.

   d) G is an algebraic group with the indiscrete topology $\tau = \{\phi, G\}$. Is it a T_0-group ? Justify your answer.

   e) Any open subgroup H of G is closed.

   f) $\mathbb{R}^n$ is a locally compact group.

   g) In a Banach algebra $\mathcal{B}$ if $x \in \mathcal{B}$ then x^{-1} is unique.

2. a) If H be a subgroup of G, then prove that $\overline{H}$ is also a subgroup. 5

   b) Prove that G/H is a T_1 quotient group if and only if H is closed in G. 5

3. a) If $f : G \to G'$ be a homomorphism from G to a topological group G' then show that f is continuous if and only if f is continuous at $e \in G$. 4

   b) When is a real-valued function defined on a topological group called uniformly continuous ? Prove that a continuous function defined on a compact group G is uniformly continuous. 2 + 4

4. a) Define the dual G^* of G; show that G^* forms an algebraic group with respect to suitable operations to be mentioned explicitly. 4

b) Explain how compact open topology is introduced in G^*; prove that with respect to this topology, G^* becomes a topological group. 2 + 4

5. a) Let C be the component of the identity e in G. Prove that C is a closed normal subgroup of G. 5

b) Hence show that $aC = Ca$ is the component of $a \in G$. 5

6. a) Define a Banach Algebra; give an example. If $\mathfrak{B}$ be such an Algebra with identity e then show that the set of invertible elements of $\mathfrak{B}$ is an open set. 5

b) Let U be a neighbourhood of e and F be a compact subset of G. Show that there exists a neighbourhood V of e such that $xVx^{-1} \subseteq U$, $x \in F$. 5

7. a) Prove that G is locally compact if and only if its identity e has a compact neighbourhood. 4

b) Let H be a closed normal subgroup of a locally compact G.

Prove that G/H is a locally compact group. Further show that if G is compact then so is G/H. 2 + 4

POST-GRADUATE COURSE

Term End Examination — December, 2014 / June, 2015

MATHEMATICS

Special Paper : Applied Mathematics

Paper - 9B(ii) : Mathematical Models In Ecology

Time : 2 Hours **Full Marks : 50**

(Weightage of Marks : 80%)

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Answer Question No. 1 and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

   a) What is the environment of a system of living organisms ? What are the parts of an environment ?

   b) Define the subject 'ecology'. What are the different parts of ecology ?

   c) Explain the modelling of a system in continuous time.

   d) What are the drawbacks of a deterministic model ?

   e) Explain the postulate of continuity of state variables in modelling ecosystems.

   f) Explain the concepts of stability and asymptotic stability of the first order differential equation of the type $\frac{dx}{dt} = f(x)$.

   g) Investigate the asymptotic stability of the equilibrium point of the model equation $\frac{dx}{dt} = -rx \log \frac{x}{k}$.

2. Derive a general continuous time model equation of growth of a single species population. Derive the Malthus growth model equation as a particular case. 10

3. Discuss the logistic model equation of population growth. Explain the concepts of carrying capacity, intra-species competition and Allee effect. 10

4. Define the stability of a fixed point x^* of the difference equation $x_{n+1} = f(x_n)$, $x(t_0) = x_0$.

   Find the condition of asymptotic stability of a fixed point x^* of the non-linear first order difference equation $x_{n+1} = f(x_n)$.

   The growth of a population satisfies the following difference equation :

   $$x_{n+1} = \frac{kx_n}{b + x_n}, b, k > 0$$

   Find the fixed point (if any). If so, is that stable ? 10

5. Solve the linear second order homogeneous difference equation $x_{n+1} = x_n + x_{n-1}$. What is Fibonacci sequence ? Find the expression of Golden mean. 10

6. What is Mutualism ? Describe the different types of mutualism. Give a simple example of mutualistic model equation explaining its different parts. 10

7. Investigate the qualitative behaviour of the solution of the system

$$\frac{dx}{dt} = x\left(1 - \frac{x}{30}\right) - \frac{xy}{x+10}$$

$$\frac{dy}{dt} = y\left(\frac{x}{x+10} - \frac{1}{3}\right)$$ 10