PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in any Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Cooperation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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NETAJI SUBHAS
OPEN UNIVERSITY

Post-Graduate : Commerce
M. Com-8

Module 1

Module 2

Unit 1 Introduction to Operations Research

Structure

1.0 Objectives

The Objectives of this unit are to :

describe the historical development of the study of Operations Research

highlight the basic features of Operations Research

illustrate the methods and tools broadly used in Operations Research study

discuss the scope of the subject.

All these will facilitate to know the nature of the subject.

1.1 Introduction

Operations Research as a separate discipline has been developed since 1950s to solve many real life decision making problems. This study implies the use of scientific, quantitative and logical methods and techniques to structure and solve dicision problems. Initially the techniques used in O.R. study were applied in a different context. But realising their importances, those techniques ae now being used and taught as a separate discipline to solve the decision problems, specially in the fields like, business, commerce, management, etc.

Unlike mathematical and statistical approaches, operations research approach to solve any decision or control problems has some special features which are to be known clearly to know the subject better. This aspect is taken into consideration here for detailed discussion. The art of model building and the tools required to handle models of operations Reserch are also analysed in this unit. The broad areas in which operations Research techniques can be applied are pointed out lastly.

1.2 Historical Development

During world war II a group of individuals and specialists from different fields like, matehmatics, statistics, economics, engineering, psychology, physical science etc. were employed first in England and then in the United States to achieve success in the war. Those academicians and professional with their joint research on military operations devised some techniques and tools for appropriate use of the military resources by which was problmes (initially the problem related to the coordination of radar equipment at gun sites) could be solved.

After the end of the war, those experts realised that the techniques which were iunitially applied to solve the war problems could also be used to solve different civilian problems. Consequently, different scholars bagan to pay their attentions to the development and applications of those techniques which were brought together under a subject coined as Operations Research (as initially the techniques were invented as a result of research on military operations). A key person in the post-war development of Operations Research was George B. Dantzig who developed first the programming technique (known as simplex method of LPP) in O.R.

A substantial progress was observed in the applications of O.R. techniques during 1950s. At present it is observed that in different areas of decision problems ranging from manufacturing sector to social service sector and from individual level to government) public administrative) level O.R. techniques are applied. In search of solving the real life problems through O.R. techniques, an O.R. club was formed also first in England and its quarterly journal was first published in 1950. Similarly an O.R. society was established in America and its journal began to be published since 1953. In India O.R. came into existence in 1949 with the establishment of an O.R. unit in Hyderabad. In 1953 Prof. P. C. Mahalanobis formed an O.R. team in calcutta and then in 1957 the O.R. Society of India was founded.

1.3 Operations Research Approach

O.R. approach implies the art of tackling any decision problem with the help of O. R. techniques. The O. R. approach has the basic four properties, namely interdisciplinary, wholistic, methodological and objectivistic in nature.

1.3.1 Definition of Operations Research

According to Churchman et al. 'Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of system so as to provide those in control of operations with optimum solutions to the problems'. So an O. R. study of a problem is its methodical and systematic study in which the given problem is to be translated first in the form of a model and then optimum solution is worked out and implemented with taking care of control if requried.

1.4 Phases of Operations Research Study

To solve any given decision problem the O.R. study is conducted and controlled through the following steps :

(i) Definition of the problem

(ii) Construction of the model

(iii) Solution of the model

(iv) Validation of the model and

(v) Implementation of the solution.

Each of these phases is explained one by one below taking an example of resource allocation problem.

1.4.1 Problem Defining Phase

The first phase of an O. R. study is to identify aecurately the decision problem than can be solved by using O. R. techniques. When a problem is placed to O. R. team for getting its solution, the team will define the problem from angles, viz. (i) description of goal or objective, (ii) identification of alternative decisions and (iii) recognition of requirements, restrictions and limitation related to the problem.

1.4.2 Model Construction Phase

In this phase, the given decision problem is expressed quantiatively. For that, first, the decision variables (on which actually decisions are to be taken) are identified. Then using the decision variables quantitative expressions for the objective function(s) and constraints of the problems are specified. In this way the model which is constructed may be either a mathematical model or a simulation model or a heuristic model; that actually depends on the nature and complexity of the problem to be studies.

1.4.3 Model Solution Phase

After the construction of the model, its solution is worked out. If the constructed model is a mathematical one, its solution is obtained by applying some well defined optimization techniques. Unlike in pure mathematics, here generally optimum solution is achieved through interactive process (i.e., by applying the solution technique repeatedly until optimum solution is obtained). However, in case of simulation or heuristic model as the concept of optimality is not well-defined, in this case solution is obtained through the technique of approximate evaluations.

1.4.4 Model Validity Phase

The fourth phase of an O.R. study is to examine the validity of the model. A model is said tto be valid if it can reasonably predict the future event related to the given decision problem. Validation of the model can be checked from two aspects. In the constructed model decision varaibles are incorporated on the basis of estimating their related parameters. Estimation of the parameters may not be accurate and in reality they may change. The strength of a model depends on how far the parameters are estimated accurately or due to change of the values of the parameters how far the solution remains effective.

The validity of the model is also checked by comparing its performance using some past avaliable data. The model will be valid if, using past data of a system, the past performance of that system can be reproduced. For a nonexisting system as past data cannot collected for comparison, the validation of the model can be checked by using data generated from a simulated model or from trial runs of the system.

1.4.5 Implementation Phase

The tested results of the model are finally implemented in the system. For proper implementation, the results of the O.R. study are to be translated into detailed operating instructions so that the personnel who actually operate and administer the system can easily understand the instructions; otherwise the study will be of no use. However, realising the ground reality the recommended results may be requred to be modified for proper implementation.

1.5 Tools and Techniques of O.R. Study

Operations research as a subject falls under the categories of both Arts and Science. As operations research mainly deals with the solutions of decision-making problems, it should take into account the human behaviour on which ultimately any decision depends. Specially for defining the decision problem, construction of the O.R. model and implementing the recommended results fruitfully the knowledges on human behaviour and human psychology are very much requred. So any O. R. study has an aspect of Arts. To deal with other two phases (namely, solution and validation) along with these three phases of the O.R. study, the knowledges on mathematics, statistics and other physical, behavioural and social sciences would be requred. So the O. R. has also the science aspect. In Operations Research the tools and techniques vary due to variation of the nature of the decision problems actually there are numerous tools and techniques in O.R. and some major of them are discussed below.

1.5.1 Allocation Techniques

When the decision problem is related to optimum allocation of resources among availahle alternative uses, the allocation techniques (alternatively known as programming techniques) are applied. In programming techniques any measure of effectiveness (expressed in the form of objective function) is optimized subject to some constraints and that measure of effectiveness may be either revenue or profit or cost or any other measure of performance. Programming techniques include linear programming, transportation, assignment, non-linear programming, integer-programming, goal programming, dynamic programming, stochastic programming technique etc. In this module some programming techniques will be analysed.

1.5.2 Inventory Control Techniques

Manufacturing and business firms generally face the problem of determining optimum level of invantory (that includes raw materials, unfinished products, finished products etc.) so that the inventory costs (i.e., cost of ordering, carrying cost and cost of shortage) which are conflicting in nature are properly managed. To deal with this problem different deterministic and probabilistic inventory control models have been evolved. Some of these model with be discussed in the other module of the subject.

1.5.3 Decision Analysis Techniques

To take decisions under risk and uncertainty and also in the competitive environment, different decision analysis techniques for different states of nature are available. In these techniques, in general, given the possible payoffs (with their associated probabilities in case of risk) an optimal course of action or optimal strategy is seldcted, that minimises probable cost or maximises probable gain. The techniques which ae applied for decision analysis are game theory techniques (when two or more players compete for the achievement of conficting goals), decision tree analysis, analysis of pay-off matrix (using the rules of minimax, maximum, minimum opportunity loss, etc.), Markov-chair analysis, etc.

1.5.4 Network Analysis Techniques

These techniques are applied to the planaing, controlling and scheduling of large projects effectively. PERT and CPM techniques are two widely used techniques in this category and with these techniques we can determine the time-cost trade-off, project completion time, optimum allocation of resources updating activity times, etc. This network analysis and game theory will be analysed in the next module. However, network analysis techniques also include the techniques like, network minimisation (to connect all the areas in a network of, say, cable connection), shortest-route algorithm, maximum-flow algorithm, etc.

1.5.5 Other O. R. Techniques

It is very difficult to give a comprehensive list of O.R. techniques as continuous researches are going on to improve the existing techniques and to devise new techniques. However, apart from the above-mentioned techniques, some other well-known O. R. tools are queuing theory (in which costs of waiting as well as casts of providing servers are minimised), simulation technique (when real situation is either complex to represent quantitatively or non-existatnt), replacement technique (to determine the time of replacement of a machine) sequening technique (applied in derterminig the sequence or order of performing a number of jobs), etc.

1.6 Scope of Operations Research

Operations Research has wide applications in the fields of management, commerce, economics, public administration, engineering, etc. Some of the managerial decision making problems which can be analysed by O. R. approach are arranged functional area-wise as follows.

(i) **Marketing management**

(a) Product selection, (b) Competitive actions, (c) Advertising and sales promotional planning, (d) Sales effort allocation and assignment, (e) size of stock determination to meet market demand etc.

(ii) **Personnel management**

(a) Recruitment policies, (b) Assignment of jobs, (c) Scheduling of training programmes, (d) Manpower planning, (e) Skill and wage balancing, etc.

(iii) **Production management**

(a) Logistics, layout, engineering design, (b) Transportation, (c) Production scheduling and sequencding, (d) Inventory management and contro, (e) Optimum product-mix determination, (f) Quality control, (g) Maintenance and replacement of machineries, (h) Project scheduling, etc.

(iv) **Finance and Accounting**

(a) Capital budgeting and rationing, (b) Cash flow analysis, (c) Dividend policies, (d) Investment and protfolio management (e) Credit policies, (f) Claim and complaint procedure, (g) Break-even analysis and so on.

(v) **Other Areas**

(a) Reliability and evaluation of alternative design, (b) Forecasting, (c) Communication of information, (d) Economic planning, (e) Solution of urban housing problem, (f) Distribution of public services, (g) Military and police personnel deploykent, (h) Pollution control, (i) Solution of traffic congestion problem and many other areas related to decision making problem.

1.7 Summary

Let us sum up the discussions of introductory unit. Operations Research originated from the researches on military operations during Wrold War-II. It was realised later on that the techniques of O.R. could also be used to solve many real life problems and as a subject O.R. came into existence from early 1950s. The main area of the O.R. study is to solve the decision-making problems quantitatively. The O.R. approach required to solve any decision problem is interdisciplinary as well as wholistic in nature. Further, the O. R. study, the decision problem is requried to be known accurately and categorically. Then the problem is to be represented in teh form of a model which is solved by applying O.R. techniques. Ultimately the solution is implemented, of course after checking the validity of the model.

There are different tools and techniques of the O.R. study and these are applied to achieve solutions of decision problems by using either non-interactive analytical method or interactive numerical method or Monte Carlo method. However, the major techniques of O. R. are programming techniques, decision analysis techniques, inventory and network analysis techniques, simulation techniques etc. These techniques can be applied to solve many real life decision making problems in different fields of business, management, economics, engineering public administraction and so on.

1.8 Exercise

1. What is O.R.? Briefly review its origin and development.
2. Briefly discuss the essential characteristics of Operations Research.
3. Mention different phases in an O. R. study.
4. Give applications of O.R. in industry.
5. Briefly discuss the major techniques of Operations Research.
6. Explain the role of Operations Research in Management.
7. Do you think that O.R. is a subject of Arts of Science? Give reasons for your answer.

1.9 References

1. V. K. Kapoor : Operations Research, Sultan Chand & Sons, New Delhi.
2. H. A. Taha : Operations Research—An Introduction, Macmillan, New York.
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Unit 2 Linear Programming

Structure

2.0 Objectives

The objectives of this unit are to introduce and explain the following issues :

Features of linear programming problems

Formulation of linear programming problems

Graphical Solution of linear programming problems

Algebraic solution (i.e., simplex method) of linear programming problems

Dual formulation of linear programming problmes

Primal-dual relationships

After knowing all these issues one will be able to take decisions independently in cases of allocation of scarce resources, choice of multiple products, etc., all of which can be put in the special format of the linear programming.

2.1 Introduction

Linear Programming (LP) is an optimization technique that was introduced by the Russian mathematician L. Kantorovich. Later on in 1947 this programming technique was developed by George B. Dantzig. In real life situations and in different fields, this programming technique can be applied to solve the decision making problem of linear type.

LP is the analysis of problems in which a linear function of a number of decision variables is maximised or minimised when those variables are subject to a number of constraints in the form of linear equalities or inequalities. In order to maximise or minimise any function subject to some constraints, we can apply classical constrained optimization technique (i) if the functions are continuous and differentiable and (ii) if the constraints are of equality type. Even if these conditions are not satisfied, one can apply LP as an optimixation technique. For instance, corresponding to an allocation problem LP can be defined as a technique concered with the 'allocation' of 'scarce resources' amongst 'competing demands' in such a way that the 'measure of performance' is 'optimised'. This unit deals with different aspects of LP technique.

2.2 Features of LP Problems

Any linear programming problem has three components : (i) decision variables, (ii) objective function and (iii) constraints. Decision variables (represented in terms of algebraic symbols) are those unknown quantities which are to be solved using LP technique. The objective function being a linear function of decision variables represents the specified goal that is to be achieved (for instance, maximisation of total profit, minimisation of total cost, etc.) The goal is to be fulfilled under certain restrictions or constraints each of which is a linear expression of decision variables with euqality or inequality signs.

Therefore, the LP problem is characterized by the following conditions :

(i) **Divisibility :** All the decision variables are perefectly divisible.

(ii) **Additivity :** The decision variables are independent and hence additive in nature.

(iii) **Non-negativity :** The variables used in LP problem should take only non-negative values. If any variable under consideration is unrestricted, that is to be transformed in non-negative nature.

(iv) **Linearity :** All the mathematical expression in LP problem are linear in forms implying thereby that the relative variations of various items are proportional to each other.

(v) **Singularity :** In LP problem only one goal can be accommodated for obtaining solution. If the given problem is related to the multiple goals, LP technique cannot be applied.

Satisfying all these conditions, the general form of LP problem having n decision variables $(x_1, x_2, ...x_n)$ and m constraints is given below :

Maximise or minimise $z = c_1x_1 + c_2x_2 + ... + c_nx_n$, (: objective function) subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n \le, =, \ge b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n \le, =, \ge b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \qquad \vdots \\ a_m1x_1 + a_m2x_2 + \dots\dots + a_{mn}x_n \le, =, \ge b_n \end{array}\right\} \text{(: constraints)}$$

and $x_1, y_2, \dots\dots x_n$ 0 [: non-negativity restrictions].

where cj $(j = 1, 2, \dots n)$, b $(i = 1, 2, \dots m)$ and a_{ij} are parameters of the LP model. It should be noted that in any specific problem each constraint may take only one of the three possible forms : (i) $\le$, (ii) $=$, (iii) $\ge$.

2.3 Formulation of LP Problems

The following three steps are taken for the formulation of linear programming problems :

Step 1 : Identify the decision variables from the given problem and represent them in terms of algebraic symbols.

Step 2 : Select the objective to be fulfilled in the given problem and represent that objective as a linear function of decision variables. This objective function is either to be maximised or minimised.

Step 3 : Recognise the constraints given in the problem and express them as liner functions of decision variables. These functions may be in the form of either equations or inequalities or both.

We illustrate the formulation of linear programming problem with two examples as follows :

Illustration : 1.

Suppose a manufacturing firm wants to procude two goods Chair and Table using two inputs labour and wood. To produce one unit of either. Chair or Table, one unit of labour is required and the total availability of labour is 5 units. Further, each unit of Chair requires 2 units of material and each unit of Table requires 3 units of wood. The total available supply of wood is 12 units.

The firm wishes to maximise profit from the production of two products Chair and Table. Profit per unit of Chair is Rs. 5 and that per unit of Table is Rs. 6. Formualte this problem in the form of an LP.

Step 1 : In this problem the decision variables are x which denotes the units of production of Chair and y that represents the units of production of Table.

Step 2 : Here the goal of the firm is to maximise total profit form production. The total profit function may be written as $z = 5x + 6y$ where 5 is the unit profit of Chair and 6 is the profit per unit Table. So the firm's objective is to

$$\text{Maximise } z = 5x + 6y.$$

Step 3 : In the problem the constraints are the limited availability of inputs—labour and wood. The requirement of labour for product Chair is $1.x$ and for product Table is 1.y. Thus the total requrement of labour is $1.x + 1.y$ which cannot exceed the total availability of labour 5 units so the labour constraint becomes :

$$x + y \leq 5$$

Similarly, the wood requirements will be $2x$ for product Chair and 3y for product Table. Thus the material constraint is given by :

$$2x + 2y \leq 12$$

Further as productions cannot be negative, so $x \geq 0$ and $y \geq 0$. Therefore, the LP formulation of the above problem is

$$\begin{aligned} \text{Maximize } z = \; & 5x + 6y, \\ \text{subject to} \quad & x + y \leq 5, \\ & 2x + 3y \leq 12, \\ & x \geq 0 \;\&m\; y \geq 0. \end{aligned}$$

Illustration : 2.

Suppose there are two types of food—F_1 and F_2. Both the foods contain two types of vitamin—V_1 and V_2. A patient requres at least 1 mg of V_1 and 50 mg of V_2. Each unit of F_1 gives 1 mg of V_1 and 100 mg of V_2. Each unit of F_2 gives 1 mg of V_1 and 10 mg of V_2. The price of one unit of F_1 is Re. 1 and price of one unit of F_2 is Rs. 2. Let the problem be the determination of the amounts of F_1 and F_2 so that the patient gets at least the minimum requrement of vitamins at the minimin cost. Give the LP formulation of this problem.

Step 1 : **Decision variables :**

x_1 denotes the amount of F_1 and

x_2 denotes the amount of F_2 to be purchased by the patient.

Step 2 : **Objective function :**

Here the objective is the minimisation of total cost and the objective function is :

Minimize $c = 1.x_1 + 2.x_2$

Step 3 : **Constraints :**

The minimum requirements of two types of vitamin from the consumption of F_1 and F_2 impose here two types of constraint. For vitamin V_1, the constraint is :

$$1.x_1 + 1.x_2 \geq 1.$$

Similarly, for vitamin V_2 the constraint is :

$$100x_1 + 10x_2 \geq 50.$$

Lastly, as the amounts of F_1 and F_2 cannot be negative, here $x_1 \geq. 0$ and $x_2 \geq 0$.

Therefore, the LP formulation of the problem is :

$$\begin{aligned} \text{Minimize } c = \; & x_1 + 2x_2, \\ \text{Subject to } \; & x_1 + x_2 \geq 1, \\ & 100x_1 + 10x_2 \geq 50, \\ & x_1 \geq 0 \text{ and } x_2 \geq 0. \end{aligned}$$

2.4 Solution of LP

LP problems can be solved in two ways depending upon the no. of variables. They are graphical method and simplex method.

2.4.1 Graphical Solution of LP Problem :

After formulation, the next step is to solve the LP problem. For the solution of LP problem, graphical method can be applied if there are only two decision variables. In graphical method, first, feasible region is identified and then a solution point within the feasible region is slected. Here feasible region implies that region where all the constraints are satisfied and solution point is that point in the feasible region where the objective function is optimized. The graphical method of solution of L.P. probkem is discussed with the help of earlier examples (Illustration 1).

⇒ **Graphical Solution of Maximisation Problems :**

Let the LP problem be

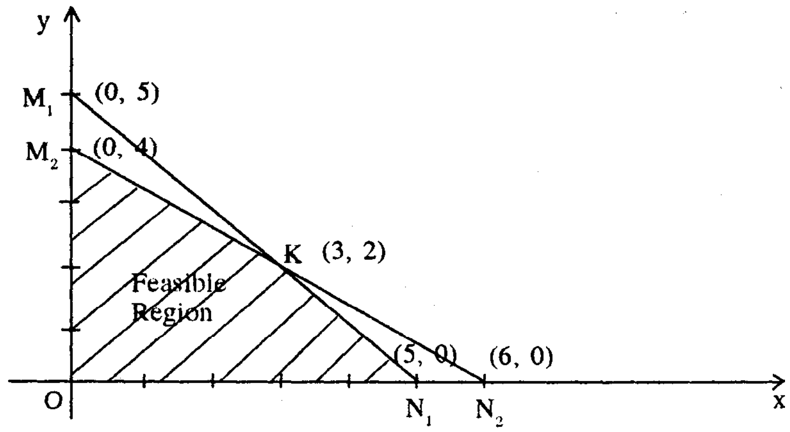
Maximize z = $5x + 6y$

Subject to $x + y \leq 5$ .... (1)

$2x + 3y \leq 12$ .... (2)

$x \geq. 0,\ y \geq. 0.$

To obtain its solution using graphical method, we plot the inequalities treating them as equalities. From (1), thus we get $x + y = 5$. So when $x = 0$, $y = 5$ and when $x = 5$, $y = 0$. Joining the points (0, 5) and (5, 0) we get line M_1N_1 in figure below :



As constraint (1) is of 'less than equal to' type, this constraint is satisfied on any points of the line M_1N_1 and also on any point in the region below M_1N_1. Similarly, from (2) we get $2x + 3y = 12$.So when $x = 0$, $y = 4$ and $x = 6$, $y = 0$. Joining points (0, 4) and (6, 0) we get the line M_2N_2 and on any points of this line and in the region below this line the constraint number (2) is satisfied with its inequality sign. Further, as $x \geq 0$ and $y \geq 0$, the solution space is restricted only to the first quadrant. Therefore, the feasible region is represented by the region OM_2KN_1.

For the solution of the LP problem, however, we need not consider all the points in the feasible region. Rather, only the corner points (like O, M_2, K and N_1) are to be wcamined for obtaining the optimum solution. Because it can be proved that optimum value will be in the extreme corner point. We know the coordinates of points 0 (0, 0), M_2(0, 4) and N_1(5, 0). As point K is the intersecting point between M_1N_1 and M_2N_2, the co-ordinate of point k is to be compared by solving the equations of these two lines, i.e., $x + y = 5$ and $2x + 3y = 12$. $x = 3$ and $y = 2$ are the solutions. So the co-ordinate of K is (3, 2).

Now we calculate the values of z(i.e., objective function at all these corner points separately. These are shown in the following table :

Corner points	Co-ordinates (x, y)	Values of z function
0	(0, 0)	$z = 5 \times 0 + 6 \times 0 = 0$
M_2	(0, 4)	$z = 5 \times 0 + 6 \times 4 = 24$
K	(3, 2)	$z = 5 \times 3 + 6 \times 2 = 27$
N_1	(5, 0)	$z = 5 \times 5 + 6 \times 0 = 25$

Here our problem is to maximize the value of z; that happens at point K. So K is the solution point and consequently the solution values of the variables related to the LP problem are :

$$\bar{x} = 3, \bar{y} = 2 \text{ and } \bar{z} = 27.$$

Thus the solution is to produce 2 Chair and 2 Tables. The maximum profit will be Rs. 27.

⇒ **Graphical Solution of Minimisation Problem**

Let us take the following LP problem of minimisation type for obtaining is solution with the help of graphical method :

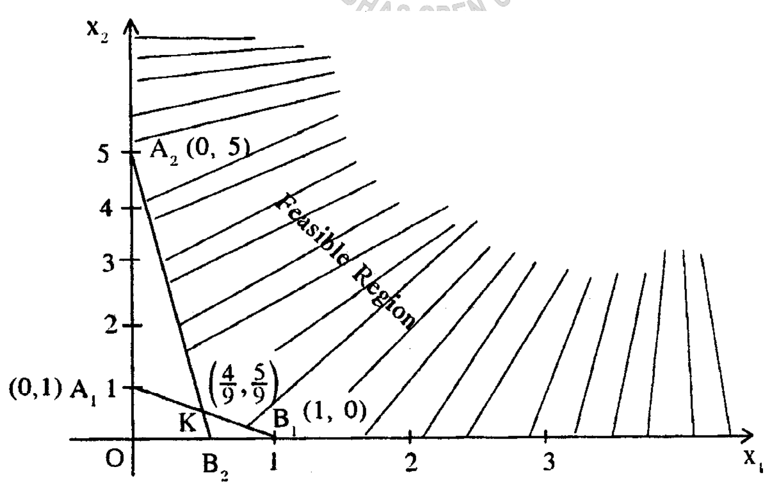
$$\text{Minimum } c = x_1 + 2x_2,$$

$$\text{Subject to} \quad x_1 + x_2 \geq 1 \text{.......... (1)}$$

$$100x_1 + 10x_2 \geq 50 \text{ .... (2)}$$

$$x_1 \geq 0, x_2 \geq 0$$

Measuring x_1 on the horizontal axis and x_2 on the vertical axis, each constraint is plotted on the following graph by treating it as a linear equation. From the equational form of constraint (1) [i.e., $x_1 + x_2 = 1$] we get the line A_1B_1 by joining the points (0, 1) and (1, 0) as before. Similarly, from the equational form of constraint (2) [i.e., $100x_1 + 10x_2 = 50$] we get two pints (0, 5) and (0.5, 0) and joining these two points A_2B_2 line is obtained. As $x_1 \geq 0$ & $x_2 \geq 0$ and the constraints (1) & (2) are of 'greater than equal to' type, the feasible region is represented here by the boundary A_2KB_1 and its upward portion (i.e., the shaded region).



We need not consider all the points on this unbounded feasible region. As the problem is to minimize the value of c, only corner points of the lower boundary (i.e., points A_2, K and B_1) are to be examined as fallows :

Corner Points	Co-ordinates (x, y)	Values of objective function
A_2	(0, 5)	$C = 1 \times 0 + 2 \times 5 = 10$
K	$\frac{4}{9}, \frac{5}{9}$ [solving equational forms of constraints (1) & (2)]	$C = 1\times\frac{4}{9} + 2 \times\frac{5}{9} = \frac{14}{9}$
B_1	(1, 0)	$C = 1\times 1 + 2\times 0 = 1$

We see that the value of C is minimum at B_1 and consequently the optimum solutions are :

$$x_1 = 1,\ x_2 = 0 \text{ and } c = 1.$$

2.4.2 Exceptional cases in LP Solution

Though in practice such small problems involving only two decision variables are usually not encountered, the graphical procedure is useful to illustrate some of the basic concepts used in solving LP problems. Further, with the help of graphical method we can clearly explain the exceptional cases that may arise in L.P. solution. These exceptional cases are explained one by one as follows.

Alternative Optima : In some LP problems there may exist moe than one set of optimum solutions. Graphically this situation will arise when the objective function happens to be parallel to any of the constraints. Alternatively, if on two corner points the value of the objective function is equally optimized, all the points on the line segment having these two corner points represent alternative optima.

Unbounded Solution : In some LP problems it is passible to find better feasible solution continuously improving the objective function values. In maximimisation case this situation arises if there remains no upper boundary in the feasible region specially in the direction of increasing values of the objective function. Similarly, in minimisation LP problem, this situation of unbounded solution arises if there is no lower boundary in the feasible region in that direction where values of the objective function can be decreased continuously. In reality LP problems become unbounded due to omission of certain constraints by mistake.

Infeasible Solution : Infeasible solution implies that in the given LP problem there is no solution which satisfies all the constraints. This situation will arise when for a given problem no feasible region can be identified. If in an LP problem there are only two constraints—one is '≥' type and other is '≤' type, the feasible region does not exist and we get infeasible solution.

Degeneracy : In the feasible region optimally is achieved by examining only the corner points. Further, a corner point is cropped up from the intersection of either (i) two constraints or (ii) one constraint and one axis or (iii) two axes. But if in any of these three cases to produce corner point unnecessarily one additional constraint remains present, this situation of degenracy will arise. Actually degeneracy creates no practical difficulty in obtaining optimum solution; this will lead to the conceptual inconvenience. The meaning of degeneracy will be explained later on.

2.5 Technical Issues in Linear Programming

If the number of decision varaibles are more than two, the graphical method fails to obtain the optimum solution. In that case algebraic method (known as simple method) would be required. Before explaining the simplex method, some technical issues related to the linear programming are analysed in this section one by one.

2.5.1 Different Forms of LP

LP problems can be represented in various forms as explained below :

General Form : Satisfying the necessary assumption of the LP, if a given problem is represented in the form of a mathematical model, that form is known as general form. An example of general form is given below :

Example 1. Maximize $z = 4x_1 + 2x_2 + 3x_3$,

subject to

$$7x_1 + 3x_2 + x_3 \leq 150 \quad \text{....... (1)}$$

$$4x_1 + 4x_2 + 2x_3 \geq 200 \quad \text{....... (2)}$$

$$3x_1 - 6x_2 - 4x_3 = -100 \quad \text{....... (3)}$$

$$x_1, x_2, x_3 \geq 0.$$

Standard Form : The standard form of an LP problem must satisfy the following conditions in addition to those of general form :

(i) All the variables must be non-negative.

(ii) All the right hand side constants of the constraints must be positive.

(iii) All constraints must be expressed as equations by adding slack variable (which represents shortage of left hand side in comparison to right hand side of a constraint which is of 'less than equal to' type) and subtracting surplus variable (that represents excess amount of left hand side compared to the right hand side of a 'greater than equal to type' constraint), if requred.

If the right hand side constant of a constraint is negative, both sides of that constraint are to be multiplied by—1. Similarly if a variable is given as negative all the coefficients of that variable in objective function and in constraints are to be multiplied by—1. On the other, if a variable (say y) is given as unrestricted, that variable is to be replaced by the subtracted form of its related two other non negative variables (say $y_1' - y_1''$). Satisfying all these conditions the standard form of the given LP problem is as follows :

Example 2. Maximise z = $4x_1 + 2x_2 + 3x_3 + OS_1 + OS_2$

Subject to

$$7x_1 + 3x_2 + x_2 + S_1 = 150 \text{ .... (1)}$$

$$4x_1 + 4x_2 + 2x_3 - S_2 = 200 \text{ .... (2)}$$

$$-3x_1 + 6x_2 + 4x_3 = 100 \text{ .... (3)}$$

$$x_1, x_2, x_3, S_1, S_2 \geq 0.$$

In this form S_1 is known as slack variable and S_2 is known as surplus variable.

Canonical Form : The canonical form of LP requires at least one basic variable in each of the constraints in addition to the requirements of standard form. Now basic variable is that variable which remains present only in one constraint with + 1 coefficient. So slack variable is a basic variable. But in case of surplus variable (i.e., when the constraint is of '≥' type) and also in case of equal to type constraint artificially basic variables are to be created and those are known as artificial variables. In our example artificial variables (A_1 and A_2) are to be inserted in the constraints (2) and (3) to get the canonical form of LP problem.

Example 3. Maximize $z = 4x_1 + 2x_2 + 3x_3 + OS_1 + OS_2 - MA_1 - MA_2$

subject to $7x_1 + 3x_2 + x_3 + S_1 = 150$ .... (1)

$4x_1 + 4x_2 + 2x_3 - S_2 + A_1 = 200$ .... (2)

$-3x_1 + 6x_2 + 4x_3 + A_2 = 100$ .... (3)

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

Thus canonial from is uded to apply simples method in LP.

[**Note :** Coefficent M will be discussed late on].

2.5.2 Different Solutions of LP

Feasible Solution : Solution values of decision variables (say, x, j = 1, 2, ...n), which satisfy all the constraints and non negativity conditions of an LP problem are said to constitute its feasible solutin.

Basic Solution : Let us consider a general LP problem having n number of decision variables and m number of constraints such that $n > m$. Setting $(n - m)$ variables equal to zero, remaining m variables are solved form m equations (obtained from m constraints). This solution of m variables is known as a basic solution.

It is to be noted that $(n - m)$ variables whose values do not appear in the solution are known as non-basic variables and the remaining m variables are called basic variables.

Basic Feasible Solution : A feasible solution to an LP proble, which is also the basic solution is known as the basic feasible solutio (B.F.S.). B.F.S. may be of two types as mentioned below :

Degenerat B.F.S. : A B.F.S. is called degenerate if at least one basic variable possesses zero value, i.e., when basic variable behaves like a non-basic variable.

Non-degenerate B.F.S. : A B.F.S. is called non-denegerate if all the m basic variables have non-zero and positive values.

Optimum B.F.S. : A B.F.S. is known as optimum B.F.S. if it optimizes (i.e., maximizes or minimizes) the objective function of the given LP problem.

2.5.3 Fundamental Theorem of LP

If an LP problem has an optimal solution, then that optimal solution must coincide with at least one basic feasible solution of that LP problem. In other words, the optimal solution, if exists, to an LP problem must be a B.F.S. Due to its fundamental importance, this theorem is called as such.

2.6 Simplex Method

If the number of decision variables in a given LP problem are more than two, simplx method is required for obtaining its solution. For simplex method, the given LP problem is to be transformed into its canonical form and then the slack and artificial varaibles are taken initially as basic varaibels (having non-zero & positive values) and all other variables as non-basic variables. Next, this initial basic feasible solution is improved with the help of iterative procedure until the optimum B.F.S. is achieved. In each iteration the value of the objective function is improved by creating each time a new basis in which one non-basic varaiabel turns into a basic variables (known as the entering variable) and simultaneously one basic variable turn into a non-basic variable (known as the leaving variable).

Entering variable and leaving variable are selected on the basis of respectively optimality and feasibility conditions which are discussed below.

Optimality Condition : In the case of maximisation (minimisation) if all the non-basic variables have non-positive (non-negative) values in teh net-evaluation row of the current tableau, the current solution is said to be optimal. Otherwise, the non-basic variable with the most positive (negative) is selected as the entering variable. The procedure of calculating the values of net evaluation (i.e., $C_J - Z_j$) row will be discussed shortly. Further, if tie arises in selecting entering variable (also in case of leaving variable), that would be broken arbitrarily.

Feasibility Condition : Feasibility Condition implies that the solution values of the basic variables must not be negative. In simplex tableu this condition will be satisfied if the leaving variable is selected on the basis of minimum ratio in all cases. Minimum ratio is selected amongst the ratio which are calculated dividing the solution values by their corresponding non-negative and non-zero values of the key row (which will be discussed later on).

With the help of simple examples, the simplex method is illustrated below.

2.6.1 Use of Simplex Method in Maximisation Problem

Let us take the earlier example of LP problem of maximisation type for the illustration of simplex method :

$$\begin{aligned} \text{Maximize } z = \; & 5x + 6y, \\ \text{subject to} \; & x + y \le 5, \\ & 2x + 3y \le 12, \\ & x, \; y \ge 0 \end{aligned}$$

Its canonical form is :

$$\begin{aligned} \text{Maximize } z = \; & 5x + 6y + OS_1 + OS_2, \\ \text{subject to} \; & x + y + S_1 + OS_2 = 5, \\ & 2x + 3y + OS_1 + S_2 = 12 \\ & x, \; y, \; S_1, \; S_2 \ge 0. \end{aligned}$$

In order to simplify the handling of the equations in the problem, they can be represented in a special tabular form known as simplex tableau. The initial simplex tableau is as follows :

Simplex Tableau I : Initial Step

		$C_j \rightarrow$	5	6	0	0		
Line No.	C_{Bj}	Basis	x	y	S_1	S_2	Solution	Ratio
L_1	0	S_1	1	1	1	0	5	$\frac{5}{1}=5$
L_2	0	S_2	2	3	0	1	12	$\frac{12}{3}=4 \leftarrow$
		Z_j	0	0	0	0		
		$C_j - Z_j$	5	$6\uparrow$	0	0	$Z = 0$	

The values of C_j row are the coefficients of the variables in the objective function. The basic varaibles (here slack variables) are written under the column 'Basis' and the values of C_{Bi} column denote the contributions of the basic variables in the objective function. As x and y being non-basic variables are equal to zero, solution of $S_1 = 5$ and solution of $S_2 = 12$; those are written under solution column. The main body of the tableu is constructed by taking left hand side coefficients of the constraints. The values of Z_j row are calculated multiplying the values of each column related to a variabel by the corresponding values of C_{Bi} and then summing them together. Subracing the values of Z_j from the corresponding values of C_j, the values of net evaluation row $(C_j - Z_j)$ are computed. From the initial table it is observed that the net evaluation of y is highest positive. So y is selected as entering variable and its corresponding column is known as key column (marked by vertical arrow). Dividing the values of solution column by thier corresponding values in key column ratios are calculated. As ratio $\frac{12}{3}$ is lower, S_2 is selected as leaving varaibale and its corresponding row (marked again by a horizontal arrow) is known as key row. The elemnt which lies in the intersection of key row and key column is known as key element put within a circle). In the next table S_2 will be replaced by y and row operations are to be performed for that table using the following two formulae :

$$\text{New} = \frac{\text{old key row}}{\text{key element}}$$

and New non-key row = old non-key row—New Key row × corresponding key column element.

For the convenience of using these formulae, each row is marked by the line number. For simple tableau II.

Elements of new key row corresponding to $y = L_4 = \dfrac{L_2}{3}$ and

Elements of new non-key row corresponding to $S_1 = L_3 = L_1 - L_4 \times 1$

Taking all these steps into account simplex tableau II is prepared belw :

Simplex Tableau II

Line No.	C_{Bj}	$C_j \rightarrow$ Basis	5 x	6 y	0 S_1	0 S_2	Solution	Ratio
L_3	0	S_1	$\left(\frac{1}{3}\right)$	0	1	$-\frac{1}{3}$	1	$\frac{1}{\left(\frac{1}{3}\right)} = 3 \leftarrow$
L_4	6	y	$\frac{2}{3}$	1	0	$\frac{1}{3}$	4	$\frac{1}{\left(\frac{2}{3}\right)} = 6$
		Z_j	4	6	0	2		
		$C_j - Z_j$	1↑	0	0	–2	Z = 24	

Simplex Tableau II does not give optimum solution as there is one positive element in $C_j - Z_j$ row. Applying the earlier steps, x is selected as entering variables, S_1 is chosen as leaving variable, $\left(\frac{1}{3}\right)$ is the key element and the new basis is formed in simplex Tableau III using the following formulae :

$$L_5 = \frac{L_3}{\left(\frac{1}{3}\right)} = 3\,L_3 \text{ and}$$

$$L_6 = L_4 - L_5 \times \frac{2}{3}$$

Simplex Tableau III

		$C_j \rightarrow$	5	6	0	0	
Line No.	C_{Bj}	Basis	x	y	S_1	S_2	Solution
L_5	5	x	1	0	3	–1	3
L_6	6	y	0	1	-2	1	2
		Z_j	5	6	3	1	
		$C_j - Z_j$	0	0	–3	–1	$z = 27$

As there is no positive elements in $C_j - Z_j$ row, Simplex Tableau II represents optimum table and optimum values are :

$$\bar{x} = 3,\ \bar{y} = 2 \text{ and } \bar{z} = 27$$

[Compare these values with the values obtained by using the graphical method].

2.6.2 Use of Simplex Method in Minimisation Problem

Let us consider the simplex method for solving a minimisation problem. The minimisation technique is almost similar to the maximisation technique with a very few differences. With the help of the following example let us explain the minimisation technique :

Minimise $C = 5u + 12v,$

Subject to $u + 2v \geq 5,$

$u + 3v \geq 6,$

$u \ \&\ v \geq 0,$

First convert this problem into its canonical form by introducing surplus and artificial variables as follows :

$$\text{Minimise } C = 5u + 12v + 0S_1 + 0S_2 + MA_1 + MA_2 ,$$

$$\text{subject to} \quad u + 2v - S_1 + 0S_2 + A_1 + 0A_2 = 5,$$

$$u + 3v + 0S_1 - S_2 + 0A_1 + A_2 = 6,$$

$$u, v, S_1, S_2, A_1 \ \& \ A_2 \geq 0.$$

Here S_1 & S_2 denote surplus variables and A_1 & A_2 are the artificial varaibels. M is defined as an infinitely large number. The rationale for attaching such large coefficients ($+M$) to artificial variables lies in the fact that these variables are very likely to leave the basis as soon as possible. If at least one of them appears in the solution even with one unit value, the value of the objective function will be infinitely large in this minimisation problem. This method of assigning a very large positive eoefficient to an artificial varaibale in the objective function of a minimisation problem is known as penalty method. It should be noted in this connection that in case of maximisation problem the artificial variable (if arises) is penalised in the objective function with $-M$ coefficient (other steps will remain same).

Next prepare the initial sixplex tableau as in the maximisation problem and in that tableau basic variables (which remain in the basis) will be those variables which have +1 coefficients in the constraints of canonical form and each of which only persents in one constraint (i.e, either slackor artificial variables will be the basic variables in the initial tableau).

Simplex Tableau I

		$C_j \rightarrow$	5	12	0	0	M	M		
Line No.	C_{Bj}	Basis	u	v	S_1	S_2	A_1	A_2	Solution	Ratio
L_1	M	A_1	1	2	–1	0	1	0	5	$\frac{5}{2} = 2 \cdot 5$
L_2	M	A_2	1	3	0	–1	0	1	6	$\frac{6}{3} = 2 \leftarrow$
		Z_j	$2M$	$5M$	$-M$	$-M$	M	M	$Z = 11\ M$	
		$C_j - Z_j$	5-2M	12-5$M\uparrow$	M	M	0	0		

The elements in $C_j - Z_j$ row are calculated as before. In case of minimisation proble, the current solution will be optimum if all the elements in $C_j - Z_j$ row are non-negative; otherwise, solution is non-optimal and optimality is to be achieved through iteration. In each iteration one variable will enter into the basis, which has negative net evalution (i.e., element in $C_j–Z_j$ row) with highest magnitude. Here v will first enter into the basis. Leaving variable is selected by applying feasibility condition which is same as in case of maximisation problem. Here A_2 will leave the basis having minimum ratio and consequently the key element is 3. Next row operations are to be performed just like maximisation case. Successive iterations to obtain optimum solution are shown in the following tables:

Line No.	C_{Bj}	Basis	u	v	S_1	S_2	A_1	A_2	Solution	Ratio
II		$C_j \rightarrow$	5	12	0	0	M	M		
$L_3 = L_1 - 2 \times L_4 \Rightarrow$	M	A_1	$\frac{1}{3}$	0	-1	$\frac{2}{3}$	1	$-\frac{2}{3}$	1	$1/\left(\frac{2}{3}\right) = \frac{2}{3} \leftarrow$
$L_4 = L_{2/3} = \Rightarrow$	12	v	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2	—
		Z_j	$\frac{1}{3}M+4$	12	$-M$	$\frac{2}{3}M-4$	M	$-\frac{2}{3}M+4$	$Z = M + 24$	
		$C_j–Z_j$	$-5\frac{1}{3}M+4$	0	M	$-\frac{2}{3}M+4$	0	$\frac{5}{3}M-4$		
III										
$L_5 = L_{3/(2/3)} \Rightarrow$	0	S_2	$\frac{1}{2}$	0	$-\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	$\frac{3}{2}\Big/\frac{1}{2} = 3 \leftarrow$
$L_6 = L_4 - L_{5(-1/3)} \Rightarrow$	12	v	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{5}{2}$	$\frac{5}{2}\Big/\frac{1}{2} = 5$
		Z_j	6	12	-6	0	6	0	Z=30	
		$C_j - Z_j$	$-1\uparrow$	0	6	0	M-6	M		
IV										
$L_7 = L_{5/(1/2)} \Rightarrow$	5	u	1	0	–3	2	3	–2	3	
$L_8 = L_6 - L_{7(1/2)} \Rightarrow$	12	v	0	1	1	–1	–1	1	1	
		Z_j	5	12	-3	-2	3	2	Z=27	
		$C_j - Z_j$	0	0	3	2	M-3	M-2		

It is observed that the optimality is reached in simplex tableau no. IV and the values related to the optimal solution are :

$\mu = 3$, $\nu = 1$ and $z = 27$.

2.6.3 Some Special Cases in Simplex Method

While applying the simplex method for the solution of an LP problem, some special situations may arise. These special situations are pointed out below :

(i) If any artificial variable remains present in the basis of the final tableau where optimality condition is satisifed, then that type of solution is known as in feasible solutio. As artificial variable is meaningless and it has no real existence, the optimum solution with its presene implies infeasible solution.

(ii) By definition, the non-basic variable's value is taken as zero and the basic variable's value is positive. But if basic variable behaves like a non-basic variable i.e., if the solution value of the basic variable is zero, that problem is known as degeneracy which may arise either in the final tableau or at the iterative stage.

(iii) If all the elements of key column happen to be either zero and negative and the current solution is not optimum, then the situation of unbounded solution arises. In this situation, no ratio can be computed i.e, no entering variable can be selected, though one basic variable satisfies the conditon to leave the basis. In this case solution can be improved indefinitely in the direction of the leaving variable.

(iv) Another special case in *LP* problem is the presence of alternative optima. In applying simplex method the situation of alternative optima arise when corresponding to the final tableau, the net evaluation (i.e, the element in $C_j - Z_j$ row) of any non-basic variable is zero. If that non-basic variable enters into the basis, the value of the objective function will nto change and we get another optimum solution.

2.7 Duality in Linear Programming

For every LP problem there is a corresponding opposite problem called the dual problem.The original problem is known as the primal problem. It is sometimes easier to find the solution of a programming problem by first solving its associated dual problem. The calculation of the dual also allows us to check on the accuracy of the primalproblem. Although every LP problem has a dual problem, the interpretation and interrelationship of the solutions of the primal and the dual are not every straight forward. In this last section shall consider the dual formulation, primal-dual relationship and the improtant on uality.

2.7.1 Dual Formulation

For the dual formualtion of a primal problem the following steps are to be taken :

1. Transform the primal problem in its standard form

2. For every primal constraint (except non-negative constraints) create one dual variable.

3. Objective function coefficients of dual variables are the repective right hand side constrants of the primal constraints.

4. For each primal variable create one dual constraint whose left hand side coefficients are the coefficients of that primal variable in primal constraints (i.e., for each primal column create one dual row) and whose right hand side constant is the coefficient of that primal variable in primal objective function.

5. If primal is a maximisation problem, dual will be a minimisation prblem and vice versa.

6. If the dual is of maximisation (minimisation) type, the signs of all dual constraints are $\leq$ ($\geq$) type.

7. If otherwise nothing is specified, the dual variables are unrestricted in nature.

consider the following primal problem :

$$\begin{aligned}
\text{Maximise } Z = \; & 2x_1 + 3x_2 + 4x_3 \\
\text{subject to } \; & x_1 + x_2 + x_3 \leq 8, \\
& -2x_1 + x_2 - 3x_3 \geq -7, \\
& x_1 + 2x_2 + 4x_3 = 15 \\
& x_1, x_3 \geq 0 \text{ and } x_2 \text{ is unrestricted.}
\end{aligned}$$

Replacing x_2 by $x'_2 - x''_2$ (where $x'_2 \geq 0$ and $x''_2 \geq 0$) we get its standard form as follows :

$$\begin{aligned}
\text{Maximise } Z = \; & 2x_1 + 3x'_2 - 3x''_2 + 4x_3 + 0S_1 + 0S_2 \\
& x_1 + x'_2 - x''_2 + x_3 + S_1 + 0S_2 = 8 \qquad \text{........ (1)} \\
& 2x_1 - x'_2 - x''_2 + 3x_3 + 0S_1 + S_2 = 7 \qquad \text{........ (2)}
\end{aligned}$$

$$x_1 + 2x'_2 - 2x''_2 + 4x_3 + 0S_1 + 0S_2 = 15 \qquad \text{........ (3)}$$

$$x_1,\ x'_2,\ x''_2,\ x_3,\ S_1,\ S_2 \geq 0.$$

In this standard form all the varaibles are non-negatives, constraints are to equal to type and right hand side constants of the constraints are positive. As in the primal problem there are 3 constraints, we have to crate 3 dual variables, namely y_1, y_2 and y_3 and following the above mentioned steps the dual problem is :

Minimise $C =$ $8y_1 + 7y_2 + 15y_3$,

subject to $y_1 + 2y_2 + y_3 \geq 2$,

$y_1 - y_2 + 2y_3 \geq 3$,

$-y_1 + y_2 - 2y_3 \geq -3$,

$y_1 + 3y_2 + 4y_3 \geq 4$,

$y_1 \geq 0$,

$y_2 \geq 0$,

y_3 is unrestrocted.

Combining second and third constraint the final form of the dual problem is:

Minimise $C =$ $8y_1 + 7y_2 + ...$

subject to $y_1 + 2y_2 + ...2.$

$y_1 - 2y_3 - 3$,

$y_1 + 3y_2 + 4 \geq 4$,

$y_1 y_2 \geq 0$ and y_3 is unrestricted.

It can be checked that the dual of the dual is the primal problem. For that, first consider the standard form of the dual problem as follows (replacing y_3 by $y'_3 - y''_3$) :

Minimise $C =$ $8y_1 + 7y_2 + 15y'_3 - 15y''_3 + 0S_1 + 0S_3$,

subject to $y_1 + 2y_2 + y'_3 - y''_3 - S_1 + S_2 = 2$,

$y_1 - y_2 + 2y'_3 - 2y''_3 + 0S_1 + 0S_3 = 3$,

$y_1 + 3y_2 + 4y'_3 - 4y''_3 + 0S_1 - S_3 = 4$,

$y_1, y_2, y'_3, y''_3, S_1$ and $S_2 \geq 0$.

To maintain parity let us assume that the variables related with first, second and third constraints are x_1, x_2 and x_3 rrspectively. Therefore, the dual of this dual problem is :

$$\begin{aligned} \text{Maximise } Z = \; & 2x_1 + 3x_2 + 4x_3, \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 8, \\ & 2x_1 - x_2 + 3x_3 \le 7, \\ & x_1 + 2x_2 + 4x_3 \le 15, \\ & -x_1 - 2x_2 - 4x_3 \le -15, \\ & -x_1 \le 0, \\ & -x_3 \le 0 \text{ and } x_2 \text{ is unrestricted.} \end{aligned}$$

Combining thrid and fourth constraints, we get its final dual form as follows :

$$\begin{aligned} \text{Maximise } Z = \; & 2x_1 + 3x_2 + 4x_3, \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 8, \\ & 2x_1 - x_2 + 3x_3 \le 7, \\ & x_1 + 2x_2 + 4x_3 = 15, \\ & x_1, x_3 \ge 0 \text{ and } x_2 \text{ is unrestricted.} \end{aligned}$$

This is nothing but the primal problem.

2.7.2 Important Theorems on Duality

Apart from the throrem that the dual of the dual is a primal problem, there are also many other important theorems on duality, which are stated bleow (without giving any proof) :

(i) If either the primal or the dual problem has a finite optimum solution, then the other problem has also a finite optimum solution.

(ii) The optimal values of the primal and the dual objective functions are always identical.

(iii) If the primal (dual) has an infeasible solution, it dual (primal) solution will be unbounded.

(iv) If a certain decision variable in a linear programing problem is optimally non-zero, the corresponding dummy variable (slack or surplus) in the counterpart programming problem must be optimally zero. On the other, if a certain dummy variable (slack or surplus) in a linear programming problem is optimally non-zero, the corresponding decision variable in the counterpart programming problem must be optimally zero. Thereofre, at the optimal stage, the product between the dual (primal) decision variable and its related primal (dual) dummy variable is always equal to zero. This theorem is known as complementary slackness theorem.

(v) At the non-optimal stage, the value of the primal objective function is less (greater) than the value of the dual objective function if the primal problem is the problme of maximisation (minimisation).

2.7.3 Primal-Dual Relationship

Already some primal-dual relations are pointed out in temrs of the theorems in the preceding sub-section. Along with those, some other relations can be established with the help of example as follows:

Primal problem		**Dual problem**	
Max. $Z =$	$5x + 6y,$	Min. $C =$	$5u + 12v,$
sub. to	$x + y \le 5,$	sub. to	$u + 2v \ge 5,$
	$2x + 3y \le 12,$		$u + 3v \ge 6,$
	$x,\ y \ge 0$		$u,\ v \ge 0$

[These two problems have alrady been solved in section 2.6]

The final tables of both these problems are as follows :

Final Table of Primal Problem

C_{Bi}	Basis	x	y	S_1	S_2	Solution
5	x	1	0	3	–1	3
6	y	0	1	–2	1	2
	Z_j	5	6	3	1	$Z = 27$
	$C_j - Z_j$	0	0	–3	–1	

Final Table of Dual Problem

C_{Bj}	Basis	u	v	t_1	t_2	A_1	A_2	Solutuion
5	u	1	0	–3	2	3	–2	3
12	v	0	1	1	–1	–1	1	1
	Z_j	5	12	–3	–2	3	2	C = 27
	$C_j - Z_j$	0	0	3	2	M-3	M-2	

[where t_1 and t_2 denote dual surplus variales].

Comparing these two optimal tables it is observed that

(i) $\bar{z} = \overline{C} = 27$ [i.e., optimal solutions of primal and dual objective functions are same].

(ii) $\bar{x}$ = 3 = element of $C_j - Z_j$ row related to t_1 column.

similarly $\bar{y}$ = 2 = element of $C_j - Z_j$ row related to t_2 column. [i.e., the optimum values of the primal variables can be obtained from the net evaluations of the related dual slack or surplus variables (ingnoring sign) or artificial variables (in case of '=' type constraints, putting M = 0 and then ignoring sing)].

(iii) $\bar{u}$ = 3 = magnitude of the element of $C_j - Z_j$ row related to the S_1 column. Similarly, $\bar{v}$ =1 = magnitude of the element of C_j – Z$_j$ row related to the S_2 column.

[i.e., The optimum values of the dual varaibles can similarly be obtained from the net evaluations of the related primal dummy variables ignoring sing and putting M = 0, requried].

(iv) Further it is observed that

$\bar{x} = 3$ and $\bar{t}_1 = 0 \Rightarrow \bar{x} \cdot \bar{t}_1 = 0$

$\bar{y} = 2$ and $\bar{t}_2 = 0 \Rightarrow \bar{y} \cdot \bar{t}_2 = 0$

$\bar{u} = 3$ and $\bar{s}_1 = 0 \Rightarrow \bar{u} \cdot \bar{s}_1 = 0$

$\bar{v} = 1$ and $\bar{s}_2 = 0 \Rightarrow \bar{v} \cdot \bar{s}_2 = 0$

All these establish the complementary slackness theorem. It should be mentioned in this connection that if x and y represent two prducts and primal constraints are the resource constraints then u and v (i.e., the dual variables) represent the shadow prices of the resources (say, labour, capital) and dual objective function denotes the total cost of using resources in the production procesws. In this way one can also given the economic interpretation of duality.

2.8 Summary

Let us conclude the analysis of Linear Programming in the following lines. The LP has wide applications, specially in allocation related descision making problems each of which can be formulated in form of a linear objective function and some linear constraints. If the number of decison variables areonly two, the graphical method can be applied to solve an LP problem, But the simplex method can be applied to solve the LP problem haivng any number of decision variables. In simplex method through limited number of iterations optimal solution of an LP problem is worked out from non-optimal situation, maintaing always the feasibility condition. Apart from its usual nature, the solution of an LP problem may be unbounded, infeasible, multiple (i.e., not unique) and irregular (i.e., basic variable may take zero value which is known as degeneracy). An LP problem can also be transformed into its dual form which has speical economic meaning. Furtherm, as primal and dual solutions are very much related, one can be used to obtain the other.

2.9 Exercise

1. A furniture maker has 6 units of wood and 28 hours of free time, by which he will make decorative screens. Two models are to be produced by the furniture maker. He estimates that model I requires 2 units of wood and 7 hours of time for one unit production, while one unit of model II requres 1 unit of wood and 8 hours of time. The prices of the models are Rs. 120 and Rs. 80 respectively. How many screens of each mnodel should the furnitues maker assemble if the wishes maximize his sales revenue?
2. Solve the problem (1) using graphical method.
3. Solve the following LP problem using simplex method :

$$\text{Max. } Z = x_1 + 9x_2 + x,$$

$$\text{sub to } \quad x_1 + 2x_2 + 3x_3 \le 9,$$

$$3x_1 + 2x_2 + 2x_3 \le 15,$$

$$x_1, x_2, x_3 \ge 0.$$

4. Solve the following LP problem using simplex method :

$$\text{Max. } Z = 5x_1 + 2x_2,$$

$$\text{sub t0} \quad 6x_1 + x_2 \ge 6,$$

$$4x_1 + 3x_2 \ge 12.$$

$$x_1, x_2 \ge 0.$$

5. Solve the following :

$$\text{Minimize } C = 2x_1 + 7x_2,$$

$$\text{subject to} \quad \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \end{matrix} \quad \begin{matrix} 8 \\ 3 \end{matrix},$$

$x1$, $x2$ 0.

6. Write the dual for the following LP problem :

$$\text{Max } Z = 2x_1 + 4x_2$$

$$\text{sub to} \quad x_1 + x_2 \ge 8,$$

$$x_1 - x_2 \le 5$$

$$2x_1 + 3x_2 = 16$$

$$x_1 + 3x_2 \le 14,$$

$$x_1, x_2 \ge 0$$

Show that the dual of the dual is primal. Solve the dual problem and find out the optimal values of the primal decision variables.

7. Give the economic interpretaion of the duality. State the important theorems on duality.

8. Write a short note on the special cases of simplex method.

2.10 References

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2. Paik : Quantitative Method for Managerial Decisions, Tata McGraw Hill Co. Ltd.
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Unit 3 Transportation Problem

Structure

3.0 Objectives

The objectives of this unit are to discuss the following topics which will help to solve many real life problems related to transportation :

Nature of a transportation problem

Transportation Alogirithm for finding initial solution

Transportation Algorithm for obtaining optimum solution

Degeneracy in transportation solution.

3.1 Introduction

Transportation problem refers to the problem of determining the minimum cost for distributing a product from several supply points to several demand points. Practically business and manufacturing units have to decide how many units of a product should be transported from different origins (i.e., factories, plants, warehouses etc.) to different destinations (i.e., markets, sales depots, etc.) so that the total tranportation cost becomes the minimum.

Transportation problem is a special type of linear Programming problem and due to its special character a separate algorithm (known as transportation algorithm) has been developed on the assumption that transportation routes are given. This unit mainly covers the mathematical formulation of a typical tranportation problem and the step to be adopted for obtaining its solution in normal situation and also in the situation of degeneracy.

3.2 Mathematical Formulation of Transportation Problem

A transportation problem is generally expressed in a tabular form, known as transportation tableau. Transportation tableau representes various transportation costs per unit of product transported from various origins to different destinations. The actual form of the transportation tableau is given below :

Transportation Tableau
Destinations

	D_1	D_2	...	D_j	...	D_m	Supplies
O_1	C_{11} X_{11}	C_{12} x_{12}		C_{1j} x_{1j}		C_{1m} X_{1m}	a_1
O_2	C_{21} X_{21}	C_{22} X_{22}		C_{2j} X_{2j}		C_{1m} X_{2m}	a_2
: :							: :
Origins O_i	C_{i1} X_{i1}	C_{i2} X_{i2}		C_{ij} X_{ij}		C_{im} X_{im}	a_i
: :							: :
O_n	C_{n1} X_{n1}	C_{n2} X_{n2}		C_{nj} X_{nj}		C_{nm} X_{nm}	a_n
Demands	b_1	b_2	...	b_j	...	b_m	$\sum_i a_i = \sum_j b_j$

Different symbols used in the above table hae the following meanings :

C_{ij} denotes per unit transportation cost of the product from origin $i(i = 1, 2, ...n)$ to destination $j(j = 1, 2, ... m)$.

O_i denoted ith origin and D_j denotes jth destination.

a_j refers to total supply from ith origin and b_j refers to total demand required for jth destination.

x_{ij} is the decision variable that represents the number of units of the product to be transported from ith origin to jth destination; Xij may be tiehr zero (if no transportation takes place) or positive integer (if transportation takes place).

In the transportation problem it is taken that total supply of the product is equal to the total demand for that product $\left(\text{i.e} \sum_{i=j}^{n} a_i = \sum_{j=i}^{m} b_j\right)$. This condition is both necessary and sufficient for obtaining basic feasible solution. However in a problem total supply of the product is greater (lower) than the total deman for that product, to make them balance a dummy column (row) is to be creted whose demand (supply) will be difference between $\sum_i a_j$ and $\sum_i b_j$ Further, due to this equality of total demand and total supply, in a transportation problem the number of basic cells (i.e., the cells having positive values of X_{ij}s.) is $m + n - 1$ (i.e., total column + total row – 1) and all other cells are non-basic cells (i.e., in these cells, X_{ij} are zero). It is to be noted that a cell represents the combination of a supply point and a demand point.

From this transportation tableau, we can give the mathematical formulation of a transportaion problem as follows :

Minimise $z = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} X_{ij}$,

subject to $\sum_{j=1}^{m} X_{ij} = a_j$, $i = 1, 2, \ldots n$ [Supply Constraints],

$\sum_{i=1}^{n} X_{ij} = b_j$, $j = 1, 2, \ldots m$ [Demand Constraints],

$X_{ij} \geq 0$ and integer

Supply constraints should be of '≤' type and demand constraints should be of '≥' type. But in a transportation problem as the equality between total supply and total demand is always mainteained, the constraints are only of 'equal to' type.

3.3 Transportation Methods for finding Initial Solution

A solution is known as feasible solution if it satisfies the demand and supply conditions (i.e., the rim requirements) and if it contains $(m + n - 1)$ number of basic cells and rest cells as nob-basic cells. To obtain optimum solution we have to start from a initial basic feasible solution and then that solution is to be improved. In this section we discuss three methods for obtaining initial basic feasible solution and the improvement of that initial solution will be explained in the next section.

3.3.1 North-West Corner Method

Under this method for finding initial solution the following steps are to be taken.

Step 1 : Select the Cell which lies at the north-west corner of the transportation tableau and allocate as much as possible in that cell (i.e., the minimum of demand and supply corresponding to that cell). After this allocation if demand is exhausted, cross out the column and subtract the allocated amount from the supply of that row to get residual supply. On the other, after cell allocation if supply is exhausted, cross out the row and compute the residual demand to be met. If the demand and supply corresponding to the north-west corner Cell are equal in amount, then only one of them is crossed out and other's adjusted quantity will be zero (which is to be treated here as like a positive quantity).

Step 2 : Again select the north-west corner cell among the uncrossed cells and allocate the minimum of demand and supply to that cell. Cross out the satisfied column (or row) and adjust the amount of supply (or demand).

Step 3 : Repeat step 2 until we get single uncrossed row or coumn which allocations are to be given on the basis of rim conditions.

This method is very simple, but less efficient in the sense that the initial solution obtained by using this method remains relatively far away from the optimum solution. Let us consider the following example for the use of this method.

Illustration 1.

Suppose a company has factories at four different places (denoted by F_1, F_2, F_3, and F_4) which supply warehouses *A, B, C, D* and *E*. Monthly factory capacities are 40, 30, 20 and 10 respectively. Monthyly warehouse requrements are 30, 30, 15, 20 and 5 respectively. Unit shipping costs (in rupees) are given below. Determine the optimum distribution to minimize total shipping cost.

Warehouses

From \ To		W_1	W_2	W_3	W_4	W_5	Supply
Factories	F_1	7	6	4	5	9	40
	F_2	8	5	6	7	8	30
	F_3	6	8	9	6	5	20
	F_4	5	7	7	8	6	10
Demand		30	30	15	20	5	100

This is a balanced transportation problem as total demand = total supply = 100. So there is no need of inserting dummy row or dummy column (in which cell-costs are all taken as zero) in the transportation tableau. Here we like to determien initial solution of this problem using North-West Corner method and that is shown in Table-1 below.

Table-1

Warehouses

From \ To		W_1	W_2	W_3	W_4	W_5	Supply		
Factories	F_1	7 (30)	6 (10)	4	5	9	~~40~~	~~10~~	II
	F_2	8	5 (20)	6 (10)	7	8	~~30~~	~~10~~	IV
	F_3	6	8	9 (5)	6 (15)	5	~~20~~	~~15~~	VI
	F_4	5	7	7	8 (5)	6 (5)	10		
Demand		30	~~30~~	~~15~~	~~20~~	5	100		
			20	5	5				
		I	II	V					

(F_1, W_1) cell lies in the north-west corner of Table-1 and corresponding to this cell supply is 40 and demand is 30. So allocated amount in this cell is 30. So allocated amount in this cell is 30. Due to this allocation as demand is fulfilled, the first column has been crossed out and remaining supply is 10 (=40 – 30). Next among uncrossed cells, (F_1, W_2) cell lies in the north-west corner where allocated amount is 10 (minimum of remaing supply and demand corresponding to that cell). After this allocation as supply is exhausted, first row of the table has been crossed out and the residual demand is 20. These steps are repeated till we get last one uncrossed row F_4 where allocations are given on the basis of rim requirements. Allocated amounts are encircled and it is observed that the total no. of allocations are 8(= 5 + 4 – 1 = m + n – 1).

3.3.2 Least Cost Method

With the help of least cost method also one can find out the initial basic feasible solution of a transportation problem. For the least cost method the following steps are taken.

Step 1 : Select that cell in the transportation tableau which has least cost and in that cell allocate as much as possible (i.e., minimum of corresponding demand and supply). If there is any tie in selecting least cost, that should be broken arbitrarily.

Step 2 : After this allocation the satisfied row or column is crossed out and other's (column or row's) quantity (i.e., demand or supply) is adjusted accordingly. It is to be noted that if both row and column are exhausted, only one of them is to be crossed out and other's adjusted quantity will be zero (which is to be treated here as positive quantity for further allocation).

Step 3 : From the uncrossed cells, again choose the least cost cell (i.e., the cell having least cost) and in that cell allocate the minimum of demand and supply corresponding to that cell.

Step 4 : All the above mentioned steps are to be repeated until we get single row or single column, where allocations are to be made on the basis of rim requirements.

Illustration 2 :

With the help of same example as mentioned in Illustration 1, we can explain the least cost method. Using the least cost method, the initial solution that we obtain is shown in Table-2 below :

Table-2

Warehouses

To From		W_1	W_2	W_3	W_4	W_5	Supply
	F_1	7 5	6	4 15	5 20	9	~~40~~ ~~25~~ 5
	F_2	8 0	5 30	6	7	8	~~30~~ 0
Factories	F_3	6 15	8	9	6	5 5	~~20~~ 15
	F_4	5 10	7	7	8	6	10
Demand		30	30	15	20	5	100
			II	I	III	IV	

In (F_1, W_3) cell, cost is lowest and so in that cell first allocation is given to the amount of 15. As demand is fulfilled due to this allocation, the corresponding column is crossed out and the remaining supply is 25. Next least cost is 5 which is observed in four cells; among them (F_2, W_2) cell is arbitrarily chosen for allocation to the amount of 30. Due to this allocation, both demand and supply are exhausted. But here column is crossed out and the remaining supply is taken as zero. Similarly in third time, (F_1, W_4) cell is chosen and in that cell maximum possible allocation is 20. So column foru is to be cross out and the adjusted supply corresponding to F_1 is 5. Next (F_3, W_5) cell is selted for allocation and consquently fifth column is crossed out. Lastly in the cells of first column allocations are given on the basis of rim requirements. Taking zero allocation into account in (F_2, W_1) cell, here also the total number of basic cells are 8.

3.3.3 Vogel's Approximation Method (VAM)

It is observed that the VAM is most efficient method for obtaining initial basic feasible solution. This method is discussed step wise as follows :

Step 1 : Calculate penalities of all rows and columns. Penalty of a row (or of a column) is the positive difference between two least costs in that row (or that column). Among the calculated penalities of rows and columns select the highest one (tie is to be broken here arbitrarily) for allocation in the corresponding row /column.

Step 2 : Allocation is made in the least cost cell of the row/column corresponding to which the penalty is highest. Due to this allocation if row condition is satisfied (i.e., supply is exhausted), row is crossed out or if column condition (i.e., demand requirement) is fulfilled, column is crossed out or if both are satisfied any one of them is crossed out and other;s quantity is adjusted accordingly.

Step 3 : For the uncrossed rows and columns again calculate the penalities and follow the above mentioned steps repeatedly until we get either single row or single column, where allocatins are to be determined on the basis of rim conditions.

In this way we shall get the initial solution having ($m + n - 1$) basic cells.

Illustrations 3 :

With the help of the same example of illustration 1 we explain all these steps of the vogel's approximation method and the calculation along with allocations are shown in the following Table-3 :

Table-3
Warehouses

	From \ To	W_1	W_2	W_3	W_4	W_5	Supply	
	F_1	7 (5)	6	4 (15)	5 (30)	9	~~25~~ 5 ~~40~~	✓✓ 1 1 1 1 1 (IV)
	F_2	8 (0)	5 (30)	6	7	8	~~30~~ 0	1 ✓2 1 0 0
Factories	F_3	6 (15)	8	9	6	5 (5)	~~20~~ 15	1 1 1 1 ✓1 (orbitarity chosen)
	F_4	5 (10)	7	7	8	6	10	1 1 1 1 1
Demand		~~30~~ 25	30	15	20	5	100	
Penalties		1 1 1 1 1	1 1 (II)	✓2 (I)	1 1 1 (III)	1 1 1 1 1 (V)		

In the first round, among the calculated penalties, the penalty of W_3 column is highest (i.e.,2) and in that column as (F_1, W_3) cell has the least cost, allocation (to the amount of 15) is given in that cell. Consequently, column three is crossed out and the remaining supply of F_1 is 25. Again the penalties are calculated and in this second round F_2 row having highest penality is chosen for allocation. In (F_2, W_2) cell cost is least and so in that cell allocation is given as much as possible (i.e., 30); crossing out the column (related to W_2) the remaining supply of F_2 is zero. All these steps are takn repeatedly to allocate successively in cells (F_1, W_1) and (F_3, W_5) to the amounts of 5 and 5 respectively. Lastly in the cells of first column allocations are made on the basis of rim conditions. Here also, the number of basic cells are 8(i.e., $m + n - 1 = 5 + 4 - 1$).

To obtain optimum solution one can start from any of these three initial solutions which are obtained by applying three methods (namely, North-West corner method, least cost method and VAM).

3.4 Transportation Algorithm for Obtaining Optimum Solution

After obtaining the initial basic feasible solution of a transportation problem, the optimality of that solution is checked. If the solution is found to be non-optimal, that solution is improved through reallocation (with the help of loop formation) until the optimality of the current solution is reached. All these are discussed here.

3.4.1 Test for Optimality

For checking the optimality of the current basic feasible solution, $u - v$ method (alternatively known as Modi method) is applied as follows :

For the ith row u_i ($i = 1, 2, \ldots n$) and for the jth column v_j ($j = 1, 2, \mathrm{m}$) are to be calculated using the following formula corresponding to only basic cells [say, (i, j) th basic cell] :

$u_j + v_j = c_{ij}$

Here number of unknowns u_is and v_js $= m + n$. But the number of equations

= number of basic cells = $m + n - 1$. That's why, any one v_j or any one u_i is taken arbitrarily as zero and that will not affect the net evaluations (similar to *LPP*) of the non-basic cells.

The net evaluations of the non-basic cells [say, (p, q)th non-basic cell] are calculated using the following formula :

Net evaluation of (p, q)th non-basic cell = d_{pq}

$= C_{pq} - (u_p + v_q)$.

If all the net evaluations of the non-basic cells are zero or positive, the current solution is optimum. Otherwise the current solution is non-optimal and there remains the scope for further improvement of the solution through reallocation.

3.4.2 Dual of the Transportation Model

Rationality behind *'u – v'* or Modi (modified distribution) method can be explained on the basis of (i) dual formulation of a transportation prblem and (ii) the complementary slackness theorem (which has already been discussed in relation of Linear Programming).

Let us consider the mathematical form of the generalised transportation problem (having m destinations and n origins) as follows :

$$\text{Minimize } z = \sum_{i=1}^{n} \sum_{j=1}^{m} C_{ij} X_{ij},$$

subject to $\sum_{j=1}^{n} X_{ij} = a_i$ [i = 1, 2, ... n] : Supply Constraints,

$\sum_{i=1}^{n} X_{ij} = b_j$ [j = 1, 2, ...m] Demand Constraints & X_{ij}s ≥ 0

Let the dual variables be

u_i for the ith supply constraint (i = 1, 2, ... n)

and v_j for the jth demand constraint (j = 1, 2, ... m).

Therefore, the dual formulation of a generalised transpoprtation problem is:

$$\text{Maximize } v = \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j . v_j,$$

Subject to $u_i + v_j \le C_{ij}$ for all i, j and u_j, v_j are all unrestricted.

Now from complementary slackness theroem we know that the optimum value of the primal variable X its corresponding dual slack = 0. For any (i, j) th basic cell as $X_{ij} > 0$, its corresponding dual slack = 0. This implies that its corresponding dual constraint will be of euality type i.e.,

$$u_j + v_j = C_{ij}$$

Again for any (p, q)th non-basic cell, $X_{pq} = 0$. Therefore, according to complementary slackness theorem its corresponding dual slack ≥ 0 i.e.,

$u_p + v_q > C_{pq}$ (when dual slack is negative)

or $u_p + v_q = C_{pq}$ (when dual slack is zero)

or, $u_p + v_q > C_{pq}$ (when dual slack is positive).

Now u_p denotes the contribution (or shadow price or locational rent) of the pth origin and v_q denotes the contribution (or shadow price) of the qth destination.

If imputed total contribution > cost

i.e., $u_p + v_q > C_{pq}$ i.e., $C_{pq} - (u_p + v_q) < 0$

i.e., $d_{pq} < 0$, it is profitable to reallocate in the (p, q) th non-basic cell. Otherwise, no reallocation is profitable. These are the conditions on the basis of which '$u - v$' method has been developed.

3.4.3 Loop in Transportation Table

If the net evaluations of the non-basic cells are not all euqal to zero or positive, the current solution can be improved through reallocation. Reallocation implies one basic cell will turn into a non-basic one and one non-basic cell will turn into a basic one. That non-basic cell will turn into a basic one (i.e., will get positve allocation) whose net-evaluation is negative and highest in magnitude (in case of tie, that is broken arbitrarily).

The leaving basic cell (i.e., the basic ell which will turn into non-basic) is determined on the basis of loop formation. A loop is constructed through horizontal and vertical (not diagonal) lines which passes through the cells of the transportation tableau. Further, loop is formed starting from that non-basic cell which will get new positive allocation and ending to that non-basic cell. In all other corner points of the loop there must be basic cells. It is to be noted that for each non-basic cell, there is only one loop (i.e., loop is unique).

After the formation of the loop, positive sign is given to that corner point where the non-basic cell lies and after that the sign will alternate for all other corner points. Next select that basic cell among all the basic cells which lie on the corner points of the loop, whose allocation is minimum (lie is to be broken arbitrarily, if arises) and allotted sign is negative. This minimum value of allocation of the selected basic cell is to be adjusted for reallocation in the cells (tht lie on the corner points of the loop) according to their allotted signs.

After reallocation, the optimality is checked. If the current solution is found to be non-optimal, all the above-mentioned steps are to be repeated until the optimality is reached.

Illustration 4 : To apply all the steps discussed in this section, the same example of illustration 1 is taken into consideration. Further, its initial basic feasible method is taken for checking the optimality condition (as shown in Table-4 below) :

Table-4

Warehouses

	To / From	W_1	W_2	W_3	W_4	W_5	u_i
	F_1	7 − (30)	6 + (20)	4 [−3]	5 [1]	9 [7]	0(u_1)
	F_2	8 [2]	5 − (20)	6 + (10)	7 [4]	8 [7]	- 1(u_2)
Factories	F_3	6 [−3]	8 0	9 − (5)	6 + (15)	5 1	2(u_3)
	F_4	5 + [−6]	7 [−3]	7 [−4]	8 − (5)	6 5	4 (u_4)
	v_j	7 (v_1)	6 (v_2)	7 (v_3)	4 (v_4)	2 (v_5)	

Arbitrarily taking $u_1 = 0$, all other u_is and v_js are calculated corresponding to each of the basic cells such that $u_i + v_j = C_{ij}$. For instance, corresponding to (F_2, W_2) basic cell u_2 is (–1) and v_2 is 6 suc-that (–1) + 6 = 5 [i.e., $u_2 + v_2 = C_{22}$].

After computing all the $u_j s$ and $v_j s$, next, the net evaluation of non-basic cells are calculated uisng the formula $d_{pq} = C_{pq} - (u_p + v_q)$. For instance, corresponding to the (F_1, W_3) non-basic cell, the net evaluation is –3 [= 4 – (0 + 7) i.e., $d_{13} = c_{13} - (u_1 + v_3)$]. It is observed from Table-4 that the current solution is not optimal as the net evaluations of the non-basic cells are not all non-negative. For the imporvement the current solution of (F_4, W_1) non-basic cell is selected for getting new allocation (ie.e., entering cell) as this cell has negative net evaluation with highest magnitude (i.e., – 6). Next to determine the leaving cell among basic cells, the loop is formed (as shown by the bold circuit in Table-4) and alternatively plus and minus signs are allotted to the cell which lie on the corner points of the loop [starting with plus sign allotted to the non-basic cell (F_4, W_1)]. Next as 5 is the lowest value of the transported amounts of the basic cells in which negative signs are allotted, this amount is adjusted for reallocation on the basis of allotted signs in the cells of the corner points of the loop and the new allocations are shown in Table 5. Due to this new allocation, it is observed that (F_3, W_3) cell becomes the leaving cell and the allocated amount of (F_4, W_4) cell is taken as zero [because at a time only one basic cell can leave the basis].

Table-5
Warehouses

	From \ To	W_1	W_2	W_3	W_4	W_5
	F_1	7 / 25	6 / 15	4 / –3	5 / 1	9 / 7
	F_2	8	5 / 15	6 / 15	7	8
Factories	F_3	6	8	9	6 / 20	5
	F_4	5 / 5	7	7	8 / 0	6 / 5

The whole process is to be repeated to obtain its optimum solution.

3.5 Degeneracy in Transportation Problem

Degeneracy in transportation problem arises when basic cell behaves like a non-basic cell, i.e., when the allocation of the basic cell is zero. This problem may arise either in the initial stage or in the interative stage. In the initial stage of obtaining initial basic feasible solution, the degeneracy problem arises if due to a cell allocation if both total demand and total supply corresponding to that cell are exhausted at a time. Check Table-2 of illustration 2 and Table-3 of illustration 3; in both these one basic cell [(F_2, W_1) cell] gets zero allocation in the initial stage.

On the other, the degeneracy problem arises in the interative stage [as observed in Table-4 and Table-5 of illustration 4] if in the constructed loop tie is observed in selecting the leaving basic cell which should have minimum allocation with negative allotted sign. Howver, if the degeneracy problem arises in a transportation problem then the zero allocation(s) of the basic cell(s) is (are) to be replaced by a very small positive quantity $\in$ such that.

(i) $X_{ij} + \in = X_{ij}$,

(ii) $X_{ij} - \in = X_{ij}$,

(iii) $\in + \in = \in$,

(iv) $\in - \in = 0$.

(v) $0 + \in = \in$ and

(vi) $0 - \in = -\in$.

Replacing each zero allocation by $\in$, the usual transportation algorithm is applied to obtain the optimum solution.

Illustration 5. We have started from Table-5 of illustration 4 and replacing zero allocation of cell (F_4, W_4) by $\in$, Table-6 is constructed below. Next the earlier mentioned steps (namely, test of optimality followed by loop formation and reallocation, if required) are applied sequentially in the following tables (without giving any further explanation) to obtain optimum solution.

Table-6

Warehouses

To / From		W_1	W_2	W_3	W_4	W_5	u_i
	F_1	7 (25) –	6 (15)	4 $\boxed{-3}$	5 + $\boxed{-5}$	9 $\boxed{1}$	2
	F_2	8 $\boxed{2}$	5 (15)	6 (15)	7 $\boxed{-2}$	8 $\boxed{1}$	1
Factories	F_3	6 $\boxed{3}$	8 $\boxed{6}$	9 $\boxed{6}$	6 (20)	5 $\boxed{1}$	–2
	F_4	5 + (5)	7 $\boxed{3}$	7 $\boxed{2}$	8 – (∈)	6 (5)	0
	v_j	5	4	5	8	6	

Table-7

Warehouses

To / From		W_1	W_2	W_3	W_4	W_5	u_i
	F_1	+ – (25)	6 (15)	4 $\boxed{-3}$	5 (∈) +	9 $\boxed{1}$	0
	F_2	8 $\boxed{2}$	5 (15)	6 (15)	7 $\boxed{-3}$	8 $\boxed{1}$	–1
Factories	F_3	6 $\boxed{-2}$	8 $\boxed{-1}$	9 $\boxed{1}$	6 (20) –	5 + $\boxed{-4}$	1
	F_4	5 (5) +	7 $\boxed{3}$	7 $\boxed{2}$	8 5	6 – (5)	–2
	v_j	7	6	7	5	8	

Table-8
Warehouses

From \ To		W_1	W_2	W_3	W_4	W_5	u_i
	F	7 20	6 15 −	4 + $\boxed{-3}$	5 5	9 $\boxed{5}$	0
	F_2	8 $\boxed{2}$	5 15 +	6 − 15	7 $\boxed{3}$	8 $\boxed{5}$	−1
Factories	F_3	6 $\boxed{0}$	8 $\boxed{3}$	9 $\boxed{3}$	6 15	5 5	1
	F_4	5 10	7 $\boxed{3}$	7 $\boxed{2}$	8 $\boxed{5}$	6 $\boxed{4}$	−2
	v_j	7	6	7	5	4	

Table-9
Warehouses

From \ To		W_1	W_2	W_3	W_4	W_5	u_i
	F_1	7 20 −	6 ∈	4 15	5 5 +	9 $\boxed{5}$	0
	F_2	8 $\boxed{2}$	5 30	6 $\boxed{3}$	7 $\boxed{3}$	8 $\boxed{5}$	−1
Factories	F_3	6 + $\boxed{-2}$	8 $\boxed{1}$	9 $\boxed{4}$	6 − 15	5 5	1
	F_4	5 10	7 $\boxed{3}$	7 $\boxed{5}$	8 $\boxed{5}$	6 $\boxed{4}$	−2
	ı	7	6	4	5	4	

Table-10

Warehouses

To / From		W_1	W_2	W_3	W_4	W_5	u_i
	F_1	7 5	6 $\in$	4 15	5 20	9 $\boxed{3}$	0
	F_2	8 $\boxed{2}$	5 30	6 $\boxed{3}$	7 $\boxed{3}$	8 $\boxed{3}$	–1
Factories	F_3	6 15	8 $\boxed{3}$	9 $\boxed{6}$	6 $\boxed{2}$	5 5	–1
	F_4	5 15	7 $\boxed{3}$	7 $\boxed{5}$	8 $\boxed{5}$	6 $\boxed{2}$	–2
	v_j	7	6	4	5	6	

In Table-10 it is observed that all the net evaluations of the non-basic cells are non-negative. So this solution of Table-10 is optimal. The optimum transported quantitites from factories to warehouses are as follows :

Factories		**Warehouses**		**Transported amount**
F_1	$\longrightarrow$	W_1, W_3, W_4	$\longrightarrow$	5, 15, 20
F_2	$\longrightarrow$	W_2	$\longrightarrow$	30
F_3	$\longrightarrow$	W_1, W_5	$\longrightarrow$	15, 5
F_4	$\longrightarrow$	W_1	$\longrightarrow$	10

The total cost of transportation = $5 \times 7 + 15 \times 4 + 5 \times 20 + 5 \times 30 + 6 \times 15 + 5 \times 5 + 10 \times 5$ = Rs. 510. The readers can check that the initial solutions obtained by applying least cost method and VAM are directly optimal. It is observed that if one starts solution using the VAM, generally few iterations will be required to obtain optimum solution.

3.6 Summary

In a typical transportation problem goods are to be transported from different origins to different destinations so that the total cost of transportation is minimised, given (i) the supplies of the origins, (ii) the demands of the destinations and (iii) unit costs of transportation from different origins to different destinations. The transportation problem is a special form of linear programming problem. To solve the transportation problem, a separate algorithm has been developed. In regard to transportation algorithm, first, initial basic feasible solution is found out from the transportation tableau with the help of either noth-west corner rule or least cost method or vogel's approximation method (which is, however, more efficient than others). Then the optimality of the solution is checked using '$u - v$' (i.e., Modi) method. If the solution is not optimal then with the help of loop formation reallocation is made to improve the solution. In this process optimality of the transportation solution is reached. The usual transportation problem and its solution may differ in different directions. Two such special cases, namely unbalanced transportation and degeneracy in transportation problem have been discussed here briefly.

3.7 Exercises

1. What is meant by the transportation problem? Give the matehmatical formulation of the transportation problem.
2. State the basic two theorems of the transportation problem.
3. Briefly discuss the vogel's Approximation Method for finding initial solution.
4. What is meant by the Modi Method? Explain the rationality behind this method with the help of dual formulation of the transportation problem.
5. Briefly discuss the transportation algorithm. How is this algorithm modified (i) if the transportation problem is unbalanced one and (ii) if degeneracy problem arises?

6. Solve the transportation problem for which the unit transportation costs, origin availabilities and destination requirements are given in the following table :

Destinations

		D	**E**	**F**	**G**	**H**	**Supply**
	A	2	11	10	3	7	4
Origins	B	1	4	7	2	1	8
	C	3	9	4	8	12	9
Demand		3	3	4	5	6	

7. Given below is the unit costs array with supplies a_i, $i = 1, 2, 3, 4$ and demands b_j, $j = 1, 2, 3, 4, 5$.

Markets

Warehouses

		M_1	M_2	M_3	M_4	M_5	a_i
	W_1	5	3	4	4	2	50
	W_2	9	2	2	3	6	30
Warehouses	W_3	6	1	6	1	4	40
	W_4	4	7	3	5	3	20
	b_j	40	30	20	50	10	

3.8 References

1. Sharma : Operations Research, Kedar Nath & Ram Nath.
2. Vohra : Quantitative Techniques in Management, Tata McGraw Hill.
3. Kapoor : Operations Research, Sultan Chand & Sons.
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Unit 4 Assignment Problem

Structure

4.0 Objectives

The purpose of this unit is to facilitate knowing the following topics related to the assignment problem :

Nature of a usual assignment problem

Assignment algorithm (known as Hungarian Method)

Some special cases of the usual assignment problem

Travelling salesman problem and its solution.

After knowing all these one will be able to solve many real life assignment related problems.

4.1 Introductions

A usual assignment problem implies the choice of assigning a number of workers to an equal number of jobs so that the total cost of performing the jobs is minimised. The assignment problem is a special form of transportation problem. In the assignment problem (i) total number of rows (workers) = total number of columns (jobs), (ii) supply of each row is 1 unit and demand for each column is also equal to 1 unit and (iii) number of basic variables is equal to the number of column or row. Due to these special features a separate algorithm (known as Hungarian method) has been developed for the solution of an assignment problem. Apart from the usual assignment of workers to the jobs, this type of problem is also observed in case of flight scheduling processing of products, etc. Further, an assignment problem can also be formulated by taking some special cases into account. One such special case is travelling salesman problem. All these issues related to the assignment problem are discussed in this unit.

4.2 Methematical Statement of Assignment Problem

Suppose n jobs are performed by n workers such that (i) one worker can perform only one job and (ii) any worker can perform any job. Further, costs (or units of time) of doing jobs differ for different workers and here the objective is to minimise the total cost (or time) incurred for completing the jobs. Like transportation problem, assignment problem can also be represented with the help of the following table :

Assignment Table

Workers \ Jobs	J_1	J_2 ...		J_j ...		J_n	Supplies
W_1	C_{11} X_{11}	C_{12} X_{12}		C_{ij} X_{1j}		C_{1n} X_{1n}	1
W_1	C_{21} X_{21}	C_{22} X_{22}		C_{2j} X_{2j}		C_{2n} X_{2n}	1
: :							: :
W_1	C_{i1} X_{i1}	C_{12} X_{i2}		C_{ij} X_{ij}		C_{in} X_{in}	1
: :							: :
W_1	C_{n1} X_{n1}	C_{n2} X_{n2}		C_{nj} X_{nj}		C_{nn} X_{nn}	1
Demands	1	1		1		1	n

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Where C_{ij} denotes cost of doing jth job by ith worker and X_{ij} is the variable corresponding to (i, j)th cell which is equal to 1(0) when jth job is performed (not performed) by ith worker; $i, j = 1, 2, \ldots n$.

From this assignment table we can easily represent the assignment problem in the following mathematical form :

$$\text{Minimize } C = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij},$$

Subject to $\sum_{j=1}^{n} X_{ij} = 1,\ i = 1, 2, \ldots n$ [supply constraints]

$\sum_{i=1}^{n} X_{ij} = 1,\ j = 1, 2, \ldots n$ [demand constraints]

and $X_{ij} = 0$ or 1 (as defined earlier).

4.3 Hungarian Method of Solution

For the solution of an assignment problem D. Konig (resident of Hungary) has developed an algorithm which is known as Hungarian method. The steps of this method are pointed out below.

Step 1. : Prepare the table (as mentioned earlier) for the given assignment problem and this table will be of the form of square matrix having equal number of rows and columns. In each cell of this table, cost of assignment is given.

Step 2 : Apply the row operation and then the column operation. The row operation means that the minimum cost of each row is subtracted from all other costs of that row so that in each row at least one zero is observed. Similarly, column operation implies the subtraction of minimum cost of each column from all other costs of that column by which at least one zero is cropped up in each column.

Step 3 : Next draw the minimum number of vertical and horizontal (not diagonal) lines to cross out all the zeros in the assignment table. If the minimum number of these lines is equal to the order (i.e., number or rows) of the matrix, the table is ready for getting optimum solution. Otherwise, go to step 4.

Step 4 : Select the lowerst cost amongst the uncrossed costs on the table (after adopting step 2 and step 3). This lowest cost is to be (i) added with those costs which are in the junction of vertical and horizontal lines and (ii) subtracted from all uncrossed costs; all other costs will remain unchanged.

Step 5 : Repeat step 3 and step 4 until the optimality condition is satisfied. Here optimality condition implies the equality between the number of minimum lines required to cross zero and the order of the matrix. After the fulfillment of the optimality condition, final assignments are to be determined on the basis of following rules which may also be used for drawing the minimum number of lines to cross out all the zeros :

Rule 1 : Check all the rows one by one and select those rows each of which has single zero. Make assignments on the basis of those single zeros and in each time after determining assignment (denoted by tick mark) draw vertical line (to cross out all zeros of the corresponding column).

Rule 2 : After completing the row checking, examine all the columns one by one and select those columns each of which has single zero. Make assignment on the basis of those single zeros and in these cases after making assignments draw horizontal lines (to cross out zeros of the corresonding rows).

Rule 3 : Repeat the above two rules sequentially until all the assignments are determined. However, if all the jobs cannot be assigned on the basis of selecting single zeros, then arbitrarily give assignment to any cell containing uncrossed zero and draw vertical and horizontal lines through theat cell. After this arbitrary choice again follow the above rules.

If rule 3 is required for determining assignment, it signifies the existence of multiple solutions because in that situation there remains the choice of alternative assignment. With the help of an example we can explain the Hungarian method.

Illustration 1.

A firm employes typists on hourly piece-rate basis for their daily work. There are five typists and their charges and speed are different. According to an earlier understanding only one job is given to one typist and the typist is paid for a full hour even if he works for a fraction of an hour. Find the least cost allocation for the following data :

Typist	Rate per hour (Rs)	No. of pages typed/hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of Pages
P	199
Q	175
R	145
S	298
T	178

Solution : Applying step 1 of the Hungarian method the following assignment table (Table-1) is prepared. The elements of Table-1 represent costs to be incurred due to assignment of different jobs to different typists. For instance, to perform job P by employing typist A, cost = $(199 \div 12) \times 5$ = (17 approximately) $\times 5$ = Rs. 85.

Table-1

Typists \ Jobs	P	Q	R	S	T
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Applying the row operation as mentioned in step 2 we get the following reduced cost matrix shown in Table-2.

Table-2

Typists \ Jobs	P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Similarly applying the column operation (step-2), we obtain the reduced cost matrix as shown in Table-3.

Table-3

Typists \ Jobs	P	Q	R	S	T
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Next step 3 is applied in Table-4 where minimum number of lines (horizontal and vertical) are drawn to cross all the zeros.

Table-4

Typists \ Jobs	P	Q	R	S	T
A	2	2	0	4	0√
B	6	4	0√	10	2
C	0√	1	0	1	2
D	2	4	0	4	2
E	2	0√	0	0	2

The zero on the basis of which lines are drawn are identified by tick mark and also sequaence of drawing the lines is marked by Roman numbers. From Table-4 we see that number of lines is 4 whcih is less than the order of the matrix (5). So step 4 is to be applied and that is shown in Table-5.

Table-5

Typists \ Jobs	P	Q	R	S	T
A	3	2	1	4	0
B	6	3	0	9	1
C	0	0	0	0	1
D	2	3	0	3	1
E	3	0	1	0	2

In Table-5 least cost element 1 (of Table-4) is added to the elements which are in the junctions and subtracted from the uncrossed elements (keeping all other dlements in tact). Next to cross all the zeros of Table-5, the minimum number of lines are drawn in Table-6 (just like Table-4).

Table-6

Typists \ Jobs	P	Q	R	S	T
A	3	2	1	4	0√
B	6	3	0√	9	1
C	0√	0	0	0	1
D	2	3	0	3	1
E	3	0√	1	0	2

III IV II I

From Table-6 again we see that the number of lines is less than the order of the matrix. So step-4 is again to be applied selecting 2 as the least cost element. After applying step-4, the adjusted cost matrix is shown in Table-7.

Table-7

Typists \ Jobs	P	Q	R	S	T
A	1	0	1	2	0√
B	4	1	0√	7	1
C	0	0√	2	0	3
D	0√	1	0	1	1
E	3	0	3	0√	4

III II IV I V

In Table-7 also step-3 is again applied i.e., to cross all the zeros, minimum number of lines are drawn. First three lines are drawn on the basis of single remaining zero either in row or in column and line (IV) is drawn arbitrarily by slecting zero of *(C, Q)* cell. Howver, from Table-7 we see that the minimum number of lines = the order of the matrix = 5. So here optimality condition is satisfied. Next on the basis of tick marks (√) in Table-7, we can determine the following optimum assignment (applying step 5) :

Jobs		Typists
T	→	A
R	→	B
Q	→	C
P	→	D
S	→	E

The total minimum cost of assignment = 75 + 66 = 66 + 80 + 112 = Rs. 399.

The learners can verify that with the same level of cost of Rs. 399, an alternative optimum solution for this problem is as follows :

Jobs		Typists
T	→	A
R	→	B
Q	→	C
P	→	D
S	→	E

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4.4 Maximisation Case in Assignment Problem

In a usual assignment problem either total cost or total time to be requred for completing the jobs is minimised. But if the assignment problem is given in terms of productivity or return or profit matrix (in place of cost matrix), the problem will be of maximisation type. For instance, a company wants to employ four salesmen to four different markets for selling its product. One salesman is to be employed for each market. Suppose the efficiency of salesmen differ for different markets and efficiencies of the salesmen measured in terms of their volume of slaes which are estimated *apriori*. So here the company will determine the assignment of salesman in such a way that total olume of sales is maximised.

In a maximising type of assignment problem we can use the same Hungarian emthod which is applied to solve an assignment problem of minimisation type by only transforming the efficiency matrix into an opportunity cost matrix. Transformation can be made with the help of any of the following two ways :

(i) Each element of profit or return (i.e., efficiency) matrix is multiplied by minus one and due to this maximisation problem will be changed to a minimisation problem.

(ii) All the elements of the efficiency matrix are changed by subtracting those from the highest (in value) element of that matrix. With this efficiency matrix will be turned into an opportunity cost matrix.

Illustration 2 :

Suppose the owner of a small machine shop has four machinists available to assign to jobs for the day. Four jobs are offered with the expected profit in rupees for each machinist on each job being as follows. Find the assignment of machinists to jobs that will result in a miximum profit.

		Jobs			
		A	B	C	D
Machinists	1	6.20	7.80	5.00	10.10
	2	7.10	8.40	6.10	7.30
	3	8.70	9.20	11.10	7.10
	4	4.80	6.40	8.70	7.70

Solution : In the given profit matrix the highest element is Rs. 11.10 from which all other unit profits are subtracted and simultaneously replaced to get the following opportunity cost matrix :

		Jobs			
		A	B	C	D
Machinists	1	4.90	3.30	6.10	1.00
	2	4.00	2.70	5.00	3.80
	3	2.40	1.90	0	4.00
	4	6.30	4.70	2.40	3.40

On this transformed opportunity cost matrix, the Hungarian method is to be applied to obtain its optimum solution. The readers can check that the optimum solution to this problem will be as follows :

1 → D, 2 → B, 3 → A, 4→ C; Z_{Max} =Rs. 35.90. One can obtain the same solution if the given maximisation problem is changed to the minimisation problem with the following matrix :

		Jobs			
		A	B	C	D
Machinists	1	–6.20	–7.80	–5.00	–10.10
	2	–7.10	–8.40	–6.10	–7.30
	3	–8.70	–9.20	–11.10	–7.10
	4	–4.80	–6.40	–8.70	–7.70

4.5 Unbalanced Assignment Problem

To obtain optimum solution by Hungarian method the necessary condition is that the assignment problem must be a balanced one i.e., there must be equal number of rows (i.e., workers) and equal number of columns (i.e., jobs) in the assignment matrix. But if the cost matrix of the given assignment problem is not a square matrix having unequal number of rows and columns, the problem is termed as an unbalanced one. In that situation, by incorporating dummy row or dummy column the assignment problem is transformed into a balanced problem and then the Hungarian method can be applied : More specificily, when the number of rows (i.e., workers) are greater than number of columns (i.e., jobs), dummy column is to be added. On the other, if the number of columns (i.e., jobs) are greater than number of rows (i.e., workers), dummy row is to be added. All the cell elements of the dummy row of column are considered as zero. After obtaining final solution with the help of the Hungarian method, the assignments related to dummy rows/columns are to be ignored as those have no real existence. In other words, if there is dummy row, then one job will not be finished. Similarly, if there is dummy column, then one worker will remain idle. With the help of following two examples the insertion of dummy row/ dummy column has explained.

Illustration 3 : Suppose a company has 3 machines (denoted by *X, Y* and *Z*) to perform four jobs (denoted by *A, B, C* and *D*). Each job can be assigned to each machine. The following cost matrix is given :

Jobs

		A	B	C	D
	X	10	7	15	4
Machines	Y	12	6	14	3
	Z	9	8	17	5

This cost matrix is not a square matrix. Hence this is an unbalanced assignment problem and to make it a balance one, dummy row (ie.e, dummy machine) is to be inserted as follows :

Jobs

		A	B	C	D
	X	10	7	15	4
Machines	Y	12	6	14	3
	Z	9	8	17	5
Dummy		0	0	0	0

Illustration 4 : Suppose a company faces the problem of assigning five workers to three different jobs. To perform each of the jobs, any one of the five workers is sufficient. The estimated costs of assignment are given in the table below :

Jobs

		1	2	3
	1	7	6	8
	2	6	5	3
Workers	3	4	6	7
	4	5	3	4
	5	8	9	7

This cost matrix also representes an unbalanced assignment problem and to transform it into the balanced form two dummy columns (i.e., two dummy jobs) are to be created as follows :

Jobs

		1	2	3	Dummy_1	Dummy_2
	1	7	6	8	0	0
	2	6	5	3	0	0
Workers	3	4	6	7	0	0
	4	5	3	4	0	0
	5	8	9	7	0	0

The learners are asked to apply the Hungarian method for obtaining final solutions in both these cases of unbalanced problem [illustrations 3 & 4] and to identfy the job that will remain unfinished in illustration 3 and the two workers who will remain idle in illustration 4.

Illustration 5 : Solve the following assignment problem where the fourth worker can perform two jobs :

Jobs

		J_1	J_2	J_3	J_4	J_5
	W_1	7	6	10	4	9
	W_2	8	5	11	6	12
Workers	W_3	9	4	10	7	8
	W_4	7	6	12	5	14

Apparently this problem books like an unbalanced problem. But this is not so. Here as W_4 can perform two jobs, so W_4 shoujd be effectively treated as two workers. Hence the assignment table will be as follows :

	Jobs					
		J_1	J_2	J_3	J_4	J_5
	W_1	7	6	10	4	9
	W_2	8	5	11	6	12
Workers	W_3	9	4	10	7	8
	W_4^1	7	6	12	5	14
	W_4^2	7	6	12	5	14

Therefore, actually this assignment problem is a balanced one. Similarly, no dummy column may be required in an apparently unbalanced assignment problem, if to perform a job more workers are necessary.

4.6 Restriction on Assignment

In an assignment problem, some cells may be restricted or prohibited in the sense that no assignments are to be allowed corresponding to thos ecells. This situation arises when some workers (or resources) are not suitable or feasible or practically possible to perform some jobs (or activities). In case of restricted assignment, the original assignment costs related to the prohibited cells are to be replaced each by a very high positive quantity *M* (similar to Big *M* of *LPP*). After this change of cost matrix, if the usual assignment algorithm is applied, then in the final solution no assignments would be made corresponding to the prohibited cells.

Illustration 6 : Consider the following cost matrix related to the assignment of five operators to five machines :

	Machines					
		M_1	M_2	M_3	M_4	M_5
	1	6	7	8	9	6
	2	4	5	6	7	4
Operators	3	3	2	1	4	5
	4	7	8	9	10	6
	5	8	9	11	10	3

Suppose operator 2 cannot be assigned to M_3 and operator 5 cannot be assigned to M_4. To satisfy these restrictions, the transformed cost matrix is :

Machines

		M_1	M_2	M_3	M_4	M_5
	1	6	7	8	9	6
	2	4	5	M	7	4
Operators	3	3	2	1	4	5
	4	7	8	9	10	6
	5	8	9	11	M	3

The learner can check that if the usual Hungarian method is applied on this changed cost matrix, the restrictions imposed on teh assignment will be satisfied.

4.7 Travelling Salesman Problem

Travelling salesman Problem is a special type of assignment problem. In this problem it is required that a salesman has to travel *n* number of cirites in round trip (known as a tour).A tour implies that the salesman starting from any city travel each of the cities only once in a continuous trip and then come back to the starting city. The costs or times required for travelling or distance between all pairs of cities are given in this problem and here the objective is to determine that tour (term as optimum tour) which leads to the minimisation of total cost or total time or total distance covred in travellling. This type of problem can also be framed in case of determination of optimum sequence of machine setup.

Actually no definite algorithm is available for the solution of a travelling salesman problem. Different scholars have prescribed different methods which are mostly enumerative and approximate in nature. Here we discuss one such approximate method known as the 'method of second best solution' whose steps are mentioned below :

Step 1 : Solve the problem as an assignment problem using the Hungarian method. If the final assignment solution given us a tour, then that willbe an optimum solution. Otherwise go to step-2.

Step 2 : As assignments on the basis of zero selection do not give a tour, select the smallest non-zero element for further assignment. If there are more than one smallest non-zero elements, each is to be selected one by one and in each case the corresponding row and column are to be deleted. Then on the truncated matrix, the Hungarian method is to be applied to get final solution. If that final solution gives a tour, then that would be optimum solutin. Otherwise, go to step-3.

Step 3 : Select the next minimum element (other than zero and the smallest element of step-2) for assignment and follow the sma procedure as mentioned for smallest non-zero element in step-2. This step is to be repeated (by selecting next to the minimum element for assignment) until a tour is obtained.

With the help of an exmaple, this algorithm has further been explained as follows.

Illustration 7 : Let us consider the solutin of a travelling salesman problem whose related cost matrix is given below :

Table-8

From city \ To city	1	2	3	4	5
1	∞	12	13	14	10
2	10	∞	12	8	12
3	14	13	∞	16	11
4	16	12	14	∞	8
5	13	14	13	10	∞

N.B. :- (i) As travelling from city *i* to city *i* here meaningless, its cost is taken as ∞.

(ii) Costs from city *i* to city *j* and from city *j* to city *i* may not be equal specially in the presence of one way traffic rule.

The learners can check that the final assignment solution to this problem is as presented in Table-9 by the tick mnarks (√).

Table-9

To city / From city	1	2	3	4	5
1	∞	0	0√	4	0
2	0√	∞	1	0	4
3	1	0√	∞	5	0
4	6	2	3	∞	0√
5	1	2	0	0√	∞

The final assignment solution of Table-9 gives us two sub-tours, viz $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ and $4 \rightarrow 5 \rightarrow 4$. To get a complete tour, let us select the smallest non-zero element 1 for further assignment. But in Table-9 we get 1 three times; all of them are selected one-by-one for obtaining an optimum tour.

If the route from city 2 to city 3 is selected for assignment, the turncated cost matrix will be as shown in Table-10.

Table-10

To city / From city	1	2	4	5
1	∞	0√	4	0
3	0√	0	5	0
4	5	2	∞	0√
5	0	2	0√	∞

Assignments in Table-10 on the basis of zero selection (represented by tick marks) along with assignment from city 2 to city 3 do not give any tour; rather we get two sub-tours : $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ and $4 \rightarrow 5 \rightarrow 4$. In this way, Table-11 and Table-12 are constructed giving second best assignment to the route from city 3 to city 1 and from city 5 to city 1 respectively.

Table 11

To city / From city	2	3	4	5
1	0√	0	4	0
2	∞	1	0√	4
4	2	3	∞	0√
5	2	0√	0	∞

Table 12

To city / From city	2	3	4	5
2	0	0√	4	0
2	∞	1	0√	4
3	0√	∞	5	0
4	2	3	∞	0√

It is observed from Table-11 and Table-12 that the solutions are feasible and the respective tours are :

(i) $1 \to 2 \to 4 \to 5 \to 3 \to 1$ and (ii) $1 \to 3 \to 2 \to 4 \to 5 \to 1$.

So this problem has two alternative solutions and the minimum total distance to be travelled by the salesman is 55.

4.8 Summary

Let us sum the discussion of this unit. In a usual assignment problem; workers are assignee to jobs in such a way that one worker performs one job and total cost of assignment is minimised. To solve an assignment problem the Hungarian method is applied. For the use of the Hungarian method, the assignment problem must be a balanced problem having equal number of workers and jobs. But if the assignment problem is an unbalanced on,e dummy row/column is to be added to make it balanced. Further if some cells of an assignment table are prohibited, then before applying the Hungarian method the costs of those cells are to be replaced by M (a very big value). We have also analysed two other special cases of assignment problem : maximisation problem and travelling salesman problem. In case of maximisation problem, the profit matrix is changed into opportunity cost matrix before applying the usual algorithm of assignment problem. In the travelling salesman problem, a salesman has to determine an optimum tour (i.e. a continuous round trip) and that can be solved by using the modified form of assignment algorithm (known as method of second best solution).

4.7 Exercise

Along with the exercises given in the text, the learner should try to solve the following problems to check their knowledge.

1. Describe the difference between a transportation problem and an assignment problem.
2. Give the mathematical formulation of a typical assignment problem.
3. Briefly discuss the Hungarian method for obtaining solution of an assignment problem.
4. Suppose a project consists of four major jobs for which four contractors have submitted tenders. The tender amounts quoted in lakhs of rupees are given in the following matrix. Each contractor has to be assigned one job. Find the optimum assignment and the associated minimum cost of the project.

Jobs

$$\begin{array}{ll c} & & \begin{array}{cccc} a & b & c & d \end{array} \\ & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 10 & 24 & 30 & 15 \\ 16 & 22 & 28 & 12 \\ 12 & 20 & 32 & 10 \\ 9 & 26 & 34 & 16 \end{bmatrix} \end{array}$$

Contractors (rows 1–4)

5. An airline that operates between Kolkata and Mumbai has the time table as shown below. Crews must have a minimum layover of 4 hours between two flights. Obtain the pairing of flights and also the base of the crews that minimizes total layover time.

Time-Table

Kolkata-Mumbai			Mumbai-Kolkata		
Flight No	**Departure**	**Arrival**	**Flight No.**	**Departure**	**Arrival**
1.	06.00 A.M.	08.00 A.M.	101.	07.30. A.M.	09.45 A.M
2.	07.30 A.M.	09.30 A.M.	102.	09.00 A.M.	11.15 A.M.
3.	10.30 A.M.	12.30 P.M.	102.	11.30 A.M.	01.45 P.M.
4.	06.00 P.M.	08.15 P.M.	104.	07.30 P.M.	09.00 P.M.

6. Products 1, 2, 3, 4 and 5 are to be processed on a machine. The set up costs in rupees per change depend on the product presnetly on the machine and the setup to be made and are given by the following data :

   $C_{12} = 12$, $C_{13} = 8$, $C_{14} = 16$, $C_{15} = 7$, $C_{23} = 6$, $C_{24} = 5$, $C_{25} = 8$, $C_{34} = 4$, $C_{35} = 11$, $C_{45} = 18$; $C_{ij} = C_{ij}$ and $C_{ii} = \infty$ [i, j = 1, 2, 3, 4, 5].

Find the optimum sequence of products in order to minimise the total set up cost. [Hints : Apply the travelling salesman algorithm].

4.10 References

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3. Sharma : Operations Research, Kedar Nath & Ram Nath.

Unit 5 Theory of Games

Structure

5.0 Objectives

Competition is a common word in our modern life. In every sector, we have to face competition to meet/reach our objectives/aims. A competitive situation exists if two or more individuals make decisions in a situation that involves conflicting interests and in which the outcome is controlled by the decision of all the concerned parties. Such a competitive situation is called a game. The term game represents a conflict between two or more parties. We are familiar with party games like bridge, cards, pohers, checkers and chess. In games, each player selects and executes a strategy which he believes, will win him the game. Each player applies deductive and inductive logic to develop appropriate strategy to win the game. In view of this, we are interested to know the mathematics of games. This mathematics of games is popularly known as theory of Games.

The Theory of Games is a mathematical theory that deals with the general features of competitive situations. The theory of Games started in the 20th century. The mathematical treatment of games was developed, when John von Newman and Morgenstem published their work "Theory of Games and Economic Behaviour" in 1944. The approach to competitive problems developed by J. Van Newmann (known as father of game theory) wilizes the minimax principle which involve the fundamental idea of minimization of the maximum loss or the maximization of the minimum gain. The game theory is capable of analysing very simple competitive situations, it can not handle all the competitive situation that may arise.

In this unit, we shall discuss the minimax-maximin principle, solution of game with saddle point in case of pure strategy. Also, we shall discuss the solution of game with mixed strategy. The game with mixed strategy can be solved by different methods. Here, we shall discuss only the algebraic and graphical method. To reduce the size of the payoff matrix of a game, dominance and modified dominance properties are discussed.

5.1 Introduction

In many practical problems, it is required to take the decision in a situation where there are two or more opposite parties with conflicting interests and the action of one depends upon the action which is taken by the oponent. Such a situation is termed as competitive situation. A great variety of competitive situation is commonly seen in every day life e.g. in military battles, political campaigns, elections, advertising etc.

A competitive situation is called a game if it has the following properties :

(i) The number of competition (participants), called players is finite.

(ii) There is a conflict in interests between the participants.

(iii) Each of the participant has a finite set of possible courses of action.

(iv) The rules governing these choices are specified and known to all players, a play of the game results when each of the players chooses a single course of action from the list of courses available to him.

(v) The outcome of the game is affected by the choices made by all the players.

(vi) The outcome for all specific set of choices by all of the players is known in advance and numerically defined.

(vii) Every play i.e. combination of courses of action determines an outcome (which may be money or point) which determines a set of payments (+ve, –ve or zero) one to each player.

5.1.1 Finite and Infinite Games

Here we shall use the word move to mean a point in a game at which one of the players picks out an alternative for some set of alternatives.

A game is said to be a finite game if it has a finite number of moves, each of which involves only a finite number of alternatives.

A game which is not a finite game is called an infinite game.

5.1.2 Two-person zero-sum games

When there are two competitors playing a game, it is called a two person game. In a game, if the number of competitors are more than two, say n, the game is referred to as n-person game.

If, in a game, the algebraic sum of the payments to all the competitors is zero for every possible outcome of the game, then the game is said to be non zero sum game.

A game with only two players in which the gains of one player is exactly equal to the losses of another player, called a two person zero sum game. It is also called a rectangular game because their payoff matrix is in the rectangular form.

5.2 Some Basic Terms

Player : The competitors in the game are known as players. A player may be individual or group of individuals or an organisation.

Strategy : A strategy for a player is defined as a set of rules or alternative courses of action available to him/her in advance, by which player decides the course of action that he should adopt, strategy may be of two types :

(i) Pure strategy (ii) Mixed strategy.

Pure strategy : If the players select the same strategy each time, then it is referred to as pure strategy. In this case, each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

Mixed strategy : When the players use a combination of strategies and each player always kept gressing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus, there is a probabilistic situation and objective of the player is to maximize the expected gains or to minimize the expected losses.

Mathematically, a mized strategy to any player is a set S of m non-negative real numbers whose sum is unity. These m non-negative real numbers represent the probabilities in which each course of action (pure strategy) should be selected, m being the number of pure strategies of the player.

Thus if x_i be the probability of choosing the course i, then $S = \{x_1, x_2, ..., x_m\}$ where $x_i \geq 0$, $i = 1, 2, ... , m.$ and $\sum_{i=1}^{m} x_i = 1$

If $x_r = 1$ and $x_i = 0$ for $i \neq r$, then the mixed strategy indicates the rth pure strategy. Thus a pure strategy is a special case of a mixed strategy.

Payoff matrix : Payoff is the outcome of the playing the game. When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called payoff matrix. Since the game is zero sum, therefore the gain of one player is exactly equal to the loss of other and vice-versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff for one of the players.

Let player A has m strategies, say, A_1, A_2, ... , A_m and player B has n strategies, say, B_1, B_2, ... , B_n. Here, it is assumed that each player has his choices from amongst the pure strategies. Also, it is assumed that player A is always the gainer and player B is always the loser. That is, all payoffs are assumed in terms of player A. Let a_{ij} be the payoff which is the gain of player A from player B if player chooses Strategy A_i whereas player B chooses B_j. Then the payoff matrix of player A is

Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \text{ i.e. } [a_{ij}]_{m \times n}$$

The payoff matrix of player B is $[-a_{ij}]_{m\times n}$.

Example 1 : Consider a two person coin tossing game. Each player tosses an unbiased coin simultaneously. Player B pays Rs. 7 to A if the outcomes of both tossing be head and Rs. 4 if both the outcomes be tail, otherwise player A pays Rs. 3 to player B. This two person game is a zero sum game since the winning of one player are the losses for the other. Each player has choices from amongst two pure strategies H and T.

In that case, A's payoff matrix will be

Player B

$$\text{Player } A \quad \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{bmatrix} 7 & -3 \\ -3 & 4 \end{bmatrix} \end{array}$$

5.3 Maximin-Minimax principle or Maximin-minimax criteria of optimality

This principle is used for the selection of optimal strategies by two players.

It states that "if a player lists his worst possible outcomes of all his potential strategies then he will choose that strategy which corresponds to the best of these worst outcomes".

Let the player A's payoff matrix be

Player B

$$
\begin{array}{ll}
 & \begin{array}{cccccc} B_1 & B_2 & \cdots & B_j & \cdots & B_n \end{array} \\
\text{Player A} \quad \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}
\end{array}
$$

If player A takes the strategy A_i then surely he will get atleast $\underset{1\le j\le n.}{m}\, a_{ij}$ $(i=1, 2, \cdots, m)$ for taking any strategy by the oponent player B.

Thus by the maximin-minimax criteria of optimality, the player A will choose that strategy which corresponds to the best of these worst outcomes $\underset{j}{min}\, a_{1j}\ \underset{j}{min}\, a_{2j'} \cdots, \underset{j}{min}\, a_{mj'}$

Thus the maximin value for player A is given by $\underset{i}{\max}\, [\underset{j}{\min}\, a_{ij}]$

Similarly, player B will choose that strategy which corresponds to the best (minimum) of the worst outcomes (maximum losses)

$$\underset{i}{\max}\, a_{i1}, \underset{i}{\max}\, a_{i2}, \cdots, \underset{i}{\max}\, a_{in}$$

Thus the minimax value for player B is given by $\underset{j}{\min}\, [\underset{i}{\max}\, a_{ij}]$

Let $\max_i [\min_j a_{ij}] = a_{pq}$ .... (1)

and $\min_j [\max_i a_{ij}] = a_{rs}$ .... (2)

From (1), it follows that a_{pq} is the minimum element in the p-th row

i.e. $a_{pq} \le a_{ps}$ .... (3)

Where a_{ps} is another element of p-th row. Again, from (2), it follows that a_{rs} is the maximum element in the s-th column.

i.e., $a_{ps} \le a_{rs}$ .... (4)

where a_{ps} is another element of s-th column.

From (3) and (4), we have

$$a_{pq} \le a_{rs}$$

i.e, $\max_i [\min_j a_{ij}] \le \min_j [\max_i a_{ij}]$

i.e., maximin for $A \le$ minimax for B $\max_i [\min_j a_{ij}]$ i.e., maximin for A is called the lower value of the game and is denoted by $\underline{v}$ and $\min_j [\max_i a_{ij}]$ i.e., minimax for B is called the upper value of the game and denoted by $\overline{v}$.

(ii) A game is said to be strictly determinable if maximin value = minimax value = the value of the game $\ne$ 0

i.e. $\underline{v} = v = \overline{v} \ (\ne 0)$.

Example 2 : Solve the game whose payoff matrix is given by

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 3 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Solution : To find out the saddle point, the row minima and column maxima are found out and displayed them in the right side of the corresponding row and in the bottom of the corresponding column respectively.

$$
\begin{array}{llccccc}
 & & & \text{Player B} & & & \\
 & & B_1 & B_2 & B_3 & & \text{Row minima} \\
 & A_1 & 1 & 3 & 3 & & 1 \\
\text{Player A} & A_2 & 0 & -4 & -3 & & -4 \\
 & A_3 & 1 & 5 & -1 & & -1 \\
\text{Column maxima} & & 1 & 5 & 3 & &
\end{array}
$$

Hence maximin value of the payoff matrix i.e., $\underline{v} = \max\{1, -4, -1\} = 1$ and minimax value of the payoff matrix

i.e., $\overline{v} = \min\{1, 5, 3\} = 1$

If v is the value of the game then it will always satisfy the inequality maximin for $A \le v \le$ minimax for B

i.e., $\underline{v} \le v \le \overline{v}$

If for a game $\underline{v} = \overline{v} = a_{lk}$, then the game possesses a solution given by

(i) optimal strategy for player A is the strategy A_l

(ii) optimal strategy for player B is the strategy B_k

(iii) the value of the game is $v = a_{lk}$

Such a game is called with saddle point for the case of pure strategy.

Saddle point : A Saddle point is a position in the payoff matrix here the maximum of row minima coincides with the minimum of column maxima. The cell entry (or payoff) at the saddle position is called the value of the game.

A game for which maximin for A = minimax for B called a game with saddle point. Thus in a game with saddle point the players use pure strategies i.e., they choose the same course of action throughout the game.

Note :-

(i) A game is said to be fair if $\overline{v} = \underline{v} = 0$

$\therefore \quad \underline{v} = 1 = \overline{v}$

Hence the pay off matrix has a saddle point at the position (1, 1). The solution is given by

(i) the optimal strategy for player A is A_1.

(ii) the optimal strategy for player B is B_1.

(iii) the value of the game is 1.

Example 3 : For what value of λ, the game with the following pay off matrix is strictly determinable?

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

Solution : Ignoring the value of λ, we shall the maximin and minimax values of the pay off. For this purpose, we have

$$\begin{array}{cccc} & & \text{Player B} & \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row minima} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} & \begin{array}{c} 2 \\ -7 \\ -2 \end{array} \\ \text{Column maxima} & & \begin{array}{ccc} -1 & 6 & 2 \end{array} & \end{array} \quad \therefore \underline{v} = 2, \overline{v} = -1$$

The game is strictly determinable if $\underline{v} = \overline{v} = v \neq 0$

Hence $-1 \leq \lambda \leq 2 (\lambda \neq 0)$.

5.4 Games with mixed Strategy

In some cases, a game can not be solved with pure strategy i.e., for such a game, no saddle point exists in the case of pure strategy.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

The value of the game obtained by the use of mixed strategies represents the best payoff in which player A can expect to gain and the least in which player B can loose. The expected pay off to a player in a game with arbitrary pay off matrix $A = [a_{ij}]$ of $m \times n$ is defined as

$$E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

$$= p^T A q \text{ (in matrix notation)}$$

where $p = (p_1, p_2, ..., p_m)$ and $q = (q_1, q_2, ..., q_n)$ denotes the mixed strategies for players A and B respectively.

Also $p_1 + p_2 + ... + p_m = 1$ and $q_1 + q_2 + ... + q_n = 1$, $p_i \geq 0$, $a_j \geq 0$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

In particular, if player B takes his pure j-th move or j-th pure strategy then the expected gain of A is given by

$$E_j(p) = \sum_{i=1}^{m} a_{ij}\, p_i,\ j=1, 2, \cdots n$$

Similarly, for particular i-th pure move of player A only, the expected loss of B is given by

$$E_i(q) = \sum_{j=1}^{n} a_{ij} a_j,\ i=1, 2, \cdots m$$

A mixed strategic game or game without saddle point can be solved by different solution methods such as

(i) Algebraic method

(ii) Analytical or calculus method

(iii) Graphical method

(iv) Matrix method

(v) L.P.P. method

(vi) Interactive method.

5.5 Minimax and Saddle point Properties

Let us consider a two person zero sum game with p and q as the set of strategies for player A and player B respectively. If a player A chooses a strategy $p \in p$ and player B chooses a strategy $q \in Q$, then $E(p, q)$ represents the expected payoff to player A and $- E(p, q)$ is the expected pay off to player B.

If p^o be the mixed strategy (announced earlier) of player A, then player B will select his best mixed strategy q^o to minimize the losses so that

$$E(p^o, q^o) = \min_{q \in Q} E(p^o, q^o)$$

where $\min\limits_{q \in Q} E(p^o, q^o) = \max\limits_{p \in q} \min\limits_{q \in Q} E(p, q) = \underline{v}$

where $\underline{v}$ represents the maximin value of the game i.e., by selecting p^o the least that A can expect to gain is $\underline{v}$.

Similarly, if the player B announced in advance a mixed strategy q^o then obviously player A will select his best strategy p^o to maximize gain so that

$$E(p^o, q^o) = \max_{p \in p} E(p^o, q^o) = \min_{q \in Q} \max_{p \in p} E(p, q) = \bar{v}$$

where $\bar{v}$ represents the minimax value of the game i.e., by selecting q^o the most the B can expect to loose is $\bar{v}$. Regarding this, we shall prove the following theorem.

Theorem : Let E(p, q) be such that both $\min\limits_{q \in Q} \max\limits_{p \in p} E(p, q)$ and $\max\limits_{q \in q} \min\limits_{q \in Q} E(p, q)$ exist, then

$$\min_{q \in Q} \max_{p \in P} E(p, q) \geq \max_{p \in P} \min_{q \in Q} E(p, q)$$

Proof : Let p^o and q^o be two arbitrarily chosen mixed strategy for the player A and B respectively.

Then, for every $p \varepsilon p$, we have

$$\max_{p \in P} E(p, q^o) \geq E(p^o, q^o)$$

and for every $q \varepsilon Q$, we have

$$\min_{q \in Q} E(p^o, q) \leq E(p^o, q^o)$$

Hence, from (1 and (2), the inequality

$$\min_{q \in Q} E(p^o, q) \le \max_{p \in P} E(p, q^o)$$

or, $$\max_{p \in P} E(p, q^o) \ge \min_{q \in Q} E(p^o, q)$$

holds for all p and q.

Since q^o is arbitrarily chosen mixed strategy, hence the above inequality holds for all values of q. Hence, if q^o be such a strategy for which $\max\limits_{p} E(p, q)$ has the maximum value, the inequality remains true. Therefore,

$$\max_{p} E(p, q) \ge E(p^o, q)$$

Again, since p^o is any strategy, the above inequality holds even if we select p^o which gives the maximum value of $E(p, q)$.

Therefore, $\max E(p, q) \ge \max\limits_{p} \min\limits_{q} E(p, q)$

This proves the theorem.

5.6 Saddle point of a Function

Let $E(p, q)$ be a function of two variables (vectors) p and q in E^m and E^n respectively. The point (p^o, q^o), $p^o \in E^m$, $q^o \in E^n$ is said to be the saddle point of the function $E(p, q)$ if

$$E(p, q^o) \le E(p^o, q^o) \le E(p^o, q).$$

Now we shall discuss a theorem regarding the existence of the saddle point of a function.

Theorem : Let $E(p, q)$ be a function of two variables $p \in E^m$ and $q \in E^n$, such that $\max\limits_{p} \min\limits_{q} E(p, q)$ and $\min\limits_{q} \max\limits_{p} E(p, q)$ exist. Then the necessary and sufficient condition for the existence of a saddle point (p^o, q^o) of-$E(p, q)$ is that

$$E(p^o, q^o) = \max_{p} \min_{q} E(p, q) = \min_{q} \max_{p} E(p, q)$$

Proof : The condition is necessary i.e., the point (p^o, q^o) is the saddle point of $E(p, q)$.

Hence from the definition of saddle point, we have

$$E(p, q^o) \le E(p^o, q^o) \le E(p^o, q) \qquad \text{......... (1)}$$

for all $p \in E^m$ and $p \in E^n$.

From (1), clearly, $\max\limits_p E(p, q^o) \le E(p^o, q^o)$ holds for all $p \in E^m$

$$\text{Hence, } \min_q \max_p E(p, q) \le E(p^o, q^o) \qquad \text{......... (2)}$$

Similarly, from (1), we have

$$E(p^o, q^o) \le \min_q E(p^o, q)$$

$$\text{i.e., } E(p^o, q^o) \le \max_p \min_q E(p^o, q) \qquad \text{......... (3)}$$

From (2) and (3), we have

$$\min_q \max_p E(p, q) \le \max_p \min_q E(p, q) \qquad \text{......... (4)}$$

But, we know that

$$\min_q \max_p E(p, q) \ge \max_p \min_q E(p, q) \qquad \text{......... (5)}$$

Hence from (4) and (5), we have

$$E(p^o, q^o) = \max_p \min_q E(p, q) \ge \min_q \max_p E(p, q)$$

The condition is sufficient.

Let the point (p^o, q^o) be such that

$$\max_p \min_q E(p, q) = \min_q \max_p E(p, q) \qquad \text{........ (6)}$$

Also, let $\max\limits_p \min\limits_q E(p, q) = \min\limits_q E(p, q)$

and $\min\limits_q \max\limits_p E(p, q) = \max\limits_p E(p, q^0)$

Hence from (6), we have

$$\min_q E(p^0, q) = \max_p E(p, q^o) \qquad \text{....... (7)}$$

Now from the definition of maximum and minimum, we have

$$\min_{q} E(p^{o}, q) \leq E(p^{o}, q^{o}) \qquad \text{......... (8)}$$

and $E(p^{o}, q^{o}) \leq \max_{p} E(p, q^{o})$

From (7) and (8), we have $\max_{p} E(p, q^{o}) \leq E(p^{o}, q^{o})$

which implies

$$E(p, q^{o}) \leq E(p^{o}, q^{o}) \text{ for all } p \in E^{m} \qquad \text{.......... (10)}$$

Again, from (7) and (9), we have

$$E(p^{o}, q^{o}) \leq \min_{q} E(p^{o}, q)$$

which implies

$$E(p^{o}, q^{o}) \leq E(p^{o}, q) \text{ for all } q \in E^{n}$$

Combining (10) and (11), we have

$$E(p, q^{o}) \leq E(p^{o}, q^{o}) \leq E(p^{o}, q)$$

which implies (p^{o}, q^{o}) is a saddle point of $E(p, q)$.

5.7 Maximin-minimax Criterian

Let $p = (p_1, p_2, ..., p_m)$ and $q = (q_1, q_2, ..., q_n)$ be two mixed strategies of two players A and B where $p_1, p_2, ..., p_m$ be the probabilities of which player A will take his strategies $A_1, A_2, ..., A_m$ respectively and $q_1, q_2, ..., q_n$ be the probabilities of which player B will take his strategies $B_1, B_2, ..., B_n$ respectively.

So, $p_i \geq 0$, $q_j \geq 0$ for all $i = 1, 2, ..., m$ and $j = 2, ..., n$

and $\sum_{i=1}^{m} p_i = 1, \ \sum_{j=1}^{n} q_j = 1.$

Let $[a_{ij}]_{m \times n}$ be the pay off matrix of player A.

Then the expected pay off of player A is given by

$$E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

Now, if maximin-minimax criterian for a mixed strategy game is applied to $E(p, q)$, then we have for player A, $\underline{v} = \max_{p} \min_{q} E(p, q)$

$$= \max_{p} \left[\min_{j} \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_{p} \left[\min \left\{ \sum_{i=1}^{m} p_i a_{ij}, \sum_{i=1}^{m} p_i a_{i2}, \ldots\ldots. \sum_{i=1}^{m} p_i a_{in}, \right\} \right]$$

Here $\min_{j} \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the least expected gain to player A if the player B uses his j-th strategy i.e., B_j. For player B,

$\bar{v} = \min_{q} \left[\max_{i} \left\{ \sum_{j=1}^{m} a_{ij} q_j \right\} \right]$ denotes the expected loss to player B if the player **A** uses his i-th strategy i.e., A_i.

Now, in general, however, when p_i and q_j correspond to the optimal strategy then the equality sign holds and in that case i.e., when $\underline{v} = \bar{v}$, then the pair of strategies (p, q) is called a saddle point of $E(p, q)$.

The above result is called maximin-minimax property.

Theorem : For any two person zero sum game when the optimal strategies are not pure (without saddle point) for which the pay off matrix for player A is

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

The optimal mixed strategies (p_1, p_2) of A and (q_1, q_2) of B are determined

$$\frac{p_1}{p_2}=\frac{a_{22}-a_{21}}{a_{11}-a_{12}};\quad \frac{q_1}{q_2}=\frac{a_{22}-a_{12}}{a_{11}-a_{21}}$$

where $p_1 + p_2 = 1$, $q_1 + q_2 = 1$ and the value of game is given by

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}a_{22}-(a_{12}+a_{21})}$$

Proof : Since (p_1, p_2) and (q_1, q_2) are the mixed strategies for players A and B respectively.

Then, $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, $p_1, p_2 \geq 0$ and $q_1, q_2 \geq 0$

Now the expected gain of player A can be calculated as follows :

$E_1(p) = a_{11}p_1 + a_{21}p_2$ when player B uses strategy B_1 and

$E_2(p) = a_{12}p_2 + a_{22}p_2$ when player B uses strategy B_2

Similarly, the expected loss of player B can be computed as follows :

$E_1(q) = a_{11}q_1 + a_{12}q_2$ if player A uses strategy A_1 and

$E_2(q) = a_{21}q_1 + a_{22}q_2$ if player A uses strategy A_2

If v be the value of the game, then player A expects a least possible gain v.

Then, $E_1(p) \geq v$

and $E_2(p) \geq v$

Again, as the maximum possible loss of player B is v, then

$$E_1(q) \leq v \text{ and } E_2(q) \leq v$$

Now, considering the above inequalities as strict equalities, we have

$E_1(p) = v$, $E_2(p) = v$ and $E_1(q) = v$, $E_2(q) = v$

i.e., $a_{11}p_1 + a_{21}p_2 = v$ .... (1)

$a_{12}p_1 + a_{22}p_2 = v$ .... (2)

and $a_{11}q_1 + a_{12}q_2 = v$ .... (3)

$a_{21}q_1 + a_{22}q_2 = v$ .... (4)

From (1) and (2), we have

$$a_{11}p_1 + a_{21}p_2 = a_{12}p_1 + a_{22}p_2$$

or, $(a_{11} - a_{12})p_1 = (a_{22} - a_{21})p_2$

or, $\dfrac{p_1}{p_2} = \dfrac{a_{22} - a_{21}}{a_{11} - a_{12}}$

Now, $p_2 \;\; = \dfrac{p_2}{p_1 + p_2} = [\because \; p_1 + p_2 = 1]$

$$= \frac{1}{\dfrac{p_1}{p_2} + 1} = \frac{1}{\dfrac{a_{22} - a_{21}}{a_{11} - a_{12}} + 1} = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$\therefore \quad p_1 = 1 - p_2 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Then the value of the game is given by

$$v = E_1(p) \;\; = a_{11}p_1 + a_{21}p_2$$

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Again, from (3) and (4), we have

$$a_{11}q_1 + a_{12}q_2 = a_{21}q_1 + a_{22}q_2$$

or, $(a_{11} - a_{21})q_1 = (a_{22} - a_{12})q_2$

or, $\dfrac{q_1}{q_2} = \dfrac{a_{22} - a_{12}}{a_{11} - a_{21}}$

This proves the theorem.

This method (described in the above theorem) for finding the values p_1, p_2, q_1, q_2 and the value of the game is known as the algebraic method. By this method, it is not possible to solve easily any game without sadle point whose pay off matrix is of $m \times n$ order ($m, n > 2$).

5.8 The rules of Dominance

Sometimes, in a rectangular game, it is seen that one or more pure strategy (or strategies) of a player are inferior to at least one of the remaining strategies. In such cases, this inferior strategy is never used. In other words, we can say that this inferior pure strategy is dominated by a superior pure strategy. In such cases of dominance, we can reduce the size of pay off matrix by removing the pure strategies which are dominated by other strategies. Hence the rules of dominance are used to reduce the size of the payoff matrix. This rules are specially used for solving two person zero sum game without saddle point.

Rule 1 : If each element in a row (say *i*-th row i.e., R_i) of a pay off matrix is either less than or equal to the corresponding element in another row, say *j*-th row i.e., R_j, then the i-th strategy is dominated by *j*-th strategy and can be deleted that row (i.e., *i*-th row) from the pay off matrix.

In other words, player *A* will never use that strategy because if player *A* chooses such strategy then he will gain less payoff.

Rule 2 : If each element in a column, say *i*-th column i.e., C_i of a payoff matrix is either greater than or equal to the corresponding element in another column, say *j*-th column i.e., C_j, then the i-th strategy is dominanted by *j*-th strategy and can be deleted that column i.e., *i*-th column from the payoff matrix.

Rule 3 : A strategy *k* can also be dominated if it is inferior to a convex combination of several other strategies. In this case, a row or column corresponding this strategy can be deleted. This domination will be decided as per rules 1 and 2 stated earlier.

Rule 3 is called the modified dominance property. It is pointed out that the element or probability value corresponding to the deleted strategy by the rules of dominance is taken as zero.

Example 4 : Solve the game whose payoff matrix is given by

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} \quad \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \\ -2 & 2 & 0 & -3 \end{bmatrix} \end{array}$$

Solution : First of all, we shall find out the maximin and minimax value in case of pure strategy.

$$\begin{array}{ccc}
 & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} & \text{Row minima} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \\ -2 & 2 & 0 & -3 \end{bmatrix} & \begin{array}{c} -2 \\ 1 \\ -1 \\ -3 \end{array} \\
\text{column maxima} & \begin{array}{cccc} 3 & 3 & 2 & 3 \end{array} &
\end{array}$$

In case of pure strategy, maximin value = 1 and minimax value = 2. As maximin value ≠ minimax value, this game has no saddle point in case of pure strategy. Hence this game can be solved for the mixed strategy. For this purpose, we shall try to reduce the size of the payoff matrix by using the dominance rule.

Since every element of fourth row is less than the corresponding elements of third row, therefore from player *A*'s point of view, fourth strategy i.e., A_4 is dominated by 3rd strategy i.e., A_3 and hence we can delete the fourth row. In this situation, the optimal strategy will not be affected. Now deleting 4-th row, we get the reduced payoff matrix as follows :

$$\begin{array}{cc}
 & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \end{bmatrix}
\end{array}$$

Again, from player *B*'s point of view, 4-th strategy i.e., B_4 is dominated by B_1 as every element of 4-th column is either greater or equal to the corresponding element of 1-st column. Then by deleting the 4-th column, we get the reduced payoff matrix as follows :

$$\begin{array}{cc}
 & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 3 & 3 \end{bmatrix}
\end{array}$$

From the reduced payoff matrix, it is seen that none of the pure strategies of players *A* and *B* is inferior to any of their other strategies. However, the covex combination due to strategies A_2 and A_3 (i.e., the average of 2nd & 3rd row i.e., 1, 2, 2) is superior than the payoff due to the strategy A_1. Thus strategy A_1 may be deleted and we get the reduced payoff matrix as follows :

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_2 \\ A_3 \end{array} & \begin{bmatrix} 3 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix} \end{array}$$

In this reduced matrix, the convex combination due to strategies B_1 and B_2 is superior than the payoff due to strategy B_2. Thus, strategy B_3 may be deleted and we get the reduced 2 × 2 pay off matrix as follows :

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{B}_1 & \mathrm{B}_2 \end{array} \\ \begin{array}{c} \mathrm{A}_2 \\ \mathrm{A}_3 \end{array} & \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \end{array}$$

Now, we have to solve this reduced game whose payoff matrix is of 2×2 order. Clearly, this game has no saddle point and can not be reduced further. Therefore, the optimal strategies will be mixed strategies.

Let the player A chooses his strategies A_2 and A_3 with probabilies p_2 and p_3 and that of B is q_1 and q_2.

$\therefore \quad p_2 + p_3 = 1,\ q_1 + q_2 = 1$ and $p_2, p_3, q_1, q_2 > 0$ and v be the value of the game.

To determine the optimal values of p_1 and p_2, we have

$$3p_2 - p_3 = p_2 + 3p_3 = v$$

which implies $p_2 = \frac{2}{3} \quad \therefore\ p_1 = 1 - p_2 = \frac{1}{3}$

To determine the optimal values of q_1 and q_2, we have

$$3q_1 + q_2 = -q_1 + 3q_2$$

which implies $q_1 = \frac{1}{3} \quad \therefore\ q_2 = \frac{2}{3}$

The value of the game $v = 3p_2 - p_3 = 3.\frac{2}{3} - \frac{1}{3} = \frac{5}{3}$

Hence the solution of the game is as follows :

Optimal strategies are $p^* = \left(0, \frac{2}{3}, \frac{1}{3}, 0\right)$, $q^* = \left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$ and the value of the game is $\frac{5}{3}$.

5.9 Graphical Solution of 2 × *n* or *m* × 2 games

Any rectangular game of order 2 × *n* or *m* × 2 can be reduced to a 2 × 2 game by using graphical method and then the reduced game of order 2 × 2 can be solved by algebraic method.

Let us consider a 2 × *n* game without saddle point. Its payoff matrix is as follows :

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots\cdots & B_n \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \end{bmatrix} \end{array}$$

Let $p = (p_1, p_2)$ and $q = (q_1, q_2, \ldots\ldots, q_n,)$ be the mixed strategies for players A and B respectively. When player B uses his pure strategy B_j, then the expected gain of player A is given by

$E_j(p) = a_{ij}p_1 + a_{2j}p_2 = a_{1j}p_1 + a_{2j}(1 - p_1)$, $j = 1, 2, \ldots\ldots, n(1)$ as $p_1 + p_2 = 1$

Clearly, both p_1 and p_2 must lie in the open interval (0, 1) [because if either p_1 = or $p_2 = 1$, the game reduces to a game of pure strategy which is against our assumption]. Hence $E_j(p)$ is a linear function of either p_1 or p_2. Considering $E_j(p)$ as linear function of p_1 (say), we have

$$\begin{aligned} E_j(p) &= a_{2j} \text{ for } p_1 = 0 \\ &= a_{1j} \text{ for } p_1 = 1 \end{aligned}$$

Hence $E_j(p)$ represents a line segment joining the points $(0, a_{2j})$ and $(1, a_{1j})$.

Now player A expects a least possible gain v. Therefore, $E_j(p) \geq v$ for all j. Now considering the strict equations for inequalities and with the help of graphical method, we shall find out two particular moves or choices of B which will maximize the minimum gain of A.

Let us draw two parallel lines $p_1 = 0$ and $p_1 = 1$ at unit distance apart. Now draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, j = 1, 2, ..., n. The lower envelope (or lower boundary) of these line segments (indicate it by thick line segment) will give the minimum expected gain of A as a function of p_1. Now the highest point of lower envelope will give the maximum of minimum gain of A. The line segments passing through the point corresponding to B's two pure moves, say, B_k and B_l are the critical moves for B which will maximize the minimum expected gain of A. Now the 2 × 2 payoff matrix corresponding to A's moves A_1, A_2 and B's moves B_k, B_l will produce the required result. Thus solving the 2 × 2 game algebraically, we can find the value of the game.

If there are more than two line segments passing through the highest (maximin) point there are ties for the optimum mixed strategies for the player B. Thus any two such lines with opposite sign slopes will determine an alternative optimum for B.

Again, if there are more than one maximin point, alternative optimum exist corresponding to these points.

Example 5 : Solve the following 2 × 4 game graphically :

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

Solution : Clearly, the given problem does not possess any saddle point in case of pure strategy. Hence this problem can be solved with mixed strategy.

Let the player A play with mixed strategy $p = (p_1, p_2)$ where $p_1 + p_2 = 1$ and both p_1 and p_2 lie in the open interval (0, 1).

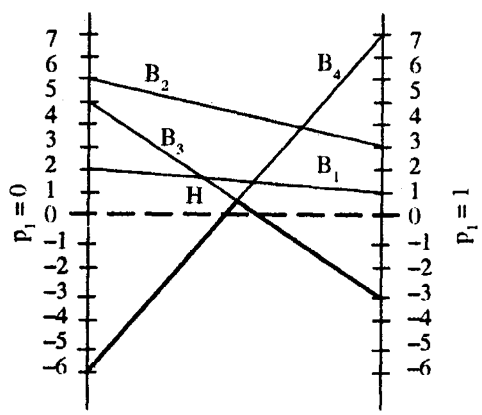

Fig. : 5.1

Then player A's expected gains against B's pure moves are given be

B's pure move	A's expected gain $E(p_1)$
B_1	$E_1(p_1) = p_1 + 2p_2 = p_1 + 2(1 - p_1)$
B_2	$E_2(p_1) = 3p_1 + 5(1 - p_1)$
B_3	$E_3(p_1) = - 3p_1 + 4(1 - p_1)$
B_4	$E_4(p_1) = 7p_1 - 6(1 - p_1)$

draw two vertical lines $p_1 = 0$ and $p_1 = 1$ at unit distance apart. Make the lines $p_1 = 0$ and $p_1 = 1$ by using the same scale as given in the figure. Now draw the line segments for the expected gain equations between two vertical lines $p_1 = 0$ and $p_1 = 1$. These line segments represent A's expected gain due to B's pure move. We denote these line segments as B_1, B_2, B_3, B_4. Since the player A wishes to maximize his minimum expected gain. The highest point of intersection H of two line segments B_3 and B_4 represents the maxinin expected value of the game for player A. Hence the solution to the original game reduces to that of the simpler game with 2×2 payoff matrix given below.

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{bmatrix} -3 & 7 \\ 4 & -6 \end{bmatrix} \end{array}$$

Now, if $p = (p_1, p_2)$ and $q = (q_3, q_4)$ be the optimum strategies for players A and B, then we have

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{-6 - 4}{-3 - 6 - (7 + 4)} = \frac{-10}{-20} = \frac{1}{2}$$

$$\therefore\ p_2 = 1 - p_1 = \frac{1}{2}$$

Again, $q_3 = \dfrac{-6 - 7}{-3 - 6 - (7 + 4)} = \dfrac{13}{20}$, $\quad q_4 = 1 - q_3 = \dfrac{7}{20}$

and the value of the game is given by

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{18 - 28}{-20} = \frac{1}{2}$$

Hence the solution of the game is as follows :

(i) Optimal strategies $p^* = \left(\frac{1}{2}, \frac{1}{2}\right),\ q^* = \left(0, 0, \frac{13}{20}, \frac{7}{20}\right)$

(ii) the value of the game = $\frac{1}{2}$

Example 6 : By graphical method, solve the game whose payoff matrix is given below.

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array}\!\!\begin{array}{c} \text{Player } B \\ \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

Solution : In a similar way, draw the graph. Here three lines pass at the highest point of the lower envelope. Thus, accordingly, we get ${}^3C_2$ i.e. 3 square matrics of order 2 and three optimal solutions. But actually we shall have to select such pair of lines which have the slope opposite in sign. Thus, we get two reduced game with payoff matrix of order 2×2 as follows :

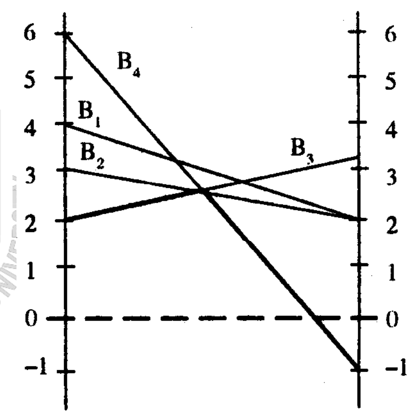

Fig. : 5.2

$$\text{Player A} \quad \begin{array}{c} \text{Player B} \\ \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \end{array} \quad \text{and Player A} \quad \begin{array}{c} \text{Player B} \\ \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} \end{array}$$

Solving these, we get the value of the game as $\frac{5}{2}$ and the optimal strategies as

$p^* = \left(\frac{1}{2}, \frac{1}{2}\right),\quad q^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ for first case and

$p^* = \left(\frac{1}{2}, \frac{1}{2}\right),\quad q^* = \left(0, 0, \frac{7}{8}, \frac{1}{8}\right)$ for 2nd case.

Any $m \times 2$ game problem can be solved by using B's mixed strategy $q = (q_1, q_2)$ where $q_1 + q_2 = 1$ and both lie in the open interval (0, 1), and A's particular critical moves can be determined graphically. In this case, the problem is to determine the minimum of the maximum expected loss of B. Thus we shall have to select the lowest point (minimax) of upper envelope and A's critical moves are the moves line segments corresponding to which pass through the minimax point. Now selecting a 2×2 payoff matrix, the value of the game can be determined easily.

Example 7 : solve the game whose payoff matrix is as follows :

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \text{Player } B \\ \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 2 & -3 \\ -2 & 5 \\ 0 & -1 \end{bmatrix} \end{array}$$

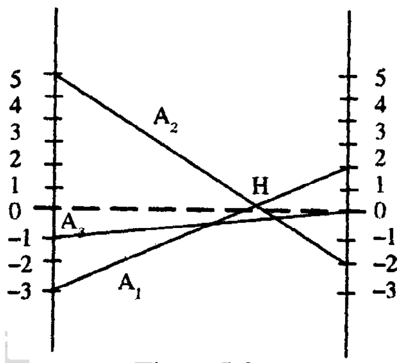

Fig. : 5.3

Here H be the lowest point of the upper envelope. As the point H be the point of intersection of two line segments A_1, A_2, the payoff matrix of the reduced game will be

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix} \end{array}$$

Now solving the reduced game by algebraic method, easily we shall find the optimal strategies and the value of the game.

For this problem, optimal strategies will be $p^* \left(\frac{7}{12}, \frac{5}{12}, 0\right)$, $p^* \left(\frac{2}{3}, \frac{1}{3}\right)$ and the value of the game is $\frac{1}{3}$.

5.10 Summary

In this unit, two person zero sum games and its solution procedure have been discussed. Particularly, the solution procedure of games with pure strategy have been discussed by maximin-minimax criteria whereas Games with mixed strategies by two wellknown methods viz. Algebraic method and graphical method. Also, to reduce the size of the payoff matrix of a game, the dominance and modified dominance properties have been discussed. To illustrate the different methods for solving the games with/without saddle point, some examples have been presented.

5.11 Exercises

1. Consider the game with the following payoff matrix :

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix} \end{array}$$

(a) Show that the game is strictly determinable whatever μ may be

(b) Determine the value of the game.

2. Solve the games whose payoff matrics are given below :

(a) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} -3 & -2 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix} \end{array}$$

(b) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 2 & 3 & 5 \\ -2 & -1 & 4 & -3 \\ 5 & 2 & 3 & 3 \\ 4 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(c) Player B

$$\text{Player } A \quad \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \end{array}$$

3. Solve the following 2 × 2 games using mixed strategies :

(a) Player B

$$\text{Plyer } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 6 & -4 \\ -1 & 2 \end{bmatrix} \end{array}$$

(b) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} \end{array}$$

4. Use graphical method in solving the following games

(a) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 1 & 2 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{bmatrix} \end{array}$$

(b) Player B

$$\text{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

5.12 References

1. P. M. Karak, *Linear Programming and Theory of Games*, New Central Book Agency (P) Ltd.
2. H. A. Taha, *Operations Research, An Introduction*, PHI, India.
3. Swarup, Gupta and Mohan, *Operations Research*, Sultan Chand & Sons, India.
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Unit 6 Project Management PERT and CPM

Structure

6.0 Obectives

Project is a set of activities which are related to each other and to be completed to signal the end of the given project. Project management is different form manufacture, sales and marketing and yet, it involves every one of them.

Setting up a factory is a project, so also building a bridge or developing technology for a new telephone network in a township. This involves activities like scheduling, sequencing and forecasting. This also calls for managerial functions like planning, organising, coordinating, directing and staffing. This also involves transformation of resources into goods and services. Project management is a one time task. It is an organising and structuring concept to obtain optimum utilization of resources employeed. It is the manifestatin of systems concept in management.

In this unit, we shall discuss the main objectives of the project management, with its different terminologies. Hence also we shall present two popular and powerful techniques used in network of a project. These are (i) **PERT (Programme Evaluation and Review Technique)** and (ii) **CPM (Critical Path Method).** In these techniques, only the duration of the activities of a project will be considered. Finally, we shall discuss the time-cost trade off and its algorithm including the cost aspects in the project scheduling.

6.1 Introduction

A project is a well defined set of jobs, tasks or activities, all of which must be competed to finish the project. Construction of a highway, power plant, production and marketing of a new product, research and development work are the examples of project. Such projects involve large number of inter-related activities (or tasks) which must be completed in a specified time, in a specified sequence (or order) and require resources such as personnel, money, materials, facilities and/or space. The main objective before starting any project is to schedule the required activities in an efficient manner so as to

(i) complete it on or before a specified time limit

(ii) minimize the total time

(iii) minimize the time for a prescribed cost

(iv) minimize the cost for a specified time

(v) minimize the total cost

(vi) minimize the idle resources.

Therefore, before starting any project, it is very much essential to prepare a plan for scheduling and controling the vairous activities involved in the project. The techniques of O.R. used for planning, scheduling and controling large and complex projects are very often referred to as network analysis, network planning for network scheduling techniques. IN all these techniques, a project is broken down into various activities whcih are arranged in logical sequence in the form of network. This approach assists managers to visualize a project a s a numnber of tasks which can be easily defined in terms of its duration, cost, starting time. The sequence of activities are also defined. There are two basic planning and control techniques that utilize a network to complete a predetermined project or schedule. There are PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method). PERT network was developed in 19546-58 by a research team of US Navy's Polaris Nuclear Submarine Missile development project. Since 1958, this technique has been used to plan in all most all types of projects. At the same time but independently, CPM was developed jointly by two companies : E. I. Dupont Comnany and Remington Rand Corporation. Other network techniques were PET (Performance Evaluation Programme), LCES (Least Cost Estimating and Scheduling), SCANS (Scheduling and Control by automated Network System).

The work involved in a project can be dividee into three phases corresponding to the management functions of planning, scheduling and control.

Planning : This phase involves setting the objectives of the project and the assumptions to be made. Also it involves the listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials requred for the project in addition to the estimates of costs and duration of the various activities of the project are also determined.

Scheduling : This consists of laying the activities according to the precedence order and determining.

(i) the starting and finishing times for each activity

(ii) the ciritical path on which the activities requre special attention and

(iii) the slack and float for the non ciritcal paths.

Controlling : This phase is exercised after planning and scheduling which involves the following :

(i) Making periodical progress reports

(ii) Revieuing the progress

(iii) Analysing the status of the project

(iv) Management decisios regarding updating, crashing and resource allocation,etc.

6.2 Advantages of Network Analysis

(i) It shows interrelationships of all jobs in the project.

(ii) It gives a clear picture of relationship controlling the order of performance of various activities than a typical bar chart.

(iii) It helps in communication of ideas. The pictorial approach helps to clarify the verbal instructions.

(iv) It provides time schedule containing much more information than other methods like Barcharts etc.

(v) It identifies jobs which are critical for a project completion data.

(vi) It permits an accure forecast of resource requirement.

(vii) It provides a method of resource allocation to meet the limiting condition and to maintain or to minimize the overall costs.

(viii)It integrates all elements of a program to whatever detail is desired by the management.

(ix) It relates time to costs which allows a rupee value to be placed on proposed changes.

6.3 Basic Components

There are two basic components in network. These are

(i) Event/ Node (ii) Activity

Event/Node : A node/event is a particular instant in time showing the end or beginning of one or more activities. It is a point of accomplishment or decision. The starting and end points of an activitiy are thus described by two events usually known as the tail event and head event respectively. An event is generally represented by a circle, rectangle, hexagon or some other geometric shapes. These geometric shapes ae numbered for distinguishing an activity from antoher one. The occurance of an event indicates that the work has been accomplished upto that point.

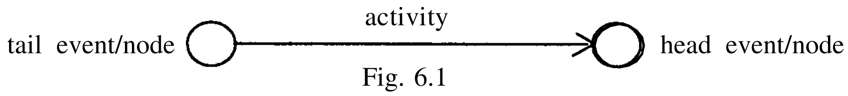


Fig. 6.1

Merge and burst events : It is necessary for an event to be the ending event of only one activity but can be the ending event of two or more activities. Such event is defined as merge event.

If the event happens to be the beginning event of two or more activities it is defined as a burst event.

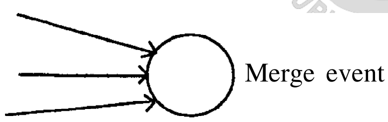


Fig. 6.2(a)

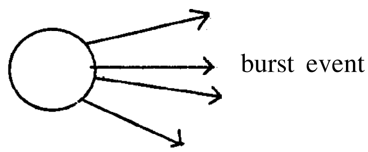
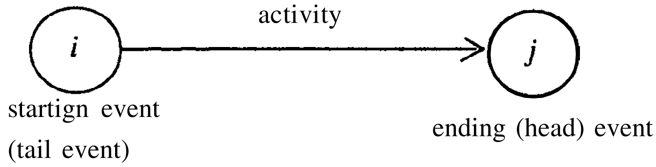


Fig. 6.2(b)

Activity : An activity is a task or item of work to be done that consumes time, effort, money or other resources. Activities are represented by arrows.

Activities are identified by the numbers of their starting (tail) event and ending (head) event. Generally, an ordered pair (i, j) represents an activity where and events i and j represent the starting and ending of the activity respectively. Activities are also denoted by capital alphabets.



The activities can be further classified into different categories;

(i) **Predecessor activity :** An activity which must be completed before one or more other activities stsart is known as predecessor activity.

(ii) **Successor activity :** An activity which started immediately after one or more of other activities are completed is known as successor activity.

(iii) **Dummy activity :** In connecting events by activities showing their inter dependencies, very often a situation arises where a certain event j can not occur until another event i has taken place but, the activity connecting i and j does not involve any time or expenditure of toehr resources. In such a case, the activity is called the dummy activity. It is depicted by dotted line in the network diagram.

Let us consider the example of a car taken to a garage for cleaning. Inside as well as outside of the car is to be cleaned before it is taken away from the garage. The events can be put doen as follows :

Event 1 : Start the car from house

2 : Park the car in garage

3 : Compete outside cleaning

4 : Compete inside cleaning

5 : Take car from garage

6 : Park the car in house

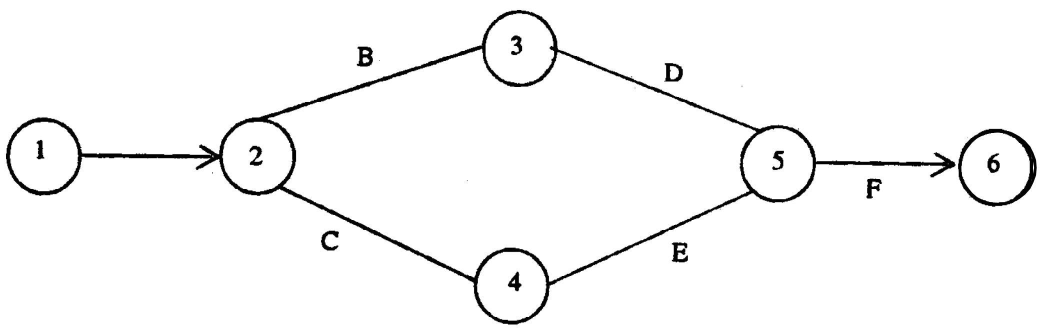


Fig. 6.4

It is assumed that inside cleaning and outside cleaning can be done concurrently by two assistants of the garage. Activities B and C represent these cleaning operations. What do activities D and E stand for? Their time consumpotions are zero but ehey express the condition that events 3 and 4 must occur before the event 5 can take place. Activities D and E are called the dummy activities.

Network : It is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called arrow diagram.

Path : An unbroken chain of activity arrows connecting the initial event to some other event is called a path.

6.4 Common Errors

There are three common errors in a network construction.

Looping (cycling) : In a network diagram looping error is also known as cycling error. Drawing an endless loop in a network is knwon as error of looping. A looping network is given below.

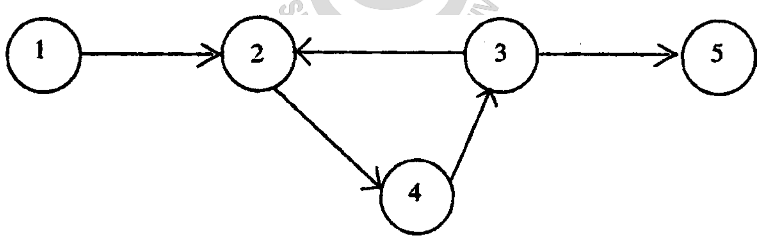


Fig. 6.5

It should be avoided in construction of a network.

Dangling : To disconnect an activity before the completion of all the activities in a network diagram is known as dangling. It should be avoided.

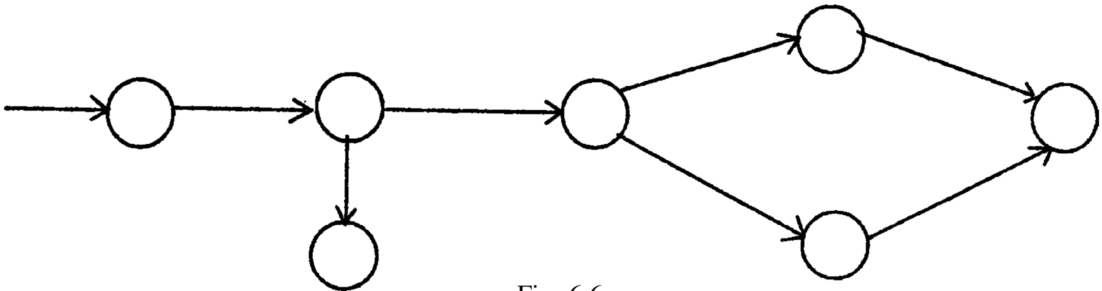


Fig. 6.6

In that case, a dummy activity is introduced in order to maintain the continuity of the system.

Redundancy : If a dummy activity is the only activity emanating from an event and which can be eliminated is known as redundancy.

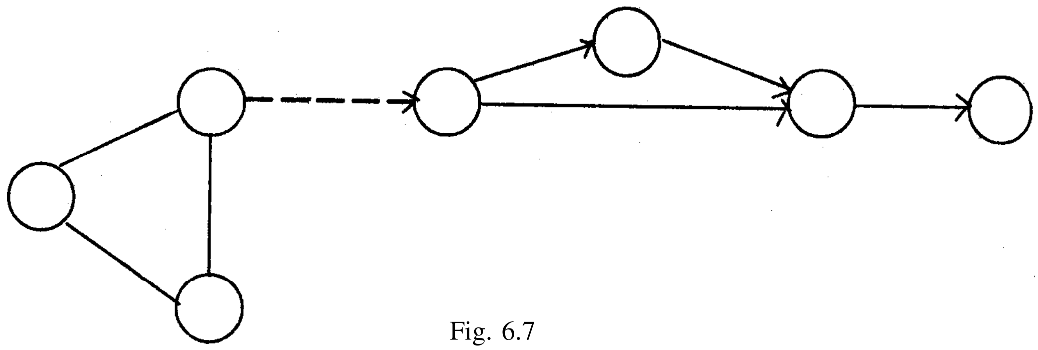


Fig. 6.7

6.5 Rules of Network Construction

For the construction of a network, generally, the following rules are followed :

(i) Each activity is represented by one and only one arrow.

(ii) Crossing an arrow and curved arrows should be avoided, only straight arrows are to be used.

(iii) Each activity must be identified by its starting and ending node.

(iv) No event can occur until every activity preceeding it has been completed.

(v) An event can not occur twice i.e. there must be no loops.

(vi) An activity succeeding an event can not be started until that event has occured.

(vii) Events are numbered to identity an activity uniquely. The number of tail event (starting event) should be lower than that of the head (ending) event of an activity.

(viii) Between any pair of nodes (events), there should be one an only one activity. However, more than one activity may emanate from a node or terminate to a node.

(ix) Dummy activities should be introduced if it extremely necessary.

(x) The network has only one entry point called the starting event and one point of emergence called the end or terminal event.

6.6 Numbering the Events

After the network is drawn in a logical sequence every event is assigned a number. The number sequence must be such so as to reflect the flow of the network. In numbering the events, Fulkerson's (D. R. Fulkerson) rules are used. These rules are as folows :

(i) Event number should be unique.

(ii) Event numbering should be carried out on a sequential basis from left to right.

(iii) An initial event is one which has all outgoing arrows with no incoming arrow. In any network, there will be one such event. Nukmber it 1.

(iv) Delete all arrows emerging from event – 1. This will crete atleast one more initial event.

(v) Number these intial events as 2, 3, ..., etc.

(vi) Delete all emerging arrows from these numbered events which will create new initial events.

(vii) Repeat step-(v) & (vi) until the last event is obtained which has no arrow emerging from it. Number the last event.

Esmaple 1 : Construct a network of each of the projects whose activities and their precedence relationships are given below. Then numebr the events.

Activity	A	B	C	D	E	F	G	H	I
Immediate Predecasser	—	A	A	—	D	B, C, E	F	D	G, H

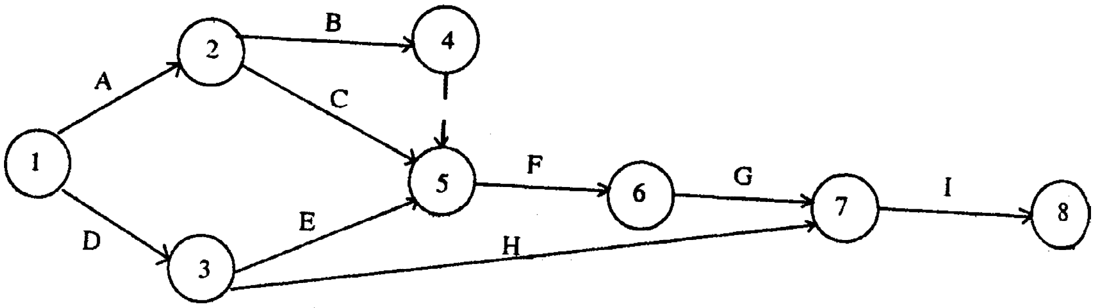


Fig. 6.8

6.7 Critical Path Analysis

Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. The main objective of the time analysis is to prepare a planning schedule of the project. The planning schedule should include the following factors :

(i) Total completion time for the prohject.

(ii) Earliest time when each activity can start.

(iii) Latest time when each activity can be started without delaying the total project.

(iv) Float for each activity i.e. the duration of time by which the completion of an activity can be delayed without delaying the total project completion.

(v) Identification of critical activities and critical path.

Notations : The following notations are used in this analysis.

E_i = Earliest occurance time of event i i.e., it is the earliest time at which the event i can occur without affecting the total project duration.

L_i = Latest allowable occurance time of event i. It is the latest allowable time at which an event can occur without affecting the total project duration.

t_{ij} = Duration of activity (i, j)

ES_{ij} = Earliest starting time of activity (i, j)

LS_{ij} = Latest starting time of activity (i, j)

EF_{ij} = Earliest finishing time of activity (i, j)

LF_{ij} = Latest finishing time of activity (i, j)

The critical path calculations are done in the following two ways :

(a) Forward Pass Calculations method

(b) Backward Pass Calculations method

6.7.1 Forward Pass Calculations Method

In this method, calculations begin from the initial event, proceed through the events in an increasing order of event number and end at the final event of the network. At each node (event), the earliest starting and finishing times are calculated for each activity. The method may be summarized as follows :

Step 1 : Set $E_j = 0,\ i = 1$

Step 2 : Calculate the earliest starting time ES_{ij} for each activity that begins at event i i.e. $ES_{ij} = E_i$ for all activities (i, j) that start at node i.

Step 3 : Calculate the earliest finishing time EF_{ij} of each activity that begins at event i by adding the earliest starting time of the activity with the duration of the activity thus $EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$

Step 4 : Go to next event (node), say event $j(j > i)$ and compute the earliest occurance time for event j. This is the maximum of the earliest finishing times of all activities ending into that event i.e.,

$E_j = \underset{i}{Max}\ EF_{ij} \quad \underset{i}{Max}\ E_i \quad t_{ij}$ for all immediate predecessor activities.

Step 5 : If $i = n$ (final event number), then the earliest finishing time for the project is given by

$En = \underset{i}{Max}\ EF_{ij} \quad \underset{i}{Max}\ E_i \quad t_{ij}$ for all terminal activites.

6.7.2 Backward Pass Calculations Method

In this method, calculations begin from the terminal event, proceed through the events in a decreasing order of event numbers and end at the initial event of the network. All each node (event), the latest starting and finishing times are calculated for each activity. The method may be summarized as follows :

Step 1 : Set $E_n = 0, j = n$

Step 2 : Calculate the latest finishng time LF_{ij} for each activity that ends at event j i.e, $LF_{ij} = L_j$ for all activities (i, j) that end at node j.

Step 3 : Calculate the latest starting time LS_{ij} of each activity that ens at event j by sutracting the duration of each activity from the latest finishing time fo the activity. Thus

$$LS_{ij} = LF_{ij} - t_{ij} = L_j - t_{ij}$$

Step 4 : Proceed backward to the node in the sequence that decrease j by 1. Also compute the latest occurance time of node i $(i < j)$.This is the minimum of the latest starting times of all activities starting from taht event i.e.

$$Lj = \min_j \{LS_{ij}\} \quad \min_j\{L_j \quad t_{ij}\}$$ for all immediate successor activities.

Step 5 : If $i = 1$ (initial node), then

$$L_1 = \min_j \{LS_{ij}\} \quad \min_j\{L_j \quad t_{ij}\}$$ for all initial activities.

6.7.3 Determination of Floats and Slack times

When the network diagram is completely draw, properly labelled, earliest and latest event times are computed, then the next object is to determine the floats of each activity and slack time of each event.

The float of an activity is the amount of time by which it is possible to delay its completion time without affecting the total project completion time. There are three types of activity floats :

(i) Total float, (ii) Free float, (iii) Independent float

Total float : The float of an activity represnets the amount of time by which an activity can be delayed without delay in the project completion time.

Mathematically, the total float of an activity (i, j) is the difference between the latest start time and earliest start time of that activity (or the difference between the earliest finish time and latest finish time). Hence the total float for an activity (i, j) is denoted by TF_{ij} and is computed by the formula.

$$TF_{ij} = LS_{ij} - ES_{ij} \text{ or } TF_{ij} = LF_{ij} - EF_{ij}$$

or $$TF_{ij} = L_j - (E_j + t_{ij})$$

Free float : Sometimes, it may be needed to know how much an activity's completion time may be delayed without causing any delay in its immediate successor activities. This amount of float is called free float. Mathematically, the free flat for an activity (i,j) is denoted by FF_{ij} and is computed by,

$$FF_{ij} = E_j - E_i - t_{ij}$$

As $TF_{ij} = L_j - E_j - t_{ij}$ and $L_j \geq E_j$

$\therefore$ $TF_{ij} \geq E_j - E_j - t_{ij}$ i.e. $TF_{ij} \geq FF_{ij}$

Hence for all activities, free float can take values from zero upto total float but it can nto exceed total float.

Again, free float is very useful for rescheduling the activities with minimum disruption of earlier plans.

Independent float : In some cases, the delay in the completion of an activity neither affects its predecessor nor the successor activities. This amount of delay is called independent float. Mathematically, independent of an activity (i, j) denoted by IF_{ij} is computed by the formula,

$$IF_{ij} = E_i - L_i - t_{ij}$$

The negative independent float is always taken as zero.

Event slack or Event float : The slack of an event is the difference between its latest time and its earliest time. Hence for an event i,

slack $= L_i - E_i$

Critical Event : An event is said to be critical if its slack is zero ie., $L_i = E_i$ for i-th event.

Critical activity : An activity is critical if its total float is zero i.e. $LS_{ij} = ES_{ij}$ or $LF_{ij} = EF_{ij}$ for an activity (i, j).

Otherwise, as activity is called non-critical.

Critical Path : The continuous chain or sequence of critical activities in a network diagram is called the ciritcal path. This path is the longest path in the network from starting event to ending event and is shown by a dark line or double lines to make distinction from other non-critical path.

The length of the critical path is the sum of the individual times of all critical activities lying on it and define the minimum time required to complete the project.

The ciritical path on a network diagram can be identified as

(i) For all activities (i, j) lying on the critical path the E-values and L-values for tail and head events are equal i.e. $E_i = L_j$ & $E_i = L_i$

(ii) On the critical path, $E_j - E_j = L_j - L_i = t_{ij}$

Main features of the critical path

The critical path has two main features :

(i) If the project has to be shortened, then some of the activities on that path must be shortened. The application of additional resources on other activities will not give the desired results unless that critical path is shortened first.

(ii) The variation in actual performance from the expected activity duration time will be completely reflected in one-to-one fashion in the anticipated completion of the whole project.

Example 2 : Determine the critical path, minimum time of completion of the project whose network diagram is shown below. Find also the different floats.

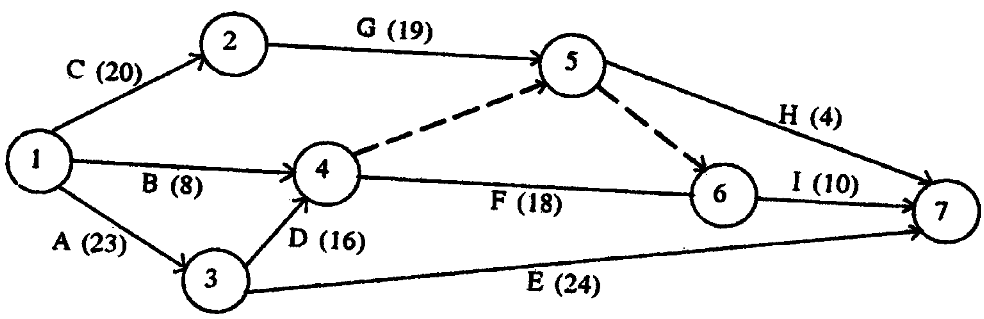


Fig. 6.9

Solution : Forward pass calculations

At node 1 : Set $E_1 = 0$

At node 2 : $E_2 = E_1 + t_{12} = 0 + 20 = 20$

At node 3 : $E_3 = E_1 + t_{13} = 0 + 23 = 23$

At node 4 : $E_4 = \max\limits_{i\ 1,3} \ E_i \quad t_{i4} \quad = \max \{E_1 + t_{14}), E_3 + t_{34}\}$

$= \max \{0 + 8, 23 + 16\} = 39$

At node 5 : $E_5 = \max\limits_{i\ 2,4} \ E_i \quad t_{i5} \quad = \max \{E_2 + t_{25,} E_4 + t_{45}\}$

$= \max \{20 + 19, 39 + 0\} = 39$

At node 6 : $E_6 = \max\limits_{i\ 4,5} \ E_i \quad t_{i6} \quad = \max \{E_4 + t_{46,} E_5 + t_{56}\}$

$= \max \{39 + 18, 39 + 0\} = 57$

At node 7 : $E_7 = \max\limits_{i\ 3,5,6} \ E_i \quad t_{i7} \quad = \max \{E_3 + t_{37,} E_5 + t_{57,} \ E_6 + t_{61}\}$

$= \max \{23 + 24, 39 + 4, 57 + 10\} = 67$

Backward pass calculations

At node 7 : Set $L_7 = E_7 = 67$

At node 6 : $L_6 = L_7 - t_{67} = 67 - 10 = 57$

At node 5 : $L_5 = \min\limits_{j\ 6,7} \ L_j \quad t_{5j} \quad = \min (L_6 - t_{56}, \ t_7 - t_{57})$

$= \min \{57 - 0, 67 - 4\} = 57$

At node 4 : $L_4 = \min\limits_{j\ 5,6} \ L_j \quad t_{4j} \quad = \min \{L_5 - t_{45}, L_6 - t_{46}\}$

$= \min \{57 - 0, 57 - 18\} = 39$

At node 3 : $L_3 = \min\limits_{j=4,7} \left\{L_j - t_{3j}\right\} = \min \{L_4 - t_{34}, L_7 - t_{37}\}$

$= \min \{39 - 16, 67 - 24\} = 23$

At node 2 : $L_2 = L_5 - t_{25} = 57 - 19 = 38$

At node 1 : $L_1 = \min\limits_{j\ 2,3,4} \ E_j \quad t_{1j} \quad = \min \{L_2 - t_{12}, L_3 - t_{13}, L_4 - t_{14}\}$

$= \min \{38 - 20, 23 - 23, 39 - 8\} = 0$

To find the critical activities and different floats, He construct the following table.

Activity	Duration of activity	Earliest time		Latest time		Float			Critical activity
		Start (E_i)	finish (E_i + t_{ij})	start (L_i + t_{ij})	finish (L_i)	Total L_j–E_i – t_{ij}	Free E_j–E_i – t_{ij}	Independent E_{ij} L_i – t_{ij}	
(1, 2)	20	0	20	18	38	18	0	0	
(1,3)	23	0	23	0	23	0	0	0	(1, 3)
(1, 4)	8	0	8	31	39	31	31	31	
(2, 5)	19	20	39	38	57	18	0	0	
(3, 4)	16	23	39	23	39	0	0	0	(3, 4)
(3, 7)	24	23	47	43	67	20	20	20	
(4, 5)	0	39	39	57	57	18	0	0	
(4, 6)	18	39	57	39	57	0	0	0	(4, 6)
(5, 6)	0	39	39	57	57	18	18	0	
(5, 7)	4	39	43	63	67	24	24	16	
(6, 7)	10	57	67	57	67	0	0	0	(6, 7)

From the above table, it is clear that the critical activites (zero total float) are (1, 3), (3, 4), (4, 6) and (6, 7). Hence the critical path is 1 – 3 – 4 – 6 – 7 and duration of the project is 67 time units (as $E_7 = L_7 = 67$).

6.8 PERT Anlysis

Time estimates : It is very difficult to estimate time required for the execution of each activity or job because of various uncertainties. Taking the uncertainties into account, three types of time estimates are generally obtained.

The PERT system is based on these three time estimates of the performance time of an activity.

(i) Optimistic time (t_0) : This is the estimate of the shortest possible time in which an activity can be completed under ideal conditions.

(ii) Pessimistic time (t_p) : This is the maximum time which is required to perform the activity under extremely bad conditions. However such conditions donot include labour strike or acts of nature (like flood, earth quake, tornedo etc.).

(iii) Most likely time (t_m) : This is the estimate of the normal time in which an activity would take. This time estimate lies between the optimistic and pessimistic time estimates. Statistically, it is the modal value of duration of the activity.

The range specified by the optimistic time (t_0) and pessimistic time (t_p) estimates supposedly must encluse every possible estimate of duration of the activity. The most likely time (t_m) estimate may not coincide with the mid point $t_{\text{mid}} = \frac{1}{2}(t_0 + t_p)$ and may occur to its left or right as shown in figure 6.10.

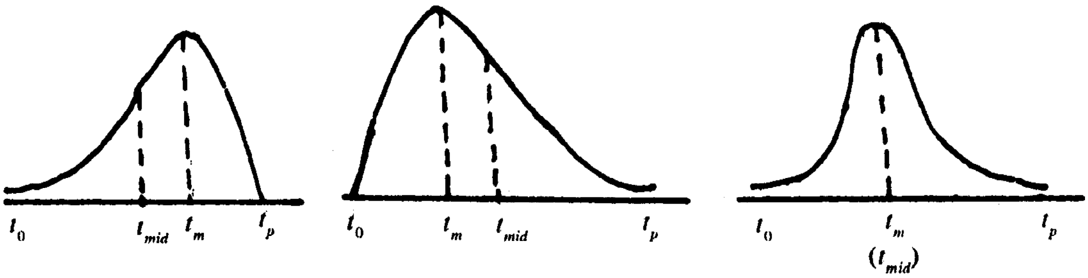


Fig. 6.10

Keeping in view of the above mentioned properties, it may be justified to assume that the duration of each activity may follow Beta (β) distribution with its unimodal point occuring at t_m and its end points at t_0 and t_p.

The expected or mean value of an activity duration can be approximated by a linear combination of three time estimates or by the weighted average of three time estimates t_0, t_p and t_m, i.e. $t_e = (t_0 + 4t_m + t_p)/6$

again, to determine the activity duration variance in PERT, the unimodal property of β-distribution is used. However in PERT, the standard deviation is expressed as

$$\frac{1}{6}\; t_p \quad t_0$$

or, variance $^2 \quad \frac{t_p \quad t_0}{6}^{\,2}$

It is noted that in PERT analysis, Beta distribution is assumed because it is unimodal, has non-negative end points and is approximately symmetric.

6.9 Probability of meeting the schedule time

After identifying the critical path and the occurance time of all activities, there arises a question what is the probability that a particular event will occur on or before the schedule date? This particular event may be any event in the network.

Let us recall that the expected time of an activity is the weighted average of three time estimates t_0, t_p and t_m,

$$\text{i.e.} \quad t_e \quad \frac{1}{6}(t_0 \quad 4t_m \quad t_p)$$

The probability that the activity $(i - j)$ will be completed in time t_e is 0.5 i.e. the chance of completion of that activity is 50%. In the frequecny distribution curve, for the activity $(i - j)$ the vertical line through t_c will devide the area under the curve in two equal parts as shown in the following figure.

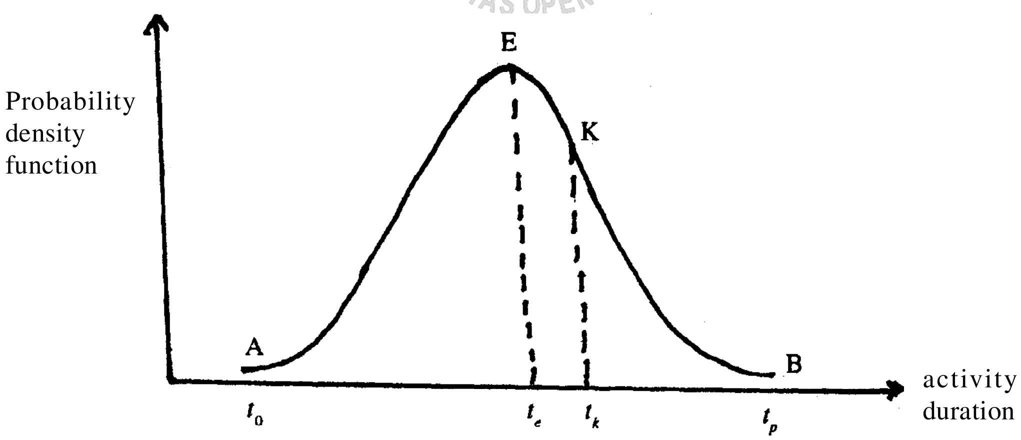


Fig. 6.11

For completing the activity in any other time t_k, the probability will be

$$p = \frac{\text{Area under AEK}}{\text{Area under AEB}}$$

A project consists of a number of activities. All activities as we know are independent random variables and hence the length of the project upto a certain event through a certain path is also a random variable. But the point of difference is the expected project length T_e does not have the same probability distribution as the expected activity time t_e. While a Beta distribution curve approximately represents the activity time probability distribution, the project expected time T_e follows approximately a standard normal distribution. This standard normal distribution curve has an area equal to unity and standard deviation 1 and is symmetrical about the mean as follows :

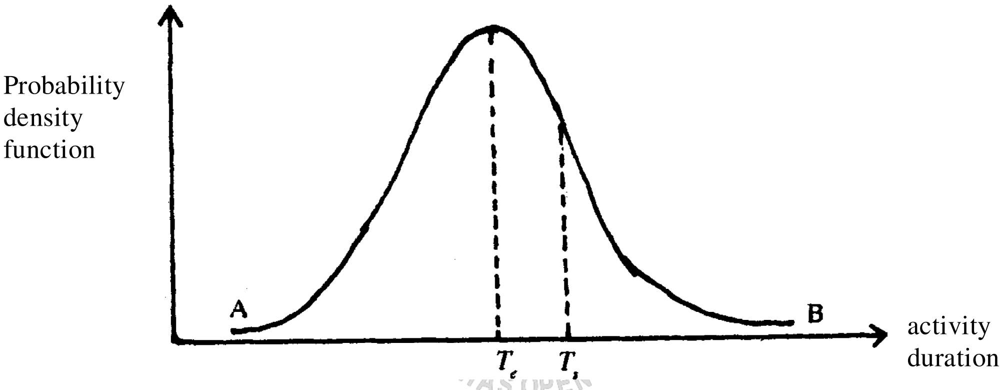


Fig. 6.12

The probility of completing a project in time T_s is given by

$$p(T_s) = \frac{\text{Area under AEK}}{\text{Area under AEB}}$$

The $p(T_s)$ depends upon the location of T_s. Taking T_e as reference point the distance $T_s - T_e$ can be expressed in terms of standard deviation for a network is calculated as

Standard deviation for a network

$$= \ {}_{e} \ \sqrt{\text{sum of the variances along the critical path}}$$

i.e. σ for a network = $\sqrt{\quad {}^{2}_{ij}}$

where ${}^{2}_{ij}$ for an activity $(i - j)$ = $\dfrac{t_p \quad t_0}{\ }^{\,2}$

Since the standard deviation for a standar normal curve is unity, the strandard deviation σ_e, calculated above, is used as scale factor for calculating the normal deviate.

The normal deviation $D = \dfrac{T_s - T_e}{\sigma_e}$

Hence the probability of completing the project by scheduling time (T_s) is given by

$P(Z \leq D)$ where $D = \dfrac{T_s \quad T_e}{e}$ and Z is the standard normal variate.

The values of the probabilities for a normal distribution curve corresponding to the different values of normal deviate are available from the table of standard normal curve.

Example 2 : A small project is composed of seven activities whose time estimates (in weeks) are listed in the following table :

Activity	1-2	1-3	1-4	2-5	3-5	4-6	5-6
t_0	1	1	2	1	2	2	3
t_m	1	4	2	1	5	5	6
t_p	7	7	8	1	14	8	15

(a) Draw the project network.

(b) Find the expected duration and variance of each activity.

(c) Calculate the earliest and latest occurance time for each event and the expected project length.

(d) Calculate the variance the standard deviation of project length.

(e) What is the probability that the project will be completed—

(i) at least 4 weeks earlier than expected?

(ii) Not more than 4 weeks later than expected?

(f) If the project due date is 19 weeks, what is the probability of meeting the due date?

(g) Find also the schedule time on which the project wil be compleed with a probability 0.90.

Solution :

(a)

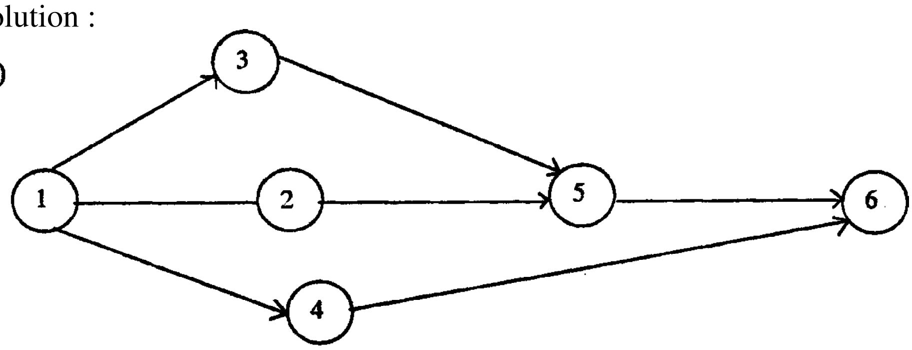


Fig. 6.13

(b) The expected time and variance of each activity is computed and displayed in the following table.

Activity	t_0	t_m	t_p	t_e $\dfrac{t_0 \quad 4t_m \quad t_p}{6}$	^{2} $\left(\dfrac{t_p \quad t_0}{6}\right)^2$
1–2	1	1	7	2	1
1-3	1	4	7	4	1
1-4	2	2	8	3	1
2-5	1	1	1	1	0
3-5	2	5	14	6	4
4-6	2	5	8	5	1
5-6	3	6	15	7	4

(c) Forward Pass calculations

Let E_i be the earliest occurance time of event i.

Set $E_1 = 0$

$E_2 = E_1 + t_2 = 0 + 2 = 2$

$E_3 = E_1 + t_{13} = 0 + 4 = 4$

$E_4 = E_1 + t_{14} = 0 + 3 = 3$

$E_5 = \underset{i \ 2,3}{\max} = \max \{E_2 + t_{25}, E_3 + t_{35}\} = \max \{2 + 1, 4 + 6\} = 10$

$E_6 = \underset{i \ 4,5}{\max} \{E_i + t_{i6}\} = \max \{E_4 + t_{46}, E_5 + t_{56}\} = \max \{3 + 5, 10 + 7\} = 17$

Backward pass calculations

Let L_j be the latest occurance time of event j

Set $L_6 = E_6 = 17$

$L_5 = L_6 - t_{56} = 17 - 7 = 10$

$L_4 = L_6 - t_{46} = 17 - 5 = 12$

$L_3 = L_5 - t_{35} = 10 - 6 = 4$

$L_2 = L_5 - t_{25} = 10 - 1 = 9$

$$L_1 = \underset{j\ \ 2,3,4}{Min} \quad L_j \quad t_{1j} \;=\; \min\,\{L_2 - t_{12,}\ L_3 - t_{13,}\ L_4 - t_{14}\}$$

$$= \min\,\{9 - 2,\ 4 - 4,\ 12 - 3\} = 0$$

From the above calculations, it is seen that

$E_1 = L_1,\ F_3 = L_3,\ E_5 = L_5,\ E_6 = L_6$

Hence the critical events are 1, 3, 5, 6 and the critical path is 1 –3 – 5 –6.

Also, the expected project length = $E_6 = L_6 = 17$ weeks

(d) Variance of the project length is given by

$^2_e = 1 + 4 + 4 = 9$ or, $\sigma_e = 3$

(e) The standard normal deviate is given by

$$D = \frac{\text{schedual time- expected time of completion}}{\sqrt{\text{variance or standard deviation}}}$$

(i) Now, the probability that the project will be completed at least 4 weeks earlier than expected is given by

$P(Z \leq D)$ where $D = \dfrac{17\text{-}4 \quad 17}{3} \quad \dfrac{4}{3} - 1.33$ (approx.)

$= P\,(Z \leq -1.33)$

$= 0.5 - P\,(0 < Z \leq 1.33)$

$= 0.5 - \phi\,(1.33)$ [From the table of area under standard nermal curvel]

$= 0.5 - 0.4082$

$= 0.0918$

(ii) Again, the probability that the project will be completed not more than 4 weeks later than expected = the probability that the project will be completed within 17 + 4 = 21 weeks

$$= P\ (Z \leq \ D) \text{ where } D = \frac{21 \quad 17}{3} \quad \frac{4}{3} \quad 1.33$$

$= P\ (Z \leq 1.33)$

$= 0.5 + \phi\ (1.33)$

$= 0.5 + 0.4082$

$= 0.9082$

(f) When the due date is 19 weeks, $D = \frac{19 \quad 17}{3} \quad \frac{2}{3} \quad 0.67$

Then the probability of meeting the due dat is given by

$P\ (Z \leq 0.67) \quad = 0.5 + \phi\ (0.67)$

$\qquad\qquad\qquad = 0.5 + 0.2514$

$\qquad\qquad\qquad = 0.7514$

(g) Since the probability for the completion of the project is 0.90, $P\ (Z \leq D)$ $= 0.90$ where $D = \frac{T_s \quad T_e}{}$ & T_s be the schedule time As $P(Z \leq 1.29) = 0.90$

$\therefore\ D = 1.29$ which implies $\frac{T_s \quad 17}{3} \quad 1.29$

i.e. $T_s = 17 + 3 \times 1.29$

i.e. $T_s = 20.87$

6.10 Difference between PERT and CPM

PERT	CPM
1. This technique was developed in connection with R & D (Research & Development) works, therefore it had to scope with the uncertainty which ae associated with R & D activities. In this case, total project duration is regarded as a random variabel. As a result, multiple time estimates are made to calculate the probability of completing the project within the schedule time. Therefore, it is a probabilistic model.	1. This technique was developed in connection with a construction and maintenance project which consists of routine tasks or jobs whose resource requirement and duration is knwon with certainty. Therefore, it is basically a determinnistic model.
2. It is used for projects involving activities of non-repetitive nature,	2. It is used for projects involving activities or repetitive nature.
3. It is event oriented technuque because the results of analysis are expressed in terms of events or distinct points in time indicative of progress.	3. It is actively oriented technique as the results of calculations are considered in terms of activities.
4. It incorporates satistical analysis thereby enables the determination of probabilities cocerning the time by which each activity and the entire project would be completed.	4. It does not incorporate statistical analysis in determining the estimates because time is precise and known.
5. It serves a useful control device as it assists the management in controlling a project by calling attention through constant review to such delays in activities which might lead to a delay in the project completion date.	5. It is difficult to use this technique as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time the changes ae introduced into the network.

6.11 Project Time-cost trade off

In this section, the cost of resources consumed by activities are taken into consideration. The project completion time can be reduced (crashing) the normal completion time of ciritical activities. The reduction in normal time of completion will increase the total budget of the project. However, the decision-maker will always look for trade-off between total cost of the project and total time required to complete it.

Project Cost : In order to include the cost aspects in project scheduling we have to find out the cost duration relationships for various activities in the project. The total cost of any project comprises direct and indirect costs.

Direct Cost : This cost is directly dependent upon the amount of resources in the execution of individual activities such as manpower loading, material consumed etc. The direct cost increases if the activity duration is to be reduced.

Indirect Cost : This cost is associated with expenditure which can not be allocated to individual activities of the project. This cost may include managerial services, loss of revenue, fixed overheads etc. The indirect cost is computed on a per day, per week or per month basis. This cost decreaes if the activity duration is to be reduced.

The network diagram can be used to identify the activities whose duration should be shortened so that the completion time of the project can be shortened in the most economic manner. The process or reducing the activity duration by putting on extra effort is called crashing the activity.

The crash time (T_c) represents the minimum activity duration time that is possible and any attempts to further crash would only raise the activity cost without reducing the time. The activity cost corresponding to the crash time is called the crash cost (C_c) which is the minimum direct cost requred to achieve the crash Performance time.

The normal cost (C_n) is equal to the absolute minimum of the direct cost required to perform an activity. The corresponding time duration taken by an activity is known as the normal time (T_n).

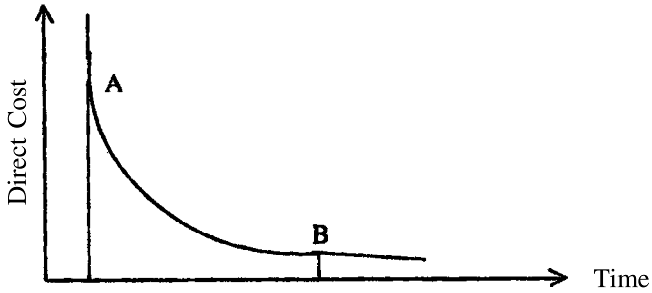


Fig. 6.14

The direct cost curve (from the relationship of direct cost and tiem) are shown in the figure. The point *B* denotes the normal time for completion of an activity whereas the point *A* denotes the crash time whch indicates the least duration in which activity can be completed. The cost curve is non-linear and asymptotic nature. But, for the sake of simplicity, it can be approximated by a straight line whose slope (in magnitude) is given by

$$\text{cost slope} = \frac{\text{Crash cost - Normal cost}}{\text{Normal time - Crash time}} = \frac{C_c - C_n}{T_n - T_c}$$

It is also called as crash cost slope or crash cost per unit time. This cost slope represents the rate of increase in the cost of performing the activity per unit reduction in time and is called cost/time trade off. It varies from activity to activity. After assessing the direct and indirect project costs, this total project cost which is the sum of direct and indirect cost can be found out.

Time-cost optimization algorithm/Time-cost trade off procedure

The following are the steps involved in the project crashing.

Step 1. Consedering normal times of all activities, identify the critical activities and find the critical path.

Step 2. Calculate the cost slope for different activities and rank the activities in the ascending order of cost slope.

Step 3. Crash the activities on the critical path as per ranking i.e. activity having lower cost slope would be crashed first to the maximum ectent possible (For the crashing of lower cost slope i.e. for the reduction of activity duration time, the direct cost of the project would increased very slowly.)

Step 4. Due to the reduction of ciritical path durtion by crashing in step 3, other path may also become critical i.e. we get parallel critical paths. In such cases, the project duration can be reduced by crashing of activities simultaneously in the parallel critical paths.

Step 5. Repeat the process until all the ciritical activities are fully crashed or no further crashing is possible.

In the case of indirect cost, the process of crashing is repeated until the total cost is minimum beyond which it may increase. The minimum cost is called the optimum project cost and the corresponding time, the optimum project time.

example 3.

The following table shows activities, their nomal time and cost and crash time and cost for a project.

Activity	Normal time (days)	Cost (Rs)	Crash time (days)	Cost (Rs.)
1–2	6	1400	4	1900
1-3	8	2000	5	2800
2-3	4	1100	2	1500
2-4	3	800	2	1400
3-4	Dummy	—	—	—
3–5	6	900	3	1600
4–6	10	2500	6	3500
5–6	3	500	2	800

Indirect cost for the project is Rs. 300 per day.

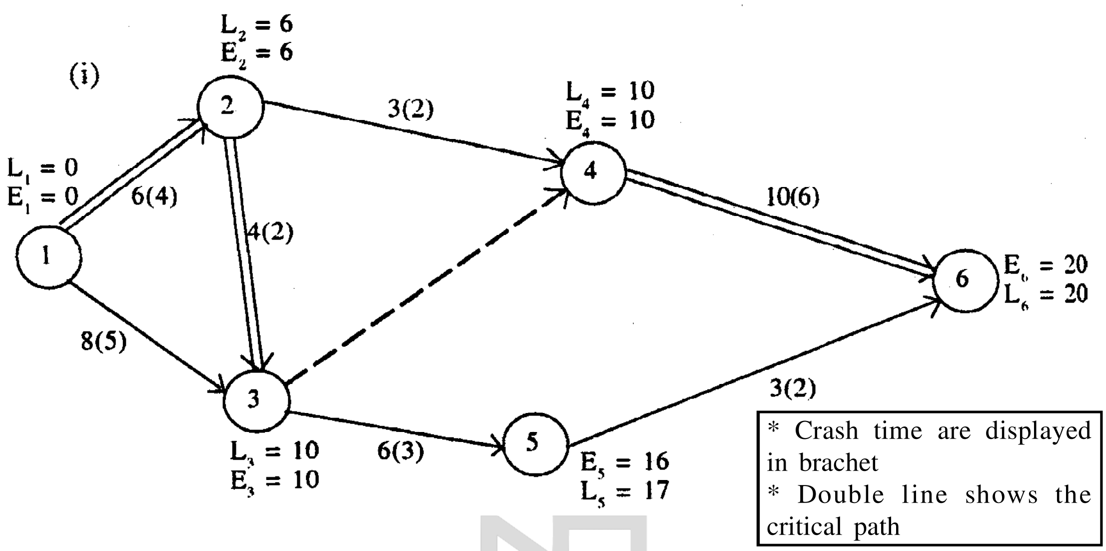
(i) Draw the network of the project.

(ii) What are the normal duration and associated cost of the project?

(iii) What will be the least project duration and corresponding cost?

(iv) Find the optimum duration and minimum project cost.

(i)



(ii) Using the normal time duration of each activity, the earliest and latest occurence time at various nodes are computed and displayed in Fig. 6.15 of the network.

From the network, it is seen that *L*-values and *E*-values at nodes 1, 2, 3, 4, 6 are sam,e. This means that the critical path is 1–2–3–4–6 and the normal duration of the project is 20 days. The associated cost of the project

= Direct normal cost + indirect cost for 20 days

= Rs. [(1400 + 2000 + 1100 + 800 + 900 + 2500 + 500) + 20 × 300]

= Rs. [9200 + 6000] = Rs. 15200

(iii) The cost slope of different activities is computed by using the formula

cost slope = (Crash Cost – Normal cost)/ (Normal time – Crash time) and these are shown in the following table.

Activity	1–2	1–3	2–3	2–4	3–5	4–6	5–6
Slope	250	267	200	600	233	250	300

Now we construct the following table for finding the optimal cost with duration and minimum project duration with cost.

Critical Path (s)	See Figure	Activities crashed & time	Project length (days)	Normal direct cost (Rs.) (A)	Crashing cost (Rs.) (B)	Indirect cost (Rs. 300/day) (C)	Total cost (Rs.) (A+B+C)
1-2-3-4-6	Fig. 6.15	—	20	9,200	—	300 × 20	15200
1-2-3-4-6	Do	2–3 (1)	19	9,200	200 × 1 = 200	300 × 19	15100
1-2-3-4-6 1-2-4-6	Fig. 6.16	1-2(1)	18	9, 200	200 + 250 × .1 = 450	300 × 18	15050
1-2-3-4-6 1-2-4-6 1-3-4-6	Fig. 6.17	4-6(1)	17	9,200	450 + 250 × 1=700	300 × 17	15000
1-2-3-4-6 1-2-4-6 1-3-4-6 & 1-3-5-6	Fig. 6.18	3-5(1) 4-6(1)	16	9,200	700+233 × 1 + 250 ×1 = 1183	300 × 16	15183
Do	Fig. 6.19	3-5(1) 4-6(1)	15	9,200	1183 + 233×1+250×1 = 1626	300 × 15	15366
Do	Fig. 6.20	3–5(1) 4-6(1)	14	9,200	1626+233 ×1+250×1 = 1626	300 × 14	15549
Do	Fig. 6.21	1–2(1) 1-3(1)	13	9,200	2109+250×1+267 × 1 = 2626	300 × 13	15766
Do	Fig. 6.22	2–3(1) 2–4(1) 1-3(1)	12	9,200	2626+250×1+600 × 1 = 2626	300 × 12	16493

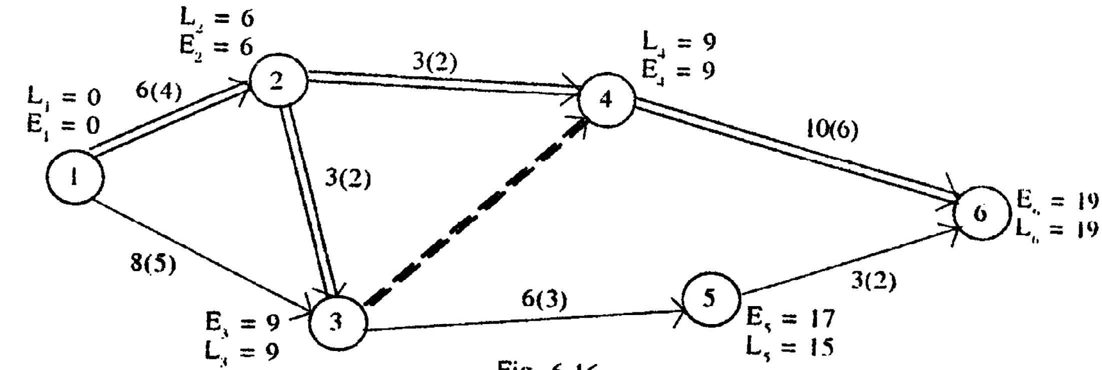


Fig. 6.16

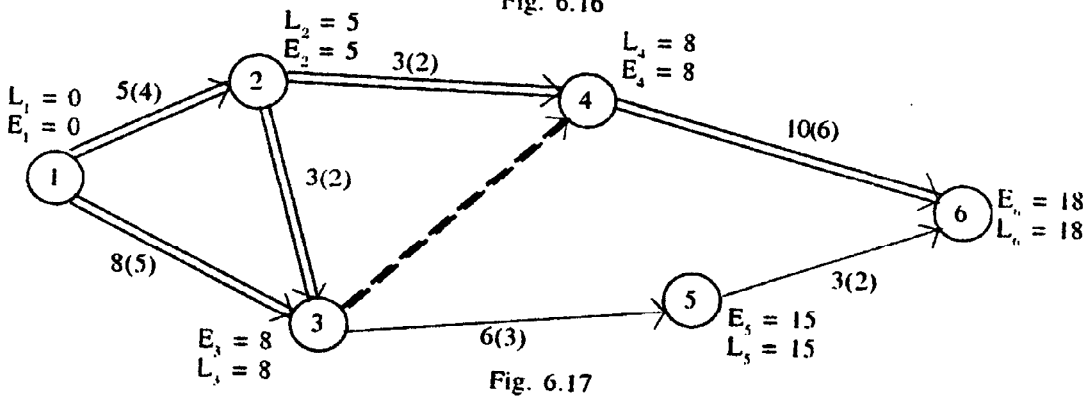


Fig. 6.17

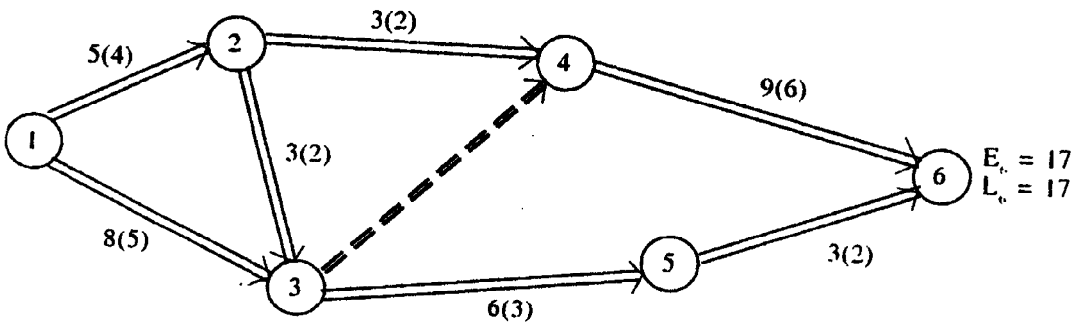


Fig. 6.18

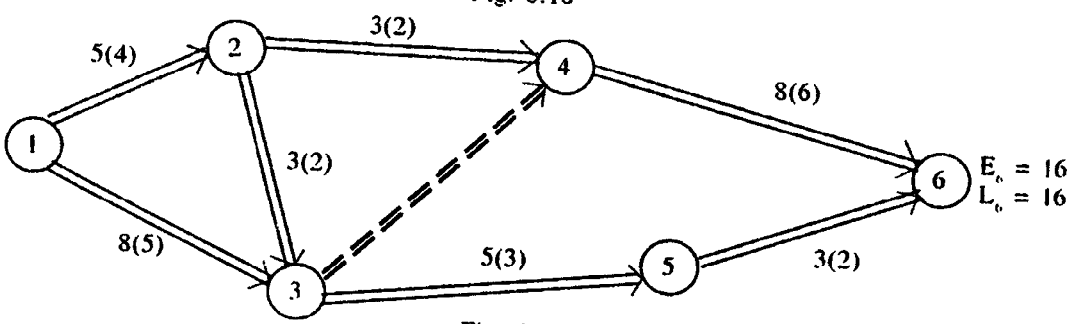


Fig. 6.19

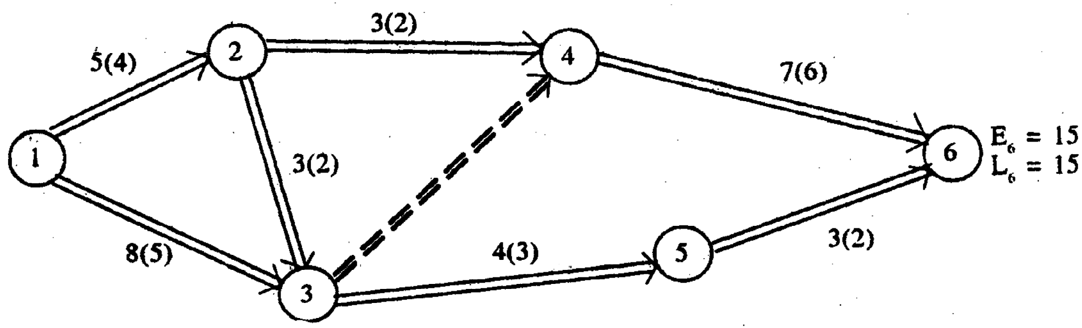


Fig. 6.20

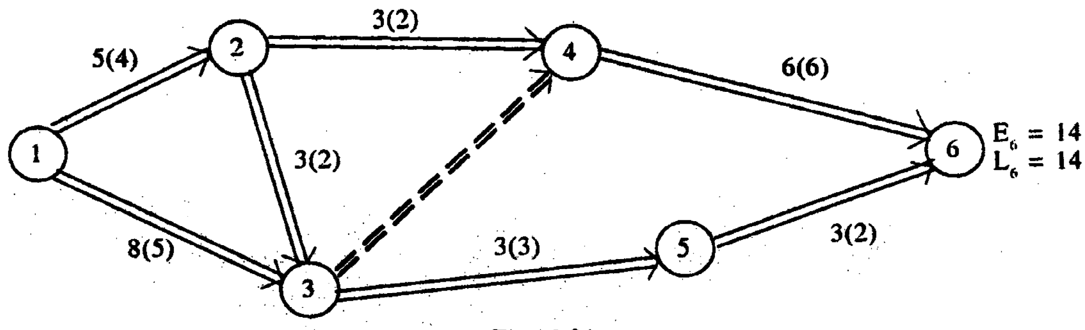


Fig. 6.21

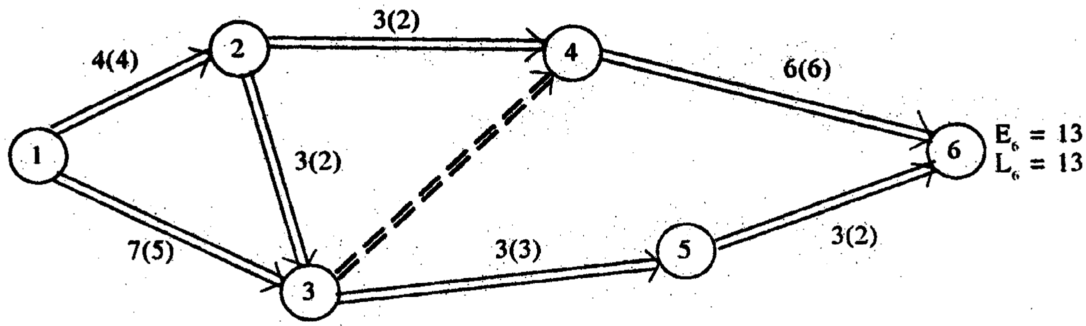


Fig. 6.22

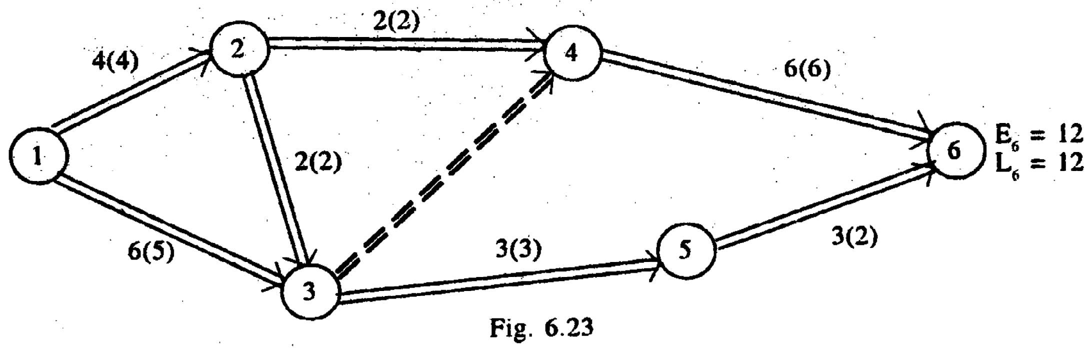


Fig. 6.23

From Fig. 6.23, it is seen that no further crashing is possible beyond 12 days. Hence the least project duration is 12 days and the corresponding cost of the project will be Rs. 16493.00

(iv) As the minimum cost occurs for 17 days schedule, optimum duration of the project is 17 days and the minimum project cost is Rs. 15000.00

6.12 Summary

In this unit, the network analysis of a project has been discussed. For this purpose, two very popular techniques i.e. CPM and PERT analysis have been explained clearly. Finally, considering the considering the cost aspects in a project, time cost trade off algorithm has been focussed. To illustrat the different techniques, some numerical examples have been solved showing the different steps.

6.13 Exercise

1. A project consists of a series of tasks lebellel A, B, ..., H, I, with the following relationships (W < X, Y means X and Y can not start until W is completed; X, Y < @ means W can not start until both X and Y are completed). With this notation construct the network diagram having the following constraints:

A < D, E; B, D < F; C < G; B, G < H; F, G < 1

Find also the minimum time of completion of the project, when the time (in days) of completion of each task is as follows :

Task :	A	B	C	D	E	F	G	H	I
Time :	23	8	20	16	24	18	19	4	10

2. The following are the detains of estimated times of activities of a certain project.

Activity :	A	B	C	D	E	F
Immediate Predecessor :	—	A	A	B,C	—	E
Estimated time : (weeks)	2	3	4	6	2	8

(a) Find the critical path and the expected time of the project.

(b) Calculate the earliest start time and earliest finish time for each activity.

(c) Calculate the slack for each activity.

3. Draw the network for the following project anc compute the earliest and latest times for each event and also find the ciritcal path :

Activity :	1-2	1-3	2-4	3-4	4-5	4-6	5-7	6-7	7-8
Immediate Predecessor :	—	—	1-2	1-3	2-4	2-4 & 3-4	4-5	4-6	6-7&5-7
Time (days) :	5	4	6	2	1	7	8	4	3

4. A project consists of eight activities with the following relevant information :

Activity	Immediate prodecessor	Estimated duration (days) Optimistic	Most likely	Pessimistic
A	—	1	1	7
B	—	1	4	7
C	—	2	2	8
D	A	1	1	1
E	B	2	5	14
F	C	2	5	14
G	D, E	3	6	15
H	F, G	1	2	3

(i) Draw the network and fined out the expected project completion time.

(ii) What duration will have 95% confidence for project completion?

(iii) If the average duration for activity F increases to 14 days, what will be its effectts on the expected project completion time which will have 95% confidence?

5. A small project consists of seven activities, the details of which are given below :

Activity	Time estimates			Predecessor
	t_0	t_m	t_p	
A	3	6	9	None
B	2	5	8	None
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C, D
G	1	5	15	E

Find the critical path. What is the probability that the project will be competed by 18 weeks?

6. The following table gives data or normal time-cost and crash time-cost for a project :

Activity	Normal		Crash	
	Time (days)	Cost (Rs.)	Time (days)	Cost (Rs.)
1–2	6	650	4	1000
1–3	4	600	2	2000
2–4	5	500	3	1500
2–5	3	450	1	650
3–4	6	900	4	2000
4–6	8	800	4	3000
5–6	4	400	2	1000
6–7	3	450	2	800

7. The following table gives the activities in a construction project and other relevant information.

Activity	Immediate Predecessor	Time (days) Normal	Time (days) Crash	Direct cost (Rs.) Normal	Direct cost (Rs.) Crash
A	—	4	3	60	90
B	—	6	4	150	250
C	—	2	1	38	60
D	A	5	3	150	250
E	C	2	2	100	100
F	A	7	5	115	175
G	B, D, E	4	2	100	240

Indirect costs vary as follows :

Days :	15	14	13	12	11	10	9	8	7	6
Cost (Rs.) :	600	500	400	250	175	100	75	50	35	25

(i) Draw an arrow diagram for the project.

(ii) Determine the project duration which will result in minimum total project cost.

6.14 References

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Unit 7 Inventory Management

Structure

7.0 Objectives

In real life, it is observed that a small retailer knows roughly the demand of his/her customes, in a year or a month or a week and accordingly, places orders on the whosesaleer to meet the demand of his/her customers. But, this is not possible for the authority/management of large production firms/deparmental stores/ shops.

The reason behind this is that the stocking of items in such cases depends upon the various factors e.g., demand of an item, ordering time, time lag between orders and actual receiving (i.e., Lead time) etc. So the problem for managers/ retailers is to have a compromise between overstocking and under-stocking. The study of such type of problems is known as Inventory Control (IC). The IC may be defined as the function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finished goods in orderly mannered to meet up the objectives of maximum customer-service with minimum investmnet and efficient (low-cost) plant operation.

In this unit, we shall introduce only the inventory models of elementary type for deterministic and probabilistic cases. These are single item purchasing and manufacturing model with/without shortages, price breaks models, multi-item purchasing model with different constraints like investment, average inventory and space constraints. Also, we shall introduce probabilistic models for discrete (well known News boy problem) and continuous cases. This discussion will help us to derive the real life purchasing/manufacturing inventory models.

7.1 Introduction

In broad sense, inventory is defined as an idle resource of an enterprise/ company/manufacturing firm. It can be defined as a stock of physical goods, commodities or other economic resources which are used to meet up the customer's demand or requirement of production. This means that the inventory acts a buffer stock between a supplier and a customer.

The inventory or stock of goods may be kept in any one of the following forms.

(i) raw materials

(ii) semi-finished goods (work-in-process inventory)

(iii) finished (or produced) goods

(iv) maintenance, repair and operating supplies (MRO) items.

In any sector of an economy, the control and maintenance of inventory is a problem common to all organisations. Inventories of physical goods are maintained in governemnt and non-government establishments e.g., Agriculture, Industry, Military, Business, etc. Some reasons for maintaing inventories are as follows :

(i) to conduct smooth and efficient running of business.

(ii) to provide the customer service by meeting their demands from stock without delay.

(iii) to earn price discount for bulk purchasing.

(iv) to maintain more stable operating and/or work force levels.

(v) to take the financial advantage of transporting/shipping economics.

(vi) to plan overall operating strategy through decoupling of succesive stages in the chain of acquireing goods, preparing products, shipping to branch ware houses and finally serving the customers.

(vii) to motivate the customers to purchase more by displaying large number of goods in the showroom/shop.

(viii) to take the advantages in purchashing of some raw materials and some commonly used physical goods (such as paddy, wheat etc.) whose prices seasonally fluctuate. In this connection, it is more profitable to procure a sufficient quantity of these raw materials/commonly used physical goods when their prices are low to be used later during the high price season or when need arises.

Production/Inventory planning and control is essentially concerned with the desing-operation and control of an inventroy system in any sector of a given economy. The problem of inventory control is primarily concerned with the following fundamental questions :

(i) whick items should be carried in stock? or which items should be produced?

(ii) How much of each of these items should be ordered/produced?

(iii) when should an order be placed? or when to produce?

(iv) what type of inventory control system should be used?

In practice, it is a formidable task to determine a suitable inventory policy. Regarding the above mentioned questions, an inventory problem is a problem of making optimal decisions. In other words, an inventory problem deals with decisions that optimize either the cost function (total or average cost) or the profit function (total or average profit) of the inventory system. However, there are certain types of problems, such as those relating to the storage of water in a dam in which one has no control over the replenishment of inventory. The supply of inventory of water in a dam depends on rainfall and the organisation operating the dam, has no contorl over it.

Our aim is to formulate mathematical models of different inventory control systems and to solve those using different mathematical analysis. For this purpose, our task is to construct a mathematical model of the inventory system. However, this type of model is biasc on different assumptions and approximations. It is difficult both to devise and operate with an exact/accurate model. We do not know what the real world is. Therefore, it is almost impossible to construct a realistic model with complete accuracy. For this reason, some approximations and simplifications must be used during the model building process. Again, the solution of the inventory problem is a set of specific values of variables that minimizes the total (or average) cost of the system or maximizes the total (or average) profit of the system.

7.2 Types of Inventory

There are five types of inventory, namely :

(i) Transportation inventories, (ii) Fluctuaton inventoreis.

(iii) Anticipation inventories, (iv) De-coupling inventoreis, (v) Lot-size inventories.

Transportation inventories : This arises due to transportaion of inventory items to various distribution centres and customers from the various production centres, when the transportation time is long, the items under transport can not be served to customers. These inventories exist solely because of transportation time.

Fluctuation inventories : These have to be carried because sales and production times can not be predicted accurately. In real-life problems, there are fluctuations in the demand and lead-times that affect the production of items.

Anticipation inventoreis : These are build up in advance by anticipating or foreseeing the future demand for the season of large sales, a promotion programme or a plant shout-down period.

De-coupling inventories : The inventories used to reduce the inter dependance of various stages of production system are known as de-coupling inventories.

Lot-size inventories : Generally, the rate of consumption is different from the rate of production or purchasing. Thereofre, items are produced in larger quantities which result in lot-size, also called as cycle inventories.

7.3 Basic concepts and Terminologies in Inventory

The inventory systems depend on several system factors and parameters such as demand, replenishment rate, shortages, constraints, various types of costs etc.

7.3.1 Demand

Demand is defined as the number of units of an item required by the customer in a unit time and has the dimension of a quantity. It may be known exactly or known in terms of probabilities or may be completely unknown.

The demand pattern of items may be either deterministic or probabilistic. Problem in which demand is known and fixed are called deterministic problem. Whereas those problems in which the demand is assumed to be a random variable are called stochastic or probabilistic problems.

In case of deterministic demand it is assumed that the quantities needed over subsequent periods of time are known exactly. Further, the known demand may be fixed or variable with time or stock-level or selling price of an item etc.

Probabilistic demand occurs when requirements over a certain period of time are not known with certainty but their pattern can be described by a known probability distribution.

In some cases, demand may also be represented by uncertain data in non-stochastic sense i.e., by vague/imprecise data. This type of demand is termed as fuzzy demand and the system as a fuzzy system.

7.3.2 Replenishment

Replenishment refers to the amount of quantities that are scheduled to be put into inventories, at the time when decisions are made about ordering these quantities or to the time when they ae actually added to stock. It can be categorized according to size, pattern and lead time. Replenishment size may be constant or variable, depending upon the type of the inventory system. It may depend on time, demand and/or on-hand inventory level. The replenishment patterns are usually instantaneous, uniform or in batch. The replenishment quantity again may be probabilistic or fuzzy in nature.

7.3.3 Constraints

Constraints are the limitations imposed on the inventory system. It may be imposed on the amount of investment, available space, the amount of inventory held, average instantaneous expenditure, number of orders, etc.

7.3.4 Fully backlogged/ Partially backlogged Shortages

During stock out period, the sales or goodwill may be lost either by a delay or complete refusal in meeting the demand. If the unfulfilled demand for the goods is satisfied completely at a later date, then it is a case of fully backlogged shortage i.e. it is assumed that no customer balk away during this period and the deamnd of all these waiting customers is met up at the beginning of the next period gradually after the commencement of next production.

Again, it is normally observed that during the stock out period, some of the customers wait for the product and toher balk away. When this happens, the phenomenon is called partially backlogged shortages.

7.3.5 Lead time

The time gap between the time of placing an order or production start and the time of arrival of goods in stock is called the lead time. If may be a constant or a variable. Again, variable lead time may be probabilistic or imprecise.

7.3.6 Planning/Time Horizon

The time period over which the inventory level will be controlled is called the time/planning horizon. It may be finite or infinite depending upon the nature of the inventory system ofr the commodity.

7.3.7 Deterioration/Damagability/Perishability

Deterioration is defined as decay, evaporation, obsolescence and loss of utility or marginal value of a commodity that results in the decreasing usefulness from the original condition. Vegetables, food grains and semiconductor chips, etc. are exmaples of such products.

Damagability is defined by the damage when the items are broken or loose their utility due to the accumulated stress, bad handling, hostile environment etc. The amount of damage by the stess varies with the size of stock and the duration for which the stress is applied. Items made of glass, china-clay, ceramic, mud ec. are examples of such products.

Perishable items are those which have finite life time (fixed or random). Fixed life time product (e.g., human blood, etc.) has a deterministic self life while the random life time scenario is closely related to the case of an inventory which experiences continuous physical depletion due to deterioration or decay.

7.3.8 Various types of Inventory Costs

Inventory costs are the costs associated ith the operation of an inventory system and result from action or lack of action on the part of management in establishing the system. They are basic economic parameters to any inventory decision model.

Purchase or Unit cost : The purchase or unit cost of an item is the unit purchase price to obtain the item from an external source or the unit production cost for the internal production. It may also depend upon the demand. When production is done in large quantities, it results in reduction of production cost per unit. Also, when quantity discounts are allowed for bulk orders, unit price is reduced and dependent on the quantity purchased or ordered.

Ordering/set up cost : The ordering or set up cost originates from the experience of issuing a purchase order to an outside supplier or from an internal production set up costs. The ordering cost includes clerical and administrative costs, telephone charges, telegrams, transportation cost, loading and unloading cost etc. Generally, this cost is assumed to be independent of the quantity ordered for or produced. In some cases, it may depend on the quantity of goods purchased because of price break or quanitity discounts or transportation cost, etc.

Holding or carrying cost : The holding or carrying cost is the cost associated with the strorage of the inventory until its use or sale. It is directly proportional to the amount/ quantity in the inventory and the time for which the stocks are held. This cost generally includes the costs such as insurance, taxes, obsolescence, deterioration, rent of warehouse, light, heat, maintenance and interest on the money locked up.

Shortage cost or stock-out cost : The shortage cost or stock-out cost is the penalty incurred for being unable to meet up a demand when it occours. This cost arises due to shortage of goods, lost sales for delay in meeting up the demand or total inability to meet up the demand. In the case, where the unfulfilled demand for the goods can be satisfied a latter date (back logging case), this cost depends on the shortage quantity and delaying time both. On the other hand, if the unfulfilled demand is lost (no backlogging case), shortage cost becomes proportional to the shortage quantity only. In both cases, there is a loss of goodwill which can not be quantified for the development of mathematical model.

Disposal cost : When an amount of some units of an item remains excess at the end of inventory cycle and if this amount is sold at a lower price in the next cycle to derive some advantages like clearing the stock, winding up the business, etc., the revenue earned through such a process is called the disposal cost.

Salvage value : During storage, some units are partially spoiled or damanged ie., some units loose their utility partially. In a developing country, it is normally observed that some of these are sold at a reduced price (less than the pruchase price) to a section of customers and this gives some revenue to the management. This revenue is called salvage value.

7.4 Classification of inventory Models

The inventory problems (models) may be classified into two categories.

(i) **Deterministic inventory models :** These are the inventory models in which demand is assumed to be known constant or variable (dependent on time, stock-level, selling price of the item, etc.). Here, we shall consider deterministic inventory models for known constant demand. Such models are usually referred to as beconomic lot-size models or Economic order Quantity (EOQ) models.

There are four types of models under this category, namely

(a) Purchasing inventory model with no shortages.

(b) Manufacturing inventory model with no shortage.

(c) Purchasing inventory model with shortages.

(d) Manufacturing model with shortages.

(ii) **Probabilistic inventory models :** These are the inventory models in which the demand is a random variable having a known probability distribution. Here, the future demand is determined by collecting data from the past experience.

7.4.1 Model-1 : The economic lot-size model (EOQ model with no shortages & instantaneous production) or Purchasing inventory model with no shortages.

In this models, we want to derive the formula for the optimum order quantity per cycle of a single product so as to minimize the total average cost under the following assumptions and notations :

(i) Demand is deterministic and uniform at a rate D units of quantity per unit time.

(ii) Production is instantaneous (i.e., production rate is infinite)

(iii) Shortages are not allowed.

(iv) Lead time is zero.

(v) The inventory planning horizon is infinite and the inventory system involves only one item and one stocking point.

(vi) Only a single order will be placed at the beginning of each cycle and the entire lot is delivered in one batch.

(vii) The inventory carrying cost, C_1 per unit quantity per unit time, the ordering cost, C_3 per order are known and constant.

(viii) T be cycle length and Q be the ordering quantiy per cycle.

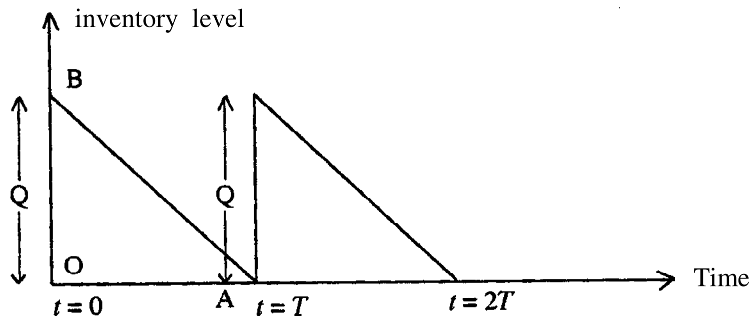

Fig. 7.1

Let us assume that an enterprise purchases an amount of Q units of item at time $t = 0$. This amount will be depleted to meet up the customer's demand. Ultimately, the stock level reaches to zero at time $t = T$. The inventory situation is shown in the Fig.1.

$$\text{Clearly, } Q = DT \qquad \text{......... (1)}$$

Now, the inventory carrying cost for the entire cycle T is $C_1 \times$ (area of ΔAOB)

$= C_1 .\ QT = \frac{1}{2}\ C_1QT$ and the ordering cost for the said cycle T is C_3.

Hence the total cost for time T is given by

$$X = C_3 + \frac{1}{2} C_1QT$$

Therefore, the total average cost is given by

$$C(Q) = \frac{X}{T}$$

$$\text{or,} \qquad C\,(Q) = \frac{C_3}{T} + \frac{1}{2}\ QC_1$$

$$\text{or,} \qquad C(Q) = \frac{C_3 D}{Q} \quad \frac{1}{2} C_1 Q \qquad \begin{array}{l} \because \ Q \quad DT \\ \quad T \quad \dfrac{Q}{D} \end{array} \qquad \text{...... (2)}$$

The optimum value of Q which minimizes $C(Q)$ is obtained by equating the first derivative of $C(Q)$ with respect to Q to zero

$$\text{i.e.,} \qquad \frac{dC}{dQ} \quad 0 \ \text{ or, } \ \frac{1}{2} \ C_1 - \frac{C_3 D}{Q^2} \quad 0$$

$$\text{or,} \qquad Q \quad \sqrt{\frac{2C_3 D}{Q_1}}$$

Again, $\dfrac{d^2 C(Q)}{dQ^2} \quad \dfrac{2C_3 D}{Q}$ which is +ve for $Q \quad \sqrt{\dfrac{2C_3 D}{Q_1}}$.

Hence $C(Q)$ is minimum for which the optimum value of Q is

$$Q^* = \sqrt{\frac{2C_3 D}{C_1}} \qquad \text{...... (3)}$$

This is known as economic lot size formula or *EOQ*. The corresponding optimum time interval is $T^* = \dfrac{Q^*}{D} \quad \sqrt{\dfrac{2C_3}{C_1 D}}$ and the minimum cost per unit time is given by

$$C_{\min} \quad \frac{C_3 D}{Q^*} \quad \frac{1}{2} C_1 Q^* \quad \sqrt{2C_1 C_3 D}$$

This model was first developed by Ford Haris of the westing house corporation, USA, in the year 1915. He derived the well-known classical lot size formula (3). This formula was also developed independently by R. H. Wilson after few years and it has been named as Haris-Wilson formula.

Remark :

(i) The total inventory time units for the entire cycle T is $\dfrac{1}{2}$ QT, so the average inventory at any time is $\dfrac{1}{2} \quad \dfrac{QT}{T} \quad \dfrac{1}{2} Q$.

(ii) Since $C_1 > 0$ from $f(Q) = \dfrac{1}{2}$ $C_1 Q$ it is obvious that the inventory carrying cost is a linear function of Q with a + ve slope i.e., for smaller average inventory, the inventory carrying costs are lower. In contrast $g(Q) = \dfrac{C_3 D}{Q}$ i.e., ordering cost increases as Q decreases.

(iii) In the above model, if we always maintain an inventory B on hand as buffer stock, then the average inventory at any time is $\frac{1}{2}$ $Q + B$. Therefore, the total cost per unit time is

$$C(Q) = \frac{1}{2}Q \quad B \;\; C_1 \quad \frac{C_3 D}{Q}$$

As before, we obtain $Q = Q^* = \sqrt{\dfrac{2C_3 D}{C_1}}$ and $T = T^* = \sqrt{\dfrac{2C_3}{DC_1}}$

(iv) In the above model, if the ordering cost is taken as $C_3 + bQ$, (where b is the purchase cost per unit quantity) instead of fixed ordering cost then there is no change in the optimum order quantity.

Proof : In this case, the average cost is given by

$$C(Q) = \frac{1}{2}\; C_1 Q + \frac{D}{Q}\; (C_3 + bQ) \qquad \text{....... (4)}$$

The necessary condition for the optimum of $C\ (Q)$ in (4), we have

$$c'\ (Q) = 0 \text{ implies } Q = \sqrt{\frac{2C_3 D}{C_1}}$$

and $\quad c''\ (Q) > 0$. Hence $Q^* = \sqrt{\dfrac{2C_3 D}{C_1}}$

This shows that there is no change in Q^* in spite of change in the ordering cost.

Example 1. An engineering factory consumes 5000 units of a component per year. The ordering, receiving and handling costs are Rs. 300 per order while the trucing cost are Rs. 1200 per order. Interest cost Rs. 0.06 per unit per year, Deterioration and obsolence cost Rs. 0.004 per unit per year, storage cost Rs. 1000.00 per year for 5000 units. Calculate the economic order quantity.

Solution : In the given proble, we have demand (D) = 5000 units

Ordering cost / Replenishment cost = Ordering, receiving, handling costs and trucing costs = Rs. (300 + 1200) = Rs. 1500 per order.

Inventory carrying cost = interest costs + Deterioration and obsolence costs stroage costs = $0.06 \quad 0.004 \quad \frac{1000}{5000}$ rupees per unit per year.

= 0.264 rupees per unit per year.

Hence the economic order quanitity is given by

$$Q^* = \sqrt{\frac{2C_3D}{C_1}} \quad \sqrt{\frac{2 \quad 1500 \quad 5000}{0.264}}$$

= 7538 units (approx.)

7.4.2 Model-2 : Manufacturing model with no shortages or Economic lot-size model with finite rate of replenishment and without shortages.

In this model, we shall derive formula for the optimum prodution quantity (Economic lot-size) per cycle of a single product so as to minimize the total average cost under the following assumptions and notations :

(i) Demand is deterministic and uniform at a rate D units of quantity per unit time.

(ii) Shortages are not allowed.

(iii) Lead time is zero.

(iv) The production rate or replenishment rate is finite, say, K units per unit time ($K > D$).

(v) The production-inventory planning horizon is infinite and the production system involves only one items and one stocking point.

(vi) The inventory carrying cost, C_1 per unit quantity per unit time, the setup cost C_3 per production cycle are known and constant.

(vii) T be the cycle length and Q be economic lot-size.

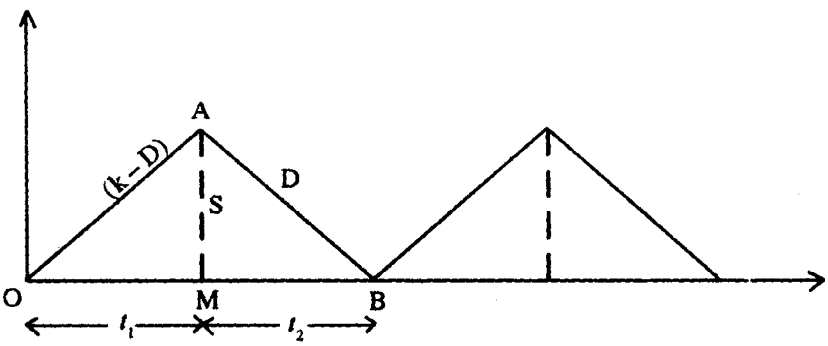

Fig. 7.1

In this model, each production cycle time T consists of two parts t_1 and t_2 where

(i) t_1 is the period during which the stock is growing up a constant rate K-D units per unit time.

(ii) t_2 is the period during which there is no replenishment (or production) but inventory is decreasing at the rate of D units per unit time.

Further, it is assumed that S is the stock available at the end of time t_1 which is expected to be consumed during the remaining period t_2 at the consumption rate D.

Therefore, $(K - D)\ t_1 = \mathrm{S}$

$$\text{or,} \quad t_1 = \frac{S}{K \quad D}$$

Since the total quantity produced during the production period t_1 is Q.

$$Q = Kt_1$$

$$\text{or,} \quad Q = K.\frac{S}{K \quad D}$$

which implies $S = \dfrac{K \quad D}{K} Q$

Again, $Q = DT$ i.e., $T = \dfrac{Q}{D}$

Now the inventory carrying cost for the entire cycle T is $(\Delta OAB).\ C_1$

$$= \frac{1}{2}\ TS\ C_1$$

and the set-up cost for time period T is C_3. Therefore, the total cost for the entire cycle T is given by $X = C_3 + \frac{1}{2} C_1 \frac{1}{2} \; ST$.

Therefore the total average cost is given by

$$C(Q) = \frac{X}{T}$$

$$\text{or,} \quad C\ (Q) = \frac{C_3}{T} \quad \frac{1}{2} C_1 S$$

$$\text{or,} \quad C\ (Q) = \frac{C_3 D}{Q} \quad \frac{1}{2} C_1 \frac{K \quad D}{K} Q \qquad \begin{matrix} Q \quad DTand \\ S \quad \frac{K \quad D}{K} Q \end{matrix}$$

The optimum value of Q which minimizes $C(Q)$ is obtained by equation the first derivative of $C\ (Q)$ with respect to Q to zero

$$\text{i.e.,} \quad \frac{dC}{dQ} \quad 0$$

$$\Rightarrow \qquad \frac{C_3 D}{Q^2} \quad \frac{1}{2} C_1 \frac{K \quad D}{K} \quad 0$$

$$\Rightarrow \qquad Q \quad \sqrt{\frac{2C_3}{C_1} \cdot \frac{DK}{K \quad D}}$$

Again, $\frac{d^2 C}{dQ^2} \quad \frac{2C_3 D}{Q^3} = +$ ve quantity for $Q = \sqrt{\frac{2C_3}{C_1} \cdot \frac{DK}{K \quad D}}$

Hence $C\ (Q)$ is minimum for which the optimum value of Q is

$$Q^* = \sqrt{\frac{2C_3}{C_1} \cdot \frac{DK}{K \quad D}}$$

The corresponding time interval is

$$T^* = \frac{Q^*}{D} \quad \sqrt{\frac{2C_3}{C_1} \cdot \frac{DK}{K \quad D}} / D \quad \sqrt{\frac{2C_3 K}{C_1 D (K \quad D)}}$$

and the minimum average cost is given by

$$C_{\min} = \frac{1}{2}\frac{K \quad D}{K} C_1\, Q^* \quad \frac{C_3 D}{Q^*}$$

$$= \sqrt{2C_1 C_3 D \frac{K \quad D}{K}}$$

Remark : (i) For this model, Q^*, T^* and C_{MIN} can between in the following form :

$$Q^* = \frac{\sqrt{2C_3 D}}{C_1} \frac{1}{1 \quad \frac{D}{K}}, \quad T^* = \sqrt{\frac{2C_3}{DC_1} \cdot \frac{1}{1 \quad \frac{D}{k}}}$$

and $C_{\min} = \sqrt{2C_1 C_3 D \;\; 1 \quad \frac{D}{K}}$

If $K \rightarrow \propto$ i.e., the production rate is infinite, this model reduces to model 1. When $k \rightarrow \propto$, then Q^*, T^* and $C_{\min}$ reduce to the expression for Q^*, T^* and $C_{\min}$ of model-1.

Example : A contractor has to supply 20,000 units per day. He can produce 30,000 units per day. The cost of holding a unit in stock is Rs. 3.00 per year and the setup cost per run is Rs. 50.00. How frequently and of what size, the production runs be made?

Solution : For this problem, it is given that

D = 20,000 units / day

K = 30,000 units / day

C_1 = Rs. 3.00 per year = Rs. $\frac{3}{365}$ per day

C_1 = Rs. 50.00 per run

Let Q^* be the optimum lot-size.

$$\therefore \quad Q^* = \sqrt{\frac{2C_3 D}{C_1} \frac{1}{1 \quad \frac{D}{K}}}$$

$$= \sqrt{\frac{2 \quad 50 \quad 20000}{\frac{3}{365}} \frac{1}{1 \quad \frac{20000}{30000}}} \text{ units}$$

$$= \sqrt{\frac{2 \quad 50 \quad 20000 \quad 365 \quad 3}{3}} = \sqrt{2 \quad 50 \quad 20000 \quad 365} \text{ units}$$

$$= 27019 \text{ units}$$

Let T^* be the optimal cycle length

$$\therefore \ T^* = \frac{Q^*}{D} \quad \frac{27019}{20000} \text{ days} = 1.35 \text{ days}$$

$$\text{Length of production cycle} = \frac{27019}{20000} = 0.9 \text{ days.}$$

Thus the production cycle starts at an interval of 1.35 days and production continues for 0.9 days so that in each cycle a batch of 27019 units is produced.

7.4.3 Model-3 : Purchasing model with shortages

In this model, we shall derive the optimal order level and the minimum average cost under the following assumptions and notations :

(i) Demand is deterministic and uniform at a rate D units of quantity per unit time.

(ii) Production is instantaneous (i.e., production rate is infinite).

(iii) Shortages are allowed and fully backlogged.

(iv) Lead time is zero.

(v) The inventory planning horizon is infinite and the inventory system involves only one items and one stocking point.

(vi) Only a single order will be placed at the beginning of each cycle and the entire lot is delivered in one batch.

(vii) The inventory carrying cost, C_1 per unit quantity per unit time, the shortage cost, C_2 per unit quantity per unit time, the ordering cost, C_3 per order are known and constant.

(viii) Q be the lot-size per cycle where as S is the initial inventory level after fulfiling the backlogged quantity of previous cycle and $Q - S$ be the maximum shortage level.

(ix) T be the cycle length or scheduling period whereas t_1 be the no shortage period.

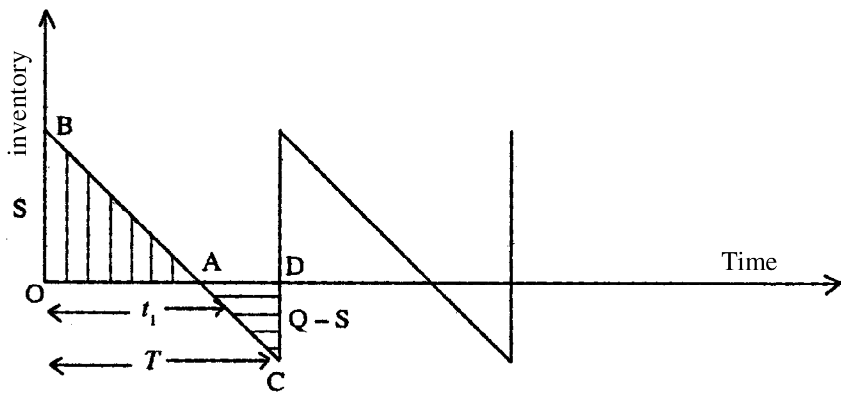

Fig. 7.3

According to the assumption of (viii) & (ix), we have $Q \; = DT$. Regarding the cycle length or scheduling period of the inventory system, two cases may arise :

Case-1 : Cycle length or scheduling peirod T is constant.

Case-2 : Cycle length or scheduling period T is a variable.

Case-3 : In this case, T is constant i.e., inventory is to be replenished after every fixed time period T.

As t_1 be the shortage peirod, $S = Dt_1$ in $t_1 = \frac{S}{D}$. Now, the inventory carrying cost during the period O to t_1 is

$$C_1 \text{ (Area of } \Delta OAB)$$

$$= \frac{1}{2} C_1 S t_1 = \frac{1}{2} \; C_1 \; \frac{S^2}{D}$$

Again the shortage cost during the interval (t_1, T) is

$$C_2 \text{ (Area of } \Delta ACD)$$

$$= \frac{1}{2} \; C_2 \; (Q - S) \; (T - t_1)$$

$$= \frac{1}{2} \; C_2 \; (Q - S)^2 \; /D \qquad \because T \quad t_1 \quad \frac{Q \quad S}{D}$$

Hence the total average cost of the system is given by

$$C = \quad C_3 \quad \frac{1}{2}C_1\frac{S^2}{D} \quad \frac{1}{2}C_2\frac{Q \quad S}{D}^{2} \quad /T \qquad \text{........ (1)}$$

Since T is constant, $Q = DT$ is also constant. Hence the above expression i.e., the expression for average cost is a function of single variable S. So, we can easily minimize the above expression (1) with respect to S like model-1.

$$\text{In this case, } S^* \quad = \frac{C_2Q}{C_1 \quad C_2} \quad \frac{C_2DT}{C_1 \quad C_2}$$

$$\text{and } C_{\min} \quad = \frac{C_1C_2Q}{C_1 \quad C_2} \quad \frac{C_1C_2DT}{C_1 \quad C_2}$$

Case-2 : In this case, cycle length or scheduling period T is a variable. Like case-1, in this case, the average cost of the inventory system will be

$$C = C = \quad C_3 \quad \frac{1}{2}C_1\frac{S^2}{D} \quad \frac{1}{2}C_2\frac{Q \quad S}{D}^{2} \quad /T \qquad \text{..... (2)}$$

where $Q \quad = DT$

Here, the average cost C is a function of two independent variables T and S_1.

Now, for optimal value of C, we have

$$\frac{c}{s_1} = 0 \text{ and } \frac{c}{T} = 0$$

(ii) If $C_2 \to \propto$ and $C_1 > 0$. then shortages are prohibited. In this case, $S^*_1 = Q^*$ $= \sqrt{2C_3\frac{D}{C_1}}$ and each batch Q^* is used entirely for inventory.

(iii) If shortage costs are negligible, then $C_1 > 0$ and $C_2 \to \propto$. 0. In this case, $S^*_1 \to$ and $Q^* \to \propto$.

(iii) If inventory carrying costs are negiligible, then $C_1 > 0$ and $C_2 > 0$. In this case, $Q^* \to \propto$.

(iv) If inventory carrying costs are negligible, then $C_1 \to 0$ and $C_2 \to 0$. In this case, $Q^* \to \propto$ and $S^*_1 \to \propto$ i.e., $S^*_1 \to Q^*$. Thus, due to very very small inventory carrying costs, large lot-size should be ordered and used to meet up the future demand.

(v) When in every carrying costs and shortage costs are equal i.e., when $C_1 = C_2$, $\frac{C_1}{C_1 \quad C_2} \quad \frac{1}{2}$.

In this case, $Q^* \; = \; \sqrt{2}\sqrt{2\frac{C_3 D}{C_1}}$

Whick shows that the lot-size is √2 times of the lot size of case-1 model i.e., when no shortages are allowed.

Example-2 : The demand for an item is 18000 units per year. The inventory carrying cost is Rs. 1.20 per unit time and the cost of shortage is Rs. 5.00. The ordering cost is Rs. 400.00. Assuming that the replenishment rate is instantaneous determine the optimum order quanity, shortage quantity and cycle length.

Solution : For the prolem, It is given that elemtnf D = 180000 units per year, carrying cost C_1 = Rs. 1.20 per unit, shortage cost C_2 = Rs. 5.000, ordering cost C_3 = Rs. 400 per order.

$$\frac{C}{S_1} \quad 0 \text{ gives } S_1 = C_2\frac{DT}{(C_1 \quad C_2)} \qquad \text{..... (3)}$$

Again, $\frac{C}{T} = 0$

$$\frac{C_1}{2D}\frac{S_1^2}{T^2} \quad C_2\frac{DT \quad S_1}{T} \quad \frac{C_2}{2D}\frac{DT \quad S_1}{T^2}^{2} \quad \frac{C_3}{T^2} \quad 0$$

putting $S_1 = C_2\frac{DT}{(C_1 \quad C_2)}$ in above and simplifying, we have

$$T = T^* = \sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1 C_2 D}} \qquad \text{..... (4)}$$

Then $S_1 = S^*_1 = \sqrt{\frac{2C_2 C_3 D}{C_1(C_1 \quad C_2)}}$ ..... (5)

Obviously, for the volumes of T and S_1 given by (4) & (5),

$$\frac{{}^2C}{S_1^2} \quad 0, \frac{{}^2S}{T^2} \quad 0 \text{ and } \frac{{}^2C}{S_1}\frac{{}^2C}{T} \quad \frac{{}^2C}{S_1 \; T}^{2} \quad 0$$

Hence C is minimum for the values of T & S_1 given by (4) and (5).

Therefore the optimum order quantity for minimum cost is given by

$$Q^* = DT^* = D\sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1C_2D}} \quad \sqrt{\frac{2C_3(C_1 \quad C_2)D}{C_1C_2}} \qquad \text{..... (6)}$$

$$\text{and } C_{\min} = C^* = \sqrt{\frac{2C_1C_2C_3D}{(C_1 \quad C_2)}} \qquad \text{..... (7)}$$

Remark :

(i) If $C_1 \to \propto$ and $C_2 > 0$, inventories are prohibited. In this $S^*_1 = 0$ and $Q^* = \sqrt{\frac{2C_3D}{C_2}}$ and each lot-size is used to fill the back orders.

The optimum order quantity Q^* is given by

$$Q^* = \sqrt{\frac{2C_3(C_1 \quad C_2)D}{C_1C_2}} \quad \sqrt{\frac{2 \quad 400 \quad (1.2 \quad 5) \quad 18000}{1.2 \quad 5}} \quad 3857 \text{ units}$$

Again, the optimum shortage quantity $Q^* - S^*_1 = 3857 - \sqrt{\frac{2C_2C_3D}{C_1(C_1 \quad C_2)}}$

$$= 3857 - \sqrt{\frac{2 \quad 5 \quad 400 \quad 18000}{1.2 \quad (1.2 \quad 5)}}$$

$= 746$ units.

Optimal cycle length $T^* = \frac{Q^*}{D} \quad \frac{3857}{18000} \quad 0.214$ year.

Example-3 : The demand for an item is deterministic and constant over time and it is equal to 600 units per year. The per unit cost of the item is Rs. 50.00 while the cost of placing an order is Rs. 5.00> The inventory carrying cost is 20% of the unit cost of the item and the shortage cost per month is Rs. 1. Find the optimal ordering quantity. If shortages are not allowed, what would be the loss of the company.

Solution : It is given that $D = 600$ units / year

$C_1 = 20\%$ of Rs. $50.00 =$ Rs. 10.00

C_2 = Re 1 per month i.e., Rs. 12 per year.

C_3 = Rs. 5 per order.

(i) when shortages are allowed.

The optimal ordering quantity Q^* is given by

$$Q^* = \sqrt{\frac{2C_3(C_1 \quad C_2)D}{C_1C_2}} = 33 \text{ units.}$$

and the minimum cost per year is

$$C(Q^*) \quad \sqrt{\frac{2C_1C_2C_3D}{(C_1 \quad C_2)}} \quad \text{Rs. } 180.91$$

If shortages ae not allowed, then the optical order quantity is

$$Q^* = \sqrt{\frac{2C_3D}{C_1}} \quad 24.5 \text{ units}$$

and the relevant average cost is given by

$$C(Q^*) = \text{Rs. } \sqrt{2C_1C_3D} = \text{Rs. } 244.95$$

Therefore, if shortages are not allowed, the loss of the company will be Rs. (244.95 – 180.91) i.e., Rs. 64.04.

7.4.4 Model-4 : Manufacturing model with shortages or Economic lot-size model with finite rate of replenishment and shortages.

In this model, we shall derive the formula for the optimum production quantity, shortage quantity anc cycle length of a single product by minimizing the average cost of the production system under the following assumptions and notations :

(i) The production rate or replenishment rate is finite, say K units per unit time (K>D).

(ii) The production-inventory planning horizon is infinite and the production system involves only one item and one stockig point.

(iii) Demand of the item is deterministic and uniform at a rate D units of quantity per unit time.

(iv) Shortages are allowed.

(v) Lead time is zero.

(vi) The inventory carrying cost, C_1 per unit quantity per unit time, the shortage cost, C_2 per unit quantity per unit time and the set up cost, C_3 per set up are known and constant.

(vii) T be the cycle length of the system i.e., T be the interval between production cycles.

(viii) Q be the economic lot-size.

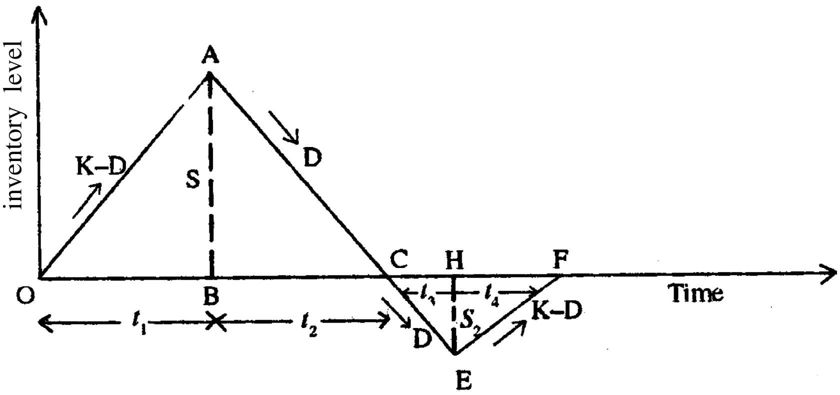

Fig. 7.4

Let us assume that each production cycle of length T consists of two parts t_{12} and t_{34} which are further subdivided inti $t1$ and t_2; t_3 and t_4 where (i) inventory is building up at a constant rate $K - D$ units per unit time during the interval $[0, t_1]$ (ii) at time $t = t_1$, the production is stopped and the stock level decreases due to meet up the customers, demand only upto the time $t = t_1 + t_2$. (iii) shortages are accumulated at a constant rate of D units per unit time during the time t_3 i.e., during the interval $[t_{12}, t_{12}, + t_3]$, (iv) shortages are being filled up immediately at a constant rate $K - D$ units per unit time during the time t_4 i.e., during the interval $[t_{12} + t_3, t_4]$ (v) The production cycle then repeats itself after the time $T = t_1 + t_2 + t_3 + t_4$.

Again, let at the end of t_1, the inventory level is S_1 end at the end of time $t = t_1 + t_2$, the stock level becomes nil. Now shortages start and suppose that the shortages are build up of quantity S_2 at time $t = t_1 + t_2 + t_3$ and then these shortages be filled up upto the time $t = t_1 + t_2 + t_3 + t_4$. The pictorial representation of the inventory situation is given in Fig. 7.4.

Now our objectives are to find the optimal value of Q, S_1, S_2, t_1, t_2, t_3, t_4. and T with the minimum average total cost.

Now the inventory carrying cost over the time period T is given by

$$C_h = C_1 \times \Delta OAC \qquad = C_1 \frac{1}{2} \; OC.AB$$

$$= \frac{1}{2} C_1 (t_1 + t_2) \; S_1$$

and the shortage cost over time T is given by

$$C_s = C_2 \times \Delta CEF$$

$$= C_2 \cdot \frac{1}{2} \; CF. \; EH$$

$$= \frac{1}{2} C_2 \cdot (t_3 + t_4) \; S_2$$

Hence the total average cost of the production system is given by

$$C = [C_3 + C_h + C_s] \; / \; T$$

From Fig.7.4, it is clear that

$$S_1 = (K - D)t_1 \text{ or, } t_1 \quad \frac{S_1}{K \quad D}$$

Again, $S_1 = Dt_2$ or, $t_2 \quad \frac{S_1}{D}$

Now, in stock-out situation,

$$S_2 = Dt_3 \text{ or, } t_3 \quad \frac{S_2}{D}$$

and $S_2 = (K - D) \; t_4$ or, $t_4 \quad \frac{S_2}{K \quad D}$

Since the total quantity produced over the time period T is Q.

$\therefore \; Q = Dt$ where D is the demand rate

or, $D \; (t_1 + t_2 + t_3 + t_4) = Q$

or, $D \quad \frac{S_1}{K \quad D} \quad \frac{S_1}{D} \quad \frac{S_2}{D} \quad \frac{S_2}{K \quad D} \qquad Q$

After simplification, we have

$$S_1 + S_2 = \frac{K \quad D}{K} Q \qquad \text{...... (2)}$$

Again, $t_1 + t_2 = \dfrac{K}{D(K \quad D)} S_1$

and $t_3 + t_4 = \dfrac{K}{D(K \quad D)} S_2$

Now substituting the value of $t_1 + t_2$, $t_3 + t_4$ and $T = \dfrac{Q}{D}$ in (1), we have

$$C\,(Q,\, S_1,\, S_2) = \frac{1}{2Q} \frac{K}{K \quad D} (C_1 S_1^2 \quad C_2 S_2^2) \quad \frac{DC_3}{Q}$$

Using (2), the above reduces to

$$C(Q, S_2) \quad \frac{1}{2Q} \frac{K}{K \quad D} \quad C_1 \quad \frac{K \quad D}{K} Q \quad S_2 \quad {}^2 \qquad \frac{DC_3}{Q} \qquad \text{...... (3)}$$

Now, for the extreme calues of $C(Q,\, S_2)$, we have

$$\frac{C}{Q} \quad 0, \qquad \frac{C}{S_2} \quad 0$$

$$\frac{C}{Q} \quad 0 \text{ implies } S_2 = C_1 \frac{K \quad D}{K} \frac{Q}{(C_1 \quad C_2)} \qquad \text{...... (4)}$$

Again, $\dfrac{C}{S_2} \quad 0$ gives $Q = \sqrt{\dfrac{2C_3(C_1 \quad C_2)}{C_1 C_2}} . \sqrt{\dfrac{KD}{K \quad D}}$ ...... (5)

For these values of Q and S_2 given in (5) & (4), it can easily be verified that

$$\frac{{}^2C}{Q^2} \quad 0, \quad \frac{{}^2C}{S_2^2} \quad 0 \text{ and } \frac{{}^2C}{Q^2} \frac{{}^2C}{S_2^2} \quad \frac{{}^2C}{Q \;\, S_2} \quad {}^2 \quad 0.$$

Hence $C\,(Q,\, S_2)$ is minimum and the optimal values of Q and S_2 are given

$$\text{by } Q^* = \sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1C_2}}\sqrt{\frac{KD}{K \quad D}} \qquad \text{...... (6)}$$

$$\text{and } S^*_2 = \sqrt{\frac{2C_1 \quad C_3}{C_2(C_1 \quad C_2)}}.\sqrt{\frac{D(K \quad D)}{K}} \qquad \text{...... (7)}$$

$$T^* = \frac{Q^*}{D} \quad \sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1C_2}}.\sqrt{\frac{K}{D(K \quad D)}} \qquad \text{...... (8)}$$

$$S^*_1 = \frac{K \quad D}{K} \; Q^* - S^*_2 = \sqrt{\frac{2C_2C_3}{C_1(C_1 \quad C_2)}}.\sqrt{\frac{D \;\; K \quad D}{K}} \qquad \text{...... (9)}$$

$$\text{Now } C_{\min} = C\,(Q^*, S^*_2) = \sqrt{\frac{2C_1C_2C_3}{C_1 \quad C_2}}.\sqrt{\frac{D \;\; K \quad D}{K}} \qquad \text{...... (10)}$$

Remarks :

(i) In this model, if we assume that the production rate is infinite i.e., $K \rightarrow \propto$, then the optimal quantities by takin $K \rightarrow \propto$ in (6), (8) and (10) are

$$Q^* = \quad \frac{Q^*}{D} \quad \frac{4489}{1500}$$

$$T^* = \sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1C_2D}}$$

$$\text{and} \qquad C_{\min} = \sqrt{\frac{2C_1C_2C_3D}{C_1 \quad C_2}}$$

This means that model-4 reduces to model-3, if $K \rightarrow \propto$.

(ii) If shortages are not allowed in model-4, then it reduce to model-3. Therefore by taking $C_2 \rightarrow \propto$ in (6), (8) and (10) we obtain the required expression of model-3 which are

$$Q^* = \sqrt{\frac{2C_3KD}{C_1(K \quad D)}},$$

$$T^* = \sqrt{\frac{2C_3K}{C_1D(K \quad D)}}$$

and $\quad C_{\min} = \sqrt{\dfrac{2C_1C_3D(K \quad D)}{K}}$

Example-4. The demand for an item in a company is 18000 units per year. The company can produce the item at a rate of 30000 per month. The cost of one set-up is Rs. 500 and the holding cost of one unit per month is Rs. 0.15. The shortage cost of one unit is Rs. 20 per month. Determine the optimum manufacturing quantity and the shortage quantity. Also determine the manufacturing time and the time between set-ups.

Solution :

For this problem, it is given that

C_1 = Rs. 0.15 per month

C_2 = Rs.20 per month

C_3 = Rs. 500.00 per set-up

K = 3000 per month

D = 18000 units per year i.e., 1500 units per month.

The optimum manufacturing quantity Q^* is given by

$$Q^* \quad = \sqrt{\frac{2C_3(C_1 \quad C_2)}{C_1C_2}}\sqrt{\frac{KD}{(K \quad D)}}$$

$$= \sqrt{\frac{2 \quad 500 \quad 0.15 \quad 20}{0.15 \quad 20}}\sqrt{\frac{3000 \quad 1500}{3000 \quad 1500}} \text{ units}$$

= 4489 units (approx.)

The optimum shortage quantity S^*_2 is given by

$$S^*_2 \quad = c_1 \quad \frac{K \quad D}{K}\frac{Q^*}{(C_1 \quad C_2)}$$

= 17 units (approx.)

Manufacturing time = $\dfrac{Q^*}{K} \quad \dfrac{4489}{3000}$ = 1.5 months and the time between set-ups $\dfrac{Q^*}{D} \quad \dfrac{4489}{1500}$ = 3 months (approx).

Now we construct the following table for finding the optimal cost with duration and minimum project duration with cost.

Critical Path (s)	See Figure	Activities crashed & time	Project length (days)	Normal direct cost (Rs.) (A)	Crashing cost (Rs.) (B)	Indirect cost (Rs. 300/day) (C)	Total cost (Rs.) (A+B+C)
1-2-3-4-6	Fig. 6.15	—	20	9,200	—	300 × 20	15200
1-2-3-4-6	Do	2–3 (1)	19	9,200	200 × 1 = 200	300 × 19	15100
1-2-3-4-6 1-2-4-6	Fig. 6.16	1-2(1)	18	9, 200	200 + 250 × .1 = 450	300 × 18	15050
1-2-3-4-6 1-2-4-6 1-3-4-6	Fig. 6.17	4-6(1)	17	9,200	450 + 250 × 1=700	300 × 17	15000
1-2-3-4-6 1-2-4-6 1-3-4-6 & 1-3-5-6	Fig. 6.18	3-5(1) 4-6(1)	16	9,200	700+233 × 1 + 250 ×1 = 1183	300 × 16	15183
Do	Fig. 6.19	3-5(1) 4-6(1)	15	9,200	1183 + 233×1+250×1 = 1626	300 × 15	15366
Do	Fig. 6.20	3–5(1) 4-6(1)	14	9,200	1626+233 ×1+250×1 = 1626	300 × 14	15549
Do	Fig. 6.21	1–2(1) 1-3(1)	13	9,200	2109+250×1+267 × 1 = 2626	300 × 13	15766
Do	Fig. 6.22	2–3(1) 2–4(1) 1-3(1)	12	9,200	2626+250×1+600 × 1 = 2626	300 × 12	16493

7.5 Multi-item Deterministic Problem

So far we have considered the models for single item or each item separately but if there exists a relationship among the items under some limitations then it is not possible to considr then separately. Thus after constructing the average cost expresion in such models, we shall use the method of Lagrange multiplier to minimize the average cost.

In all such problems, first of all we shall solve the problem ignoring the limitations and then consider the effect of limitations.

Now, we shall develop multi-item inventory model under the following assumptions and notations :

(i) There are n items with instantaneous production i.e., the production rate of each item is infinite.

(ii) Shortages are not allowed.

For i-th (i = 1, 2, ..., n) item.

(iii) D_i be the uniform demand rate.

(iv) The inventory carrying cost, Cli per unit quantity per unit time and the ordering cost, $c3i$ per order are known and constant.

(v) T_i be the cycle length.

Let Qi be the ordering quantity of i-th item.

Then, $Q_i = D_i T_i$ or, $T_i = \dfrac{Q_i}{D_i}$

Now, the total inventory time units for the i-th item is $\frac{1}{2} Q_i T_i$.

Hence the inventory carrying cost for i-th item over the inventory cycle is $\frac{1}{2} C1iQ_iT_i$.

Therefore, the average cost for the i-th item is

$$C_i = \quad C_{3i} \quad \frac{1}{2} C_{li} Q_i T_i \quad / T_i$$

or, $\quad C_i = C_{3i}\,\frac{D_i}{Q_i} \quad \frac{1}{2}C1iQ_i$

Hence the total average cost for n items is given by

$$C \quad \underset{i\ l}{\overset{n}{}}\ C_i$$

i.e., $$C \quad \underset{i\ 1}{\overset{n}{}}\ C_{3i}\frac{D_i}{Q_i} \quad \frac{1}{2}C_{1i}Q_i$$

Here C is a function of Q_1, Q_2, ... Q_n

For optimum values of Q_i (i = 1, 2, ..., n), we must have $\frac{c}{Q_i} \quad 0$

i.e., $\frac{1}{2}C1\mathrm{i} - C_{3i}\frac{D_i}{Q_i^2} \quad 0$

Or, $Q_i = \sqrt{\frac{2C_{3i}D_i}{C_{1i}}}$

Hence the optimum value of Q_i is

$$Q_i = \sqrt{\frac{2C_{3i}D_i}{C_{1i}}}$$

7.6 Limitation on Inventories

If there is a limitation on inventories that requires that the average number of all units in inventory should not exceed K units of all types, then the problem is to minimize the cost C subject to the condition that

$\frac{1}{2}\underset{i\ 1}{\overset{n}{}} Q_i \quad K$ [since the average number of at any time for an item is $\frac{1}{2}Qi$]

or, $\underset{i\ 1}{\overset{n}{}} Q_i \quad 2k \quad 0$ ...... (1)

Now two cases may arise :

Case-I : when $\frac{1}{2}\underset{i\ 1}{\overset{n}{}} Q{*}i \quad K$

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In this case, the optimal values Q^*_i (i = 1, 2, ... n) given by

$$Q^*_i = \sqrt{\frac{2C_{3i}D_i}{C_{li}}} \text{ satisfy the constraints directly.}$$

Case-II : when $\frac{1}{2}\sum_{i=1}^{n} Q^*i > K$

In this case, we have to solve the following problem :

Minimize $C = \sum_{i=1}^{n}\left(\frac{1}{2}C_{1i}Q_i + C_{3i}\frac{D_i}{Q_i}\right)$ subject to the constraint (2).

To solve it, we shall use the Lagrange multiplier method and the corresponding Lagrange function is

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}C_{1i}Q_i + C_{3i}\frac{D_i}{Q_i}\right) + \lambda\left(\sum_{i=1}^{n} Q_i - 2K\right)$$

where λ (>0) be the Lagrange multiplier.

The necesary condition for L to be minimum is

$$\frac{\partial L}{\partial Q_i} = 0, \quad i = 1, 2, ..., n$$

and $\quad \frac{\partial L}{\partial \lambda} = 0$

Now, from $\frac{\partial L}{\partial Q_i} = 0$ we have

$$\frac{1}{2}C_{1i} - \frac{C_{3i}D_i}{Q_i^2} + \lambda = 0, \quad i = 1, 2, ..., n$$

or, $\quad Q_i = \left(\frac{2C_{3i}D_i}{C_{1i} + 2\lambda}\right)^{\frac{1}{2}}$

Again from $\frac{\partial L}{\partial \lambda} = 0$ we have

$$\sum_{i=1}^{n} Q_i - 2K = 0$$

or, $\sum_{i=1}^{n} Q_i = 2K$

Hence the optimum value of Q_i is

$$Q^*_i = \left(\frac{2C_{3i}D_i}{C_{1i} + 2\lambda^*}\right)^{\frac{1}{2}} \qquad \text{...... (3)}$$

and $\sum_{i=1}^{n} Q_i = 2K$ ...... (4)

To obtain the values of Q^*_i from (3) we find the optimal value of λ^* of λ by successive trial and error method of linear interpolation method, subject to the condition given by (4). This equation (4) implies that Q^*_i must satisfy the inventory constraint in equality sense.

7.6.1 Limitation of floor space (or Warehouse Capacity)

Hence we shall discuss the multi-item inventory model with the limitation of warehouse floor space. Let A be the maximum storage area available for the n different items, a_i be the storage area required per unit of i-th item, Q_i be the amount ordered for the i-th item.

Thus the storage requirement constraint becomes

$$\sum_{i=1}^{n} a_iQ_i \le A, \quad Q_i > 0$$

$$\text{or,} \quad \sum_{i=1}^{n} a_iQ_i - A \le 0 \qquad \text{..... (5)}$$

Now two possibilities may arise :

Case-1 : when $\sum_{i=1}^{n} a_iQ^*_i \le A$

In this case, the optimal value Q^*_i (i = 1, 2, ..., n) given by

$Q^*_i = \sqrt{\frac{2C_{3i}D_i}{C_{1i}}}$ satisfy the constraint directly. Hence these optimal values Q^*_i are the required values.

Case-2 : When $\sum_{i=1}^{n} a_i Q^*i > A$

In this case, we have to solve the problem as follows :

$$\text{Minimize } \sum_{i=1}^{n} \left(\frac{1}{2} C_{1i} Q_i + C_{3i} \frac{D_i}{Q_i} \right) \text{ subject to the constraint (5).}$$

To solve it, we shall use the Lagrange multiplier method and the corresponding Lagragian function is

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} C_{1i} Q_i + C_{3i} \frac{D_i}{Q_i} \right) + \lambda \left(\sum_{i=1}^{n} a_i Q_i - A \right)$$

where λ (> 0) is the Lagrange multiplier.

The necessary condition for L to be minimum is

$$\frac{\partial L}{\partial Q_i} = 0, \; i = 1, 2, \ldots\ldots, n$$

$$\text{and} \quad \frac{\partial L}{\partial \lambda} = 0$$

Now from $\frac{\partial L}{\partial Q_i} = 0$ we have

$$\frac{1}{2} C_{1i} - \frac{C_{3i} D_i}{Q_i^2} + \lambda a_i = 0 \qquad \ldots\ldots \text{(6a)}$$

Again, from we have $\frac{\partial L}{\partial \lambda} = 0$

$$\sum_{i=1}^{n} a_i Q_i - A = 0 \qquad \ldots\ldots \text{(6b)}$$

Solving (6a) and (6b), we have the optimal values of Q_i is

$$Q^*_i = \left(\frac{2 C_{3i} D_i}{C_{li} + 2\lambda^* a_i} \right)^{\frac{1}{2}}, \; i = 1, 2, \ldots\ldots, n \qquad \ldots\ldots \text{(7)}$$

$$\text{and} \quad \sum_{i=1}^{n} a_i Q^*_i = A \qquad \ldots\ldots \text{(8)}$$

To obtain the values of Q^*_i from (7) we find the optimal value λ^* of λ by successive trial and error method or linear interpolation method subject to the condition given by (8). The equation (8) implies that Q^*_i must satisfy the inventory constrain in equality sense.

7.6.2 Limitation on Investment

In this case, there is an upper limit M on the amount to be invested on inventory. Let C_{4i} be the unit price of the i-th them then

$$\underset{i\ \ 1}{\overset{n}{}} C_{4i}Q_i \quad M \qquad \text{...... (9)}$$

Now two possibilities may arise :

Case-I : when $\underset{i\ \ 1}{\overset{n}{}} C_{4i}Q^*_i \quad M$

In this case, the constraint is satisfied by Q^*_i automatically. Hence the optimal values of Q^*_i are given by

$$Q^*_i = \sqrt{\frac{2C_{3i}D_i}{C_{1i}}} \qquad \frac{1}{2}C_1.200+\frac{C_3D}{200}$$

Case-II : when $\sum_{i=1}^{n} C_{4i}Q^*i > M$

In this case, our problem is as follows :

Minimize $C \quad \underset{i\ \ 1}{\overset{n}{}} \frac{1}{2}C_{1i}Q_i \quad C_{3i}\frac{D_i}{Q_i}$ subject to the containt (9).

To solve it, we shall use the Languale multipler method and the corresponding Lagrangian function is

$$L = \underset{i\ \ 1}{\overset{n}{}} \frac{1}{2}C_{1i}Q_i \quad C_{3i}\frac{D_i}{Q_i} \qquad \underset{i\ \ 1}{\overset{n}{}} C_{4i}Q_i \quad M$$

where λ (>0) is the Lagrange multiplier.

The necessary condition for L to be minimum is

$$\frac{L}{Q_i} \quad 0 \;\; i = 1, 2, \text{......}, n \;\; \text{and} \;\; \frac{L}{} \quad 0$$

Now from $\frac{\partial L}{\partial Q_i} = 0$ we have

$$Q_i = \left(\frac{2C_{3i}D_i}{C_{1i} + 2\lambda C_{4i}}\right)^{\frac{1}{2}}, \; i = 1, 2, \ldots\ldots, n$$

Again, from $\frac{\partial L}{\partial \lambda} = 0$ we have

$$\sum_{i=1}^{n} C_{4i}Q_i - M = 0 \text{ or, } \sum_{i=1}^{n} C_{4i}Q_i = M$$

Hence the optimum value of Q_i is

$$Q^*i = \left(\frac{2C_{3i}D_i}{C_{1i} + 2\lambda C_{4i}}\right)^{\frac{1}{2}} \qquad \ldots\ldots (10)$$

$$\text{and} \quad \sum_{i=1}^{n} C_{4i}Q^*_i \le M \qquad \ldots\ldots (11)$$

Thus the values of Q^*_i are obtained from (10) subject to the condition given by (11) where the optimal vlaue λ^* of λ is found by successive trial and error method or linear interpolation method.

Example 5 : A workshop produces three machine parts *A, B, C,* the total storage space available is 640 sq. meters. Obtain the optimal lot-size for each item from the following data :

	Items		
	A	*B*	*C*
Cost per unit (Rs.)	10	15	5
Storage space required (sq.meter/unit)	0.60	0.80	0.45
Procedurement cost (Rs.) (C_3)	100	200	75
No. of units required/year	5000	2000	10,000

The carrying charge on each item is 20% of unit cost.

Solutions : Considering one year as one unit of time, we have, the carrying charge of A is C_{11} = Rs. (20% of 10) = Rs. 2

Carrying charge of B is C_{12} = Rs.(20% of 15) = Rs. 3

Carrying charge of C is C_{13} = Rs.(20% of 5) = Rs. 1

Now, without considering the effect of restriction on storage space availability, the optimal value Q^*_i of i-th item is given by

$$Q^*_i = \sqrt{\frac{2C_{3i}D_i}{C1_i}}, \quad i = 1, 2, 3$$

$$\therefore \quad Q^*_1 = \sqrt{\frac{2D_iC_{31}}{C_{11}}} \quad \sqrt{\frac{2 \quad 5000 \quad 100}{2}} \quad 707$$

$$Q^*_2 = \sqrt{\frac{2.2000.200}{3}} \quad 516$$

$$Q^*_3 = \sqrt{\frac{2 \quad 10000 \quad 75}{1}} \quad 1225$$

Then the total storage space required for the above values of Q^*_i (i = 1, 2, 3) is

$$\sum_{i \; 1}^{3} a_iQ^*_i = a_1Q^*_1 + a_2Q^*_2 + a_3Q^*_3$$

$$= (.60 \times 707 + .80 \times 516 + .45 \times 1225) \text{ sq. meter}$$

$$= 1388.25 \text{ sq. meters.}$$

This storage space is greter than the available storage space 640 sq. meters. Therefore, we shall try to find the suitable value of λ by trial and error method for computing Q^*_i by using

$$Q^*_i \quad \frac{2C_{3i}D_i}{C_{1i} \quad 2 \quad * a_1}^{\frac{1}{2}}$$

and $$\sum_{i \; 1}^{3} a_iQ^*_i \quad 640$$

If we take λ*= 5 then

$$Q^*_1 = \sqrt{\frac{2C_{31}D_i}{C_{1i} \quad 2 \quad 5 \quad a_1}} \quad \sqrt{\frac{2 \quad 5000 \quad 100}{2 \quad 2 \quad 5 \quad .60}} \quad 354$$

$$Q^*_2 = \sqrt{\frac{2 \quad 2000 \quad 200}{3 \quad 2 \quad 5 \quad .80}} \quad 270$$

$$Q^*_3 = \sqrt{\frac{2 \quad 1000 \quad 75}{1 \quad 2 \quad 5 \ \cdot 45}} \quad 522$$

Hence the corresponding storage space is .60 × 354 + .80 × 270 + .45 × 522 = 663.3 sq. meters. This storage space is greater than the availabe storage space 640 sq. meters.

If we take $\lambda^* = 6$, then

$$Q^*_1 = \sqrt{\frac{2 \quad 5000 \quad 100}{2 \quad 2 \quad 6 \ \cdot 60}} \quad 330$$

$$Q^*_2 = \sqrt{\frac{2 \quad 2000 \quad 200}{3 \quad 2 \quad 6 \ \cdot 80}} \quad 252$$

$$Q^*_3 = \sqrt{\frac{2 \quad 1000 \quad 75}{1 \quad 2 \quad 6 \ \cdot 45}} \quad 484$$

Hence the corresponding storage space is .60 × 330 + .80 × 352 + .45 × 484 = 617.4 sq. meters. Which is less than the available storage space 640 sq. meters.

Hence it is clear that the most suitable values of λ lies between 5 and 6.

Let us assume that the required storage space wil be 640 sq. meters for $\lambda^* = x$.

Now considering the linear relationship between the value of λ and the required stroage space, we have

$$\frac{x \quad 6}{640 \quad 617.4} \quad \frac{5 \quad 6}{663.3 \quad 617.4}$$

or, $x \quad 6 \quad \dfrac{22.6}{45.9}$ or, $x = 5.5$ (approx.)

or, $\lambda^* = 5.5$

For this value of λ^*,

$$Q^*_1 = \sqrt{\frac{2 \quad 5000 \quad 100}{2 \quad 2 \quad 5.5 \quad 0.60}} \quad 341$$

$$Q^*_2 = \sqrt{\frac{2 \quad 2000 \quad 200}{2 \quad 2 \quad 5.5 \quad 0.80}} \quad 272$$

$$Q^*_3 = \sqrt{\frac{2 \quad 1000 \quad 75}{1 \quad 2 \quad 5.5 \ \cdot 45}} \quad 502$$

Hence the optimal lot-size of three machine parts *A, B, C,* are Q^*1 =341 units, Q^*_2 = 272 units and Q^*_3 = 502 units.

Example 6 : A company producing three items has a limited inventories of averagely 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is given :

Product	1	2	3
Holding cost (Rs.)	0.05	0.02	0.04
Set-up cost (Rs.)	50	40	60
Demand	100	120	75

Solution : Neglecting the restriction of the total value of inventory level, we get the optimal values Q^*_i for *i*-th item which is given by

$$Q^*i = \sqrt{\frac{2C_{3i}D_i}{C_{1i}}}, i \quad 1,2.3$$

$$\therefore \quad Q^*_1 = \sqrt{\frac{2 \quad 50 \quad 100}{0.05}} \quad 447$$

$$Q^*_2 = \sqrt{\frac{2 \quad 40 \quad 120}{0.02}} \quad 693$$

$$Q^*_3 = \sqrt{\frac{2 \quad 60 \quad 75}{0.04}} \quad 474$$

Therefore, the total average inventory is (447 + 693 + 474) / 2 units = 807 units.

But the average inventory is 750 units. Therefore, we have to determine the value of parameter λ by trial and error method for computing Q^*_i by using

$$Q^*_i = \sqrt{\frac{2C_{3i}D}{C_{1i} + 2\lambda^*}} \text{ and } \frac{1}{2}\sum Q^*_i = 750$$

Now, for λ = 0.005,

$$Q^*_1 = \sqrt{\frac{2 \quad 50 \quad 100}{0.05 \quad 2 \quad 0.005}} \quad 408$$

Q^*_2 = 566, Q^*_3 = 424

Therefore, the total average inventory is $\frac{1}{2}$ (408+566+424) = 699 units which is less than the given average inventory of items.

Again, for $\lambda = 0.003$,

$$Q^*_1 = 423,\ Q^*_2 = 608,\ Q^*_3 = 442$$

and average inventory = $\frac{1}{2}$ (423 + 608 + 442) = 737 which is less than 750 units.

Again, for $\lambda = 0.002$ then $Q^*_1 = 430$, $Q^*_2 = 632$, $Q^*_3 = 452$

and average inventory = $\frac{1}{2}$(430 + 632 + 452) = 757 which is greater than 750 units.

Therefore, the most suitale value of λ lies between 0.002 and 0.003.

Let us assume that for $\lambda^* = x$, the average inventory will be 750.

Now, considering the linear relationship between λ and average inventory, we have

$$\frac{x \quad 0.003}{750 \quad 737} \qquad \frac{\cdot 003 \quad 0\cdot 002}{737 \quad 757}$$

$$\text{or,} \qquad x - \cdot 003 = \frac{13 \quad \cdot 001}{20}$$

$$\text{or,} \qquad x = \cdot 00235 \text{ or, } \lambda^* = 0\cdot 00235$$

$$\text{For,} \qquad \lambda^* = 0\cdot 00235,$$

$$Q^*_1 = 428,\ Q^*_2 = 623,\ Q^*_3 = 449$$

7.7 Inventory Models with price breaks

So far we have assumed that the unit production cost or unit purchase cost is constant in the earlier discussion. So, we need not consider this cost in the analysis. However, in the real world, it is not always true that the nit cost of an item is independent of the quantity procured or produced. Again, discounts are offered by the supplier or wholesaler or manufacturer for the pruchase of large quantities. Such discounts are referred to as quantity discoutns or price breaks.

In this section, we shall consider a class of inventory in which cost is a variable factor when items are purchase in bulk, some discount price is usually offered by the supplier.

Let us assume that the unit purchase cost of an item is p_j when the purchased quanitity lies between b_j - 1 and b_j (j = 1, 2, ..., m). Explicitly, we have

quantity purchased	unit price
$b_0\ Q < b_1$	p_1
$b_1\ Q < b_2$	p_2
$b_2\ Q < b_3$	p_3
....	....
$b_j - 1 \leq Q < b_j$	P_j
....	....
$b_m - 1 \leq Q < b_m$	p_m

In general, $b_0 = 0$ and $b_m = \propto$ and $p_1 > p_2 > ... > p_j > ... > p_{m.}$

The values b_1, b_2, b_3 ...... b_{m-1} are termed as price breaks as unit price falls at these values.

Our problem is to determine an economic order quantity Q which minimizes the total cost. In the present model, the purchasing price is to be include in the total cost.

In these models, the assumptions are—

(i) Demand rate is known and uniform.

(ii) Shortages are not permitted.

(iii) Production for supply of commodities is instantaneous.

(iv) Lead time is zero.

7.7.1 Purchasing inventory model with single price break

Let D be the demand rate, C_1 be the holding cost per unit quantity per unit time, C_3 be the fixed ordering cost per order. Also, let p_1 be the purchasing cost per unit quantity if the ordered quantity is less than b and $p_2 (p_2 < p_1)$ be the purchasing cost per unit quantity if the ordered quantity is greater or equal to b quantities.

i.e., purchase cost	Range of quantity
p_1	$0 < Q < b$
p_2	$b \leq Q$

Hence the total average cost $C(Q)$ is given by

$C(Q)$ = ordering cost + purchasing cost + holding cost

i.e., $$C(Q) = \begin{cases} C'(Q) \ for\ 0 < Q < b \\ C''(Q) \ for\ Q \geq b \end{cases}$$

where $$C'(Q) = C_3 \frac{D}{Q} + p_1 D + \frac{1}{2} C_1 Q$$

and $$C''(Q) = C_3 \frac{D}{Q} + p_2 D + \frac{1}{2} C_1 Q$$

Thus $C(Q)$ has a discontinuity at $Q = b$ and it may be shown that the minimum value of $C(Q)$ occurs either where $\frac{dc(Q)}{dQ} = 0$ or, at the point of discontinuity.

We have $\frac{dc(Q)}{dQ} = -\frac{C_3 D}{Q^2} + \frac{1}{2} C_1$ except at $Q = b$ where it is not defined. Thus the optimal value of Q is given by

$$Q^* = \sqrt{\frac{2C_3 D}{C_1}} \qquad \text{...... (1)}$$

Now we consider the case in which $Q^* \geq b$ and $Q^* < b$.

(i) If Q^* [given by (1)] > b than the optimal lot-size Q^* is obtained by (1) and in this vase, the minimum total average cost given by

$$C_{\min}(Q^*) = \frac{C_3 D}{\sqrt{\frac{2C_3 D}{C_1}}} + p_2 D + \frac{1}{2} C_1 \sqrt{\frac{2C_3 D}{C_1}}$$

$$= p_2 D + \sqrt{2C_1 C_3 D}$$

(ii) If $Q^* < b$ then there may arise two cases as follows :

Case-1 : $C''(b) < C'(Q^*)$ for $Q^* < b$

Case-2 : C″ (b) > C′ (Q^*) for $Q^* < b$

Now C′ (Q^*) = $p_1 D + \sqrt{2C_1C_3D}$

and $C''(b) = p_2D + \frac{C_3D}{b} \; \frac{1}{2} \; C_1 b$

Hence, if $C''(b) > C'(Q^*)$ then Q^* given by (1) is the optimum order quantity. Otherwise, if $C''(b) < C'(Q^*)$, then b is the optimum order quantity.

Working rule :

Step-1 : Compute Q^* by the formula $Q^* = \sqrt{\frac{2C_3D}{C_1}}$ for the case $Q \geq b$ and then compare this Q^* with the value of b.

(i) If $Q^* \geq b$ then the optimum lot-size is Q^*.

(ii) If $Q^* < b$, then go to step-2.

Step-2 : Evaluate Q^* by the formula $Q^* = \sqrt{\frac{2C_3D}{C_1}}$ for the case $Q < b$ and evaluate $C'(Q^*)$ and $C''(b)$.

(i) If $C'(Q^*) < C''(b)$, then Q^* is the optimum lot-size.

(ii) Ohter wise, b is the optimal lot-size.

7.7.2 Purchase inventory model with two price breaks

Purchase cost per unit quantity	Range of Quantity to be purchased
p_1	$0 < Q < b_1$
p_2	$b_1 \leq Q < b_2$
p_3	$b_2 \leq Q$

The procedure used involving one price break is extended to the cases with two price breaks.

Working rule :

Step-1 : Compute Q^* for $Q \geq b_2$ (say Q^*_3). If $Q^*_3 \geq b_2$ then the optimal lot size is Q^*_3, otherwise, go to step-2.

Step-2 : Compute Q^* for $b_1 \leq Q^* < b_2$ (say Q^*_2). Since $Q^*_3 < b_2$ then Q^*_2 is also less than b_2. In this case, there are two possibilities i.e., either $Q^*_2 \geq b_1$ or $Q^*_2 < b_1$. If $Q^*_2 \geq b_1$ then compare the cost $C(Q^*_2)$ and $C(b_2)$ to obtain the lot-size. The quantity with lower cost will naturally be the optimum one. If $Q^*_2 < b_1$, then go to step-3.

Step-3 : If $Q^*_2 < b_1$, then compute Q^*_1 for the case $0 \leq Q < b1$ and compare the cost $C(Q^*_1)$, $C(b_1)$ and $C(b_2)$ to determine the optimal lot-size. The quantity with lower cost will naturally be the optimum one.

Example 7 : Find the optimum order quantity for a product for which the price breaks are as follows :

Q	price/unit (Rs.) (p)
$0<Q<100$	20
$100 \leq Q < 200$	18
$200 \leq Q$	16

The monthly demand for the product is 400 units. The stroage cost is 20% of the unit cost of the product and the cost of ordering is Rs. 25.00 per order.

Solution : Here D = 400 units/month, C_3 = Rs. 25.00 per order C_1 = 20% of purchase cost per unit = 0.2 time of purchase cost per unit.

$$\text{Let } Q^*_3 = \sqrt{\frac{2C_3D}{C_1}} \text{ for } Q \geq 200$$

$$= \sqrt{\frac{2 \quad 25 \quad 400}{.2 \quad 16}} \quad 79$$

Since $Q^*_3 < 200$, Q^*_3 is not the optimum order quantity. Therefore we have to proceed to calculate Q^*_2.

$$\text{Now } Q^*_2 = \sqrt{\frac{2C_3D}{C_1}} \text{ for } 100 \leq Q < 200$$

$$= \sqrt{\frac{2 \quad 25 \quad 400}{.2 \quad 18}} \quad 75$$

Again, since $Q^*_2 < 100$, therefore, Q^*_2 is not optimum order quantity.

Now we have to proceed to calculate Q^*_1

$$Q^*_1 = \sqrt{\frac{2C_3D}{C_1}} \text{ for } 0 < Q < 100$$

$$= \sqrt{\frac{2 \quad 25 \quad 400}{.2 \quad 20}} \quad 71$$

Since, $0 < Q^*_1 < 100$, so we have to compute $C\ (Q^*_1)$, C (100), C (200).

Now, $C(Q^*_1) = C'(Q^*_1) \quad = p_1D + \frac{1}{2} C_1Q^*_1 + \frac{C_3D}{Q^*1}$

$$20 \quad 400 \quad \frac{1}{2} \quad 0.2 \quad 20 \quad 71 \quad \frac{25 \quad 400}{71}$$

$$= \text{Rs. } 8282.85$$

C' (100) = C''(100) $\quad = p_2D \quad \frac{1}{2}C_1.100 \frac{C_3D}{100}$

$$= 18 \times 400 + \frac{1}{2} \times \cdot 2 \times 18 \times 100 + \frac{25 \times 400}{100}$$

$$= \text{Rs. } 7480.00$$

and $C(200) = C'''(200) = p_3D + \frac{1}{2}C_1.200 + \frac{C_3D}{200}$

$$16 \quad 400 \quad \frac{1}{2} \quad .2 \quad 18 \quad 100 \quad \frac{25 \quad 400}{100}$$

$$= \text{Rs. } 6770.$$

Since $C(200) < C\ (100) < C\ (Q^*_1)$, then the optimal order quantity is 200 i.e., $Q^* = 200$.

7.8 Probabilistic Inventory Model

Now we consider the situations when the demand is not known exactly but the probability distribution of demand is somehow known. The control variable in such cases is assumed to be either the scheduling period or the order level or boty. The optimum order levels will thus be derived byu minimizing the total expected cost rather than the actual cost involved.

7.8.1 Suingle period model with uniform demand (No set-up cost model)

In this model, we have to find the optimum order quantity so as to minimize the totoal expected cost with the following assumtpions :

(i) Scheduling peirod T is fixed and known. Hence it is a proscribed constant, so we do not include the set-up in our derication as C_3 is a constant.

(ii) Production is instantaneous.

(iii) Lead time is zero.

(iv) The demand is uniformly didstibuted over the period.

(v) Shortages are allowed and fully backlogged.

(vi) The holding cost, C_1 per unit quantity per unit time and the shortage cost, C_2 per unit quantity per unit time are known and constant.

Let x be the amount 9on hand hefore an oerder is placed and C_3 be the purchaising cost per unit quantity.

Let Q be the level of inventory in the beginning of each period and r units is the demand oper time peirod. Depending on the amount D, we amy have two cases :

Case 1 : $r \leq Q$ Case 2 : $r > Q$

In both cases, the inventory situation is shown in Fig. 7.5 and Fig. 7.6 respectively.

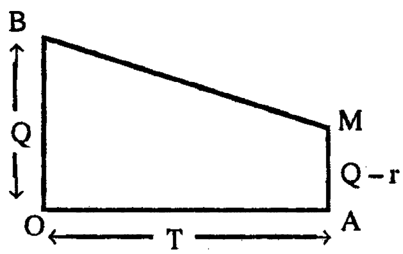

Fig. 7.5

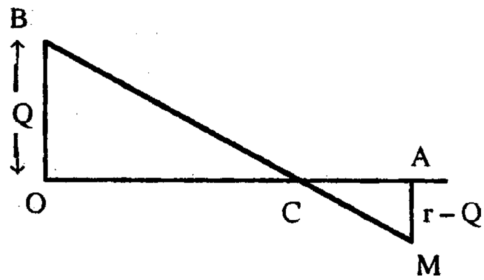

Fig. 7.6

In the first case $r \leq Q$ as shown in Fig. 7.5, no shortage occurs. In the second cases, $r > Q$ as shown in Fig. 7.6, both the cost are involved.

Discreted case : when r is a discreate random variable.

Let the demand for D unikts be estimated at a discontinuous rate with probability $p(r)$, $r = 1, 2, ..., n, ...$ That is, we may expect demand for one unit with probability $P(1)$, 2 units p with probability $p(2)$ and so on. Since all the possibilitis are to be taken care of, we must have $\sum_{r=1} p(r) = 1$ and $p(r) \geq 0$.

We also assume that r is only non-negative integer.

Case 1 : In this case, $r \leq Q$. So, there is no shortage and the total inventory is represented by the total area $OAMB = \frac{1}{2}(Q + Q - r)\, T = \left(Q - \frac{r}{2}\right) T$

Hence the holding cost for the time period T is $\left(Q - \frac{r}{2}\right) TC_1$. This is the holding cost when r $(\leq Q)$ units are demanded rate in one period. But, the probability of the demand of r units is $p(r)$. Hence the expected value of this cost is $C_1\left(Q - \frac{r}{2}\right) Tp(r)$.

Now, r can have only values less than Q. Hence the total expected cost where $r < Q$ is equal to

$$\sum_{r=0}^{Q} C_1\left(Q - \frac{r}{2}\right) Tp(r)$$

Case 2 : In this case, $r > Q$, both holding and shortage costs are involved.

Here, the holding cost is $\frac{1}{2} C1Qt_1$ and the shortage cost is $\frac{1}{2} C_2(r - Q)t_2$ where t_1 and t_2 represent the no-shortage and shortage case and $t_1 + t_2 = T$.

From the similar triange OBC and ACM in Fig. 7.6, we have

$$\frac{t_1}{t_2} = \frac{Q}{r - Q} \qquad \text{or,} \quad \frac{t_1}{Q} = \frac{t_2}{r - Q}$$

$$\text{or,} \quad \frac{t_1}{Q} = \frac{t_2}{r - Q} = \frac{t_1 + t_2}{r}$$

$$\text{or,} \quad \frac{t_1}{Q} = \frac{t_2}{r - Q} = \frac{T}{r}$$

$$\text{or,} \quad t_1 = \frac{QT}{r} \text{ and } t_2 = \frac{(r - Q)T}{r}$$

Hence the expected cost in this case $(r > Q)$ is

$$\sum_{r=Q+1}^{\infty}\left[\frac{1}{2}C_1 Q t_1 p(r) + \frac{1}{2}(r-Q)\,C_2 t_2 p(r)\right]$$

$$= \sum_{r=Q+1}^{\infty}\left[C_1\frac{Q^2}{2r}Tp(r) + C_2\frac{(r-Q)^2}{2r}Tp(r)\right]$$

Therefore the average expected cost is given by

$$TEC(Q) = C_1\sum_{r=0}^{Q}\left(Q-\frac{r}{2}\right)p(r) + C_1\sum_{r=Q+1}^{\infty}\frac{Q^2}{2r}p(r)$$

$$+\; C_2\sum_{r=Q+1}^{\infty}\frac{(r-Q)^2}{2r}p(r) + C_3(Q-x) \qquad \text{..... (1)}$$

The problem is now to find Q, so as to minimize $TEC(Q)$. Let an amount $Q+1$ instead of Q be produced. Then the average total expected cost is

$$TEC(Q+1) = C_1\sum_{r=0}^{Q+1}\left(Q+1-\frac{r}{2}\right)p(r) + C_1\sum_{r=Q+2}^{\infty}\frac{(Q+1)^2}{2r}p(r)$$

$$+\; C_2\sum_{r=Q+2}^{\infty}\frac{(r-Q-1)^2}{2r}p(r) + C_3(Q+1-x)$$

But $C_1\sum_{r=0}^{Q+1}\left(Q+1-\frac{r}{2}\right)p(r) = C_1\sum_{r=0}^{Q}\left(Q+1-\frac{r}{2}\right)p(r)$

$$+\; C_1\left(Q+1-\frac{Q+1}{2}\right)p(Q+1)$$

$$= C_1\sum_{r=0}^{Q}\left(Q-\frac{r}{2}\right)p(r) + C_1\sum_{r=0}^{Q}p(r) + C_1\frac{Q+1}{2}p(Q+1)$$

Again, $C_1\sum_{r=Q+2}^{\infty}\frac{(Q+1)^2}{2r}p(r) = C_1\sum_{r=Q+1}^{\infty}\frac{(Q+1)^2}{2r}p(r) - C_1\frac{(Q+1)^2}{2(Q+1)}p(Q+1)$

$$= C_1\sum_{r=Q+1}^{\infty}\frac{Q^2+2Q+1}{2r}p(r) - C_1\frac{Q+1}{2}p(Q+1)$$

$$= C_1\sum_{r=Q+1}^{\infty}\frac{Q^2}{2r}p(r) + C_1\sum_{r=Q+1}^{\infty}\frac{Q}{r}p(r) + \frac{C_1}{2}\sum_{r=Q+1}^{\infty}\frac{p(r)}{r} - \frac{C_1}{2}(Q+1)p(Q+1)$$

Similarly, $C_2\sum_{r=Q+2}^{\infty}\frac{(r-Q-1)^2}{2r}p(r)$

$$C_2 \sum_{r=Q+1}^{\infty} \frac{(r-Q-1)^2}{2r} p(r) + 0$$

$$= C_2 \sum_{r=Q+1}^{\infty} \frac{(r-Q)^2}{2r} p(r) - C_2 \sum_{r=Q+1}^{\infty} p(r) + C_2 \sum_{r=Q+1}^{\infty} \frac{Q}{r} p(r) + \frac{C_2}{2} \sum_{r=Q+1}^{\infty} \frac{p(r)}{r}$$

Substituting these values in TEC (Q+1) and then simplifying, we get

$$TEC(Q+1) = TEC(Q) + (C_1 + C_2)\left[\sum_{r=0}^{Q} p(r) + \left(Q + \frac{1}{2}\right) \sum_{r=Q+1}^{\infty} \frac{p(r)}{r}\right] - C_2 + C_3$$

if we put $\sum_{r=0}^{Q} p(r) + \left(Q + \frac{1}{2}\right) \sum_{r=Q+1}^{\infty} \frac{p(r)}{r} = L(Q)$ .... (2)

then $TEC(Q+1) = TEC(Q) + (C_1 + C_2) L(Q) - C_2 + C_3$ .... (3)

Similarly, putting $Q - 1$ in place of Q in (1),

we have

$TEC(Q-1) = TEC(Q) - (C_1 + C_2) L(Q - 1) + C_2 - C$ .... (4)

For optimal Q, we must have

$TEC(Q+1) - TEC(Q) > 0$

i.e. $(C_1 + C_2) L(Q^*) - C_2 + C_3 > 0$ [From (3)]

or, $L(Q^*) > \frac{C_2 - C_3}{C_1 + C_2}$ .... (5)

Again, for optimal Q, we have

$TEC(Q-1) - TEC(Q) > 0$

or, $-(C_1 + C_2) L(Q^* - 1) + C_2 - C > 0$

or, $L(Q^* - 1) < \frac{C_2 - C}{C_1 + C_2}$ .... (6)

Combining (5) and (6) for optimal value of Q^*, we have

$$L(Q^* - 1) \le \frac{C_2 - C}{C_1 + C_2} \le L(Q^*)$$ .... (7)

where $L(Q) = \sum_{r=0}^{Q} p(r) + \left(Q + \frac{1}{2}\right) \sum_{r=0}^{Q} \frac{p(r)}{r}$

Using the relation (7), we find the range of optimum value of Q. In these cases Q^* need not be unique. If $\frac{C_2 \quad C}{C_1 \quad C_2} \quad L(Q^*)$ then both Q^* and $Q^* + 1$ are the optimal values. Similarly, if $\frac{C_2 \quad C}{C_1 \quad C_2} \quad L(Q^* \quad 1)$ then both Q^* and $Q^* - 1$ are the optimal values.

Continuous case : when r is a continuous random variable when uncertain demand is estimated as a continuous random variable, the cost expression of inventory holding and shortage costs involves integrals instread of summation signs.

Let $f(r)$ be the probability density function for demand r which is known. The discrete point probabilities $p(r)$ are replacecd by the probability defferential $f(r)\,dr$ for small interval, say $r \quad \frac{dr}{2}, r \quad \frac{dr}{2}$. In this case, we have

$$\int_0 f(r)\,dr \quad 1 \text{ and } f(r) \geq 0$$

Let x be the amount on hand before an order is placed and C_3 be the purchasing cost per unit quantity.

Case 1 : When $r \leq Q$

Proceding as before for $r \leq Q$, the holding cost is $C_1 \quad Q \quad \frac{r}{2} \quad t$ and there is no shortage cost.

Case 2 : Where $r > Q$

Proceeding as before for $r > Q$, the holding cost is $C_1Q^2t/2r$ and the shortage cost is $C_2(r-Q)^2t/2r$.

Proceeding as before, the total expected cost per unit time is given by

$$TEC(Q) \quad \int_0^Q C_1 \quad Q \quad \frac{r}{2} \quad f(r)\,dr \quad \int_Q C_1\frac{Q^2}{2r} \quad C_2\frac{(r \quad Q)^2}{2r} \quad f(r)\,dr \quad C_3(Q \quad x)$$

$$\frac{dTEC(Q)***}{dQ} \quad C_1\int_0^Q f(r)dr \quad C_1 \quad Q \quad \frac{r}{2} \quad f(r)\frac{dr}{dQ} \quad {}_0^Q$$

$${}_{Q}\ \frac{C_1}{2r}.2Q \quad \frac{C_2}{2r}2(r \quad Q) \quad f(r)dr$$

$$\frac{C_1Q^2}{2r} \quad \frac{C_2(r \quad Q)^2}{2r} \quad f(r)\frac{dr}{dQ}\ {}_{Q} \qquad C_3$$

After simplification, we have

$$\frac{dTEC(Q)}{dQ} \quad C_1 \quad C_2 \quad {}_{0}^{Q}\ f(r)\,dr \quad (C_1 \quad C_2) \quad {}_{Q}\ \frac{Qf(r)}{r}dr \quad C_2 \quad C \qquad \text{....... (8)}$$

The necessary condition for *TEC (Q)* to be optimum is

$$\frac{dTEC(Q)}{dQ} \quad 0 \ \text{ for } Q = Q^*$$

i.e. $C_1 \quad C_2 \quad {}_{0}^{Q^*}\ f(r)dr \quad C_1 \quad C_2 \quad {}_{Q^*}\ \frac{Q^* f(r)}{r} \quad C_2 \quad C \quad 0$

or, $${}_{0}^{Q^*}\ f(r)dr \quad {}_{Q^*}\ \frac{Q^* f(r)}{r} \quad \frac{C_2 \quad C}{C_1 \quad C_2} \qquad \text{....... (9)}$$

Again from (8),

$$\frac{d^2TEC(Q)}{dQ} \quad C_1 \quad C_2 \quad {}_{0}^{Q}\ \frac{f(r)}{Q}dr \quad C_1 \quad C_2 \quad f(r)\frac{dr}{dQ}\ {}_{0}^{Q}$$

$$C_1 \quad C_2 \quad {}_{Q}\ \frac{}{Q} \quad \frac{Qf(r)}{r}\ dr \quad C_1 \quad C_2 \quad \frac{Qf(r)}{r}.\frac{dr}{dQ}\ {}_{Q}$$

other simplification, we have

$$\frac{d^2TEC(Q)}{dQ} \quad (C_1 \quad C_2) \quad {}_{Q}\ \frac{f(r)}{r}dr$$

= +ve quantity.

Hence the equation (9) gives the opinion value of Q for minimum expected average cost.

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* Remember that if $F(z)$ = ${}_{a(z)}^{b(z)}\ f(x,z)$ then

$$\frac{dF}{dz} \quad {}_{a(z)}^{b(z)}\ \frac{f}{dz} \quad f[b(z),z]\frac{d(z)}{dz} \quad f[a(z),z]\frac{da(z)}{dz}$$

It is called "Differentiation under the integral sign."

Example 8 : A contractor of second heand motor trucks uses to maintain a stock of trucks every month. The demand of the trucks occurs at a relatively constant rate but not in a constratnt size. The demand follows the following probability distributions :

Demand r :	0	1	2	3	4	5	6 or more
Probability $p(r)$:	·40	·24	·20	·10	·05	·01	0.0

The holding cost of an old truck in stock for one month is Rs. 100.00 and the penalty for a truck if not stupplied on the demand, is Rs. 1000.00. Determine the optimal size of the stock for the contractor.

Solution :

Q	r	$p(r)$	$\frac{p(r)}{r}$	$_{r\;Q\;1}\frac{p(r)}{r}$	$Q \quad \frac{1}{2}$	$(Q \quad \frac{1}{2}) \; _{r\;Q\;1}\frac{p(r)}{r}$	$^{Q}_{r\;0}\; p(r)$	$L(Q) \quad Q \; \frac{1}{2} \; _{r\;Q\;1}\frac{p(r)}{r} \quad ^{0}_{r\;0}\; p(r)$
0	0	.40		0.3878	0.5	0.1939	0.40	0.5937
1	1	.24	.2400	0.1478	1.5	0.2217	0.64	0.8617
2	2	.20	.1000	0.0478	2,5	0.1195	0.84	0.9595
3	3	.10	.0333	0.0145	3.5	0.05075	0.94	0.99075
4	4	.05	.0125	0.0020	4.5	0.0090	0.99	0.99900
5	5	.01	.0020	0.0	5.5	0.0	1.0	1.0
6 or more	6 or more	0.0	0.0	0.0	6.5	0.0	1.0	1.0

Here C_1 = Rs. 100.00, C_2 = Rs. 1000.00

$$\therefore \quad \frac{C_2 \quad C}{C_1 \quad C_2} \quad \frac{1000 \quad 0}{100 \quad 1000} \quad \frac{1000}{1100} \quad \frac{10}{11} \quad .9090$$

Now for optimal value of Q (say, Q^*) we must have

$$L \;\; Q^* \;\; 1 \quad \frac{C_2 \quad C}{C_1 \quad C_2} \quad L(Q^*)$$

where $L\ Q \quad \overset{Q}{\underset{r\ 0}{}}\ p(r) \quad Q\ \frac{1}{2} \underset{r\ Q\ 1}{} \frac{p(r)}{r}$

or, $L\,(Q^* - 1) < 0.9090 < L\,(Q^*)$

From the table, it is clear that for $Q = 2$, the above inequality satisfied

i.e. $L(1) < .9090 < L\,(2)$ or, $L\,(2\text{-}1) < .9090 < L\,(2)$

Hence $Q^* = 2$ i.e., optimum stock level of truck is 2.

7.8.2 Single period inventory model with discontinuous demand (no set up cost model)

In this model, we have to find the optimum order quantity which minimizes the total expected cost under the following assumptions :

(i) T is the constant interval between orders (T may also be considered as unit e.g. daily, weekly, monthly etc.)

(ii) Q is the stock level at the beginning of each period T.

(iii) Lead time is zero.

(iv) The holding cost, C_1 per unit quantity per unit time, the shortage cost, C_2 per unit quantity per unit time are known and constant.

(v) r is the demand at each interval T.

Solution : In this model it is assumed that the total demand is filled up at the beginning of the period.

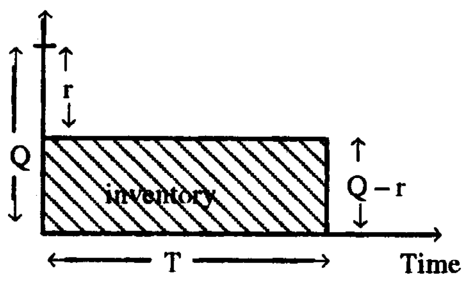

Fig. 7.7

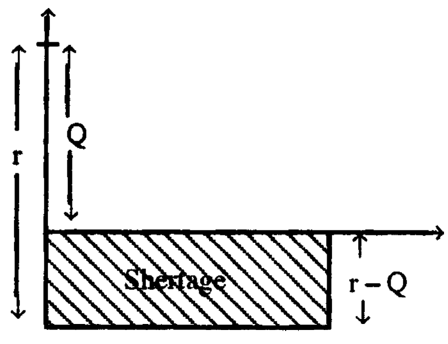

Fig. 7.8

Thus depending on the amount r demanded, the inventory position just after the demand occurs may be either positive (surplus) or negative (shortage) i.e., there are two cases : Case 1 : $r \leq Q$, case 2 : $r > Q$. The corresponsing inventory situations are shown in Fig. 7.7 and Fig. 7.8.

Discrete case *L* when r is discrete

Let r be the estimated demand at a discontinuion rate with probabilities $p(r)$. Then there is only inventory cost and no shortage cost.

Hence the holding cost is $C_1 (Q-r)$.

In the second case, the demand r is filled up at the beginning of the period. There is only shortage cost, no holding cost. Therefore, the shortage cost in this case is $C_2(r-Q)$.

Let n be the amount on hand before an order is placed and C_3 be the purchasing cost per unit quantity.

Therefore, the total expected cost for this model is

$$TEC(Q) = C_1 \sum_{r=0}^{Q} (Q-r)p(r) + C_2 \sum_{r=Q+1}^{\infty} (r-Q)p(r) + C_3(Q-x) \qquad \text{...... (1)}$$

Our problem is now to find Q, so that $TEC(Q)$ is minimum. Let an amount $Q+1$ instead of Q be ordered then the total exprcted cost given in (1) reduces to

$$TEC(Q+1) = C_1 \sum_{r=0}^{Q} (Q+1-r)p(r) + C_2 \sum_{r=Q+2}^{\infty} (r-Q-1)p(r) + C(Q+1-x)$$

on simplification, we have

$$TEC(Q+1) = C_1 \sum_{r=0}^{Q} (Q-r)p(r) + C_2 \sum_{r=Q+1}^{\infty} (r-Q)p(r) + C_1 \sum_{r=0}^{Q} p(r)$$
$$- C_2 \sum_{r=Q+1}^{\infty} p(r) + C(Q-x) + C$$

$$= TEC(Q) + (C_1 + C_2) \sum_{r=0}^{Q} p(r) - C_2 + C \quad \text{as} \quad \sum_{r=Q+1}^{\infty} p(r) = 1 - \sum_{r=0}^{Q} p(r)$$

$$TEC(Q+1) - TEC(Q) = (C_1 + C_2) \sum_{r=0}^{Q} p(r) - C_2 + C \qquad \text{..... (2)}$$

Similarly when an amount $Q - 1$ instead of Q is ordered, then we have

$$TEC(Q-1) - TEC(Q) = -(C_1 + C_2)\sum_{r=0}^{Q-1} p(r) + C_2 - C$$

or, $$TEC(Q-1) - TEC(Q) = -\left[(C_1 + C_2)\sum_{r=0}^{Q-1} p(r) - C_2 + C\right] \qquad \text{..... (3)}$$

For optimal Q^*, we must have

$$TEC\ (Q^* + 1) - TEC(Q^*) > 0$$

From (2), we have

$$(C_1 + C_2)\sum_{r=0}^{Q^*} p(r) - C_2 + C > 0$$

or, $$\sum_{r=0}^{Q^*} p(r) > \frac{C_2 - C}{C_1 + C_2} \qquad \text{..... (4)}$$

Similarly for optimal Q^*, $TEC\ (Q^*-1) - TEC\ (Q^*) > 0$

From (3), we have,

$$(C_1 + C_2)\sum_{r=0}^{Q^*-1} p(r) - C_2 + C < 0$$

or, $$\sum_{r=0}^{Q^*-1} p(r) < \frac{C_2 - C}{C_1 + C_2} \qquad \text{..... (5)}$$

Thus combining (4) and (5), we have

$$\sum_{r=0}^{Q^*-1} p(r) < \frac{C_2 - C}{C_1 + C_2} < \sum_{r=0}^{Q^*} p(r) \qquad \text{..... (6)}$$

or, $$p(r \le Q^* - 1) < \frac{C_2 - C}{C_1 + C_2} < p(r \le Q^*)$$

where $p\ (r \le Q^*)$ represents the probability for $r \le Q^*$.

Continuous case : Where r is a Continuous variable.

Let x be the on hand amount before placing the order and C_3 be the purchasing cost per unit quantity.

Let the demand r is a continuous variable with probability density function $f(r)$. then proceeding as before, the total expected cost for this model is

$$TEC(Q) = C_1 \int_0^Q (Q - r) f(r)dr + C_2 \int_Q^{\infty} (r - Q) f(r)dr + C_3(Q - x)$$

$$\therefore \quad \frac{dTEC(Q)}{dQ} = C_1 \left[\int_0^Q f(r)dr + (Q - r) f(r) \frac{dr}{dQ} \right]_0^Q$$

$$+ C_2 \left[-\int_Q^{\infty} f(r)dr + (r - Q) f(r) \frac{dr}{dQ} \right]_0 + C_3$$

$$= C_1 \int_0^Q f(r)dr - C_2 \int_Q^{\infty} f(r)dr + C$$

$$= C_1 \int_0^Q f(r)dr - C_2 \left[\int_Q^{\infty} f(r)dr - \int_0^Q f(r)dr \right] + C$$

$$= C_3 - C_2 + (C_1 + C_2) \int_0^Q f(r)dr$$

For optimum value of Q, we must have

$$\frac{dTEC(Q)}{dQ} = 0$$

i.e., $(C_1 + C_2) \int_0^Q f(r)dr = C_2 - C_3$

or, $\int_0^Q f(r)dr = \frac{C_2 - C_3}{C_1 + C_2}$ ..... (7)

More over, it can be proved that

$$\frac{d^2TEC(Q)}{dQ^2} = (C_1 + C_2) f(Q) \geq 0$$

Therefore the optimum value of Q i.e., Q^* is given by (7).

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