P.G.
(Mathematics)
SYLLABUS
General Topology VI A (Marks : 50)
PGMT-II

Topological spaces, Examples, Base for a Topology, Sub-subbase, Neighbourhood system of a point, Neighbourhood base, Limit point of a set, Closed sets, Closure of a set, Kuratowski closure operator; Interior and boundary of a set, Sub-space Topology, First and Second Countable spaces. Continuous function over a Topological space. Homeomorphism; Nets, Filters, Their convergence, Product space, Projection function, Open and Closed function, Quotient spaces.

Separation axioms T_0, T_1, T_2, T_3, T_4 in Topological spaces, Product of T_2-spaces, Regular spaces, Normal spaces, Completely regular spaces, Tychonoff spaces, Urysohns Lemma in Normal spaces, Tietze extension Theorem, Embedding in cube, Embedding Lemma, Urysohn's metrization Lemma.

Open cover, Sub-cover, Compactness, Countable open cover, Lindelöff space, Compact sets, Finite Intersection property, Tychonoff Theorem on product of compact spaces, Continuous image of a compact spaces, Locally compact spaces, One point compactification.

Connected spaces, Separated sets, Disconnection of a space, Union of connected sets, Closure of a connected set, Connected sets of reals, Continuous image of a connected space, Topological product of connected spaces, components, Totally disconnected spaces, Locally connected spaces.

Uniformity in a set, Base, Sub-base of a Uniformity, Uniform space, Uniform Topology, T_2-property of a Uniformity, Interior and closure of a set in terms of uniformity, Uniformly continuous function, Product Uniformity.

Unit 1 □ Topological Spaces

(**Topological spaces, Examples, Comparison of Topologies, Base for a Topology, its properties, Sub-base of a Topology, neighbourhood of a point, Neighbourhood system at a point, neighbourhood base, limit point if a set, closed sets, derived sets, closure of a set, Kuratowski closure operator, Interior, boundary of a set, Sub-space Topology, first and second Countable spaces**).

§1. Let X be a non-empty set. Then the family $\rho(X)$ of all subsets of X, including the empty set is called the **Power set of X.**

For example if X = {a, b, c}, then its power set $\rho(X)$ = {ϕ, X, {a}, {b}, {c}, {a, b}, {b, c} {c, a}}, contains 8 members. In General, if X consists of n (distinct) members, then its Power set $\rho(X)$ consists of 2^n member. This result is due to cantor.

Definition 1.1. A sub-family τ of $\rho(X)$ is called a Topology on X if
(T.1) ϕ, X $\in \tau$;

(T.2) Union of any number of members of τ is a member of τ; and
(T.3) Intersection of any two members of τ is a member of τ.

If τ is a Topology on X, the pair (X, τ) is called a Topological space.

Explanation : For any non-empty set X, the power $\rho(X)$ satisfies all conditions (T.1) – (T.3) and forms a Topology on X; This Topology is called the **Discrete Topology on X.** Similarly, the sub-family of $\rho(X)$ consisting of ϕ and X only also forms a Topology on X called **Indiscrete Topology on X.** But any sub-family of $\rho(X)$ does not form a Topology on X. For example, the sub-family comprising of empty set only does not form a Topology on X.

Example 1.1. Let X = {a, b, c} and τ = {ϕ, X, {a}}. Then τ is a Topology on X.

Definition 1.2. If (X, τ) is a Topological space, members of τ are called open sets in X.

Explanation : In a Topological space (X, τ) qualification for a subset to be on open set is exclusively its membership in τ. The more numerous is

τ in members the more open sets are there in X. Thus in discrete topology every subset, including singletons, is an open set; but open sets are scarce in Indiscrete Topology in X.

§2. Comparison of Topologies :

If τ_1 and τ_2 are two topologies on X, then τ_1 is said to be weaker than τ_2 (or, τ_2 stronger than τ_1 or, τ_1 is courser than τ_2 or, τ_2 is finer than τ_1) **if every member of τ_1 is a member of τ_2.**

or, in symbol $\tau_1 \subset \tau_2$.

So Indiscrete Topology is the weakest and Discrete topology the strongest Topology on X, and **any other Topology τ on X** satisfies **Indiscrete Topology** $\bar{\in} \tau \subset$ **Discrete Topology.**

Example 2.1. Cofinite Topology on X.

Let X be an infinite set and τ consists of empty set ϕ and those subsets A of X such that X\A is a finite set. Then (T.1) axiom is satisfied. Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of members of τ, and A $= \underset{\alpha \in \Delta}{\cup} A_\alpha$; Then X\A $= X \backslash \underset{\alpha \in \Delta}{\cup} A_\alpha = \underset{\alpha \in \Delta}{\cap}(X \backslash A_\alpha) \subset (X \backslash A_\alpha)$ for every α, and r h s is a finite set; So X\A is a finite set. Thus A $\in \tau$. Thus (T.2) condition is satisfied. Finally, Let A_1, A_2 $\in \tau$, then $X/(A_1 \cap A_2) = (X \backslash A_1) \cup (X \backslash A_2) = $ a Union of two finite sets = a finite set. Hence $A_1 \cap A_2$ is a member of τ. Hence τ is a Topology on X. This Topology is called Co-finite Topology on X.

Theorem 2.1. Intersection of any number of Topologies on X is a Topology on X.

Proof : Let $\{\tau_\alpha\}_{\alpha \in \Delta}$ be a collection of Topologies on X and put $\tau = \underset{\alpha \in \Delta}{\cap} \tau_\alpha$. Clearly (T.1) axion is O.K. for τ; and so is the case with (T.2). For (T.3) take U and V as two members of τ, and then they are members of each τ_α for $\alpha \in \Delta$. Since τ_α is a Topology on X, we have U $\cap$ V belongs to τ_α for every member α of Δ; and hence U $\cap$ V is a member of $\tau = \underset{\alpha \in \Delta}{\cap} \tau_\alpha$. So, τ is a Topology on X.

Remark : Theorem 2.1 is not true if word "intersection" is replaced by "Union". Following example supports this contention.

Example 2.2. : Take $X = (a, b, c)$ and $\tau_1 = \{\phi, X, (a), (a, b)\}$ and $\tau_2 = \{\phi, X, (c), (b, c)\}$. Then τ_1 and τ_2 are topologies on X such that $\tau_1 \cup \tau_2 = \{\phi, X, (a), (c), (a, b), (b, c)\}$. As $(a) \cup (c) = (a, c)$ is not a member of $\tau_1 \cup \tau_2$ we find $\tau_1 \cup \tau_2$ is not a Topology on X i.e. **Union of Two topologies may not be a Topology.** To solve a problem in a Topological space (X, τ) sometimes it suffices to know and use a part of τ called a base for Topology τ that we presently define.

Definition 2.1. : A family $\wp$ of member of τ in a Topological space (X, τ) is called **a base of the Topology** τ if and only if **every member of** τ **is a Union of some members of** $\wp$.

Members of $\wp$ are called basic open sets.

For example, the family of all singletons is a base of the discrete Topology on X.

Theorem 2.2. : In a Topological space (X, τ) a sub-family $\wp$ of τ is a base of τ if and only if for any open set G in τ, and for any member $u \in G$, there is a member $B \in \wp$ to satisfy $u \in B \subset G$.

Proof : Condition is necessary : Suppose $\wp$ is a base for τ and G is a member of τ. So G is a Union of some members of $\wp$, say, $G = G = \bigcup_{\alpha \in \Delta} \{B_\alpha : B_\alpha \in \wp\}$. If $u \in G$, there is a member, say B_{α_0} for some $\alpha_0 \in \Delta$ such that $u \in B_{\alpha_0} \in \wp$; clearly $u \in B_{\alpha_0} \subset G$.

Condition is sufficient : Suppose condition as stipulated holds. Take a member G in τ. Then for every $p \in G$, we find a member, say, B_p from $\wp$ such that $p \in B_p \subset G$. So we can write $G \subset \bigcup_{p \in G} B_p$ and, of course, converse is true i.e. $\bigcup_{p \in G} B_p \subset G$. Combining them we have $G = \bigcup_{p \in G} B_p$ = a Union of members of $\wp$. Thus $\wp$ is a basis of τ.

Remark : A Given Topology on X may have Different bases. Also there is a caution. Not any family of subsets of X is a base for a Topology. For example, take $X = (a, b, c)$ and then the family φ consisting of (a, b), (b, c),

3

X and empty set ϕ **fails to form a base for a Topology on X.** Because if φ is a base for some topology on X, Unions of members of φ shall constitute a Topology on X, and this is not the case here. Intersection property is failing here $(\mathbf{a, b}) \cap (\mathbf{b, c}) \notin \varphi)$.

We have following Theorem in this connection.

Theorem 2.3. A family $\wp$ of subsets of X forms a base for some topology on X if and only if

(i) $\phi \in \wp$ (ii) X is a union of members of $\wp$ and (iii) Given any two members B_1 and B_2 in $\wp$, and $x \in (B_1 \cap B_2)$, there is a member $B_3 \in \wp$ such that $x \in B_3 \subset (B_1 \cap B_2)$.

Proof : Necessary part follows from the Definition 2.1 and Theorem 2.2. For sufficient part suppose the $\wp$ satisfies the stated conditions, and let τ be the family of all posible unions (finite or infinite) of members of $\wp$. We check that τ is a Topology on X. For that purpose we atonce see that (T.1) is O.K.; and (T.2) is also clear in favour of τ. For (T.3) take two members C and D in τ; If $x \in (C \cap D)$, since C is a union of members of $\wp$, we find $U \in \wp$ such that

$$x \in U \subset C. \qquad ..... \ (1)$$

Similarly find a member $V \in \wp$ such that

$$x \in U \subset D. \qquad ..... \ (2)$$

From (1) and (2) and by hypothesis, we find a member $W \in \wp$ satisfying $x \in W (U \cap V) \subset (C \cap D)$. That means, we can write $C \cap D$ as a Union of members of $\wp$ showing that $(C \cap D) \in \tau$. Hence τ is a topology on X and $\wp$ is a base of τ. The proof is complete.

Example 2.3. The family of all open intervals along with ϕ forms a base for a Topology on the set R of reals known as usual or Euclidean Topology of R.

Solution : We assume ϕ as a member of this family. Take $n = 1, 2, ...$ we see that the union $(-1, 1) \cup (-2, 2) \cup ... \cup (-n, n) \cup ... = \bigcup_{n=1}^{\infty}(-n, n)$ is equal

4

to R; Further if (a, b) and (c, d) are two open intervals and $x \in (a, b) \cap$ (c, d), then open intervals intersect. Making n appropriately large, we make open interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ so small that $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset (a, b)$ and $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ $\subset (c, d)$ i.e. $\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \subset (a, b) \cap (c, d)$. Hence all conditions of Theorem 2.3 are fulfilled.

Theorem 2.4. A Topology τ_1 with a base $\wp_1$ is stronger than a Topology τ_2 with a base $\wp_2$ if and only if for $p \in X$ and for $V_2 \in \wp_2$ with $p \in V_2$, there is a member $V_1 \in \wp_1$ such that $p \in V_1 \subset V_2$.

Proof : The condition is sufficient : Let G be any member of τ_2 and $p \in G$. By base property we find a member $V_2 \in \wp_2$ with $p \in V_2 \subset G$. By stated condition we find a member $V_1 \in \wp_1$ such that

$$p \in V_1 \subset G.$$

So we can write $G = U\{V_1 \in \wp_2 : V_1 \subset G\}$. Hence $G \in \tau_1$. Thus $\tau_2 \subset \tau_1$ i.e. τ_1 is stronger than τ_2.

The condition is necessary : Let $\tau_2 \subset \tau_1$; So every member V_2 of $\wp_2$ being a member of τ_2 is a member of τ_1 whose base is $\wp_1$. So for $p \in V_2$, we find a member $V_1 \in \wp_1$ such that

$$p \in V_1 \subset V_2.$$

Example 2.4. All left-open (and right closed) intervals like **(a, b] (a, b $\in$ R and a $<$ b)** along with ϕ form **a base** for a Topology called **the upper limit Topology of R which is stronger than usual topology of R.**

Solution : Here $\overset{\infty}{\underset{n=1}{U}}(-n, n]$ equals to R. If (a, b] and (c, d] are two such intervals and $u \in (a, b] \cap (c, d]$, then left-open intervals do intersect. Then taking n appropriately large, we make left-open interval $(u - \frac{1}{n}, u]$ so small that $(u - \frac{1}{n}, u] \subset (a, b] \cap (c, d]$; and therefore Theorem 2.3 applies. Further if (a, b) is an interval as a base member of usual Topology, and $p \in (a, b)$; we

5

find a base member (a, c] of upper limit Topology such that p ∈(a, c] ⊂ (a, b). So Theorem 2.4 applies for desired conclusion.

Example 2.5. All right-open intervals like [a, b) a < b, together with φ form a base for a Topology called the lower limit Topology of R which is stronger than usual Topology of R.

Solution : Similar to that of Example 2.4.

Definition 2.2. A family $S\wp$ of subsets of X is said to form a sub-base for a Topology τ of X, if and only if the family $\wp$ of all finite intersections of subsects in $S\wp$ forms a base for τ.

Members of $S\wp$ are called sub-basic open sets.

Example 2.5. Let $S\wp$ consists of all half rays like (−∞, a) and (a, ∞) as a ∈R. Show that $S\wp$ forms a sub-base for a Topology of R(which Topology?).

Theorem 2.5. A Collection $S\wp$ of subsects of X is a sub-base for a Topology on X if and only if (a) φ ∈$S\wp$ and (b) X is the Union of some members of $S\wp$.

Proof : If $S\wp$ is a sub-base for a Topology, then ofcourse (a) and (b) hold. Conversely, let a family $S\wp$ of subsets of X obeys (a) and (b), and let $\wp$ denote the family of all finite intersections of members of $S\wp$. Then we have φ ∈ $\wp$ and X is a Union of members of $\wp$. Further, if B_1, B_2 are any two members of $\wp$, let $B_1 = U_1 \cap U_2 \cap ... \cap U_{n_1}$ where U_i ∈$S\wp$ and $B_2 = V_1 \cap V_2 \cap ... \cap V_{n_2}$ where V_i ∈$S\wp$.

If x ∈($B_1 \cap B_2$); Putting $B_3 = B_1 \cap B_2$, we find B_3 as a finite intersection of members from $S\wp$; and B_3 ∈ $\wp$, satisfying x ∈B_3 ⊂ ($B_1 \cap B_2$). Then Theorem 2.3 applies to complete the proof.

Remark : The Topology τ referred to in **Theorem 2.5 is the smallest Topology on X** containing members of $S\wp$, in the sense that τ **is weaker than every Topology on X containing members of** $S\wp$.

6

§3. Neighbourhood of a point, Neighbourhood system.

Let (X, τ) be a Topological space and $x \in X$.

Definition 3.1. A subset N_x of X is called a neighbourbood (or simply Nbd) of x if there is **an open set O_x in τ** such that

$$x \in O_x \subset N_x.$$

An open set O containing x can also be regarded as a nbd of x.

Explanation : A nbd. N_x of x is always non-empty because $x \in N_x$. Also whole set X is a nbd. of each if its points. If X is infinite, then X is the only nbd. of a given point x in X when τ is Indiscrete Topology, while there are many nbds. of x in X when τ is the discrete Topology. In the real number space R with usual topology a point x has neighbourhoods like open intervals containing x.

Theorem 3.1. A subset O of X is an open set if and only if O is a nbd. of each if its points.

Proof : Let O be an open set in (X, τ) i.e. $O \in \tau$, and $x \in O$. Put $N_x = O$ and we find $x \in O \subset N_x$, that confirms N_x as a nbd. of x.

Conversely, let a subset G enjoy the property as stated; and if $x \in G$, we find a nbd. G of x : So, there exists an open set, say, $O_x \in \tau$ such that $x \in O_x \subset G$. Then we write,

$$G = \bigcup_{x \in G} O_x$$

= a Union of some members of τ, and hence G is an open set.

Notation : If $x \in X$, let $\mathcal{N}_x$ denote the family of all nbds of x in (X, τ). $\mathcal{N}_x$ is also termed as neighbourhood system at x.

Properties of $\mathcal{N}_x$.

(a) If $N_x \in \mathcal{N}_x$, then $N_x \neq \phi$;

(b) If $N_x \in \mathcal{N}_x$ and $N_x \subset H$, then $H \in \mathcal{N}_x$ (H is a member of $\mathcal{N}_x$);

(c) Intersection of two members of $\mathcal{N}_x$ is a member of $\mathcal{N}_x$;

(d) If $N_x \in \mathcal{N}_x$, there is a member $N^* \in \mathcal{N}_x$ such that $N_x \in \mathcal{N}_u$ for every member $u \in N^*$.

Properties (a) – (c) are very much evident. We need not give proof. For (d), since N_x is a nbd. of x there is an open set, say, G satisfying

$$x \in G \subset N_x \qquad \dots \qquad \dots \qquad (1)$$

Since G is open, Theorem 3.1 says that G is a nbd. of each of its points i.e. $G \in \mathcal{N}_u$ for every member $u \in G$ and by (1) it follows that $N_x \in \mathcal{N}_u$ for $u \in G$ (here $G = N^*$).

Definition 3.2. A sub-family $\wp_x$ of $\mathcal{N}_x$ is said to be **a nbd. base of x** if for every nbd. N_x of x, there is a member $B_x \in \wp_x$ such that

$$\mathbf{B_x \subset N_x.}$$

Explanation : A Given point of a Topological space (X, τ) may have more than one nbd. base. For example, in the space R of reals with usual topology a point x has a nbd. base consisting of all open intervals like $(x - \frac{1}{n}, x + \frac{1}{n})$, $n = 1, 2, \dots$; also corresponding closed intervals constitute a nbd. base at x. In Euclidean 2-space R^2 with usual Topology, we find that every point $(x, y) \in R^2$ has a nbd. base consisting of all open oriented rectangles centred at (x, y). Also all open circular discs centred at (x, y) shall form a nbd. base at (x, y).

Topology from neighbourhood axioms :

Given $X \neq \phi$ if each point x in X is associated with a family of subsets under constraints of so called nbd. axioms, one can then derived a Topology on X. Following is a Theorem in this connection.

Theorem 3.2. Let each x in X be associated with a non-empty family $\mathcal{N}_x$ of subsets N_x of X satisfying

(a) $N_x \neq \phi$, and $x \in N_x$ for every member $N_x \in \mathcal{N}_x$

(b) if $N_x \in \mathcal{N}_x$ satisfies $N_x \subset W$, then $W \in \mathcal{N}_x$

8

(c) if $N_x^{(1)}, N_x^{(2)} \in \mathcal{N}_x$, then $(N_x^{(1)} \cap N_x^{(2)}) \in \mathcal{N}_x$

(d) if $N_x \in \mathcal{N}_x$, there is a member $N^* \in \mathcal{N}_x$ such that $N_x \in \mathcal{N}_u$ for every member $u \in N^*$.

For the proof which is lengthy, any standard book may be consulted.

§4. Limit point of a set. Closed sets.

Let A be a non-empty set in a Topological space (X, τ).

Definition 4.1. An element (point) p of X is called **a limit point of** $A \neq \phi$ **if every nbd** N_p **of p** meets A at **a point other than p.** Equivalently if $N_p \cap (A\backslash\{p\}) \neq \phi$.

If p is not a limit of A, then p is said to be an Isolated print of A. In that case we find a nbd. N_p **of p such** that $N_p \cap (A\backslash\{p\}) \neq \phi$ or equivalently $N_p \cap A$ **is either** ϕ **or** $\{p\}$.

Explanation : A limit point A may or may not be a point of A. It attracts every nbd. to intersect A at a point other than p. Naturally, the more are nbds. of p, the less is the chance of p to be a limit point of A and the less are nbds. of p, the more is the chance of p to be a limit point of A. Thus in (X, τ) with τ as discrete topology, a given non-empty set A possesses no limit point in X, because open sets are too numerous; If τ is the Indiscrete Topology, the subset A attracts every member of X as its limit point; Here only non-empty open set is X only.

Example 4.1. Obtain limit points (if any) of following sets of reals in the space R of reals with usual Topology.

(a) $A = (1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...)$, (b) The set Q of all rationals in R. (c) The set N of all antural numbers and (d) A finite subset of reals.

Definition .4.2. The set of all limit points of A in (X, τ) is called the derived set of A; it is denoted by A'.

Explanation : The set A' may be empty; for example, take a finite subset A of reals with usual topology. Here A has no limit points at all i.e. $A' = \phi$.

A′ may be disjoint with A. A′ may be a part of A; for example take A = the closed unit interval [0, 1] of reals with usual Topology. Here A′ = A. The set A′ may be strictly larger than A. For example take A as the set of all rationals in real number space R with usual topology. There A′ = R which is strictly larger than A.

Definition 4.3. A subset F in topological space (X, τ) is said to be **a closed set if F′ ⊂ F** i.e. **if every limit point of F is a point of F.**

For example, every closed interval, every finite subset of reals and the set N of all natural numbers are each a closed set in R with usual Topology.

Theorem 4.1. A subset F in (X, τ) is closed if and only if X/F (Complement of F in X) is an open set in X i.e. if and only if $(X \backslash F) \in \tau$.

Proof : The condition is necessary : Let F be a closed set in (X, τ). If F is empty, then its complement X is of course an open set $\in \tau$. Let F he non-empty and $u \in (X \backslash F)$. Then u is not a limit point of F and we find a nbd, and hence an open nbd (say) N_u of u such that $N_u \cap F = \phi$; this shows that $N_u \subset (X \backslash F)$, and $(X \backslash F)$ is rendered a nbd. of u; Thus $(X \backslash F)$ becomes a nbd. of each of its points, and so $(X \backslash F)$ is an open set in X.

The condition is sufficient : Suppose X\F is an open set in X and x is a limit point of F. It possible, let $x \notin F$. So $x \in (X \backslash F)$. Thus $(X \backslash F)$ is a nbd. of x such that

$$(X \backslash F) \cap F = \phi.$$

That contradicts the assumption that u is a limit point of F. Hence the proof is complete.

Notation : In (X, τ) denote **the family of all closed sets in X by** $\mathscr{F}$.

Remark : By De-Mongan's Laws following statements are evident.

(1) $X, \phi \in \mathscr{F}$

(2) Intersection of any number of members of $\mathscr{F}$ is a member of $\mathscr{F}$

(3) Union of two members of $\mathscr{F}$ is a member of $\mathscr{F}$.

However, union of an infinite number of members of $\mathscr{F}$ may not be a member of $\mathscr{F}$. For example, in the real number space R with usual topology let us take the closed intervals.

$I_n = [\frac{1}{n}, 1]$ as $n = 1, 2, ...$ Then each I_n is a closed set of R such that

$\overset{\infty}{\underset{n=1}{\bigcup}} I_n = (0,1]$ **which is not a closed set in R.**

§5. Closure of a set in (X, τ).

Given a subset A of X, its closure denoted by $\overline{A}$ is defined as

$\overline{A} = \cap \{F \subset X : F$ **is a closed set in** $X \supset A\}.$

The R.H.S. being intersection of a number of closed sets in (X, τ) is always a closed set in X. Thus $\overline{A}$ is always **a closed set containing A, and it is the smallest closed set to contain A.**

Explanation : Closure of A = $\overline{A}$ is a closed set no matter if A is closed or not. For example, if A = open unit interval (0, 1) of reals with usual topology we find its closure $\overline{A}$ = closed interval [0, 1] which is a closed set; but A = (0, 1) is not a closed set. Clearly if A is it self closed, then $\overline{A}$ = A and converse is also true.

Theorem 5.1. $\overline{A} = A \cup \overline{A}$.

Proof : Here $A \subset \overline{A}$; So $A' \subset \overline{A}' \subset \overline{A}$ because $\overline{A}$ is closed. Hence

$$(A \cup A') \subset \overline{A} \qquad ... (1)$$

Again if $x \in \overline{A}$, then every nbd. of x meets A. non-vaculously. For, if $x \notin \overline{A}$ then there is an open nbd. $N_2 = X \backslash \overline{A}$ of x which does not meet A.

Thus x is either in A or a limit point of A.

Accordingly, $x \in A \cup A'$.

Hence $$\overline{A} \subset (A \cup A') \qquad ... (2)$$

Combining (1) and (2) we get $\overline{A} = A \cup A'$.

Definition 5.1. (a) A subset G in (X, τ) is said to be **everywhere dense** or simply **dense** in X if $\overline{G} = X$.

11

(b) Topological space (X, τ) is said to be **separable if there is a countable dense subset in X.**

For example, the set Q of all rationals in R is dense in R with respect to usual Topology of R, because of the fact that between any two reals there are many rationals.

Example 5.1. Let X be an uncountable set, and τ is a family of subsets of X consisting of ϕ, and complements of finite subsets of X. Show that (X, τ) is a Topological space where every infinite subset of X is dense.

Proof : By a routine exercise one checks that axioms of Topology T.1 – T.3 are satisfied and (X, τ) is a Topological space. Let A be an infinite set. If $x \in X$ and N_x be a nbd. of x ; We may assume N_x to be open. If $N_x \cap (A \backslash \{x\}) = \phi$, then we have $(A \backslash \{x\}) \subset (X \backslash N_x)$ or $A \subset (X \backslash N_x) \cup \{x\}$, r.h.s. being a finite set it implies that A is a finite set which is not so. Therefore, $N_x \cap (A \backslash \{x\}) \neq \phi$, that means x is a limit point of A; i.e. $\overline{A} = X$.

Kuratowski closure operator :

An operator that assigns to each subset A of X another subset of X denoted by A^c is called a Kuratouski closure operator if following four axioms known as Kuratowski closure axioms hold :

(K.1) $\phi^c = \phi$

(K.2) For any subset A of X, $A \subset A^c$

(K.3) For any two subsets A and B of X $(A \cup B)^c = A^c \cup B^c$

and (K.4) For any subset A, $(A^c)^c = A^c$.

Theorem 5.2. Let c be a Kuratowski closure operator on a non-empty set X, and Let $\mathscr{F}$ be the family of all subsets A of X for which $A^c = A$; and $\tau = \{G \subset X : (X \backslash G) \in \mathscr{F}\}$. Then τ is a Topology on X such that $A^c = \tau$-closure of A for every subset A of X.

Proof : ϕ and X belong to τ because their complements X and ϕ are respectively members of $\mathscr{F}$. Let $\{G_\alpha\}_{\alpha \in \Delta}$ be an arbitrary sub-family of τ and

12

put $G = \bigcup\limits_{\alpha \in \Delta} G_\alpha$. Then $X \backslash G = X \backslash \bigcup\limits_{\alpha \in \Delta} G_\alpha = \bigcap\limits_{\alpha \in \Delta}(X \backslash G_\alpha) \subset (X \backslash G_\alpha)$. Then $(X \backslash G)^c \subset$
$(X \backslash G_\alpha)^c = (X \backslash G_\alpha)$ because $(X \backslash G_\alpha)$ is a member of $\mathscr{F}$. This is true for every index
$\alpha \in \Delta$, and therefore $(X \backslash G)^c \subset \bigcap\limits_{\alpha \in \Delta}(X \backslash G_\alpha) = X \backslash \bigcup\limits_{\alpha \in \Delta} G_\alpha = X \backslash G$. That means
$(X \backslash G)^c = X \backslash G$ and it is a member of $\mathscr{F}$ and hence $G \in \tau$. By a similar reasoning
intersection of any two members of τ is member of τ. So τ is a topology on
X. It remains to show that for every subset A of X A^c equals to $\overline{A}$ ($= \tau$-closure
of A).

Now $\overline{A}$ = Intersection of all τ-closed sets each containing A

$\quad\quad$ = Intersection of members of $\mathscr{F}$ each containing A.

and, therefore, $\overline{A}$ is a member of $\mathscr{F}$ with $A \subset \overline{A}$.

Using Kuratowski closure axiom, $A^c \subset (\overline{A})^c = \overline{A}$ $\quad\quad$ ... (1)

Again by Kuratowski axiom $(A^c)^c = A^c \supset A$.

That means, A^c is a member of $\mathscr{F}$ with $A^c \supset A$.

This gives, $\overline{A} \subset A^c$ $\quad\quad$ ... (2)

Combining (1) and (2) we have $\overline{A} = A^c$ and the proof is complete.

Definition 5.2. (a) Given a non-empty set A of (X, τ) a point $x \in A$ is
said to be an Interior point of A if there is an open set O in τ such that
$x \in O \subset A$.

(b) Interior of A or simply Int $A = \{x \in A : x$ is an interior point of $A\}$.

Example 5.2. If A = Open interval $(0, 1) \cup \{2, 3, 4, ..., n, ...\}$

Find Int A (w.r.t. usual topology of reals).

Solution : Here every member of open unit interval $(0, 1)$ is an interior point
of A and none of its points like 2, 3, ... is an interior point; because member
2, 3, ... individually do not attract a whole open interval containing it, but
contained in A. So Int $A = (0, 1)$.

Example 5.3. In (X, τ) if G is an open set and A is any subset of X, then
$G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$ (bar denoting closure).

Solution : If $G \cap \overline{A} = \phi$, then $G \cap A = \phi$ because $A \subset \overline{A}$.

Conversely, let $G \cap A = \phi$ and $G \cap \overline{A} \neq \phi$. Take a member $u \in (G \cap \overline{A})$. Clearly $u \notin A$; So u is a limit point of A. Since $u \in G$ and G is open; G is a nbd. of u such that $G \cap A = \phi$ — that is not true. So conclusion remains valid as wanted.

Remark : Int $A \subset A \subset$ closure A where Int A is the largest open subset of A and closure A is the smallest closed (so called) superset of A to satisfy the inclusion.

Theorem 5.3. For any non-empty subset A of (X, τ).

(a) Int A is an open set

(b) A is open if and only if $A =$ Int A.

(c) If $A \subset B$, then Int $A \subset$ Int B.

(d) Int $A = X\backslash(\overline{X\backslash A})$.

Proof : The proofs for (a) – (c) are easy and left out. For (d) $\overline{X\backslash A}$ is a closed set containing $(X\backslash A)$; So complement $X\backslash(\overline{X\backslash A})$ is an open subset of A, and therefore we have

$$X\backslash(\overline{X\backslash A}) \subset \text{Int } (A) \qquad \dots \ (1)$$

Again, $A \supset$ Int A gives $(X\backslash A) \subset X\backslash$Int A, which is a closed set containing $(X\backslash A)$; and therefore,

$$(\overline{X\backslash A}) \subset (X\backslash\text{Int } A)$$

So, $X\backslash(\overline{X\backslash A}) \supset$ Int A (Taking Complement) $\qquad \dots \ (2)$

Combining (1) and (2) we produce

Int $A = X\backslash(\overline{X\backslash A})$.

Allied to closure and Interior of a set A in (X, τ), there is another operator called Boundary of A, denoted by Bdr (A) that we presently define.

Definition 5.3. Bdr $(A) =$ closure $(A)\backslash$Int (A).

$(= \overline{A} - \mathring{A}, \mathring{A}$ denoting Int $A)$.

14

For example, in Enclidean 2-space R^2 with usual topology i: A is taken an open circular disc given by

$$A = \{(x, y) ; x^2 + y^2 < r^2\}.$$

Then $\overline{A}$ = the closed circular disc = $\{(x, y) : x^2 + y^2 \leq r^2\}$, and $\mathring{A} = A$, and therefore

Bdr A = $\overline{A} \backslash \mathring{A}$

= $\overline{A} \backslash A$

= $\{(x, y) : x^2 + y^2 = r^2\}$, namely it represents the circle with radius r centred at (0, 0).

Definition 5.4. A subset G in (X, τ) is said to be nowhere dense in X if **Int (Closure G) = ϕ**

$$\text{(i.e. } (\overline{G})^\circ = \phi).$$

For Example, every finite subset of reals w.r.t. usual topology is a nowhere dense set. An infinite subset of reals may or may not be a nowhere dense set. **The set N of all natural numbers is, of course, nowhere dense.** And the set **Q of all rationals, an infinite set of reals is not nowhere dense set; because $\overline{Q}$ = whole space R of reals with usual topology.**

§6. Sub-space of (X, τ).

Let A be a subset of a topological space (X, τ). Let us put $\tau_A = \{\mathcal{U} \cap A : u \in \tau\}$.

Then it is a routine exercise to check that the collection τ_A of subsets of A forms a Topology on A as per Definition 1.1. The pair (A, τ_A) is called a sub-space of (X, τ).

We have the following Theorem in this connection.

Theorem 6.1. Let (A, τ_A) be a sub-space of (X, τ). Then (a) if $\mathcal{B}$ is a base of τ, then $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$ is a base of τ_A.

(b) A subset H is a neighbourhood (nbd) of a point $x \in A$ if and only if $H = A \cap N_x$ where N_x is a nbd. of x in (X, τ).

(c) A subset C is closed in (A, τ_A) if and only if $C = A \cap G$, where G is a closed set in (X, τ).

Proof : (a) Take $x \in A$, since $\mathcal{B}$ is a base of τ, we find a member $B \in \mathcal{B}$ such that $x \in B$; Thus $x \in (B \cap A)$, where $(B \cap A) \in \mathcal{B}_A$. Suppose $V_1, V_2 \in \mathcal{B}_A$ and $x \in (V_1 \cap V_2)$; Then $V_1 = B_1 \cap A$, and $V_2 = B_2 \cap A$, where $B_1, B_2 \in \mathcal{B}$. Since $\mathcal{B}$ is a base of τ, and $x \in (B_1 \cap B_2)$, by property of base there is a member $B_3 \in \mathcal{B}$ to satisfy $x \in B_3 \subset (B_1 \cap B_2)$; Therefore $x \in V_3 \subset (V_1 \cap V_2)$ where $V_3 = (B_3 \cap A) \in \mathcal{B}_A$; Therefore $\mathcal{B}_A$ is a base of τ_A.

(b) Let H be a nbd. of x in (A, τ_A). Thus there is a member $V \in \tau_A$ to satisfy $x \in V \subset H$. But $V = U \cap A$ for some $U \in \tau$. Put $N_x = (U \cup H)$. So $x \in U \subset N_x$, wherefrom we find N_x is a nbd. of x in (X, τ) and also $H = A \cap N_x$; The converse goes by a similar argument.

(c) Let C be a closed set in (A, τ_A). Then A\C is open in (A, τ_A) i.e. (A\C) $\in \tau_A$ and take (A\C) $= A \cap O$ where $O \in \tau$. Then $G = (X\backslash O)$ is closed in (X, τ), such that $A \cap G = A \cap (X\backslash O) = A\backslash(A \cap O) = A\backslash(A\backslash C) = C$.

Conversely, let $C = A \cap G$ where G is a closed set in (X, τ). Then $(X\backslash G)$ is open in (X, τ), and therefore $A \cap (X\backslash G) \in \tau_A$. Now $A \cap (X\backslash G) = A\backslash(A \cap G) = A\backslash C$; showing that (A\C) $\in \tau_A$ and therefore C is closed in (A, τ_A).

Example 6.1. Let G be a closed set in (X, τ), then a subset of G is closed in (X, τ) if and only if it is closed in (G, τ_G).

Solution : We know that a subset D of G given to be closed in (X, τ) is closed in (G, τ_G) if and only if $D = G \cap H$ where H is closed in (X, τ). R.H.S. set is intersection of two closed sets, and is closed in (X, τ) ; So LHS is a closed set in (X, τ).

Conversely, let a subset D of G is closed in (X, τ). Then $D = G \cap D$ is closed in (G, τ_G).

Example 6.2. Let A be an open set in (X, τ). Then a subset of A is open in (A, τ_A) if and only if it is open in (X, τ).

16

Solution : Given A as open set in (X, τ), let a subset B of A be open in (X, τ_A); So let $B = A \cap O$ where O is open in (X, τ). Now r.h.s. = Intersection of two open sets in (X, τ), and hence is an open set in (X, τ); Thus l.h.s = B is open in (X, τ).

Conversely, let B be a subset of A and B be an open set in (X, τ). As B $= A \cap B$, so B is open in (A, τ_A).

§7. First and Second Countable spaces :

Definition 7.1. A Topological space (X, d) is said to be a Second countable space if there is a countable base of the topology τ of X.

Example 7.1. The real number space R with usual topology τ is second countable.

Solution : Consider the family $\mathcal{B}_r$ of **all open intervals with end points as rational numbers.** Since the set of all rationals is countable, so is the family $\mathcal{B}_r$. As every open interval is a member of τ, we have

$$\mathcal{B}_r \subset \tau.$$

Let $x \in R$, and G is an open set $\in \tau$ with $x \in G$, we find an open interval, say (a, b) (a, b are reals) such that

$$x \in (a, b) \subset G$$

Since the set of **all** rational numbers is everywhere dense in R, we find two rationals u, v satisfying

$$a < u < x < v < b.$$

Clearly, (u, v) is an open interval with rational end points and is a member of $\mathcal{B}_r$ such that $x \in (u, v) \subset (a, b) \subset G$. Hence $\mathcal{B}_r$ is a countable base for τ, and (R, τ) is second countable.

Theorem 7.1. Every second countable Topological space is separable.

Proof : Let (X, τ) be a second countable space and let $\{B_1, B_2, ..., B_n, ...\}$ be a countable open base of τ.

Take $b_i \in B_i$, $i = 1, 2, \ldots$ and put $B = (b_1, b_2, \ldots, b_n, \ldots)$; Then B is a countable subset of X, and we show that B is dense in X i.e. $\overline{B} = X$. Take a member x $\in (X \backslash B)$, and G is an open set containing x. By base property we find a member B_i such that

$$x \in B_i \subset G.$$

So $x \neq b_i$. Thus G intersects B at a point other than x. That is to say, x is a limit point of B. So a point of X is either a point of B or a limit point of B; So $\overline{B} = X$.

Remark : The Converse of Theorem 7.1. is false. Example 7.2 supports the statement.

Example 7.2. The real number space R with lower limit Topology generated by ϕ and all right-open intervals like [a, b), a < b is separable without being second countable.

Solution : Let Q be the set of all rationals. The Q is countable; Because between any two reals there are many rationals, basic open sets like intervals [a, b), a < b includes members of Q, and hence Q is dense in R with lower limit Topology. But **this topology is not second countable.**

If possible, let $[a_i, b_i)$, $i = 1, 2, \ldots$ be a countable base for this topology $(a_i < b_i)$; Let u be a real $\neq a_i$ $(i = 1, 2, \ldots)$ and v > u; and then [u, v) is an open set of lower limit Topology such that none of $[a_i, b_i)$ satisfies $u \in [a_i, b_i) \subset [u, v)$. Because otherwise $u \leq a_i \leq p$ gives $a_i = u$ which is not the case. Hence conclusion as desired is valid in Example 7.2.

It is time to say when a Topological space is first countable.

Definition 7.2. A Topological space is said to be first countable if the nbd. system if each of its points has a countable base.

Explanation. Concerned Definitions tell us that **if a Topological space (X, τ) is second countable, then it is first countable.** But converse is not true. Because let X be uncountable, and τ is the discrete Topology. Then (X, τ) is first countable. For if $x \in X$, single nbd $\{x\}$ constitutes nbd. base

of the nbd. system $\mathcal{N}_x$ in (X, τ). But (X, τ) is not second countable. Because $\{x\}_{x \in X}$ = a family of open sets in τ possesses no countable sub-family whose union is X.

Example 7.3. The real number space **R with lower limit Topology (See Example 7.2) is a first countable space.**

Solution : Let $x \in R$, and put $\mathcal{B}_x = \{[x, r) : r \in Q$ which is the set of all rationals$\}$. Here $x < r$. Then $\mathcal{B}_x$ is a countable sub-family of the nbd. system at x in (R, τ_e), τ_e denoting the lower limit topology on R. $\mathcal{B}_x$ is a nbd. base at x; because if N_x is any nbd. of x relative to τ_e, there is a right-open interval like $[a, b)$ such that

$$x \in [a, b) \subset N_x.$$

Clearly $a \le x < b$. Take a rational r such that $x < r < b$; Then $[x, r) \in \mathcal{B}_x$ such that $x \in [x, r) \subset N_x$. Our argument is over and Example 7.2 stands.

Exercise - A
Short answer type Questions

1. Given a non-empty set X any two Topologies are Comparable. **Either** prove it **or** give a counter example.

2. If $A = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...)$ obtain limit points, if any, of A if A is given (i) the discrete Topology (ii) the indiscrete Topology and (iii) the usual topology of reals.

3. In a Topological space if for any two open sets U and V we have $U \cap V = \phi$, show that $\overline{U} \cap V = U \cap \overline{V} = \phi$, bar denoting closure.

4. Show that if X is infinite and τ is the co-finite Topology of X, any two non-empty open sets have a non-empty intersection.

5. In a Topological space (X, τ) if $A \subset X$, show that Bdr (A) = ϕ if and only if A is clo-open.

6. Find two subsets A and B of reals with usual Topology such that Int (A) $\cup$ Int (B) $\ne$ Int (A $\cup$ B).

Exercise - B

1. Let X be an infinite set with $x_0 \in X$. If φ is the family consisting of all sets that do not contain x_0 and all sets (X\F) where F is a finite subset of X. Show that (X, φ) is a Topological space where every singleton other that $\{x_0\}$ is clo-open. Also show that $\{x_0\}$ is closed but not open.

2. Show that Interior operator in a Topological space (X, τ) is subject to following conditions :

   (i) Int (X) = X   (ii) Int (A) $\subset$ A   (iii) Int (A $\cap$ B) = Int (A) $\cap$ Int (B)

   (iv) Int (Int (A)) = Int (A) for any subsets A and B of X.

3. In a Topological space (X, τ) of A $\subset$ X, show that

   (a) Int A = A\Bdr(A)   (b) $\overline{A}$ = A $\cup$ Bdr (A), bar denoting the closure and

   (c) Int A $\cap$ Bdr A = ϕ.

4. Let X be an infinite set, and for any subset A of X, let

   A^c = A when A is a finite subset of X,

   = X when A is an infinite subset of X.

   Verify that A^c satisfies Kuratowski closure axioms, and the resulting topology is the co-finite Topology on X.

5. In a Topology space (X, τ) show that (a) if G and H are open sets in X, then Int $(\overline{G \cap H})$ = Int $(\overline{G} \cup \overline{H})$, bar denoting the closure. (b) if G is open and H is dense, then $\overline{G \cap H} = \overline{G}$, bar denoting the closure.

6. If $X \neq \phi$, show that for any collection Topologies for X there is a unique largest Topology which is smaller than each member of the collection, and a unique smallest Topology which is larger than each member of the collection.

7. For any A $\subset$ X, prove that (a) $(\overline{A})^c$ = Int (A^c) and (b) $(Int A)^c = \overline{A^c}$, where "$\varsigma$" indicates complementation.

Unit 2 ❏ Continuous Functions Over Topological Spaces

> (Continuous function over Topological space, Homeomorphism, Their Characterisations, Continuity of Characteristic function, Nets, Filters, Their convergence, Mutual implications, Product spaces, Projection functions, Their properties; Open functions, Closed functions, Quotient spaces).

§1. Let (X, τ), (Y, U) denote Topological spaces.

Definition 1.1. A function $f : (X, \tau) \to (Y, U)$ is said to be continuous if **for every open set $u \in U$ in Y, $f^{-1}(u)$ is an open set $\in \tau$ in X.**

Definition 1.1(a). If $x = x_0 \in X$, then $f : (X, \tau) \to (Y, U)$ is said to be continuous at x_0 if corresponding to any nbd. W of $f(x_0) \in Y$, there is a nbd. V of x_0 in X such that $f(V) \subset W$.

If f is continuous at every point of X, then f is said to be continuous over X, or smiply $f : (X, \tau) \to (Y, U)$ is continuous.

Definition 1.2. A function $f : (X, \tau) \to (Y, U)$ is said to **be an open function if it sends open sets into open sets i.e. if $O \in \tau$, $f(O) \in U$.**

It is called **a closed function if it sends closed sets into closed sets i.e. if F is a closed set in X, then f(F) is a closed set in Y.**

Definition 1.3. A 1–1 and onto (bijective) function $f : (X, \tau) \to (Y, U)$ is said to be a **Homeomorphism if f and f^{-1} are each continuous.**

Explanation : Let (X, τ) be a discrete space, then any function $f : (X, \tau) \to (Y, U)$ becomes a continuous, because given any $u \in U$, $f^{-1}(u)$ is always an open set in discrete Topology on X. If τ_1 and τ_2 are two topologies on X such that τ_1 is stronger than τ_2. Then identity function $I : (X, \tau_1) \to (X, \tau_2)$ i.e. $I(x) = x$ for $x \in X$ is a continuous function. If f is a real-valued function of a real variable i.e., $f : R \to R$ then taking R with usual topology of reals and remembering that open intervals form a base if the Topology we

get from Definition above that f is continuous at a point $x_0 \in R$ if corresponding to $\epsilon > 0$ there is a +ve δ such that $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$ i.e. $| f(x) - f(x_0) | < \epsilon$ whenever $x_0 - \delta < x < x_0 + \delta$ i.e. $| x - x_0 | < \delta$. Thus definition of continuity of a function f as above is in agreement with $(\epsilon - \delta)$. Definition of continuity of f in classical analysis.

Example 1.1 : Let R be taken with usual topology and $f : R \to R$ be taken as

$$f(x) = 0 \text{ if } x \le 0$$
$$= x \text{ if } 0 \ x \le 1$$
$$= 1 \text{ if } x \ge 1.$$

Then f is a closed function without being open.

Theorem 1.1. Let $f : (X, \tau) \to (Y, U)$. Then following statements are equivalent (That is to say, each implies the other).

(a) f is continuous.

(b) If F is a closed set in Y, then $f^{-1}(F)$ is closed in X.

(c) If SB be a sub-base for U in Y, and $G \in SB$, then $f^{-1}(G) \in \tau$.

(d) for each nbd. W of f(x) in Y as $x \in X$, there is a nbd. V of x in X such that $f(V) \subset W$.

(e) $f(\overline{A}) \subset \overline{f(A)}$, for every subset A of X, bar denoting closure.

(f) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset B of X, bar denoting closure.

(g) f^{-1} (Int (C)) $\subset$ Int $(f^{-1}(C))$ for every subset C of Y.

Proof : (a) $\Rightarrow$ (b); so f^{-1} (an open set in Y) is an open set in X. For (b) Let F be a closed set in Y; then (Y\F) is open in Y, and by (a), f^{-1}(Y\F) is open in X i.e. $X \backslash f^{-1}(F)$ is open in X, and hence $f^{-1}(F)$ is closed in X.

(b) $\Rightarrow$ (c); Let G be a sub-basic open set in Y, then (X\G) is a closed set in Y; by (b) f^{-1}(Y\G) = $X \backslash f^{-1}(G)$ is a closed set in X and therefore its complement $f^{-1}(G)$ is an open set in X. So (c) holds.

(c) $\Rightarrow$ (d); Let $x \in X$ and Let W be a nbd. of $f(x)$, and $U \in \mathcal{U}$ such that $f(x) \in U \subset W$. Without loss of generality take U as a basic open set in Y and therefore $U = W_1 \cap W_2 \cap \ldots \cap W_k$ as a finite intersection of sub-basic open sets W_1, W_2, ..., W_k is $\mathcal{SB}$. Let $f(x) \in W_i \subset U \subset W$. Now $f^{-1}(W_i)$ is an open set in X by (c) with $x \in f^{-1}(W_i)$; Put $V = f^{-1}(W_i)$ which is a nbd. of x such that $f(V) = W_i \subset W$. So (d) holds.

(d) $\Rightarrow$ (e); Let A be any subset of X, and $v \in f(\overline{A})$. Take $u \in \overline{A}$ so that $f(u) = v$; If $u \in A$ we have $v = f(u) \in f(A) \subset \overline{f(A)}$ and we have finished. So assume $u \notin A$ but u is a limit point of A; Let W be a nbd. of $f(u)$ in Y, by (d) there is a nbd. V of u such that $f(V) \subset W$. Now V cuts A, as u is a limit point of A, nonvacuously and so, $V \cap (A\backslash\{u\} \neq \phi$.

Now $f(V \cap (A\backslash\{u\})) \neq \phi$; or $f(V) \cap f(A\backslash\{u\}) \neq \phi$ and hence $W \cap (f(A)\backslash f(u)) \neq \phi$, since $f(V) \subset W$.

That means, $f(u)$ is a limit point of $f(A)$.

or, $v = f(u)$ is a limit point of $f(A)$

i.e. $v \in \overline{f(A)}$.

Therefore, we see $f(\overline{A}) \subset \overline{f(A)}$. So (e) holds.

(e) $\Rightarrow$ (f); Let $B \subset Y$, Put $A = f^{-1}(B) \subset X$; So, $f(A) = B$ by (e)

$f(\overline{A}) \subset \overline{f(A)}$ or, $f\left(\overline{f^{-1}(B)}\right) \subset \overline{B}$

i.e., $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$; So (f) holds.

(f) $\Rightarrow$ (g); Take $C \subset Y$.

Then we have $f^{-1}(\text{Int } C) = f^{-1}\left(Y \backslash (\overline{Y \backslash C})\right)$ (by relation between closure and Interior);

$$= X \backslash f^{-1}(\overline{Y \backslash C})$$
$$\subset X \backslash \overline{f^{-1}(Y \backslash C)} \quad \text{from (f)}$$
$$= X \backslash \overline{X \backslash f^{-1}(C)}$$
$$= \text{Int } f^{-1}(C).$$

Thus (g) holds.

(g) $\Rightarrow$ (a); Let G be an open set in Y; Then Int G = G, and by (g) $f^{-1}(G)$ = f^{-1} (Int G) $\subset$ Int $f^{-1}(G)$. That means, $f^{-1}(G)$ is an open set in X; and hence f is continuous. The proof is complete.

Theorem 1.2. For a bijective (1–1 and onto) continuous function f : $(X, \tau) \rightarrow (Y, U)$ following statements are equivalent;

(i) f is a homeomorphism

(ii) f is closed

(iii) f is open.

Proof : The proof is a routine verification based on the fact that for every subset A of X, $(f^{-1})^{-1}$ (A) = f(A); details are left out.

Example 1.2. In every Topological space (X, τ), the identity function I : $X \rightarrow X$ where I(x) = x, $x \in X$ is a homeomorphism.

Example 1.3. If the space R of reals is taken with usual Topology then for a > 0, the function f : $R \rightarrow R$ where f(x) = ax, $x \in R$ is a homeomorphism.

Definition 1.4. Topological spaces (X, τ) and (Y, U) are called homeomorphic if there is a homeomorphism h : $(X, \tau) \rightarrow (Y, U)$.

Explanation : Suppose (X, τ) and (Y, U) are homeomorphic spaces. A homeomorphism h : $(X, \tau) \rightarrow (Y, U)$ establishes 1–1 correspondence between elements of X and Y and between open sets of the two spaces. Thus a property of a Topological space X defined by means of open sets is transferred to the space Y that is homeomorphic to X. Examination of these properties, called **Toplogical properties** is essentially the **subject of 'TOPOLOGY'**.

Example 1.4. Let A be a non-empty proper subset of a Topological space (X, τ) and be R be the space of reals with usual Topology; then characteristic function $\chi_A : X \rightarrow R$ is continuous if and only if A is clo-open in X.

Solution : Let χ_A be the characteristic function of $A \subset X$, and so

$$\chi_A(x) = 1 \qquad \text{if } x \in A$$
$$= 0 \qquad \text{if } x \notin A.$$

Suppose χ_A is continuous and take an open set $G \subset R$, then

$$\chi_A^{-1}(G) = X \qquad \text{if G contains 0 and 1,} \qquad \ldots\ldots \text{(i)}$$
$$= A \qquad \text{if G contains 1 but not 0,} \qquad \ldots\ldots \text{(ii)}$$
$$= (X\backslash A) \qquad \text{if G contains 0 but not 1,} \qquad \ldots\ldots \text{(iii)}$$
$$= \phi \qquad \text{if G contains none of 0 and 1.} \qquad \ldots\ldots \text{(iv)}$$

Since $\chi_A^{-1}(G)$ is always open in X it follows from (i) – (iv) that A is clo-open in X. The converse is also true.

§2. Nets and Filters :

Definition 2.1. A binary relation denoted by $\geq$ is said to direct a non-empty set D if

(i) m, n and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$ ($\geq$ is transitive)

(ii) $m \in D$, then $m \geq m$ ($\geq$ is reflexive)

(iii) m and n are members of D, then there is a member p in D such that $p \geq m$ and $p \geq n$.

If $\geq$ directs D, then the pair $(D, \geq)$ is said to be a directed set.

For example the set $I\mathcal{N}$ of all natural numbers is a directed set with usual order of reals.

Explanation : A very common example of a directed set is the set N of all natural number with usual arithmetic ordering. So $(N, \geq)$ is a directed set. If (X, τ) is a Topological space and $\mathcal{N}_x$ is the family of all neighbourhoods of a point $x \in X$ is a directed set directed by set inclusion relation $\subset$ (being a subset of). Note that a directed set need not be a partially ordered set, since Definition 2.1 does not invite antisymmetry.

Definition 2.2. If S : $(D, \geq) \to X$ is a function where $(D, \geq)$ is a directed set, and X is a non-empty set, then S is called a net in X;

or, **equivalently, $S_n(= S(n)) \in X$ as $n \in D$.**

or, in symbol, a net $\{S_n; n \in D, \geq\}$ is **in X.**

25

A net $\{S_n : n \in D, \geq\}$ is said to be **eventually in X** if and only if there is a member $N \in D$ such that $S_n \in X$ if $n \geq N$.

A net $\{S_n : n \in D, \geq\}$ is said to be **Frequently in X if and only if for each** $m \in D$, **there is a member** $n \in D$ **such that** $n \geq m$ **and** $S_n \in X$.

Remark : A Sequence $\{x_n\}$ **is a net with** $I\mathcal{N}$ **as a directed set.**

Explanation : If a net $\{S_n; n \in D, \geq\}$ is Frequently in X, Let $E = \{n \in D : S_n \in X\}$; Then for each member $m \in D$, we find a member $p \in E$ with $p \geq$ n. **Such a subset E of D is called cofinal. Cofinal subsets of a directed set are used in theory of subsets of a net.**

Definition 2.3. : A net $\{S_n, n \in D, \geq\}$ in a Topological space (X, τ) is said **to converge to an element** $u \in X$, **if and only if the net** $\{S_n : n \in D, \geq\}$ **is eventually in every neighbourhood of u in X.**

For example, if (X, τ) is a discrete space (τ is the discrete Topology), then a net $\{S_n, n \in D, \subset\}$ coverges to a point u iff $\{S_n, n \in D, \subset\}$ is eventual in $\{u\}$; That is to say, from some point on S_n's are all equal to u. On the other extreme case if (X, τ) is indiscrete, (only open sets are ϕ and X), then every net $\{S_n, n \in D, \subset\}$ converges to every point of X. Cousequently a given net may converge to several distinct points.

Theorem 2.1. In a Topological space (X, τ) (i) A point **u in X is a limit point of a subset A of X, if** and **only if there is a net in A\{u}** such that the net converges to u.

(ii) **A point u belongs to closure of a non empty set A i.e.** $u \in \overline{A}$ **if and** only if there is **a net in A converging to u.**

(iii) A subset A in X is **closed if and only if no net in A converges to a member of (X\A).**

Proof : (i) Let $u \in X$ be a limit point of a subset A if X. Then every nbd. N_u of u cuts A in a point other than u;

$$\text{i.e. } N_u \cap (A\backslash\{u\}) \neq \phi; \text{ Take } x_{N_u} \in N_u \cap (A\backslash\{u\}).$$

26

We know that the family $\mathcal{N}_u$ of **all** nbds. N_u of u forms a directed set with respect to set inclusion relation $\subset$.

Then $\{x_{N_u}, \mathcal{N}_u; \subset\}$ is a net in $A\backslash\{u\}$ such that if N_u and N'_u are nbds. of u with $N'_u \subset \mathcal{N}_u$, then by construction $x_{N'_u} \in N'_u \subset N_u$. Hence the net $\{x_{N_u}, \mathcal{N}_u; \subset\}$ is eventually in every nbd of u, implying that the net converges to u.

The converse is obviously straight forward.

(ii) $\overline{A}$ = closure of A = A $\cup$ derived set A′ (of A). For each member u $\in$ A′, there is a net (by preceding part) in A converging to u; For each member u $\in$ A, any net $\{S_n, n \in D, \geq\}$. Such that $S_n = u$ for all n $\in$ D becomes convergent net converging to u. Therefore, each point in $\overline{A}$ attracts a net in A that converges at that point.

(iii) This part is now clear because a set is closed if and only if $\overline{A}$ = A.

Theorem 2.2. A function f : $(X, \tau) \rightarrow (Y, U)$ is continuous at c $\in$ X if and only if every net $\{S_n, n \in D, \geq\}$ in X that converges to c, the net $\{f(S_n)\}$: n $\in$ D, $\geq\}$ converges to f(c) in Y.

In symbol, $f\left(\lim_n S_n\right) = \lim_n f(S_n)$.

Proof : The condition is necessary : Suppose f is continuous at x = c $\in$ X. Let $\{S_n, n \in D, \geq\}$ be a net in X that converges to c. So it is **eventually in every nbd. of c in X.** Take $N_{f(c)}$ be a nbd. of f(c) in Y. By continunity if f at c, we have $f^{-1}(N_{f(c)})$ is a nbd. of c in X. So $\{S_n, n \in D, \geq\}$ is eventually in $f^{-1}(N_{f(c)})$. That is to say, $\{f(S_n), n \in D, \geq\}$ is eventually in $N_{f(c)}$, and the net $\{f(S_n), n \in D, \geq\}$ converges to f(c) in Y.

The condition is sufficient : Let the condition hold. If possible, let f be not continuous at c in X. We seek a contradiction.

Failure of continuity of f at c invites **a nbd. (say)** $N_{f(c)}$ of f(c) in Y such that $f^{-1}(N_{f(c)})$ is not a nbd. of c.

Thus for every member N_c of nbd. system $\mathcal{N}_c$ of c in X, we find a point x_{N_c} (say) $\in N_c$ such that $x_{N_c} \notin f^{-1}(N_{f(c)})$.

i.e. $f(x_{N_c}) \notin N_{f(c)}$ ... ... (∗)

Put D = Collection of all such ordered pairs

$$= \{(x_{N_c}, N_c)\}, \ N_c \in \mathcal{N}_c.$$

Then consider the Directed set $D = \{(x_{N_c}, N_c), \ N_c \in \mathcal{N}_c, \subset\}$.

Define $S(x_{N_c}, N_c) = x_{N_c}$; Then the net $\{S, D, \subset\}$ has the property that it is eventually in **every nbd.** N_c of c; i.e. it converges to c in X, but the image net $\{f(S), D, \subset\}$ becomes such that $f(S(x_{N_c}, N_c)) = f(x_{N_c}) \notin N_{f(c)}$ (From ∗).

i.e. the net is not eventually in $N_{f(c)}$ = a nbd. of f(c) in Y. So the image not $\{f(S), D, \subset\}$ does not converge to f(c), although the net $\{S, D, \subset\}$ converges to c in X—a contradiction. The proof is complete.

Definition 2.4. A filter in a Topological space (X, τ) is a family $\mathcal{F}$ of subsets of X satisfying

(F.1)   If $A \in \mathcal{F}$, and $A \subset B$, then $B \in \mathcal{F}$,

(F.2)   If $A_1, A_2 \in \mathcal{F}$, then $(A_1 \cap A_2) \in \mathcal{F}$,

(F.3) $\phi \notin \mathcal{F}$.

For example, the nbd. system $\mathcal{N}_x$ if a point x in X is a Filter. Because above conditions (F.1 – F.3) are all O.K. in favour of the family $\mathcal{N}_x$. That is why $\mathcal{N}_x$ is often termed as nbd. Filter.

Definition 2.5. A Filter $\mathcal{F}$ is said to converge to a point $x \in X$, if and only if each nbd N_x of x is a member of $\mathcal{F}$ (that is, the nbd system $\mathcal{N}_x$ at x is a sub-family of $\mathcal{F}$).

Theorem 2.3. A function $f : (X, \tau) \to (Y, U)$ is continuous at $x = c \in X$ if and only if for every Filter $\mathcal{F}$ on X that converges to c, the image Filter $f(\mathcal{F})$ converges to f(c) in Y.

Proof : The condition is necessary : Let f he continuous at x = c ∈ X, and let $\mathscr{F}$ be a filter in X converging to c. Then the nbd. system $\mathscr{N}_c$ at x = c is a sub-family of F i.e. $\mathscr{N}_c \subset \mathscr{F}$. We show that image Filter $f(\mathscr{F})$ in Y converges to f(c); That is to show the nbd. system $\mathscr{N}_{f(c)}$ at f(c) as a sub-family of $f(\mathscr{F})$ or

$$\mathscr{N}_{f(c)} \subset f(\mathscr{F}). \qquad ..... (1)$$

Let $N_{f(c)}$ be a nbd. of f(c) in Y. i.e. $N_{f(c)} \in \mathscr{F}_{f(c)}$. By continuity of f at x = c, we find that $f^{-1}(N_{f(c)})$ is a nbd. of c in X and hence by assumption $f^{-1}(N_{f(c)}) \in \mathscr{F}$; That means $N_{f(c)} \in f(\mathscr{F})$ and (1) is verified.

The condition is sufficient; Suppose the condition holds. and f is not continuous at x = c. We derive a contradiction. We find a nbd $N_{f(c)}$ of f(c) in Y such that no nbd N_c of c in X satisfies $N_c \subset f^{-1}(N_{f(c)})$.

i.e. no $f(N_c) \subset N_{f(c)}$, showing that $N_{f(c)} \notin f(\mathscr{N}_c)$.

Therefore image Filter $f(\mathscr{N}_c)$ does not converge to f(c) in Y, though nbd. filter $\mathscr{N}_c$ coverge to c in X—a contradiction. The proof is now complete.

Theorem 2.4. A point x is a limit point of a subset A of X, if and only if A\{x} is a member of some Filter converging to x.

Proof : Let x be a limit point of A ⊂ X. Then if $N_x \in \mathscr{N}_x$ we have $N_x \cap (A\backslash\{x\} \neq \phi$.

If we put $\mathscr{G} = \{G \subset X : N_x \cap (A\backslash\{x\}) \subset G, N_x \in \mathscr{N}_x\}$, then it is not difficult ot cheek that $\mathscr{G}$ is a Filter generated by members $N_x \cap (A\backslash\{x\})$, $N_x \in \mathscr{N}_x$. Further $\mathscr{N}_x \subset \mathscr{G}$, and, hence, $\mathscr{G}$ is a filter converging to x in X. Of course, by construction, A\{x} ∈ $\mathscr{G}$.

Conversely, Let A\{x} be a member of a filter (say) $\mathscr{H}$ that converges to x in X; That is, to say,

$$A\backslash\{x\} \in \mathscr{H}. \qquad ..... (1)$$

Also the nbd. system $\mathscr{N}_x$ at x ⊂ H; That is to say, every nbd. N_x of x is a member of $\mathscr{H}$.

$$\text{or, } N_x \in \mathscr{H} \qquad ..... (2)$$

Since $\mathscr{H}$ is Filter, we deduce from (1) and (2)

$$N_x \cap (A\backslash\{x\}) \neq \phi.$$

That means **x is a limit point of A.**

Nets and Filters Lead to essentially equivalent theories. Grounds for this suspicion rest in following Theorem.

Theorem 2.5. In a Topological space (X, τ). If $\{S_n, n \in D, \geq\}$ is a net converging to $u \in X$, they there is a Filter in X converging to u; and vice-versa.

Proof : Let the net $\{S_n, n \in D, \geq\}$ converge to $u \in X$.

For each $n \in D$, put $A_n = \{S_m, m \in D; m \geq n\}$. Since Intersection of any two such members contains another such member $\{A_n\}$ generate a Filter $\mathscr{F}$ in X. Now given any nbd. N_u of u, we know that the net is eventual in N_u; That is to say $A_n \subset N_u$ for some n. That means N_u is also a member of $\mathscr{F}$ i.e. the nbd. system $\mathscr{N}_u$ at u satisfies $\mathscr{N}_u \subset \mathscr{F}$; **Hence $\mathscr{F}$ converges to $u \in X$.**

Conversely, Let $\mathscr{F}$ be a Filter converging to $u \in X$. Put $D_{\mathscr{F}} = \{(x, F) : x \in F$ and $F \in \mathscr{F}\}$. Then $D_{\mathscr{F}}$ is a directed set with order $\geq$ by agreeing to $(Y, G) \geq (X, F)$ if and only if $G \subset F$. Consider the net $\{f(x, F) ; (x, F) \in D_{\mathscr{F}}, \geq\}$.

Where $f(x, F) = x$. Then it is a routine excercise to check that this net is eventual in every nbd N_u of u, and hence the net converges to $u \in X$.

§3. Product spaces.

There is a technique to construct a new Topological space out of a given number of Topological spaces. **That leads to the concept of product spaces.** Let (X_1, τ) and (X_2, V) be given Topological spaces, and let $X = X_1 \times X_2$. Consider a family $\mathscr{SB}$ of all subsets of $X_1 \times X_2 = X$ like $G_1 \times X_2$ and $X_1 \times G_2$ where $G_1 \in \tau$ and $G_2 \in V$. Then it is a routine exercise to see that $\mathscr{SB}$ is a sub-base for a Topology on the Cartesian Product X whose base B consisits

of all members of the form $G_1 \times G_2$, and the Topology generated by the base $\mathcal{B}$ (or by sub-base $S\mathcal{B}$) on X is called the Product Topology on X. Then a subset W of $X_1 \times X_2 = X$ shall be an open set in the Product Topology if and only if to each member $(x_1, x_2) \in W$ there correspond open nbds U_1 of x_1 in X_1 and U_2 of x_2 in X_2 such that

$$U_1 \times U_2 \subset W.$$

The given spaces X_1 and X_2 are often called co-ordinate spaces and two functions p_{r_1} and p_{r_2} that carry a member (x_1, x_2) of $X_1 \times X_2$ into x_1 and into x_2 respectively

i.e. $p_{r_1} : X_1 \times X_2 \to X_1$ and $p_{r_2} : X_1 \times X_2 \to X_2$ where

$p_{r_1}(x_1, x_2) = x_1$ and $p_{r_1}(x_1, x_2) = x_2$ respectively are called projections into Co-ordinate spaces.

Projection functions p_{r_1} and p_{r_2} are here continuous functions. Because if U_1 is an open set in X_1 we see $p_{r_1}^{-1}(U_1) = U_1 \times Y$, and that is an open set in $X = X_1 \times X_2$.

Now suppose τ is a Topology for $X_1 \times X_2$ that make projection functions continuous. Then if U is open in X_1 and V is open in X_2, then $U \times V$ is open relative to τ, because $U \times V = p_{r_1}^{-1}(U) \cap p_{r_2}^{-1}(V)$, and this set is open relative to τ because p_{r_i}'s are continuous. Therefore τ is stronger than the product, topology, and we conclude that the product topology is the weakest Topology on X to make projections continuous.

It is now a routine matter to extend the Definition of Product Topology for the cartesian product $X = X_1 \times X_2 \times ... \times X_n$ where each co-ordinate X_i is a given Topological space (X_i, τ_i). Thus a base for the Product Topology on X consists of all members of the from $U_1 \times U_2 \times ... \times U_n$ where each U_i in open in (X_i, τ_i), $i = 1, ..., n$. In particular, the real number space R with usual topology gives rise to the product space $R^n = R \times R \times ... \times R$ (n times) with Product topology; and R^n is more commonly known as the Euclidean n-space.

Here we see that each member of R^n is an ordered n-tuple of reals like $(x_1, x_2, ... x_n)$; $x_i \in R$. We look at it as a real function x on the set $(1, 2, ..., n)$ where value at i as x_i (= x(i)).

Suppose we have an arbitrary family of Topological spaces (X_α, τ_α), $\alpha \in \Lambda$- an Index set. Then the cartesian product $X = X\{X_\alpha : \alpha \in \Lambda)$ may be looked as the set of all functions x on Λ such that $x_\alpha \in X_\alpha$ for each index $\alpha \in \Lambda$. In that case X_α is α-th Co-ordinate space and corresponding projection function $p_{r_\alpha} : X \to X_\alpha$ is given by p_{r_α} $(x \in X) = x_\alpha \in X_\alpha$.

Consider the family $\{p_{r_\alpha}^{-1}(U) :$ where U is an open set is $(X_\alpha, \tau_\alpha)\}$. It is easy to check that this family of subsets of X is a sub-base for a Topology on X, and it is the smallest Topology on X to make each projection function continuous. This Topology on X is called the Product Topology and X is the Product space of given Topology spaces (X_α, τ_α). as $\alpha \in \Lambda$.

Theorem 3.1. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a family of Topological spaces and $X = X\{X_\alpha : \alpha \in \Lambda\}$ be the product space with Product Topology. Then each projection function $p_{r_\alpha} : X \to X_\alpha$ is an open function i.e. it sends open sets into open sets.

Proof : Let $p_{r_\alpha} : X \to X_\alpha$ be a projection function. Let G be an open set in X. If $x \in G$, we find a member of defining open base member, say, D of the product Topology such that $x \in D \subset G$. Then we have

$$p_{r_\alpha}(x) \in p_{r_\alpha}(D) \subset p_{r_\alpha}(G) \qquad ..... (1)$$

We know that D looks like

$D = X\{G_\alpha : \alpha \in \Lambda\}$, where G_α is open in X_α and $G_\alpha \equiv X_\alpha$ for all α except a finite number of α's (say) = $\alpha_1, \alpha_2, ..., \alpha_n$.

Now, if α = one such α_k, we find

$$p_{r_{\alpha_k}}(D) = \{p_{r_{\alpha_k}}(y) : y \in D\} = \{y_{\alpha_k} : y_{\alpha_k} \in G_{\alpha_k}\} = G_{\alpha_k} \qquad ..... (2)$$

32

And if α is none of $\alpha_1, \alpha_2, ..., \alpha_n$ we know

$p_{r_\alpha}(D) = X_\alpha$. Now (2) and (3) tell us $p_{r_\alpha}(D)$ is an open set and from (1) we conclude that $p_{r_\alpha}(G)$ is open.

Remark : A projection function may not send a closed set into a closed set. For example, in R^2, the closed set $\{(x, y) \in R^2 : xy = 1\}$ has a non-closed Projection on each Co-ordinate space.

Theorem 3.2. Let (Y, V) a topological space and $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a family of Topological space; function $f : Y \to X = X\{X_\alpha : \alpha \in \Lambda\}$ is continuous if and only if the composition function $p_{r_\alpha^0} f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$.

Proof : Let $f : Y \to X$ be a continuous function. Since X is the product space with product topology and we know that each projection function p_{r_α} on X is continuous, it follows that the composition $p_{r_\alpha^0} f$ is a continuous function.

Conversely, suppose the condition holds i.e. $p_{r_\alpha^0} f$ is continuous for each α, then for each open set U of X_α, we have $(p_{r_\alpha^0} f)^{-1}(U) = f^{-1}(p_{r_\alpha}^{-1}(U))$ and this is an open set by assumption; and product topology of X says that $p_{r_\alpha}^{-1}(U)$ is a member of sub-base for the product topology. This inverse image under f of a (any) member of defining sub-base member of the product Topology becomes an open set. That means, f is continuous.

§4. Quotient spaces.

Let (X, τ) be a Topological space, and Let $f : (X, \tau) \to Y$ be an onto function.

Let $\mathcal{U} = \{U \subset Y : f^{-1}(U) \in \tau\}$.

Then $\mathcal{U}$ forms a Topology on Y. Because

(i) $\phi, Y \in U$;

(ii) It $\{G_\alpha\}_{\alpha \in \Lambda}$ is a sub-family of $\mathscr{U}$, then $G = \underset{\alpha \in \Lambda}{U} G_\alpha \subset Y$ such that $f^{-1}(G) = f^{-1}(\underset{\alpha \in \Lambda}{U} G_\alpha) = \underset{\alpha \in \Lambda}{U} f^{1}(G_\alpha) \in \tau$, because each $f^{-1}(G_\alpha) \in \tau$. So $G \in \mathscr{U}$;

(iii) Let G, H $\in \mathscr{U}$, and we have $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$ which is a member of τ became $f^{-1}(G)$ and $f^{-1}(H)$ are so. Thus $G \cap H$ is a member of $\mathscr{U}$.

Thus (Y, $\mathscr{U}$) is a Topological space, called **quotient space.**

Theorem 4.1. The quotient Topology on Y is the largest topology such that $f : (X, \tau) \to Y$ is continuous.

Proof : Let V be a Topology on Y such that $f : (X, \tau) \to (Y, V)$ is continuous. Take $G \in V$, then by continuity of f we have $f^{-1}(G)$ is open in X i.e. $f^{-1}(G) \in \tau$, and by definition of quotient Topology G is a member of the quotient Topology on Y. Hence $V \subset$ Quotient Topology. The proof is complete.

Definition 4.1. A function f from one topological space into another is said to be a closed function if f sends each closed set into a closed set.

Theorem 4.2. If $f : (X, \tau) \to (Y, \mathscr{U})$ is continuous and onto such that f is either open or closed function, then $\mathscr{U}$ is the quotient Topology on Y.

Proof : Let $(X, \tau) \to (Y, \mathscr{U})$ be continuous and onto, and U be a subset of Y such that $f^{-1}(U)$ is open. Here $U = f (f^{-1}(U)$ is open (f is open) in $\mathscr{U}$. So every open set in quotient Topology is open in $\mathscr{U}$. If f is continuous and open, since quotient Topology is the largest Topology on Y for which f is continuous, $\mathscr{U}$ becomes the quotient Topology on Y.

If f is a closed function, we need to replace 'open' by 'closed' in argument above for desired conclusion.

Example 4.1. Let $X = \{(x, y) \in R^2 : x = 0 \text{ or } y = 0\}$. Show that projection function $P(x, y) = x$ for $(x, y) \in X$ is closed but not open.

Solution : Take any closed subset of X; it is mapped by P into singleton $\{0\}$ which is closed but not open.

Let (X, τ) be a given Topological space, and let R be an equivalence, relation on X. Then X is partitioned into disjoint equivalent classes; denote the set formed by these classes as X/R, called quotient set.

If $a \in X$, let D_a denote the 'equivalent' class containing a. Thus $D_a = D_b$ if and only if a and b enter the same 'equivalent' class i.e. if and only if $(a, b) \in R(a \; R \; b)$.

Now define $h : X \to (X/R)$ where $h(x) = D_x$, $x \in X$. Then h is onto. Also construct a family τ_R of subsets of (X/R) by the rule :-

$$\tau_R = \{K \subset (X \backslash R) : h^{-1}(K) \text{ is an open set in X}\}$$

That is to say, $\tau_R = \{K \subset (X \backslash R) : h^{-1}(K) \in \tau\}$.

Then we verify that τ_R is, indeed, a Topology on (X/R).

Because (i) $\phi, (X \backslash R) \in \tau_R$.

(ii) If $\{K_\alpha\}_{\alpha \in \Delta}$ be a sub-family of τ_R, and $K = \bigcup\limits_{\alpha \in \Delta} K_\alpha$.

Then $h^{-1}(K) = h^{-1}(\bigcup\limits_{\alpha \in \Delta} K_\alpha) = \bigcup\limits_{\alpha \in \Delta} h^{-1}(K_\alpha) \in \tau$ since each member $h^{-1}(K_\alpha) \in \tau$ which is a Topology on X. That shows $K \in \tau_R$.

(iii) Similarly we can show that intersection of two members of τ_R is a member of τ_R.

This topology τ_R on quotient set (X/R) is called the quotient Topology and Topological space (X\R) is said to be the quotient space.

Theorem 4.3. If π denote projection of (X, τ) onto the quotient space (X\R), then following statements are equivalent.

(a) π is an open function and (b) If G is open in X, R[G] is open.

Proof. For each subset A of X, we have $R[A] = \pi^{-1}(\pi[A])$.

(a) $\Rightarrow$ (b); Let π be open. If G is open in X, by continuity of projection function π. We have $\pi^{-1}(\pi[G])$ is open i.e. R[G] is open. Thus (b) holds.

(b) $\Rightarrow$ (a); Let (b) hold. If $R[G]' = \pi^{-1}(\pi[G])$ is open, then, by Definition of the quotient Topology, $\pi[G]$ is open, and so π is open.

Exercise - A

Short answer type questions

1. In $I\mathcal{N}$ is the set of all natural numbers, construct a co-finite Topology τ_{IN} on $I\mathcal{N}$. Is τ_{IN} a filter? Give reasons.

2. Show that a sequence $\{x_n\}$ is a net.

3. Give an example of a Projection function that is open, but not closed.

4. Let $P = (a = x_0 < x_1 < ... < x_n = b)$ be a Partition of a closed interval $[a, b]$. Let ξ denote the family of complements of all partitions of $[a, b]$. Show that ξ is a Directed set.

5. Let I and J be two non-degenerate intervals of reals with usual topology. Show that any homeomorphism $h : I \to J$ is monotonic.

6. If (X, τ) is a Topological space, and $f : X \to R$ of reals with usual topology, is continuous, show that the set $\{x \in X : -1 < f(x) < +1\}$ is an open set in X.

7. If $\varphi : (-1, 1) \to$ Reals with usual Topology is given by

$$\varphi(x) = \frac{x}{1 - |x|}; \text{ as } -1 < x < +1.$$

Examine if φ is a Homeomorphism.

Exercise - B

1. In a Topological space (X, τ) a subset G is open if and only if G is a member if every Filter that converges to a point of G.

2. Show that all limit points of a net in a Topological space form a closed set.

3. In a Topological space X let Φ_x denote the collection of all Filters each of which converges to $x \in X$; Show that $\cap\{\mathscr{F} : \mathscr{F} \in \Phi_x\}$ is equal to the nbd. system $\mathscr{N}_x$ at x.

4. Let $f : (X, \tau) \to (Y, V)$ be a continuous function. If $A \subset X$, show that restriction f_A is a continuous function on A. Is the converse true? Give reasons.

5. If (X, τ) and (Y, V) are Topological spaces show that a bijective (1–1 and onto) function $f : X \to Y$ is a homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$ for every subset A of X, bar denoting the closure.

6. Show that the family of all subsets of a non-empty set X each of which contains a given element $x_0 \in X$ is a Filter on X. Examine its maximality.

Unit 3 ❏ Separation Axioms in Topological Spaces

> (Separation axioms T_0, T_1, T_2, T_3, T_4 in Topological spaces, Their implications and characterisations, Product of T_2-space, Regular spaces, Normal spaces, Completely regular spaces, Tychonoff spaces, Urysohns Lemma in Normal space, Tietze extension Theorem, Embedding in cube, Embedding Lemma and Theorem, Metrization Urysohn's metrization Lemma).

§1. If there are too many or too few open sets in a Topological space (X, τ), analysis thereupon may not be interesting and useful. For example, every function over (X, τ) with τ as discrete Topology becomes continuous. There arise several separation axioms in (X, τ) in terms of availability of open sets in X. These axioms are presented below in graded style : which one is weaker or stronger than the other. Let (X, τ) be a Topological space.

**Definition 1.1.** (X, τ) is called a **To-space** if given two distinct points in X, **there is an open set containing any one but not the other.**

For example, real number space R with usual topology is a T_0-space; because given x, y $\in$ R with x $\neq$ y, one can find an open interval containing x leaving y outside. Also there is a Topological space like (X, τ) where X = (a, b, c) (a, b and c are distinct) and $\tau = \{\phi, X, (a), (b, c)\}$, such that (X, τ) is not T_0; because distinct points b and c have no T_0-separation i.e. we do not find any open set in X to contain one without containing the other.
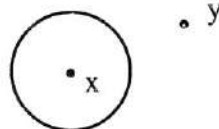
Definition 1.2. (X, τ) is called a **T_1-space** if given any two distinct elements in X, **there is an open set to contain each one of them without containing the other.**

Explanation : There are many T_1-spaces; for example, space R of reals with usual Topology. Consider a Topological space (X, τ) where X = (a, b, c), a, b are c are distinct, and $\tau = \{\phi, X, (a), (a, b)\}$. Here a and b have no attracting open sets in τ as per T_1-stipulation. So (X, τ) is not T_1. Definitions above have been so framed that $T_1 \Rightarrow T_0$ i.e. every T_1-space is a T_0-space; opposite implication is however false.
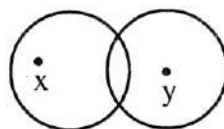
Example 1.1. The space (X, τ) where $X = \{a, b\}$, $a \neq b$; and $\tau = \{\phi, X, (a)\}$ is T_0 without being T_1.

Solution : Here for distinct elements a and b we have an open set (a) containg a without containing b; and this pair does not have a T_1- separation; because only open set to cover b is (a, b) that cuts {a}.

Definition 1.3. (X, τ) is called a T_2-space or a Hausdorff space if given any two distinct members x and y in X, there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Explanation : Here $T_2 \Rightarrow T_1$; and if (X, τ) is a Topological space where $X = \{a, b\}$ $a \neq b$; and $\tau = \{\phi, x, (a), (b)\}$. Then (X, τ) is T_2. Example 1.2 shows there is a space (X, τ) that is T_1 without being T_2.

Example 1.2. Let X be an infinite set and let the collection $\mathscr{G}$ of subsets of X be as $\mathscr{G} = \{G \subset X : (X \backslash G)$ is a finite set (may be empty)$\}$. Then we verify that $\mathscr{G}$ is a Topology on X, **very often named as co-finite Topology on X. This Topological space $(X, \mathscr{G})$ is T_1 without being T_2.** Take two members u, v $\in$ X with u $\neq$ v. Then $U = X \backslash \{v\}$ and $V = X \backslash \{u\}$ are two members of $\mathscr{G}$ containing u and v respectively such that v is outside U and u is not in V. **Hence $(X, \mathscr{G})$ is T_1. If possible, let any two distinct points x, y in X have T_2 separation.** So we find two open sets H and K in X such that $x \in H$, $y \in K$ and $H \cap K = \phi$. Clearly $(X \backslash H)$ and $(X \backslash K)$ are each finite subsets of X; and so.

$(X \backslash H) \cup (X \backslash K)$ is a finite set in X,

i.e. $X \backslash (H \cap K)$ is a finite set in X,

i.e.   X   is a finite set, because $H \cap K = \phi$.

This is a contradiction. Hence our assertion stands.

Theorem 1.1. If (X, τ) is T_0, then closures of distinct points in X are distinct.

Proof : Let (X, τ) be a T_0-space; and let x, y $\in$ X with x $\neq$ y. We show that either $x \notin \{\overline{y}\}$ or $y \notin \{\overline{x}\}$, bar denoting the closure. By T_0-separation we obtain an open set U containing (say) x without containing y. That is to say,

$$x \in U \text{ and } y \notin U;$$

Thus a nbd. of x does not cut the singleton {y}; So x is not a dimit point of {y}. As x ≠ y, we have x ∉ {$\overline{y}$}. Of course x ∈ {$\overline{x}$}. That means {$\overline{x}$} ≠ {$\overline{y}$}.

Theorem 1.2. (X,) is **T$_1$ if and only if every singleton is closed.**

Proof : Let (X, τ be T$_1$ and x ∈ X. If y ∈ X and y ≠ x, by T$_1$-separation we find an open set U such that

$$x \in U \text{ and } y \notin U.$$

Clearly then y is not a limit point of {x} i.e. y ∉ {x}′ (derived set of {x}). Of course x ∉ {x}′. That means no member of X is a member of {x}′. Hence {x}′ = φ; So

$$\{\overline{x}\} = \{x\} \cup \{x\}' = \{x\}$$

i.e. {$\overline{x}$} is closed.

Conversely, suppose every singleton in (X, τ) is closed. Take x, y ∈ X with x ≠ y. So singleton {x} is closed, and hence X\{x} is an open set containing y (without containing x); and similarly X\{y} is an open set containing x (without containing y). Thus (X, τ) is T$_1$.

Theorem 1.3. (X, τ) is **T$_2$ if and only if every net in the space converges to atmost one point in X.**

Proof : The condition is necessary. Let (X, τ) be T$_2$ (Hausdroff). If x and y are two distinct points in X, they attract disjoint neighbourhoods U and V containing x and y respectively by T$_2$ separation. Since a net can not be eventually in each of two disjoint sets at the same time it follows that no net in X converges to both x and y simultaneously.

The condition is sufficient. Here assume that the condition holds. If possible let (X, τ) be not a T$_2$-space, and let us suppose a and b are two distinct members of X such that every nbd. of a intersects every nbd. of b. Now nbd. system $\mathcal{N}_a$ at a is a directed set and so is the nbd. system $\mathcal{N}_b$ at b. Let us order their product (Cartesian) $\mathcal{N}_a \times \mathcal{N}_b$ by agreeing that (T, U) ≥ (V, W) if and only if T ⊂ V and U ⊂ W. For each member (T, U) ∈ $\mathcal{N}_a \times \mathcal{N}_b$ we have T ∩ U ≠ φ. Take a point x$_{(T,U)}$ ∈ (T ∩ U). Thus if (V, W) ≥ (T, U), then x$_{(V,W)}$ ∈ (V ∩ W) ⊂ (T ∩ U), and in consequence the net {x$_{(T,U)}$, ∈ (T, U) $\mathcal{N}_a \times \mathcal{N}_b$, ≥} converges to both a and b simultaneously—**a contradiction.**

40

Theorem 1.4. A product of T_2-spaces is a T_2-space.

Proof : If x and y are two distinct points in product $X\{X_\alpha : \alpha \in \Delta\}$, then $x_\alpha \neq y_\alpha$ for some α in Δ. If each co-ordinate space is T_2, we find disjoint open nbds U and V of x_α and y_α respectively, and $P_{r_\alpha}^{-1}(U)$ and $P_{r_\alpha}^{-1}(V)$ become disjoint open nbds of x and y respectively in the product space $X\{X_\alpha : \alpha \in \Delta\}$ with product Topology.

Definition 1.4(a). (X, τ) is said to be a regular space if given any closed set F in X, and an outside point x in X ($\notin$ F), there are open sets U and V in X such that

$$x \in U \text{ and } F \subset V \text{ with } U \cap V = \phi.$$

(b) A regular space which is also T_1 is called a T_3-space.

Explanation : Take X = (a, b, c) and a Topology $\tau = \{\phi, X, (a), (b, c)\}$ in X. Here only closed sets are X, ϕ, (b, c) and (a). We verify that (X, τ) is a regular space, and this regular space is not T_1; because singleton (c) is not a closed set in (X, τ).

$$\text{Further } T_3 \Rightarrow T_2 \text{ (and hence } \Rightarrow T_1 \Rightarrow T_0).$$

As singletons are closed sets in T_1-space, we have $T_3 \Rightarrow T_2$.

Definition 1.5(a). (X, τ) is called a Normal space if given a pair of disjoint closed sets F and G in X, there are disjoint open sets U and V such that $F \subset U$ and $G \subset V$.

(b) A normal space which is also T_1 is said to be a T_4-space.

Explanation : Take X = (a, b, c, d, e, f) and $\tau = (\phi, X, (e), (f), (e, f), (a, b, e), (c, d, f), (a, b, e, f), (c, d, e, f)\}$; Then (X, τ) is a normal space. There are only four pairs of disjoint non-empty closed sets ; they are ((a, b), (c, d)), ((a, b), (c, d, f)), ((a, b, e), (c, d)) and ((a, b, e), (c, d, f)). Each pair is separated by pair of disjoint open sets ((a, b, c), (c, d, f)).

This normal space (X, τ) is not regular; Because (a, b) is a closed set in X with e $\notin$(a, b); there is no disjoint pair of open sets in X to separate them.

Further $T_4 \Rightarrow T_3$. Because if F is a closed set, and x is a point $\notin$ F in a T_4-space (X, τ), we see that singleton $\{x\}$ is a closed set in X. So by normality of (X, τ) desired separation is immediate. So (X, τ) is a T_3-space.

Definition 1.6(a). (X, τ) is said to be **completely regular** if given any closed set F and a point $x \notin F$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(u) = 1$ for $u \in F$.

(b) **A completely regular space which is also T_1 is called a Tychonoff space** often designated as $T_{3\frac{1}{2}}$-space.

Theorem 1.4. A space (X, τ) is **Normal if and only if** for every closed set F and open set H containing F there is an open set G such that

$$F \subset G \subset \overline{G} \subset H.$$

(Pushing a pair of open and closed sets $(G, \overline{G})$ in between a given pair of closed and open sets (F, H).

Proof : The condition is necessary : Let (X, τ) be a normal space and Let F and H be a pair of closed and open sets in X such that

$$F \subset H.$$

Then complement of $H = H^c$ is a closed set with $F \cap H^c = \phi$. By normality axiom we get a pair of disjoint open sets G and M satisfying

$$F \subset G \text{ and } H^c \subset M \text{ and } G \cap M = \phi.$$

Thus $G \subset M^c$; and $H^c \subset M$ gives $M^c \subset (H^c)^c = H$. As M^c is a closed set, we get

$$F \subset G \subset \overline{G} \subset M^c \subset H.$$

That is, $F \subset G \subset \overline{G} \subset H.$

The condition is sufficient. Suppose the condition holds. Let F_1 and F_2 be a pair of disjoint closed sets in X.

Then we have $F_1 \subset F_2{}^c$ which is open.

Hence by the condition assumed we find an open set G such that

$$F_1 \subset G \subset \overline{G} \subset F_2{}^c.$$

Now $\overline{G} \subset F_2{}^c$ gives $F_2 \subset \overline{G}{}^c$, and, of course, $G \subset \overline{G}$. So,

$$G \cap \overline{G}{}^c = \phi.$$

Thus $F_1 \subset G$ and $F_2 \subset \overline{G}{}^c$ where G and $\overline{G}{}^c$ is a pair of disjoint open sets in X. **Hence (X, τ) is normal.**

Theorem 1.5. (Urysohn's Lemma) : In a normal space X if A and B are disjoint closed sets in X, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.

Proof : Let (X, τ) be normal, and A, B be a pair of disjoint closed sets i.e. $A \cap B = \phi$; So $A \subset B^c$ which is open, B^c denoting the complement of B. So Theorem 1.4. applies and we get an open set (say) $= G_{\frac{1}{2}}$ such that

$$A \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset B^c \tag{1}$$

Now pair $(A, G_{\frac{1}{2}})$ is a pair of closed and open set with $A \subset G_{\frac{1}{2}}$, and, as before, calls for an open set, say $= G_{\frac{1}{4}}$ such that

$$A \subset G_{\frac{1}{4}} \subset \overline{G}_{\frac{1}{4}} \subset G_{\frac{1}{2}} \tag{2a}$$

and by a similar reasoning there is an open set (say) $= G_{\frac{3}{4}}$ such that

$$\overline{G}_{\frac{1}{2}} \subset G_{\frac{3}{4}} \subset \overline{G}_{\frac{3}{4}} \subset B^c \tag{2b}$$

So (2a) and (2b) give

$$A \subset G_{\frac{1}{4}} \subset \overline{G}_{\frac{1}{4}} \subset G_{\frac{1}{2}} \subset \overline{G}_{\frac{1}{2}} \subset G_{\frac{3}{4}} \subset \overline{G}_{\frac{3}{4}} \subset B^c \tag{2}$$

We continue this chain and for each dyadic rational of the form $\frac{m}{2^n}$ ($n = 1, 2, ...$ and $m = 1, 2, ..., 2^{n-1}$) in [0, 1], we designate an open set G_l with property that for any two dyadic rationals l_1 and l_2 in [0, 1], $l_1 < l_2$ gives

$$A \subset G_{l_1} \subset \overline{G}_{l_1} \subset G_{l_2} \subset \overline{G}_{l_2} \subset B^c.$$

We know that collection of all such dyadic rationals in [0, 1] is dense in [0, 1].

Let us define $f : X \to [0, 1]$ by the rule :
$$f(x) = 0 \text{ if } x \in \text{ every member } G_l$$
$$= \sup \{l : x \notin G_l\}.$$

So $0 \leq f(x) \leq 1$ for $x \in X$; Further from definition above we have $f(x) = 0$ if $x \in A$, and if $u \in B$; as $G_l \subset \overline{G}_l \subset B^c$, we see that $u \notin G_l$ for every

l. By dense property of the set of all *l*'s we deduce from $f(u) = \sup \{l : u \notin G_l\} = 1$.

It remains to work that f is continuous. Since intervals like [0, a) and (b, 1] where $0 < a, b < 1$ form sub-basic open sets in [0, 1] with usual Topology of reals. it suffices to show that $f^{-1}[0, a)$ and $f^{-1}[b, 1)$ are open sets in X. Now we can cheek

$$f^{-1}[0, a) = \{x \in X : 0 \le f(x) < a\}$$
$$= \{x \in X : x \in G_l \text{ form some } l < a\}$$
$$= \bigcup_{l < a} G_l \text{ which is an open set in X.}$$

Also $\quad f^{-1}(b, 1) = \{x \in X : b < f(x) \le 1\}$
$$= \{x \in \overline{G}_l^c \text{ for some } l > b.$$
$$= \bigcup_{l > b} \overline{G}_l^c, \text{ a union of open sets} = \text{an open set.}$$

The proof is now complete.

Corollary 1.1. Let X be a normal space, and let A and B be a pair of disjoint closed sets in X. Then there is a continuous function $g : X \to [a, b]$ such that $g(x) = a$ for $x \in A$ and $g(x) = b$ for $x \in B$.

Take $g = (b - a)f + a$ as f appearing in Urysohn's Lemma.

Example 1.3. In (X, τ) if for any pair of disjoint closed sets A and B there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$, then (X, τ) is Normal.

Solution : Under the given hypothesis, put
$$G = \left\{x \in X : f(x) < \frac{1}{2}\right\} = f^{-1}\left[0, \frac{1}{2}\right) \quad \text{and} \quad H = \left\{x \in X : f(x) > \frac{1}{2}\right\} = f^{-1}\left(\frac{1}{2}, 1\right].$$

Since $\left[0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right]$ are open sets in [0, 1] with usual topology of reals, and since f is continuous it follows that G and H are a pair of disjoint open sets in X satisfying

$$A \subset G \text{ and } B \subset H.$$

So (X, τ) is normal.

Theorem 1.6. A product of Tychonoff spaces is a Tychonoff space.

Proof : Given a Topological space (X, τ), and a continuous function $f : X \to [0, 1]$ with usual topology of reals, let us make a convention :—

If $x \in X$, and U is a nbd. of x, we say that f is for a pair (x, U) if $f(x) = 0$ and $f(u) = 1$ if $u \in (X/U)$.

Then if $f_1, f_2, ..., f_n$ are functions for $(x, U_1), ... (x, U_n)$, (n = a +ve integer), and if $g(x) = \underset{1 \le i \le n}{\text{Sup}}\{f_i(x)\}$, $x \in X$, we see atonce that g is for the pair $\left(x, \bigcap_{i=1}^{n} U_i \right)$. Therefore, the space (X, τ) is completely regular if for each x and for each nbd. U of x (one may take U as a sub-basic member of τ), there is a function for the pair (x, U).

Suppose X = the product $X\{X_\alpha : \alpha \in \Delta\}$ of Tychonoff spaces X_α and take $x \in X$; Let U_α be a nbd. of x_α in X_α. If f is a function for (x_α, U_α), then $f_0 P_{r_\alpha}$ ($P_{r_\alpha} \equiv \alpha$th Projection function) is a function for $\left(x, P_{r_\alpha}^{-1}(U_\alpha) \right)$. Now family of sets like $P_{r_\alpha}^{-1}(U_\alpha)$ constitute a sub-base for the product Topology ; and therefore the product space is completely regular. Since product of T_1-spaces is again a T_1-space, we have proved Theorem.

Theorem 1.7. (Tietze Extension Theorem) Let (X, τ) be a normal space, and F be a closed sub-space of X, and let $f : F \to [a, b]$ be a continuous function. Then f has an extension h over X with values in $[a, b]$.

Proof : Invoking corollary 1.1, we may assume that $a = -1$ and $b = 1$. Put $f_0 = f$ and $A_0 = \left\{x \in F : f_0(x) \le -\frac{1}{3}\right\}$, $B_0 = \left\{x : f_0(x) \ge \frac{1}{3}\right\}$. So A_0 and B_0 are a pair of non-empty disjoint closed subsets in F, and hence are also closed in X, because F is closed. Now apply again corollary 1.1 to find a continuous function $g_0 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $g_0(A_0) = \left\{-\frac{1}{3}\right\}$ and $g_0(B_0) = \left\{\frac{1}{3}\right\}$. Define $f_1 = f_0 - g_0$ on F; then $|f_1(x)| \le \frac{2}{3}$ for all $x \in F$.

If $A_1 = \left\{x : f_1(x) \le \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$, and

45

$$B_1 = \left\{ x : f_1(x) \geq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) \right\}.$$

Then by reasoning as above, there is a continuous function $g_1 : X \to$ $\left[\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right), \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right]$ such that $g_1(A_1) = \left\{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$ and $g_1(B_1) = \left\{\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\right\}$.

Next, define f_2 on F by $f_2 = f_1 - g_1 = f_0 - (g_0 + g_1)$, and see that $|f_2(x)| \leq \left(\frac{2}{3}\right)^2$. We continue this process to obtain a sequence $\{f_0, f_1, \ldots\}$ defined on F such that $|f_x(x)| \leq \left(\frac{2}{3}\right)^n$ and a sequence $\{g_0, g_1, \ldots\}$ defined on X such that $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ with property that on F we have $f_n = f_0 - (g_0 + g_1 + \ldots + g_{n-1})$. Define $S_n = g_0 + g_1 + \ldots + g_{n-1}$, we treat S_n as n-th partial sum of an infinite series of functions (real-valued) over X. We know that the space $C(X, R)$ of all real-valued continuous functions with sup norm becomes complete;

As $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ and since $\sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1$, We conclude that $\{S_n\}$ converges uniformly to a bounded continuous h such that $|h(x)| \leq 1$ over X. We may conclude our proof by the observation that $|f_n(x)| \leq \left(\frac{2}{3}\right)^n$ and $\{S_n\}$ converges uniformly over F to f_0 which is equal to f, and that h equals to f over F, and h is a continuous extension of f over whole space X which has the desired property.

Remark : Theorem 1.7 is not true if we remove assumption that F is closed. For example, take X to be the closed Unit interval $[0, 1]$, and take F $\subset$ X as F $= (0, 1]$. Then look at $f : F \to [-1, 1]$ where $f(x) = \sin\frac{1}{x}$. Then f is continuous, but f does not admit a continuous extension over $[0, 1]$.

§2. Embedding in Cube.

The Cartesian product of closed unit intervals $Q = [0, 1]$ with product topology is called a cube.

So we look upon a cube as the set Q^A of all functions on A to Q. Suppose F is a family of functions such that each $f \in F$ acts on a Topological space X to a space Y_f (Y_f may be different for different $f \in F$). Put $X\{Y_f : f \in F\}$. There is a function $e : X \rightarrow$ the Product, often named as evaluation function where $e(x \in X)_f = f(x)$. It is seen that if members of F are continuous, then evaluation function e is continuous. F is said to distinguish points of X, if given x, $y \in X$ with $x \neq y$, there is a member $f \in F$ such that $f(x) \neq f(y)$. F is said to distinguish points and closed sets if for each closed set A in X and $x \in (X \backslash A)$, we find a member $f \in F$, such that $f(x) \notin \overline{f(A)}$.

Theorem 2.1. (Embedding Lemma). Suppose F is a family of continuous functions $f : X \rightarrow Y_f$ (X, Y_f are Topological spaces); then

(a) Evaluation function $e : X \rightarrow X \{Y_f : f \in F\}$ is continuous.

(b) e is an open function of X onto e [X] if F distinguishes points and closed sets. ·

(c) e is $1 - 1$ if and only if F distinguishes points.

Proof : (a) we see that composition of evaluation function e with projection function p_{r_f} i.e. $p_{r_f}.e$ is equal to f; and here we know that $p_{r_f} \circ e$ is continuous, because $\left(p_{r_f^o} e\right)(x) = f(x)$. Consequently e is continuous, by Theorem 3.2 (Unit II).

(b) It suffices to show that image under e of an open nbd. U of x contains $e[X] \cap$ a nbd. of e(x) in product. Take a member $f \in F$ such that $f(x) \notin$ closure of f(X/U). The set of all y in the product such that $y_f \notin \overline{f(X \backslash U)}$ is open, and its intersection with e[x] is a subset of e[U]. So, e is an open function of X onto e[X]. Now (c) is clear.

47

Theorem 2.2. (Embedding Theorem). (X, τ) is a **Tychonoff space if and only if** it is **homeomorphic** to a sub-space of **a cube.**

Proof : It is a routine verification that the closed unit interval with usual topology of reals is a completely regular space and it is also T_1. So it is a Tychonoff space. As product if Tychonoff spaces is a Tychonoff space. So a cube is a tychonoff space. Each sub-space of a cube is therefore a Tychonoff space. We now observe that if X is a Tychonoff space and F is is the collection of all continuous functions on X to closed unit interval, then by Embedding Lemma Evaluation function is a homeomorphism of X into the cube $Q^{\mathscr{F}}$.

§3. Metrization :

Metrization problem in Topology deals with obtaining necessary and sufficient conditions for a Topological space to be metrizable. A partial answer to the problem had been given by Urysohn as early as 1924 through a Theorem, better known as Urysohn's Lemma that we present below. We know that the sequence space l_2 of all real sequences $x = \{x_1, x_2, \ldots x_n, \ldots\}$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$ is a metric space with a metric d given by $d(x, y)$

$$= \left(\sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{1/2},$$ where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in l_2$. This

sequence space l_2 is a known as a real Hilbert space.

Theorem 3.1. (Urysohn's Metrization Lemma)

If (X, τ) is a secon countable normal T_1-space, then it is homeomorphic onto a subspace of l_2, and hence is metrizable.

Proof : Without loss of generality assume that X is infinite. Since (X, τ) is second countable, it has a countable open base, say $\mathscr{B} = (G_1, G_2, \ldots, G_n, \ldots)$, each G_i being different from ϕ and X. If $p \in G_k$, since X is Normal and T_1, we find a member G_i (say) of $\mathscr{B}$ such that

$$p \in G_i \subset \overline{G}_i \subset G_k .$$ 　　　　　　　　　... (1)

48

Put $Q = \{(G_i, G_k) : \overline{G}_i \subset G_k\}$, clearly Q is countable.

and let us write $Q = \{D_1, D_2, ...\}$.

For each ordered pair $D_n = (G_i, G_k)$, (say) with $\overline{G}_i \subset G_k$, we find the pair $(\overline{G}_i, G_k^c)$, ('c' meaning complement) as a pair of disjoint closed sets in normal space (X, τ), and hence it invites a continuous function $f_n : X \to [0, 1]$ such that

$$f_n(\overline{G}_i) = \{0\} \text{ and } f_n(G_k^c) = \{1\} \qquad \cdots \qquad (2)$$

Define a function $f(x)$ on X by

$$f(x) = \left\{ f_1(x), \frac{1}{2} f_2(x), ..., \frac{1}{n} f_n(x), ... \right\}.$$

For each natural number n, we have $0 \le f_n(x) \le 1$, and hence $\left(\frac{f_n(x)}{n} \right)^2 \le \frac{1}{n^2}$, and we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Therefore $\sum_{n=1}^{\infty} \left(\frac{f_n(x)}{n} \right)^2 < \infty$ as $x \in X$. So $f(x)$ is a member in l_2.

i.e. $f : X \to l_2$. Now **f has following properties** :-

(a) f is 1–1. Let $x, y \in X$ with $x \ne y$. By T_1-separation we find $G_k \in B$ such that $x \in G_k$ and $y \notin G_k$. Then from (1) we find $G_i \in B$ such that

$$x \in G_i \subset \overline{G}_i \subset G_k.$$

From (2) $f_n(x) = 0$, $x \in \overline{G}_i$ and $f_n(y) = 1$; That is to say, nth term of the sequence $f(x) = 0$, but nth term of the sequence $f(y)$ shall be $\frac{1}{n}$. Thus $f(x) \ne f(y)$. Hence f is 1–1.

(b) f is continuous. Let $x_o \in X$, and $\in > 0$ be given. Take $y \in X$ with $y \ne x_0$. Then

$$f(y) = \left\{ f_,(y), \frac{1}{2} f_2(y), ..., \frac{1}{n} f_n(y), ... \right\}$$

49

and $f(x_0) = \left\{ f_1(x_0), \dfrac{1}{2} f_2(x_0), ..., \dfrac{1}{n} f_n(x_0), ... \right\}$,

Since value of f_n lie in $[0, 1]$, we have

$$\frac{\left| f_n(y) - f_n(x_0) \right|^2}{n^2} \le \frac{1}{n^2} \qquad \qquad ... \qquad (3)$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we find an integer n_0 (independent of y and x_0), such that

$$\sum_{n=n_0+1}^{\infty} \frac{1}{n^2} < \frac{1}{2} \in^2 \qquad \qquad ... \qquad (4)$$

By continuity of f_n at x_0, for $n = 1, 2, ..., n_0$ we take open sets V_n containing x_0 such that

$$\frac{\left| f_n(y) - f_n(x_0) \right|^2}{n^2} < \frac{\in^2}{2 n_0} \quad \text{whenever } y \in V_n \qquad \qquad ... \qquad (5)$$

Taking $V = \displaystyle\bigcap_{n=1}^{n_0} V_n$ we see V as an open set containing x_0 such that for

$y \in V$, $d(f(y), f(x_0))^2 = \displaystyle\sum_{n=1}^{\infty} \frac{\left| f_n(y) - f_n(x_0) \right|^2}{n^2}$

$$< \sum_{n=1}^{n_0} \frac{\left| f_n(y) - f_n(x_0) \right|^2}{n^2} + \frac{1}{2} \in^2 \quad \text{from (3) and (4)}$$

$$< \frac{\in^2}{2 n_0} \cdot n_0 + \frac{\in^2}{2} \quad \text{by (5)}$$

$$= \in^2.$$

So, $d(f(y) - f(x_0)) < \in$ whenever $y \in V$.

Hence f is continuous of x_0.

(c) $f^{-1} : f(X) \to X$ is continuous.

Take $x_0 \in X$. There is $G_i \in B$ such that $x_0 \in G_i, \subset \overline{G}_i \subset G_k$ by (1). Let D_{n_0} = pair (G_i, G_k). Choose $\in$ such that $0 < \in < \dfrac{1}{2 n_0}$.

Then $d(f(y), f(x_0)) < \epsilon$ means

$$\sum_{n=1}^{\infty} \frac{\left| f_n(y) - f_n(x_0) \right|^2}{n^2} < \left(\frac{1}{2n_0} \right)^2$$

and hence $\left| f_{n_0}(y) - f_{n_0}(x_0) \right|^2 < \frac{1}{4}$.

Since $x_0 \in \overline{G}_i$, $f_{n_0}(x_0) = 0$, and hence $f_{n_0}(y) < \frac{1}{2}$.

Since $f_{n_0}\left(G_k^c \right) = \{1\}$, We see at once that $y \notin G_k^c$. Hence $y \in G_k$. Therefore $d(f(y), f(x_0)) < \epsilon$ implies $y \in G_k$. So f^{-1} is continuous at $f(x_0) \in f(X)$. From properties (a), (b) and (c) it is shown that f is a homeomorphism of X onto a sub-space of l_2 i.e. X is homeomorphic to a sub-space of metric space l_2. The proof is now complete.

Exercise - A

Short answer type questions

1. Give an example with reason of a Topological space on which each real-valued continuous function is constant.

2. Show that each normal T_1-space is a Tychonoff space.

3. Show that every T_3-space is a T_2-space.

4. Show that a compact (see unit IV) subset of T_2-space is closed.

5. Show that a finite Topological space that is T_1 has discrete Topology.

6. In a Hausdorff space X if $x \in X$, show that

    $\cap \{\overline{N}_x : N_x \in \mathcal{N}_x\} = \{x\}$, bar denoting the closure, and $\mathcal{N}_x$ denoting the nbd. system at x.

Exercise - B

1. If X is an infinite set, Show that smallest T_1-Topology in X is its Co-finite Topology.

2. Let f, g : $X \to Y$ be continuous function where X is a Topological space and Y is Hausdorff, Show that the subset = $\{x \in X : f(x) = g(x)\}$ is a closed set in X.

3. Let X be a Topological sapce. Prove that following statements are equivalent :–

   (a) X is T_1.

   (b) Every singleton of X is closed.

   (c) Every finite subset of X is closed.

   (d) If $x \in X$, $\cap \{N_x : N_x \in \mathcal{N}_x\} = \{x\}$.

4. Show that every metric space is Normal.

5. Let X be a Hausdorff space and $f : X \to X$ be a continuous function. Show that the set $\{x \in X : f(x) \neq x\}$ is an open set in X.

6. Show that a Compact (see unit IV) T_2-space is metrizable if it is second countable.

7. Show that a homeomorphic image of a Hausdorff space is Hausdorff.

Unit 4 ❑ Compactness in Topological Spaces

(Open cover, Sub-cover, Compactness, Countable open cover, Lindelöff space, Lindelöff Theorem, Properties of compact sets, Finite intersection property, Its relation with compactness, Tychonoff Theorem on Product of Compact spaces, Continuous image of a compact space, Locally compact spaces, I-point compactification).

§1. Heine-Borel Theorem is a wellknown phenomenon in real analysis. The essence of this lies in Compactness in a Topological space (X, τ).

Definition 1.1(a). A family $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Lambda}$ of **open sets** G_α in X is said to be **an open Cover for X if** $X = \underset{\alpha \in \Lambda}{U} G_\alpha$.

(b) A sub-family $\mathscr{L} \subset \mathscr{G}$ is said to be an **open sub-cover for X** if $\mathscr{L}$ is itself an open cover for X.

For example, the family θ of all **open intervals like (a, b)**, a, bar reals with $a < b$ forms **an open cover** for the space R of reals with usual topology ; because each open interval (a, b) is an open set, and the sub-family θ_0 of θ consisting of all members like **(−n, n)**, n = 1, 2, ... also **forms an open cover for R,** and θ_0 is a sub-cover for R.

Definition 1.2. (X, τ) is said to be compact if each open cover for X has a finite open sub-cover for X.

Explanation : The real number space R with usual topology is not compact. Because open cover consisting of **all** open intervals (−n, n) does not have a finite open sub-cover for R. Ofcourse, **there are many compact spaces ;** for instance every finite Topological space is compact.

Example 1.1. Let X be an infinite set and τ be the Co-finite Topology for X ; Then (X, τ) is compact.

Solution : Let $\mathscr{G}$ be an open cover for X and fix a member G_{α_0} of $\mathscr{G}$. Then $(X \backslash G_{\alpha_0})$ is a finite set, say, $= (x_1, x_2, ..., x_m)$ in X. Since $\mathscr{G}$ is an open cover

for X i.e. $X = U\{G_\alpha : \alpha \in \Lambda\}$; there are members G_{α_1}, G_{α_2}, ..., G_{α_m} of $\mathscr{G}$ such that $x_i \in G_{\alpha_i}$. $i = 1, 2, ..., m$. Then G_{α_1}, G_{α_2}, ..., G_{α_m} phus G_{α_0} all together form a finite sub-family of $\mathscr{G}$ such that $X = G_{\alpha_1} \cup G_{\alpha_2} \cup ... \cup G_{\alpha_m} \cup G_{\alpha_0}$. Hence (X, τ) is compact.

Definition 1.2. A subset E of (X, τ) is said to be compact if as a sub-space with relativised Topology E becomes a compact space.

Definition 1.3. (X, τ) is called a Lindelöff space if every open cover of X has a countable open sub-cover.

For example, it follows from Definitions that every compact space is a Lindelöff space.

Theorem 1.1. Every second countable space (X, τ) is Lindelöff.

Proof : Let (X, τ) be a second countable space with $(W_1, W_2, ..., W_n, ...)$ as a countable open base for τ. Now take $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for X. Corresponding to any $x \in X$, there exists some $\alpha \in \Delta$ such that $x \in G_\alpha$; we find a base member, W such that $x \in W \subset G_\alpha$. The corresponding base members $\{W\}$ form a part of $\{W_1, W_2, ..., W_n, ...\}$ and is a countable sub-family of $(W_1, W_2, ...\}$, say, $\{W_{n_1}, W_{n_2}, ..., W_{n_k}, ...\}$; Now each $W_{n_i} \subset G_i$—some member of $\mathscr{G}$, $i = 1, 2, ...$ Then $\{G_1, G_2, ...\}$ is a countable sub-family of $\mathscr{G}$ to act as a sub-cover for X; and (X, τ) is a Lindelöff space.

Corollary 1.1. The space R of all reals is a second countable space and is therefore a Lindelöff space, without being compact, with usual topology with basis consisting of intervals (a, b), a and b rationals. Thus a Lindelöff space need not be a compact space.

Remark : Theorem 1.1. is often named as Lindelöff Theorem.

Example 1.2. Continuous image of a Lindeloff space is a Lindelöff space.

Solution : Let $f : (X, \tau) \to (Y, V)$ be a continuous function where (X, τ) is Lindelöff, and (Y, V) is any Topological space. We show that $f(X)$ is a Lindelöff sub-space of (Y, V).

Take $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for f(X); put $G_\alpha = f(X) \cap H_\alpha$ where H_α is open in (Y, V) for each $\alpha \ \varepsilon \ \Delta$. By continuity of f we have $f^{-1}(H_\alpha)$ is open in X for each $\alpha \ \varepsilon \ \Delta$.

Now $f(X) = \bigcup_{\alpha \in \Delta} G_\alpha \subset \bigcup_{\alpha \in \Delta} (H_\alpha)$;

So $X \subset f^{-1} \left(\bigcup_{\alpha \in \Delta} H_\alpha \right) = \bigcup_{\alpha \in \Delta} f^{-1} (H_\alpha)$.

That is to say, $X = \bigcup_{\alpha \in \Delta} f^{-1} (H_\alpha)$; Hence the family $\{f^{-1} (H_\alpha)\}$ forms an open cover for X. As (X, τ) is Lindeloff, there is a countable open sub-cover for X, say,

$$X = f^{-1}(H_1) \cup f^{-1}(H_2) \cup ... \cup f^{-1}(H_n) \cup ...$$

So, $f(X) = \bigcup_{n=1}^{\infty} H_n$.

Thus $f(X) = f(X) \cap \left(\bigcup_{n=1}^{\infty} H_n \right) = \bigcup_{n=1}^{\infty} f(X) \cap H_n = \bigcup_{n=1}^{\infty} G_n$.

So, $\{G_n\}$ becomes a countable sub-family of $\{G_\alpha\}_{\alpha \in \Delta}$ to cover f(X). Hence f(X) is a Lindelöf sub-space of (Y, V).

Theorem 1.2. Every closed set in a compact space is compact.

Proof : Let (X, τ) be a compact space and E be a closed subset in X.

So its complement (X\E) is open in X. Suppose $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for E i.e. $E \subset \bigcup_{\alpha \in \Delta} G_\alpha$, each G_α being open in X. So the family $\{G_\alpha\}_{\alpha \in \Delta}$ and (X\E) form an open cover for X, by compactness of which we find a finite sub-family of this enlarged family as an open sub cover for X. Let $G_{\alpha_1}, G_{\alpha_2}, ...,$ G_{α_n} and possibly (X\E) form a finite open sub-cover for X, and hence from a finite open sub-cover for E. Clearly $\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_n}\}$ is a finite open sub-cover for E, and E is compact.

Remark : Coverse of Theorem 1.2 is not true. For example, take X = a, b with topology τ = {ϕ, X, (a)} Then the subset (a) is not closed in (X, τ). However it is compact in (X, τ).

Theorem 1.3. Every compact subset of a T_2-space is closed.

Proof : Let (X, τ) be a T_2-space and E be a compact subset of X. It suffices to show that (X\E) is open in X; without loss of generality take (X\E) $\neq$ ϕ, and x $\in$ (X\E). If y $\in$ E, we have x $\neq$ y, and by T_2-separation we find two disjoint open sets V_y and W_y such that x $\in$ V_y and y $\in$ W_y with $V_y \cap W_y$ = ϕ ; So

$$V_y \subset (X\backslash W_y) \qquad \cdots \qquad (1)$$

Now $E \subset \underset{y \in E}{\cup} W_y$; showing that the family $\{W_y\}_{y \in E}$ is an open cover for E, by compactness of which we obtain a finite number of members, say, W_{y_1}, W_{y_2}, ..., W_{y_n} out of this family such that

$$E \subset \overset{n}{\underset{k=1}{\cup}} W_{y_k} \qquad \cdots \qquad (2)$$

Look at the corresponding open sets V_{y_1}, V_{y_2}, ..., V_{y_n} each containing x, and put

$$G = V_{y_1} \cap V_{y_2} \cap ... \cap V_{y_n} . \qquad \cdots \qquad (3)$$

Clearly G is an open set containing x such that (from (1) and (3))

$$G \subset (X \backslash W_{y_1}) \cap (X \backslash W_{y_2}) \cap ... \cap (X \backslash W_{y_n})$$

$$= X(W_{y_1} \cup W_{y_2} \cup ... \cup W_{y_n})$$

$$\subset (X\backslash E) \text{ by (2)}.$$

Thus for each x $\in$ (X\E), we find an open set G containing x such that G $\in$ (X\E); Hence (X\E) is open in X.

Theorem 1.4. Show that a subset E of reals with usual Topology is compact if and only if E is closed and bounded.

Proof : Let E be a bounded and closed set of reals in respect of usual Topology of reals. Suppose [a, b] (a < b) be a closed interval such that

$E \subset [a, b]$, since E is bounded. We appeal to Heine-Borel Theorem to say that $[a, b]$ is compact ; and Now Theorem 1.2 applies to conclude that E is compact.

Conversely, suppose E is a compact set of reals with usual topology which is T_2; we apply Theorem 1.3 to see that E is closed. Finally, we see that $E \subset \bigcup\limits_{n=1}^{\infty}(-n, n)$; So the family $\{(-n, n)\}_{n = 1, 2, \dots}$ is an open cover for E, by compactness of which we find a finite number of members $(-n_1, n_1)$, ..., $(-n_k, n_k)$ with $n_1 < n_2 < \dots < n_k$ such that $E \subset \bigcup\limits_{i=1}^{k}(-n_i, n_i) = (-n_k, n_k)$ and hence E is shown to be bounded.

§2. Definition 2.1.

A family $\mathscr{F}$ of subsets $(F_\alpha)_{\alpha \in \Delta}$ in a topological space (X, τ) is said to have the finite intersection property (F.I.P) if every finite sub-family $\{F_{\alpha_1}, F_{\alpha_2}, \dots, F_{\alpha_n}\}$ of $\mathscr{F}$ has non-empty intersection i.e.

$$\bigcap\limits_{i=1}^{n} F_{\alpha_i} \neq \phi.$$

For example, every decreasing sequence $\{A_n\}$ of non-empty subsets A_n of X has F.I.P. ; because every finite sub-family of $\{A_n\}$ has the smallest member ($\neq \phi$) as its intersection.

Theorem 2.1. A topological space (X, τ) is compact if and only if each family $F = \{F_\alpha\}_{\alpha \in \Delta}$ of closed sets with F.I.P has a non-empty intersection.

Proof : Let (X, τ) be compact and $F = \{F_\alpha\}_{\alpha \in \Delta}$ be a family of closed sets with F.I.P. If possible, Let $\bigcap\limits_{\alpha \in \Delta} F_\alpha = \phi$. That means $X = \bigcup\limits_{\alpha \in \Delta}(X \backslash F_\alpha)$ where each $(X \backslash F_\alpha)$ is open ; and hence the family $\{(X \backslash F_\alpha) : F_\alpha \in F\}$ is an open cover for X. By compactness of (X, τ), there is a finite family, say, $(X \backslash F_1)$, $(X \backslash F_2)$, ..., $(X \backslash F_n)$ of this family to cover X.

So we have $X \subset \bigcup\limits_{i=1}^{n}(X \backslash Fi)$, and taking the complement, we duduce that $\bigcap\limits_{i=1}^{n} F_i^{i=1} = \phi -$ a contradiction that F has F.I.P.

Conversely, suppose the stated condition holds without (X, τ) being compact. We seek a contradiction. Let $\mathcal{G} = \{G_j\}_{j\in\Delta}$ be an open cover of X that has no finite sub-cover for X. Now $X = \underset{j\in\Delta}{\cup} G_j$; taking $F_j = (X\backslash G_j)$ we see that

$\phi = \underset{j\in\Delta}{\cap}(X\backslash G_j)$; hence the family $\mathcal{F}$ consisting of closed sets $F_j = (X\backslash G_j)$ as

$j \in \Lambda$ is such that $\phi = \cap\{F_j : F_j \cdot \in \mathcal{F}\}$—a contradiction; however by **assumption F does have F.I.P.** Therefore, there is a finite sub-family say $F_1, F_2, ..., F_n$ of F such that $\overset{n}{\underset{i=1}{\cap}} F_i = \phi$ i.e. $X = \overset{n}{\underset{i=1}{\cup}}(X|F_i) = \overset{n}{\underset{i=1}{\cup}} G_i$; that means $\mathcal{G}$ admits of a finite sub-cover for X, and (X, τ) is compact.

Theorem 2.2. (Tychonoff Theorem on Product) The Cartesian product of any number of compact spaces is compact w.r.t product Topology.

Proof : Let (X_α, τ_α) be compact spaces and : Let $X = X\{X_\alpha : \alpha \in \Delta\}$. We show that with product topology X is compact.

Put $\mathcal{SB} = \left\{p^{-1}_{r_\alpha}(u_\alpha) : u_\alpha \in \tau_\alpha ; \alpha \in \Delta\right\}$ Then the family $\mathcal{SB}$ forms a sub-base for the product topology for X, p_{r_α} denoting the projection function : $X \to X_\alpha$ as $\alpha \in \Delta$ now X will be compact if each sub-family φ of $\mathcal{SB}$ such that no finite sub-collection of φ forms a covering of X, fails to form a covering of X.

For each $\alpha \in \Delta$, Let $\mathcal{B}_\alpha$ denote the family of all those open set $u_\alpha \in \tau_\alpha$ for which $p^{-1}_{r_\alpha}(u_\alpha) \in \varphi$. Then no finite sub-family of $\mathcal{B}_\alpha$ forms a covering for X_α; and hence ashole family can not form a covering of X_α since (X_α, τ_α) is compact. So there is a point (say) $x_\alpha \in X_\alpha$ which is missing in any open set $u_\alpha \in \mathcal{B}_\alpha$. Then the point $x \in X$ with αth co-ordinate x_α, does not belong to any member of φ—meaning that φ does not form a cover for X.

Corollary : Each cube as the product of closed unit intervals is compact.

Definition 2.2. A subset H in (X, τ) is said to be **nowhere dense in X if Int** $\left(\overline{H}\right) = \phi$, bar denoting the closure.

For example every finite subset of reals with usual Topology is a nowhere dense set of reals.

Theorem 2.2(a). If **an infinite number of** Co-ordinate **spaces** are **not compact, then each compact set** in their **product** with product Topology **is no-where dense.**

Proof : Let (X_α, τ_α) be Topological spaces for each α of an infinite index set Δ be $X = X\{X_\alpha : \alpha \in \Delta\}$ with the product Topology.

Take G be a compact set in the product space X. Suppose u is an interior point of G. Take N as a nbd. of u with $N \subset G$; Without loss of generality take N as a member of the defining base. Thus N is of a form :-

$N = \cap\{p_{r_\alpha}^{-1} [V_\alpha] : \alpha \in F\}$, where **F is a finite subset of Δ,** and V_α is open in X_α. If $\beta \in (\Delta \backslash F)$, then $p_{r_\beta}(G) = X_\beta$, and X_β being continuous image of compact space shall be compact. Therefore all but a finite number of co-ordinate spaces are compact.

Theorem 2.3. Continuous image of a compact space is compact.

Proof : Let (X, τ) be a compact space and (Y, V) be a topological space, and $f : X \to Y$ be a continuous function. We show that $f(X)$ is compact in Y. Take $\mathscr{G} = \{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for $f(X)$. So

$$f(X) \subset \cup\{G_\alpha : G_\alpha \in \mathscr{G}\}.$$

Therefore $\quad X \subset f^{-1}\left(\underset{\alpha \in \Delta}{\cup} G_\alpha\right) = \underset{\alpha \in \Delta}{\cup} f^{-1}(G_\alpha)$ $\qquad$ . . . $\quad$ (1)

By continuity of f each $f^{-1}(G_\alpha)$ is open in X, and (1) shows that the family $\{f^{-1}(G_\alpha)\}_{\alpha \in \Delta}$ is an open cover for (X, τ) by compactness of which it follows that there are a finite number of members, say, $f^{-1}(G_1), f^{-1}(G_2), ..., f^{-1}(G_n)$ such that

$$X \subset f^{-1}(G_1) \cup f^{-1}(G_2) \cup ... \cup f^{-1}(G_m) = f^{-1}\left(\underset{i=1}{\overset{n}{\cup}} G_i\right)$$

So $f(X) \subset \left(\underset{i=1}{\overset{n}{\cup}} G_i\right)$.

So given open cover G for $f(X)$ has a finite sub-cover for $f(X)$.

Thus f(X) is compact.

Corollary (1) If f is continuous function of a compact space onto a topological space Y, then Y is compact.

(2) A homeomorphic image of a compact space is compact.

(3) A real-valued continuous function on a compact space is bounded and attains its bounds.

Because, Let $f : (X, \tau) \to$ Reals be continuous where (X, τ) is compact. So f(X) is a compact set of reals with usual topology, and therefore f(X) is bounded and closed.

Theorem 2.4. A 1-1 continuous function of a compact space onto a T_2-space is a homeomorphism.

Proof : Let $f : (X, \tau) \to (Y, V)$ be a continuous function that is 1-1, where (X, τ) is compact and (Y, V) is T_2. It suffices to show that f is an open function or equivalently, we show that if F is a closed set in X, the image f(F) is closed in Y. We know that every closed set in compact space is compact ; So F is a compast set in X, and f sends F to a compact set i.e. f(F) is a compact set in Y-which is T_2. Hence f(F) is closed in Y.

Theorem 2.5. If A is a compact set of a T_2-space (X, τ) and $x \in (X\backslash A)$, there are open sets V and W such that $x \in V$ and $A \subset W$ and $(U \cap W) = \phi$.

Proof : Let A be a compact set in X which is T_2, and take $x \in (X\backslash A)$. If $y \in A$, then $x \neq y$, and by T_2-separation there are opon sets V_y and W_y such that $x \in V_y$, $y \in W_y$ and $V_y \cap W_y = \phi$; and, therefore, $x \notin \overline{W}_y$. Now the family $\{W_y : y \in A\}$ becomes an open cover for A which is compact. So there are a finite number of members (say) $W_{y_1}, W_{y_2}, ..., W_{y_n}$ such that

$A \subset W_{y_1} \cup W_{y_2} \cup ... \cup W_{y_n}$. Put $W = \bigcup_{i=1}^{n} W_{y_i}$. Then W is an open set such

that $A \subset W$ and $x \notin W_{y_i}$ (i = 1, 2, ..., n) and hence $x \notin W \subset \overline{W}$. Take $V = X \setminus \overline{W}$. Then V and W are open sets to satisfy $x \in V$, $A \subset W$ and $V \cap W) = \phi$.

Corollary each compact set in a T_2-space is closed.

Theorem 2.6 Let A and B be two compact sets in a T_2-space X such that $A \cap B = \phi$. Then there are two open set V and W such that $A \subset V$ and $B \subset W$ with $V \cap W = \phi$.

(Consequently, a compact T_2-space is Normal).

Proof : For each $x \in A$, we have B is a compact in X with $x \notin B$ i.e. $(X \backslash B)$; Theorem 2.5 says there are open sets V_x and W_x satisfying $x \in V_x$ and $B \subset W_x$ with $V_x \cap W_x = \phi$. Also $B \cap \overline{V}_x = \phi$. Now family $\{V_x : x \in A\}$ of open sets is an open cover for A which is compact. So there are a finite number of members V_1, V_2, ..., V_n of this family to cover A. Put $V = V_1 \cup V_2 \cup ... \cup V_n$. Then $A \subset V$ and $B \cap \overline{V}_i = \phi$ $(i = 1, 2, ..., n)$ gives $B \cap \overline{V} = \phi$. Let us take $W = (X \backslash \overline{V})$. Then V and W are open sets in X such that $A \subset V$ and $B \subset W$ with $V \cap W = \phi$.

(Consequential statement is clear).

§3. Locally Compact Spaces :

Definition 3.1. A topological space (X, τ) is said to be locally compact if each point in X has a compact neighbourhood (nbd).

Explanation : If (X, τ) is compact, then of course it is locally compact ; one may take X itself as a compact nbd. of each of its points. If τ is discrete and X is infinite then (X, τ) is Locally compact without being compact ; because for each $x \in X$, the singleton $\{x\}$ is a compact nbd. of x. However, family of all singletons is an open cover for X that has no finite sub-cover for X.

Example 3.1. The real number space R with usual topology is not compact. It is Locally Compact.

Solution : R being not bounded, it is not compact. However if $x \in R$, then a closed interval like $[x - \delta, x + \delta]$, $\delta > 0$ is a nbd. of x which is a closed and bounded set of reals and hence it is compact.

Theorem 3.1. Let (X, τ) be Locally compact T_2-space then the family of all closed compact nbds of a point in X is a nbd. base there.

Proof : Let (X, τ) be Locally compact adn a T_2-space. Take $x \in X$. Let C be a compact nbd. of x; suppose U is **any** nbd. if x.

$$\text{Put } W = \text{Int} (U \cap C).$$

Then $\overline{W}$ is a compact T_2-sub-space of X. So W contains a closed compact set V which is a nbd. of x in $\overline{W}$; but V is also a nbd. of x in W (relative to topology in W), and is therefore a nbd. of x in X.

Theorem 3.2. Every closed sub-space of a Locally compact space is locally compact.

Proof : Let E be a closed set of (X, τ) which is Locally compact. We show that (E, τ_E) (where τ_E is the relativised topology of τ on E) is Locally compact. Take $x \in E$. Since X is Locally compact, we find a compact nbd., say, G of x in X. Put $M = G \cap E$. So M is a nbd. of x in E. As M is a closed set in compact space G, we see that M is a compact subset of G i.e. M is a compact nbd. of x in (E, τ_E). Hence (E, τ_E) is Locally compact.

§4. One point compactification :

Let (X, τ) be a Locally compact T_2-space, and u is an element outside X $(u \notin X)$. Put $X_u = X \cup \{u\}$. Define a Topology τ_u on X_u as :–

(i) All open sets in X as subsets of X_u are in τ_u.

(ii) All complements in X_u of compact sets in X are in τ_u and

(iii) $X_u \in \tau_u$.

We now check that above prescribed family is indeed a Topology in X_u. From (i) $\tau \subset \tau_u$, and by (iii) $X_u \in \tau_u$.

FOR UNION : It is O.K. in respect of members of τ (i).

Consider a family $\{(X_u \backslash A_\alpha) : A_\alpha$ is a compact set in X which is $T_2\}_{\alpha \in \Delta}$

Hence A_α is closed in X, and hence $\bigcap_{\alpha \in \Delta} A_\alpha$ is compact in X, and we

have $\bigcup_{\alpha \in \Delta}(X_u/A_\alpha) = \left(X_u / \bigcap_{\alpha \in \Delta} A_\alpha\right)$, and by (ii) we have the union is a member

of τ_u. Also if $G \in \tau$ and $(X_u \backslash H) \in \tau_u$ as in (ii) where H is a compact set

in X, then

$$G \cup (X_u / H) = X_u \setminus ((X \setminus G) \cap H)$$

Since $(X \backslash G)$ is closed in X, and H is compact in X. Therefore $(X \backslash G) \cap H$
is compact in X, and from (ii) r.h.s. set is a member of τ_u.

FOR INTERSECTION : It is O.K. in respect of (i). Take two members
$(X_u \backslash D)$ and $(X_u \backslash C)$ where C and D are compact sets in X as members from (ii).
Then we have $(X_u \backslash D) \cap (X_u \backslash C) = (X_u \backslash (C \cup D))$. Where $C \cup D$ is also compact
in X. Hence r.h.s member $\in \tau_u$ as in (ii). Also if, $A \in \tau_u$ as in (i) and $(X_u \backslash B)$,
B is compact in X as in (ii), then B is also closed in X; because X is T_2.
Therefore $(X_u \backslash B) \backslash (u)$ is open in X.

Now $A \cap (X_u \backslash B)$

$= A \cap \{(X_u \backslash B) \backslash (u)\}$, and hence is a member of τ as in (i).

Our verification is complete, and (X_u, τ_u) is a Topological space. It remains
to show that τ **is actually equal to relativised Topology of τ_u on X.**

If $G \in \tau$, then $G \in \tau_u$ and $G = G \cap X$. Thus

$$\tau \subset \text{relativised Topology } \tau_u \text{ on X} \qquad \dots (1)$$

Let H be an open in relatised Topology τ_u in X. So we write $H = X \cap A$
for some member $A \in \tau_u$.

In case A is of type as in (i) we have H as a member of τ.

In case A is of type as in (ii), put $H = X \cap (X_u \backslash B)$ for some compact set
B in X. Because X is T_2, we find B as a closed set in X. So $(X \backslash B)$ is a member
of τ. Therefore from above

$H = X \cap (X_u \backslash B) = X \cap (X \backslash B)$, and so, $H \in \tau$. Therefore

$$\text{relativised topology } \tau_u \text{ in } X \subset \tau \qquad \dots (2)$$

Combining (1) and (2) proof is complete.

Theorem 4.1. (X_u, τ_u) is a compact T_2-space.

Proof : Take $x, y \in X_u$ with $x \neq y$. If x, y are members of X, we have finished. So let us suppose one of them equals u, say $u = y$. Now (X, τ) is Locally compact, so, there is a compact nbd. N_x of x in X. As X is T_2 we take N_x to be closed. Then

Int $(N_x) \cap (X_u \backslash N_x) = \phi$, where Int (N_x) and $X_u \backslash N_x$ are open sets in X_u containing x and u respectively. So (X_u, τ_u) is T_2.

To show that (X_u, τ_u) is compact, suppose $\{G_\alpha\}_{\alpha \in \Delta}$ be an open cover of X_u. Let $u \in G_{\alpha_0}$ for some $\alpha_0 \in \Delta$, and therefore G_{α_0} is such that $G_{\alpha_0} = (X_u \backslash D)$ where D is a compact set in X.

$$\text{i.e. } (X_u \backslash G_{\alpha_0}) = D \qquad \qquad \dots\dots \ (*)$$

Now $(X_u \backslash G_{\alpha_0}) \subset X = \bigcup_{\alpha \in \Delta} G_\alpha$, giving

$$(X_u \backslash G_{\alpha_0}) \subset \bigcup_{\alpha \in \Delta} (X \cap G_\alpha);$$

This shows that the family $\{(X \cap G_\alpha) : \alpha \in \Delta\}$ is an open cover for $(X_u \backslash G_{\alpha_0})$. (*) says that $(X_u \backslash G_{\alpha_0})$ is a compact set in X, and hence there is a finite sub-cover of this open cover for $(X_u \backslash G_{\alpha_0})$ say $(X \cap G_{\alpha_1})$, $(X \cap G_{\alpha_2})$, ..., $(X \cap G_{\alpha_n})$.

Thus $(X_u \backslash G_{\alpha_0}) \subset (X \cap G_{\alpha_1}) \cup (X \cap G_{\alpha_2}) \cup \dots \cup (X \cap G_{\alpha_n})$

i.e. $(X_u \backslash G_{\alpha_0}) \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

So $X_u = G_{\alpha_0} \cup (X_u \backslash G_{\alpha_0})$

$$\subset G_{\alpha_0} \cup G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}.$$

That means Given open cover admits of a finite sub-cover for X_u, and (X_u, τ_u) is compact.

Remark : Compact T_2-space (X_u, τ_u) arising out of a given Locally compact T_2-space (X, τ) in the manner described above is called "**One point compactification**" of X, and external point **u** is designated **as the point at infinity**.

Exercise - A

Short answer type questions

1. Show that a finite Topological space (X, τ) is compact.

2. Show that no open interval of reals with usual topology is compact.

3. Which one of followings subsets of reals with usual topology is compact?

   (a) $[1, 2] \cup [3, 4]$

   (b) The set N of all natural numbers. Give reasons.

4. In a Topological space (X, τ) any set E consisting of points of a convergent sequence together with its converging limit is compact. Prove it.

5. Show that union of a finite number of compact sets is a compact set in a Topological space. Is the union of an infinite number of compact sets a compact set? Give reasons.

6. In a compact T_2-space families of closed sets and compact sets are identical. Prove it by quoting relevant Theorems.

7. Show that an infinite discrete space is Locally compact without being compact.

Exercise - B

1. Show that every regular Lindelöff space is normal.

2. Show that a compact T_2-space is T_4.

3. Show that a sub-space of a compact space need not be compact, and verify that every closed sub-space of a compact space is compact.

4. Show that every real-valued continuous function on a compact space X there are points x and y in X such that $f(x) = \sup_{X} f$ and $f(y) = \inf_{X} f$.

5. Show that intersection of two compact sets in a Topological space X may not be compact. If A is a closed set and B is a compact set in X, show that $A \cap B$ is compact.

6. Show that a continuous image of a locally compact space need not be Locally compact.

Unit 5 ❑ Connectedness

> (Connected spaces, Separated sets, Disconnection of a space, Union of connected sets, Closure of a connected set, Connected sets of reals, Continuous image of a connected space, Topological product of connected spaces, Components, Their properties, Totally disconnected spaces, Locally connected spaces).

§1. A Connected Topological space (X, τ), roughly speaking, is such a **strong piece of objects** that it does not allow its partition into two **non-empty disjoint open (or closed) subsets.** To be more precise we present the following definitions.

Let (X, τ) denote a Topological space.

Definition 1.1. Two subsets A and B of X are said to be separated if $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$.

Explanation : Without loss of generality, assume $A \neq \phi$, $B \neq \phi$. Subsets A and B when separated are, of course disjoint. But there are more things to look at. Neither A nor B contains a limit point of other. In relative topology for $A \cup B$, both A and B are regarded as closed in $(A \cup B)$; or equivalently, A or B is taken as (each) open and hence each has status of a clo-open set in $(A \cup B)$. Take for example, open intervals $(0, 1)$ and $(1, 2)$ of reals with usual topology. They are disjoint subsets. But they are not separated as per Definition 1.1 above because number 1 belongs to closure of each.

Theorem 1.1. If A and B are subsets if (X, τ), and both A and B are closed (or both are open), then $(A \setminus B)$ and $(B \setminus A)$ are separated.

Proof : Let A and B be closed subsets in X. Then relative to $(A \cup B)$, A and B are closed, and therefore $(A \setminus B) = ((A \cup B) \setminus B))$ and $(B \setminus A)$ are open in $(A \setminus B) \cup (B \setminus A)$ and since they are complements relative to $((A \setminus B) \cup (B \setminus A))$, both become closed in $((A \setminus B) \cup (B \setminus A))$. Hence $(A \setminus B)$ and $(B \setminus A)$ are separated.

In case A and B are both open, proof is done by a similar dual arguments.

Definition 1.2(a). (X, τ) is sais to be **connected** if **X is not a union of two non-empty disjoint open sets in X.**

or equivalently, if X is **not a union of two non-empty disjoint closed sets in X**.

(b) A subject Y of X is said to be connected if Y as a sub-space of (X, τ) with relativised topology becomes connected.

or equivalently, Y is not a union of two non-empty disjoint open subsets (closed subsets) of Y in relativised topology on Y.

(c) (X, τ) is said to be disconnected if it is not connected i.e., if X admits of a decomposition like

$$X = G \cup H,$$

Where G and H are a pair of non-empty disjoint open (or closed) subsets of X. Such a decomposition of X is called a disconnected of X.

Explanation : Definitions say that (X, τ) is connected if and only if only clo-open sets are ϕ and X. A subset Y of (X, τ) is disconnected if Y has a partition like $Y = P \cup Q$ where $P \neq \phi$, $Q \neq \phi$, and P and Q are disjoint open subsets of Y. So we put $P = G \cap Y$ and $Q = H \cap Y$ where G and H are open sets in X. Therefore, $Y = (G \cap Y) \cup (H \cap Y)$ where r.h.s. members are non-empty and disjoint.

Conversely if $Y = (G \cap Y) \cup (H \cap Y)$ where G and H are open sets in (X, τ) whose intersections with Y are non-empty and disjoint, then Y is not connected.

Example 1.1. The subset Q of all rationals in real number space with usual topology is disconnected.

Solution : If x, $y \varepsilon Q$ with $x < y$, take an irrational number α such that $x < \alpha < y$.

Then we write $Q = \{(-\infty, \alpha) \cap Q\} \cup \{(\alpha, \infty) \cap Q\}$

Where r.h.s. members are each open sets relative to sub-space Q such that each is non-empty because $x \varepsilon (-\infty, \alpha) \cap Q$ and $y \varepsilon (\alpha, \infty) \cap Q$ and they are disjoint. Hence Q is disconnected.

Remark 1. : Above decomposition of Q is a disconnection for Q. By a different choice of α one gets another disconnection for Q. As the choices are many, there are many disconnections for Q.

Remark 2. : By a similar reasoning one sees that the set E of all irrationals in the real number spece R with usual topology is also disconnected, and there are many disconnections for E.

Remark 3. : We shall presently see that the real number space R with usual topology is a connected space. Thus we at once conclude that **Union of two disconnected sets may be a connected set.**

Theorem 1.2. If A and B are two non-empty separated sets in (X, τ), then $A \cup B$ is disconnected.

Proof : Let A and B be a pair of separated sets (non-empty) in (X, τ).

Then we have $A \cap \overline{B} = \phi = \overline{A} \cap B$. Put $G = (X \setminus \overline{B})$ and $H = (X \setminus \overline{A})$.

Then G and H are disjoint open sets of X such that

$A \cup B = ((A \cup B) \cap G) \cup ((A \cup B) \cap H)$, which is a disconnection for $(A \cup B)$.

Theorem 1.3. A Subset Y of (X, τ) is disconnected if and only if Y is a union of two non-empty separated sets.

Proof : The condition is sufficient : This part follows from Theorem 1.2.

The condition is necessary : Let Y be a dis connected subset of (χ, τ). So Y admits of a partition like

$$Y = (Y \cap G) \cup (Y \cap H) \tag{*}$$

Where G and H are open sets in X such that their intersections with Y are non-empty and disjoint. We check that members in r.h.s. of (*) are separated sets. It suffices only to verify that neither $(Y \cap G)$ nor $(Y \cap H)$ contains a limit point of the others. We use method of contradiction and let

68

u be a limit point of $(Y \cap G)$ and $u \varepsilon (Y \cap H)$. Since H is an open set containing u, so

$$H \cap ((Y \cap G) \setminus \{u\}) \neq \phi \qquad \qquad \ldots (1)$$

But $(Y \cap G) \cap H = (Y \cap G) \cap (Y \cap H) = \phi$, by hypothesis $\qquad \ldots (2)$

Now (1) and (2) are contradictory, and Theorem is proved.

Example 1.2. The Union $(0, 1) \cup (3, 4)$ is not a connected set of reals with usual topology.

Solution : Take a real number a such that $1 < a < 3$.

If $E = (0, 1) \cup (3, 4)$, we may write

$$E = ((-\infty, a) \cap E) \cup ((a, \infty) \cap E)$$

Where r.h.s. members are each non-empty open sets in E (with relativised topology) and they are disjoint. Hence E is disconnected.

Remark : We shall soon see that **every interval** of reals is **a connected set of reals** with usual topology; and Example 1.2 says that **Union of two connected sets may not be connected**. However, we have the following Theorem.

Theorem 1.4. In (X, τ) if $\{A_\alpha\}_{\alpha \varepsilon \Delta}$ be a family of connected sets such that $\bigcap_{\alpha \varepsilon \Delta} A_\alpha \neq \phi$, then their Union $= \bigcup_{\alpha \varepsilon \Delta} A_\alpha$ is connected.

Proof : Let us assume the contrary and let $A = \bigcup_{\alpha \varepsilon \Delta} A_\alpha$ be disconnected. We seek a contradiction. Suppose A has a disconnection :

$$A = (A \cap G) \cup (A \cap H),$$

Where G and H are two open sets in X such that each of $(A \cap G)$ and $(A \cap H)$ is non-empty and $(A \cap G) \cup (A \cap H) = \phi$. Since $A = \bigcup_{\alpha \varepsilon \Delta} A_\alpha$, we have

$$A_\alpha \subset (G \cup H) \qquad \qquad \ldots (*)$$

Now A_α lies entirely either in G or in H; ... (*) otherwise, $(A_\alpha \cap G) \neq \phi$, $(A_\alpha \cap H) \neq \phi$ and $A_\alpha = (A_\alpha \cap G) \cup (A_\alpha \cap H)$ gives rise to a disconnection of

A_α. Thus statement in (*) is valid for each member A_α. Further, if $A_\alpha \subset G$ and $A_\beta \subset H$ with $\alpha \neq \beta$, we see that $u \varepsilon \bigcap_{\alpha \varepsilon \Delta} A_\alpha \subset (A_\alpha \cap A_\beta) \subset (G \cap H)$. Ofcourse, $u \varepsilon A$. Hence $u \varepsilon A \cap (G \cap H) = (A \overset{\alpha \varepsilon \Delta}{\cap} G) \cap (A \cap H)$— which is a contradiction. Therefore, members A_α enblock lie either in G or in H, and hence $\bigcup_{\alpha \varepsilon \Delta} A_\alpha = A$ lies either in G or in H—again a contradiction a desired. We have completed the proof.

Example 1.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{a, b, c\}, \{a, b\}\}$. Show that topological space (X, τ) is connected.

Solution : Here, the family of closed sets is $\{X, \phi, \{d\}, \{c, d\}, \{b, c, d\}\}$ and we check that only clo-open sets are ϕ and X. Therefore (X, τ) is connected.

Example 1.2. Let (X, τ) be a topological space where $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Examine if (X, τ) is connected.

Solution : Here $X = \{a\} \cup \{b, c\}$ which is a disconnection for X, and hence (X, τ) is disconnected.

Theorem 1.5. In a topological space (X, τ) let A be a connected set in X and B be a subset in X satisfying $A \subset B \subset \overline{A}$, then B is connected.

Proof : Assume the contrary. Let B have a disconnection like

$$B = (B \cap G) \cup (B \cap H) \qquad \ldots (1)$$

Where G and H are open sets in X such that $(B \cap G)$ and $(B \cap H)$ are non-empty and

$$(B \cap G) \cap (B \cap H) = \phi \qquad \ldots (2)$$

Now $A \subset B$; from (1) we see that $A \subset (G \cup H)$.

Now $A \cap G \neq \phi$ and $A \cap H \neq \phi$ give a disconnection of A as $A = (A \cap G) \cup (A \cap H)$ —making A disconnected. Hence A lies entirely either in G or in H. Suppose, $A \cap H = \phi$. Because H is open, we have $\overline{A} \cap H = \phi$. That

means $B \cap H = \phi$ which is again a contradiction. We thus have reached the desired contradiction and the proof is complete.

Corollary 1.1. : Closure of a connected set is connected in a Topological space.

Theorem 1.6. A Topological space (X, τ) is connected if and only if given any two distinct points in X there is a connected sub-space of X containing both.

Proof : **The condition is necessary :** Let (X, τ) be connected. Given any two distinct points in X, the space X itself takes care of them as desired.

The condition is sufficient : Suppose the condition holds but X is disconnected. We derive a contradiction.

Let $X = C \cup D$ be a disconnections of X, where C and D are a pair of non-empty disjoint open sets in X. Take $c \varepsilon C$ and $d \varepsilon D$; so $c \neq d$ in X. By hypothesis there is a connected subspace G of X containing c and d. Clearly $G \subset X = C \cup D$; Because G is connected, either $G \subset C$ or $G \subset D$. Let $G \subset C$; then c, d ε $G \subset C$ and so $d \varepsilon (C \cap D) = \phi$. This is absurd. Hence we have proved theorem.

§2. Consider the real number space R with usual topology. Here intervals are of various types like $(a, b) = \{x \varepsilon R : a < x < b\}$, $[a, b) = \{x \varepsilon R : a \leq x < b\}$, Similarly $(a, b]$, $(-\infty, a) = \{x \varepsilon R : x < a\}$; $(a, \infty) = \{x \varepsilon R; x > a\}$, and similarly $(-\infty, a]$, $[a, \infty)$ and $(-\infty, \infty) = \{x : x \varepsilon R\} = R$. An interval I of R may be characterised by the property :—

$$a, b \varepsilon I \text{ means the closed interval } [a, b] \subset I.$$

Theorem 2.1. A subset of R with usual topology is connected if and only if it is an interval.

Proof : **The condition is necessary :** Suppose E is a connected subset if R without being an interval. Then we find a pair of distinct members a, b ε E

71

such that [a, b] $\not\subset$ E. Thus there is a member u such that a < u < b and u $\notin$ E..
Then write, $E = ((-\infty, u) \cap E) \cup ((u, \infty) \cap E)$ and that is a disconnection for E
—a contradiction. Hence necessary part is proved.

The condition is sufficient : Let I be an interval of reals. If possible, let
I have a disconnection

$$I = A \cup B,$$

where A and B are a pair of **non-empty disjoint closed sets in I**.

Take x ε A and z ε B; since A $\cap$ B = ϕ we have x ≠ z, and without loss of
generality, assume x < z. Because I is an interval we have the closed interval
[x, z] $\subset$ I. Thus

$$[x, z] \subset (A \cup B)$$

Put y = sup([x, z] $\cap$ A). Then x ≤ y ≤ z; so y ε I. Since A is closed in I, we
have

$$y \varepsilon A \qquad \qquad \dots (1)$$

Therefore, y ≠ z and we have y < z.

By property of supremum, for large natural numbers n, all numbers $y + \frac{1}{n}$
belong to B; and since B is closed, passing on limit as n $\to \infty$, we have
$\lim_{n \to \infty} \left(y + \frac{1}{n} \right) = y \varepsilon B$; this contradicts (1) because A $\cap$ B = ϕ. The proof is now
complete.

Corollary : 1. The real number space R with usual topology is connected.

2. The only non-empty clo-open set in R is R itself.

§3. Theorem 3.1. Continuous image of a connected space is connected.

Proof : Let (χ, τ) be a connected space and $f : (X, \tau) \to (Y, \mu)$ be a continuous
function where (Y, μ) is topological space. We show that f(x) is connected in
Y. If possible, let f(X) be disconnected and let

$$f(X) = (G \cap f(X)) \cup (H \cap f(X))$$

72

Where G and H are open sets in Y such that $G \cap f(X)$ and $H \cap f(X)$ are a pair of non-empty disjoint open sets in $f(X)$.

But continuity of f we know that $f^{-1}(G)$ and $f^{-1}(H)$ are open sets in X, and from (1) we find

$$X = f^{-1}(G) \cup f^{-1}(H),$$

Where members on r.h.s. are non-empty and disjoint. That means X is disconnected —a contradiction that X is connected. The proof is now complete.

Corollary : 3.1. Let (X, τ) and $(Y, \mathcal{V})$ be two Homeomorphic spaces. If X is connected, then Y is connected.

3.2. Any real valued continuous function over a closed interval [a, b] of reals possesses Intermediate-value property.

Because, f[a, b] = image of [a, b] under f is connected, and hence it is an interval of reals, by theorem 2, $f(a)$, $f(b)$ ε f [a, b]; and if $f(a) < f(b)$, we have the closed inverval $[f(a), f(b)] \subset f[a, b]$.

Now if $f(a) < \mu < f(b)$, there is a member c between a and b such that $\mu = f(c)$.

Remark : In Corollary 3.2 domain of the continuous function is taken to be a closed interval. Statement is valid in respect of any interval.

Let us consider a topological space consisting of two mambers 0 and 1 with discrete topology. This discrete two pointic space {0, 1} is disconnected, because {0, 1} = {0} $\cup$ {1} is a disconnection. One can characterise a disconnected space with the help of this discrete space {0, 1}.

Theorem 3.2. A space (X, τ) is disconnected if and only if there is a continuous function f : $(X, \tau) \rightarrow$ {0, 1}, which is onto.

Proof : The condition is necessary : Let (X, τ) be disconnected. X admits of a decomposition like $X = G \cup H$, where G and H are non-empty disjoint open sets in X.

73

Define $f : X \to \{0, 1\}$ by the rule :—

$\qquad f(x) = 0$ if $x \varepsilon G$

and $\qquad = 1$ if $x \varepsilon H$.

Clearly f is an onto function. To check continuity of f we see that ϕ, $(0, 1\}$, $\{0\}$ and $\{1\}$ are the only open sets in the discrete space $\{0, 1\}$ and that $f^{-1}(0)$ is always an open set in X, 0 being any open set in $\{0, 1\}$.

The condition is sufficient : Suppose the condition holds. Then X must be disconnected; because continuous image of a connected space is connected and here the discrete space $\{0, 1\}$ is disconnected.

Theorem 3.3. Topological product of two connected spaces is connected and conversely.

Proof : Let (X, τ) and (Y, v) be two topological spaces, and let $Z = X \times Y$ denote the topological product of X and Y with product topology. Suppose Z is connected. Then consider the projection functions $p_{r_1} : Z \to X$ and $p_{r_2} : Z \to Y$ that are each continuous; and $p_{r_1}(Z) = X$ and $p_{r_2}(Z) = Y$. Since continuous images of connected spaces ae connected it follows that each of X and Y is connected.

Conversely, assume that X and Y are connected spaces. Take a fixed memeber $y_0 \varepsilon Y$, and put $X_{y_0} = X_x\{y_0\}$. Then X and X_{y_0} are homeomorphic. Since X is connected it follows that X_{y_0} is connected. If $x \varepsilon X$, put $Y_x = \{x\} \times Y$. By a similar reasoning Y_x is connected, because Y is connected. As $(x_1 y_0) \varepsilon X_{y_0} \cap Y_x$, it follows that $X_{y_0} \cup Y_x$ is connected by Theorem 1.4. Finally, we write $X \times Y = \bigcup_{x \varepsilon X} (X_{y_0} \cup Y_x)$ and observe that $X_{y_0} \subset (X_{y_0} \cup Y_x)$ for every member $x \varepsilon X$, and because every individual member of r.h.s. is connected we finally see that $X \times Y$ is connected.

Remark : Theorem 3.3 remains valid for an arbitrary number of connected spaces.

Definition 3.1. A connected set C of (X, τ) is said to be a **component** in X if it is **a maximal connected sub set**; that is, C is not properly contained in any other connected set of X.

For example, in a **connected space (X, τ), X is itself a component.**

Example 3.1. Let (X, τ) be a topological space where $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. **Find all the components in X.**

Solution : Here X is not connected; because $X = \{a\} \cup \{b, c, d, e\}$ is a disconnection of X. However, there are connected sets in X. For example $\{a\}$ is a connected set; and we find $\{a\}$ and $\{b, c, d, e\}$ are the only components in X.

Theorem 3.4. Let (X, τ) be a Topological space. Then—

    (a) Each point of X is contained in a component of X.

    (b) The components of X determine a partition for X.

    (c) Each connected set in X is contained in a component of X.

    (d) A connected set in X that is both open and closed is a component of X.

    (e) Each component of X is closed.

Proof : (a) Let $x \varepsilon X$. Put $C_x = \cup \{C_i : C_i$ is a connected set in X containing $x\}$. Because $x \varepsilon C_i$ for every i, it follows that C_x is a connected set. We now show that C_x is maximal. Let $C_x \subset D$ where D is a connected set containing x in X. By construction of C_x we have $D \subset C_x$; and therefore, $C_x = D$. Hence C_x is a component in X containing x.

(b) For each $x \varepsilon X$, construct C_x as in (a). Put $\zeta = \{C_x : x \varepsilon X\}$. We now verify that ζ becomes a partition of X. By construction, $X = \bigcup_{x \varepsilon X} C_x$. Now suppose $C_x \cap C_y \neq \phi$. Take $p \varepsilon C_x \cap C_y$, then we have $C_x \subset C_p$ and $C_y \subset C_p$. Now C_x and C_y are each connected set containing p, by maximality of C_p we

have $C_p = C_x = C_y$. Therefore any two members of ζ are either disjoint or coincident. Hence (b) follows.

(c) Let A be connected set in X and take x ε A; by construction of C_x we find $A \subset C_x$.

(d) Let E be a connected set in X and let E be both open and closed. Now by (c) there is a component C of X such that $E \subset C$. Suppose E is a proper subset of C. Then we write $C = (C \cap E) \cup (C \cap E^c)$, "c" denoting coplement of E. Because E is assumed to be clo-open, this decomposition is a disconnection of C— a contradiction, for a component C is connected. Hence result is E = C = a component.

(e) Let C be a component of X without being closed and $\overline{C}$ is strictly larger than C; Now $C \subset \overline{C}$ shows that $\overline{C}$ is connected because C is so— by corollary 1.1 that contradicts maximality of C which is a component. Hence C is closed.

Example 3.2. Give an example of a Topological space where components are not open.

Solution : Consider the sub-space Q of all rationals in real number space R with relativised topology in respect of usual topology on R. Here each component in Q is a singleton and this is not an open set in Q.

§4. Definition 4.1. : A topological space (X, τ) is said to be totally disconnected if for each pair of distinct points x and y in X, X has a disconnection $X = G \cup H$ with x ε G and y ε H.

Explanation : A totally disconnected space (X, τ) is a T_2-space; because each pair of distinct points of X attracts a disjoint pair of open sets containing them individually. Ofcourse, a totally disconnected space is disconnected.

Example 4.1. The real number space R with upper limit Topology is totally disconnected.

Solution : We know that upper limit Topology for R is generated by left-open intervals like (a, b], a, b ε R with a < b. Let x, y ε R with x ≠ y, x < y; then we write

$$R = (-\infty, x] \cup (x, \infty),$$

where sets in r.h.s. are a pair of disjoint open sets in upper limit Topology containing x and y respectively. So R is totally disconnected.

Remark : With respect to usual topology R is a connected space.

Theorem 4.1. The components of a totally disconnected space are its singletons.

Proof : Let X be a totally disconnected space and let C be a component in X. We show that C does not have more than one point. Let x, y ε C with x ≠ y; as X is totally disconnected, X has a disconnection like X = G ∪ H where G, H are non-empty open disjoint sets with x ε G and y ε H.

We write C = C ∩ X

$$= C \cap (G \cup H) = (C \cap G) \cup (C \cap H),$$

showing that C is disconnected which is not the case. Hence theorem is proved.

Theorem 4.2. The product of totally disconnected spaces as totally disconnected in product topology.

Proof : Let $\{(X_i, \tau_i)\}_{i \varepsilon \Delta}$ be a family of totally disconnected spaces and let X = X$\{X_i : i \varepsilon \Delta\}$ be the product space with product topology. Take two distinct points x = (x_i), and y = (y_i) in X. Therefore, for some index i = $i_0 \varepsilon \Delta$, we have $x_{i_0} \neq y_{i_0}$ in co-ordinate space X_{i_0} which is assumed to be totally disconnected. There we find two disjoint open sets (say) G_{i_0} and H_{i_0} in X_{i_0} such that $x_{i_0} \varepsilon G_{i_0}$ and $y_{i_0} \varepsilon H_{i_0}$ and $X_{i_0} = G_{i_0} \cup H_{i_0}$. Take G = X$\{G_i :$ where $G_i = X_i$ for all i except i_0 in $\Delta\}$ and H = X$\{H_i :$ where $H_i = X_i$ for all i except

i_0 in Δ}. Then we see that $x \varepsilon G$, $y \varepsilon H$ with $G \cap H = \phi$, and G and H are rendered open sets of X in product topology such that

$$X = G \cup H$$

That means X is totally disconnected.

§5. Locally connected spaces :

Definition 5.1. (a) A topological space (X, τ) is said to be locally connected at $x \varepsilon X$, **if every nbd. of x contains a open connected nbd. of x.**

or equivalently, **if open connected nbds of x form a base for the nbd. system at x.**

(b) (X, τ) is said to be locally connected if it is locally connected at each point of X.

Explanation : Unlike the relationship of compactness and local compactness of a space, local connectedness neither implies connectedness nor is implied by connectedness of the space. we have following examples in support of our contention.

Example 5.1. If $X = (0, 1) \cup (2, 3)$ is taken as a topological space with usual topology of reals, then X is locally connected without being connected.

Solution : Taking a real number α with $1 < \alpha < 2$, we write $X = ((-\infty, \alpha) \cap X) \cup ((\alpha, \infty) \cap X)$, and this is a disconnection for X. So X is not connected.

On the other hand if $u \varepsilon X$, say, $0 < u < 1$, and given any nbd. N_u of u in X, we can find on open interval like $(u - \delta, u + \delta)$, $\delta > 0$ such that $(u - \delta, u + \delta) \subset N_u$; as an open interval of reals is connected, if follows that N_u contains an open connected nbd. of u, and X is locally connected at u. If $1 < u < 2$, then also similar conclusion holds.

Example 5.2. Take $X = A \cup B$ as a sub-space of the Eudidean 2-space with usual topology where

$$A = \left\{ (x, y) : 0 < x \le 1; \text{ and } y = \sin \frac{1}{x} \right\},$$

and $\quad B = \{ (0, y) : -1 \le y \le 1 \}$.

Then X is connected without being locally connected.

Solution : Consider a function $f : (0, 1] \to R^2$ where $f(x) = \left(x, \sin \frac{1}{x} \right)$, $0 < x \le 1$.

Then f is continuous, and hence the image $A = f(0, 1]$ is connected, because the interval $(0, 1]$ is connected. Now we check that $X = \overline{A}$ and therefore **X is connected** because $\overline{A}$ is so.

However, **X is not locally connected at (0, 1)** ε **X**. Because open circular disc centred at $(0, 1)$ with radius, say, $= \frac{1}{4}$ does not contain any connected open set containing the point $(0, 1)$.

Theorem 5.1. (X, τ) is locally connected if and only if components of each open subspace of X are openin X.

Proof : The condition is necessary : Let (X, τ) be locally connected, and let Y be an open sub-space of X. Suppose C is a component of Y. Take $x \varepsilon C$. Since X is locally connected at x, there is an open cunnected set U in X such that $x \varepsilon U \subset Y$. Now $x \varepsilon C \cap U$ where U and C are connected; therefore $C \cup U$ is connected and $C \cup U \subset Y$. Since C is a component, by maximality of C, we have $(C \cup U) = C$ or $U \subset C$. That is $x \varepsilon U \subset C$; as x is an arbitrary member of C, we conclude that C is open.

The condition is sufficient : Suppose the condition holds. Let $x \varepsilon X$, and let N_x be an open nbd. of x in X. Take C as a component such that $x \varepsilon C \subset N_x$. By assumed condition C is open; this shows that there is an open connected nbd. C of x such that $C \subset N_x$. Thus X is locally connected at x; otherwords, X is locally connected.

Example 5.3. Continuous image of a locally connected space may not be locally connected.

Solution : Take $X = \{0, 1, 2, 3, \ldots\}$ with discrete topologies and $Y = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ as a sub-space of reals with usual topology. Consider a function

$\phi : X \rightarrow Y$ where $\phi(0) = 0$, and $\phi(n) = \frac{1}{n}$, $n = 1, 2, \ldots$

The function ϕ is an $1-1$ and onto continuous function, such that X is locally connected; but $Y = f(x)$ is not locally connected, because induced topology on Y is not discrete, but singletons in Y are connected.

EXERCISE - A
Short answer type Questions

1. Show that a topological space (X, τ) with indiscrete topology τ is connected.

2. If X has more than two members, show that (X, τ) with discrete topology τ is disconnected.

3. Give an example to show that connectedness is not a hereditary property.

4. Examine if the real number space R with lower limit topology is connected.

5. Show that the set of all irrational numbers with topology of reals is a disconnected set.

6. Give an example of a topological spece with a non-discrete topology where each singleton is a component.

   (Hint : Take the sub-space Q of all rationals with usual topology of reals).

EXERCISE - B

1. If a Hausdorff space (X, τ) has an open base whose members are closed, show that X is totally disconnected.

2. Give an example to show that product space of an arbitrary number of locally connected spaces may not be locally connected.

3. Show that image of a locally connected space X under a continuous function which is an open function is locally connected.

4. Show that a compact locally connected space has a finite number of components.

5. Show that components of a totally disconnected space X are singletons of X.

6. Given a topological space (X, τ), a binary relation ρ in X is defined by $x \rho y$ (x, $y \varepsilon X$) holds if and only if x and y belong to a connected set in X. Show that ρ is an equivalence relation on X and verify that ρ-equivalent classes are all the components of X.

Unit 6 ❑ Uniform Spaces

> (Symmetric sets and composition of sets in XxX; uniformity $\mathcal{U}$ in X, base, sub-base for $\mathcal{U}$, uniform space (X, $\mathcal{U}$), uniform topology $\tau_{\mathcal{U}}$ for X, T_2-property of $\tau_{\mathcal{U}}$; Interior and closure of A in terms of member of $\mathcal{U}$, uniformly continuous function, Product Uniformity, Uniform continuty of a metric in (X, $\mathcal{U}$).

§1. Let X be a non-empty set. So XxX $\neq \phi$. If A $\subset$ (XxX), then A^{-1} is defined as $A^{-1} = \{(y, x) : (x, y) \, \varepsilon \, A\}$. So that $(A^{-1})^{-1} = A$. If A and B are two subsets of XxX, then their composition $A_0B = \{(x, z) \, \varepsilon \, (XxX) : (x, y) \, \varepsilon \, B$ and $(y, z) \, \varepsilon \, A$ for some $y \, \varepsilon \, X\}$.

Now A_0B may not be the same as B_0A; however, composition is associative i.e. $A_0(B_0C) = (A_0B)_0C$ and also $(A_0B)^{-1} = B^{-1}_0A^{-1}$. The set of all pairs (x, x) as $x \, \varepsilon \, X$ is called the Diagonal, often denoted by Δ, of XxX. Also if U $\subset$ (XxX) and K $\subset$ X, the set $U[K] = \{y \, \varepsilon \, X : (x, y) \, \varepsilon \, U$ for some $x \, \varepsilon \, K\}$.

In particular, if K is a singleton, say $= \{x\}$ in X, we have

$$U[x] = \{y \, \varepsilon \, X : (x, y) \, \varepsilon \, U\}$$

With these preliminaries we are ready to proceed further.

Definition 1.1. A subset U of XxX is said to be symmetric if $U = U^{-1}$.

Explanation : Let X = The set R of all reals, and U $= \{(x, y) \, \varepsilon \, RxR : |x - y| < 1\}$; then U $= U^{-1}$, hence U is symmetric because if a pair (x, y) satisfies $|x - y| < 1$, then $|y - x| = |-(x - y)| = |x - y| < 1$; So (x, y) ε U if and only if (x, y) $\varepsilon \, U^{-1}$. However, if

$$H = \{(x, y) \varepsilon \, RxR : y - x < 1\}$$
$$= \{(x, y) \varepsilon \, R^2 : y < x + 1\}$$

Then H is no symmetric; for (0, 2) ε H, but (−2, 0) $\notin$ H, although (−2, 0) $\varepsilon \, H^{-1}$. Geometrically, H denotes the lower half-space of R^2 bounded above by the line y = x + 1; and

$$H^{-1} = \{(x,y) \varepsilon R^2 : (y,x) \varepsilon H\}$$
$$= \{(x,y) \varepsilon R^2 : x < y+1\}$$
$$= \{(x,y) \varepsilon R^2 : y > x-1\}$$
$$= \text{upper half - space bounded below by the line } y = x-1.$$

And further

$$H \cap H^{-1} = \{(x,y) \varepsilon R^2 : x-1 < y < x+1\}$$
$$= \{(x,y) \varepsilon R^2 : |x-y| < 1\} = U$$
$$= \text{Open strip in } R^2 \text{ bounded by the lines } y = x-1 \text{ and } y = x+1.$$

Theorem 1.1. If U and V are two subsets of XxX such that V is symmetric; then $V \circ U \circ V = U\{V[x] \times V[y]\} : (x,y) \varepsilon U$.

Proof : Here, $V \circ U \circ V =$ **the set of all pairs (u, v) such that (u, x) ε V, (x, y) ε U and (y, v) ε V for some x and some y.** Since V is symmetric; this is the collection of all (u, v) such that u ε V[x] and v ε V[y] for some (x, y) ε U. But u ε V[x] and v ε V[y] if and only if (u, v) ε V[x] × V[y], and hence $V \circ U \circ V$

$$= \{(u,v) : (u,v) \varepsilon V[x] \times V[y] \text{ for some } (x,y) \varepsilon U\}$$
$$= U\{V[x] \times V[y] : (x,y) \varepsilon U\}$$

Definition 1.2. A Uniformity $\mathcal{U}$ on X is a non-empty family of subsets of XxX satisfying the following conditions known as axioms of Uniformity :

    (u.1) Each member of $\mathcal{U}$ contains the diagonal Δ;

    (u.2) If u ε $\mathcal{U}$, then u^{-1} ε $\mathcal{U}$;

    (u.3) If u ε $\mathcal{U}$, there is a member V ε $\mathcal{U}$ such that $V \circ V \subset U$;

    (u.4) If u and v are two members of $\mathcal{U}$, then $(U \cap V)$ ε $\mathcal{U}$; and

    (u.5) If u ε $\mathcal{U}$, and $U \subset V \subset XxX$, then v ε $\mathcal{U}$.

If $\mathcal{U}$ is a Uniformity on X, then the pair (X, $\mathcal{U}$) is called a Uniform space.

Definition 1.3. **(a)** A sub-family $\mathcal{B}$ of a Uniformity $\mathcal{U}$ on X is called **a base** for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

(b) A sub-family $S\mathcal{B}$ of $\mathcal{U}$ is said to be a sub-base for $\mathcal{U}$ if and only if the family of all finite intersections of members of $S\mathcal{B}$ forms a base for $\mathcal{U}$.

Example 1.1. Every metric space is a Uniform space.

Solution : Let (X, d) be a metric space. For each +ve r let

$$V_{d,r} = \{(x, y) \varepsilon (X \times X) : d(x, y) < r\}.$$

Then we note that $V_{d,r}^{-1} = V_{d,r} ; V_{d,r} \cap V_{d,t} = V_{d,u}$ where u = min[r, t]; and $V_{d,r} \circ V_{d,r} \subset V_{d,2r}$.

Now it is a routine exercise to verify that the family of all sets of form $V_{d,r}$ forms a base for a uniformity for X; and X becomes a Uniform space.

Analogous to base and sub-base for a Topology in X, we have the following Theorem that is easy to prove and the proof is left out as such.

Theorem 1.2. A family $\mathcal{B}$ of subsets of $X \times X$ is a base for a Uniformity for X if and only if

    (a) Each member of $\mathcal{B}$ contains Δ;

    (b) If $B \varepsilon \mathcal{B}$, then B^{-1} contains a member of $\mathcal{B}$;

    (c) If $B \varepsilon \mathcal{B}$, there is a member C in $\mathcal{B}$ such that $C \circ C \subset B$; and

    (d) If B_1, $B_2 \varepsilon \mathcal{B}$, then there is a member $B_3 \varepsilon \mathcal{B}$ such that $B_3 \subset (B_1 \cap B_2)$.

Proof : Let (X, $\mathcal{U}$) be a Uniform space.

Define $\tau = \{G \subset X :$ for each $x \varepsilon G$, there is a member $U \varepsilon \mathcal{U}$ such that $U[x] \subset G\}$. We now verify that τ is, indeed, a Topology in X. Definition says that union of members of τ is a member of τ. Let G, H be two members of τ; and let $x \varepsilon (G \cap H)$. So, there are members U and V in $\mathcal{U}$ such that $U[x] \subset G$ and $V[x] \subset H$. Therefore, $(U \cap V)[x] \subset (G \cap H)$; since $(U \cap V)$ is a member of $\mathcal{U}$, it follows that $G \cap H$ is a member of τ, and (X, τ) is a Topological space. This topology τ is rather abbreviated as $\tau_{\mathcal{U}}$ since it is being induced by the uniformity $\mathcal{U}$, and very often named as a Uniform Topology on X.

Theorem 1.3. Let $(X, \mathcal{U})$ be a Uniform space with uniform Topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$. If $A \subset X$, then

$\tau_{\mathcal{U}}$-Int$A = \{x \varepsilon A :$ there is a member $U \varepsilon \mathcal{U}$ such that $U[x] \subset A\}$. (or, simply, we write Int A instead $\tau_{\mathcal{U}}$–Int A).

Proof : Let $B = \{x \in A : U[x] \subset A$ for some member $U \in \mathcal{U}\}$. We show that $B \in \tau_{\mathcal{U}}$. Take $x \in B$, then we find a member $U \in \mathcal{U}$ such that $U[x] \subset A$; Also we find a member $V \in \mathcal{U}$ such that $V_0V \subset U$. If $u \in V[x]$ and $y \in V[u]$, we have $(u, y) \in V$ and $(x, u) \in V$; Therefore, $(x, y) \in V_0V$ i.e. $y \in V_0V[x]$.

$$\text{or, } V[u] \subset V{\circ}V[x] \subset U[x] \subset A.$$

Therefore $u \in B$. In otherword, $V[x] \subset B$. Hence B is open. Further, B contains every open subset of A, and consequently it is the Largest open subset of A i.e. $B = $ Int A.

Remark 1. If $U \in \mathcal{U}$, $U[x]$ is a nbd. of x in $(X, \tau_{\mathcal{U}})$,

2. The family of all set $U[x]$ as U comes from $\mathcal{U}$ is a base for the nbd. system at x.

Theorem 1.4. Let $(X, \mathcal{U})$ be a Uniform space with a Uniformity $\mathcal{U}$ and $\tau_{\mathcal{U}}$ be the Uniform Topology on X induced by $\mathcal{U}$. If $A \subset X$, then $\tau_{\mathcal{U}}$-closure of A (or simply Closure of A)

$$= \cap \{U[A] : U \in \mathcal{U}\}.$$

Proof : A point $x \in \overline{A}$ (Closure of A w.r.t. $\tau_{\mathcal{U}}$) if and only if for each $U \in \mathcal{U}$, $U[x]$ cuts A non-vacuously. Now $U[x]$ intersects A non-vacuously if and only if $x \in U^{-1}[A]$.

Because, $U[x] \cap A \neq \phi$ iff $p \in U[x] \cap A$;

i.e. iff $p \in U[x]$ and $p \in A$;

i.e. iff $(x, p) \in U$ and $p \in A$;

i.e. iff $(p, x) \in U^{-1}$ and $p \in A$;

i.e. iff $x \in U^{-1}[A]$.

Since each member of $\mathcal{U}$ contains a symmetric member $\in \mathcal{U}$,

We conclude, $x \in \bar{A}$ iff $x \in U[A]$ for each $U \in \mathcal{U}$.

The proof is now complete.

Given a uniformity $\mathcal{U}$ in X, the uniform Topology $\tau_{\mathcal{U}}$ may be used to construct the product Topology for $X \times X$. Members of $\mathcal{U}$ have an intimate relation with this product Topology. We shall presently see that the family of all open symmatric members of $\mathcal{U}$ is a base for $\mathcal{U}$. To that end, we need the following Lemma :

Lemma 1.1. If $U \in \mathcal{U}$ there is a symmetric member $V \in \mathcal{U}$ such that

(i) $V \circ V \circ V \subset U$

and (ii) $V \circ V \circ V = U\{V[x] \times V[y] : (x, y) \in V\}$.

Proof : (i) By axion of Uniformity we find a member $H \in \mathcal{U}$ such that

$$H \circ H \subset U \qquad\qquad ..... \quad (1)$$

Take $W = H \cap H^{-1}$; so W is a symmetric member of $\mathcal{U}$ such that

$$W \circ W \subset H \circ H \subset U \text{ from (1).}$$

Taking W in place of U, we obtain a symmetric member $V \in \mathcal{U}$ such that $V \circ V \subset W$; therefore $V \circ V \circ V \circ V \subset W \circ W$; and $V \circ V \circ V \subset V \circ V \circ V \circ V \subset W \circ W \subset U$ from above.

Theorem 1.1. now applies; we take $U = V$ in Theorem 1.1 and find atonce that $V \circ V \circ V = U\{V[x] \times V[y] : (x, y) \in V\}$.

This is (ii), and the proof of the Lemma is complete.

Theorem 1.5. If $U \in \mathcal{U}$, then Int (U) $\in \mathcal{U}$; and the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

Proof : If $E \subset X \times X$, we have Int $E = \{(x, y) : U[x] \times V[y] \subset E$ for some $U, V \in \mathcal{U}\}$. By axion of University $(U \cap V) \in \mathcal{U}$; so

Int $E = \{(x, y) : V[x] \times V[y] \subset E$ for some $V \in \mathcal{U}\}$. Now Lemma 1.1 says that there is a symmetric member $V \in \mathcal{U}$ such that $V \circ V \circ V \subset U$, and

$V \circ V \circ V = U\{V[x] \times V[y] : (x, y) \in V\}$. Hence every point of $V \in$ Int U i.e. $V \subset$ Int U, and by Uniformity axiom Int $U \in \mathcal{U}$; and in consequence the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

By Theorem 1.4 for $x \in X$ we have $\tau_\mathcal{U}$-closure $\{x\} = \cap \{U[x] : U \in \mathcal{U}\}$. So $(X, \tau_\mathcal{U})$ is T_2 (Hansdorff) if and only if $\cap \{U : U \in \mathcal{U}\}$ is equal to Diagonal Δ. In that case $(X, \tau_\mathcal{U})$ is also said to be separated.

§2. Let $(X, \tau_\mathcal{U})$ and $(Y, \tau_\mathcal{V})$ be two uniform spaces with uniform topologies $\tau_\mathcal{U}$ and $(X, \tau_\mathcal{V})$ respectively induced by given uniformities $\mathcal{U}$ and $\mathcal{V}$, and let $f : (X, \tau_\mathcal{U}) \to (Y, \tau_\mathcal{V})$ be a function.

Definition 2.1. $f : X \to Y$ is said to be uniformly continuous if and only if for each member $V \in \mathcal{V}$, the set $\{(x, y) \in X \times X : (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$.

The above statement may be re-phrased like :-

If $S\mathcal{B}(\mathcal{V})$ is a subbase for $\mathcal{V}$, then f is said to be uniformly continuous iff $f_2^{-1}(V) \in \mathcal{U}$ for each member $V \in S\mathcal{B}(\mathcal{V})$, where $f_2(x, y) = (f(x), f(y))$.

Explanation : Let $X = R$, and for each +ve r let $V_r = \{(x, y) \in R \times R : |x - y| < r\}$; Then the family $\{V_r\}_{r>0}$ of subsets of $R \times R$ forms a base for a Uniformity $\mathcal{U}$ known as usual uniformity for reals. So, the induced uniform topology $\tau_\mathcal{U}$ shall consist of members like $I_r = V_r[x] = \{y \in R : (x, y) \in V_r\} = \{y \in R : |x - y| < r\} = \{y \in R : x - r < y < x + r\}$ = an open interval $(x - r, x + r)$ as $x \in R$; These members act as basic open set in $\tau_\mathcal{U}$ — confirming that $\tau_\mathcal{U}$ is the usual topological of reals. So, as per Definition 2.1 above, a real-valued function f of a real variable is uniformly continuous if given a $\in > 0$, there is a +ve δ such that $|f(x) - f(y)| < \in$ whenever $|x - y| < \delta$. This is in agreement with usual and familiar notion of uniform continuity of f. We also know that uniform continuity of f implies its continuity. The same is also true in a general uniform space.

Theorem 2.1. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces with induced uniform topology $\tau_\mathcal{U}$ and $\tau_\mathcal{V}$ respectively, then every uniformly continuous function $: X \to Y$ is continuous relative to uniform Topology.

Proof : Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous function ; Take H to be a nbd. of $f(x)$ in $(Y, \tau_\mathcal{V})$, $x \in X$. So we find a member $V \in \mathcal{V}$ such that

$$V[f(x)] \subset H,$$

$$\text{and } f^{-1}\{V[f(x)]\}$$

$$= \{y \in X : f(y) \in V[f(x)]\}$$

$$= \{y \in X : (f(x), f(y)) \in V\}$$

$$= f_2^{-1}(V) \cdot [x] \text{ where } f_2(x, y) = (f(x), f(y))$$

$$= \text{a nbd. of } x \text{ in } (X, \tau_{\mathcal{U}}).$$

That means $f^{-1}(H)$ becomes a nbd. of x, and f is rendered continuous at $x \in X$. Since x is any arbitrary member of X, we have proved Theorem.

§3. Suppose for each member α in an Index set Δ, $(X_\alpha, \mathcal{U}_\alpha)$ is a uniform space ; then **the product Uniformity for $X\{X_\alpha : \alpha \in \Delta\}$ is the smallest Uniformity such that projection functions from the Product $X\{X_\alpha : \alpha \in \Delta\}$ into each co-ordinate space $(X_\alpha, \mathcal{U}_\alpha)$ are uniformly continuous.**

The family of sets of the form $\{(x, y) : (X_\alpha, Y_\alpha) \in U\}$, for $\alpha \in \Delta$ and $U \in \mathcal{U}_\alpha$ forms a sub-base for the product Uniformity. If $x \in X\{X_\alpha : \alpha \in \Delta\}$, then sub-basic members of the nbd. system at x (with respect to the Product Uniform Topology), shall be obtained from the sub-basic members for the Product uniformity. Thus the family of all sets like $\{y : (X_\alpha, Y_\alpha) \in U\}$ becomes a sub-base for the Nbd. system at x. Clearly, a base for the Nbd. system at x with respect to induced topology from the product Uniformity is the family of all finite intersections of sets of the form $\{Y : Y_\alpha \in U[X_\alpha]\}$ for $\alpha \in \Delta$ and $U \in \mathcal{U}_\alpha$. But this family is also a base for the Nbd. system at x with respect to the product Topology; and therefore the product topology is the topology of the product Uniformity.

Theorem 3.1. A function f on a Uniform space to a product of Uniform spaces is uniformly continuous if and only if composition of f with each projection function into a Co-ordinate space is uniformly continuous.

Proof : If f is a uniformly continuous function with values in the product $X\{X_\alpha : \alpha \in \Delta\}$, then each projection function p_{r_α} is uniformly continuous, and we know that composition $p_{r_\alpha} \circ f$ is again uniformly continuous.

Conversely, if each $p_{r_\alpha} \circ f$ is uniformly continuous for each $\alpha \in \Delta$, and if U is a member of $\mathcal{U}_\alpha$ in X_α, then $\{(u, v) : (p_{r_\alpha} \circ f(u), p_{r_\alpha} \circ f(v)) \in U\}$ is a member of the uniformity $\mathcal{V}$ of domain f. Now we write this set as $f_2^{-1} [\{(x, y) : (x_\alpha, y_\alpha) \in U\}]$. So, inverse under f_2 of each member of a subbase for the product Uniformity is a member of $\mathcal{V}$, and therefore f becomes uniformly continuous.

Theorem 3.2. Let $(X, \mathcal{U})$ be a Uniform space and let d be a metric for X. $d : X \times X \to R$ is uniformly continuous if and only if $\{(x, y) : d(x, y) < r\}$ is a member of $\mathcal{U}$ for each $r > 0$.

Proof : For each $r > 0$, take $V_{d,r} = \{(x, y) : d(x, y) < r\}$. It suffices to show that $V_{d,r} \in \mathcal{U}$ if and only if d is uniformly continuous. Let $U \in \mathcal{U}$, then sets $\{(x, y), (u, v)) : (x, u) \in U\}$ and $\{((x, y), (u, v)) : (y, v) \in U\}$ belong to the product uniformity, and we find that the family of sets of form $\{((x, y), (u, v)) : (x, u) \in U \text{ and } (y, v) \in U\}$ is a base for the product Uniformity. If d is uniformly continuous, then for each $r > 0$, there is $U \in \mathcal{U}$ such that if (x, u) and (y, v) belong to U, then $| d(x, y) - d(u, v) | < r$. Say, in particular $(u, v) = (y, y)$, then it follows that if $(x, y) \in U$, then $d(x, y) < r$. Hence $U \subset V_{d,r}$ and therefore $V_{d,r} \in \mathcal{U}$.

For converse part, if both (x, u) and (y, v) belong to $V_{d,r}$ then $| d(x, y) - d(u, v) | < 2r$ because $d(x, y) \leq d(x, u) + d(u, v) + d(y, v)$ and $d(u, v) \leq d(x, u) + d(x, y) + d(y, v)$. It follows that if $V_{d,r} \in \mathcal{U}$ for each +ve r, then d is uniformly continuous.

Theorem 3.2. Opens the gate to develop relation between Uniformities and metries (or pseudometrics). The reader may see the Literature as in Kelley's book in Chapter of Uniform spaces.

EXERCISE - A
Short answer type questions

1. Construct a Uniformity $\mathcal{U}$ for the space R of reals to induce the usual topology for R.

2. Show that a metric space is a uniform space.

3. Over a non-empty set X obtain (a) the largest uniformity and (b) the smallest uniformity for X.

EXERCISE - B

1. Show that the family of closed symmetric members of a Uniformity $\mathcal{U}$ is a base for $\mathcal{U}$.

2. Describe the product Uniformity in the product $X\{X_\alpha : \alpha \in \Delta\}$ where each $(X_\alpha, \mathcal{U}_\alpha)$ as $\alpha \in \Delta$ is a uniform space.

3. Prove that a continuous function of a compact Uniform space into a Uniform space is uniformly continuous.

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By Hahn-Banach Theorem we find a member $x^* \in X^*$ with $\| x^* \| = 1$ and $\| x^*(x) \| = \| x \|$.

Therefore $\| x \| \leq \sup\limits_{\| x^* \| \leq 1} \{ | x^*(x) | \}$ .... (1)

Again $\| x^*(x) \| \leq \| x^* \| \| x \| \leq \| x \|$ when $\| x^* \| \leq 1$

Therefore $\| x \| \geq | x^*(x) |$ when $\| x^* \| \leq 1$

Thus $\| x \| \geq \sup\limits_{\| x^* \| \leq 1} | x^*(x) |$. .... (2)

From (1) and (2) we have

$$\| x \| = \sup\{ | x^*(x) | : x^* \in X^* \text{ with } \| x^* \| \leq 1 \}.$$

which is $= \sup\{ | \hat{x}(x^*) | : x^* \in X^* \text{ with } \| x^* \| \leq 1 \}$

$$= \| \hat{x} \|.$$

It shows that $\hat{x}$ is a bounded linear functional over X^* with $\| \hat{x} \| = \| x \|$.

Finally, let $x_1, x_2 \in X$ and $x^* \in X^*$, then

$$\widehat{(x_1 + x_2)}(x^*) = x^*(x_1 + x_2)$$

$$= x^*(x_1) + x^*(x_2)$$

$$= \hat{x}_1(x^*) + \hat{x}_2(x^*).$$

Similarly for any scalar α we have $\widehat{(\alpha x_1)}(x^*) = x^*(\alpha x_1)$

$$= \alpha x^*(x_1)$$

$$= \alpha \hat{x}_1(x^*)$$

Therefore the mapping $x \to \hat{x}$ is linear; and since $\| \hat{x} \| = \| x \|$, this mapping is Isometry.

That is, $x \to \hat{x}$ is a Linear Isometry of X onto the linear sub-space $\{ \hat{x} : x \in X^* \}$ of X^{**}.

Definition 4.3.1. Given a *NLS* $(X, \| \, \|)$, Linear Isometry $x \to \hat{x}$ is called the Canonical mapping of X into its second conjugate space X^{**}.