PG (MT) 06 : Group B

Functional Analysis

UNIT 1

(***Contents*** : Metric spaces, metric Topology, convergent and Cauchy sequences, completeness, metric space of all real sequences, complete metric spaces l_p, $C[a,b]$; Metric sub-spaces, separable metric space, continuous functions, Homeomorphism, Isometry, Compact metric spaces, Sequential compactness, Arzela-Ascoli Theorem)

§ 1.1 METRIC SPACES :

Let X be a non-empty set; so the Cartesian product $X_x X$ of all ordered pairs (x, y) of elements $x, y \in X$ is also non-empty.

Definition 1.1.1. A function $d : X_x X \to R$ (reals) is called a metric or a distance function over X if it satisfies following conditions, known as metric or distance axioms :

(M.1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$. (Property of non-negativity),

(M.2) $d(x, y) = d(y, x)$ for all $x, y \in X$. (Property of symmetry).

(M.3) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y and $z \in X$. (Property of triangle inequality).

If d is a metric on X, then the pair (X, d) is called a metric space. In a metric space (X, d) if $x_0 \in X$ and r is *a* +ve real, we have

Definition 1.1.2. The subset $\{x \in X : d(x_0, x) < r\}$ of X denoted by $B_r(x_0)$ is called an open ball in X, centred at x_0 with radius = r.

For example, if $d(x, y) = |x - y|$ for any two reals $x, y \in R$, then (R, d) is a metric space and for $x_0 \in R$ and r any +ve r, open ball $B_r(x_0) = \{x \in R : |x - x_0| < r\}$

$= \{x \in R : x_0 - r < x < x_0 + r\}$

= an open interval $(x_0 - r; x_0 + r)$ with mid point x_0 and length = $2r$.

Similarly, in the metric space ¢ of all complex numbers with usual metric we find an open ball $B_r(z_0)$ looks like an open circular disc with centre at $z_0 \in$ ¢ having radius = r.

Definition 1.1.3. The subset $\{x \in X : d(x_0, x) \leq r\}$ of a metric space (X, d) is called a closed ball centred at x_0 with radius = r.

The subset $\{x \in X : d(x_0, x) = r\}$ of X is called a sphere centred at x_0 with radius = r. It is also called boundary (Bdr) of open (closed) ball centred at x_0 having radius = r.

The open balls in a metric space (X, d) form a base for a Topology, called metric Topology τ_d (induced by the metric d) on X. So every metric space (X, d) is a topological space with metric topology τ_d. This metric topology τ_d is Hausdorff (T_2).

Definition 1.1.4. A sequence $\{x_n\}$ in (X, d) is said to be a convergent sequence if there is a member $u \in X$ such that, $\lim_{n\to\infty} d(u, x_n) = 0$.

Or, equivalently, given any +ve ε, there is an index N such that $d(u, x_n) < \varepsilon$, when $n \geq N$.

If $\{x_n\}$ is a convergent sequence in (X, d) with $u \in X$ and $\lim_{n\to\infty} d(u, x_n) = 0$, we write $\lim_n x_n = u \in X$, and u is a unique member of X, because metric space is Hausdorff.

Definition 1.1.5. A sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, d) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Or, equivalently, given any +ve ε, there is an index N satisfying $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

It is an easy exercise to see that in a metric space every convergent sequence is cauchy, but converse is false.

Definition 1.1.6. A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) is convergent in X.

For example, real number space R with usual metric $d(x, y) = |x - y|;\ x, y \in R$ is a complete metric space. This is what is known as Cauchy's General Principle of convergence; and essentially by same reason the Euclidean n-space R^n consisting of all n tuples of reals like $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n), x_i \in R$ is also a complete metric space with usual/Euclidean metric d where $d^2(\underset{\sim}{x}, \underset{\sim}{y})$

$$= \sum_{i=1}^{n} |x_i - y_i|^2;\ \underset{\sim}{x} = (x_1, x_2, \ldots, x_n),\ \underset{\sim}{y} = (y_1, y_2, \ldots, y_n) \in R^n$$

Example 1.1.1. The collection S of all sequences of reals is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$, where $\underset{\sim}{x} = (\xi_1, \xi_2, \ldots\ldots)$,

$\underset{\sim}{y} = (\eta_1, \eta_2, ...) \in S$. The r.h.s. series is convergent because each term is domoninated by a corresponding term of a convergent geometric series. Here is a routine exercise to see that metric axioms are all satisfied. For completeness part we remark on passing that if $a_{n,m} \geq 0$, then $a_{n,m} \to 0$ if and only if $\dfrac{a_{n,m}}{1+a_{n,m}} \to 0$ as $n, m \to \infty$.

Take $\{\underset{\sim}{x}_n\}$ as a Cauchy sequence of elements in S

where $\underset{\sim}{x}_n = (\xi_1^{(n)}, \xi_2^{(n)}, ... \xi_i^{(n)}, ...)$.

Corresponding to a + ve ε we find an index N such that

$\rho(\underset{\sim}{x}_n, \underset{\sim}{x}_m) < \varepsilon$ for all $n, m \geq N$

or, $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(m)}|}{1+|\xi_i^{(n)} - \xi_i^{(m)}|} < \varepsilon$ for all $n, m \geq N$ ....(1.1.7)

As individual term in series above is ≥ 0, we appeal to the remark made earlier to say that $|\xi_i^{(n)} - \xi_i^{(m)}| \to 0$ as $n, m \to \infty$. And hence for each co-ordinate i by Cauchy's General Principle of Convergence, $\{\xi_i^{(n)}\}$ is convergent.

Put $\lim\limits_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, i = 1, 2, ...

Taking $\underset{\sim}{x}_0 = \left(\xi_1^{(0)}, \xi_2^{(0)}, ...\right)$ we find $\underset{\sim}{x} \in S$ and passing on limit as $m \to \infty$ in (1.1.7) we have

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(0)}|}{1+|\xi_i^{(n)} - \xi_i^{(0)}|} \leq \varepsilon \text{ for } n \geq N.$$

That means, $\lim\limits_{n\to\infty} \rho(\underset{\sim}{x}_n, \underset{\sim}{x}_0) = 0$

or $\lim\limits_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0 \in S$

So the sequence space S becomes a complete metric space.

Remark : The convergence of sequence of elements in S as shown above is known as co-ordinatewise convergence; that is to say, $\lim\limits_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0$ in S,

where $\underset{\sim}{x}_n = \left\{\xi_i^{(n)}\right\}$ and $\underset{\sim}{x}_0 = \left\{\xi_i^{(0)}\right\}$, if and only if $\lim\limits_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, for i = 1, 2, 3, ...; **The convergence is not necessarily uniform.**

Example 1.1.2. The sequence space l_p ($1 < p < \infty$) consisting of all sequences $\underset{\sim}{x} = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ of reals with $\sum_{i=1}^{\infty} |\xi_i|^p < +\infty$ is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p \right)^{1/p}$, for $\underset{\sim}{x} = (\xi_1, \xi_2, \ldots), \underset{\sim}{y} = (\eta_1, \eta_2, \ldots) \in l_p$.

Example 1.1.3. The function space $C[a,b]$ consisting of all real valued continuous functions over the closed interval $[a, b]$ is a complete metric space with sup metric $\rho(f, g) = \sup\limits_{a \le t \le b} |f(t) - g(t)|$, as $f, g \in C[a,b]$

The last two examples appear in Book PGMT 2A. They are referred to there.

§ 1.2 SUB-SPACES :

Let Y be a non-empty subset of a metric space (X, d). There is a natural metric, namely the restriction d_Y of d to $Y \times y$.

Definition 1.2.1. The metric space (Y, dy) is called a sub-space of (X, d).

Theorem 1.2.1. A subset A in Y is open in (Y, dy) if and only if there is a subset A_1 in X that is open in (X, d) such that $A = Y \cap A_1$.

Proof : Let $x \in X$ and $y \in Y$ and r be a +ve number n and let $B_X(x, r)$ and $B_Y(y, r)$ denote open balls centred at x and at y respectively with radius = r in (X, d) and in (Y, dy).

Then we have $B_Y(y,r) = Y \cap B_x(y,r)$ for all $y \in Y$, and $r > 0$ ....(1.2.1)

Take A as an open set in (Y, d_Y), then we know that A is a Union of some open balls of (Y, d_Y); say of $\{B_Y(y,r)\}$ as $y \in A$ and $r > 0$.

Thus $A = UB_Y(y,r)$

$= U\{Y \cap B_X(y,r)\}$ by (1.2.1)

$= Y \cap \{UB_X(y,r)\}$

$= Y \cap A_1$ (say)

where A_1 is a union of open balls in (X, d) and A_1 is an open set in (X, d).

Conversely, let $A = Y \cap A_1$, where A_1 is an open set in (X, d). For $y \in A$, there is an open ball $B_X(y,r) \subset A_1$, and hence $B_Y(y,r) = Y \cap B_X(y,r) \subset (Y \cap A_1) = A$. So every member of A attracts an open ball in (Y, d_Y) *i.e.* A is an open set in (Y, d_Y). The proof is complete.

Corollary : A is closed in (Y, d_Y) if and only if there is a subset A_1 of X that is closed in (X, d) such that $A = Y \cap A_1$. (If $A = Y \cap A_1$, we have $Y \setminus A = Y \cap (X \setminus A_1)$, and now proceed).

Definition 1.2.2. A metric space (X, d) is said to be separable if and only if there is a countable subset D of X such that D is dense in (X, d) (or equivalently, $\bar{D}$ (closure of D) = X).

For example, real number space R with usual metric is separable, because the set Q of all rationals if R is dense in R, where we know that Q is countable.

Theorem 1.2.2. A sub-space of a separable metric space is separable.

Proof : Let (Y, d_Y) be a sub-space of (X, d) which is a separable metric space. Let $A = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable set in X such that $\bar{A} = X$.

If $y \in Y$, then for each +ve integer m the open ball $B\left(y, \frac{1}{m}\right)$ meets A at some point, say = x_n.

Thus $x_n \in \left\{A \cap B\left(y, \frac{1}{m}\right)\right\}$.

So, Open ball $B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi$

Put $\Delta = \left\{(n, m) : B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi\right\}$. Thus $\Delta \neq \phi$. For each $(n, m) \in \Delta$, take a member $y_{n,m} \in \left\{B\left(x_n, \frac{1}{m}\right) \cap Y\right\}$, and put $B = \left\{y_{n,m} : (n, m) \in \Delta\right\}$. Therefore B is a countable subset of Y because Δ is so. We now verify that B is dense in (Y, d_Y). Take $y \in Y$ and $r > 0$; choose +ve integer m so that $\frac{1}{m} \leq \frac{1}{2} r$. As said above there is an integer n such that $x_n \in B\left(y, \frac{1}{m}\right)$. Then $(n, m) \in \Delta$, and we have

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \leq r.$$

That means, $y_{n,m} \in B(y, r)$. Therefore $y \in \bar{B}$ in (Y, d_Y), or, B is dense in (Y, d_Y).

§ 1.3 CONTINUOUS FUNCTIONS :

Let (X, d) and (Y, ρ) be two metric spaces.

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be continuous at a point $c\in X$, if and only if given a +ve ε, there is a +ve δ (depending on ε and c) such that $e(f(x),f(c))<\varepsilon$ whenever $d(x,c)>\delta$.

or equivalently, $f(B(c,\delta))\subset B(f(c),\in)$.

f is said to be a continuous function if f remains continuous each point of X.

Further details on continuous functions over metric spaces may be seen in Book PGMT 2A.

Homeomorphism, Isometry :

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be a homeomorphism if f is 1–1, onto (bijective) and both f and f^{-1} are continuous functions.

If there is a homeomorphism between (X, d) and (y, ρ), then two metric spaces (X, d) and (y, ρ) are called homeomorphic.

Explanation : If f is a homeomorphism of X onto Y, then f^{-1} is so between Y and X. Also it is a routine matter to see that composition of two homeomorphisms is again a homeomorphism; thus in the family of all metric spaces the binary relation 'of being homeomorphic' is an equivalence relation.

Example 1.3.1. Consider the metric space R of reals with usual metric and a function $T:R\to R$ given by $T(x)=x+a$, where a is a fixed real number, and $x\in R$. Then this translation function (equals to Identity function when a = 0) is a homeomorphism; here $T^{-1}:R\to R$ is given by $T^{-1}(x)=x-a;\ x\in R$. Similarly one shows that for any non-zero real λ, multiplication function $M_\lambda:R\to R$ given by $M_\lambda(x)=\lambda x, x\in R$ is a homeomorphism, where $M_\lambda^{-1}=M_{\lambda^{-1}}$.

We know that family of all open sets in (X, d) forms a Topology, called metric topology τ_d on X induced by d. Any property in a metric space (X, d) that can be formulated entirely in terms of members of τ_d (open sets) is known as **a Topological property.**

Consequently, homeomorphic metric spaces have the same topological properties like convergence of sequences in the space and continuity of functions over the space. Following example shows **completness is not a topological property in a metric space.**

Example 1.3.1. Take $X = \{1, 2, 3, ...\}$ and $Y = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. Regarded as a subspace of the space R of reals with usual metric we find that spaces X and Y are discrete metric spaces (every subset being both open and closed); thus the function $h: X \to Y$ where $h(n) = n^{-1}$ is a homeomorphism of X onto Y. Since X is a closed subset of R which is a complete metric space, the space X is complete. On the other hand Y is not complete.

Definition 1.3.2. A function $f: Y \to Y$ that is onto (surjective) is said to be an Isometry if $e(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Explanation : Identity function on X is an Isometry of X onto itself. Also a transformation of rotation like $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$ is an Isometry of Eudidean 2-space R^2 onto itself with usual metric. Also an Isometry is a homeomorphism. Thus two metric spaces that are isometric are indistinguishable in respect of their metric properties.

Example 1.3.2. In metric space (X, d) take $x_0 \in X$.

For $x \in X$, Let $f_x : x \to R$ (space of reals with usual metric) be given as

$f_x(y) = d(y, x) - d(y, x_0)$ for $y \in X$.

Then show that $x \to f_x$ is an isometry of X into $C(X)$ where $C(X)$ is metric space of all real valued continuous functions over X with sup metric

$$\| f - g \| = \sup_{y \in X} | f(y) - g(y) | < \infty.$$

As distance function d is continuous, it follows that f_x is continuous for all $x \in X$.

Solution : Take $u, v \in X$; so we have

$$\left.\begin{array}{l} f_u(y) = d(y, u) - d(y, x_0) \\ \text{and } f_v(y) = d(y, v) - d(y, x_0) \quad \text{for all } y \in X \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1.3.1)$$

So, $| f_u(y) - f_v(y) | = | d(y, u) - d(y, v) | \le d(u, v)$ which is independent of $y \in X$; taking the $\sup_{y \in X}$ over L.H.S. we obtain

$$\sup_{y \in X} | f_u(y) - f_v(y) | \le d(u,v)$$

or $\| f_u - f_v \| \le d(u,v)$ ....1.3.2)

Putting $y = u$ in (1.3.1) we have,

$$f_u(u) = -d(y, x_v) \text{ and } f_v(u) = d(u,v) - d(y, x_0)$$

So, $| f_u(u) - f_v(u) | = d(u,v)$

Now $\sup_{y \in X} | f_u(y) - f_v(y) | \ge | f_u(u) - f_v(u) | = d(u,v)$ .... (1.3.3)

from (1.3.2) and (1.3.3) we obtain

$$\| f_u - f_v \| = d(u,v).$$

Thus $x \to f_x$ invites an Isometry of X into $C(X)$.

§ 1.4 COMPACT METRIC SPACES :

Some important properties of reals as we encounter in real analysis had motivated more important concepts in a metric space like completeness and compactness. Cauchy's General Principle of Convergence is the driving force behind completeness in a metric space. Essence of Heine-Borel Theorem could be found in concept of compactness in a metric space.

In consequence, it had been an inevitable task with urgency to identify compact subsets in a metric space. Russian Mathematicians like Alexandrov and Urysohn had been responsible to put forward notion of compactness via 'open cover' in the space; on the other hand close to Bolzano-Weirstrass property is classical analysis concept of sequential compactness owed to Frechet in a metric space. And now we know for certain that these two routes are equivalent in describing compactness in a metric space. For details in this context see the book PGMT 2A.

It has been possible to discover that a subset in Euclidean n-space R^n with usual metric is compact if and only if the subset is a bounded and closed set in R^n.

Given a metric space X it is often hard to decide which subsets of X are compact, and which are not. Our present task is the job of identifying compact subsets of a very important and useful function space of some continuous functions that we presently discuss below. The concerned target theorem in this connection is Ascoli-Arzela Theorem.

Definition 1.4.1. Let (X, d) denote a metric space.

(a) A family $Q = \{A_1\}_{i \in \Delta}$ of open sets A_i in (X, d) is said to be an open cover for X if every element of X belongs to at least one member A_i of the family Q. That is to say, $X \subset \underset{i \in \Delta}{U} A_i$.

(b) A sub-family of an open cover for X which by itself is an open cover for X is called sub-cover for X.

(c) (X, d) is said to be a compact metric space if every open cover for X has **a finite sub-cover for X.**

Explanation : By a finite sub-cover we mean that the sub-cover consists of a finite number of members only. Consider a family $\{(-n, n)\}_{n \in N}$ (N = set of all natural numbers). Its members are open intervals, and hence open sets in the metric space R of reals with usual metric. It is an open cover for R; because $R = \underset{n=1}{U}(-n, n)$. Clearly, this open cover possesses no finite sub-cover for R. That is why, R is not compact.

Definition 1.4.2. A subset G of (X, d) is said to be compact if as a sub-space of (X, d) it is compact under definition 1.4.1.

For example, although R is not compact with usual metric any finite subset of R becomes compact.

Definition 1.4.3. (X, d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence in X.

It is a bit lengthy exercise to conclude that a metric space is compact if and only if it is a sequentially compact. See book PGMT 2A.

The function space $C[a, b]$ of all real-vaheed continuous functions over a closed interval $[a, b]$.

We know that $C[a, b]$ is a complete metric space with respect to sub metric $\rho(f, g) = \sup_{a \le t \le b} |f(t) - g(t)|$, $f, g \in C[a, b]$. But $C[a, b]$ is not compact with respect to sub metric, because $C[a,b]$ is not bounded; for all constant functions like $f_n(t) = n$ for $a \le t \le b$ satisfy $\rho(f_n, 0) = n \to \infty$ as $n \to \infty$. However there are compact sets in $C[a, b]$. In searching then we need some Definitions.

Definition 1.4.1. (a) A subset M of $C[a, b]$ is said to be **uniformly bounded** if there is a +ve constant K such that $|x(t)| \le K$ for all t in $a \le t \le b$ and for all members $x \in M$.

(b) Subset M is said to be **equi-continuous** if given any +ve ε, there is a +ve δ (depending on ε only) such that $|x(t_1) - x(t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta (t_1, t_2 \in [a,b])$ for all members $x \in M$.

Example 1.4.1. Show that the subset $\{f_n\} \subset C[0,1]$ is equibounded where $f_n(t) = 1 + \frac{t}{n}; 0 \le t \le 1$.

Solution : Here $|f_n(t)| = |1 + \frac{t}{n}| \le 1 + |\frac{t}{n}| \le 1 + \frac{1}{n} \le 2$ for all n and for all t in $0 \le t \le 1$. So the conclusion stands.

Theorem 1.4.1 (Arzela-Ascoli Theorem) : A subset M of $C[a,b]$ is compact if and only if M is uniformly bounded and equi continuous.

Proof : The condition is necessary : Let M be a compact subset of $C[a,b]$ (*w.r.t.* sup metric). Then M is bounded, because a compact set in a metric space is bounded and closed. Thus we find a closed ball say $\overline{B}_r(x_0)$ centred at $x_0 \in C[a,b]$ with radius = r, such that

$$M \subset \overline{B}_r(x_0)$$

Thus $\sup\limits_{a \le t \le b} |x(t) - x_0(t)| \le r$

Now $x(t) = x(t) - x_0(t) + x_0(t)$ and

$$\sup_{a \le t \le b} |x(t)| \le \sup_{a \le t \le b} |x(t) - x_0(t)| + \sup_{a \le t \le b} |x_0(t)| \quad \le r + k, \text{ say,}$$

where $k = \sup\limits_{a \le t \le b} |x_0(t)|$.

That means $|x(t)| \le (r + R) = K$ (say) for all t in $a \le t \le b$ and for all $x \in M$. Hence M is uniformly bounded.

For equi-continuity take a +ve ε.

Since M is compact, we find an $\frac{\varepsilon}{3} - net = (x_1(t), x_2(3), ..., x_n(t))$ for M.

Since every real-valued continuous function over a closed interval is uniformly continuous. So here each of the members $x_1, x_2, ... x_k$ of $C[a,b]$ is uniformly continuous in $[a,b]$.

So, for each $x_i(t)$ we find a +ve δ_i such that

$$|x_i(t_1)-x_i(t_2)|<\tfrac{\varepsilon}{3} \quad \text{whenever } |t_1-t_2|<\delta_i,\ t_1,t_2\in[a,b].$$

Now take a +ve $\delta = \min_{1\le i\le k}\{d_i\}$. Then we have

$$|x_i(t_1)-x_i(t_2)|<\frac{\varepsilon}{3} \text{ whenever } |t_1-t_2|<\delta, t_1,t_2\in[a,b] \text{ for all } i = 1,2,\ldots, k.$$

Now for every member $x \in M$, we find a member, say, x_i from $\frac{\varepsilon}{3}$–net, such that

$$\rho(x,x_i)<\frac{\varepsilon}{3} \quad (\rho = \text{sup-metric of } C[a,b]).$$

If $t_1,t_2\in[a,b]$ and $|t_1-t_2|<\delta$ we have

$$|x(t_1)-x(t_2)|\le x(t_1)-x_i(t_1)|+|x_i(t_1)-x_i(t_2)|+|x_i(t_2)-x(t_2)|$$

$$\le \sup_{a\le t\le b}|x(t)-x_i(t)|+|(x_i(t_1)-x_i(t_2)|+\sup_{a\le t\le b}|x_i(t)-x(t)|$$

$$<\rho(x,x_i)+\frac{\varepsilon}{3}+\rho(x,x_i)<\varepsilon.$$

This inequality holds for all $t_1,t_2\in[a,b]$, with $|t_1-t_2|<\delta$ and for all members $x\in M$. So M is equi-continuous.

The condition is sufficient : Suppose M is uniformly bounded and equi-continuous ; we show that M is compact. Because $C[a,b]$ is complete and so is M; It suffices to show that every sequence in M has a Cauchy subsequence. Let $D = (t_2, t_3, t_4, \ldots)$ be a countable dense set of reals in $[a,b]$.

Suppose $S_1=(f_{11},f_{12},f_{13},\ldots)$ be any sequence of elements in M. By uniform boundedness property of M. We find a +ve K such that

$|f(t)|\le K$ for all t in $a\le t\le b$ and for all $f\in M$. .... (1.4.6)

Let us examine real sequence

$$\{f_{11}(t_2), f_{12}(t_2), f_{13}(t_2),\ldots, f_{1m}(t_2),\ldots\}$$

From (1.4.6) it is clear that this is a bounded sequence of reals and has a convergent subsequence.

Let $S_2=(f_{21}, f_{22}, f_{23},\ldots\}$ be a sub-sequence of S_1 above such that $\{f_{21}(t_2), f_{22}(t_2), f_{23}(t_2),\ldots\}$ converges.

Now examine real sequence $\{f_{21}(t_3), f_{22}(t_3), f_{23}(t_3),.....\}$, and by similar reasoning as above, we have

$S_3 = \{f_{31}, f_{32}, f_{33},.....\}$ as a subsequence of S_2 such that

$\{f_{31}(t_3), f_{32}(t_3), f_{33}(t_3),.....\}$ is convergent.

We continue this chain to construct $S_1, S_2, S_3, \ldots$ of sequences of functions like :

$$S_1 = \{f_{11}, f_{12}, f_{13},.....\}$$
$$S_2 = \{f_{21}, f_{22}, f_{23},.....\}$$
$$S_3 = \{f_{31}, f_{32}, f_{33},.....\}$$
$$\ldots \quad \ldots \quad \ldots$$

where S_m constitutes a subsequence of $S_{m-1} (m = 2,3,....)$ with the property that $\{f_{n1}(t_n), f_{n2}(t_n)\ f_{n3}(t_n),....\}$ is a convergent sequence of reals.

Now put $f_n = f_{nn}$ $(n = 2, 3, 4,...)$ then $\{f_1, f_2, f_3,....\}$ is the diagonal subsequence of S_1. From mode of construction

$x_n \in D$ and $\{f_1(t_n), f_2(t_n), ..., f_i(t_n),...\}$ is a convergent real sequence.

If $i > k$, consider $|f_{ii}(t_n) - f_{kk}(t_n)|$ for $i > k > n$ and knowing that both $f_{ii}(t_n), f_{kk}(t_n)$ are members of convergent real sequence

$$\{f_{n1}(t_n), f_{n2}(t_n), (f_{n3}(t_n),....\}$$

We have $|f_i(t_n) - f_k(t_n)| \to 0$ as $i,k \to \infty$. Thus $\{f_1(t_n), f_2(t_n), f_3(t_n),....\}$ is a Cauchy sequence of reals.

Finally, take any +ve ε. Since M is equi-continous and $S \subset S_1 \subset M$, we find a +ve δ such that $|f_n(t) - f_n(t') < \frac{\varepsilon}{3}$ whenever $|t - t'| < \delta$, $t, t' \in [a,b]$ for all members $f_n \in S$.

Now consider the family $\{t_n - \delta, t_n + \delta)\}$ of open intervals with mid point $t_n \in D$.

It is routine verification with dense property of D in $[a,b]$ that this family of open intervals becomes an open cover for $[a,b]$. By compactness of $[a,b]$ we obtain a finite sub-over, say

$$[a,b] = \underset{t_n \in D}{U} (t_n - \delta, t_n + \delta) \text{ and } 2 \le n \le n_0$$

Again $\{f_1(t_n), f_2(t_n), ...\}$ is Cauchy; thus a +ve integer K_0 is there such that

$$|f_i(t_n) - f_k(t_n)| < \frac{\varepsilon}{3} \text{ for all } 2 \le n \le n_0$$

If t is any position of $[a, b]$, we find n with $2 \le n \le n_0$ so that $t_n - \delta < t < t_n + \delta$ and for $i, K \ge K_0$ we have

$$|f_i(t) - f_k(t)| \le |f_i(t) - f_i(t_n)| + |f_i(t_n) - f_k(t_n)|$$

$$+ |f_k(t_n) - f_k(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

That means $\sup\limits_{a \le t \le b} |f_i(t) - f_k(t)| \le \varepsilon$ for $i, k \ge k_0$

or, $\rho(f_i, f_k) \le \in$ for $i, k \ge k_0$

or, $S = \{f_1, f_2, ...\}$ is a Cauchy subsequence of S_1.

The proof is now complete.

EXERCISE A

Short-answer type questions :

1. Show that compactness is not a heriditary property in a metric space.
2. Give an example to show that a closed bounded set in a metric space may not be compact.
3. Show that $f(x) = x + a$ or $= -x + a$ where a is a fixed real is an Isometry on the space R of reals with usual metric.
4. Prove that any bounded sequence of reals has a convergent subsequence.
5. In a metric space (X, d) if $\lim\limits_{n \to \infty} x_n = x \in X$, show that $\{x_n\} \cup \{x\}$ is copact.

EXERCISE B

Broad questions

1. Show that the closed ball $\tilde{B} = \left\{x : \sup\limits_{0 \le t \le 1} |x(t) \le 1\right\}$ of $C[0, 1]$ with supmetric is not compact.

2. Prove that only Isometries of the space R of reals with usual metric are $f(x) = x + a$ and $f(x) = -x + a$ where a is a real number.
3. Give an example of a Homeomorphism that is not an Isometry.
4. Let f be a real-valued function on a compact metric space (X, d), show that f assumes its maximum and minimum on X.
5. Verify that closed Unit ball in sequence space l_2 is bounded without being totally bounded.
6. Let X denote the metric space of all real polynomials $p(t)$ in $0 \leq t \leq 1$; show that X is not a complete metric space with respect to sup metric.

UNIT 2

(***Contents*** : Linear spaces, Dimension of a linear space, Normed linear space (NLS), Banach space, $C[a,b]$ as a Banach space, Quotient space of a NLS, Convex sets, their algebra, Bounded linear operator; its continuity, Unbounded linear operator, Norm ||T|| of a bounded linear operator T; Formulae for ||T||.)

§ 2.1 LINEAR SPACES

Definition 2.1.1. Let R (q) denote the field of reals (complex numbers) that are also called scalars. A linear space (Vector space) V is a collection of objects called vectors satisfying following conditions :

I. V is additively an Abelian (commutative) Group, the identity element of which is called the Zero vector denoted by 0.

II. For every pair (α v), α being a scalar and $v \in V$, there is a vector, denoted by $\alpha.v$ (**not** $v\alpha$), called a scalar multiple of v such that

(a) $1.v = v$ for all $v \in V$.

(b) $\alpha.(u+v) = \alpha.u + \alpha.v$ for all scalars α and for all vectors $u, v \in V$.

(c) $(\alpha+\beta).v = \alpha.v + \beta.v$ for all scalars α and β and for all vectors $v \in V$.

(d) $\alpha.(\beta.v) = (\alpha.\beta).v$ for all scalars α and β and for all $v \in V$.

Example 2.1.1. Let R^n be the collection of all n tuples of reals like $x = (x_1, x_2, \ldots x_n)$; x_i being reals. Then R^n becomes a linear space with real scalar field where addition of vectors and scalar multiplication of vectors are defined as

$$x + y = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \text{ and}$$

$\alpha.x = \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$; $x, y \in R^n$ and α any real scalar.

Here R^n is also called Euclidean n-space with the zero vector $\underline{0} = (0, 0, \ldots, 0)$ (all co-ordinates are zero), and it is a real Linear space.

Example 2.1.2. Let $C[a,b]$ denote the collection of all real valued continuous functions over a closed interval $[a,b]$. Then $C[a,b]$ is a real linear space (associated scalar field being that of reals) where vector sum and scalar multiplication are defined as under :

$$(f+g)(t) = f(t) + g(t); \quad a \le t \le b, \text{ and } f, g \in C[a,b]$$

and $(\alpha f)(t) = \alpha f(t)$; $a \le t \le b$ and α any real scalar.

As we know that sum of two continuous functions is a continuous function and so is a scalar multiple of a continuous function, we see that $f+g$ and αf are members of $C[a,b]$ where $f,g \in C[a,b]$ and α is any scalar. Here the zero vector equals to the zero function $\left(0(t)=0; a \le t \le b\right)$ over the closed interval $[a,b]$.

There are many other linear spaces like the sequence spaces $l_p(1 < p < \infty)$, polynomial space $\rho[a,b]$, function space $L_2[a,b]$, that we encounter in our discussion to follow.

Definition 2.1.2. (a) If A and B are subsets of a linear space V then $A+B=\{a+b : a \in A \text{ and } b \in B\}$.

(b) For any scalar λ,

$$\lambda A = \{\lambda a : a \in A\}$$

The subset $A-B = A+(-1)B$; and taking λ = zero scalar we find $0A = \{\underline{0}\}$. Further we see that A + B = $B + A$, because vector addition is commutative, However $A-B \ne B-A$. Taking A and B as singleton and $A=\{(1,0)\}, B=\{(0,0\}$ in Euclidean 2-space R^2, we find $A-B=\{(1,0)\}$ and $B-A=\{(-1,0)\}$.

Further for any scalar α we have $\alpha A = \{\alpha a : a \in A\}$.

Here is a caution. In general, $A+A \ne 2A$.

Because take $A=\{(1,0),(0,1)\}$; Then we have

$2A=\{(2,0),(0,2)\}$ which is not equal to $A+A$

where $A+A=\{(2,0),(0,2),(1,1)\}$.

Given a fixed member $a \in V$, the subset $a+B=\{a+b : b \in B\}$ is called a translate of B.

§ 2.2. Let X denote a linear space over reals/complex scalars. Given $x_1, x_2, \ldots, x_n$ in X, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ as scalars, the vector $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ of X is called a linear combination of $x_1, x_2, \ldots, x_n$.

A subset E of X is said to span (generate) X if and only if every member of X is a linear combination of some elements of E.

Elements $x_1, x_2, \ldots, x_n$ of E are said to be linearly dependent if and only if there are corresponding number of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \underline{0}$$

A finite number of elements $x_1, x_2, ..., x_k$ of X are said to be linearly independent if they are not linearly dependent. This amounts to say that if

$$\sum_{i=1}^{k} \alpha_i x_i = \underline{0} \text{ implies } \alpha_1 = \alpha_2 = ... = \alpha_k = 0.$$

An arbitrary system of elements of X is called linearly independent if every finite subset of the given system becomes linearly independent.

Observe that if a set of vectors in X contains a linearly dependent subset, whole set becomes linearly dependent. Also note that a linearly independent set of vectors does not contain the zero vector.

Definition 2.2.1. A non-empty sub-set L of a linear space X is called a sub-space of X if $x + y$ is in L whenever x and y are both in L, and also αx is in L, whenever x is in L and α is any scalar.

Example 2.2.1. Let S be any non-empty subset of X. Let L = the set of all linear combinations of elements of S. Then L is sub-space of X, called the sub-space spanned (generated) by S.

The subset = $\{\underline{0}\}$ is a sub-space, called the Null-space.

Theorem 2.2.1. Let $x_1, x_2, ..., x_n$ be a set of vectors of X with $x_1 \neq \underline{0}$. This set is linearly dependent if and only if some one of vectors $x_2, ..., x_n$, say x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Proof : Suppose the given set of vectors is linearly dependent. There is a smallest k with $2 \leq k \leq n$ such that $x_1, x_2, ..., x_k$ is linearly dependent; and we have $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = 0$ with not all α's are zero scalars. Necessarily, we have $\alpha_k \neq 0$; otherwise $x_1, x_2, ..., x_{k-1}$ would form a linearly dependent set.

In consequence $x_k = -\frac{\alpha_1}{\alpha_k} x_1 - \frac{\alpha_2}{\alpha_k} x_2 - .... - \frac{\alpha_{k-1}}{\alpha_k} x_{k-1}$.

That means x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Conversely, if one assumes that some x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$; then we have

$$x_k = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_{k-1} x_{k-1}$$

That means $x_1, x_2, \ldots, x_k$ are linearly dependent, and in turn we have the set $(x_1, x_2, \ldots, x_k)$ as linearly dependent.

Definition 2.2.2. In a linear space X suppose there is a +ve integer n such that X contains **a set of *n* vectors** that are **linearly independent,** while **every set of *n* + 1** vectors in X is **linearly dependent,** then X is called finite dimensional and ***n*** is called **dimension of *X*** {Dim(X)}.

The Null-space is finite dimensional of **dimension 0.**

If X is **not** finite dimensional it is called **infinite dimensional.**

Definition 2.2.3. A finite set B in linear space X is called a basis of X if B is linearly independent, and f the sub-space spanned (generated) by B is all of X.

Explanation : If $x_1, x_2, \ldots, x_n$ is a basis for X, every member $x \in X$ can be expressed as $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ where scalar coefficients α_i's are uniquely determined; so x does not have a **different** linear combination of basis members.

Suppose $\text{Dim}(X) = n$ $(n \geq 1)$. Then X has a basis consisting of n members; For, X certainly contains vectors $x_1, x_2, \ldots, x_n$ that form a linearly independent set. Now for any member $x \in X$, the set of vectors $x_1, x_2, \ldots, x_n$ plus x u of n + 1 vectors must be linearly dependent. Now Theorem 2.2.1 applies to conclude that x is in the sub-space generated by $x_1, x_2, \ldots, x_n$. Hence $x_1, x_2, \ldots, x_n$ form a basis of X.

§ 2.3 NORMED LINEAR SPACES :

Definition 2.3.1. A linear space X is called a Normed Linear Space (*NLS*) if there is a non-negative real valued function denoted by $\| \; \|$, called a norm on X whose value at $x \in X$ denoted by $\| x \|$ satisfies following conditions (N.1) – (N.3), called norm axioms :–

(N.1) $\quad \| x \| \geq 0$, and $\| x \| = 0$ if and only if $x = \underline{0}$.

(N.2) $\quad |\alpha x \| = |\alpha| \| x \|$ for any scalar α and for any $x \in X$.

(N.3) $\quad \| x + y \| \leq \| x \| + \| y \|$ for any two members x and y in X.

If $\| \; \|$ is a norm on X, the ordered pair (X, $\| \; \|$) is designated as a *NLS*. If norm changes, *NLS* also changes.

In a *NLS* (X, $\| \; \|$) one can define a metric ρ by the rule : $\rho(x, y) = \| x - y \|$ for all $x, y \in X$. It is an easy task to check that ρ satisfies all metric axioms; and (X,ρ) becomes **a metric space** with the metric topology called **Norm Topology** because of its induction from norm || ||. We write $\lim_{n\to\infty} x_n = x$ in X off $\lim_{n\to\infty} \| x_n - x \| = 0$; this convergence in *NLS X* is known as convergence in Norm. Similarly, we define a Cauchy sequence in *NLS X*.

A subset B in a *NLS X* is said to be bounded if there is a +ve K such that $\| x \| \le K$ for all $x \in B$

Let $x_0 \in X$, and take a +ve number r. Then in NLS X, the set $\{x \in X : \| x - x_0 \| < r\}$ is called an open ball denoted by $B_r(x_0)$ centred at x_0 having radius = r. Similarly, we have a closed ball $\overline{B}_r(x_0) = \{x \in x : \| x - x_0 \| \le r\}$; and in agreement with usual open sphere we encounter in Co-ordinate Geometry we have a sphere $S_r(x_0) = \{x \in X : \| x - x_0 \| = r$ centred at x_0 with radius = r.

Definition 2.3.2. A *NLS* (X, || ||) is said to be a Banach space if it is a complete metric space with metric induced from the norm function || || on X.

Example 2.3.1. The space $C[a,b]$ of all real-valued continuous functions over closed interval $[a,b]$ is a Banach space with supnorm $\| f \| = \sup_{a \le t \le b} | f(t) |; f \in C[a,b]$.

Solution : It is routine exercise to see that $C[a,b]$ is a real linear space in respect of usual addition and scalar multiplication of continuous functions.

Now put $\| f \| = \sup_{a \le t \le b} | f(t) |$ for $f \in C[a,b]$ wherein we recall that $|f|$ is also continuous function over closed interval $[a,b]$ with a finite sup value $= \| f \| \ge 0$. Also $\| f \| = 0$ if and only f equals to the zero function. So (N.1) axiom is satisfied; For (N.2) take α any scalar (real), then we have for $f \in C[a,b]$,

$$\| \alpha f \| = \sup_{a \le t \le b} \| (\alpha f)(t) | = \sup_{a \le t \le b} \| \alpha\, f(t) | = | \alpha | \sup_{a \le t \le b} | f(t) | = | \alpha | \| f \|.$$

Also, if $f, g \in C[a,b]$ we have $\| f + g \| = \sup_{a \le t \le b} | (f + g)(t) |$

$$= \sup_{a \le t \le b} | f(t) + g(t) | \le \sup_{a \le t \le b} | f(t) | + \sup_{a \le t \le b} | g(t) | = \| f \| + \| g \|.$$

Thus $C[a,b]$ is a NLS; Now take $\{f_n\}$ as a Cauchy sequence in $C[a,b]$; So $\| f_n - f_m \| \to 0$ as, $n, m \to \infty$. Give a $\varepsilon > 0$, we find an index N satisfying

$$\| f_n - f_m \| < \varepsilon \text{ whenever } n, m \ge N.$$

That is, $\sup_{a \le t \le b} |f_n(t) - f_m(t)| < \varepsilon$ ....

Thus for $a \le t \le b$, we have $|f_n(t) - f_m(t)| \le \sup_{a \le t \le b} |f_n(t) - f_m(t)| < \varepsilon$ whenever $n, m \ge N$. Above inequality shows that the sequence $\{f_n\}$ of continuous functions over the closed interval $[a,b]$ converges uniformly to a function say f over $[a,b]$ and also f becomes a continuous function over $[a,b]$. So $f \in C[a,b]$. Taking $m \to \infty$ in (2.3.1) we find

$|f_n(t) - f(t)| \le \varepsilon$ whenever $n \ge N$ and for all t in $a \le t \le b$.

This gives $\sup_{a \le t \le b} |f_n(t) - f(t)| \le \varepsilon$ whenever $n \ge N$

or, $\|f_n - f\| \le \varepsilon$ for $n \ge N$

That means, $\lim_{n \to \infty} f_n = f \in C[a,b]$. Thus $C[a,b]$ is a Banach space.

Theorem 2.3.1. Let X be a *NLS* with norm $\| \;\|$. Then

(a) $\big| \|x\| - \|y\| \big| \le \|x - y\|$ for any two members $x, y \in X$.

(b) $\| \;\| : X \to$ Reals is a continuous function.

Proof : (a) We write $\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$

or, $\|x\| - \|y\| \le \|x - y\|$ .... (2.3.1)

Interchanging x and y we have

$\|y\| - \|x\| = \|y - x\| = \|x - y\|$ .... (2.3.2)

From (2.3.1) and (2.3.2) we write

$$\pm (\|x\| - \|y\|) \le \|x - y\|$$

or, $\big| \|x\| - \|y\| \big| \le \|x - y\|$

(b) Let $\{x_n\}$ be a sequence of elements in X converge to x_0.

So $\|x_n - x_0\| \to 0$ as $x \to \infty$. By (a) we have

$$\big| \|x_n\| - \|x_0\| \big| \le \|x_n - x_0\| \to 0 \text{ as } n \to \infty.$$

That means, $\lim_{n \to \infty} \|x_n\| = \|x_0\|$. Hence norm function $\| \;\|$ is continuous at x_0; As x_0 may be taken as any point in X, (b) follows.

Remark : If $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} y_n = y_0$ in *NLS X*, then

(a) $\lim_{n\to\infty}(x_n \pm y_n) = x_0 \neq y_0$

(b) $\lim_{n\to\infty}(\lambda x_n) = \lambda x_0$ for any scalar λ.

Definition 2.3.3. Two norms $\| \|_1$ and $\| \|_2$ in a linear space X are said to be equivalent if there two +ve constants a and b such that

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \text{ for all } z \in X.$$

Example 2.3.2. Consider $NLS = R^2$ (Euclidean 2-space) with two norms $\| \|_1$ and $\| \|_2$ defined by $\|x, y\|_1 = \sqrt{x^2 + y^2}$ and $\|x, y\|_2 = \max(|x|, |y|)$ for $(x, y) \in R^2$. Show that two norms are equivalent.

Solution : We have for $(x, y) \in R^2$, $|x|^2 \leq |x|^2 + |y|^2$ and $|y|^2 \leq |x|^2 + |y|^2$

Thus $\|(x, y)\|_2 = \max(|x|, |y|) \leq \sqrt{|x|^2 + |y|^2} = \sqrt{x^2 + y^2} = \|(x, y)\|_1$

or, $\|(x, y)\|_2 \leq \|(x, y)\|_1$ (2.3.1)

Again $\|(x, y)_1{}^2 = x^2 + y^2 = |x|^2 + |y|^2 \leq 2\{\max(|x|, |y|)\}^2 = 2\|(x, y)\|_2$

or, $\|(x, y)_1 \leq \sqrt{2}\|(x, y)\|_2$ (2.3.2)

Combining (2.3.1) and (2.3.2) we produce

$$\|(x, y)\|_2 \leq \|(x, y)\|_1 \leq \sqrt{2}\|(x, y)\|_2$$

Therefore two norms as given are equivalent in $NLS = R^2$.

Explanation : If two norms $\| \|_1$ and $\| \|_2$ are equivalent in a NLS X, then identify function : $(X, \| \|_1) \to (X, \| \|_2)$ is a homeomorphism. (In fact, it is a linear homeomorphism).

§ 2.4 QUOTIENT SPACE :

Let $(X, \| \|)$ be a NLS and F be a linear sub-space of X.

If $x \in X$, let $x + F = \{x + y : y \in F\}$.

These subsets $x + F$ as $x \in X$ are cosets of F in X.

Put $X/F = \{x + F : x \in X\}$.

One observes that $F = \underline{0} + F$, $x_1 + F = x_2 + F$ if and only if $x_1 - x_2 \in F$, and as a result, for each pair $x_1, x_2 \in X$, either $(x_1 + F) \cap (x_2 + F) = \Phi$

or, $x_1 + F = x_2 + F$

Further, if $x_1, x_2, y_1, y_2 \in X$, and $(x_1 - x_2) \in F$, $(y_1 - y_2) \in F$, then

$(x_1 + y_1) - (x_2 + y_2) \in F$, and for any scalar α $(\alpha x_1 - \alpha x_2) \in F$ because F is Linear sub-space.

We define two operations in $X \backslash F$ by the following rule :-

(i) $(X \backslash F \times (X \backslash F) \to (X|F)$

where $(x + F, y + F) \to (x + F) + (y + F) = (x + y) + F$

and (ii) $R(¢) \times (X \backslash F) \to (X|F)$

where $(\alpha, x + F) \to \alpha(x + F) = \alpha x + F$

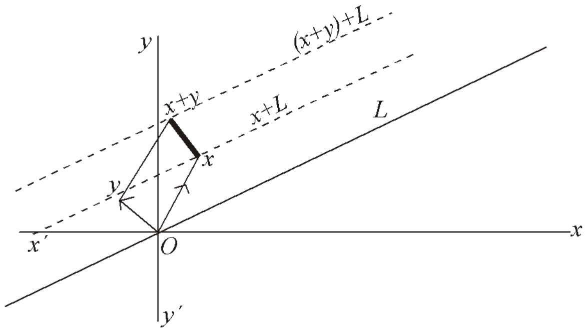
for all $x, y \in X$ and α any scalar.

It is now a routine exercise to verify that $(X \backslash F)$ is a linear space in respect of above 'addition' and 'scalar multiplication'. Note that zero vector of this Linear space $(X|F)$ equals to F.

Definition 2.4.1. The linear space $X \backslash L$ where L is a linear subspace of *NLS X* is called the quotient space (or quotient space of X modulo L).

Example 2.4.1. Geometrically describe the quotient space R^2/L where R^2 = the Euclidean 2-space and L is the sub-space represented by a line through origin $(0, 0) \in R^2$.

Solution : Given a sub-space L as represented by a line through $(0, 0) \in R^2$, $\underset{\sim}{x}$ is any position of R^2, then $\underset{\sim}{x} + L$ geometrically represents a straight line through $\underset{\sim}{x}$ parallel to the line represented by L; that is say that $\underset{\sim}{x} + L$ is a translate of L through



x. Further if y is any other position of R^2, then by Law of parallelogram we obtain the position $x + y$ and here $(x+L)+(y+L)=(x+y)+L$ is re-presented by the straight line through $x + y$ and it is parallel to L; that is—it is the translate of L through $(x + y)$ in R^2.

Example 2.4.2. Obtain the quotient space $C[0,1]/L$ where $C[0,1]$ is the linear space of all real valued continuous functions over the closed interval $[0,1]$ and L consists of those members $f \in C[0,1]$ with $f(1) = 0$, *i.e.* vanishing at $t = 1$.

Solution : If $f, g \in L$, then $f(1) = g(1) = 0$; Now $(f+g)(1) = f(1)+g(1) = 0$; So $f+g \in L$ (note that sum of two continuous functions over $[0,1]$ is again a continuous functions over $[0,1]$), and for any scalar α we have $\alpha f \in L$ when $f \in L$. Therefore L is a sub-space of $C[0,1]$.

Let us look at members of $C[0,1]\backslash L$. Take $f \in C[0,1]$ where $f(1) = a$ (say). Then for any other member $g \in C[0,1]$ sharing the value a at $t = 1$, *i.e.* $g(1) = a$; we note that $(g-f) \in C[0,1]$ such that $(g-f)(1) \in g(1) - f(1) = a - a = 0$; showing that $(g-f) \in L$ *i.e.* $g \in f + L$. So these members g plus f all belong to $f + L$.

Now if $h \in C[0,1]$ with $h \notin (f+L)$ (2.4.1)

So, $h - f \notin L$

i.e. h and f differ at $t = 1$.

i.e. $h(1) \neq f(1) = a$

We similarly construct a member $(h + L)$ of $C\backslash L$, where

$$(h+L) \cap (f+L) = \phi \qquad (2.4.2$$

or else, we find a member ϕ in both implying

$$\varphi - h \in L \qquad \text{and } \varphi - f \in L$$

therefore $\varphi(1) - h(1) = 0$ and $\varphi(1) - f(1) = 0$

i.e. $\varphi(1) = h(1)$ and $\varphi(1) = f(1)$

i.e. $h(1) = f(1)$

that means $h \in (f+L)$, which is not the case by (2.4.1).

Theorem 2.4.1. Let L be a closed linear sub-space of *NLS X*, and let $\| x+L \| = Inf\{\| x+y \| : y \in L\}$, for all $x \in X$, then above is a norm function on the quotient space $(X \setminus L)$. Further if X is Banach space, so will be $(X \setminus L)$.

Proof : For any member $x + L$ of $X \backslash L$, from definition we have

$$\| x+L \| \geq 0 \text{ for any } x \in X.$$

Now assume that $\| x+L \| = 0$ for some $x \in X$.

i.e. $Inf\{\| x+y \| : y \in L\} = 0$

As $y \in L$ if and only if $-y \in L$, we have

$$Inf\{\| x-y \| : y \in L\} = 0.$$

Since L is closed, $x \in L$ (distance of x from L is zero);

That means $x+F=F=$ the zero vector of the quotient space X/L.

For verification (N.2) take α any non-zero scalar. Then

$$\| \alpha(x+L) \| = \| \alpha x + L \|$$
$$= Inf\{\| \alpha x + y \| : y \in L\}$$
$$= Inf\{\| \alpha(x + \tfrac{y}{\alpha}) \| : y \in L\}$$
$$= |\alpha| \, Inf\{\| x + (\tfrac{1}{\alpha})y \| : y \in L\}$$
$$= |\alpha| \| x+L \|$$, because L is a linear sub-space of X.

For triangle inequality (N.3) take $x, y \in L$

Then $\| (x+t)+(y+L) \| = \| (x+y)+L \|$ (L is a linear sub-space).

$$= Inf\{\| x+y+u \| : u \in L\}$$
$$= Inf\{\| x+y+\tfrac{u}{2}+\tfrac{u}{2} \| : u \in L\}$$
$$\leq Inf\{\| x+\tfrac{u}{2} \| + \| y+\tfrac{u}{2} \| : u \in L\}$$
$$\leq Inf\{\| x+\tfrac{u}{2} \| : u \in L\} + Inf \| y+\tfrac{u}{2} \| : u \in L\}$$
$$= Inf\{\| x+h \| : u \in L\} + Inf \| y+K \| : K \in L\};$$ L is a sub-space.
$$= \| x+L \| + \| y+L \|$$

Thus quotient space $X\backslash L$ is a *NLS.*

Now suppose X is a Banach space. We show that the quotient space X/L is so. Let $\{x_n + L\}$ be a Cauchy sequence in (X/L). So corresponding to each +ve integer k we find an index N_k such that

$$\| x_n - x_m + L \| < \frac{1}{2^k}, \text{ whenever } m,n \geq N_k \qquad (2.4.1)$$

We define by Induction a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\| x_{n_k} - x_{n_{k+1}} + L \| < \frac{1}{2^k}$$

Take $n_1 = N_1$, and suppose $n_2, n_3, \ldots, n_k$ have been so defined that $n_1 < n_2 < \ldots < n_k$ and $N_j \leq n_j$ (j = 1,2,....,k).

Let $n_{k+1} = \max\{N_{k+1}, n_k + 1\}$. This enables one to obtain an increasing sequence $\{n_k\}$ and (*) follows from (2.4.1)

Put $y_k = x_{n_k}$. Then by induction we define a sequence $\{z_k\}$ in L such that $z_k \in (y_k + L)$ and $\| z_k - z_{k+1} \| < \frac{1}{2^{k+1}}$, k = 1,2, ...

Choose $z_1 \in (y_1 + L)$, suppose $z_2, \ldots, z_k$ have been so chosen to satisfy above condition. Then $y_k + L = z_k + L$ and by (2.4.1) we have $\| z_k - y_{k+1} + F \| < \frac{1}{2^k}$. By definition of norm in $(X\backslash L)$

we find $z_{k+1} \in (y_{k+1} + L)$ such that $\| z_k + z_{k+1} \| \leq z_k - y_{k+1} + L \| + \frac{1}{2^k}$.

Then $\| z_k + z_{k+1} \| < \frac{1}{2^{k+1}}$ as wanted.

That means $\sum_{k=1}^{\infty} \| z_k - z_{k+1} \|$ is convergent, and hence $\sum_{k=1}^{\infty} (z_k - z_{k+1})$ is convergent.

But $\sum_{k=1}^{\infty} (z_k - z_{k+1}) = (z_1 - z_2) + (z_2 - z_3) + \ldots + (z_m - z_{m+1}) = z_1 - z_{m+1}$.

So, $\{z_m\}$ is convergent; Put $\lim_{k\to\infty} z_k = z$; since $z_k \in (y_k + L)$ we have $\|(z+L)-(y_k+L)\| = \|z - y_k + L\| \le \|z - z_k\|$.

That means $\lim_{k\to\infty}\{y_k + L\} = z + L$. Thus given Cauchy sequence $\{x_n + L\}$ has a convergent subsequence $\{x_{n_k} + F\}$.

Hence $\{x_n + L\}$ is convergent in $(X\backslash L)$. This proves that $(X\backslash L)$ is a Banach space.

§ 2.5 CONVEX SETS IN *NLS* :

Let $(X, \|\ \|)$ be a *NLS*, and C be a non-empty subset of X.

Definition 2.5.1. C is said to be a convex set if for any real scalar α in $0 \le \alpha \le 1$, and any two members $x_1, x_2 \in C$ we have $\alpha x_1 + (1-\alpha)x_2$ is a member of C.

Or, equivalently, for any two reals α, β with $0 \le \alpha, \beta \le 1$ $\alpha + \beta = 1$, $(\alpha x_1 + \beta x_2) \in C$.

Or, equivalently, the segment consisting of members $tx_1 + (1-t)x_2$ $(0 \le t \le 1)$ is a part of C.

For example, in an Euclidean space like R^n, cubes, ball, sub-spaces are all examples of convex sets in R^n.

Theorem 2.5.1. Intersection of any number of convex sets in a *NLS* is a convex set, but their union may not be so,

Proof : Suppose $\{C_\alpha\}_{\alpha\in\Delta}$ be a family of convex set in *NLS* $(X, \|\ \|)$ and put $C = \bigcap_{\alpha\in\Delta} C_\alpha$; Let $C \ne \phi$ and let $x, y \in C$ take $0 \le \alpha \le 1$. Now $x, y \in \bigcap_{x\in\Delta} C_\alpha$, so for every α, x and y are members of C_α which is convex, thus, $(\alpha x + (1-\alpha)y \in C_\alpha$. Therefore $\alpha x + (1-\alpha)y$ is a member of every C_α and hence is a member of $\bigcap_{x\in\Delta} C_\alpha = C$. Thus C is shown to be a convex set in X.

Union of two convex sets may not be a convex set. Every triangular region in Euclidean plane is a convex set but the figure ⧖ as a union of two such convex sets fails to be a convex set.

Theorem 2.5.2. A subset C in a *NLS* is convex if and only if $sC + tC = (s+t)C$ for all +ve scalars s and t.

Proof : For all scalars s and t we have

$$(s+t)C \subset sC + tC \quad \ldots\ldots (2.5.1)$$

If C is convex and s, t are +ve scalars we have

$$\frac{s}{s+t}C + \frac{t}{s+t}C \subset C$$

Or $\quad sC + tC \subset (s+t)C \quad \ldots\ldots (2.5.2)$

Combining (2.5.1) and (2.5.2) we have

$$sC + tC = (s+t)C$$

Conversely, suppose $(s+t)C = sC + tC$ holds for all +ve scalars; If $0 \le \alpha \le 1$, take $s = \alpha$ and $t = 1-\alpha$ and then we find $\alpha C + (1-\alpha)C \subset C$. So C is convex.

Theorem 2.5.3. A ball (open or closed) of a *NLS* is a convex set.

Proof : $\overline{B}(x_0, r)$ be a closed ball in a *NLS* $(X, \|\ \|)$.

Let $x, y \in \overline{B}(x_0, r)$; So $\| x - x_0 \| \le r$ and $\| y - x_0 \| \le r$. If $0 \le t \le 1$, and $u = tx + (1-t)y$, we have

$$\| u - x_0 \| = \| tx + (1-t)y - (tx_0 + (1-t)x_0) \| = \| t(x - x_0) + (1-t)(y - y_0) \|$$

$$\le t \| x - x_0 \| + (1-t) \| y - y_0 \| \le tr + (1-t)r = r.$$

That shows $u \in \overline{B}(x_0, r)$. So, $\overline{B}(x_0, r)$ is shown to be convex. The proof for an open ball shall be similar.

Example 2.5.1. If $(X, \|\ \|)$ is a Banach space and L is a closed sub-space of X, show that L is a Banach space.

Solution : If L is a closed sub-space of X, then L becomes a closed set of a complete metric space X, the metric being induced from the norm $\|\ \|$. And we know that every closed sub-space of a complete metric space is a complete metric sub-space and hence here L is a Banach space. (as a sub-space of X).

§ 2.6 BOUNDED LINEAR OPERATORS OVER A *NLS* $(X, \|\ \|)$:

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. (Here, same notation $\|\ \|$ has been used for norm function; it is to be noted that norm functions in X and Y are, in general, different).

Definition 2.6.1. A function (or mapping or transformation) (function, map, mapping, transformation are synonyms of the same mathematical object) $T : X \to Y$ is called a linear operator if (1) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for any two members x_1 and x_2 in X, and

(2) $T(\alpha x_1) = \alpha T(x_1)$ for any scalar α and for any member $x_1 \in X$.

Explanation : For a linear operator $T : X \to Y$ condition (1) in Definition 2.6.1 is termed as linearity condition which says Image of the sum is equal to sum of the images. Condition (2) is known as that homogeneity. For example, if $X = Y = R =$ the space of reals with usual norm (Euclidean norm) and $T : R \to R$ is given by $T(x) = \alpha x$ where $x \in R$ and α is a fixed real (zero or non-zero), we verify that T is a linear operator; and we shall presently see that any linear operator : $R \to R$ shall be of the form $T(x) = \alpha x$ for some fixed scalar α for all $x \in R$.

Definition 2.6.2. The operator $T : X \to Y$ defined by $T(x) = \underline{0}$ in Y. For all X, is called **the zero operator, denoted by 0**.

Remark : (a) The zero operator : $X \to Y$ is a Linear operator.

(b) The identity operator, $I : X \to X$ where $I(x) = x$ for all $x \in X$ is a linear operator.

Theorem 2.6.1. Let $T : X \to Y$ be a linear operator. If T is continuous at one point of X, then T is continuous at every other point of X.

Proof : Suppose T is continuous at $x_0 \in X$; so given $\varepsilon > 0$, there is a +ve δ such that $\| T(x) - T(x_0) \| < \varepsilon$ whenever $\| (x) - (x_0) \| < \delta$. Suppose $x_1 (\neq x_0)$ be another point of X. Then if $\| x - x_1 \| < \delta$, we write $\| x - x_1 \| = \| x_0 - (x - x_1 + x_0) \|$.

Thus $\| (x - x_1 + x_0) \| < \delta$ shall give by virtue of continuity of T at x_0,

$$\| T(x - x_1 + x_0) - T(x_0) \| < \varepsilon$$

or, $\| T(x) - T(x_1) + T(x_0) - T(x_0) \| < \varepsilon$ because T is linear.

or, $\| T(x) - T(x_1) \| < \varepsilon$. Therefore T is continuous at $x = x_1$.

Corollary : A linear operator over a *NLS* X is continuous **either everywhere or nowhere in X.**

Definition 2.6.3. A linear operator $T : X \to Y$ is called bounded if there is a +ve constant M such that

$$\|T(x)\| \le M\|x\| \text{ for all } x \in X.$$

or equivalently $\dfrac{\|T(x)\|}{\|x\|} \le M$ for all non-zero numbers $x \in X$.

Theorem 2.6.2. Let $T : X \to Y$ be a linear operator. Then T is continuous if and only if T is bounded.

Proof : Let $T : X \to Y$ be a continuous linear operator; if possible let T be not bounded. So for every +ve integer n we find a member $x_n \in X$ such that

$$\|T(x_n)\| > n\|x_n\| \quad \text{.................................} \quad (2.6.1)$$

Now x_n is non-zero vector in X, put $u_n = \dfrac{x_n}{n\|x_n\|}$,

clearly $\|x_n\| = \dfrac{1}{n} \cdot \dfrac{1}{\|x_n\|} = \dfrac{1}{n} \to 0$ as $n \to \infty$. So we see $\lim\limits_{n\to\infty} u_n = \underline{0}$ in X; By continuity of T we have $\lim\limits_{n\to\infty} T(u_n) = T(\underline{0}) = \underline{0}$ in Y. ($T(\underline{0}) = \underline{0}$, because T is linear);

Therefore we have $\|T(u_n)\| \to 0$ as $n \to \infty$ (*)

On the other hand, $\|T(u_n)\| = \left\| T\left(\dfrac{x_n}{n\|x_n\|}\right) \right\|$,

$= \left\| \dfrac{1}{n\|x_n\|} T(x_n) \right\|$, because T is linear

$= \dfrac{1}{n} \cdot \dfrac{1}{\|x_n\|} \|T(x_n)\| > 1$ by (2.6.1)

Now $\|T(u_n)\| > 1$ and (*) are contradictory.

So, we have shown that $T : X \to Y$ is bounded.

Conversely, suppose linear operator $T : X \to Y$ is bounded. Then we find a +ve scalar such that

$$\|T(x)\| \le M\|x\|;$$

So given $\in > 0$, there is a +ve $\delta = \dfrac{\in}{2M}$ (here), such that

$$\| T(x) \| < \varepsilon \quad \text{whenever} \quad \| x \| < \delta$$

i.e. $\| T(x) - T(\underline{0}) \| < \varepsilon$ whenever $\| x - \underline{0} \| < \delta$ because $T(\underline{0}) = \underline{0}$ in *Y*. That means, *T* is continuous at $x = \underline{0}$ in *X*, and therefore Theorem 2.6.1 says that *T* is continuous at every non-zero position of *X*. The proof is now complete.

Examples of bounded and unbounded linear operators.

Example 2.6.1. Consider a transformation *T* of rotation in Euclidean 2-space R^2 given by $T(x, y) \to (x', y')$ where

$$\left. \begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \right\} \qquad (*)$$

Now it is easy to verify that $T : R^2 \to R^2$, under (*) is a linear operator in respect which rotation takes place around origin (0,0) with axes of co-ordinates being rotated through angle θ to give new axes of co-ordinates.

In *NLS* R^2 with usual norm $\| (x, y) \| = x^2 + y^2$, we see that

$$\| T(x, y) \|^2 = \| (x', y') \| = x'^2 + y'^2 = (x\cos\theta + y\sin\theta) + (-x\sin\theta + y\cos\theta)^2$$

$$= x^2 + y^2 = \| (x, y) \|^2 .$$

Thus $\| T(x, y) \| = \| (x, y) \|$; and this is true for all points (*x*, *y*) in R^2 , and we conclude that *T* is a bounded linear operator.

Example 2.6.2. Consider the Banach space *C*[0,1,] of all real-valued continuous functions over the closed interval [0,1] with respect to sup norm

$$\| f \| = \sup_{a \le t \le 1} | f(t) |; \; f \in C[0,1]$$

Let *K*(*s*,*t*) be a real-valued continuous function over the square

$$\{ 0 \le s \le t \, ; \, 0 \le t \le 1 \} .$$

Now define $T : C[0,1] \to C[0,1]$ by the rule : let $T(f) = F$

where $F(s) = \int_0^1 k(s,t) f(t) dt$; as $f \in C[0,1]$.

It is a routine exercise to check that F is continuous over $[0,1]$ and T is a linear operator.

Now, $$\|T(f)\| = \|F\| = \sup_{0\le s\le 1} |F(s)| = \sup_{0\le s\le 1} \left|\int_0^1 k(s,t)f(t)dt\right|$$

$$\le \sup_{0\le s\le 1}\int_0^1 |k(s,t)||f(t)|dt \le M\int_0^1 |f(t)|dt \text{ where } M = \sup_{0\le s\le 1,\, 0\le t\le 1} |k(s,t)|;$$

$$\le M.\sup_{0\le t\le 1}|f(t)|\int_0^1 dt = M.\|f\|.$$ This is true for every member $f \in C[0,1]$.

Therefore, T is shown to be bounded.

Example 2.6.3. Let $C^{(1)}[0,1,]$ denote the class of real-valued continuous functions that are continuously differentiable over $[0,1]$. Then $C^{(1)}[0,1]$ is a sub-space of $C[0,1]$ which is Banach space with sup norm. Consider the Differential operator $D : C^{(1)}[0,1] \to C[0,1]$ when $D(f) = \varphi$; $f \in C^{(1)}[0,1]$ and $\frac{d}{dt}f(t) = \varphi(t)$ in $0 \le t \le 1$. We can easily verify that D is a linear operator; presently we see that D is not bounded.

Let us take $f_n \in C^{(1)}[0,1]$ where $f_n(t) = \sin n\pi t$ in $0 \le t \le 1$. Then we have $Df_n = \varphi_n$ where $\varphi_n(t) = \frac{d}{dt}(\sin n\pi t) = n\pi\cos n\pi t$ in $0 \le t \le 1$.

Therefore, $\|f_n\| = \sup_{0\le t\le 1} |\sin n\pi t| = 1$ and

$$\|D(f_n)\| = \|\varphi_n\| = \sup_{0\le t\le 1} |n\pi\cos n\pi t| = n\pi$$

Here $\frac{\|D(f_n)\|}{\|f_n\|} = \frac{n\pi}{1} \to \infty$ as $n \to \infty$

That means D can not be bounded.

Definition 2.6.4. Let $T : X \to Y$ be a bounded (or equivalently, continuous) linear operator. Then the norm of T, denoted by $\|T\|$ is defined as

$$\|T\| = Inf\{M > 0 : \|T(x)\| \le M\|x\| \text{ for all } x \in X\}$$

(A set of +ve reals has always *Inf.* value).

Theorem 2.6.3. Let $T : X \to Y$ be a bounded linear operator. Then

(a) $\| T(x) \| \le \| T \| \| x \|$ for all $x \in X\}$

(b) $\| T \| = \sup_{\|x\| \le 1} \{ \| T(x) \| \}$

(c) $\| T \| = \sup_{\|x\| = 1} \{ \| T(x) \| \}$

(d) $\| T \| = \sup_{x \ne \underline{0}} \left\{ \frac{\| T(x) \|}{\| x \|} \right\}$

Proof : (a) From definition of operator norm we see that for any +ve ε we have $\| T(x) \| \le (\| T \| + \varepsilon) \| x \|$ for all $x \in X$.

Taking $\varepsilon \to 0_+$ we have $\| T(x) \| \le \| T \| \| x \|$

(b) If $\| x \| \le 1, x \in X$, we have $\| T(x) \| \le \| T \| \| x \| \le \| T \|$

Therefore $\sup_{\|x\| \le 1} \| T(x) \| \le \| T \|$ (1)

From Definition of operator norm $\| T \|$, given any +ve ε, we find $x_\varepsilon \in X$ such that $\| T(x_\varepsilon) \| > (\| T \| - \varepsilon) \| x \|$.

Take $u_\varepsilon = \frac{x_\varepsilon}{\| x_\varepsilon \|}$ we see $\| u_\varepsilon \| = 1$ such that

$$\| T(u_\varepsilon) \| = \frac{1}{\| x_\varepsilon \|} \| T(x_\varepsilon) \| > \frac{1}{\| x_\varepsilon \|} .(\| T \| - \varepsilon) \| x_\varepsilon \| = \| T \| - \varepsilon$$

As $\| u_\varepsilon \| = 1$, this gives $\sup_{\|x\| \le 1} \| T(x) \| \ge \| T(u_\varepsilon) \| > \| T \| - \varepsilon$. As $\varepsilon > 0$ is arbitrary we produce $\sup_{\|x\| \le 1} \| T(x) \| \ge \| T \|$ (2)

From (1) and (2) we have (b), namely, $\sup_{\|x\| \le 1} \| T(x) \| = \| T \|$

(c) the proof shall be like that of (b).

(d) we have $\| T(x) \| \le \| T \| \| x \|$ for all $x \in X$.

So, $\frac{\|T(x)\|}{\|x\|} \leq \|T\|$ for $x \in X$ with $x \neq 0$.

Since r.h.s does not depend on non-zero $x \in X$, we have

$$\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \leq \|T\| \tag{3}$$

Again given a +ve ε $(0 < \varepsilon < \|T\|)$ we find a member $x_\varepsilon \in X$ such that

$\|T(x_\varepsilon)\| > (\|T\| - \varepsilon)\|x_\varepsilon\|$; clearly $x_\varepsilon \neq \underset{\sim}{0}$.

Thus $\frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Therefore $\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \geq \frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Now taking $\varepsilon \to 0_+$ we find

$$\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \geq \|T\| \tag{4}$$

Combining (3) and (4) we have $\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} = \|T\|$.

EXERCISE A

Short answer type questions :

1. In a linear space X if $x \in X$ show that $-(-x) = x$.
2. If a finite set of vectors in a linear space contains the zero vector show that it is a linearly dependent set.
3. In Euclidean 2-space R^2 describe geometricaly open ball centred at (0,0) with radius = 1 in respect of (a) $\|x\|_1 = \sqrt{x_1{}^2 + x_2{}^2}$ (b) $\|x\|_2 = |x_1| + |x_2|$ and (c) $\|x\|_3 = \max\{|x_1|, |x_2|\}$ where $x = (x_1, x_2) \in R^2$.
4. Obtain a condition such that function suit and $\sin \lambda t$ are linearly independent in the space $C[0,2\pi]$.
5. Construct a basis of Euclidean 3-space R^3 containing (1,0,0) and (1,1,0).

EXERCISE B

Broad asnwer type questions

1. If $C[a,b]$ is the linear space of all real-valued continuous functions over the closed interval $[a,b]$, show that $C[a,b]$ is a Normed Linear space with respect to $\|f\| = \int_a^b |f|\,dt$, $f \in C[a,b]$. Examine if $C[a,b]$ is a Banach space with this norm.
2. In a *NLS X*. verify that for a fixed member $a \in X$, the function $f : X \to X$ given by $f(x) = x + a$; $x \in X$ is a homeomorphism. Hence deduce that translate of an open set in X is an open set.
3. Examine if the sub-space $\rho[0,1]$ of all real polynomials over the closed interval $[0,1]$ is a closed sub-space of the Banach space $C[0,1]$ with sup norm.
4. Prove that in a *NLS* the closure of the open unit ball is the closed unit ball.
5. Let $(X, \|\;\|)$ and $(Y, \|\;\|)$ be two *NLS* over the same scalars and $T : X \to Y$ be a linear operator that sends a convergent sequence in X to a bounded sequence in Y. Prove that T is a bounded linear operator.
6. Let $T : C[0,1] \to$ itself, where $C[0,1]$ is the Banach space of all real-valued continuous functions over the closed unit interval with sup norm such that $T(x) = y$ where

   $$y(t) = \int_0^t x(u)\,du; \quad x \in C[0,1] \text{ and } 0 \le t \le 1$$

   Find the range of T, and obtain $T^{-1} : (\text{range}\, T) \to C[0,1]$.

   Examine if T^{-1} is linear and bounded.

UNIT 3

(***Contents*** : Every Finite Dimensional *NLS* is a Banach space, Equivalent norms, Riesz Lemma, Finite Dimensionality of *NLS* by compact unit ball, Linear operators over finite Dimensional *NLS* and matrix representation; Isomorphism, Boundedness of linear operators over finit Dimensional *NLS*, space *Bd𝓛(X,Y)* of bounded linear operators, and its completeness).

§ 3.1 FINITE DIMENSIONAL *NLS*

Theorem 3.1.1. Every finite dimensional *NLS* is a Banach space. To prove this Theorem we need a Lemma.

Lemma 3.1.1. Let $(x_1, x_2, ..., x_n)$ be a set of linearly independent vectors in a *NLS* $(X, \|\ \|)$; then there is a +ve β such that

$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\| \geq \beta(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$ for every set of scalars $\alpha_1, \alpha_2 ..., \alpha_n$.

Proof : Put $S = \sum_{i=1}^{n} |\alpha_i|$. Without loss of generality we take $S > 0$.

Then above inequality is changed into

$$\|\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n\| \geq \beta, \text{ where } \beta_i = \frac{d_i}{S} \qquad (*)$$

and $\sum_{i=1}^{n} |\beta_i| = 1$.

If suffices to establish (*) for any set of scalars $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{i=1}^{n} |\beta_i| = 1$. We apply method of contradiction. Suppose there is a sequence $\{y_m\}$ with

$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n$; and $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ for $m = 1, 2, ...$

such that $\|y_m\| \to 0$ as $m \to \infty$

Now $|\beta_i^{(m)}| \leq \sum_{i=1}^{n} |\beta_i^{(m)}| = 1$

Hence for a fixed i the sequence $\{\beta_i^{(m)}\} = \{\beta_i^{(1)}, \beta_i^{(2)}, \ldots\}$ is bounded. So Bolzano-Weirstrass Theorem says that $\{\beta_i^{(m)}\}$ has a sub-sequence that converges to (say) β_i.

Let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ shall give a sub-sequence, say $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). We continue this process. At nth stage we produce a subsequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2}, \ldots\}$ of $\{y_m\}$ whose term

$$y_{n,m} = \sum_{i=1}^{n} \delta_i^{(m)} x_i, \quad \sum_{i=1}^{n} |\delta_i^{(m)}| = 1$$

such that $\lim_{m\to\infty} \delta_i^{(m)} = \beta_i$, Hence we see

$\lim_{m\to\infty} y_{n,m} = \sum_{i=1}^{n} \beta_i x_i = y$ (say) when $\sum_{i=1}^{n} |\beta_i| = 1$. That means all β_i's are not zero. Since $x_1, x_2, \ldots, x_n$ are linearly independent it follows that $y \neq \underset{\sim}{0}$.

Now $\lim_{m\to\infty} y_{n,m} = y$ gives

$$\lim_{m\to\infty} \| y_{n,m} \| = \| y \|.$$

Since $\{y_{n,m}\}$ is a sub-sequence of $\{y_m\}$ and $\| y_m \| \to 0$ as $m \to \infty$, So $\| y_{n,m} \| \to 0$ as $m \to \infty$ and so $\| y \| = 0$ giving $y = \underset{\sim}{0}$, a contradiction. Therefore Lemma is proved.

Proof of Theorem 3.1.1. Suppose $\{y_m\}$ be a Cauchy sequence in a finite dimensional *NLS* $(X, \| \|)$. Let Dim(X) = n, and $(e_1, e_2, \ldots, e_n)$ forms a basis in X. So each y_m has a unique representation.

$$y_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \ldots + \alpha_n^{(m)} e_n$$

Give a +ve ε, as $\{y_m\}$ is Cauchy, we find an index N such that

$$\| y_m - y_r \| < \varepsilon \text{ for } m, r \geq N.$$

Now $\varepsilon > \| y_m - y_r \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(r)}) e_i \|$

$$\geq \beta \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(r)} | \text{ by Lemma 3.1.1}$$

whenever $m, r > N$. Therefore

$$| \alpha_i^{(m)} - \alpha_i^{(r)} | \leq \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(r)} | < \frac{\varepsilon}{\beta} \text{ for } m, r > N$$

Therefore, each of the n sequences

$\{\alpha_i^{(m)}\}$ $(i = 1, 2, ..., n)$ becomes a Cauchy sequence of scalars (reals/complex), and by Cauchy's General Principle of convergence becomes a convergent sequence with, say,

$$\lim_{m \to \infty} \alpha_i^{(m)} = \alpha_i^{(0)} \text{ (say)}, \ i = 1, 2, ..., n.$$

Put $y = \alpha_1^{(0)} e_1 + \alpha_2^{(0)} e_2 + ... + \alpha_n^{(0)} e_n$; so $y \in X$.

Further, $\lim_{m \to \infty} \alpha_i^{(m)} = \alpha_i^{(0)}$ for i = 1,2, ..., n gives,

$$\| y_m - y \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(0)}) e_i \| \leq \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(0)} | \| e_i \| \to 0 \text{ as } m \to \infty.$$

i.e. $\lim_{m \to \infty} y_m = y \in X$. So given Cauchy sequence $\{y_m\}$ in X is convergent in X; and $(X, \| \ \|)$ is Banach space.

Theorem 3.2.1. Any two norms in a finite dimensional *NLS X* are equivalent.

Proof : Let Dim $(X) = n$ and $(e_1, e_2, ..., e_n)$ form a basis for X. If $x \in X$, we write $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ uniquely.

Applying Lemma 3.1.1 we find a +ve β such that

$$\| x \|_1 \geq \beta(| \alpha_1 | + | \alpha_2 | + ... + | \alpha_n |)$$

If $\mu = \max_{1 \leq i \leq n} \| e_i \|_2$; Then we have

$$\| x \|_2 \leq \sum_{i=1}^{n} | \alpha_i | \| e_i \| \leq \mu \sum_{i=1}^{n} | \alpha_i | \leq \frac{\mu}{\beta} \| x \|_1$$

or, $\beta_{\overline{\mu}} \| x \|_2 \leq \| x \|_1$, the other half of desired inequality comes by interchanging norms $\| \ \|_1$ and $\| \ \|_2$. The proof is now complete.

Theorem 3.1.3. A *NLS* $(X, \|\ \|)$ is finite Dimensional if and only if the closed unit ball (centred at $\underset{\sim}{0}$) is compact.

To prove this theorem we need support of another result popularly known as Riesz Lemma.

Lemma 3.1.2 (Riesz Lemma). Let L ($\neq X$) be a closed sub-space of a *NLS* $(X, \|\ \|)$. Given a +ve $\in$ $(0 < \in < 1)$ there is a member $y \in \left(\frac{X}{L}\right)$ with $\| y \| = 1$ such that $\| y - x \| > 1 - \varepsilon$ for all $x \in L$.

Proof : Take $y_0 \in \left(\frac{X}{L}\right)$ and put $d = dist(y_0, L)$

$$= \underset{x \in L}{Inf} \| y_0 - x \|.$$

Since L is closed and y_0 is outside L, we have $d > 0$. Given a +ve ε, choose $\eta > 0$ such that

$$\frac{\eta}{d + \eta} < \varepsilon$$

So we find a member $x_0 \in L$ such that

$$d \leq \| y_0 - x_0 \| < d + \eta$$

Take $y = \frac{y_0 - x_0}{\| y_0 - x_0 \|} (y_0 \neq x_0)$; then $\| y \| = 1$, and we have

$y_0 = x_0 + \| y_0 - x_0 \| y$. Since y_0 is outside L, we find y also outside L *i.e.* $y \in \left(\frac{X}{L}\right)$.

If $x \in L$, we have $\| y - x \| = \left\| \frac{y_0 - x_0}{\| y_0 - x_0 \|} - x \right\|$

$$= \frac{1}{\| y_0 - x_0 \|} \left\| y_0 - x_0 - x \| y_0 - x_0 \| \right\| = \frac{1}{\| y_0 - x_0 \|} \| y_0 - x' \| \text{ (say)}$$

where $x' = x_0 + \| y_0 - x_0 \| x$; clearly $x' \in L$ because $x_0, x \in L$.

Therefore, $\| y - x \| > \frac{1}{d + \eta} \| y_0 - x' \| \geq \frac{d}{d + \eta} = 1 - \frac{\eta}{d + \eta} = 1 - \varepsilon$.

The proof is now complete.

Proof of Theorem 3.1.3. First suppose that closed unit ball $\hat{B}_1(\underset{\sim}{0}) = \{x \in X : \|x\| \le 1\}$ in a *NLS* (*X*, $\|\ \|$) is compact and hence is sequentially compact. We show that $\text{Dim}(X) < \infty$.

Suppose no. take $x_1 \in X$ with $\|x_1\| = 1$ and L_1 as the sub-space spanned by $x_1(\neq \underset{\sim}{0})$. Then L_1 is a closed sub-space of X without being equal to X. So we apply Riesz Lemma (Lemma 3.1.2) when we take $\in = \frac{1}{2}$. Then we find $x_2 \in (X \setminus L_1)$ with $\|x_2\| = 1$ and $\|x_1 - x_2\| > \frac{1}{2}$.

Take L_2 as the sub-space spanned by x_1 and x_2. By the argument same as above we find L_2 as a proper closed sub-space of X and attracts Riesz Lemma. Thus there is $x_3 \in (X \setminus L_2)$ with $\|x_3\| = 1$ and $\|x_3 - x_1\| > \frac{1}{2}$, $\|x_3 - x_2\| > \frac{1}{2}$.

We continue this process to obtain a sequence $\{x_n\}$ with $\|x\| = 1$ *i.e.* $x_n \in \tilde{B}_1(\underset{\sim}{0})$ such that $\|x_n - x_m\| > \frac{1}{2}$ for $n \neq m$. That means $\{x_n\}$ does not admit if any convergent subsequence : a contradiction that $\hat{B}_i(\underset{\sim}{0})$ is sequentially compact. Hence we have shown that $\text{Dim}(X) < \infty$.

Conversely let (*X*, $\|\ \|$) be finite dimensional. Then it is a well known property that a subset in X is norm-compact if and only if that subset is bounded and closed. Here the closed unit ball $\hat{B}_i(\underset{\sim}{0})$ is bounded; and hence it must be compact. The proof is now complete.

§ 3.2 LINEAR OPERATORS OVER FINITE DIMENSIONAL SPACES :

Let R^n denote the Euclidean n-space. Then an $m \times n$ real matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ defines a Linear operator $T : R^n \to R^m$ where $T(\underset{\sim}{x}) = \underset{\sim}{y}$;

$\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n)$ and $\underset{\sim}{y} = (\eta_1, \eta_2, ..., \eta_n)$ such that

$$\sum_{j=1}^{n} a_{ij}\xi_j = \eta_i \quad i = 1, 2, ..., m.$$

Verification is an easy exercise and is left out.

Conversely, given a linear operator $T : R^n \to R^m$. We show that it is represented by an $(m \times n)$ real matrix. Let us take $(e_1, e_2, ..., e_n)$ as a basis in R^n where $e_i = \left(\underset{i\text{th place}}{\underline{0} 1 \underline{0}}\right)$, $i = 1, 2, ..., n$. And let $f_1 = \underbrace{(1, 0, 0, ...0)}_{m \text{ places}}$, $f_2 = (0, 1, 0, 0, ...0)$, $f_n = (0, 0, ..., 1)$ form the analogous basis in R^m.

Let $T(e_j) = \underset{\sim}{a}_j \in R^m$

$$= \alpha_{1j} f_1 + \alpha_{2j} f_2 + ... + \alpha_{mj} f_m \quad \text{(say)} \quad (j = 1, 2, ..., n)$$

In general, if $\underset{\sim}{x} = (\xi_1, \xi_2, ... \xi_n) \in R^m$ and if $T(\underset{\sim}{x}) = \underset{\sim}{y} \in R^m$

we have $\eta_1 f_1 + \eta_2 f_2 + ... + \eta_m f_m = \underset{\sim}{y}$ and

$$\underset{\sim}{y} = T(\underset{\sim}{x}) = T\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j T(e_j) = \sum_{j=1}^{n} \xi_j \underset{\sim}{a}_j$$

$$= \sum_{j=1}^{n} \xi_j \left(\sum_{i=1}^{m} \alpha_{ij} f_i\right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij} \xi_j\right) f_i$$

Or, $\sum_{i=1}^{m} \eta_i f_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right) f_i$ gives $\eta_i = \sum_{j=1}^{n} \alpha_{ij}\xi_j$; $i = 1, 2, ..., m.$

Therefore, T is represented by the matrix $\left((\alpha_{ji}\right)_{m \times n}$.

Remark : Given a linear operator $T : R^n \to R^m$, there is an $(m \times n)$ matrix to represent T. Entries (reals) in this matrix depend upon the choice of basis in underlying spaces. If basis changes co-efficients entering representative matrix change; However order of the matrix does not change.

Example 3.2.1. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval $[0,1]$ with degree ≤ 3. Let $D:\rho_3[0,1]\to\rho_2[0,1]$ be the differential operator. Show that D is a linear operator and obtain a representative matrix for D.

Solution : Here $\rho_3[0,1]$ (and similarly $\rho_2[0,1]$) is a real linear space with $\text{Dim}\,\rho_3[0,1]=4$ (Dim($\rho_2[0,1]=3$). Let us take (p_0, p_1, p_2, p_3) as a basis for $\rho_3[0,1]$ where $p_0(t)=1$, $p_1(t)=t$, $p_2(t)=t^2$ and $p_3(t)=t^3$ in $0\leq t\leq 1$.

Then we have $D(p_0)=0$, $D(p_1)=1$, $D(p_2)=2t$ and $D(p_3)=3t^2$; and we write

$$
\begin{aligned}
0 &= 0p_0+0p_1+0p_2\\
1 &= 1p_0+0p_1+0p_2\\
2t &= 0p_0+2p_1+0p_2\\
\text{and}\qquad 3t^2 &= 0p_0+0p_1+3p_2
\end{aligned}
$$

And therefore representative matrix $((a_{ij}))_{3\times4}$ for D is given by

$$
\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times4}
$$

Remark : Representative matrix for linear operator changes if basis is changed.

Example 3.2.2. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval $[0,1]$ with degree ≤ 3.

Let $T:\rho_3[0,1]\to\rho_3[0,1]$ be a linear operator given by

$T(a_0+a_1x+a_2x^2+a_3x^3)=a_0+a_1(x+1)+a_2(x+1)^2+a_3(x+1)^3$ for every member $a_0+a_1x+a_2x^2+a_3x^3\in\rho_3[0,1]$; obtain representative matrix for T relative to basis (i) $(1, x, x^2, x^3)$ and (ii) $(1, 1+x, 1+x^2, 1+x^3)$ of $\rho_3[0,1]$

Solution : Here $\text{Dim}\,\rho_3[0,1]=4$; So required matrix for linear operator T is of order 4×4; where $T:\rho_3[0,1]\to\rho_3[0,1]$.

Now (i) $(1, x, x^2, x^3)$ forms a basis for $\rho_3[0,1]$.

Now we have,

$T(1) = 1$, $T(x) = (x + 1)$, $T(x^2) = (x + 1)^2$ and $T(x)^3 = (x + 1)^3$. So we write with respect to basis above

$T(1) = 1 \qquad = 1.1 + 0.x + 0.x^2 + 0.x^3$

$T(x) = 1 + x \qquad = 1.1 + 1.x + 0.x^2 + 0.x^3$

$T(x^2) = (x + 1)^2 = 1.1 + 2.x + 1.x^2 + 0.x^3$

$T(x^3) = (x + 1)^3 = 1.1 + 3.x + 3.x^2 + 1.x^3$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Here basis is $(1, 1+x, 1+x^2, 1+x^3)$ of $\rho_3[0,1]$

We have $T(1) = 1$, $T(1+x) = 1 + (1 + x)$, $T(1+x^2) = 1 + (1+x)^2$ and $T(1+x^3) = 1 + (1+x)^3$

Therefore relative to basis $(1, 1+x, 1+x^2, 1+x^3)$ we write

$T(1) \quad = 1 = 1.1 + 0.(1+x) + 0.(1+x^2) + 0.(1+x^3)$

$T(1+x) = 2+x = 1.1 + 1.(1+x) + 0.(1+x^2) + 0.(1+x^3)$

$T(1+x^2) = 1+1+2x+x^2 = -1.1 + 2.(1+x) + 1.(1+x^2) + 0.(1+x^3)$

$T(1+x^3) = 1+1+3x+3x^2+x^3 = -5.1 + 3.(1+x) + 3.(1+x^2) + 1.(1+x^3)$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note : Basis taken and treated above should be termed as ordered basis. In ordered basis order of arrangement of vectors is basis in important. For example, in

Euclidean 3-space R^3 we know (e_1, e_2, e_3) is a basis in R^3, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Then each of (e_1, e_2, e_3), (e_2, e_1, e_3) and (e_1, e_3, e_2) is an ordered basis and they are different ordered basis for R^3.

§ 3.2(A) ISOMORPHIC LINEAR SPACES :

Definition 3.2.1. Two linear spaces X and Y over the same scalars are said to be isomorphic (or, linearly isomorphic) if there is a linear operator $T : X \to Y$ that is 1–1 (injective) and onto (surjective). The operator T is called an Isomorphism.

Theorem 3.2.1. Linear isomorphism between linear spaces over same scalars on the class Γ, of all such spaces is an equivalence relation.

Proof : If $X \in \Gamma$, the identity operator $I : X \to X$ is an isomorphism. So the binary relation of being isomorphic is reflexive; let $X, Y \in \Gamma$ such that X is isomorphic to Y with $\varphi : X \to Y$ as an isomorphism; Then $\varphi^{-1} : Y \to X$ is also an isomorphism. Thus Y is isomorphic to X. Hence relation of isomorphism is symmetric. Finally, if $f : X \to Y$ and $g : Y \to Z$ are isomorphism, then $(g . f) : X \to Z$ is also an isomorphism. Therefore, the relation of isomorphism is transitive. Thus it is an equivalence relation.

Theorem 3.2.2. Every real linear space X with $\dim(X) = n$ is isomorphic to the Euclidean n-space R^n.

Proof : Let $(u_1, u_2,..., u_n)$ form a basis in X. So if $u \in X$ we write

$$u = \xi_1 u_1 + \xi_2 u_2 + ... + \xi_n u_n \text{ uniquely.}$$

Define an operator $T : X \to R^n$ by the rule :

$$T(u) = (\xi_1, \xi_2, ..., \xi_n) \in R^n \text{ where } u = \xi_1 u_1 + \xi_2 u_2 + ... + \xi_n u_n \in X$$

Then it is easily verified that T is a linear operator. Furhter, if $u = \sum_{i=1}^{n} \xi_i u_i$ and $v = \sum_{i=1}^{n} \eta_i u_i$ with $u \neq v$ are members of X, then we have

$$(\xi_1, \xi_2, ..., \xi_n) \neq (\eta_1, \eta_2, ..., \eta_n) \text{ or } T(u) \neq T(u);$$

thus T is 1–1. Finally, for $(\alpha_1, \alpha_2, ..., \alpha_n) \in R^n$

We have $\sum_{i=1}^{n} \alpha_i u_i \in X$ such that $T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = (\alpha_1, \alpha_2, ..., \alpha_n)$.

So T is onto. Therefore X is isomorphic to R^n.

Notation : If two linear space X and Y are isomorphic we use the symbol $X \cong Y$.

Corollary : Any two real linear spaces of same finite dimension are isomorphic

Because if X and Y are finite dimensional real linear spaces with Dim(X) = Dim(Y), we apply Theorem 3.2.2. to say $X \cong R^n$; and hence $X \cong Y$.

Theorem 3.2.3. Every linear operator over a finite dimensional *NLS* is bounded (hence continuous).

Proof : Let (X, $\|\ \|$) and (Y, $\|\ \|$) be two *NLS* over same scalars and Dim(X) $< \infty$, say, being equal to n, and let (e_1, e_2, ..., e_n) be a basis for X. Then each member $x \in X$ has a unique representative as $x = \xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$ where ξ_i's are scalars. Let us define a norm $\|x\|'$ by the formula :

$$\|x\|' = \sum_{i=1}^{n} |\xi_i|.$$

It is an easy task to check that $\|x\|'$ is indeed a norm in X. Since X is finite dimensional, we know that any two norms in X are equivalent.

Therefore there is a +ve M satisfying

$$\|x\|' \leq M \|x\| \text{ for all } x \in X$$

$$i.e. \quad \sum_{i=1}^{n} |\xi_i| \leq M \|x\| \qquad \text{...........} (*)$$

If $T : X \to Y$ is a linear operator and $x = \sum_{i=1}^{n} \xi_i e_i \in X$, we have

$$\|T(x)\| = \left\|T\left(\sum_{i=1}^{n} \xi_i e_i\right)\right\| = \left\|\sum_{i=1}^{n} \xi_i T(e_i)\right\|$$

$$\leq \sum_{i=1}^{n} |\xi_i| \|T(e_i)\|$$

$$\leq \max(\|T(e_i)\|, \|T(e_2)\|, ..., \|T(e_n)\|).M \|x\|$$

(from (*)) $\quad = L\|x\|$, (say).

This being true for all $x \in X$, we conclude that T is bounded.

§ 3.3 SPACE OF ALL BOUNDED LINEAR OPERATORS Bd$\mathscr{L}$(X, Y)

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. Then zero operator $O : X \to Y$ where $O(x) = \underset{\sim}{0} \in Y$ *as* $x \in X$ is a bounded linear operator. Therefore $Bd\mathscr{L}(X,Y) \neq \phi$. It is a routine exercise to check that $Bd\mathscr{L}(X,Y)$ becomes a linear space with respect to addition and scalar multiplication as given by

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for all $x \in X$; and $T_1, T_2 \in Bd\mathscr{L}(X,Y)$ and $(\lambda T_1)(x) = \lambda T_1(x)$ for all $x \in X$ and for all scalars λ and $T_1 \in Bd\mathscr{L}(X,Y)$

Theorem 3.3.1. *Bd$\mathscr{L}$(X,Y)* is a Normed Linear space, and it is a Banach space when Y is so.

Proof : Let us take the norm in linear space $Bd\mathscr{L}(X,Y)$ as operator norm $\|T\|$ as $T \in Bd\mathscr{L}(X,Y)$. We verify that all norm axioms are satisfied here.

For (N.1) it is obvious that $\|T\| \geq 0$ always for any member $T \in Bd\mathscr{L}(X,Y)$; zero operator O has the norm $\|O\| = 0$.

Suppose $\|T\| = 0$ *i.e.* $\sup\limits_{\|x\|\leq 1} \|T(x)\| = 0$. So if $\|x\| \leq 1$, we have

$$\|Tx\| \leq \sup_{\|x\|\leq 1} \|T(x)\| = 0 \text{ gives } \|T(x)\| = 0 \quad \ldots\ldots\ldots\ldots (1)$$

If $\|x\| > 1$, then put $y = \dfrac{x}{\|x\|}$; Thus $\|y\| = \left\|\dfrac{x}{\|x\|}\right\| = 1$; so as got above

$\|T(y)\| = 0$ or $0 = \|T(y)\| = \left\|T\left(\dfrac{x}{\|x\|}\right)\right\| = \dfrac{1}{\|x\|}\|T(x)\|$ giving

$$\|T(x)\| = 0 \quad \ldots\ldots\ldots\ldots (2)$$

So (1) and (2) say that $T(x) = \underset{\sim}{0}$ for all $x \in X$ *i.e.* T equals to the zero operator.

For (N.2) take λ to be any scalar.

$$\text{Then } \|\lambda T\| = \sup_{\|x\|\leq 1} \|(\lambda T)(x)\|$$

$$= \sup_{\|x\|\leq 1} \|\lambda T(x)\| = \sup_{\|x\|\leq 1}\{|\lambda| \|T(x)\|\}$$

$$= |\lambda| \sup_{\|x\|\leq 1} \|T(x)\| = |\lambda| \|T\|.$$

So (N.2.) is satisfied.

For triangle inequality, if T_1, T_2 are members of $Bd\mathscr{L}(X,Y)$ we have for

$x \in X$, $\|T_1 + T_2)(x)\| = \|T_1(x) + T_2(x)\| \leq \|T_1(x)\| + \|T_2(x)\|$

$\leq \|T_1\|(x)\| + \|T_2\|\|x\| = (\|T_1\| + \|T_2\|)\|x\|$; this is true for all $x \in X$,

Therefore $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$, and that is the triangle inequality.

Therefore $Bd\mathscr{L}(X,Y)$ is a Normed Linear space (*NLS*) with respect to operator norm.

Now suppose that Y is a Banach space. We show that $Bd\mathscr{L}(X,Y)$ is so. Take $\{T_n\}$ as a Cauchy sequence in $Bd\mathscr{L}(X,Y)$ *i.e.* $\|T_n - T_m\| \to 0$, as $n, m \to \infty$

If $x \in X$, we have $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$\leq \|T_n - T_m\|\|x\| \to 0$ as $n, m \to \infty$. That means, $\{T_n(x)\}$ is a Cauchy sequence in $(Y, \|\ \|)$ which is complete.

Let $\lim_{n\to\infty} T_n(x) = y \in Y$

Let us define $T : X \to Y$ by the rule :

$$T(x) = \lim_{n\to\infty} T_n(x) \text{ as } x \in X.$$

Now it is easy to see that T is a linear operator.

Further, $\left|\|T_n\| - \|T_m\|\right| \leq \|T_n - T_n\| \to 0$ as $n, m \to \infty$.

That means $\{\|T_n\|\}$ is a sequence of non-negative reals and this is Cauchy sequence and therefore is bounded. So we find a +ve K satisfying

$$\|T_n\| \leq K \text{ for all } n.$$

So, $\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\|$

$$\leq \lim_{n\to\infty} \|T_n\|\|x\| \leq K\|x\| \text{ by above inequality.}$$

This being true for all $x \in X$, we find $T : X \to Y$ as a bounded linear operator *i.e.* $T \in Bd\mathscr{L}(X,Y)$.

Finally, from Cauchyness of $\{T_n\}$, given a +ve ε, we have

$$\| T_{n+p} - T_n \| < \varepsilon \text{ for } n \geq n_0 \text{ and } p = 1,2,\ldots\ldots$$

Take $\| x \| \leq 1$ in X, So $\| T_{n+p}(x) - T_n(x) \| = \| (T_{n+p} - T_n)(x) \|$

$$\leq \| T_{n+p} - T_n \| \| x \| \leq \| T_{n+p} - T_n \| . < \varepsilon \text{ for } n \geq n_0$$

Let us pass on limit as $p \to \infty$, then we have

$$\| T(x) - T_n(x) \| \leq \varepsilon \text{ whenever } n \geq n_0$$

This is the case whenever $\| x \| \leq 1$; taking sup we have

$$\sup_{\|x\|\leq 1} \| T(x) - T_n(x) \| \leq \varepsilon \text{ whenever } n \geq n_0$$

Now $\| T - T_n \| = \sup\limits_{\|x\|\leq 1} \| (T - T_n)(x) \|$

$$= \sup_{\|x\|\leq 1} \| (T(x) - T_n(x) \|$$

$$\leq \varepsilon \text{ whenever } n \geq n_0$$

So we obtain $\lim\limits_{n\to\infty} T_n = T \in Bd\mathscr{L}(X,Y)$ in operator norm.

The proof is now complete.

Example 3.3.1. Show $Bd\mathscr{L}(R^n, R^n)$ is finite dimensional with dimension n^2.

Solution : By matrix representation theorem we know that every member $T \in Bd\mathscr{L}(R^n, R^n)$ has a representative matrix of order $n \times n$ (*i.e.* a square matrix of size n). With respect to a fixed basis in R^n, we also see that $Bd\mathscr{L}(R^n, R^n)$ and the linear space $m_{n\times n}$ is finite dimensional with $\text{Dim}(m_{n\times n}) = n^2$.

Therefore $\text{Dim}(Bd\mathscr{L}(R^n, R^n)) = n^2$

Example 3.3.2. A *NLS* (X, $\| \|$) is a Banach space if and only if $\{x \in X : \| x \| = 1\}$ is complete.

Solution : Suppose (X, $\| \|$) is a Banach space; then the given set $\{x \in X : \| x \| = 1\}$ is a closed subset of X, and hence is complete.

Conversely, suppose $S=\{x \in X : \|x\|=1\}$ is complete. Now let $\{x_n\}$ be a Cauchy sequence in *X*, so $\|x_n - x_m\| \to 0$ as $n,m \to \infty$

Therefore $\left| \|x_n\| - \|x_m\| \right| \le \|x_n - x_m\| \to 0$ as $n,m \to \infty$. Thus scalar sequence $\{\|x_n\|\}$ is Cauchy, and by Cauchy General Principle of convergence $\{\|x_n\|\}$ is convergent; put $\lim_{n\to\infty} \|x_n\| = \alpha$. If $\alpha = 0$ we see $\{x_n\}$ to be convergent in *X* and we have finished. Or else $\alpha > 0$. Without loss of generality we assume that $\alpha = 1$. Let us put $y_n = \dfrac{x_n}{\|x_n\|}$ making $\|y_n\|=1$ *i.e.* $y_n \in S$. If possible, let $\{y_n\}$ be not Cauchy. Then there is a +ve ε_0 (say) and there are indices $n_k(\ge k), m_k(\ge k)$ such that

$$\|y_{n_k} - y_{m_k}\| \ge \varepsilon_0,\ k = 1, 2, \ldots\ldots$$

$$\text{or, } \varepsilon_0 \le \left\| \frac{x_{n_k}}{\|x_{n_k}\|} - \frac{x_{m_k}}{\|x_{m_k}\|} \right\| \le \left\| \frac{x_{n_k}}{\|x_{n_k}\|} - x_{n_k} \right\| + \|x_{n_k} - x_{m_k}\| + \left\| x_{m_k} - \frac{x_{m_k}}{\|x_{m_k}\|} \right\|$$

$$= \|x_{n_k}\| \left| 1 - \frac{1}{\|x_k\|} \right| + \|x_{m_k}\| \left| 1 - \frac{1}{\|x_{m_k}\|} \right| \to 0 \quad \text{as} \quad k \to \infty;$$ arriving at contradiction that ε_0 is +ve. Therefore we conclude that $\{y_n\}$ is Cauchy in *S* by completeness of which let $\lim_{n\to\infty} y_n = y_0 \in S$. That is $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \|x_n\| y_0 = y_0$.

Hence $\{x_n\}$ is convergent in *X* and *X* is shown as a Banach space.

EXERCISE A

Short answer type questions

1. Let *X* be the linear space spanned by *f* and *g* where $f(x) = \sin x$ and $g(x) = \cos x$. For any real θ, let $f_1(x) = \sin(x+\theta)$ and $g_1(x) = \cos(x+\theta)$. Show that f_1 and g_1 are members of *X*, and they are linearly independent.
2. Let *A* and *B* be two subsets of a *NLS X* and let $A+B=\{a+b : a \in A \text{ and } b \in B\}$. Show that if *A* or *B* is open then $A + B$ is open.
3. Let $m_{2\times2}$ be the linear space of all real 2×2 matrics and $E = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

If $T : m_{2\times 2} \to m_{2\times 2}$ is taken as $T(A) = EA$ for $A \in m_{2\times 2}$, show that T is a linear operator.

4. If C is a convex subset of a *NLS* X and $x_0 \in X$, and α is a non-zero scalar, show that $x_0 + C$ and αC are convex sets.
5. Show that $T : C[a,b] \to R$ (real space with usual norm) defined by the rule :

$$T(f) = \int_a^b tf(t)dt\,; \qquad f \in C[a,b].$$

Show that T is a bounded linear operator.

EXERCISE B

1. Let A and B be two subsets of a *NLS* X, and let $A + B = \{a + b : a \in A \text{ and } b \in B\}$. If A and B are compact, show that $A + B$ is compact.
2. Let M be a closed linear sub-space of a *NLS* $(X, \|\ \|)$, and X/M be the quotient space, and $T : X \to X/M$ where $T(x) = x + M$ for $x \in X$.

   Show that T is a bounded linear operator with $\|T\| \le 1$.
3. Show that the space of all real polynomials of degree $\le n$ is the closed interval $[a,b]$ is isomorphic to the Euclidean $(n+1)$-space R^{n+1}.
4. Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be *NLS* over same scalars and $F, T : X \to Y$ be bounded linear operators such that F and T agree over a dense set in X, show that $F \equiv T$.
5. If X is a finite Dimensional *NLS*, and Y is a proper sub-space of X, then show that there is a member $x \in X$ with $\|x\| = 1$. satisfying dist$(x, Y) = 1$.

UNIT 4

(***Contents*** : Bounded Linear functionals, sub-linear functionals, Hahn-Banach Theorem; Its applications, Conjugate spaces of a *NLS*, Canonical mapping, Embedding of a *NLS* into its second conjugate space under a linear isometry, reflexive Banach space; Open mapping theorem, Closed Graph Theorem.).

§ 4.1 LINEAR FUNCTIONALS :

Let $(X, \|\ \|)$ be a *NLS* over reals/complex numbers.

Definition 4.1.1. A Scalar-valued Linear operator f over X is called a Linear functional.

For example if X = Banach space $C[0,1]$ with sup norm, then $f : X \to$ Reals (with usual norm) is a linear functional when $f(x) = \int_0^1 x(t)dt; \quad x \in C[0,1]$.

Explanation : Linear functionals are special kind of Linear operators, and thus enjoy all the properties of Linear operators like sending dependent set of the domain into a similar such elements in range.

Let us consider the collection of all continuous (bounded) linear functionals over X *i.e.* we have the space $Bd\mathscr{L}(X,R)$ whenever X is a real *NLS*. We have seen that the space $Bd\mathscr{L}(X,R)$ is always a *NLS* with operator norm $\|f\|$; f being a member of $Bd\mathscr{L}(X,R)$. We have also seen that the *NLS* $Bd\mathscr{L}(X,R)$ is a Banach space because R is so.

Definition 4.1.2. The space $Bd\mathscr{L}(X,R)$ denoted by X^* is called first conjugate space (Dual space) of X.

Thus first conjugate space or simply conjugate space X^* of any *NLS* $(X, \|\ \|)$ is always a Banach space irrespective of X being complete or not.

By a similar construction one can produce $Bd\mathscr{L}(X^*,R)$ = the space of all bounded linear functionals over X^*; this Banach space $X^{**} = (X^*)^*$ is called second conjugate (Dual) space of X; and so on.

Most of theory of conjugate spaces rests on one single theorem, known as famous Hahn-Banach Theorem that asserts that any continuous linear functional on a linear subspace of X can be extended to a continuous linear functional over X by keeping the norm-value of the functional unchanged. The proof of Hahn-Banach Theorem is lengthy but necessarily indispensable item in Functional Analysis.

Before we take up Hahn-Banach Theorem in setting of a *NLS* we proceed as under :

Definition 4.1.2. Let X be a real linear space. Then $p : X \to$ Reals satisfying (i) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$ and (ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$, $x \in X$ is called a sub-linear functional.

Note : Condition (i) above is known as condition of sub-additivity and condition (ii) above is called positive homogeneity.

It is not difficult to see that norm function in a *NLS* X is a sub-linear functional over X.

Theorem 4.1.1. (Hahn-Banach Theorem in a linear space)

Let M be a subspace of a real linear space X, and p is a sub-linear functional over X and f is a linear functional on M such that $f(x) \le p(x)$ for all $x \in M$.

Then there is a linear functional F over X which is an extension of f (over M) such that

$$F(x) \le p(x) \text{ for all } x \in X.$$

The proof of this Theorem rests upon following Lemma.

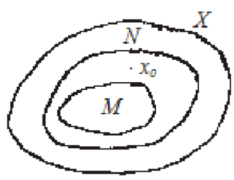
Lemma 4.1.1. Suppose M is a subspace ($\ne X$) of a real linear space X and $x_0 \in (X \backslash M)$. Let N be the subspace spanned by M and $\{x_0\}$ *i.e.* $N = [M \cup \{x_0\}]$; suppose $f : M \to R$ is a Linear functional such that

$f(x) \le p(x)$ for all $x \in M$, where $p : X \to R$ is a sub-linear functional (over X).

Then f can be extended to a linear functional F defined on N such that

$$F(x) \le p(x) \quad \text{for } x \in N.$$

Proof : Since $f(x) \le p(x)$ over M, we have for $y_1, y_2 \in M$.



$$f(y_1 - y_2) = f(y_1) - f(y_2) \le p(y_1 - y_2) = p(y_1 + x_0 - y_2 - x_0)$$

$$\le p(y_1 + x_0) + p(-y_2 - x_0)$$

or, $\quad -p(-y_2 - x_0) - f(y_2) \le p(y_1 + x_0) - f(y_1)$ .... (1)

(separation of terms involving y_1 and y_2)

Now fix y_1 and allow y_2 to change over M. From (1) we see that the set of reals $\{-p(-y_2 - x_0) - f(y_2)\}$ possesses sup.

Put $a = \sup\limits_{y_2 \in M} \{-p(-y_2 - x_0) - f(y_2)\}$; and in a similar argument, put

$b = \operatorname*{Inf}_{y_1 \in M}\{p(y_2 + x_0) - f(y_1)\}$. The relation (1) says, $a \leq b$.

Take a real c_0 between a and b *i.e.* $a \leq c_0 \leq b$

Therefore as $y \in M$ we have

$$-p(-y - x_0) - f(y) \leq c_0 \leq p(y + x_0) - f(y) \qquad \text{................ (2)}$$

Since $x_0 \notin M$, we write $x_0 \in N$ as $x = y + \alpha x_0$, and this representation is unique.

Consider $F : N \to R$ defined by the rule :

$F(y + \alpha x_0) = f(y) + \alpha c_0$, as $(y + \alpha x_0) \in N$ ($y \in M$ & α a scalar). It is easy to check that F is a linear functional over N such that $F(y) = f(y)$ as $y \in M \subset N$.

In other words F is an extension of f from M to N. We verify further that

$F(x) \leq p(x)$ for all $x \in N$. To achieve this we are to consider following two cases : When $x \in N$, we have $x = y + \alpha x_0$, where α is a scalar.

Case I . When $\alpha > 0$; we consider R.H.S. of inequality (2) with y replaced by $\frac{y}{\alpha}$; this gives $c_0 \leq p\left(\frac{y}{\alpha} + x_0\right) - f\left(\frac{y}{\alpha}\right)$.

Multiplying throughout by α and using the fact that p is sub-linear we have

$$f(y) + \alpha c_0 \leq p(y + \alpha x_0)$$

$$\text{or,} \quad F(x) \leq p(x)$$

Case II. When $\alpha < 0$, we use L.H.S. of inequality (2) with y replaced by $\frac{y}{\alpha}$. This gives rise to

$$-p\left(-\frac{y}{\alpha} - x_0\right) - f\left(\frac{y}{\alpha}\right) \leq c_0$$

$$\text{or,} \qquad -p\left(\frac{y}{\alpha} - x_0\right) \leq c_0 + f\left(\frac{y}{\alpha}\right).$$

Multiplying throughout by α and reversing the sign we have,

$$(-\alpha)p\left(-\frac{y}{\alpha} - x_0\right) \geq \alpha\, c_0 + f(y)$$

Since $-\alpha > 0$, we have $p(y+\alpha x_0) \geq \alpha c_0 + f(y)$

or, $p(x) \geq F(x)$

or, $F(x) \leq p(x)$

When $\alpha = 0$, we readily see $F(y) = f(y)$. The proof of Lemma is now complete.

Proof of Theorem 4.1.1. To prove the theorem we invite partial ordering in a set and use Zorn's Lemma which says that in a partially ordered set if every chain has an upper bound, then there is a maximal member in the set.

Here let Γ denote the collection of all linear functionals $\{\hat{f}\}$ such that each $\hat{f}$ is an extension of f such that $\hat{f}(x) \leq p(x)$ over domain of $\hat{f} \equiv D_{\hat{f}}$.

Lemma 4.1.1 tells us that Γ is non-empty. Let us partially order Γ as for $\hat{f}_1, \hat{f}_2 \in \Gamma$ we say, $\hat{f}_1 < \hat{f}_2$

if $\hat{f}_2$ is an extension of $\hat{f}_1$ with $D_{\hat{f}_2} \supset D_{\hat{f}_1}$, and $\hat{f}_2 = \hat{f}_1$ over $D_{\hat{f}_1}$.

We may verify that α is a partial order relation in Γ where we show that every chain (totally ordered subset) in Γ has an upper bound in Γ. To that goal, let $\tau = \{\hat{f}_\alpha\}$ be a totally ordered subset of Γ. We find some member $\hat{f} \in \Gamma$ to act as an upper bound for τ.

Construct $\hat{f}$ whose domain $= \bigcup_\alpha D_{\hat{f}_\alpha}$. If $x \in \bigcup_\alpha D_{\hat{f}_\alpha}$ there is a member α such that $x \in D_{\hat{f}_\alpha}$ and let $\hat{f}(x) = f_{\hat{\alpha}}(x)$

By routine work we verify that $\bigcup_\alpha D_{\hat{f}_\alpha}$ is a sub-space of X; taking $x, y \in \bigcup_\alpha D_{\hat{f}_\alpha}$ we find two indices α_1 and α_2 such that $x \in D_{\hat{f}_{\alpha_1}}$ and $y \in D_{\hat{f}_{\alpha_2}}$.

Since τ is totally ordered either $D_{\hat{f}_{\alpha_1}} \subset D_{\hat{f}_{\alpha_2}}$ or $D_{\hat{f}_{\alpha_2}} \subset D_{\hat{f}_{\alpha_1}}$, and in either of the cases we have

$(x+y) \in \bigcup_\alpha D_{\hat{f}_\alpha}$ and similarly $\alpha x \in \bigcup_\alpha D_{\hat{f}_\alpha}$ and $\bigcup_\alpha D_{\hat{f}_\alpha}$ is a sub-space of X.

Finally we show $\hat{f}$ is well-defined.

Suppose $x \in D_{\hat{f}_\alpha}$ and $x \in D_{\hat{f}_\beta}$; by definition

$$\hat{f}(x) = f_{\hat{\alpha}}(x) \text{ and } \hat{f}(x) = f_{\hat{\beta}}(x)$$

By total ordering of τ either $\hat{f}_\alpha$ is an extension of $\hat{f}_\beta$ or vice-versa.

So $\hat{f}_\alpha(x) = \hat{f}_\beta(x)$. Thus we have

$\hat{f}(x) \le p(x)$ for $x \in D_{\hat{f}}$ and for any member $\hat{f}_\alpha$ of τ, we have $\hat{f}_\alpha \alpha \hat{f}$. So $\hat{f} \in \Gamma$ is an upper bound of τ. So we apply Zorn's Lemma to obtain a maximal member (say) F in Γ. And F is the desired extension of f as a linear functional with $F(x) \le p(x)$ for all $x \in X$; that domain of F equals to X follows maximality of F; Otherwise by argument as above one can have an extension of F to some other functional—a contradiction of maximality of F. The proof of theorem is now complete.

Remark : Theorem 4.1.1 is also true for complex spaces, for which one has to furnish proof.

Theorem 4.1.2. (Hahn-Banach Theorem in a *NLS*).

Suppose f is a bounded linear functional on a sub-space M of *NLS* X. There is a bounded linear functional F which is an extension of f from M to X having the same norm as that of f.

Proof : If $x \in M$ we have $|f(x)| \le \|f\| \|x\|$.

Define $p : X \to R$ by the rule :

$$p(x) = \|f\| \|x\| \text{ for } x \in X.$$

Then we verify that p is a sub-linear functional over X.

Such that $f(x) \le p(x)$ for $x \in M$.

Now apply Theorem 4.1.1 (Hahn-Banach Theorem in real space) to get a linear functional F which is an extension of f from M to X such that

$$|F(x)| \le p(x) \quad \text{for all } x \in X.$$

$$\text{i.e. } |F(x)| \le \|f\| \|x\| \quad \text{for all } x \in X.$$

This is true for all $x \in X$; So we conclude that F is a bounded linear functional over X with $\|F\| \le \|f\|$ .... (1)

Further, over M we have $f(x) = F(x)$

So $|f(x)| = |F(x)| \le \|F\| \|x\|$ for all $x \in M$. This gives

$$\|f\| \le \|F\| \quad \text{.............. (2)}$$

Now (1) and (2) together say $\|f\| = \|F\|$

§ 4.2 SOME CONSEQUENCES OF HAHN-BANACH THEOREM :

Application I. Given a real *NLS* $(X, \|\ \|)$ and a non-zero member $x_0 \in X$. There is a bounded linear functional F over X such that $F(x_0) = \| x_0 \|$ with $\| F \| = 1$.

Proof : Consider the sub-space M of X spanned by x_0 ;

Then $M = [x_0] = \{\alpha x_0 : \alpha \text{ any real scalar}\}$

Define $f : M \to$ Reals by the rule :

$$f(\alpha x_0) = \alpha \| x_0 \|; \text{ as } (\alpha x_0) \in M.$$

Then f is a linear functional over M and $| f(x) | = | \alpha | \| x_0 \| = \| \alpha x_0 \|$ for all $x = \alpha x_0 \in M$ and hence we have $\| f \| \leq 1$. *i.e.* f is a bounded linear functional.

Further if $u = \alpha x_0$ is a member of M with $\| u \| = 1$ we have

$$| f(u) | = | \alpha | \| x_0 \| = \| \alpha x_0 \| = \| u \| = 1$$

$$\therefore \| f \| \geq | f(u) | = 1 \text{ giving } \| f \| = 1.$$

Now an application of Hahn-Banach Theorem gives a bounded linear functional F over X satisfying

$$F(x) = f(x) \quad x \in M$$

and $\quad \| F \| = \| f \| = 1$

This gives $F(x_0) = f(x_0) = \| x_0 \|$ and $\| F \| = 1$.

Corollary : For a non-null *NLS* $(X, \|\ \|)$, its conjugate space X^* is non-null.
(Hints : because F appearing in corollary is non-zero member of X^*).

Application II. For every $x \in X$, $\| x \| = \sup\limits_{f(\neq 0) \in X^*} \dfrac{| f(x) |}{\| f \|}$.

Proof : From Application I we find a non-zero bounded linear functional $f_0 \in X^*$ such that $f_0(x) = \| x \|$ and $\| f_0 \| = 1$.

Therefore, $\sup\limits_{f \neq 0 \in X^*} \dfrac{| f(x) |}{\| f \|} \geq \dfrac{| f_0(x) |}{\| f_0 \|} = \| x \|$

i.e. $\quad \sup\limits_{f(\neq 0) \in X^*} \dfrac{| f(x) |}{\| f \|} \geq \| x \| \quad$ .... (1)

On the other hand, if f is any non-zero member of X^*, we have

$$|f(x)| \le \|f\| \|x\|$$

or $$\frac{|f(x)|}{\|f\|} \le \|x\|, \text{ r.h.s. being independent of } f$$

we have, $$\sup_{f(\neq 0)\in X^*} \frac{|f(x)|}{\|f\|} \le \|x\| \qquad \ldots\ldots\ldots\ldots (2)$$

From (1) and (2) one has $\|x\| = \sup_{f(\neq 0)\in X^*} \frac{|f(x)|}{\|f\|}$.

Corollary : If $f(x) = 0$ for every non-zero bounded linear functional $f \in X^*$, then $x = \underset{\sim}{0}$ in X.

Application III. Let M be a closed subspace of X and $M \neq X_0$, if $u \in (X \setminus M)$ and $d = dist(u, M) = \underset{m\in M}{Inf} \|u - m\|$;

Then $d > 0$, and there is a bounded linear functional $f \in X^*$ such that

(i) $f(x) = 0$ for $x \in M$

(ii) $f(u) = 1$

and (iii) $\|f\| = \frac{1}{d}$.

Proof : Here M is a closed sub-space ($\neq X$); so $d > 0$.

Take N = Linear subspace spanned by M and u

i.e. $N = [M \cup \{u\}]$; So every member of N is of the form $m + tu$ where t is a real scalar, and $m \in M$.

Define $g : N \to R$ by the rule :

$$g(m + tu) = t \text{ as } (m + tu) \in N.$$

It is easy to check that g is a linear functional over N such that g vanishes over M *i.e.* $g(m) = 0$ for $m \in M$, and $g(u) = 1$ (taking $t = 1$).

Now $$|g(m + tu)| = |t| = \frac{|t| \|m + tu\|}{\|m + tu\|} = \frac{\|m + tu\|}{\|\frac{m}{t} + u\|}$$

$$= \frac{\| m+tu \|}{\| u-(\frac{m}{t}) \|} \leq \frac{\| m+tu \|}{d} = \frac{1}{d} \| m+tu \|.$$

because $d = \inf_{v \in m} \| u-v \| \leq \| u-(-\frac{m}{t}) \|$.

This is true for all member $(m+tu) \in N$; and hence g is a bounded linear functional over N with $\| g \| \leq \frac{1}{d}$.

So, $\| g \| \leq \frac{1}{d}$ .... (1)

Again from $d = \inf_{v \in m} \| u-m \|$; we find a sequence $\{m_n\}$ in M

such that $\| u-m_n \| \to d$ as $n \to \infty$

i.e. $\lim_{n \to 0} \| u-m_n \| = d$ .... (2)

Now $| g(m_n - u) | \leq \| g \| \| m_n - u \|$

or, $| g(m_n) - g(u) | \leq \| g \| \| m_n - u \|$

or, $| 0-1 | \leq \| g \| \| m_n - u \|$; ($g$ vanishing over M and $g(u) = 1$).

or, $1 \leq \| g \| \| m_n - u \|$

Now passing on limit as $n \to \infty$ we produce

$$1 \leq \| g \| .d$$

giving, $\| g \| \geq \frac{1}{d}$ .... (3)

Combining (1) and (3) we have $\| g \| = \frac{1}{d}$.

Finally, Hahn-Banach Theorem says that g has an extension f from N to the whole space X as a bounded linear functional with $\| f \| = \| g \|$; As f and g agree over $M \subset N$, we have the result as wanted.

Application IV. Let M be a sub-space of *NLS* (X, $\| \|$) and $M \neq X$; if $u \in (X \setminus M)$ such that *dist*(u, M) > 0, say = d.

Then there is a bounded linear functional $F \in X^*$ satisfying

(i) $F(x) = 0$ over M (for $x \in M$)

(ii) $F(u) = d$

and (iii) $\|F\| = 1$.

Proof : Let N = Linear sub-space spanned by M plus u, *i.e.* $N = [M \cup \{u\}]$

Now define $f : N \to$ Reals by rule :

$f(m + tu) = td$ (d as above), where $m + tu$ is a representative member of N($m \in M$, t a scalar).

Clearly f is a linear functional over N, such that for $t = 0$, f vanishes over M and $f(u) = d$ ($t = 1$).

Also for $t \neq 0$, $\|m + tu\| = \|-t\left(-\frac{m}{t} - u\right)\|$ (here $\frac{-m}{t} \in M$)

$$= |t|\ \|-\frac{m}{t} - u\| \geq |t|\, d.$$

So, $|f(m+tu) = |t|\, d \leq \|m + tu\|$; this inequality stands even for $t = 0$.

That means, f is a bounded linear functional over N with $\|f\| \leq 1$.

For $\in > 0$, we find by Infimum property, a member $m \in M$ such that $\|m - u\| < d + \in$.

Put $v = \frac{m-u}{\|m-u\|}$, making $\|v\| = 1$ and $v \in N$ (because, v is the form $m' + t'u$).

So, $|f(v)| = \frac{d}{\|m-u\|} > \frac{d}{d+\in} = \frac{d}{d+\in}\|v\|\ (\because \|v\| = 1)$

That means, $\|f\| \geq \frac{d}{d+\in}$. Now this is true for every +ve $\in$, and taking $\in \to 0_+$, we find $\|f\| \geq 1$.

i.e. $\|f\| \geq 1$ .... (2)

Combining (1) and (2) we find $\|f\| = 1$. Now we apply Hahn-Banach Theorem to find an extension F of f from N to the whole space X as a bounded linear functional over X with $\|F\| = \|f\|$; since F agrees with f over M, we have the result as desired.

§ 4.3 CONJUGATE SPACES X^*, X^{**}, ... OF A *NLS* (X, || ||) :

Let (X, || ||) be a *NLS*; then X^*, $X^{**} = (X^*)^*$, .. are first, second, ...conjugate space of X.

Theorem 4.3.1. If X^* is separable, then so is X.

Proof : Suppose D is a countable dense subset of X^*. Let D_1 be the subset of D which is dense in the surface $\{f \in X^* : \| f \| = 1\}$ of the closed unit ball of X^*; let us write $D_1 = \{f_1, f_2, ..., f_n, ...\}$ with $\| f_n \| = 1$ for all n. From $\| f_n \| = 1$, we find a member say x_n with $\| x_n \| = 1$ such that

$$| f_n(x_n) | > \frac{1}{2}.$$

Consider the linear sub-space L of X spanned by $\{x_1, x_2, ..., x_n\}$

i.e. $L = [x_1, x_2, ..., x_n .....]$ and Put $M = \overline{L}$ (closure of L). The M is also a linear sub-space of X.

Suppose, $M \neq X$ .... (1)

Take $x_0 \in (X \setminus M)$, then $d = dist(x_0, M) > 0$ because M is closed.

By application of Hahn-Banach Theorem we obtain a bounded linear functional $F \in X^*$ with $\| F \| = 1$ such that F vanishes (F = 0) over M and $F(x_0) \neq 0$.

Clearly F is a member of the set $\{f \in X^* : \| f \| = 1\}$ and $F(x_n) = 0$ for all n.

Now $f_n(x_n) = f_n(x_n) - F(x_n) + F(x_n)$ gives

$$| f_n(x_n) | \leq | f_n(x_n) - F(x_n) | + | F(x_n) |$$

$$= | (f_n - F)(x_n) |$$

Thus $\frac{1}{2} < | f_n(x_n) | \leq \| f_n - F \| \| x_n \|$

or, $\frac{1}{2} < \| f_n - F \|$ for all n; This contradicts that $\{f_1, f_2, ..., f_n, ..\}$ is dense in the set $\{f \in X^* : \| f \| = 1\}$.

So, $M = X$.

That is $\overline{L} = X$;

Now L contains that subset formed by finite linear combinations of $x_1, x_2, .., x_n, ..$ with rational coefficients; and that subset becomes countable dense in X. The proof is now complete.

Remark : Converse of Theorem 4.3.1 is not true. The *NLS* l_1 consisting of all those real sequences $\underset{\sim}{x} = (x_1, x_2 ... x_n, ...)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ with norm $\|\underset{\sim}{x}\| = \sum_{i=1}^{\infty} |x_i|$ is separable but its conjugate space l_∞ consisting of all bounded sequences of reals is not separable.

Example 4.3.1. Let $(X, \|\ \|)$ be a *NLS* over reals, and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Show that there is a bounded linear functional f over X such that $f(x_1) \neq f(x_2)$.

Solution : Here $x_1, x_2 \in X$ with $x_1 \neq x_2$ *i.e.* $x_1 - x_2 \neq \underset{\sim}{O}$ in X. So an application of Hahn-Banach Theorem there is a bounded linear functional $f \in X^*(f \neq 0)$ such that

$$f(x_1 - x_2) \neq 0$$

or, $$f(x_1) - f(x_2) \neq 0$$

or, $$f(x_1) \neq f(x_2).$$

Given a *NLS* $(X, \|\ \|)$ we show that there is a natural embedding of X in its second conjugate space X^{**} through a mapping, called the Canonical mapping that we presently define using X^*.

Theorem 4.3.2. Given $x \in X$, let $\hat{x}(x^*) = x^*(x)$ for all $x^* \in X^*$. Then $\hat{x}$ is a bounded linear functional over X^*, and the mapping $x \to \hat{x}$ is a Linear Isometry of X into X^{**}.

Proof : Let $x \in X$, $x_1^*, x_2^* \in X^*$; then we have

$$\hat{x}(x_1^* + x_2^*) = (x_1^* + x_2^*)(x) = x_1^*(x) + x_2^*(x) = \hat{x}(x_1^*) + \hat{x}(x_2^*).$$

Also if λ is any scalar we have $\hat{x}(\lambda x_1^*) = (\lambda x_1^*)(x) = \lambda x_1^*(x) = \lambda \hat{x}(x_1^*)$.

Therefore $\hat{x}$ is a linear functional over X^*.

Now we show that $\|x\| = \sup_{\|x^*\| \leq 1} \{|x^*(x)|\}$.

By Hahn-Banach Theorem we find a member $x^* \in X^*$ with $\|x^*\| = 1$ and $\|x^*(x)\| = \|x\|$.

Therefore $\|x\| \le \sup\limits_{\|x^*\| \le 1} \{|x^*(x)|\}$ .... (1)

Again $\|x^*(x)\| \le \|x^*\| \|x\| \le \|x\|$ when $\|x^*\| \le 1$

Therefore $\|x\| \ge |x^*(x)|$ when $\|x^*\| \le 1$

Thus $\|x\| \ge \sup\limits_{\|x^*\| \le 1} |x^*(x)|$. .... (2)

From (1) and (2) we have

$$\|x\| = \sup\{|x^*(x)| : x^* \in X^* \text{ with } \|x^*\| \le 1\}.$$

which is $= \sup\{|\hat{x}(x^*)| : x^* \in X^* \text{ with } \|x^*\| \le 1\}$

$= \|\hat{x}\|$.

It shows that $\hat{x}$ is a bounded linear functional over X^* with $\|\hat{x}\| = \|x\|$.

Finally, let $x_1, x_2 \in X$ and $x^* \in X^*$, then

$$\begin{aligned}(\widehat{x_1 + x_2})(x^*) &= x^*(x_1 + x_2) \\ &= x^*(x_1) + x^*(x_2) \\ &= \hat{x}_1(x^*) + \hat{x}_2(x^*).\end{aligned}$$

Similarly for any scalar α we have $(\widehat{\alpha x_1})(x^*) = x^*(\alpha x_1)$

$$= \alpha x^*(x_1)$$

$$= \alpha \hat{x}_1(x^*)$$

Therefore the mapping $x \to \hat{x}$ is linear; and since $\|\hat{x}\| = \|x\|$, this mapping is Isometry.

That is, $x \to \hat{x}$ is a Linear Isometry of X onto the linear sub-space $\{\hat{x} : x \in X^*\}$ of X^{**}.

Definition 4.3.1. Given a *NLS* (X, $\| \|$), Linear Isometry $x \to \hat{x}$ is called the Canonical mapping of X into its second conjugate space X^{**}.

Definition 4.3.2. A *NLS* $(X, \| \|)$ is called reflexive if and only if the Canonical mapping $x \to \hat{x}$ maps X onto X**.

Thus a necessary condition for X to be reflexive is that X is a Banach space. However there are Banach spaces without being reflexive.

§ 4.4 OPEN MAPPING THEOREM AND CLOSED GRAPH THEOREM :

Like a big and important theorem of Hahn-Banach we have another big theorem known as open mapping theorem in Functional analysis. There one is concerned with open mappings that send open sets into open sets. Open mapping theorem states conditions under which a bounded linear operator shall be an open mapping.

Definition 4.4.1. Let X and Y be two metric spaces. Then a mapping $f : X \to Y$ is called an open mapping if G is an open set in X, its image under $f = f(G)$ is an open set in Y.

Theorem 4.4.1. Let $(X, \| \|)$ and $(Y, \| \|)$ be two Banach spaces; and $T : X \to Y$ be a bounded linear operator which is onto (surjective). Then T is an open mapping.

The proof of the above theorem shall rest on following Lemma that we prove first.

Lemma 4.4.1 Let $T : X \to Y$ be a bounded linear operator which is onto and let $B_0 = B_1(\underset{\sim}{0})$ be the open unit ball in X, then $T(B_0)$ contains an open ball centred at $\underset{\sim}{0}$ in Y.

Proof : We may complete the proof in three stages as under :

(a) $\overline{T(B_0)}$ (closure of $T(B_0)$) contains an open ball B^*.

(b) If B_n = open ball $B_{\frac{1}{2^n}}(\underset{\sim}{0})$ in X, then $\overline{T(B_n)}$ shall contain an open ball V_n centred at $\underset{\sim}{0}$ in Y.

and (c) $T(B_0)$ contains an open ball cenred at $\underset{\sim}{0}$ in Y.

(a) Consider open ball $B_1 = B_{\frac{1}{2}}(\underset{\sim}{0}) \subset X$. If $x \in X$, we find large real k so that $x \in kB_1$. Therefore we write

$X = \overset{\infty}{\underset{k=1}{\cup}} kB_1$; Since T is onto and linear, we have

$$Y = T(X) = T\left(\overset{\infty}{\underset{k=1}{\cup}} kB_1\right) = \overset{\infty}{\underset{k=1}{\cup}} kT(B_1) = \overset{\infty}{\underset{k=1}{\cup}} \overline{kT(B_1)},$$ taking closure did not add more points to the Union = Y. As Y is a Banach space, we invite Baire Category

Theorem to conclude that one component say $\overline{kT(B_1)}$ contains an open ball. That means $\overline{T(B_1)}$ contains an open ball, say, $B^* = B(y_0, \varepsilon)$. So we write

$$B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_1)} - y_0$$

(b) We show that $B^* - y_0 \subset \overline{T(B_0)}$, where B_0 stands as appearing in theorem. This is accomplished by showing :

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

Take $y \in \overline{T(B_1)} - y_0$; then $(y + y_0) \in \overline{T(B_1)}$ and remembering that $y_0 \in \overline{T(B_1)}$ we find

$$u_n = T(w_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} u_n = y + y_0$$

$$v_n = T(z_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} v_n = y_0 .$$

Since $w_n, z_n \in B_1$ and B_1 is of radius = $\frac{1}{2}$ we have

$$\| w_n - z_n \| \le \| w_n \| + \| z_n \| < \frac{1}{2} + \frac{1}{1} = 1\text{; So that } (w_n - z_n) \in B_0 .$$

From $T(w_n - z_n) - T(w_n) - T(z_n) = u_n - v_n \to y$ as $n \to \infty$.

Therefore, $y \in \overline{T(B_0)}$. Since $y \in \left(\overline{T(B_1)} - y_0\right)$ is an arbitrary we have shown

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

From $B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_1)} - y_0$ above we have

$$B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_0)} \qquad \text{................ (1)}$$

Take $B_n = B(\underset{\sim}{0}, 2^{-n}) \subset X$. Since T is linear, we have $\overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$;
From (1) one obtains

$$V_n = B(\underset{\sim}{0}, \frac{\varepsilon}{2^n}) \subset \overline{T(B_n)} \qquad \text{.................. (2)}$$

(c) Finally, we show that $V_1 = B(\underset{\sim}{0}, \frac{1}{2}\varepsilon) \subset T(B_0)$.

Take $y \in V_1$. From (2), for $n = 1$, we have $V_1 \subset \overline{T(B_1)}$.

Hence $y \in \overline{T(B_1)}$ and we find $v \in \overline{T(B_1)}$ such that $\| y - v \| < \frac{\varepsilon}{4}$

Now $v \in \overline{T(B_1)}$ implies $v \in T(x_1)$ for some $x_1 \in B_1$.

Therefore $\| y - T(x_1) \| < \frac{\varepsilon}{4}$

Using this and (2) above with $n = 2$ we see that $(y - T(x_1)) \in V_2 \subset \overline{T(B_2)}$.

As before we find $x_2 \in B_2$ such that $\| y - T(x_1) \| - T(x_2) \| < \frac{\varepsilon}{8}$

Hence $(y - T(x_1) - T(x_2) \in V_3 \subset \overline{T(B_3)}$, and so on. In nth step we take $x_n \in B_n$ such that

$$\left\| y - \sum_{k=1}^{n} T(x_k) \right\| < \frac{\varepsilon}{2^{n+1}}, \quad n = 1, 2, \ldots \qquad \text{............... (3)}$$

Put $z_n = x_1 + x_2 + \ldots + x_n$; Since $x_k \in B_k$, we have $\| x_k \| < \frac{1}{2^k}$ that means $n > m$,

$$\| z_n - z_m \| \leq \sum_{k=m+1}^{n} \| x_k \| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \text{ which} \to 0 \text{ as } m \to \infty.$$

So $\{z_n\}$ is Cauchy; let $\lim_{n\to\infty} z_n = x$ (X is a Banach space).

Also $x \in B_0$ since B_0 has radius = 1, and

$$\sum_{k=1}^{\infty} \| x_k \| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

As T is continuous, we have $\lim_{n\to\infty} T(z_n) = T(x)$ and (3) shows that $T(x) = y$. So $y \in T(B_0)$.

Proof of Theroem 4.4.1. If A is an open set in X, we show that $T(A)$ is open in Y, by showing that every $y \in T(x) \in T(A)$ attracts an open ball centred at $y = T(x)$ within $T(A)$.

Take $y = T(x) \in T(A)$. As A is open there is an open ball centred at $x \subset A$. Hence $A - x$ contains an open ball centred at $\underset{\sim}{0} \in X$. Let radius of that open ball = r. Put $k = \frac{1}{r}$ or $r = \frac{1}{k}$. Then $k(A - x)$ contains the open unit ball $B(\underset{\sim}{0}, 1)$. Now Lemma 4.4.1 says that $T(k(A-x)) = k[T(A) - T(x)]$ contains an open ball centred at $\underset{\sim}{0}$, and so does $T(A) - T(x)$. Hence $T(A)$ contains an open ball centred at $y = T(x)$. As y is an arbitrary member of $T(A)$, we have shown that $T(A)$ is open.

Corollary : Under open mapping theorem if T is bijective, T^{-1} is bounded.

Example 4.4.1. Let $T : R^2 \to R$ be defined by $T(x,y) = x$ for $(x,y) \in R^2$. Show that T is an open mapping. Examine if $T : R^2 \to R^2$ where $T(x, y) = (x, 0)$, $(x, y) \in R^2$ is an open mapping.

Solution : Here $T : R^2 \to R$ given by $T(x, y) = x$ is a projection mapping and we know that it is a bounded linear operator such that T is onto. So we apply open mapping theorem to conclude that T is an open mapping (In fact, T sends open circular disc of R^2 onto an open interval).

If $T : R^2 \to R^2$ is given by $T(x, y) = (x, 0)$; there Image of an open circular disc under T is not like that. So T is not an open mapping.

We know that all linear operators are bounded. For instance, differential operator is an unbounded linear operator. Closed Linear operators that we introduce presently behave satisfactorily in this respect. Another important theorem, known as closed Graph Theorem states sufficient conditions under which a closed linear operator on a Banach space is bounded.

Let $(X \, || \; ||)$ and $(Y, || \; ||)$ be *NLS* with same scalars.

Definition 4.4.2. A linear operator $T : X \to Y$ is called a closed linear operator if its graph $G(T) = \{(x, y) \in (X \times Y) : y = T(x), x \in X\}$ is a closed set in *NLS* $X \times Y$ with norm $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in (X \times Y)$.

Theorem 4.4.2. Let X and Y be Banach spaces, and $T : X \to Y$ be a closed linear operator. Then T is a bounded linear operator.

Proof : First we verify that $X \times Y$ with norm $\|(x, y)\| = \|x\| + \|y\|$ as $(x, y) \in (X \times Y)$ is also a Banach space.

Let $\{z_n = (x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$.

Then $\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\|$

Thus $\|x_n - x_m\| \le \|z_n - z_m\| \to 0$ as $n, m \to \infty$ shows that $\{x_n\}$ is Cauchy in X, and since X is complete,

let $\lim_{n\to\infty} x_n = x \in X$, and similarly let $\lim_{n\to\infty} y_n = y \in Y$.

These together imply that $\lim_{n\to\infty} z_n = z = (x, y) \in (X \times Y)$. Thus we see that $X \times Y$ is a Banach space. Graph $G(T)$ being a closed set in $X \times Y$, it follows that $G(T)$ is complete (infact, $G(T)$ is a Banach space as a subspace of $X \times Y$)

Consider a mapping $p : G(T) \to X$ given by $p(x, T(x)) = x \in X$. Then p is linear operator over $G(T)$. p is also bounded, because

$$\| p(x, T(x)) \| = \| x \| \le \| x \| + \| T(x) \| = (x, T(x)) \|$$

Further, p is bijective; with p^{-1} given by

$p^{-1} : X \to G(T)$ mapping $x \to (x, T(x))$ as $x \in X$. By applying open mapping theorem we find p^{-1} to be bounded. Hence there is a +ve K such that

$$\|(x, T(x))\| \le K \| x \| \quad \text{for} \quad x \in X.$$

Therefore $\| T(x) \| \le \| T(x) \| + \| x \| = \| (x, T(x)) \| \le K \| x \|$.

That means T is bounded. The proof is complete.

Example 4.4.2. If X and Y are Banach spaces over same scalars, and $T : X \to Y$ is a linear operator. Show that Graph $G(T)$ is a subspace of $X \times Y$.

Solution : Let $(x_1, T(x_1))$ and $(x_2, T(x_2))$ be two members of $G(T)$ as $x_1, x_2 \in X$, where $G(T) = \{(x, T(x)) : x \in X\} \subset (X \times Y)$.

$$\begin{aligned} \text{Then } \; (x_1, T(x_1)) + (x_2, T(x_2)) &= (x_1 + x_2, T(x_1) + T(x_2)) \\ &= (x_1 + x_2, T(x_1 + x_2)) \quad (T \text{ is linear}) \\ &\in G(T). \end{aligned}$$

If λ is any scalar $\lambda(x_1, T(x_1)) = (\lambda x_1, \lambda T(x_1)) = (\lambda x_1, T(\lambda x_1)) \in G(T)$.

Therefore $G(T)$ is a sub-space of $(X \times Y)$.

EXERCISE A

Short answer type questions

1. Show that a norm in a linear space X is a sub-linear functional over X.
2. Show that a sub-linear functional p in a linear space X satisfies (a) $p(\underset{\sim}{0}) = 0$ and (b) $p(-x) \geq -p(x)$ for $x \in X$.
3. Show that non-null NLS X has a non-null conjugate space X^*.
4. If $f(x) = f(y)$ for every bounded linear functional on a NLS X, show that $x = y$ in X.
5. If X and Y are Banach spaces show that the Null space $N(T)$ of a closed linear operator $T : X \to Y$ is a closed sub-space of X.
6. If two non-zero linear functionals f_1 and f_2 over a linear space have the same Null space, then show that f_1 and f_2 are proportional.

EXERCISE B

1. Let X be a NLS, and $x_0 \in X$ such that $|f(x_0)| \leq c$ for all $f \in X^*$ with $\|f\| = 1$, show that $\|x_0\| \leq c$.
2. If X is a NLS which is reflexive, show that X^* is reflexive.
3. If X and Y are Banach spaces over the same scalars, and $T : X \to Y$ is a closed linear operator, then show that (a) if C is compact in X, $T(C)$ is closed in Y, and (b) if K is compact in Y, $T^{-1}(K)$ is closed in X.
4. Let f be a non-zero linear functional in a linear space X, and x_0 is a fixed element in $(X \setminus N(f))$, ($N(f)$ = Null space of $f = \{x \in X : f(x) = 0\}$), then any member x in X has a unique representation $x = \alpha x_0 + y$ where $y \in N(f)$. Prove it.
5. Show that $T : C[a,b] \to R$ defined by $T(f) = \int_a^b f dt,\ f \in C[a,b]$ is a bounded linear functional over $C[a,b]$ and find $\|T\|$.
6. Show that f defined over $C[-1,1]$ by the rule :

$$f(x) = \int_{-1}^{0} x dt - \int_{0}^{1} x dt, \quad x \in C[-1,1]$$

is a bounded linear functional over $C[-1, 1]$ and find $\|f\|$.

UNIT 5

(***Contents*** : Inner product spaces, Cauchy-Schwarz inequality, I.P. spaces as *NLS*, continuity of I.P. function, Law of parallelogram, orthogonal (orthonormal) system of vectors, Projection Theorem in Hilbert space H; Reisz Theorem for a bounded linear functional over H, Bessel's inequality, Grahm-Smidst orthogonalisation process, complete orthonormal system in H.)

§ 5.1 INNER PRODUCT SPACE

In a Normed Linear space principle operations involved are addition of vectors and scalar multiplication of vectors by scalars as in elementary vector algebra. Norm in such a space generalizes elementary idea of length of a vector. What is still more missing in an *NLS* is an analogue if well known dot product $a.b = a_1b_1 + a_2b_2 + a_3b_3$, and resulting formulas among other things like (i) length $|a| = \sqrt{a.a}$ and (ii) relation of orthogonality $a.b = 0$. These are important tools in numerous applications.

History of Inner product spaces is older than that of *NLS*. Theory had been initiated by Hilbert through his work on integral equations. An inner product space is a Linear space with an inner-product structure that we presently define.

Suppose X denotes a complex Linear space.

Definition 5.1.1. X is said to be an Inner Product space or simply I.P. space if there is a scalar-valued function known Inner product function, denoted by, $<, >$ over $X_x X$ satisfying

(I.P. 1) $<x + y, z> = <x, z> + <y, z>$ for all $x, y, z \in X$,

(I.P. 2) $<\alpha x, y> = \alpha <x, y>$ for all scalars α and for all $x, y \in X$,

(I.P. 3) $<y, x> = \overline{<x, y>}$ for all $x, y \in X$, bar denoting complex conjugate.

(I.P. 4) $<x, x> \geq 0$ for all $x \in X$ and it is = 0 if and only if $x = \underline{0}$ in X.

From I.P. axioms above one can immediately derive the following :

(a) $<x, \alpha y> = \overline{\alpha} <x, y>$ for all scalars α and $x, y \in X$.

(b) $<\lambda x + \mu y, z> = \lambda <x, z> + \mu <y, z>$ for all $x, y, z \in X$ and for all scalars λ, μ.

(c) $<x, \alpha y + \beta z> = \overline{<\alpha x + \beta z, x>} = \overline{\alpha <y, x>} + \overline{\beta <z, x>}$

$= \overline{\alpha}\,\overline{<y,x>} + \overline{\beta}\,\overline{<z,y>} = \overline{\alpha} \overline{<x,y>} + \overline{\beta}\overline{<x,z>},$ because conjugate of a complex scalar is itself.

Example 5.1.1. Unitary space $\not{c}^n = \underbrace{\not{c} \times \not{c} \times \ldots \times \not{c}}_{n\ copies}$ whose $\not{c}$ is the space of all complex number is an I.P. space with I.P. $< >$ given by

$<\underset{\sim}{z}, \underset{\sim}{w}) = z_1\overline{w}_1 + z_2\overline{w}_2 + \ldots + z_n\overline{w}_n$ where $\underset{\sim}{z} = (z_1, z_2, \ldots.. z_n)$ and $\underline{w} = (w_1, w_2, \ldots.. w_n) \in \not{c}^n$.

Solution : Here $\overline{<\underset{\sim}{z}, \underset{\sim}{w}>} = \overline{z_1\overline{w}_1 + z_2\overline{w}_2 + \ldots + z_n\overline{w}_n}$

$$= \overline{z_1\overline{w}_1} + \overline{z_2\overline{w}_2} + \ldots + \overline{z_n\overline{w}_n} = \overline{z}_1 w_1 + \overline{z}_2 w_2 + \ldots + \overline{z}_n w_n$$

$= <\underset{\sim}{w}, \underset{\sim}{z}>$; and this (I.P. 3); rest of axioms are routine check-ups.

In an I.P. space $(X, < >)$ of $x \in X$, let us define $\| x \|^2 = <x, x>$ which is always a non-negative quantity and is equal to 0 if and only if $x = \underline{0}$ in X.

Theorem 5.1.1. Every I.P. space is an *NLS*. To prove this Theorem we need help from following Lemma that is an independent proposition as well.

Lemma 5.1.1 (Cauchy-Schwarz inequality/C-S inequality)

In an I.P. space $(X, < >)$ if $x, y \in X$,

$$|<x,y> \leq \| x \| \| y \|.$$

Proof : Without loss of generality take $y \neq \underset{\sim}{0}$ in X. (taking $y = \underset{\sim}{0}$ L.H.S. = R.H.S. For any scalar λ we have

$$\| x + \lambda y \|^2 \geq 0$$

or, $<x + \lambda y, x + \lambda y> = 0$

or, $<x, y> + \lambda\overline{\lambda} <y, y> + \overline{\lambda} <x, y> + \overline{\lambda} <y, x> \geq 0$

or, $\| x \|^2 + |\lambda|^2 \| y \|^2 + \overline{\lambda} <x, y> + \lambda\overline{<x, y>} \geq 0$

Let us now choose $\lambda = -\dfrac{<x,y>}{<y,y>}$

$$= -\frac{<x,y>}{\| y \|^2}.$$

Then L.H.S. of above inequality

$$=\|x\|^2+\frac{|x,y|^2}{\|y\|^2}-\frac{|<x,y>|^2}{\|y\|^2}-\frac{|<x,y>|^2}{\|y\|^2}=\|x\|^2-\frac{|<x,y>|^2}{\|y\|^2}$$

Therefore above inequality assumes the form

$$\|x\|^2-\frac{|<x,y>|^2}{\|y\|^2}\geq 0$$

or $$|<x,y> \leq \|x\|\|y\|.$$

Proof of Theorem 5.1.1. Norm axioms (N.1) and (N.2) follow from (I.P. 4); and the fact $\|\alpha x\|^2 = <\alpha x, \alpha x> = \alpha\bar{\alpha} <x,x> = |\alpha|^2\|x\|^2$.

This gives $\|\alpha x\|^2 = |\alpha|\,\|x\|$

For triangle inequality (N.3), let x, $y \in X$, then we have

$$\|x+y\|^2 = <x+y, x+y> = \|x\|^2 + <x,y> + <y,x> + \|y\|^2.$$

Thus $$\|x+y\|^2 \leq \|x\|^2 + |<x,y> + |<y,x>| + \|y\|^2$$

$$= \|x\|^2 + 2|<x,y>| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \text{ by Lemma 5.1.1.}$$

$$= (\|x\| + \|y\|)^2.$$

Therefore $\|x+y\| \leq \|x\| + \|y\|$.

The proof is now complete.

Remark : Equality sign in C-S inequality holds if and only if $y = 0$ or $0 = \|x+\lambda y\|^2$ *i.e.* $x = -\lambda y$ or $x + \lambda y = 0$ showing that x and y to be linearly dependent.

Theorem 5.1.2. In an I.P. space (X, < >), show that I.P. function is a continuous function.

Proof : Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_n z_n = x$ and $\lim_{x\to\infty} y_n = y$ in norm. That is to say, $\lim_{n\to\infty}\|x_n - x\| = 0 = \lim_{n\to\infty}\|y_n - y\|$.

Now $$|<x_n, y_n> - <x,y>| = |<x_n, y_n> - <x_n, y> + <x_n, y> - <x,y>|$$

$$= |<x_n, y_n - y> + <x_n - x, y>|$$

$\leq |< x_n, y_n - y >| + |< x_n - x, y >|$

$\leq \| x_n \| \| y_n - y \| + \| y \| \| x_n - x \|;$

Since $\lim_{x\to\infty} x_n = x$ in norm, $\{x_n\}$ is norm bounded; So there is an M (+ve) such that $\| x_n \| \leq M$ for all n.

Therefore above inequality assumes the form

$\leq M \| y_n - y \| + \| y \| \| x_n - x \| \to 0$ as $n \to \infty$. This shows that $\lim_{n\to\infty} < x_n, y_n > = < x, y >$ and I.P. function is continuous at (x, y).

Definition 5.1.2. An I.P. space X is said to be a Hilbert space if X is a complete *NLS* with norm $\| \; \|$ as induced from I.P. function.

Thus every Hilbert space is a Banach space. But opposite is not true.

Very often a Hilbert space is denoted by H and an I.P. space is termed as a pre-Hilbert space.

Theorem 5.1.3. If x and y are two members in a Hilbert space H, then

$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$. (Law of parallelogram).

Proof : Here $\| x + y \|^2 + \| x - y \|^2 = < x + y, x + y > + < x - y, x - y >$

$= \| x \|^2 + < x, y > + < y, x > + \| y \|^2 + \| x \|^2 - < x, y > - < y, x > + \| y \|^2$

$= 2 \| x \|^2 + 2 \| y \|^2$.

Remark : In school Geometry it is known that sum of squares raised on sides of a parallelogram is equal to the sum of squares raised on its diagonals. This is exactly what is in Theorem 5.1.3 above. Hence the name is Law of parallelogram.

Example 5.1.2. The sequence space l_2 of all real sequences $\underset{\sim}{x} = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ with $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ is a real Hilbert space.

Solution : We know that l_2 is a real linear space where let us define an I.P. function $< \underset{\sim}{x}, \underset{\sim}{y} > = \sum_{i=1}^{\infty} \xi_i \eta_i$, the r.h.s. series is convergent because

$|\xi_i \eta_i| \leq \frac{1}{2}(|\xi_i|^2 + |\eta_i|^2 \; [(\underset{\sim}{x} = (\xi_1, \xi_2, \ldots), \; \underset{\sim}{y} = (\eta_1, \eta_2, \ldots) \in l_2]$. $i = 1, 2, \ldots\ldots$

By routine exercise we check that all I.P. axioms are O.K. in l_2, and l_2 is an I.P.

space with real scalars. Further, with respect to the induced norm $\|\underset{\sim}{x}\|^2 = \sum_{i=1}^{\infty} |\xi_i|^2$ it is also known that l_2 becomes a complete *NLS*. Hence l_2 is a Hilbert space.

Example 5.1.3. The sequence space $l_p(1 < p < \infty)$ consisting of all real sequences

$\underset{\sim}{x} = (\xi_1, \xi_2, ...)$ with $\left(\sum_{i=1}^{\infty} |\xi_i| p\right)^{1/p} < \infty$ is a Banach space without being a Hilbert space with I.P. function to induce Banach-space norm.

Solution : We have already seen that sequence space $l_p(1 < p > \infty)$ is a Banach space with norm $\|x\| = \left(\sum_{i=1}^{\infty} |\xi_i| p\right)^{1/p}$, as $\underset{\sim}{x} = (\xi_1, \xi_2, ...) \in l_p$. We now show that this norm does not come from an I.P. function on l_p. This is verified by showing that this norm does not satisfy Law of Parallelogram. Take $\underset{\sim}{x} = \left(1, 1, \frac{0}{\text{block}}\right)$, $\underset{\sim}{y} = \left(1, -1, \frac{0}{\text{block}}\right)$ from l_p. Then we find $\|\underset{\sim}{x}\| = \|\underset{\sim}{y}\| = 2^{1/p}$ and $\|\underset{\sim}{x} + \underset{\sim}{y}\| = 2$ $= \|\underset{\sim}{x} - \underset{\sim}{y}\|$. Therefore, if $p \neq 2$ parallelogram law fails.

§ 5.2 ORTHOGONAL ELEMENTS IN HILBERT SPACE

Let H denote a Hilbert space.

Definition 5.2.1. (a) Two members x and y in a Hilbert space H are called orthogonal if $< x, y> = 0$;

We write in this case $x \perp y$.

(b) Given a non-empty subset L of H, an element $x \in$ H is said to be orthogonal to L, denoted by $x \perp y$ if $< x, l> = 0$ for every member $l \in L$.

Theorem 5.2.1. (Pythogorian Law) If $x, y \in H$ and $x \perp y$, then

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$

Proof : (i) $\|x + y\|^2 = < x + y, x + y > = \|x\|^2 + < x, y > + < y, x > + \|y\|^2$

$= \|x\|^2 + < x, y > + \overline{< x, y >} + \|y\|^2 = \|x\|^2 + \|y\|^2$ since $<x, y> = 0$.

(ii) the proof is similar to above.

Theorem 5.2.2. Every closed convex subset of a Hilbert space H has a unique member of smallest norm.

Proof : Let C be a closed convex subset of H, and let d = Inf $\{\|x\| : x \in C\}$.

Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n\| = d$. for $x_n, x_m \in C$ we have $\frac{1}{2}(x_n + x_m) \in C$, because C is convex.

So, $\left\|\frac{x_n + x_m}{2}\right\| \geq d$ or, $\|x_n + x_m\| \geq 2d$ .... (1)

By Law of Parallelogram we have

$$\|x_n - x_n\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$

$$\leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

Since $\lim_{n\to\infty} \|x_n\| = d$ and similarly $\|x_m\| \to d$ as .... (2)

$m \to \infty$; taking limit n, $m \to \infty$ in (2) we get

$\lim_{n,m\to\infty} \|x_n - x_m\| = 0$; showing that $\{x_n\}$ is Cauchy in C.

As C is closed, Let $\lim_{n\to\infty} x_n = x \in C$. Thus $\|x\| = \lim_{n\to\infty} \|x_n\| = d$.

Hence $x \in C$ has a smallest norm. For uniqueness of x, let $x' \in C$ so that $\|x'\| = d$. By convexity of C we have $\frac{x+x'}{2} \in C$ and also $\|\frac{x+x'}{2}\| \geq d$. Again by Law of Parallelogram we have

$$\|\frac{x+x'}{2}\|^2 = \frac{\|x\|^2}{2} + \frac{\|x'\|^2}{2} - \frac{\|x-x'\|^2}{2}$$

$$< \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x'\|^2 \quad \text{if } x \neq x'$$

$= d^2$; giving $\|\frac{x+x'}{2}\| < d$—a contradiction of $\|\frac{x+x'}{2}\| \geq d$ as arrived at early. The proof is now complete.

Theorem 5.2.3 (Projection Theorem). Let L be a closed subspace of H and $L \neq H$; Then every member $x \in H$ has a unique representation $x = y + z$ where $y \in L$ and $x \perp L$.

Proof : If x is a member of $L \subset H$; we write $x = x + \underset{\sim}{0}$ when $\underset{\sim}{0} \in \perp L$.

Let us take $x \in (H \setminus L)$, and put

$d = \operatorname*{Inf}_{a \in L} \| x - a \|^2 = dist(x, L)$; Because L is closed we have $d > 0$,

and there is a sequence $\{a_n\}$ of member a_n in L such that

$$\lim_{n \to \infty} d_n = \| x - a_n \|^2 = d. \qquad \text{.............. (1)}$$

Take any non-zero member a in L. As L is a sub-space of H, we have for any scalar ε, $(a_n + \varepsilon a) \in L$ and therefore

$$\| x - (a_n + \varepsilon a) \|^2 \geq d$$

or, $$< x - a_n - \varepsilon a,\ x - a_n - \varepsilon a > \geq d$$

or, $$\| x - a_n |^2 - \bar{\varepsilon} < x - a_n, a > - \varepsilon < a, x - a_n > + | \varepsilon |^2 \| a \|^2 \geq d.$$

Now take $\varepsilon = \dfrac{< x - a_n, a >}{\| a \|^2}$; with such a choice of ε, we have

$$\| x - a_n \|^2 - \frac{|< x - a_n, a >|^2}{\| a \|^2} \geq d$$

or, $$|< x - a_n, a >|^2 \leq \| a \|^2 \, (d_n - d)$$

or, $$|< x - a_n, a >| \leq \| a \| \sqrt{d_n - d} \qquad \text{.............. (*)}$$

Inequality holds for $a = \underset{\sim}{0}$ in L; So for any $a \in L$ we have

$$|< a_n - a_m, a >| \leq |< a_n - x, a >| + |< x - a_m, a >|$$

i.e. $$|< a_n - a_m, a >| \leq \| a \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right) \qquad \text{from (*)}$$

Putting $a = a_n - a_m$, we have

$$|< a_n - a_m, a_n - a_m >| \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$$

i.e. $$\| a_n - a_m \|^2 \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$$

or, $\|a_n - a_m\| \le \left(\sqrt{d_n - d} + \sqrt{d_m - d}\right)$, where r.h.s. $\to 0$ as $n, m \to \infty$ by (1).

That means $\{a_n\}$ is Cauchy in L.

Since L is closed, let $\lim_{n\to\infty} a_n = y \in L$.

Now in $|< x - a_n, a >| \le \|a\| \sqrt{d_n - d}$, let us pass on the $\lim_{n\to\infty} a_n = y$ and get $|< x - y, a >| = 0$

i.e. $< x - y, 0 > = 0$; This is true for any member a in L; Therefore $(x - y) \perp L$. Let us put $z = x - y$.

Then we have $x = y + z$ where $y \in L$ and $z \perp L$.

For uniqueness of this representation, let $x = y + z = y' + z'$ where $y' \in L$ and $z' \perp L$. Thus y, y' come from L and $z, z' \perp L$. Clearly, $y - y' = z' - z$, and

$\|y - y'\|^2 = < y - y', y - y' > = < y - y', z' - z > = 0$ where $\|z' - z\| \perp L$.
Therefore $y = y'$ and hence $z = z'$. The proof is now complete.

Remark : In representation Theorem 5.2.3. where $x = y + z$, y is called projection of x on L. It is obvious that collection M of all elements, orthogonal to L forms a sub-space. M is also closed because of continuity of I.P. function. That is why z is called projection of x on M which is called orthogonal complement of L. Further, Hilbert space H is then sum of two orthogonal sub-spaces L and M. Here we see orthogonal sum is a special case of the Direct sum. Thus projection Theorem 5.2.3 gives a decomposition of any member in Hilbert space H into its projections onto two complementary orthogonal sub-spaces.

§ 5.3. It is important to know that the general form of a bounded Linear functional acting on a given space. Such formulae in respect of some *NLS* are known; their derivations could be much complicated. Situation is, however, surprisingly simple for a Hilbert space H.

Theorem 5.3.1 (Riesz Theorem on representation of functional over H).

Let f be a bounded linear functional over a Hilbert space H. Then $f(x) = < x, y >$ for all $x \in H$ and for some $z \in H$ uniquely determined by f such that $\|z\| = \|f\|$.

Proof : If f is the zero functional over H. We take $z = \underset{\sim}{0}$ in H to do the job. Suppose that f is a non-zero bounded linear functional over H. Consider the null-space $N(f)$ of f where

$N(f) = \{x \in H : f(x) = 0\}$. Clearly $N(f)$ is a closed linear sub-space of H without being equal to H.

Take a non-zero $z_0 \in \perp N(f)$

Let $x \in$ H. Put $v = f(x)z_0 - f(z_0)x$

So that $f(v) = f(f(x)z_0) - f(f(z_0)x)$

$= f(x)f(z_0) - f(z_0)f(x)$; (f is linear)

$= 0$

That means $v \in N(f)$; by choice z_0 is orthogonal to v

So $0 = < v, z_0 > = < f(x)z_0 - f(z_0)x, z_0 >$

$= f(x) < z_0, z_0 > - f(z_0) < x, z_0 >$

$= \| z_0 \|^2 f(x) - f(z_0) < x, z_0 >$

Giving $f(x) = \dfrac{f(z_0)}{\| z_0 \|} < x, z_0 >$

$$= \left\langle x, \ \overline{\frac{f(z_0)}{\| z_0 \|}} z_0 \right\rangle$$

$= < x, y >$ (say), where $z = \overline{\dfrac{f(z_0)}{\| z_0 \|^2}} z_0$. .... (1)

This is the representative formula for $f(x)$ as wanted.

For uniqueness of z, let $f(x) = < x, z_1 > = < x, x_2 >$ for all $x \in$ H.

Then we have $< x, z_1 > = < x, z_2 >$ or, $< x, z_1 - z_2 > = 0$

put $x = z_1 - z_2$; So $< z_1 - z_2, \ z_1 - z_2 > = 0$ or, $\| z_1 - z_2 \|^2 = 0$ or, $z_1 = z_2$.

Finally, We have $| f(x) | = |< x, z >| \le \| x \| \| z \|$

This gives $\| f \| \le \| z \|$ .... (1)

Again taking $z = x$ in (1) we have $< z, z > = f(z)$

or, $\| z \|^2 \le \| f \| \| z \|$

or, $\| z \| \le \| f \|$ .... (2)

Combining (1) and (2) we have $\| f \| \leq \| z \|$.

Converse of Theorem 5.3.1. is true. This is what Example 5.3.1 has to say.

Example 5.3.1. Let z be a fixed member in a Hilbert space H. Show that

$f(x) = <x, z>$ for all $x \in$ H is a bounded linear functional over H with $\| f \| = \| z \|$.

Solution : Here $f : H \rightarrow$Scalar such that for $x_1, x_2 \in H$.

Then $f(x_1 + x_2) = <x_1 + x_2, z> = <x_1, z> + <x_2, z> = f(x_1) + f(x_2)$.

And for any scalar α $f(\alpha x_1) = <\alpha x_1, z> = \alpha <x_1, z> = \alpha f(x_1)$.

Thus f is Linear. Further $|f(x)| = |<x, z>| \leq \| x \| \| z \|$ (by C-S inequality)

This is true for all $x \in$ H. Therefore f is a bounded linear functional such that

$$\| f \| \leq \| z \| \qquad \text{............... (1)}$$

Taking $x = z$ in $f(x) = <x, y>$ we have

$$\| z \|^2 = <z, z> = f(z) \leq \| f \| \| z \|$$

$$\text{or, } \| z \| \leq \| f \| \qquad \text{.............. (2)}$$

(1) plus (2) gives $\| f \| \leq \| z \|$.

Corollary to Theorem 5.3.1. Every Hilbert space H is reflexive.

Because by Theorem 5.3.1. together example put up above says that every bounded linear functional over H. *i.e.* every member of H^* arises out of a member of H and conversely. This correspondence gives rise to an isomorphism between H and H^*; and we say that H is self-dual and this in turn implies that here Canonical mapping between H and H^{**} is a surjection. Hence H is reflexive.

§ 5.4 ORTHONORMAL SYSTEM IN HILBERT SPACE *H*.

Definition 5.4.1. (a) A non-empty subset $\{e_i\}$ of Hilbert space H is said to be an orthonormal system if

(i) $i \neq j$, $e_i + e_j$ *i.e.* any two distinct members of $\{e_i\}$ are orthogonal.

and (ii) $\| e_i \| = 1$ for every i *i.e.* any vector of the system is non-zero unit vector in H.

(b) If an orthonormal system of H is countable, we can enumerate its elements in a sequence say it as an orthonormal sequence.

For example in Euclidean n-space R^n which is a real Hilbert space the fundamental unit vectors $e_1 = (1,0,0,...,0)$, $e_2 = (0,1,0,0,...,0)$ ...... $e_n = (1,0,...,0,1)$ form an orthonormal system of vectors in R^n.

Example 5.4.1. Let $L_2[0,2\pi]$ be the real Hilbert space of all square integrable functions f over $[0,2\pi]$ with I.P. function

$$<f,g> = \int_0^{2\pi} fg dt; \qquad f,g \in L_2 \ [0,2\pi].$$

$$\therefore \| f \| = \sqrt{\sum_0^{2\pi} f^2 dt}.$$

Then $e_0(t) = \dfrac{1}{\sqrt{2\pi}}, e_n(t) = \dfrac{\cos nt}{\sqrt{\pi}}$ $\{n = 1, 2, ...)$ and $0 \le t \le 2\pi$.

form an orthonormal sequence in $L_2[0,2\pi]$; because

$$\int_0^{2\pi} \cos mt \ \cos nt \ dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n = 1,2... \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

Theorem 5.4.1. An orthonormal system in H is linearly independent.

Proof : Let $\{e_i\}$ be an orthonormal system in H; and let for a finite subset, say, e_1, e_2, ...e_n of the system we have

$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = \underset{\sim}{0}$ where α_i 's are scalars. Then for $1 \le j \le n$ we have

$$0 = <\underset{\sim}{0}, e_j> = \left\langle \sum_{i=1}^{n} \alpha_i e_i, e_j \right\rangle = \sum_{i=1}^{n} \alpha_i, <e_i, e_j>$$

$= \alpha_j <e_j, e_j> = \alpha_j$; (other terms being zero because of mutual orthogonality). So $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. That means any finite sub-system of the given system is linearly independent. Hence proof is done.

Definition 5.4.2. Let $\{e_i\}$ be an orthonormal system in H and $x \in H$; Then scalars $c_i = <x, e_i>$ are called Fouries co-efficients of x w.r.t the system.

Theorem 5.4.2. Suppose $\{e_1, e_2, e_3, ..., e_n, ...\}$ be an orthonormal sequence in H;

then for $x \in$ H,

$$\sum_{i=1}^{\infty} |< x, e_i >|^2 \le \| x \|^2$$

(This inequality is very often termed as Bessel's inequality).

Proof : Let n be a +ve integer. If c_i are Fouries coefficients of x w.r.t. $\{e_i\}$, we have

$$0 \le \| x - \sum_{i=1}^{n} c_i e_i \|^2 = \left\langle x - \sum_{i=1}^{n} c_i e_i, x - \sum_{i=1}^{n} c_i e_i \right\rangle$$

$$= \| x \|^2 - \left\langle x, \sum_{i=1}^{n} c_i e_i \right\rangle - \left\langle \sum_{i=1}^{n} c_i e_i, x \right\rangle + \left\langle \sum_{i=1}^{n} c_i e_i, \sum_{k=1}^{n} c_k e_k \right\rangle$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c}_i < x, e_i > - \sum_{i=1}^{n} c_i < e_i, x > + \sum_{i=1}^{n} c_i < e_i, \sum_{k=1}^{n} c_k e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c}_i c_i - \sum_{i=1}^{n} c_i \overline{c}_i + \sum_{i=1}^{n} \sum_{k=1}^{n} c_i \overline{c}_k < e_i, e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} | c_i |^2 - \sum_{i=1}^{n} | c_i |^2 + \sum_{i=1}^{n} | c_i |^2 = \| x \|^2 - \sum_{i=1}^{n} | c_i |^2$$

Therefore, $\sum_{i=1}^{n} | c_i |^2 \le \| x \|^2$ or, $\sum_{i=1}^{n} |< x, e_i >|^2 \le \| x \|^2$.

This is true for any +ve integer n, and thus $\sum_{i=1}^{\infty} |< x, c_i >|^2$ is convergent and

$\sum_{i=1}^{n} |< x, c_i >|^2 \le \| x \|^2$.

Theorem 5.4.3. In a separable Hilbert space H every orthonormal system is countable.

Proof : Let $E = \{e_i\}$ be an orthonormal system in H which is separable. If $e_i \ne e_j$ we have $< e_i, e_j > = 0$ and $\| e_i \| = 1 = \| e_j \|$.

Therefore $< e_i - e_j, e_i - e_j > = \| e_i \|^2 - < e_i, e_j > - < e_j, e_i > + \| e_j \|^2 = 0+1+1 = 2$

So, $\| e_i - e_j \|^2 = 2$

or, $\| e_i - e_j \| = \sqrt{2}$.

By separability of H, we find a countable set $\{y_1, y_2, ..., y_n, ...\}$ which is dense in E. So we find two members, say, y_i and y_j such that

$$\| y_i - e_i \| < \frac{\sqrt{2}}{3} \text{ and } \| y_j - e_j \| < \frac{\sqrt{2}}{3}.$$

So
$$\sqrt{2} = \| e_i - e_j \| = \| e_i - y_i + y_i - y_j + y_j - e_j \|$$
$$\leq \| e_i - y_i \| + \| y_i - y_j \| + \| y_j - e_j \|$$
$$< \frac{2\sqrt{2}}{3} + \| y_i - y_j \|.$$

Showing $\| y_i - y_j \| > \frac{\sqrt{2}}{3}$. clearly $i \neq j$; This establishes an H correspondence between members of E with members of a subset of a countable set. Therefore E is countable.

Gram-Schmidt Orthogolisation Process : Subject is that in a Hilbert space H one can transform a linearly independent set of elements in H into an orthonormal system in H by a technique known by above name.

Let x_1, x_2,... be an independent system of vectors in H (So none is zero vector)

Put $e_1 = \frac{x_1}{\| x_1 \|}$ and let $y_2 = x_2 - c_{21}e_1$ where $c_{21} = < x_2, e_1 >$.

Next we put $e_2 = \frac{y_2}{\| y_2 \|}$; By verification we see $< e_1, e_1 > = 1$, $< e_2, e_2 > = 1$, and $< e_1, e_2 > = 0$.

Now let $y_3 = x_3 - (c_{31}e_1 + c_{32}e_2)$ where we choose $c_{31} = < x_3, e_1 >$, $c_{32} = < x_3, e_2 >$.

Next we put $e_3 = \frac{y_3}{\| y_3 \|}$, and as before we have

$$< e_3, e_3 > = 1, \; < e_3, e_2 > = 0 = < e_3, e_1 >.$$

We continue this process, if $e_1, e_2, ..., e_{k-1}$ have been constructed, let us take

$$y_k = x_k - \sum_{i=1}^{k-1} c_{ki} e_i$$

where $c_{ki} = < x_k, e_i >$; so that y_k is orthogonal to $e_1, e_2, ..., e_{k-1}$; Define $e_k = \frac{y_k}{\| y_k \|}$. Inductively, we construct e_n as a linear combination of x_1, x_2,... and x_n. This way we are led to orthonormal system (e_1, e_2,... e_n,...) from (x_1, x_2,... x_n,...).

Definition 5.4.3. In a Hilbert space H an orthonormal system E is called a complete orthonormal system if there is no orthonormal system in H to contain E as a proper subset.

For example, in Euclidean n-space R^n (a real Hilbert space) the set of all fundamental unit vectors $\{e_1, e_2, ..., e_n\}$ where $e_j = (\underline{0} \underbrace{\underline{1}}_{j\text{th place}} \underline{0})$, $j = 1, 2, ..., n$ is a complete orthonormal system in R^n.

Theorem 5.4.4. In a Hilbert space H let $\{e_1, e_2, ..., e_n ...\}$ be an orthonormal sequence in H. Then following statements are equivalent (one implies other).

(a) $\{e_i\}$ is complete.

(b) $< x, e_i > = 0$ for all i implies $x = \underset{\sim}{0}$ in H.

(c) $x = \sum_{i=1}^{\infty} < x, e_i > e_i$ for each $x \in H$.

(d) $\sum_{i=1}^{\infty} |< x, e_i >|^2 = \| x \|^2$ for every $x \in H$.

Proof : (a) $\Rightarrow$ (b); Let (a) be true. Suppose (b) is false. Then we find a non-zero x in H such that $< x, e_i > = 0$ for $i = 1,2,...$

Put $e = \frac{x}{\| x \|}$. So that $\| e \| = 1$, and $< e, e_i > = 0$ for all i,

Therefore $\{e_1, e_2, ..., .. e_n, ...\} \cup \{e\}$ becomes an orthonormal system containing given system properly—a contradiction that $\{e_1, e_2, ..., e_n\}$ is complete. Hence (b) is established.

<u>(b) $\Rightarrow$ (c)</u> Let $S_n = \sum_{i=1}^{n} < x, e_i > e_i$;

Then $\sum_{i=1}^{\infty} < x, e_i > e_i = \lim_{n\to\infty} S_n = S$ (say)

If $1 \le j \le n$, $< x, e_j > - < S_n, e_j >$

$$= < x, e_j > - < \sum_{i=1}^{n} < x, e_i > e_i, e_j >$$

$$= < x, e_j > - < x, e_j > = 0$$

Thus $< S_n, e_j > = < x, e_j >$

Now $< x - \sum_{i=1}^{n} < x, e_i > e_i, e_j > = < x - S, e_j > = < x, e_j > - < S, e_j >$

$$= < x, e_j > - < \lim_{n\to\infty} S_n, e_j > = < x, e_j > - \lim_{n\to\infty} < S_n, e_j > = < x, e_j > - < x, e_j > = 0$$

That means $e_j \perp \left(x - \sum_{i=1}^{\infty} < x, e_i > e_i \right)$; therefore from (b) we have

$$x - \sum_{i=1}^{n} < x, e_i > e_i = \underset{\sim}{0} \quad i.e. \quad x = \sum_{i=1}^{\infty} < x, e_i > e_i .$$

<u>(c) $\Rightarrow$ (d)</u>. We have $\|x\|^2 = < x, x > = < \sum_{i=1}^{\infty} < x, e_i > e_i . \sum_{j=1}^{\infty} < x, e_j > e_j >$

$$= < \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > e_i, < \lim_{n\to\infty} \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} < \sum_{i=1}^{n} < x, e_i > e_i, \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > \overline{< x, e_i >} = \lim_{n\to\infty} \sum_{i=1}^{n} |< x, e_i >|^2$$

$$= \sum_{i=1}^{\infty} |< x, e_i >|^2 .$$

(d) $\Rightarrow$ (a). Let (d) hold and if possible let $\{e_i\}$ be not complete. Then we find an orthonormal system strictly larger than $\{e_1, e_2, ..., e_n, ...\}$; say larger system looks as $\{e, e_1, e_2, ..., e_n, ...\}$ where, of course, $\| e \| = 1$ and $<e, e_i> = 0$ for $i = 1, 2, ...$ Now (d) applies (taking $x = e$), and we have

$$\| e \|^2 = \sum_{i=1}^{\infty} |< e, e_i >|^2 = 0 \text{— a contradiction. So we have proved (a).}$$

Example 5.4.2. Let $\{x_n\}$ be a sequence in Hilbert space H and $x \in H$ such that $\lim_{n\to\infty} \| x_n \| = \| x \|$, and $\lim_{n\to\infty} < x_n, x >\| = < x, x >$. Show that $\lim_{n\to\infty} x_n = x$.

Solution : Given $\lim_{n\to\infty} \| x_n \| = \| x \|$ and $\lim_{n\to\infty} < x_n, x > = < x, x > = \| x \|^2$.

Now $\| x_n - x \|^2 = < x_n - x, x_n - x > = \| x_n \|^2 - < x_n, x > - < x, x_n > + \| x \|^2$

$$= \| x_n \|^2 - < x_n, x > - \overline{< x_n, x >} + \| x \|^2$$

$$\to \| x \|^2 - \| x \|^2 - \| x \|^2 + \| x \|^2 = 0 \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty} x_n = x$

Example 5.4.3. In a real Hilbert space H if $\| x \| = \| y \|$, show that $<x + y, x - y> = 0$. Interpret the result Geometrically if H = Euclidean 2-space R^2.

Solution : Let H be a real Hilbert space and $x, y \in H$ that such $\| x \| = \| y \|$.

Now $< x + y, x - y > = < x, x > - < x, y > + < y, x > - < y, y >$

$= \| x \|^2 - < x, y > + < x, y > - \| y \|^2$ (because it is a real Hilbert space, $< x, y > = \overline{< x, y >}$)

$= 0$

That means $(x + y) \perp (x - y)$.

It Euclidean 2-space R^2, fig is an equilateral parallelogram *i.e.* a Rhombus with adjacent sides represented by x and y with $\| x \| = \| y \|$; and we know that in a Rhombus Diagonals cut at right angles.

EXERCISE A

Short answer type questions

1. If in an I.P. space $< x, u > = < x, v >$ for all x in the space, show that $u = v$.

2. Show that Banach space $c[a,b]$ with sup norm is not a Hilbert space with an I.P. to induce the sup norm.
3. If f is a bounded linear functional over Euclidean 3-space R^3, show that f can be represented by a dot product

$$f(x) = x.z = \xi_1\rho_1 + \xi_2\rho_2 + \xi_3\rho_3 .$$

4. Show that in a Hilbert space H convergence of $\sum_{j=1}^{\infty} \| x_j \|$ implies convergence of $\sum_{j=1}^{\infty} x_j$
5. If ¢ denotes the Unitary space of all complex numbers. If z_1, z_2 ∈ ¢, show that $< z_1, z_2 > = z_1, \overline{z}_2$ defines an I.P. function on ¢.

EXERCISE B

1. If x and y are two non-zero elements in a Hilbert space H, show that $\| x + y \| \leq \| x \| + \| y \|$ where equality holds if and only if $y = \alpha x$ for a suitable scalar α.
2. Let c be a convex set in a Hilbert space H, and $d = \text{Inf}\{\| x \| : x \in c\}$. If $\{x_n\}$ is a sequence in c such that $\lim_{n\to\infty} \| x_n \| = d$, show that $\{x_n\}$ is a Cauchy sequence.
3. If $\{e_n\}$ is any orthonormal sequence in a Hilbert space H and x, $y \in H$, show that

$$| \sum_{n=1}^{\infty} < x, e_n > < y, e_n > | \leq \| x \| \| y \|$$

4. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in a Hilbert space H where n is fixed. If $x \in H$ be a fixed member, show that for scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ $\| x - \sum_{i=1}^{n} \alpha_i e_i \|$ is minimum when $\alpha_i = < x, e_i >, i = 1, \ldots, n$.
5. Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space H. For $x \in H$, define $y = \sum_{k=1}^{\infty} < x, e_k > e_k$; show that $(x - y) \perp e_k$ $(k = 1, 2, \ldots)$.
6. Show that for the sequence space l_2 (a real Hilbert space) its conjugate space l_2^* is isomorphic to l_2.

UNIT 6

(***Contents*** : Adjoint of bounded linear operator in a Hilbert space H, Algebra of adjoint operators, proudct of adjoints, self-adjoint operators in H, their algebra, Norm of self-adjoint operator, space of self-adjoint operators, every bounded linear operator in H as a sum of self-adjoint operators, eigen value and eigen vectors of self-adjoint operator.)

§ 6.1 Let H be a complex Hilbert space and let $Bd\alpha(H,H)$ denote the space of all bounded linear operators $T : H \to H$. Take one such $T : H \to H$ as a bounded linear operator. Let $y \in$ H.

Define $f_y : H \to$ scalars by the rule :

$$f_y(x) = \langle T(x), y \rangle \quad \text{as } x \in H \qquad \text{.......... (1)}$$

Notice that if x_1, $x_2 \in$ H, we have

$$f_y(x_1 + x_2) = \langle T(x_1 + x_2), y \rangle = \langle T(x_1) + T(x_2), y \rangle \text{ because } T \text{ is linear}$$

$$= \langle T(x_1), y \rangle + \langle T(x_2), y \rangle \text{ using property inner product}$$

$$= f_y(x_1) + f_y(x_2);$$

Similarly $f_y(\alpha x_1) = \alpha f_y(x_1)$ for any scalar α.

That means, f_y is a linear functional over H.

Plus $|f_y(x)| = |\langle T(x), y \rangle| \leq \| T(x) \| \| y \|$ by C-S inequality,

$$\leq \| T \| \| x \| \| y \| = (\| T \| \| y \|) \| x \| \quad \text{for all } x \in H.$$

Therefore, f_y is a bounded linear functional over H, and as we had seen earlier, Riesz representation Theorem says, there is a unique member, say $y^* \in H$ such that

$$f_y(x) = \langle x, y^* \rangle \qquad \text{.......... (2)}$$

where we remember that y^* is determined by f_y. From the text as put up above one sees that given $y \in H$, there is a unique member $y^* \in H$ (via f_y).

Let us define $T^* : H \to H$ by formula :

$$T^*(y) = y^* \text{ as described above} \qquad \text{.......... (3)}$$

This operator T^* is called adjoint operator to T in H and as explained above they are connected by relation

$< T(x), y > = < x, T^*(y) >$ from (1), (2) and (3) above **for all** $x, y \in H$.

Explanation : T^* is well defined over H. Because, suppose that for all $x, y \in H$, we have simultaneously

$$< T(x), y > = < x, T^*(y) >$$

and $\quad < T(x), y > = < x, T_1^*(y) >$ for another $T_1 : H \to H$.

Therefore we see $< x, T^*(y) > = < x, T_1^*(y) >$ for all $x, y \in H$.

meaning thereby $T^*(y) = T_1^*(y)$ for $y \in H$. *i.e.* $T^* \equiv T_1^*$

Theorem 6.1.1. $T^* : H \to H$ is a bounded linear operator $(T^* \in Bd\alpha(H, H))$.

Proof : Let $x, y, z \in H$. Then $< x, T^*(y+z) > = < T(x), y+z >$

$$= < T(x), y > + < T(x), z > = < x, T^*(y) > + < x, T^*(z) >$$

$$= < x, T^*(y) + T^*(z) >.$$

Therefore, $T^*(y+z) = T^*(y) + T^*(z)$ .... (1)

Again for a scalar λ, $< x, T^*(\lambda y) > = < T(x), \lambda y >$

$$= \bar{\lambda} < T(x), y > = \bar{\lambda} < x, T^*(y) > = < x, \lambda T^*(y) >.$$

Therefore, $T^*(\lambda y) = \lambda T^*(y)$ .... (2)

(1) and (2) together say that T^* is a linear operator.

Again, for $y \in H$ we have

$$\| T^*(y) \|^2 = < T^*(y), T^*(y) > = < T T^*(y), y >$$

$$\leq \| T T^*(y) \| \| y \| \leq \| T \| \| T^*(y) \| \| y \|$$

That means, $\| T^*(y) \| \leq \| T \| \| y \|$, and therefore T^* is a bound linear operator over H with $\| T^* \| \leq \| T \|$.

Corollary 1. $T^{**} \equiv T$.

Now T^* is a bounded linear operator; and from the relation $< T(x), y > = < x, T^*(y) >$ let us put T^* in place of T to get for all $x, y \in H$,

$$< T^*(x), y > = < x, T^{**}(y) >$$

Interchange x and y to get

$$< T^*(y), x > = < y, T^{**}(x) >$$

Taking conjugates, $< T^{**}(x), y > = < x, T^*(y) > = < T(x), y >$ ....(*)

Now (*) remains true for all $y \in H$, therefore we deduce that

$TT^*(x) = T(x)$ and this being true for all $x \in H$ we finally obtain $T^{**} = T$.

Corollary 2. $\| T^* \| = \| T \|$.

We do already thave $\| T^* \| \leq \| T \|$; let us apply this in favour of T^* to get

$$\| T^{**} \| \leq \| T^* \|$$

or, $\| T \| \leq \| T^* \|$

Therefore, $\| T \| = \| T^* \|$.

§ 6.2 ALGEBRA OF ADJOINT OPERATORS IN HILBERT SPACE *H*.

Let A and B be two bounded linear operators $: H \to H$ *i.e.* $A, B \in Bd\alpha(H,H)$. Then $A + B$ and αA (α any scalar) are also members of $Bd\alpha(H,H)$.

Theorem 6.2.1. (a) $(A+B)^* = A^* + B^*$ and (b) $(\alpha A)^* = \bar{\alpha} A^*$, where A^* denotes adjoint of A.

Proof : (a) For all x, $y \in H$ we have $< A(x), y > = < x, A^*(y) >$ and $< B(x), y > = < x, B^*(y) >$.

Now $< x, (A+B)^* y > = < (a+B)(x), y >$

$$= < A(x) + B(x), y >$$

$$= < A(x), y > + < B(x), y >$$

$$= < x, A^*(y) > + < x, B^*(y) >$$

$$= < x, A^*(y) + B^*(y) >$$

$$= < x, (A^* + B^*)(y) >$$

This shows that $(A+B)^* = A^* + B^*$

(b) $<x, (\alpha A)^*(y)> = <(\alpha A)(x), y>$

$$= <\alpha A(x), y>$$
$$= \alpha < A(x), y>$$
$$= \alpha < x, A^*(y)>$$
$$= < x, \bar{\alpha} A^*(y)>$$
$$= < x, (\bar{\alpha} A^*)(y)>$$

This being true for all $x, y \in H$, we have $(\alpha A)^* = \bar{\alpha} A^*$.

For A and B belonging to $Bd\alpha(H,H)$, let us define $(AB) : H \to H$ by following rule of composition;

$(AB)(x) = A\big(B(x)\big)$ for $x \in H$. In this way $(BA) : H \to H$ is also defined. It is a routine verification that $(AB) : H \to H$ is a linear operator such that for $x \in H$,

$$\|(AB)(x)\| = \|A\big(B(x)\big)\| \le \|A\| \|B(x)\| \le \|A\| \|B\| \|x\|.$$

This is true for all $x \in H$; Therefore (AB) is also a bounded linear operator over H *i.e.* $(AB) \in Bd\alpha(H,H)$.

Theorem 6.2.2. $(AB)^* = B^* A^*$.

Proof : For $x, y \in H$, we have $< A(x), y> = < x, A^*(y)>$

and $< B(x), y> = < x, B^*(y)>$

Now $< (AB)(x), y> = < x, (AB)^*(y)>$ which is the same as,

$$< x, (AB)^*(y)> = < (AB)(x), y>$$
$$= < A(B(x)), y>$$
$$= < B(x), A^* y>$$
$$= < x, B^*(A^*(y))>$$

$= < x, (B^* A^*)(y)>$; Therefore we have $(AB)^* = B^* A^*$.

Theorem 6.2.3. For any $A \in Bd\alpha(H,H)$, $\|A^* A\| = \|A\|^2 = \|AA^*\|$.

Proof : We always have $\|A^* A\| \le \|A^*\| \|A\| = \|A\| \|A\| = \|A\|^2$ because A^* is also a member of $Bd\alpha(H,H)$

i.e. $\|A^*A\| \leq \|A\|^2$ .... (1)

Again $\|A\|^2 = \sup_{\|x\|\leq 1} \{\|A(x)\|^2\}$

$$= \sup_{\|x\|\leq 1} \{|< A(x), A(x) >|\}$$

$$= \sup_{\|x\|\leq 1} \{|< A^*(A(x)), x >|\}$$

$$= \sup_{\|x\|\leq 1} \{|< (A^*A)(x), x >|\}$$

$$\leq \sup_{\|x\|\leq 1} \{\|(A^*A)(x)\| \|x\|\}$$ form *C-S* inequality,

$$\leq \|A^*A\|.$$

That is, $\|A\|^2 \leq \|A^*A\|$. ....(2)

From (1) and (2) we have $\|A^*A\| = \|A\|^2$. Now applying this equality to A^* one obtains $\|AA^*\| = \|(A^*)^*A^*\| = \|A^*\|^2 = \|A\|^2$. The proof is now complete.

Corollary : If $A \in Bd\alpha(H,H)$ is such that $AA^* = A^*A$ (*i.e.* A and A^* commute), then $\|A^2\| = \|A\|^2$.

§ 6.3 SELF-ADJOINT OPERATORS OVER HILBERT SPACE *H*.

Definition 6.3.1. A member $T \in Bd\alpha(H,H)$ *i.e.* T being a bounded linear operator over H is called self-adjoint if $T^* = T$.

Theorem 6.3.1. (a) If T_1 and T_2 are self-adjoint operators over H, then $T_1 + T_2$ is so.

(b) If T_1 is self-adjoint and α any real scalar, then αT_1 is self-adjoint.

(c) For any member $T \in Bd\alpha(H,H)$, T^*T, TT^* and $T+T^*$ are self-adjoint.

(d) If T_1 and T_2 are self-adjoint, then T_1T_2 is self-adjoint if and only if $T_1T_2 = T_2T_1$ (T_1 and T_2 commute).

Proof : (a) $(T_1 + T_2)^* = T_1^* + T_2^* = T_1 + T_2$

(b) $(\alpha T_1)^* = \bar{\alpha}T_1^* = \bar{\alpha}T_1 = \alpha T_1$ because α is a real scalar.

(c) $(T^*T)^* = T^*\, T^{**} = T^*T$, $(TT^*)^* = T^{**}T^* = TT^*$;

and $(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$.

and finally (d) $(T_1T_2)^* = T_2^*T_1^* = T_2T_1$; Therefore $(T_1T_2)^* = T_1T_2$ if and only if $T_1T_2 = T_2T_1$.

Theorem 6.3.2. The class of all self-adjoint operators forms a closed real sub-space of $Bd\alpha(H,H)$, and hence it is a Banach space.

Proof : If 0 and I denote the zero operator and identity operator, we have 0 and I are members of $Bd\alpha(H,H)$. Further $0^* = 0$ and $I^* = I$; Now if A and B are self-adjoint operators with α and β two real scalars, we have

$$(\alpha A+\beta B)^* = \bar{\alpha}A^*+\bar{\beta}B^* = \alpha A^*+\beta B^*$$

$$= \alpha A+\beta B$$

Showing thereby that $\alpha A+\beta B$ is also self-adjoint.

Further if $\{A_n\}$ is a sequence of self-adjoint operators over H such that $\lim_{n\to\infty} A_n = A$ in operator norm, *i.e.* $\|A_n - A\| \to 0$ as $n\to\infty$. Then we know that A is a bounded linear operator on Hilbert space. So that

$$\|A - A^*\| \leq \|A - A_n\| + \|A_n - A_n^*\| + \|A_n^* - A^*\|$$

$$= \|A - A_n\| + \|(A_n - A)^*\|$$

$$= \|A - A_n\| + \|A_n - A\| \to 0 \text{ as } n\to\infty.$$

Hence $A = A^*$ and A is self-adjoint.

Theorem 6.3.3. Let A be a bounded linear operator : $H \to H$ such that for all $x, y \in H$, $< A(x), y > = 0$, then A equals to the zero operator and conversely.

Proof : For the zero operator we always have $< 0(x), y > = < \underset{\sim}{0}, y > = 0$. Conversely let for all x, $y \in H$, $<A(x), y> = 0$. Let us fix $x \in H$ and consider $< A(x), y > = 0$ for all $y \in H$. That means $A(x) = \underset{\sim}{0}$ in H; Now let x be free and we see $A(x) = \underset{\sim}{0}$ for $x \in H$; showing $A = 0$.

Corollary : If A is a bounded linear operator : $H \to H$ satisfies $< A(x), x > = 0$ for all $x \in H$, then A is the zero operator.

If $x, y \in H$ and α, β are any two scalars we have

$$0 = <(\alpha x + \beta y), \quad \alpha x + \beta y>$$

$$= <\alpha A(x) + \beta A(y), \quad \alpha x + \beta y> \quad (A \text{ is Linear})$$

$$= \alpha\overline{\alpha} < A(x), x > + \alpha\overline{\beta} < A(x), y > + \beta\overline{\alpha} < A(y), x > + \beta\overline{\beta} < A(y), y >$$

$$= \alpha\overline{\beta} < A(x), y > + \beta\overline{\alpha} < A(y), x > \text{ other terms are zero by given condition.}$$

Let us take $\alpha = 1$ and $\beta = 1$, then we have

$$< A(x), y > \div < A(y), x > = 0 \qquad \text{..........(1)}$$

Again take $\alpha = i$ and $\beta = 1$, then above gives

$$i < A(x), y > - i < A(y), x > = 0$$

or, $$< A(x), y > - < A(y), x > = 0 \qquad (2)$$

Adding (1) and (2) we deduce $< A(x), y > = 0$, and now apply Theorem 6.3.3 for desired conclusion.

Theorem 6.3.4. Let $T \in Bd\alpha(H, H)$ ($T : H \to H$ is a bounded linear operator). Then T is self-adjoint if and only if $< A(x), x >$ is a real scalar for all $x \in H$ (Hilbert space).

Proof : Suppose T is a self-adjoint operator over H, and let $x \in H$; we have

$$< \overline{T(x), x} > = < x, T(x) > = < x, T^*(x) > = < T(x), x >$$

Therefore scalar $< T(x), x >$ is a real scalar.

Conversely, let $< T(x), x >$ is real for all $x \in H$.

Then $< T(x), x > = < \overline{T(x), x} > = < \overline{x, T^*(x)} > = < T^*(x), x >$

Thus $< T(x), x > - < T^*(x), x > = 0$

or, $< T(x) - T^*(x), x > = 0$

or, $< (T - T^*)(x), x > = 0$

This being true for all x in H, we conclude that

$T - T^*$ = zero operator

or, $T = T^*$

i.e. T is a self-adjoint operator.

Theorem 6.3.5. If H is a Hilbert space and $T \in Bd\alpha(H,H)$, such that T is self-adjoint, Then $\|T\| = \sup\limits_{\|x\|=1} |<T(x), x>|$

Proof : If T is self-adjoint, it is ofcourse a bounded linear operator over H. Then for any x with $\|x\| = 1$ in H.

$$|<T(x), x>| \leq \|T(x)\| \|x\| \qquad \text{by C-S inequality,}$$

$$\leq \|T\| \|x\| . \|x\| = \|T\|.$$

Therefore, $\sup\limits_{\|x\|\leq 1} |<T(x), x>| \leq \|T\|$ ....(1)

Let K= $\sup\limits_{\|x\|\leq 1} |<T(x), x>|$.

Now we show that $\|T\| \leq K$

If $T(u) = 0$ for all u with $\|u\| = 1$ in H, then we see that $T = 0$ (zero operator), and in that case we have finished.

Otherwise for any z with $\|z\| = 1$ such that $T(z) \neq \underset{\sim}{0}$, put $v = \sqrt{\|T(z)\|}.z$ and $w = \frac{1}{\sqrt{\|T(z)\|}} T(z)$. Then $\|v\|^2 = \|w\|^2 = \|T(z)\|$. Let us now put $y_1 = v + w$ and $y_2 = v - w$. Then on straight calculation and using the fact that T is self-adjoint, we have

$$<T(y_1), y_1> - <T(y_2), y_2> = 2(<T(v), w> + <T(w), v>$$

$$= 2(<T(z), T(z)> + <T^2(z), z>) = 4\|T(z)\|^2 \qquad ....(2)$$

Now for every $y \neq \underset{\sim}{0}$, and $x = \frac{y}{\|y\|}$, we have

$$y = \|y\| x \text{ and } <T(y), y> = \|y\|^2 |<T(x), x>|$$

$$\leq \|y\|^2 \leq \sup_{\|u\|=1} |<T(u), u>| = K \|y\|^2.$$

Now $|<T(y_1), y_1> - <T(y_2), y_2>| \leq |<T(y_1), y_1>| + |<T(y_2), y_2>|$

$$\le K(\| y_1 \|^2 + \| y_2 \|^2)$$
$$= 2K(\| v \|^2 + \| w \|^2)$$
$$= 4K \| T(z) \|$$

From here and (2) we get $4 \| T(z) \|^2 \le 4K \| T(z) \|$

Hence $\| T(z) \| \le K$

So taking supremum over all z with norm | one obtains $\| T \| \le K$

together with $K \le \| T \|$ from (1) we finally get $\| T \| = K$.

Theorem 6.3.6. Let $T \in Bd\alpha(H, H)$, H being Hilbert space show that following statements are equivalent.

(a) $T^*T = I$ (Identity operator)

(b) $< T(x), T(y) > = < x, y >$ for all $x, y \in H$

(c) $\| T(x) \| = \| x \|$ for all $x \in H$

Proof : (a) $\Rightarrow$ (b). Let (a) hold. Then for all $x, y \in H$, we have

$$< T^*T(x), y > = < I(x), y > = < x, y >$$

or, $< T(x), T(y) > = < x, y >$, (b) follows.

(b) $\Rightarrow$ (c) ; suppose (b) is true. Taking $y = x$ in (b).

We have $< T(x), T(x) > = < x, x >$

or, $\| T(x) \|^2 = \| x \|^2$

or, $\| T(x) \| = \| x \|$

$(c) \Rightarrow (a)$; Then $\| T(x) \| = \| x \|$ gives $\| T(x) \|^2 = \| x \|^2$

or, $< T(x), T(x) > = < x, x >$

or, $< T^*(T(x)), x > = < x, x >$

or, $< T^*T(x), x > - < x, x > = 0$

or, $((T^*T - I)(x), x) = 0$; Here we apply corollary of Theorem 6.3.3 to conclude that $T^*T - T = 0$ or, $T^*T = I$.

§ 6.4 EIGEN VALUES AND EIGEN VECTORS OF OPERATOR ON HILBERT SPACE *H*.

Let T be a bounded Linear operator : $H \to H$ *i.e.* $T \in Bd\alpha(H,H)$.

Definition 6.4.1. A non-zero vector $x \in H$ is said to be an eigen vector corresponding to a scalar λ called an eigen value of T if

$$T(x) = \lambda x$$

or, $T(x) - \lambda I(x) = 0$ (I denoting Identity operator on H)

or, $(T - \lambda I)(x) = 0$

Theorem 6.4.1. Let $T : H \to H$ be a self-adjoint operator. Then (1) all eigen values of T (if they exist) are real, and (2) Eigen vectors corresponding to different eigen values of T are orthognonal.

Proof : (a) Let λ be an eigen value of T and x a corresponding eigen vector. Then $x \neq \underset{\sim}{0}$ and $T(x) = \lambda x$.

Since T is self-adjoint, we have

$$\lambda < x,x > = < \lambda x, x > = < T(x), x > = < x, T(x) > = < x, \lambda x >$$

$= \bar{\lambda} < x,x >$ where $< x,x > = \|x\|^2$ is +ve as $x \neq \underset{\sim}{0}$, and this gives $\lambda = \bar{\lambda}$ (since $\|x\| > 0$) and therefore λ is real.

(b) Let λ and μ be two different eigen values of T, and let x and y be eigen vectors (non-zero) corresponding to eigen values λ and μ respectively.

Then we have $T(x) = \lambda x$ and $T(y) = \mu y$. Since T is self-adjoint and eigen values are real, we have

$$\lambda < x,y > = < \lambda x, y > = < T(x), y > = < x, T(y) >$$

$$= < x, \mu y > = \mu < x,y >, \quad \mu \text{ being real.}$$

Since $\lambda \neq \mu$ we conclude that $< x, y > = 0$ or, $x \perp y$ holds.

Theorem 6.4.2. If $T \in Bd\alpha(H,H)$ such that $T^*T = TT^*$, then if x is an eigen vector of T with eigen value λ, then x is also an eigen value of T^* with eigen value $\bar{\lambda}$, and conversely.

Proof : Consider the operator $T - \lambda I$ in H. Then

$$(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda} I - \lambda T^* + |x|^2,$$

and similarly $(T - \lambda I) + (T - \lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + |\lambda|^2 I$

Given $T^*T = T^*T$. Therefore

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I) \text{ puting } T-\lambda I = S$$

We have $SS^* = S^*S$,

Thus for $x \in H$, $SS^*(x) = S^*S(x)$

or, $\quad < SS^*(x), x > = < S^*S(x), x >$

or, $\quad < S^*(x), S^*x > = < S(x), S(x) >$

or, $\quad \| S^*(x) \|^2 = \| S(x) \|^2$

or, $\quad \| (T^* - \bar{\lambda} I)(x) \|^2 = \| (T-\lambda I)(x) \|^2$

or, $\quad \| (T-\lambda I)(x) \|^2 = \| (T^* - \bar{\lambda} I)(x) \|^2$

or, $\quad \| T - \lambda x \| = \| T^* - \bar{\lambda} x \|$.

This shows that $T(x) = \lambda x$ if and only if $T^*(x) = \bar{\lambda} x$.

Example 6.4.1. Let $L_2[0,1]$ be the real Hilbert space of all square integrable functions over the closed interval [0,1] with I.P. function $< x, y > = \int_0^1 x(t)\, y(t) dt$ as $x, y \in L_2[0,1]$.

Show that $T : L_2[0,1] \to L_2[0,1]$ defined by $T(x) = y \in L_2[0,1]$ where $y(t) = t\, x(t)$ in $0 \le t \le 1$ is a bounded linear operator which is self-adjoint having no eigen values.

Solution : Here T is a linear operator because if $x, y \in L_2[0,1]$ and if $T(x+y) = z$ where $z(t) = t(x+y)(t)$, in $0 \le t \le 1$, we have

$$T(x+y)(t) = z(t) = t\left(x(t) + y(t)\right) = tx(t) + ty(t)$$

$$= T(x)(t) + T(y)(t) \qquad \text{in } 0 \le t \le 1.$$

$\therefore$ $T(x+y) = T(x) + T(y)$ and similarly for any real scalar α, $T(\alpha x) = \alpha\, T(x)$.

Further, $T(x)(t) = tx(t)$ in $0 \le t \le 1$.

$$\therefore \quad \|T(x)\|^2 = \int_0^1 t^2 x^2(t)dt \le \sup_{0\le t\le}\{t^2\}\int_0^1 x^2(t)dt$$

$$= 1.\|x\|^2;$$

Thus $\|T(x)\| \le \|x\|$; that shows that T is a bounded linear operator in $L_2[0,1]$.

***T* is self-adjoint.** Let $x, y \in L_2[0,1]$, then we have

$$< x, T(y) > = \int_0^1 x(t)ty(t)dt = \int_0^1 tx(t)y(t)dt$$

and $$< y, T(x) > = \int_0^1 y(t)t\, x(t)dt = \int_0^1 tx(t)y(t)dt$$

Therefore $< x, T(y) > = < y, T(x) >$; That shows T as self-adjoint.

If λ is an eigen value of T, and a non-zero $x \in L_2[0,1]$ is an eigen vector of T corresponding to the eigen value λ, we have

$T(x) = \lambda x$

or, $tx(t) = \lambda x(t)$ in $0 \le t \le 1$

or, $(t - \lambda)x(t) = 0$ in $0 \le t \le 1$

Since x is non-zero, we have $t = \lambda$ in $0 \le t \le 1$, which is not the case. Thus no such λ is there, *i.e.* T possesses no eigen value.

Theorem 6.4.3. Every bounded linear operator T on a Hilbert space H is equal to a sum $A + iB$ where A and B are self-adjoint operator in H.

Proof : Let us define A and B as follows :

$$A = \frac{1}{2}(T + T^*), \text{ and } B = \frac{1}{2i}(T - T^*).$$

Then $A^* = \frac{1}{2}(T^* + T) = A$ and $B^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = B$; So each of A and B is a self-adjoint operator on H such that $A + iB = T$.

Remark : Representation of T as $T = A + iB$ is unique. Because, Let $T = C + iD$ where C and D are self-adjoint operator on H; then $T^* = (C + iD)^* = C - iD$ and hence $T + T^* = 2C$ and $T - T^* = 2iD$; Thus $C = A$ and $D = B$.

EXERCISE A

Short answer type questions

1. Find the eigen values and eigen vectors of $\begin{pmatrix} a, & b \\ -b, & a \end{pmatrix}$ $b \neq 0$ and a, b are reals.
2. Examine if zero operator and Identity operator in a Hilbert space H are self-adjoint.
3. If T is a self-adjoint operator in a Hilbert space H, show that for every natural number n, T^n is self-adjoint.
4. If T is a self-adjoint operator in a Hilbert space H, and S is any bounded Linear operator in H, show that S^*TS is self-adjoint.
5. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not possess any eigen vector.

EXERCISE B

1. Given a square matrix $A = \left((a_{ji})\right)_{n \times n}$ having eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, show that kA has eigen values $k\lambda_1, k\lambda_2, ..., k\lambda_n$; and A^2 has eigen values $\lambda_1^2, \lambda_2^2, ... \lambda_n^2$.
2. Let $T : l_2 \to l_2$ be defined by $T(\xi_1, \xi_2, ...) = (0, 0, \xi_1, \xi_2, ...)$ as $(\xi_1, \xi_2, ...) \in l_2$; Examine if T is a bounded linear operator in l_2 and if T is self-adjoint in l_2.
3. Show that in a Hilbert space H, $T_1^* T_1 = T_2^* T_2$ if and only if $\| T_1(x) \| = \| T_2(x) \|$ for all $x \in H$.
4. In H if T is self-adjoint show that $T(x) = \underset{\sim}{0}$ in H if and only if $T\,T(x) = \underset{\sim}{0}$.
5. Let $T : H \to H$ and $W : H \to H$ be bounded Linear operators and $S = W^*TW$. Show that if T is self-adjoint and +ve, so will be S.

NOTES