NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS POST GRADUATE

PG (MT) VII
GROUPS A & B

Differential Equations
and
Integral Transformations

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Integral Equations

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the meterials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great part of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these to admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

NETAJI SUBHAS OPEN UNIVERSITY

PG (MT)–VII
Differential Equations and Integral Transformation, Integral Equations

Group A

Differential Equations and Integral Transformation

Group B

Integral Equations

CHAPTER 1 ❑ INTEGRAL TRANSFORM AND DIFFERENTIAL EQUATIONS

Structure

1.1 Introduction

1.2 Integral Transform

1.3 Application to Differential Equations

1.0 INTRODUCTION

Integral transforms play an essential part of mathematical background required by scientists and engineers, as these provide an easy and effective means for the solutions of many problem arising on those areas. An attempt has been made to cover the commonly used properties of those integral transforms which are currently in use the solution of differential equations. It is to be noted these transforms can fruitfully applied to find the solutions of integral equations also. There are chapters on Fourier, Laplace and Hankel transforms. Closely related topics such as finite transforms, dual integral equations etc. are also considered.

1.1 INTEGRAL TRANSFORM

The integral transform of a function $f(x)$ of real variable x denoted by I[f(x)] is defined by

$$I[f(x)] = \int_{x_1}^{x_2} K(\alpha, x) f(x) dx \quad (1.1)$$

where $K(\alpha, x)$ is called the kernel of the transform and is a function of the variables x and α, x being real and α being real or complex and where x_1, x_2 are two real constants. Obviously $I[f(x)]$ is a function of α and so we can denote $I[f(x)]$ by $\bar{f}(\alpha)$.

The integral transform can be considered as an integral operator I, which operates on $f(x)$ to produce the function $\overline{f}(\alpha)$ on α-space. This integral operator I is linear due to the following reasons :

(i) If $f(x)$ and $g(x)$ be two functions of real variable x which posses the integral transforms by the Kernel $K(\alpha, x)$, then we have

$$I[f(x)+g(x)] = \int_{x_1}^{x_2} K(\alpha,x)[f(x)+g(x)]dx$$

$$= \int_{x_1}^{x_2} K(\alpha,x)f(x)dx + \int_{x_1}^{x_2} K(\alpha,x)g(x)dx$$

$$= I[f(x)] + I[g(x)].$$

(ii) If c be a constant real or complex then,

$$I[cf(x)] = \int_{x_1}^{x_2} K(\alpha,x)cf(x)dx = c\int_{x_1}^{x_2} K(\alpha,x)f(x)dx$$

$$= cI[f(x)].$$

If there exists an operator I^{-1} which transforms the fnction $\overline{f}(\alpha)$ back to the function $f(x)$, then I^{-1} is called the inverse of the integral transform operator I. So we can write.

$$f(x) = I^{-1}[f(x)] \tag{1.2}$$

The following are some examples of integral transforms :

(a) When $K(\alpha, x) = e^{iax}$, $x_1 = -\infty$, $x_2 = \infty$, where α is real, the corresponding integral transform is called **Fourier transform**.

(b) When $K(\alpha, x) = \sin \frac{\alpha\pi x}{a}$ (or $\cos \frac{\alpha\pi x}{a}$), $x_1 = 0$, $x_2 = a$ (> 0) where α is real and positive integer, the corresponding integral transform is called **Finite Fourier sine (or consine) transform**.

(c) When $K(\alpha, x) = e^{-ax}$, $x_1 = 0$, $x_2 = \infty$, where α is complex, the corresponding integral transform is called **Laplace transform.**

(d) When $K(\alpha, \mathrm{x}) = xJ\gamma(\alpha, \mathrm{x})$, $\left(\gamma > -\frac{1}{2}\right)$, $x_1 = 0$, $x_2 = \infty$, α beng real and positive integer, the corresponding integral transform is called **Hankel transform.**

1.2 APPLICATION TO DIFFERENTIAL EQUATIONS

The application of integral transform is quite extensive. But we shall restrict our attention in solving ordinary and partial diffreential equations. By the use of integral transform, ordinary and partial differential equations can be reduced respectively to algebraic and ordinary differential equations, which are easier to solve than solving the original ones.

As an example, we consider the following Cauchy problem for ordinary linear differential equation with constant coefficients which is to be solved by the use of Laplace transform.

Let the equation is

$$a_0 y^{(n)}(t) + a_{1} y^{(n-1)}(t) + \ldots\ldots + a_n y(t) = f(t), \tag{1.3}$$

satisfying the initial conditions.

$$y(0) = y_0, y'(0) = y_1, \ldots\ldots y^{n-1}(0) = y_{n-1} \tag{1.4}$$

where y_0, y_1, ..., y_{n-1} are some given constants. The coefficients of the equation a_0, a_1, ..., a_n are constants, $f(t)$ is a given function of t and $y^{(j)}(t) = \dfrac{d^j y}{dt^j}$,

Specific examples have been taken to solve ordinary differential equations with variable coefficients by the use of Laplace transform. Fourier, Laplace and Hankel tranform have been applied in solving various initial-boundary value problems related to partial differential equations of the type.

$$\lambda \nabla^2 u - \frac{\partial u}{\partial t} = 0 \tag{1.5a}$$

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.5b}$$

$$\nabla^2 u = 0 \tag{1.5c}$$

in different geometries, which appear in numerous problems of practical applications in theoretical physics and applied mathematics.

CHAPTER 2 ❑ FOURIER TRANSFORM WITH APPLICATION TO DIFFERENTIAL EQUATIONS

Structure

2.0 INTRODUCTION

Joseph Fourier (1768–1830), a French mathematician, invented a mathematical tool, called Fourier transform, in 1801, to explain the heat flow around an anchor ring. Later, this transform has been to be a powerful effective method in diversified fields of science and engineering. Fourier transform provides a means in solving problems arising in dynamic response to electricity, heat or light, identifications of contributions to fluctuating signal in astronomy, medicine etc, designing electrical circuits, analyzing mechanical vibrations and so on, Sometimes, it is also necessary to come across finite intervals in boundary value problems for which it is natural to extend Fourier transform method for the finite range of independent variable.

In this unit, we proceed to discuss Fourier transform, infinite or finite, its properties and its applications in solving differential equations.

2.1 DEFINITION AND ELEMENTARY PROPERTIES OF FOURIER TRANSFORM

Definition : The Fourier transform of a function $f(x)$ of real variable x is a function of real variable k, which we denote by $F(k)$ or $F[f(x)]$, is defined by

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx \tag{1.1}$$

provided the integral exists.

The integral in (2.1.) exists, if $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent. Since if $f(x)$ is integrable in any finite interval then $f(x)e^{ikx}$ is also integrable in the same interval and if $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent, the integral $\int_{-\infty}^{\infty} f(x)e^{ikx}$ is also convergent due to the following inequality : $\left|\int_{-\infty}^{\infty} f(x)e^{ikx}dx\right| \le \int_{-\infty}^{\infty} |f(x)|\left|e^{ikx}\right|dx = \int_{-\infty}^{\infty}|f(x)|\,dx.$

In the subsequent development of the theory of Fourier transform, we shall often consider functions, which satisfy the following conditions, known as **Dirichlet's conditions**.

Dirichlet's conditions : A function $f(x)$ is said to satisfy Dirichlet's conditions in an interval (a, b) in which it is defined, if it satisfies any one of the following conditions :

(i) $f(x)$ is bounded in (a, b) and the interval ean be broken up into a finite number of intervals in each of which $f(x)$ is monotonic.

(ii) $f(x)$ has only a finite number of maxima and minima in (a, b) and a finite number of finite discontinuities is (a, b).

Obviosuly if the function $f(x)$ satisfies Dirichlet's conditions in $-\infty < x < \infty$ and the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent, then the integral in (1) exists and therefore the Fourier transform of $f(x)$ exists.

Some elementary properties

(a) The Fourier transform of a function, if exists, is bounded.

Proof : Let $F(k)$ be the Fourier transform of a function $f(x)$. Then

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

Since $F(k)$ exists, the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ is convergent and hence

$$\int_{-\infty}^{\infty} |f(x)|\,dx \le B, \text{ a positive constant.}$$

$$\text{Now, } |F(k)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)||e^{ikx}|dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|dx \le \frac{B}{\sqrt{2\pi}} \quad [\because |e^{ikx}| = 1]$$

Hence the Fourier transform of $f(x)$ is bounded.

(b) Fourier transform is linear. That is, if $F_1(k)$ and $F_2(k)$ are the Fourier transforms of the two functions $f_1(x)$ and $f_2(x)$ respectively, then the Fourier transform of $\alpha_1 f_1(x) + \alpha_2 f_2(x)$ is $\alpha_1 F_1(k) + \alpha_2 F_2(k_1)$, where α_1 and α_2 are two complex constants.

Proof : If we suppose that both $F_1(k)$ and $F_2(k)$ exist, then both $f_1(x)$ and $f_2(x)$ are integrable in any finite interval and the integrals.

$$\int_{-\infty}^{\infty} |f_1(x)|dx \text{ and } \int_{-\infty}^{\infty} |f_2(x)|dx$$

are convergent. This implies that the function $\alpha_1 f_1(x) + \alpha_2 f_2(x)$ is also integrable in any finite interval and

$$\int_{-\infty}^{\infty} |\alpha_1 f_1(x) + \alpha_2 f_2(x)| dx \le |\alpha_1| \int_{-\infty}^{\infty} |f_1(x)| dx + |\alpha_2| \int_{-\infty}^{\infty} |f_2(x)| dx,$$ a bounded quantity.

Hence Fourier transform of $\alpha_1 f_1(x) + \alpha_2 f_2(x)$ exists and is given by

$$F[\alpha_1 f_1(x) + \alpha_2 f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\alpha_1 f_1(x) + \alpha_2 f_1(x)] e^{ikx} dx$$

$$= \alpha_1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{ikx} dx + \alpha_2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{ikx} dx$$

$$= \alpha_2 F_1(f) + \alpha_2 F_2(k) \tag{2}$$

(c) If $F(k)$ is the Fourier transform of $f(x)$, then $F(k)\, e^{iak}$ is the Fourier transform of $f(x - a)$.

Proof. If we assume that $F(k)$ exists, then $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent. This implies that $f(x - a)$ is also integrable in any finite interval and since.

$\int_{-\infty}^{\infty} |f(x-a)| dx = \int_{-\infty}^{\infty} |f(y)| dy, y = x - a$, the integral $\int_{-\infty}^{\infty} |f(x-a)| dx$ is convergent. Hence Fourier transform of $f(x - a)$ exists and is given by

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{ik(y+a)} dy, \; y = x - a$$

$$= e^{iak} . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{iky} dy$$

$$= e^{iak} F(k). \tag{3}$$

(d) If $F(k)$ is the Fourier transform of $f(x)$, then $F(k + a)$ is the Fourier transform of $f(x)\,e^{iax}$, where a is real.

Proof. Assuming the existence of $F(k)$, we can show as in (c) above that the Fourier transform of $f(x)\,e^{iax}$ exists and is given by

$$F\left[f(x)e^{iax}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{iax}.e^{ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(k+a)x}dx = F(k+a) \tag{4}$$

since in the expression.

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

for $F(k)$ replacing K by $k + a$, we get

$$F(k+a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(k+a)x}dx$$

2.2 CONTINUITY AND DIFFERENTIABILITY OF FOURIER TRANSFORM

Theorem-1 : If the Fourier transform $F(k)$ of a function $f(x)$ exists, then $F(k)$ is a continuous function of k.

Proof. Since Fourier transform of $f(x)$ exists, $f(x)$ is integrable in any finite interval and the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent. We further assume that $f(x)$ is bounded in any finite interval.

Since the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ is convergent, corresponding to any arbitrary positive $\in$ there exists a number X (> 0) such that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}|f(x)|dx < \frac{\in}{5},\ \frac{1}{\sqrt{2\pi}}\int_{-X}^{\infty}|f(x)|dx < \frac{\in}{5} \tag{2.1}$$

Now, $F(k+h)-F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(k+h)x}dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx$

$$= G(h+k)-G(k)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X} f(x)e^{ikx}\left(e^{ihx}-1\right)dx$$

$$+\int_{X}^{\infty} f(x)e^{ikx}\left(e^{ihx}-1\right)dx \tag{2.2}$$

where $G(k)=\frac{1}{\sqrt{2\pi}}\int_{-X}^{X} f(x)e^{ikx}$ (2.3)

Since $f(x)$ is bounded and integrable in the finite interval $(-X, X)$ and e^{iks} is a continuous function of x and k in the intervals $(-X < x < X)$, $(-\infty < k < \infty)$. $G(k)$ is the continuous function of k. Therefore corresponding to the arbitrary positive $\in$, which we have already choosen, there exists a positive number δ such that

$$|G(k+h)-G(k)|<\frac{\in}{5}, \text{ whenever } |h|<\delta \tag{2.4}$$

Therefore from (2.2) we get

$$|F(k+h)-F(k)|\le|G(k+h)-G(k)|+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}|f(x)|\left|e^{ikx}\right|\left(\left|e^{ikh}\right|+1\right)dx$$

$$+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}|f(x)|\left|e^{ikx}\right|\left(\left|e^{ikh}\right|+1\right)dx$$

$$<\frac{\in}{5}+\frac{2\in}{5}+\frac{2\in}{5}=\in, \text{ [by (1) and (4)] whenever } |h|<\delta$$

Hence $F(k)$ is a continuous function of k.

Theorem-2. If the Fourier transform of a function $f(x)$ and $xf(x)$ exist, then the derivative of $F(k)$, the Fourier transform of $f(x)$, exists and is given by

$$F'(k)=F[ixf(x)] \tag{2.5}$$

Proof. Since Fourier transform of $f(x)$ exists, $f(x)$ is integrable in any finite interval. We further assume that $f(x)$ is bounded in any finite interval. Further since Fourier transform of $xf(x)$ exists, the integral $\int_{-\infty}^{\infty}|x||f(x)|dx$ is convergent, which implies that corresponding to any arbitrary positive $\in$ there exists a number X (> 0) such that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}|x||f(x)|dx < \frac{\in}{5}, \frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}|x||f(x)|dx < \frac{\in}{5} \quad ...(2.6)$$

We now show that

$$\left|\frac{F(k+h)-F(k)}{h} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ixf(x)e^{ikx}dx\right| < \in$$

wherever $|h| < \delta$, which depends on $\in$.

We have

$$\frac{F(k+h)-F(k)}{h} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ixf(x)e^{ikx}dx$$

$$= \left\{\frac{G(k+h)-G(k)}{h} - \frac{1}{\sqrt{2\pi}}\int_{-X}^{X} ixf(x)e^{ikx}dx\right\} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X} f(x)e^{i\left(k+\frac{h}{2}\right)x} \cdot \frac{2i\sin\frac{hx}{2}}{h}dx$$

$$+\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} f(x)e^{i\left(k+\frac{h}{2}\right)x} \frac{2i\sin\frac{hx}{2}}{h}dx - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X} ixf(x)e^{ikx}dx,$$

$$-\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty} ixf(x)e^{ikx}dx \quad (2.7)$$

where $G(k) = \frac{1}{\sqrt{2\pi}}\int_{-X}^{X} f(x)e^{ikx}dx$

Since $f(x)$ is bounded and integrable in the finite interval $(-X, X)$ and e^{ikx}

and its partial derivative with respect to k are continuous function of x and k in the intervals $-X < x < X$ and $-\infty < k < \infty$, the derivative of $G(k)$ exists and is equal to

$$\frac{1}{\sqrt{2\pi}}\int_{-X}^{X} ixf(x)e^{ikx}dx.$$

Therefore, corresponding to the arbitrary positive ϵ, which we have already chosen, there exists a positive number δ such that

$$\left|\frac{G(k+h)-G(k)}{h}-\frac{1}{\sqrt{2\pi}}\int_{-X}^{X} ixf(x)e^{ikx}dx\right| < \frac{\epsilon}{5} \tag{2.8}$$

whenever $|h| < \delta$.

From (2.7) we have

$$\left|\frac{F(k+h)-F(k)}{h}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ixf(x)e^{ikx}dx\right|$$

$$\leq \left|\frac{G(k+h)-G(k)}{h}-\int_{-X}^{X} ix\,f(x)e^{ikx}dx\right|$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} |f(x)|\left|e^{i\left(k+\frac{h}{2}\right)x}\right|\frac{2\left|\sin\frac{hx}{2}\right|}{|h|}dx$$

$$+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty} |f(x)|\left|e^{i\left(k+\frac{h}{2}\right)x}\right|\frac{2\left|\sin\frac{hx}{2}\right|}{|h|}dx$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X} |x||(x)|\left|e^{ikx}\right|dx+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty} |x||f(x)|\left|e^{ikx}\right|dx$$

$$< \left|\frac{G(k+h)-G(k)}{h}-\int_{-X}^{X} ix\,f(x)e^{ikx}dx\right|$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}|f(x)||x|\,dx+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}|f(x)||x|dx$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X}|x||f(x)|dx+\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty}|x||f(x)|dx,$$

$$\left(\text{Since}\left|\sin\frac{hx}{2}\right|<\frac{|h||x|}{2}\right)$$

$< \frac{\in}{5}+\frac{\in}{5}+\frac{\in}{5}+\frac{\in}{5}+\frac{\in}{5}=\in$, by (2.6) and (2.8) whenever $|h|<\delta$.

Therefore, $\lim\limits_{h\to 0}\frac{F(k+h)-F(k)}{h}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ix\,f(x)e^{ikx}dx,$

which means that the derivative of $F(k)$ exists and is given by

$$F'(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ix\,f(x)e^{ikx}dx$$

or, $F'(k)=F[ix\,f(x)]$.

This completes the proof of the theorem.

It is seen that the formula (2.5) for the derivative of Fourier transform can be obtained from the expression (1.1) for $F(k)$ by differentiating under the sign of integration. This has been justified in the proof of theorem-2. The justification of differentiating (1.1) under sign of integration can be made not only for differentiation once but also for any number of times, and the result is

$$F^{(m)}(k)=\int_{-\infty}^{\infty}(ix)^m f(x)e^{ikx}dx$$

$$\text{or}\quad F^{(m)}(k)=F\left[(ix)^m f(x)\right] \tag{2.9}$$

where by $F^{(m)}(k)$ we mean m times differentiation of $F(k)$, m being any positive integer.

(2.9) is a generalization of theorem-2 and remains vaild if Fourier transform of $x^n f(x)$ exist for $n = 0, 1, 2, ..., m$.

2.3 FOURIER TRANSFORM OF DERIVATIVES

The Fourier transform of the derivative of a function can be expressed in terms of the Fourier transform of the function. This is stated in the following theorem.

Theorem-3. If in any finite interval a function $f(x)$ is continuous and its derivative is piecewise continuous, the integrals $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f'(x)dx$ are absolutely convergent, and $f(x) \to 0$ as $|x| \to \infty$, then (2.10)

$$F[f'(x)] = ikF(k)$$

where $F(k)$ is the Fourier transform of $f(x)$.

Proof : $f(x)$ being continuous in any finite interval, is integrable in any finite interval. Further since the integral $\int_{-\infty}^{\infty} |f(x)| dx$ exists, Fourier transform of $f(x)$ exists. Similarly, it follows that Fourier transform of $f'(x)$ exists.

Obviously, $$F[f'(x)] = \lim_{X \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-X}^{X} f'(x) e^{ikx} dx \tag{2.11}$$

where the limit exists, since the Fourier transform of $f'(x)$ exists.

Since $f'(x)$ is piecewise continuous, the finite interval $(-X, X)$ can be broken up into a finite number of sub-intervals in each of which $f'(x)$ is continuous. Let there be n such subintervals and let (b_{r-1}, b_r) be the r-th sub-interval, where $b_0 = -X$ and $b_n = X$. Therefore we can write

$$\int_{-X}^{X} f'(x) e^{ikx} dx = \sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f'(x) e^{ikx} dx$$

$$= \sum_{r=1}^{n} \left\{ \left[e^{ikx} f(x) \right]_{b_{r-1}}^{b_r} - ik \int_{b_{r-1}}^{b_r} f(x) e^{ikx} dx \right\} \text{ (integrating by parts)}$$

$$= \sum_{r=1}^{n} \left[e^{ikb_r} f(b_r - 0) - e^{ikb_{r-1}} f(b_{r-1} + 0) \right] - ik \sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f(x) e^{ikx} dx$$

$$= \left[e^{ikX} f(X) - e^{ikX} f(-X)\right] - ik \int_{-X}^{X} f(x) e^{ikx} dx \tag{2.12}$$

Since $f(x)$ is continuous in $b_{r-1} \le b_r, f(b_r - 0) = f(b_r)$ and $f(b_{r-1} + 0) = f(b_{r-1})$.

Now, $\left|e^{ikx} f(X)\right| = |f(X)| \to 0$ as $X \to \infty$

and $\left|e^{-ikX} f(-X)\right| = |f(-X)| \to 0$ as $X \to -\infty$

Therefore, $e^{ikX} f(X), e^{-ikX} f(X) \to 0$ as $|X| \to \infty$ (2.13)

Since Fourier transform of $f(x)$ exists, we have

$$\lim_{X \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-X}^{X} f(x) e^{ikx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} = F(k) \tag{2.14}$$

Therefore by the use of (2.12) we can write (2.11) as

$$F[f'(x)] = \lim_{X \to \infty} \frac{1}{\sqrt{2\pi}} \left[e^{ikX} f(X) - e^{-ikX} f(-X)\right] - ik \lim_{X \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-X}^{X} f(x) e^{ikx} dx$$

$$= -ik\, F(k), \text{ by (2.13) and (2.14).}$$

This complets the proof of the theorem.

A generalization of the above theorem giving a formula for the Fourier transform of the higher derivative of a function is stated in the following theorem without proof.

Theorem-4. If a function $f(x)$ and its derivatives up to order $n - 1$ are continuous in any finite interval, its n-th derivative is piecewise continuous in any finite interval, the integrals $\int_{-\infty}^{\infty} f^{(m)}(x) dx$ are absolutely convergent for $m = 0, 1, 2, \ldots, n$ and $f^{(m)}(x) \to 0$ as $|x| \to \infty$ for $m = 0, 1, \ldots\ldots, n-1$, then

$$F\left[f^{(n)}(x)\right] = (-ik)^n F(k), \tag{2.15}$$

where by $f^{(n)}(x)$ we mean the n-th derivative of $f(x)$.

2.4 EVALUATION OF FOURIER TRANSFORM OF SOME COMMONLY OCCURING FUNCTIONS

In this section we take some examples on the evaluation of Fourier transform of some important functions. The convergence of the integral

$$\int_{-\infty}^{\infty} |f(x)| dx$$

is actually a sufficient condition for the existence of Fourier transform of the function $f(x)$. In some of the following examples we shall see that the Fourier transform of a function may exist even if the above condition is not satisfied. For rigorous treatment of Fourier transform of such function, it is necessary to introduce the concept of generalized functions, which is beyond the scope of this book.

Ex. 2.4.1. Find the Fourier transform of $e^{-a|x|}, a > 0$.

Soln. $F\left[e^{-a|x|}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-ax} e^{-ikx} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ax} e^{ikx} dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a+ik)x}}{a+ik}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-(a-ik)x}}{a-ik}\right]_{0}^{\infty} = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+ik} + \frac{1}{a-ik}\right], \text{ since } a > 0.$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{k^2 + a^2}$$

Ex. 2.4.2. Find the Fourier transform of $e^{-a^2x^2}, a > 0$.

Soln. $F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - ikx\right)} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{ik}{2a}\right)^2 - \frac{k^2}{4a^2}} dx = \frac{1}{a\sqrt{2\pi}} e^{-\frac{k^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du, u = ax - \frac{ik}{2a}$$

$$= \frac{2}{a\sqrt{2\pi}} e^{-\frac{k^2}{4a^2}} \int_{0}^{\infty} e^{-u^2} du = \frac{1}{a\sqrt{2\pi}} e^{-\frac{k^2}{4a^2}} \int_{0}^{\infty} e^{-v} v^{\frac{1}{2}-1} dv, \ v = u^2$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{k^2}{4a^2}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{a\sqrt{2\pi}} e^{-\frac{k^2}{4a^2}} \sqrt{\pi}, \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{k^2}{4a^2}}$$

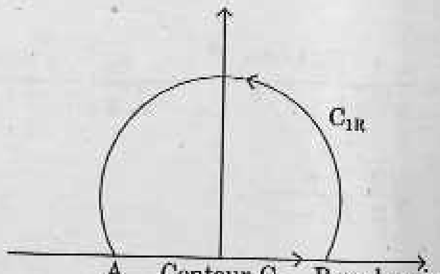
Ex. 2.4.3. Find the Fourier transform of $\frac{a}{x^2+a^2}, a > 0.$

Soln. $F\left[\frac{a}{a^2+x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{ae^{ikx}}{x^2+a^2} dx$

To evalute this integral we apply the theory of residues of complex variable. We integrate the function $f(z) = \frac{ae^{ikz}}{z^2+a^2}$ of complex variable z round the closed contour C_1 or C_2 according as $k > 0$ or < 0 in the complex z-plane. C_1 consists of straight line segment AB joining the points $-R$ to R and a semi-circular arc $C_{1R} : |z| = R, 0 < \text{agr } z < \pi$; C_2 consists of straight line segment CD joining the points R to $-R$ and a semi-circular arc $C_{2R} : |z| = R, -\pi < \arg z < 0.$

(i) For $k > 0$.

The only singularity of the function $f(z)$ that lies inside C_1 for sufficiently large R is at z = ia, which is a simple pole, and the residue of $f(z)$ at this pole is $\frac{1}{2i} e^{-ak}$. Therefore by Cauchy's residue



theorem, we have $\int_{-R}^{R} \frac{ae^{-ikx}}{x^2+a^2} dx + \int_0^{\pi} \frac{ae^{ikR(\cos\theta + i\sin 0)}}{R^2 e^{2i0} + a^2} i\,\mathrm{R}e^{i0}\, d\theta = \pi e^{-ak}$, (2.16)

(since, on AB, $z = x$ and on C_{1R}, $z = Re^{i\theta}$)

Now, $\left| \int_{C_{1R}} f(z)dz \right| \leq \int_0^{\pi} \frac{ae^{-kR\sin\theta} R}{|R^2 e^{2i0} + a^2|} d0 < \int_0^{\pi} \frac{aRd\theta}{|R^2 e^{2i0} + a^2|}$,

(since for $0 < \theta < \pi$, $\sin\theta > 0$

$$= \int_0^{\pi} \frac{aR}{R^2} d\theta = \frac{\pi a}{R} \to 0 \text{ as } R \to \infty, \qquad \text{and therefore } e^{-kh\sin\theta} < 1$$

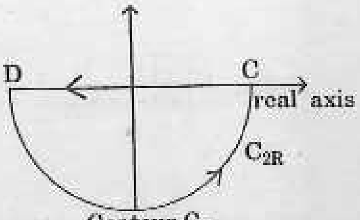
$$\therefore \quad \int_{C_R} f(z)dz \to 0 \text{ as } R \to \infty$$

Therefore from (2.16) proceeding to the limit $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{ae^{ikx}}{x^2 + a^2} dx = \pi e^{-a|k|} \tag{2.17}$$

(ii) For $k < 0$

In this case the only singularity of the function $f(z)$ that lies inside C_2 is at $z = -ia$, which is also a simple pole, and the residue of $f(z)$ at this pole is $-\frac{ae^{-|k|a}}{2ia}$ (k being negative we can write $k = -|k|$). Therefore by Cauchy's residue theorem, we have



Contour C_2

$$-\int_{-R}^{R} \frac{ae^{ikx}}{x^2 + a^2} dx + \int_{-\pi}^{0} \frac{ae^{ikR(\cos\theta + i\sin 0)}}{R^2 e^{2i\theta} + a^2} i\,\mathrm{R}e^{i0}\, d\theta = -\pi e^{-a|k|} \tag{2.18}$$

The integration on C_{2R} for large R becomes

$$\left| \int_{C_{2R}} f(z)dz \right| \approx \left| \int_{-\pi}^{0} \frac{ae^{+i|k|R(\cos 0 + i\sin\theta)}}{R^2 e^{2i\theta}} i\,\mathrm{R}e^{i\theta}\, d\theta \right|$$

$$\leq \int_{-\pi}^{0} \frac{ae^{+|k|R\sin\theta}}{R} d\theta,$$

$$= \int_0^{\pi} \frac{ae^{-|k|R\sin\varphi}}{R} d\varphi, \text{ putting } \theta = -\varphi < \frac{a\pi}{R} \to 0 \text{ as R} \to \infty, \text{ since } e^{-|k|R\sin\varphi} < 1,$$

as $\sin\varphi > 0$ for $0 < \varphi < \pi$.

Therefore from (2.18) proceeding to the limit $R \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{ae^{ikx}}{x^2+a^2}\,dx = \pi e^{-a|k|}. \tag{2.19}$$

consequently whether $k > 0$ or < 0, we have

$$\int_{-\infty}^{\infty} \frac{ae^{ikx}}{x^2+a^2} = \pi e^{-a|k|}$$

and therefore $F\left[\frac{a}{a^2+x^2}\right] = \sqrt{\frac{\pi}{2}}e^{-a|k|}$

Ex. 2.4.4. Find the Fourier transform of $\frac{1}{x}$.

Soln. $F\left[\frac{1}{x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x}e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{1}{x}e^{ikx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{1}{x}e^{ikx}dx$

$= -\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{1}{x'}e^{ikx'} + \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{1}{x}e^{ikx}dx$ (setting $x = -x'$ in the first integral)

$= -\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{e^{ikx}-e^{-ikx}}{x}dx = \frac{2i}{\sqrt{2\pi}}\int_0^{\infty}\frac{\sin kx}{x}dx$

$= \frac{2i}{\sqrt{2\pi}}\text{sgn}(k)\int_0^{\infty}\frac{\sin|k|x}{x}dx$

$= \frac{2i}{\sqrt{2\pi}}\text{sgn}(k)\int_0^{\infty}\frac{\sin y}{y}dy|k|x = y = \frac{2i}{\sqrt{2\pi}}\text{sgn}(k).\frac{\pi}{2}$

$= i\sqrt{\frac{\pi}{2}}\,\text{sgn}(k), \left(\text{since}\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}\right)$

Ex. 2.4.5. Find the Fourier transform of $\frac{\text{sgn}(x)}{|x|^{\frac{1}{2}}}$

Soln. $F\left[\frac{\text{sgn}(x)}{|x|^{\frac{1}{2}}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\text{sgn}(x)}{|x|^{\frac{1}{2}}}e^{ikx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{\text{sgn}(x)}{|x|^{\frac{1}{2}}}e^{ikx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\text{sgn}(x)}{|x|^{\frac{1}{2}}}e^{ikx}dx$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{e^{ikx}}{|x|^{\frac{1}{2}}}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{e^{ikx}}{|x|^{\frac{1}{2}}}dx$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{e^{-ikx'}}{\sqrt{x'}}dx' + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{e^{ikx}}{\sqrt{x}}dx, \text{ (setting } x = -x' \text{ in the first integral)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{e^{ikx} - e^{-ikx}}{\sqrt{x}}dx = \frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sin kx}{\sqrt{x}}dx$$

$$= \frac{2i}{\sqrt{2\pi}}\text{sgn}(k)\int_{0}^{\infty}\frac{\sin(|k|)x}{\sqrt{x}}dx = \frac{2i}{\sqrt{2\pi}}\frac{\text{sgn}(k)}{|k|^{\frac{1}{2}}}\int_{0}^{\infty}\frac{\sin u}{\sqrt{u}}du, \ (u = |k|x)$$

$$= \frac{2i}{\sqrt{2\pi}}\cdot\frac{\text{sgn}(k)}{|k|^{\frac{1}{2}}}\cdot\sqrt{\frac{\pi}{2}}, \left(\text{since}\int_{0}^{\infty}\frac{\sin x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}\right)$$

$$= i\frac{\text{sgn}(k)}{|k|^{\frac{1}{2}}}$$

2.5 INVERSE FOURIER TRANSFORM

If $F(k)$ is the Fourier transform of a function $f(x)$, then by **inverse Fourier transform** of $F(k)$ we mean a function $G(x)$ of real variable x denoted by $F^{-1}[F(k)]$ and defined by

$$G(x) = F^{-1}[F(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk \tag{2.19}$$

In this section we shall show that if $f(x)$ satisfies certain conditions then

$G(x) = f(x)$, at points of continuity of $f(x)$

$= \frac{1}{2} [f(x + 0) + f(x - 0)]$, at points of finite discontinuity of $f(x)$.

This is the content of theorem-6. To prove this theorem we require the following theorem known as Riemann-Lebesgue's theorem.

Theorem-5 (Riemann-Lebesgue's theorem) : If $f(x)$ satisfies Dirichlet's conditions in $-\infty < x < \infty$, the integral $\int_{-\infty}^{\infty} |f(x)| dx$ exists and $F(k)$ be the Fourier transform of $f(x)$, then $F(k) \to 0$ as $|k| \to \infty$, *i.e.* $\lim_{|k| \to \infty} F(k) = 0$ (2.20)

Proof. Since the integral $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent, corresponding to an arbitrary positive $\in$ there exists a positive number X such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-X} |f(x)| dx < \frac{\in}{4}, \frac{1}{\sqrt{2\pi}} \int_{X}^{\infty} |f(x)| dx < \frac{\in}{4} \tag{2.21}$$

As $f(x)$ satisfies Dirichlet's condition in $(-X, X)$, we can suppose that there are finite number of points, $a_1, a_2, \ldots\ldots a_p$ taken in order in this interval, at which $f(x)$ has either a turning value or a finite discontinuity. If we replace $-X$ by a_0 and X by a_{p+1}, then we can write.

$$\frac{1}{\sqrt{2\pi}} \int_{-X}^{X} f(x) e^{ikx} dx = \sum_{r=0}^{p} \frac{1}{\sqrt{2\pi}} \int_{a_r}^{a_{r+1}} f(x) e^{ikx} dx \tag{2.22}$$

Now in each of the intervals (a_r, a_{r+1}) $(r = 0, 1, 2, \ldots\ldots, p)$, $f(x)$ is a continuous function and is either monotonic increasing or monotonic decreasing, so that by the second mean-value theorem of integral calculus we have

$$\int_{a_r}^{a_{r+1}} f(x)e^{ikx}dx = f(a_r+0)\int_{a_r}^{\xi} e^{ikx}dx + f(a_{r+1}-0)\int_{\xi}^{a_{r+1}} e^{ikx}dx,$$

where ξ lies in the interval (a_r, a_{r+1}).

Performing the integrations on the right hand side we get

$$\int_{a_r}^{a_{r+1}} f(x)e^{ikx}dx = \frac{1}{ik}f(a_r+0)\left(e^{ik\xi} - e^{ika_r}\right) + \frac{1}{ik}f(a_{r+1}-0)\left(e^{ika_{r+1}} - e^{ik\xi}\right)$$

from which we find

$$\lim_{|k|\to\infty} \int_{a_r}^{a_{r+1}} f(x)e^{ikx}dx = 0, \; r = 0, 1, 2, \ldots, p.$$

Therefore equation (2.22) gives

$$\lim_{|k|\to\infty} \int_{-X}^{X} f(x)e^{ikx}dx = 0 \qquad (2.23)$$

This implies that corresponding to tbe arbitrary positive $\in$, which we have already chosen, there exists a number N dependent on $\in$ such that

$$\left|\frac{1}{\sqrt{2\pi}}\int_{-X}^{X} f(x)e^{ikx}dx\right| < \frac{\in}{2} \text{ whenever } |k| > N. \qquad (2.24)$$

Now, $|F(k)| = \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx\right|$

$$\le \left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-X} f(x)e^{ikx}dx\right| + \left|\frac{1}{\sqrt{2\pi}}\int_{-X}^{X} f(x)e^{ikx}dx\right| + \left|\frac{1}{\sqrt{2\pi}}\int_{X}^{\infty} f(x)e^{ikx}dx\right|$$

$$< \frac{\in}{4} + \frac{\in}{2} + \frac{\in}{4} = \in \text{ by (2.21) and (2.24) whenever } |k| > N(\in) \qquad (2.25)$$

Equation (2.25) implies that

$$\lim_{|k|\to\infty} F(k) = 0$$

This completes the proof of the theorem.

Theorem-6 (Fourier inversion theorem) : If $f(x)$ satisfies Dirichlet's conditions in $-\infty < x < \infty$ and the integral $\int_{-\infty}^{\infty}|f(x)|dx$ exists, then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk = \frac{1}{2}[f(x-0)+f(x+0)] \tag{2.26}$$

where $F(k)$ is the Fourier transform of $f(x)$.

Proof. We consider the function,

$$\varphi(v,x) = \frac{1}{\sqrt{2\pi}}\int_{-v}^{v} F(k)e^{-ikx}dk, v > 0 \tag{2.27}$$

Replacing $F(k)$ by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ikt}dt$ according to the definition of Fourier transform, we can write (2.27) as

$$\varphi(v,x) = \frac{1}{2\pi}\int_{-v}^{v} dk\, e^{-ikx}\int_{-\infty}^{\infty} dt f(t)e^{ikt} \tag{2.28}$$

To prove the theorem we shall have to show that

$$\left|\varphi(v,x) - \frac{1}{2}[f(x-0)+f(x+0)]\right|$$

can be made arbitrarily small by making v sufficiently large.

Now we express (2.28) in a different form. Since the integral $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent, the integral $\int_{-\infty}^{\infty} dt\, f(t)e^{ikt}$ is absolutely and uniformly convergent with respect to k. Therefore we can change the order to integration in (2.28) and write it as

$$\varphi(v,\ x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\int_{-v}^{v} dk e^{ik(t-x)}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin v(t-x)}{t-x}dt, \text{ (Performing the second integration).}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y+x)\frac{\sin vy}{y}\,dy, \ (y = t - x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} f(y+x)\frac{\sin vy}{y}\,dy + \int_{0}^{\infty} f(y+x)\frac{\sin vy}{y}\,dy$$

(replacing y by $-y$ in the first integral)

$$\text{or, } \varphi(v,x) = \frac{1}{\pi}\int_{0}^{\infty}[f(x-y)+f(x+y)]\frac{\sin vy}{y}\,dy$$

$$= \frac{1}{\pi}\int_{0}^{\infty}[f(x-y)+f(x+y)-f(x-0)-f(x+0)]\frac{\sin vy}{y}\,dy$$

$$+\frac{1}{2}[f(x-0)+f(x+0)], \left(\text{since} \int_{0}^{\infty}\frac{\sin vy}{y}\,dy = \frac{\pi}{2}\right)$$

Therefore,

$$\varphi(v,x) - \frac{1}{2}[f(x-0)+f(x+0)] = \frac{1}{\pi}\int_{0}^{\infty} g(y)\frac{\sin vy}{y}\,dy, \tag{2.29}$$

where $g(y) = f(x-y) + f(x+y) - f(x-0) - f(x+0)$. (2.30)

Since $f(x)$ satisfied Dirichlet's conditions in $-\infty < x < \infty$, the function $g(y)$ considered as a function of y also satisfies the same Dirichlet's conditions in $-\infty < y < \infty$. From (2.30) we find that $g(+0) = 0$ Let, a_1 be the first (starting from $y = 0$) maximum or minimum or a point of discontinuity of $g(y)$ for $y > 0$. Then since $g(y)$ is continuous in $(0, a_1)$, corresponding to an arbitrary chosen positive number $\in$, there exists a number δ (> 0) such that

$$|g(h) - g(+0)| < \frac{\in}{2}, \text{ whenever } h < \delta$$

$$\text{or, } |g(h)| < \frac{\in}{2}, \text{ whenever } h < \delta \tag{2.31}$$

Let $y_1 > 0$ satisfies the condition $y_1 < \delta$, then according to (2.31)

$$|g(y_1)| < \frac{\in}{2} \tag{2.32}$$

Dividing the interval $(0, \infty)$ into two intervals $(0, y_1)$ and (y_1, ∞), the relation (2.29) can be written as

$$\varphi(v,x) - \frac{1}{2}[f(x-0) + f(x+0)]$$

$$= \frac{1}{\pi}\int_0^{y_1} g(y)\frac{\sin vy}{y}dy + \frac{1}{\pi}\int_{y_1}^{\infty} g(y)\frac{\sin vy}{y}dy \tag{2.33}$$

Since $g(y)$ is continuous and monotonic increasing or decreasing in $(0, y_1)$ and $\frac{\sin vy}{y}$ is bounded and integrable in $(0, y_1)$ for any positive v, by the second mean-value theorem of integral calculus we have

$$\frac{1}{\pi}\int_0^{y_1} g(y)\frac{\sin vy}{y}dy = \frac{1}{\pi}[g(0+)\int_0^{\xi}\frac{\sin vy}{y}dy + g(y_1-0)\int_{\xi}^{y_1}\frac{\sin vy}{y}dy], 0 < \xi < y$$

$$= \frac{1}{\pi}g(y_1)\int_{\xi v}^{y_1 v}\frac{\sin u}{u}du, \text{ (since } g(0+) = 0 \text{ and } g(y) \text{ is continuous at } y = y_1)$$

Therefore, $$\left|\frac{1}{\pi}\int_0^{y_1} g(y)\frac{\sin vy}{y}dy\right| = \frac{1}{\pi}|g(y_1)|\left|\int_{\xi v}^{y_1 v}\frac{\sin u}{u}du\right|$$

$$< \frac{1}{\pi}.\frac{\epsilon}{2}.\pi, \text{ (by (2.32) and by the inequality } \int_p^q \frac{\sin x}{x}dx < \pi, \text{ when } 0 \le p < q)$$

or, $$\left|\frac{1}{\pi}\int_0^{y_1} g(y)\frac{\sin vy}{y}dy\right| < \frac{\epsilon}{2} \tag{2.34}$$

Now, we define a function $h(x)$ as follows

$$h(x) = 0, \text{ for } x < y_1$$

$$= \frac{1}{x}g(x), \text{ for } x \ge y_1 \tag{2.35}$$

Obviously this function satisfies Dirichlet's conditions in $-\infty < x < \infty$ and

$\int_{-\infty}^{\infty}|h(x)|dx$ is convergent. If $H(k)$ be the Fourier transform of $h(x)$, then by Riemann Lebesgue's theorem

$$\lim_{k\to\infty} H(k) = 0$$

$$\text{or,}\quad \lim_{k\to\infty}\frac{1}{\sqrt{2\pi}}\int_{y_1}^{\infty}\frac{1}{y}g(y)e^{iky}dy = 0$$

Taking imaginary part of this and then multiplying by $\sqrt{\frac{2}{\pi}}$ we get

$$\lim_{\upsilon\to\infty}\frac{1}{\pi}\int_{y_1}^{\infty}\frac{1}{y}g(y)\sin \upsilon y dy = 0. \tag{2.36}$$

where we have replaced k by υ. (2.36) implies that corresponding to the arbitrary positive $\in$, which we have already chosen, there exists a positive number. V, such that

$$\left|\frac{1}{\pi}\int_{y_1}^{\infty}g(y)\frac{\sin \upsilon y}{y}dy\right| < \frac{\in}{2}, \text{ for } \upsilon > V \tag{2.37}$$

By the use of the inequalities (2.34) and (2.37) we get the following from (2.33)

$$\left|\varphi(\upsilon,x) - \frac{1}{2}[f(x-0)+f(x+0)]\right|$$

$$\leq \left|\frac{1}{\pi}\int_{0}^{y_1}g(y)\frac{\sin \upsilon y}{y}dy\right| + \left|\frac{1}{\pi}\int_{y_1}^{\infty}g(y)\frac{\sin \upsilon y}{y}dy\right|$$

$$< \frac{\in}{2}+\frac{\in}{2} = \in, \text{ (whenever } \upsilon > V.) \tag{2.38}$$

From this it follows that

$$\lim_{\upsilon\to\infty}\varphi(\upsilon,x) = \frac{1}{2}[f(x-0)+f(x+0)]$$

$$\text{or,}\quad \lim_{\upsilon\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\upsilon}^{\upsilon}F(k)e^{-ikx}dk = \frac{1}{2}[f(x-0)+f(x+0)]$$

$$\text{or,}\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk = \frac{1}{2}[f(x-0)+f(x+0)] \tag{2.39}$$

This completes the proof of the theorem.

At points of continuity of $f(x)$, the formula (2.26) becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk = f(x) \tag{2.26a}$$

Corollary : If $f(x)$ satisfies the same conditions as in theorem-6, then

$$\frac{1}{2}[f(x-0)+f(x+0)] = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} f(t)\cos[k(t-x)]dt \tag{2.40}$$

This is known as **Fourier integral theorem.**

Proof. From Fourier inversion theorem we have

$$\frac{1}{2}[f(x-0)+f(x+0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dkF(k)e^{-ikx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{-ikx}\int_{-\infty}^{\infty} dt\, f(t)e^{ikt},\ \text{(substituting for } F(k))$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} dk\int_{-\infty}^{\infty} dt\, f(t)e^{ik(t-x)} + \frac{1}{2\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} dt\, f(t)e^{ik(t-x)}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} dt\, f(t)e^{-ik(t-x)} + \frac{1}{2\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} dt\, f(t)e^{ik(t-x)}$$

(Replacing k by $-k$ in the first integral)

$$= \frac{1}{\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} dt\, f(t)\frac{1}{2}\left[e^{ik(t-x)}+e^{-ik(t-x)}\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty} dk\int_{-\infty}^{\infty} dt\, f(t)\cos k(t-x)$$

2.6 CONVOLUTION THEOREM AND PARSEVAL'S RELATION FOR FOURIER TRANSFORM

Definition : The function,

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)g(x-y)dy \tag{2.41}$$

is called the convolution or Faltung of the two functions $f(x)$ and $g(x)$.

It can be shown that if both the functions $f(x)$ and $g(x)$ are integrable in any finite interval and the integrals $\int_{-\infty}^{\infty} |f(x)|dx$ and $\int_{-\infty}^{\infty} |g(x)|dx$ exist, then the integral in (2.41) exists.

Theorem-7 (Convolution theorem) : The Fourier transform of the convolution of the two functions $f(x)$ and $g(x)$ is equal to $F(k)G(k)$, where $F(k)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, *i.e.*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ikx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy = F(k)G(k) \tag{2.42}$$

By, Fourier inversion theorem this can be written as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)G(k)e^{-ikx}dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy \tag{2.42a}$$

Proof : Let $H(k)$ be the Fourier transform of the convolution $h(x)$ of the two functions $f(x)$ and $g(x)$, where

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Therefore, $H(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)e^{ikx}dx$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx} \int_{-\infty}^{\infty} f(x-y)g(y)dx, \text{ (subsituting for } h(x)).$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x-y)g(y)e^{ik(x-y)}e^{iky}$$

We assume that we can change the order of integration in the above. Then we have

$$H(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, g(y)e^{iky} \int_{-\infty}^{\infty} dx\, f(x-y)e^{ik(x-y)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, g(y)e^{iky} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du\, f(u)e^{iku}$$

(setting $u = x - y$ in the second integral)

$= G(k)\ F(k).$

Theorem-8. If $F(k)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |F(k)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx \tag{2.43}$$

This relation is known as Parseval's relation.

Proof : The convolution of the two functions $f(x)$ and $g(x)$ is given by

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \tag{2.44}$$

If $F(k)$, $G(k)$ and $H(k)$ be respectively the Fourier transforms of $f(x)$, $g(x)$ and $h(x)$, then according to the convolution theorem.

$$H(k) = F(k)\ G(k) \tag{2.45}$$

By Fourier inversion theorem we have

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(k)e^{-ikx}dk,$$

where we assume that both $f(x)$ and $g(x)$ are continuous functions of x and consequently $h(x)$ is also a continuous function of x.

Substituting in the above for $h(x)$ given by (2.44) and for $H(k)$ gives by (2.45) we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)G(k)e^{-ikx}dk$$

Setting $x = 0$ we get

$$\int_{-\infty}^{\infty} f(\xi)g(-\xi)d\xi = \int_{-\infty}^{\infty} F(k)G(k)dk \tag{2.46}$$

Now let $g(-\xi) = \overline{f(\xi)}$, where bar implies complex conjugate. Then

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(-u)e^{-iku}du, (x = -u)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}(u)e^{-iku}du = \overline{F}(k) \tag{2.47}$$

Taking complex conjugate of $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx$, we get

$$\overline{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}(x)e^{-ikx}dx$$

Therefore, from (2.46) we get

$$\int_{-\infty}^{\infty} f(\xi)\overline{f}(\xi)d\xi = \int_{-\infty}^{\infty} F(k)\overline{F}(k)dk$$

or, $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(k)|^2\, dk$$

Theorem-9. If $F_1(k)$ and $F_2(k)$ are the Fourier transforms of the functions $f_1(x)$ and $f_2(x)$ respectively, then

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}\,dx = \int_{-\infty}^{\infty} F_1(k)\overline{F_2(k)}\,dk \tag{2.48}$$

This is a generalization form of Parseval's relation, since Parseval's relation (2.43) can be obtained from this by setting $f_2(x) = f_1(x)$.

Proof : If $F_1(k)$ and $G_1(k)$ be the Fourier transforms of the two functions $f_1(x)$ and $g_1(x)$ respectively, then following the proof of theorem-8 we get

$$\int_{-\infty}^{\infty} f_1(\xi)g_1(-\xi)d\xi = \int_{-\infty}^{\infty} F_1(k)G_1(k)dk \text{ eqn. (3) of Th. 2} \tag{2.49}$$

Now let $g_1(-x) = \overline{f_2}(x)$, then

$$G_1(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(x)e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(-x)e^{-ikx}dx, \text{ (replacing } x \text{ by } -x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f_2}(x)e^{-ikx}dx = \overline{F_2}(k)$$

Therefore from (2.49) we get $\int_{-\infty}^{\infty} f_1(\xi)\overline{f_2}(\xi)d\xi = \int_{-\infty}^{\infty} F_1(k)\overline{F_2}(k)\,dk$

2.7 SOME EXAMPLES ON APPLICATION OF FOURIER INVERSION AND CONVOLUTION THEOREMS

In this section we take some examples to show how Fourier inversion theorem and Parseval's relation can be employed to evaluate certain definite integrals and also to show how convolution theorem can be used to evaluate Fourier inversion of certain functions.

Ex. 2.7.1. Find the Fourier transform of the function.

$f(x) = 1$, for $|x| < 1$

0, for $|x| \geq 1$.

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}dx$.

Soln. $F(k)$, the Fourier transform of $f(x)$, is

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{ikx}dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{ik}\left(e^{ik} - e^{-ik}\right) = \sqrt{\frac{2}{\pi}}\frac{\sin k}{k}$$

By Fourier inversion theorem, we get the following at points of continuity of $f(x)$:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin k}{k}e^{-ikx}dk$$

The point $x = 0$ being a point of continuity of $f(x)$, setting $x = 0$ in the above we get

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin k}{k}dk = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\sin k}{k}dk + \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin k}{k}dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin k}{k}dk + \frac{1}{\pi}\int_{0}^{\infty}\frac{\sin k}{k}dk$$ (replacing k by $-k$ in the first integral)

or, $1 = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin x}{x}dx$, (since $f(0) = 1$)

Therefore $\int_{0}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$.

Ex. 2.7.2. Find the Fourier transform of

$$f(x) = 1 - x^2, \quad \text{for } |x| \le 1$$
$$= 0, \quad \text{for } |x| > 1$$

Hence evaluate $\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx$

Soln. The Fourier transform $F(k)$ of $f(x)$ is given by

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - x^2)e^{ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}(1-x^2)\frac{e^{ikx}}{ik}\Big|_{-1}^{1} + \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 2x.\frac{e^{ikx}}{ik}dx$$

$$= \sqrt{\frac{2}{\pi}}x\frac{e^{ikx}}{-k^2}\Big|_{-1}^{1} - \sqrt{\frac{2}{\pi}}\int_{-1}^{1}\frac{e^{ikx}}{(ik)^2}dx$$

$$= -\frac{1}{k^2}\sqrt{\frac{2}{\pi}}(e^{ik}+e^{-ik}) + \sqrt{\frac{2}{\pi}}\frac{1}{ik^3}(e^{ik}-e^{-ik})$$

$$= -\sqrt{\frac{2}{\pi}}.\frac{2}{k^3}(k\cos k - \sin k)$$

By Fourier inversion theorem we have the following at points of continuity of $f(x)$.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{-ikx}dx$$

$$= -\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{2}{k^3}(k\cos k - \sin k)e^{-ikx}dk$$

Since $x = \frac{1}{2}$ is a point of continuity of $f(x)$, we have

$$f\left(\frac{1}{2}\right) = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^3}(k\cos k - \sin k)e^{-\frac{ik}{2}}dk$$

or, $$1-\frac{1}{4} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^3}(k\cos k - \sin k)\left(\cos\frac{k}{2} - i\sin\frac{k}{2}\right)dk$$

Taking real part of this we get

$$\int_{-\infty}^{\infty}\frac{1}{k^3}(k\cos k - \sin k)\cos\frac{k}{2}dk = -\frac{3\pi}{8}$$

or, $$\int_{0}^{\infty}\frac{1}{x^3}(x\cos x - \sin x)\cos\frac{x}{2}dx = -\frac{3\pi}{16},$$

(the integrand being an even function)

Ex. 2.7.3. Use Parseval's relation to show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

Soln : We consider two functions

$$f_1(x) = e^{-a|x|} \text{ and } f_2(x) = e^{-b|x|}, (a, b > 0)$$

If Fourier transform of these two functions be $F_1(k)$ and $F_2(k)$, then

$$F_1(k) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{k^2+a^2}, \; F_2(k) = \sqrt{\frac{2}{\pi}} \cdot \frac{b}{k^2+b^2}, \quad \text{(Ex. 2.4.1.)}$$

Since, k, b, x are all real $\overline{f_1}(x) = f_1(x), \overline{f_2}(x) = f_2(x)$. Therefore from the generalized form of Parseval's relation (2.48), we get

$$\int_{-\infty}^{\infty} e^{-a|x|} . e^{-b|x|} dx = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{k^2+a^2} \cdot \sqrt{\frac{2}{\pi}} \frac{b}{k^2+b^2} dk$$

or, $$\int_{-\infty}^{\infty} \frac{dk}{(k^2+a^2)(k^2+b^2)} = \frac{\pi}{2ab} . 2 \int_0^{\infty} e^{-(a+b)|x|} dx$$

(the integrand being an even function)

$$= \frac{\pi}{ab} \cdot \frac{1}{a+b}$$

or, $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

Ex. 2.7.4. Use Parseval's relation to show that

$$\int_0^{\infty} \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi a}{2}, \text{ where } 0 < a < b$$

We consider two functions $f_1(x)$ and $f_2(x)$ defined by

$$\left.\begin{aligned} f_1(x) &= 1, |x| \le a \\ &= 0, |x| > a \end{aligned}\right\} \begin{aligned} f_2(x) &= 1, |x| \le b \\ &= 0, |x| > b \end{aligned} \qquad (2.50)$$

If $F_1(k)$ and $F_2(k)$ be the Fourier transforms of these two functions, then

$$F_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ik} \left(e^{ika} - e^{-ika}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k}$$

and similarly $F_2(k) = \sqrt{\frac{2}{\pi}} \frac{\sin kb}{k}$

in the generalized form of Parsevals relation (2.48), setting for $f_1(x)$ and $f_2(x)$ given by (2.50), we get

$$\int_{-a}^{a} f_1(x)\overline{f_2(x)}\, dx = \int_{-\infty}^{\infty} F_1(k)\overline{F}_2(k)\, dk,$$

Since $(-a, a)$ is the interval within which $f_1(x)$ and $f_2(x)$ are different from 0.

$$\text{or,} \quad \int_{-a}^{a} dx = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin ka \sin kb}{k^2} dk$$

$$\text{or,} \quad 2a = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin ax \sin bx}{x^2} dx, \text{ (the integrand being an even function)}$$

$$\text{Therefore,} \quad \int_{0}^{\infty} \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi a}{2}$$

Ex. 2.7.5. Use Fourier inversion theorem to show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$$

We consider the function $f(x)$ defined by

$$f(x) = 1 - |x|, \text{ when } |x| \le 1$$
$$0, \text{ when } |x| > 1.$$

Then its Fourier transform $F(k)$ is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ikx} dx - \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (-x) e^{ikx} dx + \int_{0}^{1} x e^{ikx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{k^2} (1 - \cos k) = \frac{4}{\sqrt{2\pi}} \frac{\sin^2 \frac{k}{2}}{k^2}$$

Therefore, by Fourier inversion theorem we get the following at places of continuity of $f(x)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \frac{\sin^2 \frac{k}{2}}{k^2} e^{-ikx} dk$$

Setting $x = 0$ in this relation which is a point of continuity of $f(x)$ we get

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{k}{2}}{k^2} dk$$

$$\text{or,} \quad 1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{k}{2}}{k^2} dk, \text{ (since } f(0) = 1)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx, \text{ (setting } \frac{k}{2} = x)$$

$$\text{or,} \quad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$$

Ex. 2.7.6. Find the Fourier inversion of $|k|^{\frac{1}{2}} \overline{f}(k)$, where $\overline{f}(k)$ is the Fourier transform of $f(x)$.

Soln. $F^{-1}\left[|k|^{\frac{1}{2}} \overline{f}(k) \right] = F^{-1}\left[\frac{i}{k} |k|^{\frac{1}{2}} \left\{ -ik \overline{f}(k) \right\} \right]$

$$= F^{-1}\left[\frac{i\,\text{sgn}\,k}{|k|^{\frac{1}{2}}}\left\{-ik\bar{f}(k)\right\}\right] = F^{-1}\left[\bar{f}_1(k)\bar{f}_2(k)\right] \tag{2.51}$$

where $\bar{f}_1(k) = \frac{i\,\text{sgn}\,k}{|k|^{\frac{1}{2}}}, \bar{f}_2(k) = -ik\bar{f}(k)$

Let $\bar{f}_1(k)$ and $\bar{f}_2(k)$ be the Fourier transform of the two function $f_1(x)$ and $f_2(x)$. By the help of the formula (2.10) we get

$$f_2(x) = f'(x)$$

and by Fourier inversion theorem we get

$$f_1(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\,\text{sgn}\,k}{|k|^{\frac{1}{2}}}e^{-ikx}dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{-i}{|k|^{\frac{1}{2}}}e^{-ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{i}{|k|^{\frac{1}{2}}}e^{-ikx}dk$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{i}\int_0^{\infty}\frac{e^{ikx}}{\sqrt{k}}dk - \frac{1}{i}\int_0^{\infty}\frac{e^{-ikx}}{\sqrt{k}}dk\right] \text{ (replacing } k \text{ by } -k \text{ in the first integral)}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin kx}{\sqrt{k}}dk, \text{ (in the interval of integration } k \text{ being positive } |k| = k.)$$

$$= \sqrt{\frac{2}{\pi}}\,\text{sgn}\,x\int_0^{\infty}\frac{\sin k|x|}{\sqrt{k}}dk = \sqrt{\frac{2}{\pi}}\frac{\text{sgn}\,x}{|x|^{\frac{1}{2}}}\int_0^{\infty}\frac{\sin u}{\sqrt{u}}du, (u = |x|k)$$

$$= \sqrt{\frac{2}{\pi}}\frac{\text{sgn}\,x}{|x|^{\frac{1}{2}}}\sqrt{\frac{\pi}{2}} = \frac{\text{sgn}\,x}{|x|^{\frac{1}{2}}}$$

Therefore from (2.51) we get the following by convolution theorem.

$$F^{-1}\left[|k|^{\frac{1}{2}}\bar{f}(k)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\text{sgn}(x-\xi)}{|x-\xi|^{\frac{1}{2}}}f'(\xi)d\xi$$

2.8 FOURIER SINE AND COSINE TRANSFORM : DEFINITION, INVERSION FORMULAS AND PARSEVAL'S RELATIONS

If a function $f(x)$ is defined in the interval $(0, \infty)$, then its Fourier sine and cosine transforms denoted respectively by $F_s(k)$ and $F_c(k)$ are defined by the integrals

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin kx dx \tag{2.52}$$

and $$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos kx dx, \tag{2.53}$$

provided the integrals exist.

Since, $$\left|\int_0^{\infty} f(x)\sin kx\, dx\right| \le \int_0^{\infty} |f(x)||\sin kx|\, dx \le \int_0^{\infty} |f(x)|\, dx$$

and $$\left|\int_0^{\infty} f(x)\cos kx\, dx\right| \le \int_0^{\infty} |f(x)||\cos kx|\, dx \le \int_0^{\infty} |f(x)|\, dx,$$

the integrals (2.52) and (2.53) exist, if $f(x)$ is integrable in any subinterval of $(0, \infty)$ and the integral $\int_0^{\infty} f(x)dx$ is absolutely convergent.

In order to derive an inversion formula for Fourier sine transform from which we can construct the original function $f(x)$ from its sine transform, we define a function $g(x)$ in $(-\infty, \infty)$ as follows :

$$g(x) = f(x),\ x \ge 0$$
$$\quad = -f(-x),\ x < 0$$

If Fourier sine transform of $f(x)$ exists, then the Fourier transform $G(k)$ of $g(x)$ exists and is given by

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{ikx} = \frac{1}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0} f(-x)e^{ikx}dx + \int_0^{\infty} f(x)e^{ikx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\int_0^\infty f(x)e^{-ikx}dx + \int_0^\infty f(x)e^{ikx}dx\right]$$

(replacing x by $-x$ in the first integral)

$$= i\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\, dx - iF_s(k)$$

So, if f(x) satisfies Dirichlet's conditions in $(0, \infty)$ and $\int_0^\infty |f(x)|dx$ is convergent, then obviously the function $g(x)$ satisfies the same conditions in $(-\infty, \infty)$ and the integral $\int_{-\infty}^\infty |g(x)|dx$ is convergent.

Hence by Fourier inversion theorem we get the following for $x > 0$:

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty G(k)e^{-ikx}dk$$

$$= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^\infty F_s(k)e^{-ikx}dk = \frac{i}{\sqrt{2\pi}}\left[\int_{-\infty}^0 F_s(k)e^{-ikx}dk + \int_0^\infty F_s(k)e^{-ikx}dk\right]$$

$$= \frac{i}{\sqrt{2\pi}}\left[\int_0^\infty F_s(-k)e^{ikx}dk + \int_0^\infty F_s(k)e^{-ikx}dk\right],$$

replacing k by $-k$ in the first integral

$$= -\frac{i}{\sqrt{2\pi}}\int_0^\infty 2iF_s(k)\sin kx\, dk$$ since from (2.52, $F_s(-k) = -F_s(k)$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx dk$$

So, the inversion formula for Fourier sine transform becomes

$$\frac{1}{2}[f(x+0)+f(x-0)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kxdx \qquad (2.54)$$

At places of continuity of $f(x)$, the inversion formula becomes

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kxdk \qquad (2.54a)$$

In order to derive inversion formula for Fourier cosine transform we define a function $g(x)$ in $(-\infty, \infty)$ as follows :

$$g(x) = f(x),\ x \geq 0$$
$$= f(-x),\ x \leq 0$$

Then proceeding in the same way as in deriving the formula (2.54) for Fourier sine transform, we arrive at the following inversion formula for Fourier cosine transform.

$$\frac{1}{2}[f(x+0)+f(x-0)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)\cos kxdk \qquad (2.55)$$

At places of continuity of $f(x)$ this formula becomes

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(k)\cos kxdk \qquad (2.55a)$$

We now derive formulas equivalent to (2.42a) representing convolution theorem for Fourier sine and cosine transforms.

Let $f(x)$ and $g(x)$ be two functions both defined in the interval $(0, \infty)$ and let their Fourier sine and cosine transforms be $F_s(k)$, $G_s(k)$ and $F_c(k)$, $G_c(k)$ respectively, Then we have

(i) $\int_0^\infty F_c(k)G_c(k)\cos kxdk$

$$= \int_0^\infty dkF_c(k)\cos kx\sqrt{\frac{2}{\pi}}\int_0^\infty g(t)\cos ktdt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty dt g(t) \int_0^\infty dk F_c(k) \cos kx \cos kt,$$

(changing the order of integration assuming that it is possible.)

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, g(t) \int_0^\infty dk F_c(k) [\cos k(x+t) + \cos k(x-t)] dk$$

$$= \frac{1}{2} \int_0^\infty dt\, g(t) \left[\sqrt{\frac{2}{\pi}} \int_0^\infty dk F_c(k) \cos k(x+t) + \sqrt{\frac{2}{\pi}} \int_0^\infty dk F_c(k) \cos k|x-t| \right]$$

Therefore by the use of inversion formula (2.55a) for Fourier cosine transform we get

$$\int_0^\infty F_c(k) G_c(k) \cos kx dk = \frac{1}{2} \int_0^\infty dt g(t) [f(x+t) + f(|x-t|)] \tag{2.56}$$

(ii) $\int_0^\infty F_c(k) G_s(k) \sin kx dk$

$$= \int_0^\infty dk F_c(k) \sin kx \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin kt dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty dt g(t) \int_0^\infty dk F_c(k) \sin kx \sin kt$$

changing the order of integration assuming that it is possible.

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty dt g(t) \int_0^\infty dk\, F_c(k) [\cos k(x-t) - \cos k(x+t)]$$

$$= \frac{1}{2} \int_0^\infty dt\, g(t) \left[\sqrt{\frac{2}{\pi}} \int_0^\infty dk\, F_c(k) \cos k|x-t| - \sqrt{\frac{2}{\pi}} \int_0^\infty dk\, F_c(k) \cos k(x+t) \right]$$

This gives the following by the use of the inversion formula (2.55a)

$$\int_0^\infty F_c(k)G_s(k)\sin kx dk = \frac{1}{2}\int_0^\infty dt\, g(t)[f(|x-t|) - f(x+t)] \tag{2.57}$$

Similarly we can deduce the following formulas.

(iii) $$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \frac{1}{2}\int_0^\infty dt\, f(t)[g(|x-t|) - g(x+t)] \tag{2.58}$$

(iv) $$\int_0^\infty F_s(k)G_s(k)\cos kx dk = \frac{1}{2}\int_0^\infty dt\, f(t)[g(x+t) - \text{sgn}(x-t)g(|x-t|)] \tag{2.59}$$

The formulas (2.56)–(2.59) are the equivalent forms of convolution theorems for Fourier sine and cosine transforms.

The following relations are obtained from (2.56) and (2.59) by setting x = 0.

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(t)g(t)dt \tag{2.60}$$

$$\int_0^\infty F_s(k)G_s(k)dk = \int_0^\infty f(t)g(t)dt \tag{2.61}$$

Replacing here $g(t)$ by $\overline{g}(t)$, which is the complex conjugate of $g(t)$, we get the following generalization of Parseval's relations for Fourier sine and cosine transform.

$$\int_0^\infty F_c(k)\overline{G}_c(k)dk = \int_0^\infty f(t)\overline{g}(t)dt \tag{2.62}$$

$$\int_0^\infty F_s(k)\overline{G}_s(k)dk = \int_0^\infty f(t)\overline{g}(t)dt \tag{2.63}$$

Setting here $g(t) = f(t)$, we get the relations,

$$\int_0^\infty |F_c(k)|^2\, dk = \int_0^\infty |f(t)|^2\, dt \tag{2.64}$$

and $\int_0^\infty |F_s(k)|^2\, dk = \int_0^\infty |f(t)|^2 dt.$

which are the Parseval's relations for Fourier sine and cosine transforms.

2.9 SOME APPLICATIONS OF INVERSION FORMULAS AND PARSEVAL RELATIONS FOR FOURIER SINE AND COSINE TRANSFORMS

In this section we will show through some examples how certain definite integrals can be evaluated and some integral equations involving Fourier kernels can be solved by the use of the formulas derived in section2.8.

Ex.2.9.1. Use the inversion formula for Fourier sine and cosine transform to evaluate the following integrals.

(i) $\int_0^\infty \frac{\cos \alpha x}{\alpha^2 + b^2} d\alpha = \frac{\pi}{2b} e^{-bx}, x > 0,$ (ii) $\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} d\alpha = \frac{\pi}{2} e^{-bx}, x > 0,$

(iii) $\int_0^\infty \frac{\sin \alpha x \cos \alpha x}{\alpha} = \frac{\pi}{2}$, if $0 < x < a$.

0, if $x \geq a$

Soln. (i) The Fourier cosine transform $F_c(k)$ of the function $f(x) = e^{-bx}$, $x > 0$. is

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos kx\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{e^{-bx}(-b \cos kx + k \sin kx)}{b^2 + k^2} \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{b}{b^2 + k^2}$$

Therefore by inversion formula we get

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx\, dk = \sqrt{\frac{2}{\pi}} . \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{b \cos kx}{b^2 + k^2} dk$$

Hence, $\int_0^\infty \frac{\cos kx}{k^2 + b^2} dk = \frac{\pi}{2b} e^{-bx}, x > 0$

(ii) The Fourier sine transform $F_s(k)$ of the same function is

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin kx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-bx}(-b\sin bx - k\cos kx)}{b^2+k^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\frac{k}{b^2+k^2}$$

Therefore by inversion formula we get

$$e^{-bx} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx dk = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{k\sin kx}{b^2+k^2}dk$$

Hence $\displaystyle\int_0^\infty \frac{k\sin kx}{k^2+b^2}dk = \frac{\pi}{2}e^{-bx}, x > 0.$

(iii) The Fourier cosine transform $F_c(k)$ of the function.

$$f(x) = 1,\ 0 \le x \le a$$
$$0, \qquad x > a$$

is the following :

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^a 1.\cos kx\,dx = \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin ka$$

Therefore by inversion formula we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin ka\cos kx\,dk, x > 0$$

i.e. $\displaystyle\int_0^\infty \frac{\sin ka\cos kx}{k}dk = \frac{\pi}{2}, 0 < x < a$

$$= 0 \qquad x > a$$

Ex. 2.9.2. Use Parseval relation for Fourier cosine transform to evaluate the following integrals :

(i) $\displaystyle\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$, (ii) $\displaystyle\int_0^\infty \frac{\sin\lambda t\sin\mu t}{t^2}dt = \frac{\pi}{2}\min(\lambda,\mu)$,

(iii) $\int_0^\infty \frac{\sin(\lambda t)\,dt}{t(a^2+t^2)} = \frac{\pi}{2}\left(\frac{1-e^{\lambda a}}{a^2}\right)$

Sonl. (i) We consider the two function,

$$f(x) = e^{-ax}, x > 0 \text{ and } g(x) = e^{-bx}, x > 0$$

Let their Fourier cosine transforms be respectively $F_c(k)$ and $G_c(k)$. Then from Ex. 2.9.1 we get

$$F_c(k) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2},\quad G_c(k) = \sqrt{\frac{2}{\pi}}\frac{b}{b^2+k^2}$$

Therefore from the generalization of Parseval's relation for Fourier consine transform,

$$\int_0^\infty F_c(k)\overline{G}_c(k)dk = \int_0^\infty f(t)\overline{g}(t)dt,$$

we get $\displaystyle \frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+k^2)(b^2+k^2)}dk = \int_0^\infty e^{-ax}.e^{-bx}dx = \frac{1}{a+b}$

or, $\displaystyle \int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$

(ii) Let $F_c(k)$ and $G_c(k)$ be the Fourier cosine transform of the two functions $f(x)$ and $g(x)$ respectively defined by

$$f(x) = 1,\ 0 < x < \mu \qquad g(x) = 1,\ 0 < x < \lambda$$
$$= 0,\ x \geq \mu \qquad\qquad = 0,\ x \geq \lambda$$

Then from Ex. 2.9.1. we get

$$F_c(k) = \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin k\mu,\ G_c(k) = \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin k\lambda$$

Substituting these in the above written generalization of Parseval's relation for cosine transform we get

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin(\mu k)\sqrt{\frac{2}{\pi}}\frac{1}{k}\sin(\lambda k)dk = \int_0^{\min(\lambda,\mu)} 1\,dx = \min(\lambda,\mu)$$

Hence $\displaystyle \int_0^\infty \frac{\sin\lambda t\sin\mu t}{t^2}dt = \frac{\pi}{2}\min(\lambda,\mu)$

(iii) Let $F_c(k)$ and $G_c(k)$ be the Fourier cosine transform of the two functions $f(x)$ and $g(x)$ respectively defined by,

$$f(x) = 1,\ 0 < x < \lambda \qquad g(x) = e^{-ax},\ x > 0,\ (a > 0)$$

$$= 0,\ x \geq \lambda$$

Then from Ex. 2.2.1 we get

$$F_c(k) = \sqrt{\frac{2}{\pi}}\frac{1}{k}\ \sin(\lambda k) \text{ and } G_c(k) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}$$

Substituting these in the same Generalization of Parserval's relation for cosine transform we get,

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{k}\sin(\lambda k)\sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}dk = \int_0^\lambda 1.e^{-ax}dx = \frac{1}{a}(1-a^{-a\lambda})$$

Hence $\displaystyle\int_0^\infty \frac{\sin(\lambda t)}{a^2+t^2} = \frac{\pi}{2}.\left(\frac{1-e^{\lambda a}}{a^2}\right)$

Ex. 2.9.3. Solve the integral equation,

$$\int_0^\infty f(x)\cos\alpha x\, dx = 1-\alpha,\ 0 \leq \alpha \leq 1$$

$$= \quad 0,\ \ \alpha > 1$$

Hence show that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$

Soln. Let $F_c(k)$ be the Fourier cosine transform of $f(x)$. Then

$$F_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\, dx - \sqrt{\frac{2}{\pi}}(1-k),\ 0 \leq k \leq 1$$

$$= 0, \quad k > 1$$

according to the given integral equation.

By the inversion formula we get,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx dk$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} (1-k) \cos kx dk = \frac{2}{\pi x^2}(1 - \cos x)$$

Substituting this expression for $f(x)$ in the given integral equation we get,

$$\int_0^\infty \frac{2}{\pi x^2}(1 - \cos x) \cos \alpha x dx = 1 - \alpha, \ 0 \le \alpha \le 1$$

$$= 0, \qquad \alpha > 1$$

From this proceeding to the limit $\alpha \to 0$ we get

$$\frac{2}{\pi} \int_0^\infty \frac{1 - \cos x}{x^2} dx = 1$$

$$\text{or, } \frac{4}{\pi} \int_0^\infty \frac{\sin^2 \frac{x}{2}}{x^2} dx = 1$$

$$\text{or, } \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}, \text{ setting } \frac{x}{2} = t$$

2.10. FINITE FOURIER TRANSFORM

From the theory of Fourier sine and cosine series we know that if $f(x)$ defined in the interval $(0, a)$ satisfies Dirichlet's conditions in the interval $(0, a)$, then

$$\frac{1}{a}\bar{f}_c(0) + \frac{2}{a}\sum_{n=1}^{\infty} \bar{f}_c(n) \cos \frac{n\pi x}{a} = \frac{1}{2} [f(x-0) + f(x+0)], \text{ when } 0 < x < a$$

$$= f(0+), \text{ when } x = 0$$

$$= f(a-0), \text{ when } x = a \tag{2.65}$$

and $$\frac{2}{a}\sum_{n=1}^{\infty} \bar{f}_s(n) \sin \frac{n\pi x}{a} = \frac{1}{2} [f(x-0) + f(x+0)] \text{ when } 0 < x < a$$

$$= 0, \text{ when } x = 0, a. \tag{2.66}$$

where $\bar{f}_c(n) = \int_0^a f(x) \cos \frac{n\pi x}{a} dx$ (2.67)

$$\bar{f}_s(n) = \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad (2.68)$$

The left hand sides of (2.65) and (2.66) are respectively the Fourier cosine and sine series for the function $f(x)$ defined in the interval $(0, a)$.

The functions $\bar{f}_c(n)$ and $\bar{f}_s(n)$ defined by (2.67) and (2.68) are called respectively the **Finite Fourier cosine** and **finite Fourier sine transform** of $f(x)$. These are functions of positive integral values of n including zero. Obviously (2.65) and (2.66) are the inversion formulas for finite Fourier cosine and finite Fourier sine transform respectively. If $f(x)$ is continuous at x in the interval $0 < x < a$, then the inversion formulas (2.65) and (2.66) become as follows.

$$f(x) = \frac{1}{a}\bar{f}(0) + \frac{2}{a}\sum_{n=1}^{\infty}\bar{f}_c(n)\cos\frac{n\pi x}{a} \quad (2.69)$$

$$f(x) = \frac{2}{a}\sum_{n=1}^{\infty}f_s(n)\sin\frac{n\pi x}{a} \quad (2.70)$$

2.11. MULTIPLE FOURIER TRANSFORM

In this section we consider the Fourier transform of a function of several variables. The theory of Fourier transform of a function of single variable developed in Sec. 2.1 can be extended to functions of several variables. If $f(x, y)$ be a function of two independent variable x and y, defined in $(-\infty < x < \infty, -\infty < y < \infty)$, then $f(x, y)$ considered as a function of x has the Fourier transform $\bar{f}(k, y)$ given by

$$\bar{f}(k, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, y)e^{ikx}dx \quad (2.71)$$

where y appears as a parameter. Again this function $\bar{f}(k, y)$ considered as a function of y has the Fourier transform $F(k, l)$ given by

$$F(k,l) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(k,y) e^{ily} dy \tag{2.72}$$

Substituting here the expression for $\bar{f}(k,y)$ given by (2,71), we get the following expression for $F(k, l)$:

$$F(k,l) = \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{i(kx+ly)} dx\, dy \tag{2.73}$$

This function $F(k, l)$ defined by (2.73) is called the **two dimensional Fourier transform** of the function $f(x, y)$ of two variables x, y.

We now proceed to find the inversion formula for two dimensional Fourier transform. We suppose that $f(x, y)$ is a continuous function of x, y. Then taking inversion of (2.71) we get.

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(k,y) e^{-ikx} dx \tag{2.74}$$

and taking inversion of (2.72) we get

$$\bar{f}(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k,l) e^{-iky} dl \tag{2.75}$$

Substituting for $\bar{f}(k,y)$ from (2.75) in (2.74) we get

$$f(x,y) = \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k,l) e^{-i(kx+ly)} dk\, dl \tag{2.76}$$

This is the inversion formula for two dimensional Fourier transform, where it is supposed that $f(x, y)$ is a continuous function of (x, y).

To derive the convolution theorem for two dimensional Fourier transform we consider two functions $f(x, y)$ and $g(x, y)$ of two variables x, y, where we suppose that both the functions are continuous functions of x, y. Let their two-dimensional Fourier transform be $F(k, l)$ and $G(k, l)$. Then according to (2.76).

Fourier inversion of $F(k, l)\ G(k, l)$

$$= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k,l)\, G(k,l) e^{-i(kx+ly)} dk\, dl$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k,l) e^{-i(kx+ly)} dk dl \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v) e^{i(ku+lv)} du dv,$$

substituting for $F(k, l)$ according to (2.73)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, du\, dv \left[\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k,l) e^{-i\{k(x-u)+l(y-v)\}} dk\, dl\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, g(x-u, y-v)\, du\, dv, \text{ by (4.6)} \tag{2.77}$$

So if we define convolution of the two function $f(x, y)$ and $g(x, y)$ of two variables x and y by the function,

$$h(x,y) = \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u, y-v)\, g(u,v)\, du dv$$

$$\frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\, g(x-u, y-v)\, du dv, \tag{2.78}$$

then (2.77) can be expressed as

Fourier inversion of $F(k, l)$ $G(k, l)$ = convolution of the two function $f(x, y)$ and $g(x, y)$. (2.79)

This is the same as saying that

Fourier transform of the convolution of the two functions $f(x, y)$ and $g(x, y)$ = $F(h, k)$, $G(h, k)$. (2.80)

(2.79) and (2.80) state the convolution theorem for functions of two variables.

The above definition and formulas for Fourier transform for a function of two variables can obviously be extended for functions of several variables. Let $f(x_1, x_2, \ldots, x_n)$ be a function of n independent variables $x_1, x_2, \ldots, x_n$. Then n-dimensional Fourier transform of the functions $f(x_1, x_2, \ldots, x_n)$ is defined by the function

$$F(k_1,k_2,\ldots,k_n)=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int\int_{-\infty}^{\infty}\ldots\int f(x_1,x_2,\ldots,x_n)e^{-i(k,x)}dx_1dx_2\ldots dx_n \quad (2.81)$$

where $k.x = k_1x_1 + k_2x_2 + \ldots + k_nx_n$

If $f(x_1, x_2, \ldots, x_n)$ be a continuous function of $x_1, x_2, \ldots, x_n$ in $(-\infty < x_1 < \infty < x_2 < \infty, \ldots, -\infty < x_n < \infty)$ then the inversion formula (2.76) for two dimensional Fourier transform can easily be extended for n-dimensional Fourier transform by

$$f(x_1,x_2,\ldots,x_n)=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int\int_{-\infty}^{\infty}\ldots\int F(k_1,k_2,\ldots,k_n)e^{-i(k,x)}dk_1dk_2\ldots dk_n \quad (2.82)$$

The convolution theorems stated by (2.79) and (2.80) for two-dimensional Fourier transform can also be extended for n-dimensional Fourier transform as follows :

Let $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ be two function of n independent variables $x_1, x_2, \ldots, x_n$ and let their n-dimensional Fourier transform be $F(k_1, k_2, \ldots, k_n)$ and $G(k_1, k_2, \ldots, k_n)$ respectively. Then an extension of (2.79) to n dimensions is the following.

Fourier inversion of $F(k_1, k_2, \ldots, k_n)\, G(k_1, k_2, \ldots, k_n)$ = convolution of the two function $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$, (2.83)

where the convolution of the two function f and g is defined by the function,

$$h(x_1, x_2, \ldots, x_n)=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(x_1-u_1,x_1-u_2,\ldots,x_n-u_n)$$

$$\times g(u_1,u_2,\ldots,u_n)du_1du_2\ldots du_n=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(u_1,u_2,\ldots,u_n)$$

$$\times g(x_1-u_1, x_2-u_2, \ldots, x_n-u_n)\, du_1du_2\ldots du_n \quad (2.84)$$

An obvious extension of (2.80) to n dimensions is the following.

Fourier transform of the convolution of the two functions $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n) = F(k_1, k_2, \ldots, k_n)\, G(k_1, k_2, \ldots, k_n)$ (2.85)

2.11. SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

In this section we shall show how some partial differential equations, which appear in some classical problems of theoretical physics, can be solved by the application of Fourier transform. Taking a suitable Fourier transform, the partial differential equation of each problem can be reduced either to an ordinary differential equation or to an algebraic equation, which are obviously easier to solve than solving the original ones. For partial differential equations of infinite space dimension, exponential Fourier transforms are suitable ones. While for semi-infinite and finite space dimensions, Fourier sine-cosine transforms and finite Fourier transforms respectively are suitable.

I. Heat Conduction in Solids.

The conduction of heat in a solid is governed by following equation, which is known as the **heat equation** or **diffusion equation.**

$$\frac{\partial u}{\partial t} = \lambda \nabla^2 u + q \tag{2.86}$$

Here $u(x, y, z, t)$ denotes the temperature of the solid at the point (x, y, z) and at time t, $q(x, y, z, t)$ is proportional to the source of heat, λ is a constant called the **diffusivity** and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator. In steady state and in absence of heat source, the equation reduces to the Laplace's equation.

$$\nabla^2 u = 0 \tag{2.87}$$

We now take some examples on problems of solution of heat equation with given initial-boundary conditions.

Ex. 2.11.1. Solve the following heat conduction problem in an infinite rod :

(i) $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty,$

(ii) $u(x, 0) = f(x), -\infty < x < \infty$

(iii) $u(x, t), u_x(x, t) \to 0$ as $|x|$ ∞.

Soln. We denote the Fourier transform of $u(x, t)$ with respect to x by $\overline{u}(k, t)$. Then since

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}e^{ikx}dx = \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{ikx}dx$$

$$= \frac{\partial \overline{u}}{\partial t} \text{ and}$$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = (-ik)^2\,\overline{u}, \quad \text{[by (2.15) by the use of (iii),}$$

we get the following by taking Fourier transform of both sides of the given equation (i)

$$\frac{d\overline{u}}{dt} = -\lambda k^2\overline{u} \tag{2.88}$$

Taking Fourier transform of the initial condition (ii) we get,

$$\overline{u}(k,0) = \bar{f}(k). \tag{2.89}$$

where $\bar{f}(k)$ is the Fourier transform of $f(x)$.

The solution of equation (2.88) is

$$\overline{u}(k,t) = Ae^{-\lambda k^2 t} \tag{2.90}$$

where A is a constant, *i.e.*, independent of t, Setting $t = 0$ here we get

$$A = \overline{u}(k,0) = \bar{f}(k), \text{by (2.89)}$$

Therefore $\overline{u}(k,t)$ given by (2.90) becomes

$$\overline{u}(k,t) = \bar{f}(k)e^{-\lambda k^2 t} \tag{2.91}$$

Taking Fourier inversion of this we shall get the desired function $u(x, t)$.

Let $F[g(x)] = e^{\lambda k^2 t} = \bar{g}(k)$

Then by Fourier inversion formula we get,

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\lambda k^2 t - ikx}dx = \frac{e^{-\frac{x^2}{4\lambda t}}}{\sqrt{2\lambda t}}, \tag{2.92}$$

where integration is performed following the Ex. 2.4.2.

Therefore by convolution theorem we get the following from (2.91)

$$u(x,t) = F^{-1}[\bar{f}(k)\bar{g}(k)]$$

$$= \text{convolution of the two functions } f(x) \text{ and } g(x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha)e^{\frac{(x-\alpha)^2}{4\lambda t}} d\alpha \tag{2.93}$$

Ex. 2.11.2. Solve the following heat conduction problem in an infinite rod.

(i) $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} + q(x,t), \ -\infty < x < \infty,$

(ii) $u(x, 0) = 0, \ -\infty < x < \infty,$

(iii) $u(x, t), \ u_x(x, t) \to 0$ as $|x| \to \infty$

Soln. Taking Fourier transform of the given equation (i) we get the following equation by the use of (iii).

$$\frac{d\bar{u}}{dt} = -\lambda k^2 \bar{u} + \bar{q} \tag{2.94}$$

where $\bar{u}(k,t)$ and $\bar{q}(k,t)$ are the Fourier transform with respect to x of $u(x, t)$ and $q(x, t)$ respectively. Taking Fourier transform of the initial condition (ii), we get

$$\bar{u}(k,0) = 0 \tag{2.95}$$

Writing the equation (2.94) as

$$\frac{d}{dt}[\bar{u}e^{\lambda k^2 t}] = \bar{q}(k,t)e^{\lambda k^2 t}$$

and then replacing t by τ we get

$$\frac{d}{d\tau}[\bar{u}(k,\tau)e^{\lambda k^2 \tau}] = \bar{q}(k,\tau)e^{\lambda k^2 \tau}$$

Integrating both sides of this equation with respect of τ between the limits 0 to t and using the condition (2.95), the following equation is obtained.

$$\bar{u}(k,t) = \int_0^t \bar{q}(k,\tau)e^{-\lambda k^2 (t-\tau)} d\tau$$

Fourier inversion of this gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(k,t)e^{-ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{-ikx} \int_0^t \bar{q}(k,\tau) e^{-\lambda k^2(t-\tau)} d\tau,$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} \bar{q}(k,\tau) e^{-\lambda k^2(t-\tau)} . e^{-ikx} dk$$ (changing the order of integration)

$$= \frac{1}{\sqrt{2\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} \bar{q}(k,\tau) \bar{f}(k) e^{-ikx} dx, \tag{2.96}$$

where $\bar{f}(k) = e^{-\lambda k^2(t-\tau)} = F[f(x)]$, say

Therefore, $f(x) = F^{-1}[e^{-\lambda k^2(t-\tau)}]$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda k^2(t-\tau)-ikx} dk$$

$$= \frac{e^{-\frac{x^2}{4\lambda(t-\tau)}}}{\sqrt{2\lambda(t-\tau)}},$$ following Ex. 2.4.2.

Now, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{q}(k,\tau)\bar{f}(k) e^{-ikx} = F^{-1}[\bar{q}(k,\tau)\bar{f}(k)]$

= convolution of the two functions $q(x, \tau)$ and $f(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(\xi,\tau) \frac{e^{-\frac{(x-\xi)^2}{4\lambda(t-\tau)}}}{\sqrt{2\lambda(t-\tau)}} d\xi$$

Therefore from (2.96) we finally get

$$u(x,t) = \int_0^t d\tau \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(\xi,\tau) \frac{e^{-\frac{(x-\xi)^2}{4\lambda(t-\tau)}}}{\sqrt{2\lambda(t-\tau)}} d\xi$$

$$= \int_0^t \frac{d\tau}{\sqrt{4\pi\lambda(t-\tau)}} \int_{-\infty}^{\infty} q(\xi,\tau) e^{-\frac{(x-\xi)^2}{4\lambda(t-\tau)}} d\xi.$$

Ex. 2.11.3. Solve the following heat conduction problem in an infinite rod.

(i) $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} + q(x,t), -\infty < x < \infty$

(ii) $u(x, 0) = f(x), -\infty < x < \infty$

(iii) $u(x, t), u_x(x, t) \to 0$ as $|x| \to \infty$.

Soln. Taking Fourier transform of the given equation (i) with respect to x and using the conditions (iii), we get

$$\frac{d\overline{u}}{dt} + \lambda k^2 \overline{u} = \overline{q} \tag{2.97}$$

where $\overline{u}(k,t)$ and $\overline{q}(k,t)$ are Fourier transforms with respect of x of $u(x, t)$ and $q(x, t)$ respectively:

Taking Fourier transform of the initial condition (ii), we get

$$\overline{u}(k,0) = \bar{f}(k) \tag{2.98}$$

where $\bar{f}(k)$ is the Fourier transform of $f(x)$.

The equation (2.97) can be written as

$$\frac{d}{d\tau}[\overline{u}(k,\tau)e^{\lambda k^2 \tau}] = \overline{q}(k,\tau)e^{\lambda k^2 \tau},$$

where we have replaced t by τ. Integrating this equation with respect to τ between the limits 0 to t and using the initial condition (2.98) we get,

$$\overline{u}(k,t) = \bar{f}(k)e^{-\lambda k^2 t} + \int_0^t \overline{q}(k,\tau)e^{-\lambda k^2(t-\tau)}d\tau$$

$$= \bar{f}(k)\overline{g}(k,t) = \int_0^t \overline{q}(k,\tau)\,\overline{g}(k,t-\tau)\,d\tau$$

where $\overline{g}(k,t) = F[g(x,t)], g(x,t)$ being the function $g(x, t) = \frac{e^{-\frac{x^2}{4\lambda t}}}{\sqrt{2\lambda t}}$, according to Ex. 2.4.2.

$u(x, t)$ can now be obtained by taking the Fourier inversion of $\overline{u}(k,t)$ as follows :

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(k)\overline{g}(k,t)e^{-ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dke^{-ikx}\int_0^t \overline{q}(k,\tau)\overline{g}(k,t-\tau)d\tau$$

$$= F^{-1}[\bar{f}(k)\overline{g}(k,t)] + \int_0^t d\tau \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{q}(k,\tau)\overline{g}(k,t-\tau)e^{-ikx}dx$$

(changing the order of integrating in the second terms)

$$= F^{-1}[\bar{f}(k)\overline{g}(k,t) + \int_0^t d\tau F^{-1}[\overline{q}(k,\tau)\overline{g}(k,t-\tau)]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\alpha f(\alpha).\frac{1}{\sqrt{2\lambda t}}e^{-\frac{(x-\alpha)^2}{4\lambda t}} + \int_0^t d\tau \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} q(\alpha,\tau)\frac{1}{\sqrt{2\lambda(t-\tau)}}e^{-\frac{(x-\alpha)^2}{4\lambda(t-\tau)}}d\alpha$$

(by convolution theorem)

$$= \frac{1}{\sqrt{4\pi\lambda t}}\int_{-\infty}^{\infty} d\alpha f(\alpha)e^{-\frac{(x-\alpha)^2}{4\lambda t}} + \int_0^t \frac{d\tau}{\sqrt{4\pi\lambda(t-\tau)}}\int_{-\infty}^{\infty} q(\alpha,\tau)e^{-\frac{(x-\alpha)^2}{4\lambda(t-\tau)}}d\alpha$$

Ex. 2.11.4. Find the temperature u at time t and at a distance x from one end of a semi-infinite rod satisfying the equation,

(i) $\frac{\partial u}{\partial t} = \lambda\frac{\partial^2 u}{\partial x^2}, 0 < x < \infty,$

and the following initial-boundary conditions :

(ii) $u(x, 0) = f(x)$, $o < x < \infty$

(iii) $u(0, t) = 0,$

(iv) $u, u_x \to 0$ as $x \to 0$

Sonl. Fourier sine transform is appropriate for this example. If F_s $[f]$ denotes the Fourier sine transform of $f(x)$, then

$$F_s\left(\frac{\partial u}{\partial t}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial u}{\partial t}\sin kx dx = \frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,t)\sin kx dx$$

$$= \frac{d\overline{u}_s}{dt}(k,t),$$

where $\overline{u}_s$ (k, t) is the Fourier sine tranform of u (k, t) and

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin kx dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin kx.\frac{\partial u}{\partial x}\right]_{x=0}^{\infty} - \sqrt{\frac{2}{\pi}}k\int_0^\infty \cos kx\frac{\partial u}{\partial x}dx$$

$$= \sqrt{\frac{2}{\pi}}k[\cos kx\, u(x,t)]_{x=0}^{\infty} - \sqrt{\frac{2}{\pi}}k^2\int_0^\infty \sin kxudx$$

$= -k^2\overline{u}_s(k,t)$, by the use of the conditions (iii) and (iv).

Therefore taking Fourier sine transform of the given equation (i), we get

$$\frac{d\overline{u}_s}{dt} = -\lambda k^2\overline{u}_s \qquad (2.99)$$

Also taking Fourier sine transform of initial condition (ii) we get

$$\overline{u}_s(k,0) = \bar{f}_s(k), \qquad (2.100)$$

where $\bar{f}_s(k)$ is the Fourier sine transform of $f(x)$.

Solution of equation (2.99) is

$$\overline{u}_s(k,t) = Ae^{-\lambda k^2 t},$$

where A is a constant, *i.e*, independent of t. Setting $t = 0$ in (3) we find that

$$A = \overline{u}_s(k,0) = \bar{f}_s(k), \text{ by } (2.100)$$

Therefore the solution for $\overline{u}_s(k,t)$ is given by

$$\overline{u}_s(k,t) = \bar{f}_s(k)e^{-\lambda k^2 t}.$$

Taking Fourier sine inversion of this we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(x) e^{-\lambda k^2 t} \sin kx dk$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dk e^{-\lambda k^2 t} \sin kx \sqrt{\frac{2}{\pi}}\int_0^\infty f(\alpha)\sin k\alpha d\alpha,$$

(substituting for $\bar{f}_s(k)$ according to (2.52))

$$= \frac{2}{\pi}\int_0^\infty d\alpha f(\alpha)\int_0^\infty dk e^{-\lambda k^2 t} \sin k\alpha \sin kx \text{ (changing the order of integration.)}$$

$$= \frac{1}{\pi}\int_0^\infty d\alpha f(\alpha)\int_0^\infty dk e^{-\lambda k^2 t}[\cos k(\alpha - x) - \cos k(\alpha + x)]$$

$$= \frac{1}{2\pi}\int_0^\infty d\alpha f(\alpha)\int_0^\infty dk e^{-\lambda k^2 t}[\cos k(\alpha - x) - \cos k(\alpha + x)], \tag{2.101}$$

(being an even function of k.)

Now from Ex. 2.4.2. we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda t k^2} e^{ik(\alpha - x)} dk = \frac{1}{\sqrt{2\pi t}} e^{\frac{(\alpha-x)^2}{4\lambda t}}$$

Replacing a^2, x and k by λt, k and $\alpha - x$ respectively and taking real part we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda t k^2} \cos k(\alpha - x) dk = \frac{1}{\sqrt{2\lambda t}} e^{-\frac{(\alpha-x)^2}{4\lambda t}} \tag{2.101a}$$

Replacing here $\alpha - x$ by $\alpha + x$ we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda t k^2} \cos k(\alpha + x) dk = \frac{1}{\sqrt{2\lambda t}} e^{-\frac{(\alpha+x)^2}{4\lambda t}} \tag{2.101b}$$

Using these relations (2.101a) and (2.101b) we get the following from (2.101)

$$u(x,t)=\frac{1}{\sqrt{4\pi\lambda t}}\int_0^\infty d\alpha\, f(\alpha)\left[e^{-\frac{(x-\alpha)^2}{4\lambda t}}-e^{-\frac{(x+\alpha)^2}{4\lambda t}}\right]$$

Ex. 2.11.5. Find the temperature u at time t and at a distance x from one end of a semi-infinte rod satisfying the equation,

(i) $\frac{\partial u}{\partial t}=\lambda\frac{\partial^2 u}{\partial x^2},\ 0\le x<\infty$

and the following initial-boundary conditions :

(ii) $u\ (0,\ t) = f(t)$

(iii) $u\ (x,\ 0) = 0,\ 0 < x < \infty$

(iv) $u,\ u_x \to 0$ as $x \to \infty$.

Soln. Fourier sine transform is appropriate for this example. Using the same notation and following the same procedure as in the previous example we get

$$F_s\left[\frac{\partial u}{\partial t}\right]=\frac{d\overline{u}_s}{dt}$$

and $F_s\left[\frac{\partial^2 u}{\partial x^2}\right]=+\sqrt{\frac{2}{\pi}}ku(0,t)-k^2\overline{u}(k,t)$

$$=\sqrt{\frac{2}{\pi}}kf(t)-k^2\overline{u}(k,t),\ \text{by the use of (ii) and (iv)}$$

Therefore, if we take Fourier sine transform of the given equation (i) and the initial condition (iii) we shall get respectively the following two equations.

$$\frac{d\overline{u}_s}{dt}+\lambda k^2\overline{u}_s=\sqrt{\frac{2}{\pi}}\lambda kf(t)$$

$$\overline{u}_s(k,0)=0$$

The equation (i) can be written as

$$\frac{d}{d\tau}\left(\overline{u}_s e^{\lambda k^2\tau}\right)=\sqrt{\frac{2}{\pi}}\lambda kf(\tau)e^{\lambda k^2\tau},$$

where we have replaced t by τ. Integrating both sides of this equation with respect to τ between the limits 0 to t we get the following by the use of the initial condition

$$\overline{u}_s(k,t) = \sqrt{\frac{2}{\pi}}\lambda k \int_0^t f(\tau)e^{-\lambda k^2(t-\tau)}d\tau$$

Fourier sine inversion of this gives

$$u(x,t) = \frac{2\lambda}{\pi}\int_0^\infty dk\, k \sin kx \int_0^\infty f(\tau)e^{-\lambda k^2(t-\tau)}d\tau,$$

which becomes the following after changing the order of integration

$$u(x,t) = \frac{2\lambda}{\pi}\int_0^t d\tau\, f(\tau)\int_0^\infty dk\, ke^{-\lambda k^2(t-\tau)}\sin kx$$

$$= \frac{\lambda}{\pi}\int_0^t d\tau\, f(\tau)\int_{-\infty}^\infty dk\, ke^{-\lambda k^2(t-\tau)}\sin kx, \tag{2.102}$$

(since the integrand of the last integral in an even function of k.)

Now from Ex.2.4.2. we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\lambda k^2(t-\tau)}.e^{ikx}dk = \frac{1}{\sqrt{2\lambda(t-\tau)}}e^{-\frac{x^2}{4\lambda(t-\tau)}}$$

by replacing a^2 by $\lambda(t-\tau)$, x by k and k by x. Taking real part of this relation we get

$$\int_{-\infty}^\infty e^{-\lambda k^2(t-\tau)}\cos kxdk = \sqrt{\frac{\pi}{\lambda(t-\tau)}}e^{-\frac{x^2}{4\lambda(t-\tau)}}$$

which gives the following by differentiating with respect to x under sign of integration.

$$\int_{-\infty}^\infty ke^{-\lambda k^2(t-\tau)}\sin kxdk = \frac{\sqrt{\pi}x}{2[\lambda(t-\tau)]^{\frac{3}{2}}}e^{-\frac{x^2}{4\lambda(t-\tau)}}$$

By the use of this relation, (2.102) can be expressed as

$$u(x,t)=\frac{x}{\sqrt{4\pi\lambda}}\int_0^t \frac{f(\tau)}{(t-\tau)^{\frac{3}{2}}}e^{-\frac{x^2}{4\lambda(t-\tau)}}d\tau$$

Ex. 2.11.6. Find temperature u at time t and at a distance x from one end of a semi-infinite rod satisfying the equation.

(i) $\frac{\partial u}{\partial t}=\lambda\frac{\partial^2 u}{\partial x^2},\ 0\le x<\infty$

and the following initial-boundary conditions :

(ii) $u_x(0, t) = -f(t)$,

(iii) $u(x, 0) = 0$,

(iv) $u, u_x \to 0$ as $|x| \to \infty$

Soln. Fourier cosine transform is appropriate here. If $F_c[g]$ denotes the Fourier cosine transform of a function $g(x)$, then

$$F_c\left[\frac{\partial u}{\partial t}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos kx dx=\frac{\partial}{\partial t}\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\cos kx dx$$

$= \frac{d\overline{u}_c}{dt}(k,t)$ where $\overline{u}_c\ (k,\ t)$ is the Fourier cosine transform of $u(x, t)$ and

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos kx dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\cos kx\frac{\partial u}{\partial x}\right]_0^\infty+\sqrt{\frac{2}{\pi}}k\int_0^\infty \frac{\partial u}{\partial x}\sin kx dx$$

$$=-\sqrt{\frac{2}{\pi}}u_x(0,t)+\sqrt{\frac{2}{\pi}}[ku\sin kx]_0^\infty-\sqrt{\frac{2}{\pi}}k^2\int_0^\infty u\cos kx dx$$

$$=\sqrt{\frac{2}{\pi}}f(t)-k^2\overline{u}_c$$

Therefore taking Fourier cosine transform of the given equation (i) and the initial conditions (ii), we get

$$\frac{d\overline{u}_c}{dt} + \lambda k^2 \overline{u}_c = \lambda\sqrt{\frac{2}{\pi}}f(t)$$

$$\overline{u}_c(k,0) = 0$$

The above equation can be written as

$$\frac{d}{d\tau}\left(\overline{u}_c e^{\lambda k^2 \tau}\right) = \lambda\sqrt{\frac{2}{\pi}}f(\tau)e^{\lambda k^2 \tau}$$

where we have replaced t by τ,

Integrating this equation with respect to τ between the limits 0 to t we get

$$\overline{u}_c(k,t)e^{\lambda k^2 t} - \overline{u}_c(k,0) = \int_0^t \lambda\sqrt{\frac{2}{\pi}}f(\tau)e^{\lambda k^2 \tau}d\tau$$

$$\text{or,}\quad \overline{u}_c(k,t) = \lambda\sqrt{\frac{2}{\pi}}\int_0^t f(\tau)e^{-\lambda k^2(t-\tau)}d\tau, \text{ by (2)}$$

Taking Fourier cosine inversion of this the following expression $u(x, t)$ is obtained :

$$u(x,\ t) = \frac{2\lambda}{\pi}\int_0^\infty dk\cos kx\int_0^t f(\tau)e^{-\lambda k^2(t-\tau)}d\tau$$

$$= \frac{2\lambda}{\pi}\int_0^t d\tau f(\tau)\int_0^\infty e^{-\lambda k^2(t-\tau)}\cos kx dk$$

$$= \sqrt{\frac{\lambda}{\pi}}\int_0^t d\tau\frac{f(\tau)}{\sqrt{(t-\tau)}}e^{\frac{x^2}{4\lambda(t-\tau)}}$$

[Since taking real part of the relation,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda(t-\tau)k^2}e^{ixk}dk = \frac{1}{\sqrt{2\lambda(t-\tau)}}e^{\frac{x^2}{4\lambda(t-\tau)}}$$

obtaind from Ex. 2.4.2. by replacing a^2, k, x by $\lambda(t - \tau)$, x, k respectively, we get

$$\int_0^{\infty} e^{-\lambda(t-\tau)k^2} \cos kx dk = \sqrt{\frac{\pi}{4\lambda(t-\tau)}} e^{-\frac{x^2}{4\lambda(t-\tau)}} \Bigg]$$

Ex. 2.11.7. Solve the following heat conduction problem in a semi-infinite medium

(i) $\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}, 0 \le x < \infty$

(ii) $u(0, t) = u_0$, for $|t| \le T$
$= 0$, for, $|t| > T$

(iii) $u \to 0$ as $x \to \infty$

(iv) $u \to 0$ as $t \to \pm \infty$

Here we shall take Fourier transform with respect to t. Since.

$$F\left[\frac{\partial u}{\partial t}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{ikt} dt$$

$$= e^{ikt} u(x,t)\Big|_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty} iku(x,t)e^{ikt} dt$$

$$= -ik\overline{u}(x,k), \text{ by condition (iv)}$$

and $$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{ikt} dt = \frac{d^2}{dx^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{ikt} dt$$

$$= \frac{d^2 \overline{u}(x,k)}{dx^2},$$

so, taking Fourier transform of the given equation with respect to t, we get

$$\frac{d^2 \overline{u}}{dx^2} = -\frac{ik}{\lambda} \overline{u} \tag{2.103}$$

Also the Fourier transform of the boundary conditin (ii) with respect to t gives

$$\overline{u}(0,k)=\frac{1}{\sqrt{2\pi}}\int_{-T}^{T}u_0e^{ikt}=\frac{2u_0}{\sqrt{2\pi}}\frac{\sin kT}{k} \tag{2.103a}$$

Taking fourier transform of the boundary condition (iii) with respect to t we get

$$\overline{u}(x,k)\to 0 \text{ as } x\to\infty \tag{2.103b}$$

(a) For $k > 0$, we have $\sqrt{-\frac{ik}{\lambda}}=\sqrt{\frac{|k|}{\lambda}}\left(e^{-i\frac{\pi}{2}}\right)^{\frac{1}{2}}=\sqrt{\frac{|k|}{2\lambda}}(1-i)$, since $k = |k|$

Therefore solution of equation (2.103) in this case becomes

$$\overline{u}(x,k)=Ae^{-x\sqrt{\frac{|k|}{2\lambda}}(1-i)}+Be^{x\sqrt{\frac{|k|}{2\lambda}}(1-i)},$$

where A and B are two constant,

Due to the condition (2.103b) we must have $B = 0$. Hence, the solution of equation (2.103) becomes

$$\overline{u}(x,k)=Ae^{-x\sqrt{\frac{|k|}{2\lambda}}(1-i)}\text{for } k>0 \tag{2.103c}$$

(b) For $k < 0$, we have $\sqrt{-\frac{ik}{\lambda}}=\sqrt{\frac{|k|}{\lambda}}\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}}=\sqrt{\frac{|k|}{2\pi}}(1+i)$, since $k = -|k|$.

Therefore the solution of equation (2.103) in this case becomes

$$\overline{u}(x,k)=A'e^{-x\sqrt{\frac{|k|}{2\lambda}}(1+i)}+B'e^{x\sqrt{\frac{|k|}{2\lambda}}(1+i)},$$

where A' and B' are two constants. Due to condition (2.103b) we must have $B' = 0$. Hence the solution of equation (2.103) becomes

$$\overline{u}(x,k)=A'ex^{-\sqrt{\frac{|k|}{2\lambda}}(1+i)} \text{ for } k<0 \tag{2.103d}$$

The equation (2.103a) can be written as

$$\overline{u}(0,k)=\frac{2u_0}{\sqrt{2\pi}}\frac{\sin(|k|T)}{|k|}, \text{ for both } k>0 \text{ and } k<0$$

Setting $x = 0$ in (2.103c) we get

$$A = \overline{u}(0,k) = \frac{2u_0}{\sqrt{2\pi}}\frac{\sin(|k|T)}{|k|} \quad (6).$$

Similarly from (2.103d) we get

$$A' = \frac{2u_0}{\sqrt{2\pi}}\frac{\sin(|k|T)}{|k|}.$$

Therefore the solution of equation (2.103) becomes

$$\overline{u}(x,k) = \frac{2u_0}{\sqrt{2\pi}}\frac{\sin(|k|T)}{|k|}e^{-x\sqrt{\frac{|k|}{2\lambda}}(1-i)}, \; k > 0$$

$$= \frac{2u_0}{\sqrt{2\pi}}\frac{\sin(|k|T)}{|k|}e^{-x\sqrt{\frac{|k|}{2\lambda}}(1+i)}, \; k < 0$$

Taking Fourier inversion of this we finally get $u(x, t)$ as follows :

$$u(x,\, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{u}(x,k)e^{-ikt}dk$$

$$= \frac{u_0}{\pi}\int_{-\infty}^{0}\frac{\sin(|k|T)}{|k|}e^{-x\sqrt{\frac{|k|}{2\lambda}}(1+i)-ikt}dk + \frac{u_0}{\pi}\int_{0}^{\infty}\frac{\sin(|k|T)}{|k|}e^{-x\sqrt{\frac{|k|}{2\lambda}}(1-i)-ikt}dk$$

$$= \frac{u_0}{\pi}\int_{0}^{\infty}dk\frac{\sin(|k|T)}{|k|}e^{-x\sqrt{\frac{|k|}{2\lambda}}}[e^{i(x\sqrt{\frac{|k|}{2\lambda}}-kt)} + e^{-(x\sqrt{\frac{|k|}{2\lambda}}-kt)}]$$

$$= \frac{2u_0}{\pi}\int_{0}^{\infty}dk\frac{\sin(kT)}{k}e^{-x\sqrt{\frac{k}{2\lambda}}}\cos\sqrt{\frac{k}{2\lambda}}\left(x - \sqrt{2\lambda k}t\right)$$

since within the interval of integration $(0, \infty)$, we can write $|k| = k$.

Ex. 2.11.8. Find the solution of the following heat conduction problem in a rod of finite length.

(i) $\frac{\partial u}{\partial t} = \lambda\frac{\partial^2 u}{\partial x^2}$, $0 < x < a$, $t = 0$,

(ii) $u(0, t) = f(t)$, $t > 0$, $f(0) = 0$

(iii) $u(a, t) = 0, t > 0,$

(iv) $u(x, 0) = 0, \quad 0 \le x \le a.$

Soln. In this problem finite Fourier sine transform with respect to x is appropriate. If by $F_{fs}[g(x)]$ or g_s we mean finite Fourier sine transform of $g(x)$, then

$$(1)\ F_{fs}\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^a \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{a} dx$$

$$= \sin\frac{n\pi x}{a} \cdot \frac{\partial u}{\partial x}\bigg|_{x=0}^{a} - \frac{n\pi}{a}\int_0^a \cos\frac{n\pi x}{a}\frac{\partial u}{\partial x} dx$$

$$= -\frac{n\pi}{a}\cos\frac{n\pi x}{a} u(x,t)\bigg|_{x=0}^{a} - \frac{n^2\pi^2}{a^2}\int_0^a \sin\frac{n\pi x}{a} u dx$$

$$= \frac{n\pi}{a} f(t) - \frac{n^2\pi^2}{a^2}\overline{u}_s(n,t), \text{ since by (ii) } u(0, t) = f(t),$$

$$(2)\ F_{fs}\left[\frac{\partial u}{\partial t}\right] = \int_0^a \frac{\partial u}{\partial t}\sin\frac{n\pi x}{a} dx$$

$$= \frac{\partial}{\partial t}\int_0^a u \sin\frac{n\pi x}{a} dx = \frac{d\overline{u}_s}{dt}(n,t)$$

Therefore taking finite Fourier sine transform of the given equation (i) and the initial condition (ii) we get

$$\frac{d\overline{u}_s}{dt}(n,t) = -\frac{n^2\pi^2\lambda}{a^2}\overline{u}_s(n,t) + \frac{\lambda n\pi}{a} f(t)$$

Also taking Finite Fourier sine transform of the initial condition (iv) we get

$$\overline{u}_s(n,0) = 0$$

The above equation can be written

$$\frac{d}{d\tau}\left[e^{\frac{n^2\pi^2\lambda t}{a^2}}\,\overline{u}(n,t)\right] = \frac{\lambda n\pi}{a} f(\tau) e^{\frac{n^2\pi^2\lambda t}{a^2}},$$

where we have replaced t by τ. Now integrating this equation with respect to τ between the limits 0 to t and using the initial condition, the following expression for $\overline{u}_s(n,t)$ is obtained,

$$\overline{u}_s(n,t) = \frac{\lambda n\pi}{a}\int_0^t f(\tau)e^{-\frac{n^2\pi^2\lambda}{a^2}(t-\tau)}d\tau$$

The Fourier inversion of this according to the formula (2.70) gives

$$u(x,t) = \frac{2}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\overline{u}_s(n,t)$$

$$= \frac{2\lambda\pi}{a^2}\sum_{n=1}^{\infty} ne^{-\frac{n^2\pi^2\lambda}{a^2}t}\sin\frac{n\pi x}{a}\int_0^t d\tau f(\tau)e^{\frac{n^2\pi^2 t}{a^2}\tau}$$

Ex. 2.11.9. Solve the following problem for stationary temperature distribution in a semi-infinite body.

(i) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial u^2}=0,\ -\infty < x < \infty, y \geq 0,$

(ii) $u(x,\ 0) = f(x),\ \ -\infty < x < \infty$

(iii) $u(x,\ y) \to 0$, as $y \to \infty$

(iv) $u(x,\ y),\ u_x(x,\ y) \to 0,\ |x| \to \infty$

Taking the Fourier transform of (i), (ii) and (iii) with respect to x we get the following equations :

$$\frac{d^2\overline{u}}{dy^2} - k^2\overline{u} = 0 \tag{2.104}$$

$\overline{u}(k,0) = \bar{f}(k)$ (initial condition)

$\overline{u}(k,y) \to 0$ as $y \to \infty$ (regularity condition)

where $\overline{u}(k,y)$ and $\bar{f}(k)$ are the Fourier transforms of the function $u(x,\ y)$ and $f(x)$ respectively and inderiving (2.104) we have used the conditions (iv) and the formula (2.15)

The solution of equation (2.104) is

$$\overline{u}(k,y) = Ae^{-|k|y} + Be^{|k|y}$$

where A and B are two constants. Due to regularity condition we must have $B = 0$ and consequently the solution for $\overline{u}(k, y)$ becomes

$$\overline{u}(k, y) = Ae^{-|k|y}$$

Setting here y = 0, we get

$$A = \overline{u}(k,0) = \overline{f}(k), \text{ (by the initial condition)}$$

Hence, $\overline{u}(k, y) = \overline{f}(k)e^{-|k|y}$ (2.104a)

Let $F[g(x)] = e^{-|k|y} = \overline{g}(k)$ (2.104b)

Taking Fourier inversion of this we get

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|k|y - ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ky - ikx} dk + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ky - ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-k(y - ix)} dk \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-k(y + ix)} dk$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y - ix} + \frac{1}{y + ix} \right] = \sqrt{\frac{2}{\pi}} \frac{y}{y^2 + x^2} \quad (2.104c)$$

By the use of (2.104b), the equation (2.104a) can be written as

$$\overline{u}(k, y) = \overline{f}(k)\overline{g}(k)$$

and therefore by convolution theorem we get

$u(x, y)$ = convolution of the two functions $f(x)$ and $g(x)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x - \xi) d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) \sqrt{\frac{2}{\pi}} \frac{y}{y^2 + (x - \xi)^2} d\xi \text{ (by 2.104c)}$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (x - \xi)^2} d\xi$$

Ex. 2.11.10. The steady state temperature distribution $u(x, y)$ in any cross-section of a long square bar with one face held at a constant temperature T_0 and the other faces held at zero temperature is governed by the following equation and boundary conditions, Find $u(x, y)$.

(i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < a.$

(ii) $u(0, y) = 0, \quad u(a, y) = 0,$

(iii) $u(x, 0) = 0, \quad u(x, a) = T_0$

Soln. Finite Fourier sine transform is appropriate in this example. Using the same notation as in Ex. 2.11.18. we get

$$F_{fs}\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^a \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{a}\,dx$$

$$= \sin\frac{n\pi x}{a}\frac{\partial u}{\partial x}\bigg|_{x=0}^{a} - \int_0^a \frac{n\pi}{a}\cos\frac{n\pi x}{a}\frac{\partial u}{\partial x}\,dx$$

$$= -\frac{n\pi}{a}\cos\frac{n\pi x}{a}u(x,y)\bigg|_{x=a}^{0} - \frac{n^2\pi^2}{a^2}\int_0^a \sin\frac{n\pi x}{a}u\,dx$$

$$= -\frac{n\pi}{a}u(0,y) + \frac{n\pi}{a}(-1)^n u(a,y) - \frac{n^2\pi^2}{a^2}\overline{u}_s(n,y)$$

$$= -\frac{n^2\pi^2}{a^2}\overline{u}_s(n,y), \text{ by the boundary conditions (ii)}$$

and $$F_{fs}\left[\frac{\partial^2 u}{\partial y^2}\right] = \int_0^a \frac{\partial^2 u}{\partial y^2}\sin\frac{n\pi x}{a}\,dx - \frac{\partial^2}{\partial y^2}\int_0^a u\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{d^2\overline{u}_s}{dy^2}(n,y)$$

Therefore taking finite Fourier transform of the given equation (i) and the boundary conditions (iii) we get

$$\frac{d^2\overline{u}_s}{dy^2} - \frac{n^2\pi^2\overline{u}_s}{a^2} = 0 \tag{2.105}$$

$$\overline{u}_s(n,0) = 0 \tag{2.105a}$$

$$\overline{u}_s(n,a) = T_0\int_0^a \sin\frac{n\pi x}{a}\,dx = -\frac{T_0 a}{n\pi}\left[(-1)^n - 1\right]$$

$$= 0,\ \frac{2T_0 a}{n\pi}\ \text{(according as n is even or odd)} \tag{2.105b}$$

Solution of the above equation (2.105) is

$$\overline{u}_s(n,y) = Ae^{\frac{n\pi y}{a}} + Be^{\frac{n\pi y}{a}} \tag{2.105c}$$

where A and B are two constants.

Setting $y = 0$ in (2.105c) we get the following by the use of (2.105c)(2.105a)

$$A + B = \overline{u}_s(n,0) = 0$$

Next setting y = a in (2.105c) get the following by the use (2.105b)

$$Ae^{n\pi} + Be^{-n\pi} = \overline{u}_s(n,a) = 0 \text{ or } \frac{2T_0 a}{n\pi} \text{ according as n is even or odd.}$$

Solving A and B we get

$$\left.\begin{array}{ll} A = -B = \dfrac{T_0 a}{n\pi\sinh n\pi}, & \text{when } n \text{ is odd} \\ A = B = 0, & \text{when } n \text{ is even} \end{array}\right\}$$

Therefore, (2.105c) gives the following solution for $\overline{u}_s(n,y)$:

$$\overline{u}_s(n,y) = \frac{2T_0 a \sin h\frac{n\pi y}{a}}{n\pi\sinh n\pi},\ \text{for } n \text{ odd}$$

$$= 0,\ \text{for } n \text{ even.}$$

Taking inversion of this we get the soluton for $u(x, y)$ as given below

$$u(x,y) = \frac{2}{a}\sum_{\substack{n=1\\(odd)}}^{\infty} \frac{2T_0 a}{n\pi}\frac{\sinh\frac{n\pi y}{a}}{\sinh n\pi}$$

$$= \frac{4T_0}{\pi} \sum_{\substack{n=1 \\ (odd)}}^{\infty} \frac{\sinh \frac{n\pi y}{a}}{n \sinh n\pi}$$

Ex. 2.11.11. The end of a semi-infinite cylinder $0 \le r \le a$, $0 \le z \le \infty$ is held at a constant temperature T_0, while the cylinder surface is held at zero temperature. Show that the steady temperature is given by

$$u(r,z) = T_0\left[1 - \frac{2}{\pi}\int_0^{\infty} \frac{I_0(kr)}{I_0(ka)} \cdot \frac{\sin kz}{k}\, dk\right],$$

where $I_0\ (x)$ is modified Bessel function.

Soln. In steady state the temperature $u\ (x,\ y,\ z)$ satisfies the equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

If the axis of the cylinder is along z-axis, then obviously u is symmetric about z-axis. So in cylindrical co-ordinates this equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ 0 \le r \le a,\ 0 \le z \le \infty \tag{2.106}$$

The boundary conditions are

$$u\ (r,\ 0) = T_0, \tag{2.106a}$$

$$u\ (0,\ z) = 0 \tag{2.106b}$$

The Fourier sine transform with respect to z is appropriate in this problem

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\partial^2 u}{\partial z^2} \sin kz\, dz = \sqrt{\frac{2}{\pi}} \sin kz \frac{\partial}{\partial z}\{u(r,z)\}\Bigg|_{z=0}^{\infty} - \sqrt{\frac{2}{\pi}}k\int_0^{\infty} \cos kz \frac{\partial u}{\partial z}\, dz$$

$$= -\sqrt{\frac{2}{\pi}}k \cos kz.\, u(r,z)\Bigg|_{z=0}^{\infty} - k^2\sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin kz.\, u(r,z) dz$$

$= k\sqrt{\frac{2}{\pi}}u(r,a) - k^2\overline{u}_s(r,k) = T_0 k\sqrt{\frac{2}{\pi}} - k^2\overline{u}_s(r,k)$, by (2.106a). Here we assume that $u(r, z)$, $u_z\ (r, z) \to 0$ as $z \to \infty$.

Also $\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \frac{\partial^2}{\partial r^2}u(r,z)\sin kzdz = \frac{\partial^2}{\partial r^2}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty u(r,z)\sin kzdz$

$$= \frac{d^2\overline{u}_s}{dr^2}(r,k)$$

and $\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \frac{1}{r}\frac{\partial}{\partial r}u(r,z)\sin kz = \frac{1}{r}\frac{\partial}{\partial r}\sqrt{\frac{2}{\pi}}\int\limits_0^\infty u(r,z)\sin kzdz$

$$= \frac{1}{r}\frac{d\overline{u}_s}{dr}(r,k)$$

Therefore taking Fourier sine transform of (2.106) with respect to z we get

$$\frac{d^2\overline{u}_s}{dr^2} + \frac{1}{r}\frac{d\overline{u}_s}{dr} - k^2\overline{u}_s = -T_0 k\sqrt{\frac{2}{\pi}} \tag{2.107}$$

Taking Fourier sine transform of (2.106b) with respect to z we get

$$\overline{u}_s(a,k) = 0$$

To solve (2.107) we set

$$\overline{u}_s(r,k) = \sqrt{\frac{2}{\pi}}\frac{T_0}{k} + \overline{v}_s(r,k) \tag{2.107a}$$

Substituing this in (2.107) we find that $\overline{v}_s(r,k)$ satisfies the equation,

$$\frac{d^2\overline{v}_s}{dr^2} + \frac{1}{r}\frac{d\overline{v}_s}{dr} - k^2\overline{v}_s = 0 \tag{2.107b}$$

The condition $\overline{u}_s(a,k) = 0$ gives the following condition for $\overline{v}_s(a,k)$.

$$\overline{v}_s(a,k) = -\sqrt{\frac{2}{\pi}}\frac{T_0}{k} \tag{2.107c}$$

If we set $s = ikr$ in (2.107b), then the equation for $\overline{v}_s(r,k)$ becomes

$$\frac{d^2\overline{v}_s}{ds^2} + \frac{1}{s}\frac{d\overline{v}_s}{ds} + \overline{v}_s = 0, \tag{2.108}$$

which is the Bessel's equation of order zero and therefore its solution, which remains finite at $r = 0$ *i.e.*, at $s = 0$ is

$$\overline{v}_s(r,k) = AJ_0(ikr) = AI_0(kr)$$

where $I_0(rk)$ is the modified Bessel's function of order zero.

Due to the condition (1.207c) we have

$$-\sqrt{\frac{2}{\pi}}\frac{T_0}{\pi} = AI_0(ka)$$

Therefore, $A = -\sqrt{\frac{2}{\pi}}\frac{T_0}{k} / I_0(ka)$

and consequently

$$\overline{v}_s(r,k) = -\sqrt{\frac{2}{\pi}}\frac{T_0}{k_0}\cdot\frac{I_0(kr)}{I_0(ka)},$$

which again gives the following expression for $\overline{u}_s(r,k)$ for (2.107a) :

$$\overline{u}_s(r,k) = \sqrt{\frac{2}{\pi}}T_0\left[\frac{1}{k} - \frac{1}{k}\frac{I_0(kr)}{I_0(ka)}\right]$$

Fourier sine inversion of this gives

$$u(r,z) = \frac{2}{\pi}T_0\int_0^\infty\left[\frac{1}{k} - \frac{1}{k}\frac{I_0(kr)}{I_0(ka)}\right]\sin kz dk$$

$$= T_0\left[1 - \frac{2}{\pi}\int_0^\infty\frac{I_0(kr)}{I_0(ka)}\cdot\frac{\sin kz}{k}dk\right]$$

where we have used the result $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$

Ex. 2.11.12. Find the solution of the following heat conduction problem.

(i) $\frac{\partial u}{\partial t} = \lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), -\infty < x < \infty, -\infty < y < \infty$

(ii) $u\ (x, y, 0) = f\ (x, y)$

(iii) $u, u_x, u_y \to 0$ as $|x| \to \infty, |y| \to \infty$

Soln. We take Fourier transform both with respect to x and y according to the definition,

$$\overline{u}(l,m,t) = \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)e^{i(lx+my)}dx\,dy$$

$$\overline{f}(l,m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{i(lx+my)}dx\,dy$$

Then from equation (i) and the boundary condition (ii) and the condition (iii) we get

$$\frac{d\overline{u}}{dt} = -\lambda(l^2 + m^2)\overline{u}, \tag{2.109}$$

$$u(l,m,0) = \overline{f}(l,m) \tag{2.109a}$$

Solution of equation (2.109) is

$$\overline{u}(l,m,t) = Ae^{-\lambda\left(l^2+m^2\right)t}$$

where A is a constant. Setting here $t = 0$ we get

$$A = \overline{u}(l,m,0) = \overline{f}(l,m),$$

So that

$$\overline{u}(l,m,t) = \overline{f}(l,m,)e^{-\lambda\left(l^2+m^2\right)t}$$

$$= \overline{f}(l,m,)\overline{g}(l,m) \tag{2.109b}$$

where $\overline{g}(l,m) = e^{-\lambda\left(l^2+m^2\right)t} = F[g(x,y)]$, say

Taking Fourier inversion of this we get the function $g(x, y)$:

$$g(x,y) = \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda\left(l^2+m^2\right)t} . e^{-i(lx+my)} dl\, dm$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda t l^2 - ilx} dl . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda t m^2 - im} dm$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4\lambda t}} . \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{4\lambda t}} = \frac{1}{2\lambda t} e^{-\frac{x^2+y^2}{4\lambda t}} \quad \text{[by Ex. 2.4.1.]}$$

Now taking Fourier inversion of (2.109b) the following expression for $u(x, y, t)$ is obtained.

$u(x, y, t)$ = convolution of the two functions $f(x, y)$ and $g(x, y)$, by (2.79)

$$= \frac{1}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)\, g\,(x-\xi, y-\eta) d\xi\, d\eta$$

$$= \frac{1}{4\pi\lambda t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4\lambda t}} d\xi d\eta$$

II MECHANICAL VIBRATIONS

Mechanical vibration in a medium are governed by the following equations, which is known as **wave equation.**

$$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t} = q$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator, $u(x, y, z, t)$ is the displacement of a material particle at time t from its unperturbed position (x, y, z), $q(x, y, z, t)$ is proportional to the external force acting on the system under consideration and c is a constant equal to the velocity of wave propagation in the system.

In particular the transverse vibrations of stretched strings and membranes are governed by the equations

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = q$$

and $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = q$$

respectively.

On the other hand transverse vibrations of elastic bars and thin elastic plates are governed by the equations

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = P$$

and $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u + \frac{1}{b^2}\frac{\partial^2 u}{\partial t^2} = P$$

respectively, where a and b are two constants, u is the transverse displacement of the beam or plate and P is proportional to the vertical load acting on the beam or plate.

Ex. 2.11.13. Find the solution of the following problem of free vibration of a stretched string of infinite length.

(i) $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \; -\infty < x < \infty$

(ii) $u\ (x,\ 0) = f(x)$

(iii) $u_t\ (x,\ 0) = g(x)$

(iv) $u,\ u_x \to 0$ as $|x| \to \infty$

Soln. Using the conditions (iv) we find that the Fourier transform of the equation (i) with respect to x is

$$k^2\overline{u} + \frac{1}{c^2}\frac{d^2\overline{u}}{dt^2} = 0 \tag{2.110}$$

where we have used the formula (2.15). Here $\overline{u}(k,t)$ denotes the Fourier transform of $u(x,\ t)$ with respect to x.

Taking Fourier transform of the initial conditions (ii) and (iii) we get

$$\overline{u}(k,0) = \overline{f}(k)$$

$$\frac{d\overline{u}}{dt}(k,0) = \overline{g}(k)$$

where $\overline{f}(k)$ and $\overline{g}(k)$ denote the Fourier transform of $f(x)$ and $g(x)$ respectively

Solution of the equation (2.110) is

$$\overline{u}(k,t) = Ae^{ikct} + Be^{-ikct}$$

where A and B are two constants, and therefore

$$\overline{u}_t(k,t) = ikcAe^{ikct} - ikce^{-ikct}$$

Seeting $t = 0$ in the above we get the following two equations by the use of

$$A + B = \overline{u}(k,0) = \overline{f}(k)$$

$$ick(A - B) = \overline{u}_t(k,0) = \overline{g}(k)$$

Solving these two equations for A and B we get

$$A = \frac{1}{2}\left[\overline{f}(k) - \frac{i}{kc}(\overline{g})(k)\right], B = \frac{1}{2}\left[\overline{f}(k) + \frac{i}{kc}\overline{g}(k)\right]$$

Using these expressions for A and B we get for $\overline{u}(k,t)$ the expression,

$$\overline{u}(k,t) = \frac{1}{2}\overline{f}(k)\left(e^{ikct} + e^{-ikct}\right) - \frac{i}{2kc}\overline{g}(k)\left(e^{ikct} - e^{ikct}\right)$$

Taking Fourier inversion of this we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{u}(k,t)e^{-ikx}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f}(k)e^{-ik(x-ct)}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f}(k)e^{-ik(x+ct)}dh\right]$$

$$+ \frac{1}{2c}\left[-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\overline{g}(k)}{k}\left\{e^{-ik(x-ct)} - e^{-ik(x+ct)}\right\}dk\right] \qquad (2.110a)$$

Now in the inversion formula.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(k) e^{-ikx} dk$$

replacing x by $x - ct$ and $x + ct$ we get respectively the relations,

$$f(x - ct) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \bar{f}(k) e^{-ik(x-ct)}$$

and $$f(x + ct) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \bar{f}(k) e^{-ik(x+ct)} \quad (2.110b)$$

Next integrating the inversion formula,

$$g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(k) e^{-ik\xi} d\xi$$

with respect to ξ between the limits $x - ct$ to $x + ct$, we get

$$\int_{x-ct}^{x+ct} g(\xi) d\xi = \int_{x-ct}^{x+ct} d\xi \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(k) e^{-ik\xi} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \bar{g}(k) \int_{x-ct}^{x+ct} d\xi e^{-ik\xi} x, \text{ (changing the order of integration)}$$

$$= -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\bar{g}(k)}{k} \left[e^{-ik(x-ct)} - e^{-ik(x+ct)} \right] \quad (2.110c)$$

Using the relations (2.110b) and (2.110c), $u(x, t)$ given by (2.110a) can be expressed as

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi$$

Ex. 2.11.14. Find the solution of the following problem of free vibration of a semi-infinite stretched string.

(i) $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \ 0 \le x < \infty$

(ii) $u(x, 0) = f(x), \quad 0 \le x < \infty$

(iii) $u_t(x, 0) = g(x) \quad 0 \le x < \infty$

(iv) $u(0, t) = 0$

(v) $u, u_x \to 0$ as $x \to \infty$

Soln. Appropriate integral transform here is Fourier sine transform. Now

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin kx\,dx = \sqrt{\frac{2}{\pi}}\sin kx\frac{\partial u}{\partial x}\bigg|_0^\infty - \sqrt{\frac{2}{\pi}}k\int_0^\infty \cos kx\frac{\partial u}{\partial x}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}k\cos kx\, u(x,t)\bigg|_0^\infty - k^2\sqrt{\frac{2}{\pi}}\int_0^\infty \sin kx\, u(x,t)dx$$

$$= k\sqrt{\frac{2}{\pi}}u(0,t) - k^2\overline{u}_s(k,t) = -k^2\overline{u}_s(k,t), \text{ by (iv)}$$

and $$\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty \frac{\partial^2 u}{\partial t^2}\sin kx\,dx = \frac{\partial^2}{\partial t^2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty u(x,t)\sin kx\,dx$$

$$= \frac{d^2\overline{u}_s}{dt^2}$$

Here $\overline{u}_s(k,t)$ denotes the Fourier sine transform of $u(x, t)$. Due to these relations, we get the following equation from the equation (i), if we take its Fourier sine transform.

$$\frac{d^2\overline{u}_s}{dt^2} = -c^2k^2\overline{u}_s \tag{2.111}$$

Taking Fourier sine transform of the initial conditions (ii) and (iii) we get the following equations,

$$\overline{u}_s(k,0) = \overline{f}_s(k),$$

$$\frac{\partial\overline{u}_s}{\partial t}(k,0) = \overline{g}_s(k),$$

where $\overline{f}_s(k)$ and $\overline{g}_s(k)$ are the Fourier sine transform of $f(x)$ and $g(x)$ respectively. (2.111a)

The solution of (2.111) can be written as

$$\overline{u}_s(k,t) = A\cos kct + B\sin kct$$

where A and B are two constants. Differentiating this with respect to t we get

$$\frac{\partial \overline{u}_s}{\partial t}(k.t) = -ck\,A\sin kct + ckB\cos kct$$

Setting $t = 0$ in the above we get the following values for A and B by the use of the initial conditions :

$$A = \overline{u}_s(k,0) = \bar{f}_s(k)$$

$$B = \frac{1}{ck}\frac{\partial \overline{u}_s}{\partial t}(k,0) = \frac{1}{ck}\overline{g}_s(k)$$

Therefore the solution (2.111a) for $\overline{u}_s(k,t)$ becomes

$$\overline{u}_s(k,t) = \bar{f}_s(k)\cos kct + \frac{1}{ck}\overline{g}_s(k)\sin kct$$

Taking Fourier sine inverse of this we get

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \overline{u}_s(k,t)\sin kx\,dk$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\cos kct\sin kxdk + \frac{1}{c}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\overline{g}_s(k)\sin kct\sin kxdk$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\{\sin k(x+ct)\sin k(x-ct)dk$$

$$+\frac{1}{2c}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{k}\overline{g}_s(k)\{\cos k(x-ct) - \cos k(x+ct)\}dk \qquad (2.112)$$

Now we are to consider two cases depending on whether $x > ct$ or $x < ct$.

Case 1. $x > ct$

Now in the inversion formula,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\sin kx\,dk, x > 0.$$

replacing x by $x - ct$ and $x + ct$, we get respectively the relatons.

$$f(x-ct) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\sin k(x-ct)dk, \tag{2.113a}$$

$$f(x+ct) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\sin k(x+ct)dk$$

Also integrating the inversion formula,

$$g(\xi) - \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{g}_s(k)\sin k\xi dk,\ \xi > 0$$

with respect to ξ between the limits $x - ct$ to $x + ct$ we get

$$\int_{x-ct}^{x+ct} g(\xi)d\xi = \int_{x-ct}^{x+ct}\sqrt{\frac{2}{\pi}}d\xi\int_0^\infty \bar{g}_s(k)\sin k\xi dk$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\bar{g}_s(k)\int_{x-ct}^{x+ct}\sin k\xi d\xi, \text{ changing the order of integration}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\,\bar{g}(k)\frac{1}{k}\{\cos k(x-ct) - \cos k(x+ct)\} \tag{2.113b}$$

By the use of the relations (2.113), $u(x, t)$ given by (2.112) can be expressed as

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$$

Case II. $x < ct$

Since in this case $ct - x > 0$, replacing x by $ct - x$ in the inversion formula for $f(x)$, we get

$$f(ct-x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\sin k(ct-x)dk$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_s(k)\sin k(x-ct)dk \tag{2.114a}$$

Also in the inversion formula for $g(\xi)$ given above we integrate $g(\xi)$ with respect to ξ between the limits $ct - x$ to $ct + x$, thus we get

$$\int_{ct-x}^{ct+x} g(\xi)d\xi = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, \bar{g}_s(k)\frac{1}{k}[\cos k(x-ct) - \cos k(x+ct) \qquad (2.114b)$$

Therefore by the use of (2.114), $u(x, t)$ given by (2.112) can be expresed as

$$u(x,t) = \frac{1}{2}[f(x+ct) - f(x-ct) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(\xi)d\xi$$

Ex. 2.11.15. Find the solution of the following problem of free vibration of a stretched string of finite length.

(i) $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,\ 0 \le x \le a$

(ii) $u(x, 0) = f(x)$

(iii) $\frac{\partial}{\partial t}u(x,0) = g(x)$

(iv) $u(0, t) = 0, \quad u(a, t) = 0$

Soln. Finite Fourier sine transform is appropriate to this problem

Since, $\int_0^a \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{a}dx$

$$= \sin\frac{n\pi x}{a}u_x(x,t)\Big|_0^a - \int_0^a \frac{n\pi}{a}\cos n\frac{\pi x}{a}\frac{\partial u}{\partial x}dx$$

$$= -\frac{n\pi}{a}\cos\frac{n\pi x}{a}u(x,t)\Big|_0^a - \frac{n^2\pi^2}{a^2}\int_0^a u(x,t)\sin\frac{n\pi x}{a}dx$$

$$= -\frac{n^2\pi^2}{a^2}\bar{u}_s(n,t),\ \text{by the boundary conditions (iv)}$$

and $\int_0^a \frac{\partial^2 u}{\partial t^2}\sin\frac{n\pi x}{a}dx = \frac{\partial^2}{\partial t^2}\int_0^a u(x,t)\sin\frac{n\pi x}{a}dx$

$$= \frac{d^2}{dt^2}\overline{u}_s(n,t)$$

where $\overline{u}_s(n,t)$ is the finite Fourier sine transform of $u(x, t)$, so taking finite Fourier sine transform of the given equation (i) we get

$$-\frac{n^2\pi^2}{a^2}\overline{u}_s(n,t) - \frac{1}{c^2}\frac{d^2}{dt^2}\overline{u}_s(n,t) = 0 \text{ or, } \frac{d^2}{dt^2}\overline{u}_s(n,t) = -\frac{c^2n^2\pi^2}{a^2}\overline{u}_s(n,t) \quad (2.115)$$

Also taking finite Fourier sine transform of the initial conditions (ii) and (iii) we get

$$\overline{u}_s(n,0) = \overline{f}_s(n),\ \frac{\partial}{\partial t}\overline{u}_s(n,0) = \overline{g}_s(n) \quad (2.116)$$

where $\overline{f}_s(n)$ and $\overline{g}_s(n)$ are finite Fourier sine transform of $f(x)$ and $g(x)$ respectively.

The solution of equation (2.115) is

$$\overline{u}_s(n,t) = A_n \cos\frac{n\pi ct}{a} + B_n \sin\frac{n\pi ct}{a} \quad (2.117a)$$

where A_n and B_n are two constants. Differentiating (2.117a) with respect to t we get

$$\frac{\partial}{\partial t}\overline{u}_s(n,t) = -\frac{n\pi c}{a}A_n \sin\frac{n\pi ct}{a} + \frac{n\pi c}{a}B_n \cos\frac{n\pi ct}{a} \quad (2.117b)$$

Setting $t = 0$ in (2.117) and using (2.116) we get

$$A_n = \overline{u}_s(n,0) = \overline{f}_s(n), \frac{n\pi c}{a}B_n = \frac{\partial \overline{u}_s}{\partial t}(n,0) = \overline{g}_s$$

which give

$$A_n = \overline{f}_s(n) \text{ and } B_n = \frac{a}{n\pi c}\overline{g}_s$$

With these values of A_n and B_n the solution for $\overline{u}_s(n,t)$ given by (2.117a) becomes

$$\overline{u}_s(n,t) = \overline{f}_s(n)\cos\frac{n\pi ct}{a} + \frac{a}{n\pi c}\overline{g}_s(n)\sin\frac{n\pi ct}{a}$$

Taking inversion of this according to the formula (2.70) we get $u(x, t)$ as given below :

$$u(x, t) = \frac{2}{a}\sum_{n=1}^{\infty}\bar{u}_s(n,t)\sin\frac{n\pi x}{a}$$

$$= \frac{2}{a}\sum_{n=1}^{\infty}\left[\bar{f}_s(n)\cos\frac{n\pi ct}{a} + \frac{a}{n\pi c}\bar{g}_s(n)\sin\frac{n\pi ct}{a}\right]\sin\frac{n\pi x}{a}$$

$$= \frac{2}{a}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{a}\sin\frac{n\pi x}{a}\int_0^a f(\alpha)\sin\frac{n\pi\alpha}{a}\,d\alpha$$

$$+\frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi ct}{a}\sin\frac{n\pi x}{a}\int_0^a g(\alpha)\sin\frac{n\pi\alpha}{a}\,d\alpha$$

where we have substituted the expression for finite Fourier sine transforms $\bar{f}_s(n)$ and $\bar{g}_s(n)$ according to the definition (2.68).

Ex. 2.11.16. Solve the following vibration problem of an infinite elastic beam

(i) $\dfrac{\partial^4 u}{\partial x^4} + \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0, \ -\infty < x < \infty$

(ii) $u\ (x,\ 0) = f(x), \ -\infty < x < \infty$

(iii) $\dfrac{\partial u}{\partial t}(x,0) = \dfrac{\partial^2 g(x)}{\partial x^2}, \ -\infty < x < \infty$

(iv) $u, u_x, u_{xx}, u_{xxx} \to 0 \text{ as } |x| \to \infty$

where $f(x)$ and $g(x)$ are given prescribed functions and the function $g(x)$ is such that $g,\ g_x \to 0$ as $|x| \to \infty$

Soln. Taking Fourier transform of the given equation (i) and the initial conditions (ii), (iii) we get the following equations by the use of the conditions (iv) and the formula for the Fourier transform of the derivative of a function.

$$\frac{d^2\bar{u}}{dt^2} + c^2k^4\bar{u} = 0 \tag{2.118}$$

$$\left.\begin{aligned}&\bar{u}(k,0) = \bar{f}(k)\\ &\frac{d\bar{u}}{dt}(k,0) = -k^2\bar{g}(k)\end{aligned}\right\} \tag{2.119}$$

The solution of equation (2.118) is

$$\overline{u}(k,t) = A\cos ck^2t + B\sin ck^2t,$$

where A and B are two constants. Differentiating this with respect to t we get

$$\frac{\partial \overline{u}}{\partial t}(k,t) = -ck^2 A\sin ck^2t + ck^2 B\cos ck^2t$$

Setting $t = 0$ in the above and using the equations (2.119) we get

$$A = \bar{f}(k), B = -\frac{1}{c}\bar{g}(k),$$

Therefore the solution becomes

$$\overline{u}(k,t) = \bar{f}(k)\cos ck^2t - \frac{\bar{g}(k)}{c}\sin ck^2t$$

$$= \bar{f}(k)\overline{\varphi}(k) - \frac{1}{c}\bar{g}(k)\overline{\psi}(k) \qquad (2.119)$$

where $\overline{\varphi}(k) = \cos ck^2t = F[\varphi(x)]$, say

and $\overline{\psi}(k) = \sin ck^2t = F[\psi(x)]$, say

By inversion formula we get

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(ck^2t)e^{-ikx}dk$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{i(ck^2t-kx)} + e^{-i(ck^2t+kx)}\right]dk \qquad (2.120)$$

We now evaluate the first integral appearing in the above :

$$\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(ck^2t-kx)}dk$$

$$= \frac{1}{2\sqrt{2\pi}}e^{-\frac{ix^2}{4ct}}\int_{-\infty}^{\infty}e^{i\left(\sqrt{ct}k-\frac{x}{2\sqrt{ct}}\right)}dk$$

$$= \frac{e^{-\frac{ix^2}{4ct}}}{2\sqrt{2\pi ct}}\int_{-\infty}^{\infty}e^{i\xi^2}d\xi,\ \xi = \sqrt{ct}k - \frac{x}{2\sqrt{ct}}$$

$$= \frac{1}{\sqrt{2\pi ct}}e^{-\frac{ix^2}{4ct}}\int_{0}^{\infty}e^{i\xi^2}d\xi, \text{ the integrand being an even function}$$

$$= \frac{e^{-\frac{ix^2}{4ct}}}{2\sqrt{2\pi ct}} \int_0^{\infty} \frac{e^{iv}}{\sqrt{v}} dv = \frac{e^{-\frac{ix^2}{4ct}}}{2\sqrt{2\pi ct}} \left[\int_0^{\infty} \frac{\cos v}{\sqrt{v}} dv + i \int_0^{\infty} \frac{\sin v}{v} dv \right], \xi^2 = v$$

$$= \frac{e^{-\frac{ix^2}{4ct}}}{4\sqrt{ct}}(1+i), \text{ since } \int_0^{\infty} \frac{\cos v}{\sqrt{v}} dv = \int_0^{\infty} \frac{\sin v}{\sqrt{v}} dv = \sqrt{\frac{\pi}{2}}$$

or, $$\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(ck^2t - kx\right)} dk = \frac{1}{4\sqrt{ct}} e^{-\frac{ix^2}{4ct}}(1+i) \qquad (2.121a)$$

Similarly we can deduce that

$$\frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\left(ck^2t + kx\right)} dk = \frac{1}{4\sqrt{ct}} e^{\frac{ix^2}{4ct}}(1-i) \qquad (2.121b)$$

which can be obtained from (2.121a) by replacing i by $-i$ and x by $-x$.

Therefore from (2.120) we get

$$\varphi(x) = \frac{1}{4\sqrt{ct}} e^{-\frac{ix^2}{4ct}}(1+i) + \frac{1}{4\sqrt{ct}} e^{\frac{ix^2}{4ct}}(1-i)$$

$$= \frac{1}{2\sqrt{ct}} \left[\cos \frac{x^2}{4ct} + \sin \frac{x^2}{4ct} \right]$$

Similarely by inversion formula we get

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin\left(ck^2t\right) e^{-ikx} dk$$

$$= \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{i\left(ck^2t - kx\right)} - e^{-i\left(ck^2t + kx\right)} \right] dk$$

$$= \frac{1}{4i\sqrt{ct}} \left[e^{-\frac{ix^2}{4ct}}(1+i) - e^{\frac{ix^2}{4ct}}(1-i) \right], \text{ by (2.121) and (2.121b)}$$

$$= \frac{1}{2\sqrt{ct}} \left[\cos \frac{x^2}{4ct} - \sin \frac{x^2}{4ct} \right]$$

Therefore taking inversion of (2.119) we get the following by the use of convolution theorem.

$$u(x,t) = \frac{1}{2\sqrt{2\pi ct}} \int_{-\infty}^{\infty} (x-\xi)\left[\cos\frac{\xi^2}{4ct} + \sin\frac{\xi^2}{4ct}\right] d\xi$$

$$-\frac{1}{2c\sqrt{2\pi ct}} \int_{-\infty}^{\infty} g(x-\xi)\left[\cos\frac{\xi^2}{4ct} - \sin\frac{\xi^2}{4ct}\right] d\xi$$

III HYDRODYNAMICS

(a) Irrotational motion of a perfect fluid is governed by the equation.

$$\nabla^2\varphi = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator and φ in the velocity potential. The velocity components u, v, w of the fluid are given by

$$u = -\frac{\partial\varphi}{\partial x}, v = -\frac{\partial\varphi}{\partial y}, w = -\frac{\partial\varphi}{\partial z}$$

(b) Two-dimensional slow steady motion of a viscous fluid is governed by the equation,

$$\nabla^4\psi = 0$$

where $\nabla^4 = (\nabla^2)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$ and ψ is the stream function. The velocity components u, v of the fluid are given by

$$u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}$$

Ex. 2.11.17. Solve the following problem of two dimensional flow of a perfect fluid in a half space, where the fluid is introduced wih prescribed velocity through a slit on the boundary.

(i) $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, -\infty < x < \infty, y \geq 0$

(ii) $\frac{\partial \varphi}{\partial y} = -f(x)$, for $|x| < a$, $y = 0$,

$= 0$, for $|x| > a$, $y = 0$

(iii) $\varphi(x, y) \to 0$ as $y \to \infty$

Also find the solution in the particular case when $f(x) = U$.

Soln. Taking Fourier transform of the given equation (i) and the boundary condition (ii), (iii) with respect to x we get

$$\frac{d^2\overline{\varphi}}{dy^2} - k^2\overline{\varphi} = 0 \tag{2.122}$$

$$\left.\begin{aligned} &\frac{d}{dy}\overline{\varphi}(k,0) = -\overline{g}(k) \\ &\overline{\varphi}(k,y) \to 0 \text{ as } y \to \infty \end{aligned}\right\} \tag{2.123}$$

where $\overline{\varphi}(k,y)$ is the Fourier transform of $\overline{\varphi}(x,y)$ with respect to x and $\overline{g}(k)$ is the Fourier transform of a function $g(x)$ defined by $g(x) = f(x)$, $|x| \le a$, $= 0$, $|x| > a$.

Solution of equation (2.122) is

$$\overline{\varphi}(k,y) = Ae^{-|k|y} + Be^{|k|y}$$

Since $\varphi(k, y) \to 0$ as $y \to \infty$, we must have $B = 0$

Hence, $\overline{\varphi}(k,y) = Ae^{-|k|y}$

and therefore $\frac{d\overline{\varphi}}{dy}(k,y) = -|k|Ae^{-|k|y}$ so that setting $y = 0$ we get

$$-|k|A = \frac{d}{dy}\overline{\varphi}(k,0) = -\overline{g}(k)$$

which gives $A = \frac{1}{|k|}\overline{g}(k)$.

With this value of A the solution for $\overline{\varphi}(k,y)$ becomes

$$\overline{\varphi}(k,y) = \frac{1}{|k|}\overline{g}(k)e^{-|k|y}$$

Fourier inversion of which gives

$$\varphi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\bar{g}(k)}{|k|} e^{-(ikx+|k|y)} dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{1}{|k|} e^{-(ikx+|k|y)} \int_{-a}^{a} f(\alpha) e^{ik\alpha} d\alpha \qquad (2.124)$$

where we have sbstituted for $\bar{g}(k)$ according to the definition of Fourier transform.

Now when $f(x)$ = U, we have

$$\bar{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} U e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{U}{ik} \left[e^{ika} - e^{-ika} \right]$$

$$= \frac{2U}{\sqrt{2\pi}} \frac{\sin ka}{k}$$

Hence the solution (2.124) in this case becomes

$$\varphi(x,y) = \frac{U}{\pi} \int_{-\infty}^{\infty} \frac{\sin ka}{k} \cdot \frac{e^{-ikx}}{|k|} \cdot e^{-|k|y} dk$$

Ex. 2.11.18. Solve the following problem of steady, slow and viscous flow of a fluid in the half space $y \geq 0$, in which fluid is introduced normally with prescirbed velocity through a strip $|x| < a$ in the half plane y = 0.

(i) $\nabla^4 \psi = 0, -\infty < x < \infty, y > 0$

(ii) $\frac{\partial}{\partial y} \psi(x,0) = 0, -\infty < x < \infty$

(iii) $\frac{\partial}{\partial x} \psi(x,0) = f(x)$, when $|x| \leq a$

$= 0$, when $|x| > a$

(iv) $\psi(x, y) \to 0$ as $y \to \infty$

(v) $\psi, \psi_x, \psi_{xx}, \psi_{xxx} \to 0$ as $|x| \to \infty$

Soln. In two dimensions the equation (i) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi = 0 \text{ or, } \frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}+\frac{\partial^4\psi}{\partial y^4}=0$$

Taking Fourier transform of this equation and the boundary conditions (ii) (iii), (iv), we get the following, in which we use the conditions (v) in finding the Fourier transform of derivatives of ψ.

$$\frac{d^4\overline{\psi}}{dy^4}-2k^2\frac{d^2\overline{\psi}}{dy^2}+k^4\overline{\psi}=0 \tag{2.125}$$

$$\left.\begin{aligned} &\frac{d}{dy}\overline{\psi}(k,0)=0\\ &-ik\overline{\psi}(k,0)=\overline{g}(k)\\ &\overline{\psi}(k,y)\to 0\, as\, y\to\infty \end{aligned}\right\} \tag{2.125}$$

where $\overline{\psi}(k,y)$ is the Fourier transform of the function $\psi(x, y)$ with respect to x and $\overline{g}(k)$ is the Fourier transform of a function $g(x)$ defined by

$$\begin{aligned} g(x) &= f(x), \quad |x| \le a\\ &= 0, \quad |x| > a \end{aligned}$$

The solution of equation (2.125) is

$$\overline{\psi}(k,y)=(A+By)e^{-|k|y}+(C+Dy)e^{|k|y}$$

Due to the last condition of (2.125) we must have

$$C = D = 0$$

and therefore the solution for $\overline{\psi}(k,y)$ becomes

$$\overline{\psi}(k,y)=(A+By)e^{-|k|y} \tag{2.126}$$

Differentiating this with respect to y we get

$$\frac{d\overline{\psi}}{dy}(k,y)=-|k|(A+By)e^{-|k|y}+Be^{-|k|y}$$

Setting $y = 0$ in these relations we get the following by the use of first and second equations of (2.125)

$$A = \frac{i\overline{g}(k)}{k},\ B = \frac{i|k|}{k}\overline{g}(k)$$

Therefore, the expression for $\overline{\psi}(k, y)$ given by (2.126) becomes

$$\overline{\psi}(k, y) = \frac{i\overline{g}(k)}{k}(1 + |k|y)e^{-|k|y} = \overline{g}(k)\overline{h}(k) \tag{2.127}$$

where $\overline{h}(k) = \frac{i}{k}(1 + |k|y)e^{-|k|y} = F[h(x)]$ say

By inversion formula we get

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i}{k}e^{-i|k|y-ikx}dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}iy\frac{|k|}{k}e^{-|k|y-ikx}dk \tag{2.128}$$

Now, $\frac{iy}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{|k|}{k}e^{-|k|y-ikx}dk$

$$\left[\because |k| = -k \text{ for } k < 0 \atop k \text{ for } k > 0\right]$$

$$= \frac{iy}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{-k}{k}e^{ky-ikx}dk + \frac{iy}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{k}{k}e^{-ky-ikx}dk$$

$$= -\frac{iy}{\sqrt{2\pi}}\left[\frac{e^{ky-ikx}}{y-ix}\right]_{-\infty}^{0} + \frac{iy}{\sqrt{2\pi}}\left[-\frac{e^{-ky-ikx}}{y+ix}\right]_{0}^{\infty}$$

$$= \frac{iy}{\sqrt{2\pi}}\left[-\frac{1}{y-ix} + \frac{1}{y+ix}\right] = \frac{2xy}{\sqrt{2\pi}(x^2+y^2)}$$

And $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{ie^{-|k|y-ikx}}{k}dk$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\frac{ie^{k(y-ix)}}{k} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{ie^{-k(y+ix)}}{k}dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{i}{k} \left[-e^{-k(y-ix)} + e^{-k(y+ix)} \right] dk = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ky} \sin kx}{k} dk$$

Also, $\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ky} \sin kx dk = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{2i} \left[e^{-k(y-ix)} - e^{-k(y+ix)} \right] dk$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2i} \left[\frac{1}{y-ix} - \frac{1}{y+ix} \right] = \sqrt{\frac{2}{\pi}} \frac{x}{y^2+x^2}$$

Integrating both sides of this relations with respect to y between the limits 0 to ∞, we get

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ky} \sin kx}{k} dk = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} - \tan^{-1} \frac{y}{x} \right] = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{y}$$

Therefore, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{ie^{-|k|y-ikx}}{k} dk = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{y}$

Consequently $h(x)$ given by (2.128) becomes

$$h(x) = \sqrt{\frac{2}{\pi}} \left[\tan^{-1} \frac{x}{y} + \frac{xy}{x^2+y^2} \right]$$

So by convolution theorem we get the following form (2.127)

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) h(x-\xi) d\xi$$

$$= \frac{1}{\pi} \int_{-a}^{a} f(\xi) \left[\tan^{-1} \frac{x-\xi}{y} + \frac{y(x-\xi)}{(x-\xi)^2+y^2} \right] d\xi$$

which is the solution of the given problem.

MODEL QUESTIONS

I. Short Questions :

1. State the conditions for which the Fourier transform of a function exists.

2. Define Fourier infinite, sine and cosine, finite, multiple transforms.

3. Show that the Fourier transform of a function is bounded and linear.

4. Assuming that the Fourier transform of $f(x)$ does exist, find the Fourier transform of $f(x)e^{iax}$, where a is real.

5. Define inverse Fourier transform.

6. State Riemann-Lebesgue's theorem, Fourier inversion theorem, convolution theorem, Parseval's relation for Fourier thransform, Fourier integral theorem.

II. Broad Questions :

1. Show that the Fourier transform $F(k)$ of a function $f(x)$ is a continuous function of k.

2. Stating the necessary conditions, find the derivative of the Fourier transform of a given function.

3. Stating the necessary conditions, find the Fourier transform of the derivative of a given function.

4. State and prove Riemann-Lebesgue's theorem.

5. State and prove Fourier inversion theorem, convolution theorem, Parseval's relation, Fourier integral theorem, for Fourier infinite, sine and cosine transforms.

III. Problems :

1. If $\bar{f}(k)$ is the Fourier transform of f(x), show that

$$F\left[f(x)\cos^2 \alpha x\right] = \frac{1}{2}\bar{f}(k) + \frac{1}{4}\left[\bar{f}(k+2) + \bar{f}(k-2)\right]$$

2. If $\bar{f}(k)$ is the Fourier transform of $f(x)$, show that the Fourier inversion of

$$-i\frac{\bar{f}(k)}{k}\left\{e^{ika} - e^{-ik\alpha}\right\} \text{is} \int_{x-\alpha}^{x+\alpha} f(u)du$$

3. Find Fourier sine transform of $e^{-|x|}$

Hence show that $\int_0^{\infty} \frac{x \sin x}{1+x^2} dx = \frac{\pi e^{-x}}{2}, x > 0$

4. Assuming that $F\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}} \frac{a}{k^2+a^2}$ and $F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{k^2}{4a^2}}$, show that the solution of the integral equation $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x-\xi|} u(\xi) d(\xi) = e^{-\frac{x^2}{2}}$ is

$u(x) = \sqrt{\frac{\pi}{2}}\left(2 - x^2\right) e^{-\frac{x^2}{2}}$

5. Use Parseval's identity to show that

(i) $\int_0^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$ (ii) $\int_0^{\infty} \frac{x^2 dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

Hint : Take Fourier sine and cosine transform e^{-x}, $x > 0$.

6. Solve the equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial y^2} = 0, \; -\infty < x < \infty, \; y \geq 0$$

subject to the conditions : (i) $u(x, 0) = f(x)$, (ii) $u_y(x, 0) = 0$,

(iii) $u, u_x, u_{xx}, u_{xxx} \to 0$ as $|x| \to \infty$

Soln. $u(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} f(\xi) \cos\left\{\frac{(x-\xi)^2}{4y} - \frac{\pi}{4}\right\} d\xi$

Summary : In this chapter, Fourier transform and its inversion have been defined with special reference to infinite, sine and cosine, finite and multiple transforms, Applications of this transform to solve differential equation arising in diversified fields of science and engineering are also shown.

CHAPTER 3 ❑ LAPLACE TRANSFORM WITH APPLICATION TO DIFFERENTIAL EQUATIONS

Structure

3.0 INTRODUCTION

Pierre Simon de Laplace (1749–1827), a French mathematician and astronomer introduced a transform that bears his name as Laplace transform. This transform is one of the most fruitful methods of analysis, specially analyzing equations. The essence of the method is the replacement of the study of a function by its Laplace transform. For example, a complicated equation in $f(t)$ is converted into a simple relation in terms of its Laplace transform. This transform is the basis of most analysis and design procedures for control analysis.

In the present unit we introduce the concept of Laplace transform and consider its various properties along with its applications in solving differential equations arising in various branches in science and technology.

3.1 DEFINITION AND BASIC PROPERTIES OF LAPLACE TRANSFORM

Definition : The Laplace transform of a function $f(t)$ of real variable t defined for $t \geq 0$ is denoted by $L[f(t)]$ and is defined by the integral

$$\int_0^\infty e^{-pt} f(t)dt,$$

provided the integral exists. Here p is complex and consequently the Laplace transform of $f(t)$ is a function of complex variable p. If we denote this function by $F(p)$, then

$$F(p) = \int_0^\infty e^{-pt} f(t)dt \tag{3.1}$$

A class of functions f(t), for which their Laplace transform exists, i.e. the integral in (3.1) exists, satisfies the following properties.

(i) $f(t)$ is piecewise continuous in any finite interval of t for $t > 0$.

(ii) $f(t)$ is of exponential order at $t \to \infty$, *i.e.*, $|f(t)| \leq Me^{at}$ for $t > 0$ and for some real positive constants a and M. This a is called the **index of the order of growth** of the function $f(t)$.

In the following theorem we show that if $f(t)$ satisfies the above conditions, then the integral in (3.1) exists and $F(p)$ is analytic in a domain of compex p-plane.

Theorem-1 : If a real-valued function $f(t)$ of real variable t is piecewise continuous in any finite interval of t and is of exponential order $O(e^{at})$ at $t \to \infty$ when $t \geq 0$, then (i) the integral $\int_0^\infty e^{-pt} f(t)dt$ converges in the domain Re(p) > a and (ii) $F(p)$, the Laplace transform of $f(t)$, is analytic in the domain Re(p) > a of complex p-plane.

Proof. (i) Since $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$, there exist two real positive constants M and a such that

$$|f(t)| \leq Me^{at}$$

Therefore, $\left|\int_0^\infty e^{-pt} f(t)dt\right| \le \int_0^\infty |e^{-(x+iy)t}||f(t)|dt, (p = x + iy)$

$$\le \int_0^\infty e^{-xt}|e^{-iyt}|Me^{at}dt = M\int_0^\infty e^{-(x-a)t}dt, \text{since} |e^{-iyt}| = 1$$

which exists if $x > a$, *i.e.*, $\text{Re}(p) > a$.

Hence the integral $\int_0^\infty e^{-pt} f(t)\, dt$ exists in the domain $\text{Re}(p) > a$.

(ii) We divide the interval $(0, \infty)$ of t into intervals (t_0, t_1), (t_1, t_2), ... (t_{n-1}, t_n), ... of finite lengths, where $t_0 = 0$ and $t_n \to \infty$, and consider the sequence of functions $\{u_n(p)\}$, where

$$u_n(p) = \int_{t_{n-1}}^{t_n} e^{-pt} f(t)dt, n = 1, 2, ..., n, ...$$

Now $\left|F(p) - \sum_{k=1}^{n} u_k(p)\right| = \left|\int_{t_n}^\infty e^{-pt} f(t)dt\right|$

$$\le \int_{t_n}^\infty |e^{-(x+iy)t}||f(t)|dt$$

$$\le M\int_{t_n}^\infty e^{-(x-a)t}dt$$

Let $x \ge a + \delta$, where δ in an arbitrary positive number. Then $e^{-(x-a)t} \le e^{-\delta t}$ Therefore for $x \ge \alpha + \delta$, we have

$$\left|F(p) - \sum_{k=1}^{n} u_k(p)\right| \le M\int_{t_n}^\infty e^{-\delta t}dt$$

$$= \frac{M}{\delta}e^{-\delta t_n}$$

Since $\frac{M}{\delta}e^{-\delta t_n} \to 0$ as $n \to \infty$ ($\because$ $t_n \to \infty$ as $n \to \infty$), corresponding to an arbitrary positive $\in$ there exist a positive integer N such that

$$\frac{M}{\delta}e^{-\delta t_n} < \in for\, n > N.$$

Therefore, corresponding to an arbitrary positive $\in$ there excests a positive integer N dependent on $\in$ only and not on P such that

$$|F(p) - \sum_{k=1}^{n} u_k(p)| < \in \text{ for } n > N \text{ and for Re } (p) \geq a + \delta$$

This implies that the series $\sum_{k=1}^{\infty} u_k(p)$ coverges uniformly to $F(p)$ in the domain $\text{Re}(p) \geq a + \delta$ *i.e.* in the domain $\text{Re}(p) > a$, since δ in an arbitrary positive number.

Since for any t on the real line of complex-t plane, $e^{-pt} f(t)$ is an entire function of p, the functions.

$$u_n(p) = \int_{t_{n-1}}^{t_n} e^{-pt} f(t) dt,\ n = 1, 2, ...$$

are also entire functions. Hence the series $\sum_{n=1}^{\infty} u_n(p)$ being uniformly convergent in the domain $\text{Re}(p) > a$, it sum $F(p)$ is analytic in the same domain $\text{Re}(p) > a$.

3.2 LAPLACE TRANSFORM OF DERIVATIVES

Theorem-2 : If $f(t)$ is continuous and is of exponential order $O(e^{at})$ at $t \to \infty$ and $f'(t)$ is piecewise continuous in any finite interval of t, then the Laplace transform of $f'(t)$ exists for $\text{Re}(p) > a$ and is given by

$$L[f'(t)] = pF(p) - f(0) \tag{3.2}$$

Proof. Obviously $L[f'(t)] = \lim_{T \to \infty} \int_0^T f'(t) e^{-pt} dt,$ (3.3)

provided the limit exists.

Since $f'(t)$ is piecewise continuous in any finite interval $(0, T)$ of t, this interval can be broken up into a finite number of subintervals in each of which $f'(t)$ in continuous. Let there be n such subintervals and (b_{r-1}, b_r) be the r-th subinterval, where $b_0 = 0$ and $b_n = T$

Therefore, we can write

$$\int_0^T e^{-pt} f'(t)dt = \sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f'(t)e^{-pt}dt$$

$$= \sum_{r=1}^{n} \left\{ \left[e^{-pt} f(t)\right]_{b_{r-1}}^{b_r} + p \int_{b_{r-1}}^{b_r} f(t)e^{-pt}dt \right\}$$

$$= \sum_{r=1}^{n} \left[e^{-pb_r} f(b_r - 0) - e^{-pb_{r-1}} f(b_{r-1} + 0)\right] + p\sum_{r=1}^{n} \int_{b_{r-1}}^{b_r} f(t)e^{-pt}dt$$

$$= e^{-pT} f(T) - f(0) + p\int_0^T f(t)e^{-pt}dt \tag{3.4}$$

Since $f(t)$ is continuous, $f(b_r - 0) = f(b_r + 0) = f(b_r)$

Now $|e^{-pr} f(T)| = |e^{-(x+iy)T}||f(T)|, p = x + iy$

$\le e^{-xT} M e^{aT} = M e^{-(x-a)T} \to 0$ as $T \to \infty$,

if $x > a$ *i.e.*, $\text{Re}(p) > a$.

Therefore, $\lim\limits_{T \to \infty} e^{-pT} f(T) = 0$ for $\text{Re}(p) > a$ (3.5)

Also since $f(t)$ is continuous and is of exponential order $O(e^{at})$ at $t \to \infty$, its Laplace transform exists for $\text{Re}(p) > a$, and so the following limit exists and is equal to $F(p)$:

$$\lim_{T \to \infty} \int_0^T f(t)e^{-pt}dt = F(p) \text{ for Re(p)} > a \tag{3.6}$$

By the use of (3.4), the equation (3.3) can be written as

$$L[f'(t)] = \lim_{T\to\infty} e^{-pT} f(T) - f(0) + p \lim_{T\to\infty} \int_0^T f(t)e^{-pt}dt$$

$$= pF(p) - f(0), \text{ by (3.5) and (3.6), for } \mathrm{Re}(p) > a.$$

This completes the proof of the theorem.

A generalization of the result of this theorem concerning the Laplace transform of the n-th derivative of a function is given in the following theorem.

Theorem-3. If the $(n-1)$th derivative of a function $f(t)$ is continuous, its n-th derivative is piecewise continuous in any finite interval of t and $f(t)$, $f'(t)$, ..., $f^{(n-1)}(t)$ are each of exponential order Order $O(e^{at})$ at $t \to \infty$, then the Laplace transform of $f^{(n)}(t)$ exists for $\mathrm{Re}(p) > a$ and is given by

$$L\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \tag{3.7}$$

where F(p) is the Laplace transform of $f(t)$.

Proof. Since $f^{(n)}(t)$, the derivative of the function $f^{(n-1)}(t)$, is piecewise continuous in any finite interval of t and the function $f^{(n-1)}(t)$ is continuous and is of exponential order $O(e^{at})$ at $t \to \infty$, by the previous theorem we have

$$L\left[f^{(n)}(t)\right] = pL\left[f^{(n-1)}(t)\right] - f^{(n-1)}(0) \tag{3.8a}$$

Again since $f^{(n-1)}(t)$, the derivative of the function $f^{(n-2)}(t)$, is continuous and the function $f^{(n-2)}(t)$ is continuous and is of exponential order $O(e^{at})$ at $t \to \infty$, by the same theorem we get

$$L\left[f^{(n-1)}(t)\right] = pL\left[f^{(n-2)}(t)\right] - f^{(n-2)}(0) \tag{3.8b}$$

Continuing this argument we get the following relations

$$L\left[f^{(n-2)}(t)\right] = pL\left[f^{(n-3)}(t)\right] - f^{(n-3)}(0)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$L[f''(t)] = pL[f'(t)] - f'(0) \tag{3.8c}$$

$$L[f'(t)] = pL[f(t)] - f(0)$$

Multiplying (3.8a) to (3.8c) respectively by 1, p, ..., p^{n-1} and then adding together we get

$$L\left[f^{(n)}(t)\right] = p^n F(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

As each of the relations (3.8a) to (3.8c) hold for Re(p) > a, Laplace transform of $f^{(n)}(t)$ exists for Re(p) > a.

3.3 SOME PROPERTIES OF LAPLACE TRANSFORM

(i) If $L[f_r(t)] = F_r(p)$, which exists for Re(p) > a_r, (r = 1, 2, ...n), then

$$L[c_1f_1(t) + c_2f_2(t) + \ldots + c_nf_n(t)] = c_1F_1(p) + c_2F_2(p) + \ldots + c_nF_n(p)$$

which exists for Re(p) > max (a_1, a_2, ...a_n). In the above c_1, c_2, ..., c_n are n constants. This is known as the **linear property** of Laplace transform.

Proof. $L[c_1f_1(t) + c_2f_2(t) + \ldots + c_nf_n(t)]$

$$= \int_0^\infty e^{-pt}[c_1f_1(t) + c_2f_2(t) + \ldots + c_nf_n(t)]dt$$

$$= c_1\int_0^\infty e^{-pt}f_1(t)dt + c_2\int_0^\infty e^{-pt}f_2(t)dt + \ldots + c_n\int_0^\infty e^{-pt}f_n(t)dt$$

$$= c_1F_1(p) + c_2F_2(p) + \ldots + c_nF_n(p)$$

which exists in the common region of existence of the integrals $\int_0^\infty e^{-pt}f_r(t)dt$.

This common region is the domain of complex p-plane given be

Re(p) > max (a_1, a_2, ..., a_n)

(ii) **The similarity theorem :** If $L[f(t)] = F(p)$, which exists for Re(p) > a, then for any real positive α.

$$L[f(\alpha t)] = \frac{1}{\alpha}F\left(\frac{p}{\alpha}\right)$$

Proof. Since the Laplace transform of $f(t)$ exists for Re(p) > a, the function $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$. This implies that there exists a positive constant M such that

$$|f(t)| \le Me^{at}$$

Replacing t by αt we get

$$|f(\alpha t)| \le Me^{a\alpha t}$$

By the use of this we get

$$\left|\int_0^\infty f(\alpha t)e^{-pt}dt\right| \le \int_0^\infty |f(\alpha t)||e^{-(x+iy)t}|dt \quad (p = x + iy)$$

$$\le M\int_0^\infty e^{a\alpha t}.e^{-xt}dt = \int_0^\infty e^{-(x-a\alpha)t}dt$$

which exists for $x > a\alpha$ or $\text{Re}(p) > a\alpha$

Hence the Laplace transform of $f(\alpha t)$ exists for $\text{Re}(p) > a\alpha$.

Now, $L[f(\alpha t)] = \int_0^\infty e^{-pt}f(\alpha t)dt$, put $\alpha t = u$

$$= \int_0^\infty e^{-\frac{p}{\alpha}u} f(u)\frac{1}{\alpha}du = \frac{1}{\alpha}\int_0^\infty e^{-\frac{p}{\alpha}u} f(u)du$$

$$= \frac{1}{\alpha}F\left(\frac{p}{\alpha}\right).$$

since in the relation $\int_0^\infty f(t)e^{-pt}dt = F(p)$, replacing p by $\frac{p}{\alpha}$ we get

$$\int_0^\infty e^{-\frac{p}{\alpha}t} f(t)dt = F\left(\frac{p}{\alpha}\right).$$

(iii) **Shifting theorem :** If $L[f(t)] = F(p)$, which exists for $\text{Re}(p) > a$, then for any complex constant λ,

$$L\left[e^{-\lambda t}f(t)\right] = F(\lambda + p)$$

which exists for $\text{Re}(p) > a - \text{Re}(\lambda)$.

Proof. Let $\varphi(t) = e^{-\lambda t} f(t)$. Since Laplace transform of $f(t)$ exists for Re $(p) > a$, $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$, so there exists a positive constant M such that $|f(t)| \le Me^{at}$. Therefore,

$$|\varphi(t)| = |e^{-(\lambda_x + i\lambda_i)t}||f(t)| \le Me^{at}e^{-\lambda_r t}, \lambda = \lambda_r + i\lambda_i$$

$$= Me^{(a-\lambda_r)t}$$

and consequently $\varphi(t)$ is of exponential order $O\left(e^{(a-\lambda_r)t}\right)$. So its Laplace transform exists for $\text{Re}(p) > a - \lambda_r = a - \text{Re}(\lambda)$

$$\text{Now } L\left[e^{-\lambda t} f(t)\right] = \int_0^\infty e^{-pt}.e^{-\lambda t} f(t)dt$$

$$= \int_0^\infty e^{-(p+\lambda)t} f(t)dt = F(p+\lambda)$$

since in the relation $\int_0^\infty e^{-pt} f(t)dt = F(p)$, replacing p by $p + \lambda$, we get

$$\int_0^\infty e^{-(p+\lambda)t} f(t) = F(p+\lambda)$$

(iv) Translation property : If $L[f(t)] = F(p)$, which exists for $\text{Re}(p) > a$ and $\theta(t)$ is the unit step function, then for $\tau > 0$, $L[f(t-\tau)\ \theta\ (t-\tau)] = e^{-p\tau}\ F(p)$ which exist for $\text{Re}(p) > a$.

Proof. Obviously $|f(t)| \le Me^{at}$, where M is a real positive number, Replacing here t by $t - \tau$ we get

$$|f(t-\tau)| \le Me^{a(t-\tau)} = M'e^{at}$$

where $M' = Me^{-a\tau}$, which shows that $f(t-\tau)\ \theta\ (t-\tau)$ is also of exponential order $O(e^{at})$ at $t \to \infty$. Hence its Laplace transform exists for $\text{Re}(p) > a$ and is given by

$$L[f(t-\tau)\theta(t-\tau)] = \int_\tau^\infty e^{-pt} f(t-\tau)dt, \text{ put } t - \tau = u$$

$$= \int_0^\infty e^{-p(u+\tau)} f(u)du = e^{-p\tau} \int_0^\infty e^{-pu} f(u)du$$

$$= e^{-p\tau} F(p)$$

(v) **Convolution theorem :** If $L[f_1(t)] = F_1(p)$ and $L[f_2(t)] = F_2(p)$, which exist respectively in the domains $\text{Re}(p) > a_1$ and $\text{Re}(p) > a_2$, then

$$L[\varphi(t)] = F_1(p) F_2(p)$$

which exist in the domain $\text{Re}(p) > \max(a_1, a_2)$, where $\varphi(t)$ is the convolution of the two functions $f_1(t)$ and $f_2(t)$ defined by

$$\varphi(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_0^t f_1(t-\tau) f_2(\tau) d\tau$$

Proof. Obviosuly $f_1(t)$ and $f_2(t)$ are of exponential orders $O(e^{a_1 t})$ and $O(e^{a_2 t})$ respectively at $t \to \infty$. Therefore there exist two positive constants M_1 and M_2, such that

$$|f_1(t)| \le M_1 e^{a_1 t}, |f_2(t)| \le M_2 e^{a_2 t}.$$

By the use of these inequalities we get

$$|\varphi(t)| \le \int_0^t |f_1(\tau)||f_2(t-\tau)| d\tau \le \int_0^t M_1 e^{a_1 \tau} M_2 e^{a_2 (t-\tau)} d\tau$$

$$= M_1 M_2 e^{a_2 t} \int_0^t e^{(a_1 - a_2)\tau} d\tau = M_1 M_2 e^{a_2 t} \frac{e^{(a_1 - a_2)t} - 1}{a_1 - a_2}$$

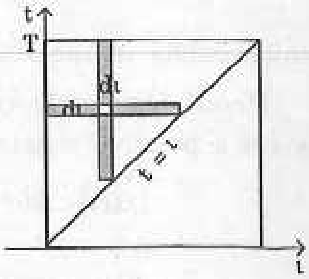
$$= M_1 M_2 \frac{e^{a_1 t} - e^{a_2 t}}{a_1 - a_2}$$

If $a_1 > a_2$, $|\varphi(t)| \le \frac{M_1 M_2}{a_1 - a_2} e^{a_1 t}$

and if $a_2 > a_1$, $|\varphi(t)| \le \frac{M_1 M_2}{a_2 - a_1} e^{a_2 t}$

Hence $|\varphi(t)| \le Me^{at}$, where $M = \dfrac{M_1 M_2}{|a_1 - a_2|}$ and $a = \max\,(a_1, a_2)$

Therefore Laplace transform of $\varphi(t)$ exists in the domain $\text{Re}(p) > a = \max\,(a_1, a_2)$ and is given by



$$L|\varphi(t)| = \int_0^\infty \varphi(t)e^{-pt}dt = \lim_{T\to\infty}\int_0^T \varphi(t)e^{-pt}dt$$

$$= \lim_{T\to\infty}\int_0^T dt e^{-pt}\int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

$$= \lim_{T\to\infty}\int_{t=0}^T dt e^{-pt}\int_{\tau=0}^t f_1(\tau)f_2(t-\tau)d\tau$$

$$= \lim_{T\to\infty}\int_{\tau=0}^T d\tau\int_{t=\tau}^T dt e^{-pt}f_1(\tau)f_2(t-\tau)$$

(changing the order of integration)

$$= \lim_{T\to\infty}\int_{\tau=0}^T d\tau f_1(\tau)\int_{t=\tau}^T dt e^{-pt}f_2(t-\tau)$$

$$= \lim_{T\to\infty}\int_{\tau=0}^T d\tau f_1(\tau)\int_{t'=0}^{T-\tau} dt' e^{-p(\tau+t')}f_2(t')$$

putting $t - \tau = t'$

$$= \lim_{T\to\infty}\int_{\tau=0}^T d\tau f_1(\tau)e^{-p\tau}\int_{t'=0}^{T-\tau} dt' f_2(t')e^{-pt'}$$

$$= \int_{\tau=0}^\infty d\tau f_1(\tau)e^{-p\tau}\int_{t'=0}^\infty dt' f_2(t')e^{-pt'}$$

$$= F_1(p)F_2(p)$$

(vi) The Laplace transform of an integral : If $L[f(t)] = F(p)$, which exists in the domain $\text{Re}(p) > a$, and $\varphi(t) = \int_0^t f(\tau)d\tau$, then

$$L[\varphi(t)] = \frac{1}{p}F(p)$$

which exists in the same domain $\text{Re}(p) > a$.

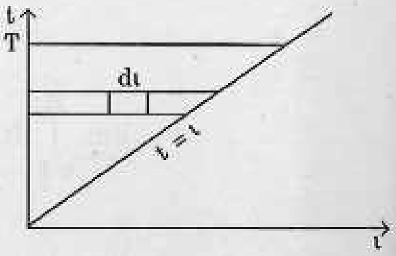
Proof. Obviously $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$. Therefore there exists a positive constant M such that

$$|f(t)| < Me^{at}$$

and so $|\varphi(t)| = |\int_0^t f(\tau)d\tau| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{a\tau}d\tau$

$$= \frac{M}{a}(e^{at} - 1) \le \frac{M}{a}e^{at}$$

Hence φ(t) is of same exponential order as f(t) and so its Laplace transform exists and is given by



$$L[\varphi(t)] = \int_0^\infty e^{-pt}dt \int_0^t f(\tau)d\tau$$

$$= \lim_{T\to\infty} \int_{t=0}^T e^{-pt}dt \int_{\tau=0}^t f(\tau)d\tau$$

$$= \lim_{T\to\infty} \int_0^T d\tau f(\tau) \int_\tau^T e^{-pt}dt \text{ (changing the order of integration)}$$

$$= \lim_{T\to\infty} \int_0^T d\tau f(\tau)\frac{1}{p}\left[e^{-p\tau} - e^{-pT}\right]$$

$$= \frac{1}{p}\int_0^\infty d\tau f(\tau)e^{-p\tau} = \frac{1}{p}F(p), \text{ which exists for } \text{Re}(p) > a$$

Since $\left|e^{-pT}\right| = \left|e^{-(x+iy)T}\right| = e^{-xT} < e^{-aT}$ for $x > a$

$$\to 0 \text{ as } T \to \infty.$$

and this implies that $e^{-pt} \to 0$ as $T \to \infty$

3.4 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

In the following $f(t) = 0$ for $t < 0$.

(a) $f(t) = e^{\lambda t}$, where λ is a complex constant.

$$L[e^{\lambda t}] = \int_0^\infty e^{\lambda t}.e^{-pt}dt = \int_0^\infty e^{-(p-\lambda)t}dt$$

$$= -\frac{1}{p-\lambda}\left[e^{-(p-\lambda)t}\right]_0^\infty = \frac{1}{p-\lambda}, \text{if } \mathrm{Re(p)} > \mathrm{Re}\,\lambda$$

$$\therefore L[e^{\lambda t}] = \frac{1}{p-\lambda} \tag{3.9}$$

(b) $f(t) = \sin \omega t$, ω is a real constant.

$$L[\sin \omega t] = L\left[\frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})\right]$$

$$= \frac{1}{2i}L[e^{i\omega t}] - \frac{1}{2i}L[e^{-i\omega t}], \text{ by (1.3)}$$

$$= \frac{1}{2i}\frac{1}{p-i\omega} - \frac{1}{2i}\frac{1}{p+i\omega}, \text{ if } \mathrm{Re}(p) > 0, \text{ by (3.9)}$$

$$= \frac{\omega}{p^2+\omega^2}$$

$$\therefore L[\sin \omega t] = \frac{\omega}{p^2+\omega^2} \tag{3.10}$$

(c) $f(t) = \cos \omega t$, ω is a real constant.

$$L[\cos \omega t] = L\left[\frac{1}{2}(e^{i\omega t} + e^{-i\omega t}\right] = \frac{1}{2}L[e^{i\omega t}] + \frac{1}{2}L[e^{i\omega t}], \text{ by (1.3)}$$

$$= \frac{1}{2} \cdot \frac{1}{p - i\omega} + \frac{1}{2} \cdot \frac{1}{p + i\omega}, \text{ if } \operatorname{Re}(p) > 0, \text{ [by (1.10)]} = \frac{p}{p^2 + \omega^2}$$

$$\therefore L[\cos \omega t] = \frac{p}{p^2 + \omega^2} \tag{3.11}$$

(d) $f(t) = t^\gamma$, $\gamma > -1$.

If $-1 < \gamma < 0$, then $f(t) \to \infty$ as $t \to 0+$. Hence $f(t)$ is not a piecewise continuous function in any finite integral of t and so the condition of existence of its Laplace transform gets violated. We now show that its Laplace transform still exists.

Since $\left| \int_0^\infty t^\gamma e^{-p\lambda} dt \right| = \left| \int_0^\infty t^\gamma e^{-(x+iy)t} dt \right| \le \int_0^\infty t^\gamma e^{-xt} dt$, which is known to converge for $\gamma > -1$ and $x > 0$, the Laplace transform of t^γ exists for $\operatorname{Re}(p) > 0$. We next proceed to evaluate the value of this integral. Let $F(p) = \int_0^\infty t^\gamma e^{-pt} dt$. It is known that F(p) is analytic in the domain $\operatorname{Re}(p) > 0$.

On positive real axis, *i.e.* for $p = x > 0$ (real) the value of the integral becomes

$$\int_0^\infty t^\gamma e^{-xt} = \int_0^\infty \frac{\tau^\gamma}{x^\gamma} e^{-\tau} \frac{d\tau}{x}, \text{ by the substitution } \tau = xt.$$

$$= \frac{1}{x^{\gamma+1}} \int_0^\infty \tau^{(\gamma+1)-1} e^{-\tau} d\tau = \frac{\Gamma(\gamma+1)}{x^{\gamma+1}}$$

So on positive real axis

$$F(p) = \frac{\Gamma(v+1)}{x^{\gamma+1}}, \text{ if } p = x \ (> 0) \text{ real.}$$

The function $\frac{\Gamma(\gamma+1)}{p^{\gamma+1}}$ is analytic in the domain $\operatorname{Re}(p) > 0$ and coincides with F(p) on positive real axis. So by uniqueness property the analytic function, F(p), must be equal to $\frac{\Gamma(\gamma+1)}{p^{\gamma+1}}$ in domain $\operatorname{Re}(p) > 0$, i.e.

$$F(p) = \frac{\Gamma(\gamma+1)}{p^{\gamma+1}} = \int_0^\infty t^\gamma e^{-pt} dt = L[t^\gamma] \text{ for } \mathrm{Re}(p) > 0$$

Therefore the Laplace transform of t^γ for $\gamma > -1$ exists and is given by

$$L[t^\gamma] = \frac{\Gamma(\gamma+1)}{p^{\gamma+1}} \text{ for } \mathrm{Re}(p) > 0 \tag{3.12}$$

When $\gamma = n$, a positive integer, the above formula gives

$$L[t^n] = \frac{n!}{p^{n+1}} \tag{3.12a}$$

In particular when n = 0, we get

$$L[1] = \frac{1}{p} \tag{3.12b}$$

(e) $f(t) = t^n e^{\lambda t}$,

$$L[t^n e^{\lambda t}] = \frac{n!}{(p-\lambda)^{n+1}}, \text{ by shifting and similarity theorems} \tag{3.13}$$

(f) $f(t) = \sin \omega t e^{\lambda t}$,

$$L[\sin \omega t\, e^{\lambda t}] = \frac{\omega}{(p-\lambda)^2 + \omega^2}, \text{ by shifting theorem and (3.10)} \tag{3.14a}$$

(g) $f(t) = \cos \omega t e^{\lambda t}$,

$$L[\cos \omega t\, e^{\lambda t}] = \frac{p-\lambda}{(p-\lambda)^2 + \omega^2}, \text{ by shifting theorem and (3.11)} \tag{3.14b}$$

(h) $f(t) = t \sin \omega t$

$$L[t \sin \omega t] = L\frac{1}{2i}\left[t(e^{i\omega t} - e^{-i\omega t})\right]$$

$$= \frac{1}{2i}L[te^{i\omega t}] - \frac{1}{2i}L[te^{-i\omega t}] \text{ by (1.3)}$$

$$= \frac{1}{2i}\frac{1}{(p-i\omega)^2} - \frac{1}{2i}\frac{1}{(p+i\omega)^2}, \text{ by shifting theorem and (3.12a)}$$

$$= \frac{2\omega p}{(p^2+\omega^2)^2}$$

$$\therefore L[t \sin \omega t] = \frac{2\omega p}{(p^2+\omega^2)^2} \tag{3.15a}$$

(i) $f(t) = t \cos \omega t$

By the same method as in the derivation for the formula (3.15a) we get

$$L[t \cos \omega t] = \frac{p^2 - \omega^2}{(p^2+\omega^2)^2} \tag{3.15b}$$

3.5 ASYMPTOTIC PROPERTIES OF LAPLACE TRANSFORM

Theorem 4. If F(p) is the Laplace transform of a function $f(t)$, which is piecewise continuous in any finite interval of t and is of exponential order $O(e^{at})$ at $t \to \infty$, then

(i) $\lim_{p\to\infty} F(p) = 0$ (3.16a)

and (ii) $\lim_{p\to\infty} pF(p) = f(0)$, [Initial value-theorem] (3.16b)

Proof : (i) $F(p) = \int_0^\infty f(t)e^{-pt}dt$

Let $|f(t)| \le Me^{at}$

Therefore, $|F(p)| \le \int_0^\infty |f(t)| \left|e^{-i(x+iy)t}\right| dt,\ p = x + iy$

$$\le M\int_0^\infty e^{-(x-a)t}dt = \frac{M}{x-a} \text{ for } x > a$$

$$\to 0 \text{ as } x \to \infty.$$

Hence, $F(p) \to 0$ as $x \to \infty$.

But $x \to \infty$ implies $p \to \infty$ and so

$$\lim_{p\to\infty} F(p) = 0$$

(ii) We suppose that $f'(t)$ is piecewise continuous in any finite interval of t, and $f(t)$ is continuous and of exponential order $O(e^{at})$ at $t \to \infty$. Then the Laplace transform of $f'(t)$ as given by (3.2) is

$$L[f'(t)] = pL[f(t)] - f(0)$$

According to what we have proved in (i)

$$\lim_{p\to\infty} L[f(t)] = 0$$

Therefore proceeding to the limit as $p \to \infty$ in the above relation we get

$$\lim_{p\to\infty} pL[\, f(t)\,] = f(0)$$

Theorem 5 : Final value theorem : If f(t) is continuous and is of exponential order $O(e^{at})$ at $t \to \infty$. and $f'(t)$ is piecewise continues in any finite interval of t, then

$$\lim_{p\to\infty} pF(p) = \lim_{t\to\infty} f(t) = f(\infty) \tag{3.17}$$

Proof. Let $G(p)$ be the Laplace transform of $f'(t)$, Then

$$G(p) = \int_0^\infty e^{-pt} f'(t)$$

Due to the conditions imposed on $f(t)$ and $f'(t)$, it can be shown that $G(p)$ is a continuous function of p. Therefore,

$$\lim_{p\to 0} G(p) = G(0) = \int_0^\infty f'(t)\,dt$$

$$\text{or,}\quad \lim_{p\to 0} G(p) = f(\infty) - f(0) \tag{3.18a}$$

Again, $G(p) = L[f'(t)] = pF(p) - f(0)$

From this proceeding to the limit $p \to 0$ we get

$$\lim_{p\to 0} G(p) = \lim_{p\to 0} pF(p) - f(0) \tag{3.18b}$$

From (3.18a) and (3.18b) we get

$$\lim_{p\to 0} pF(p) = f(\infty)$$

3.6 DIFFERENTIATION AND INTEGRATION OF LAPLACE TRANSFORM

Theorem 6. If $f(t)$ is piecewise continuous in any finite interval of t and is of exponential order $O(e^{at})$ at $t \to \infty$, then its Laplace transform is differentiable any number of times and

$$F^{(n)}(p) = (-1)^n \int_0^\infty e^{-pt} t^n f(t)\,dt = (-1)^n L[t^n f(t)], \text{ for } \operatorname{Re}(p) > a \qquad (3.19)$$

Proof. $\left| \int_0^\infty t f(t) e^{-pt} dt \right| \le \int_0^\infty t |f(t)| \left| e^{-(x+iy)t} \right| dt, \; p = x + iy$

$$\le M \int_0^\infty t e^{-(x-a)t} dt, \text{ since } |f(x)| \le M e^{at}, \; M > 0$$

$$= \frac{Mte^{-(x-a)t}}{-(x-a)} \Bigg|_0^\infty + M \int_0^\infty \frac{e^{-(x-a)t}}{x-a} dt$$

$$= \frac{M}{(x-a)^2}, \text{ for } x > a$$

Hence Laplace transform of $tf(t)$ exists in the domain $\operatorname{Re}(p) > a$, which is also the domain of existence of the Laplace transform of $f(t)$.

3.1 It has been proved in § 3.1 that $F(p)$, the Laplace transform of $f(t)$, is analytic in the domain $\operatorname{Re}(p) > a$. Hence its derivative exists and is given by

$$F'(p) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad (3.20)$$

where $F(p) = u(x, y) + iv(x, y) = \int_0^\infty e^{-(x+iy)t} f(t)\, dt.$

Separating this into real and imaginary parts we get

$$u = \int_0^\infty e^{-xt} \cos yt\, f(t) dt, \; v = \int_0^\infty e^{-xt} \sin yt\, f(t)\, dt$$

Now it is seen that all the following conditions for differentiability of μ under sign of integration with respect to x are satisfied :

(i) $\varphi(x,t) = e^{-xt} \cos yt$ is a continuous function of x and t for any fixed y.

(ii) $\int_0^\infty e^{-xt} \cos yt\, f(t)dt$ converges uniformly for $x > a$ and $-\infty < y < \infty$.

(iii) $f(t)$ is bounded and integrable in any finite interval of t.

(iv) The integral, $\int_0^\infty \frac{\partial}{\partial x}\left[e^{-xt} \cos yt\, f(t)\right]dt = -\int_0^\infty te^{-xt} \cos yt\, f(t)\, dt$ converges uniformly for $x > a$ and $-\infty < y < \infty$.

Therefore we get,

$$\frac{\partial u}{\partial x} = -\int_0^\infty te^{-xt} \cos yt\ f(t)\ dt$$

and by similar argument we get,

$$\frac{\partial v}{\partial x} = \int_0^\infty te^{-xt} \sin yt\ f(t)\ dt$$

Substituting these in (3.20) we get

$$F'(p) - \int_0^\infty t\, f(t)e^{-pt}, \text{ for } \mathrm{Re}(p) > a$$

$$\text{or, } F'(p) = (-1)\ L[t\ f(t)], \text{ for } \mathrm{Re}(p) > a \qquad (3.21)$$

Just as we have shown the existence of the Laplace transform $tf(t)$, we can show that the Laplace transform of $t^2 f(t)$ exists in the domain $\mathrm{Re}(p) > a$. Now $-F'(p)$, the Laplace transform of $t\ f(t)$, being analytic in $\mathrm{Re}(p) > a$ following similar steps of deriving (3.21), we find that

$$-F''(p) = (-1)L[t^2 f(t)], \text{ for } \mathrm{Re}(p) > a$$

$$\text{or, } F''(p) = (-1)^2 L[t^2 f(t)], \ \mathrm{Re}(p) > a$$

Continuing this process we get the following after n steps.

$$F^{(n)}(p) = (-1)^n L[t^n f(t)], \text{ for } \mathrm{Re}(p) > a$$

Theorem 6. If a function $\frac{f(t)}{t}$ satisfies the existence conditions of its Laplace transform and $L[f(t)] = F(p)$, which exists in the domain $\mathrm{Re}(p) > a$, then

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(q)\, dq \qquad (3.22)$$

Proof. Let $G(p)$ be the Laplace transform of $g(t) = \frac{f(t)}{t}$, *i.e.*

$$G(p) = L\left[\frac{f(t)}{t}\right] = \int_0^\infty \frac{f(t)}{t} e^{-pt} dt$$

Then by the above Theorem-5, $G'(p)$ exists and is given by

$$G'(p) = \int_0^\infty f(t) e^{-pt} dt = F(p)$$

Further both $G(p)$ and $F(p)$ are analytic in the domain $\text{Re}(p) > a$, which is the domain of existent of $F(p)$. Integrating the above with respect to p between the points p and ∞ along any contour in the complex p-plane lying in the domain $\text{Re}(p) > a$, we get

$$G(p) - G(\infty) = \int_p^\infty F(p) dp$$

Now $G(p)$ being the Laplace transform of a function, $G(\infty) = 0$ by (3.16a). Therefore the above relation becomes

$$G(p) = \int_p^\infty F(p) dp$$

$$\text{or,} \quad L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(p)\, dp$$

where the integral on the right hand side does not depend on the path of integration, since $F(p)$ is analytic.

3.7 THE INVERSE LAPLACE TRANSFORM

So far we have considered only the problem of finding the Laplace transform $F(p)$ of a given function $f(t)$ and have derived some properties of $F(p)$. But in application of Laplace transform to practical problems it is necessary to solve the inverse problem of finding the original function $f(t)$ from its Laplace transform $F(p)$. This is called the inverse Laplace transform and is denoted by L^{-1}, i.e.

$$f(t) = L^{-1}[f(p)] \tag{3.23}$$

In many cases the original function $f(t)$ can be determined from its Laplace transform $F(p)$ from the existing table of Laplace transform of some elementary function like given in the formulas (3.9) to (3.15) by application of the properties (i)—(vi) of Laplace transform given in Sec. 33. This is possible due to the following uniqueness property of Laplace transform.

If the Laplace transform of a given function exists, it is determined uniquely. On the other hand, it can be shown that two functions $f_1(t)$ and $f_2(t)$ having the same Laplace transform cannot differ over any interval of t, though they may differ at some isolated points. Since this is of no importance in applications, we may say that the inverse of a given Laplace transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical.

As the method of constructing the original function from its Laplace transform stated above is a trial and error method, it may not always be possible to construct the original function following this method. In such cases we are to depend on the complex inversion integral formula given in the last subsection of the present section.

I. INVERSION BY THE USE OF LINEAR AND SHIFTING PROPERTY

By linear property and shifting theorem, we get

$$L^{-1}[C_1F_1(p) + C_2F_2(p) + \dots + C_nF_n(p)]$$
$$= C_1L^{-1}[F_1(p)] + C_2L^{-1}[F_2(p)] + \dots + C_nL^{-1}[F_n(p)]$$
$$L^{-1}[F(\lambda + p)] = e^{-\lambda\tau} L^{-1}[F(p)]$$

Ex. 3.6.1 Find $L^{-1}\left[\dfrac{p-7}{p^2+6p+17}\right]$

Soln. We can write $L^{-1}\left[\dfrac{p-7}{p^2+6p+17}\right] = L^{-1}\left[\dfrac{(p+3)-10}{(p+3)^2+8}\right]$

$$= e^{-3t}L^{-1}\left[\frac{p-10}{p^2+8}\right], \text{ by shifting property}$$

$$= e^{-3t}L^{-1}\left[\frac{p}{p^2+8} - \frac{10}{\sqrt{8}}.\frac{\sqrt{8}}{p^2+\left(\sqrt{8}\right)^2}\right]$$

$$= e^{-3t}\left\{L^{-1}\left[\frac{p}{p^2+8}\right] - \frac{10}{\sqrt{8}}L^{-1}\left[\frac{\sqrt{8}}{p^2+\left(\sqrt{8}\right)^2}\right]\right\}, \text{ by linear property}$$

$$= e^{-3t}\left[\cos\sqrt{8}t - \frac{5}{\sqrt{2}}\sin\sqrt{8}t\right], \text{ by (3.10) and (3.11).}$$

Ex. 3.6.2. Find $L^{-1}\left[\dfrac{p^2-2}{(p-1)^4}\right]$

Soln. We can write $L^{-1}\left[\dfrac{p^2-2}{(p-1)^4}\right] = L^{-1}\left[\dfrac{(p-1)^2+2(p-1)-1}{(p-1)^4}\right]$

$$= e^t L^{-1}\left[\frac{p^2+2p-1}{p^4}\right], \text{ by shifting property}$$

$$= e^t L^{-1}\left[\frac{1}{p^2} + 2.\frac{1}{p^3} - \frac{1}{p^4}\right]$$

$$= e^t\left\{L^{-1}\left[\frac{1}{p^2}\right] + 2L^{-1}\left[\frac{1}{p^3}\right] - L^{-1}\left[\frac{1}{p^4}\right]\right\} \text{ by linear property}$$

$$= e^t\left[\frac{1}{1!}t + 2.\frac{1}{2!}t^2 - \frac{1}{3!}t^3\right], \text{ by (3.12a)}$$

II. INVERSION BY THE USE OF FORMULAS FOR THE DERIVATIVE AND INTEGRAL OF A LAPLACE TRANSFORM

From (3.14) and (3.22) we get,

$$L^{-1}\left[F^{(n)}(p)\right] = (-1)^n t^n L^{-1}[F(p)] \tag{3.24}$$

$$L^{-1}\left[\int_p^\infty F(q)dq = \frac{1}{t}L^{-1}F(p)\right] \tag{3.25}$$

Ex. 3.6.3. Find $L^{-1}\left[ln\dfrac{p^2+1}{p(p+1)}\right]$

Soln. Let $f(t) = L^{-1}\left[ln\dfrac{p^2+1}{p(p+1)}\right] = L^{-1}[F(q)]$

$$\therefore -tL^{-1}[F(p)] = L^{-1}[F'(p)], \text{ by } (2.4)$$

$$= L^{-1}\left[\frac{d}{dp}\left\{ln\left(p^2+1\right) - lnp - ln(p+1)\right\}\right]$$

$$= 2L^{-1}\left[\frac{p}{p^2+1}\right] - L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{1}{p+1}\right], \text{ by linear property}$$

$$= 2 \cos t - 1 - e^{-t}$$

$$\text{or, } -tL^{-1}\left[ln\frac{p^2+1}{p(p+1)}\right] = -2\cos t - 1 - e^{t}$$

$$\therefore L^{-1}\left[ln\frac{p^2+1}{p(p+1)}\right] = -\frac{2}{t}\cos t + \frac{1}{t} + \frac{1}{t}e^{-1}$$

3.6.4. Find $L^{-1}\left[\tan^{-1}\frac{2}{p}\right]$

Soln. Let $f(t) = L^{-1}\left[\tan^{-1}\frac{2}{p}\right] = L^{-1}[F(p)]$

$$\therefore -tL^{-1}[F(p)] = L^{-1}[F'(p)], \text{ by } (2.4)$$

$$= L^{-1}\left[\frac{1}{1+\frac{4}{p^2}}\left(-\frac{2}{p^2}\right)\right] = L^{-1}\left[-\frac{2}{p^2+4}\right]$$

$$= -\sin 2t.$$

$$\therefore L^{-1}[F(p)] = \frac{1}{t}\sin 2t, \quad \text{or, } L^{-1}\left[\tan^{-1}\frac{2}{p}\right] = \frac{1}{t}\sin 2t$$

3.6.5. Find $L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right]$

Soln. Let $f(t) = L^{-1}\left[\frac{p}{(p^2+a^2)}2\right] = L^{-1}[F(p)]$

$$\therefore \frac{1}{t}f(t) = \frac{1}{t}L^{-1}[F(p)] = L^{-1}\left[\int_p^{\infty} F(q)dq\right], \text{ by (2.5)}$$

$$= L^{-1}\left[\int_y^{\infty}\frac{q}{(q^2+a^2)^2}dq\right] = L^{-1}\left[-\frac{1}{2}\frac{1}{q^2+a^2}\Bigg|_p^{\infty}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{p^2+a^2}\right] = \frac{1}{2a}L^{-1}\left[\frac{a}{p^2+a^2}\right] = \frac{1}{2a}\sin at$$

$$\therefore L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

III. INVERSION BY THE USE CONVOLUTION THEOREM

From convolution theorem we get,

$$L^{-1}[F_1(p)F_2(p)] = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau \qquad (3.26)$$

where $f_1(t) = L^{-1}[F_1(p)]$ and $f_2(t) = L^{-1}[F_2(p)]$

Ex. 3.6.6. Find $L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right]$

Soln. Let $L^{-1}\left[\frac{p}{p^2+a^2}\frac{1}{p^2+a^2}\right] = L^{-1}[F_1(p)F_2(p)]$

where $F_1(p) = \frac{p}{p^2+a^2}$, $F_2(p) = \frac{1}{a}.\frac{a}{p^2+a^2}$

Now, $f_1(t) = L^{-1}\left[\frac{p}{p^2+a^2}\right] = \cos at$, by (3.11)

$$f_2(t) = L^{-1}\left[\frac{1}{a}.\frac{p}{p^2+a^2}\right] = \frac{1}{a}\sin at, \text{ by (3.10)}$$

$$\therefore L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right] = \int_0^t \cos a\tau.\frac{1}{a}\sin a(t-\tau)d\tau, \text{ by (3.26)}$$

$$= \frac{1}{2a}\int_0^t [t\sin at + \sin a(t-2\tau)]d\tau$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos a(t-2\tau)\Big|_0^t\right]$$

$$= \frac{1}{2a}t\sin at$$

Ex. 3.6.7. Find $L^{-1}\left[\frac{1}{\sqrt{p}}.\frac{1}{p-1}\right] = L^{-1}[F_1(p)F_2(p)]$,

where $F_1(p) = \frac{1}{\sqrt{p}}$ and $F_2(p) = \frac{1}{p-1}$

Soln. Here $f_1(t) = L^{-1}\left[\frac{1}{p^{-\frac{1}{2}+1}}\right] = \frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi t^{\frac{1}{2}}}}$, by (3.12)

$$f_2(t) = L^{-1}\left[\frac{1}{p-1}\right] = e^t L^{-1}\left[\frac{1}{p}\right] = e^t, \text{ by shifting theorem and (3.12b)}$$

$$\therefore L^{-1}\left[\frac{1}{\sqrt{p}(p-1)}\right] = \int_0^t \frac{1}{\sqrt{\pi\tau^{\frac{1}{2}}}}e^{t-\tau}d\tau \text{ by (3.26)}$$

$$= \frac{e^t}{\sqrt{\pi}}\int_0^t \tau^{-\frac{1}{2}}e^{-\tau}d\tau$$

IV. INVERSION BY BREAKING UP IN PARTIAL FRACTIONS :

We consider here a method of finding $L^{-1}\left[\frac{Q_m(p)}{P_n(p)}\right]$, where $P_n(p)$ and $Q_m(p)$

are polynomials in p of degree n and m respectively. We note that since $Q_m(p)/P_n(p)$ is the Laplace transform of a function, which we are going to determine, according to a property stated in Theorem-5 of Section (3.5) $Q_m(p)/P_n(p) \to 0$ as $P \to \infty$. Therefore the degree of the polynomial $Q_m(p)$ is less than that of $P_n(p)$, *i.e.* $m < n$ and consequently $Q_m(p)/P_n(p)$ is a proper rational fraction.Now to find the Laplace inversion of such a proper rational fraction, we are to first decompose it into partial fractions and then find Laplace inversion of each of these partial fractions.

Now to decompose $Q_m(p) / P_n(p)$ into partial fractions we first factorize $P_n(p)$ into real factors, which may be of four types : (i) non-repeated real linear factor (ii) repeated real linear factor (iii) non-repeated real quadratic factor (iv) repeated real quadratic factor. To each of these four types of factors, the partial fractions that arise in the expression for $Q_m(p) / P_n(p)$ are the following : (i) Non-repeated real linear factor : To each non-repeated real linear factor $p - a$ there appears a partial fraction $\frac{A}{p-a}$ in the decomposition of $Q_m(p) / P_n(p)$ into partial fractions.

(ii) Repeated real linear factor : To each repeated real linear factor $(x - b)^r$, there appear the partial fractions

$$\sum_{j=1}^{r} \frac{Bj}{(p-b)^j}$$

in the decomposition of $Q_m(p) / P_n(p)$ into partial fractions.

(iii) Non-repeated real quadratic factor : To each non-repeated real quadratic factor $p^2 + cp + d$, there appears the partial fraction.

$$\frac{Cp + D}{p^2 + cp + d}$$

in the decomposition of $Q_m(p) / P_n(\text{p})$ into partial fractions.

(iv) Repeated real quadratic factor : To each repeated real quadratic factor $(p^2 + c'p + d')^r$, there appear the partial fractions

$$\sum_{j=1}^{r} \frac{C_j p + D_j}{(p^2 + c'p + d')j}$$

in the decomposition of $Q_m(p) / P_n(p)$ into partial fractions.

The constants A, B_j's, C, D, C_j's, D_j's can be determined by the standard existing method.

The determination of Laplace inversion of a rational fraction $Q_m(p)/P_n(p)$ then reduces to the determination of the same for the partial fractions to which the rational fraction has been resolved. Finally the Laplace inversion of these partial fractions can be evaluated by the methods given in sections. I to III.

Ex. 3.6.8. Find $L^{-1}\left[\dfrac{4p^4 - 14p^3 + 30p^2 - 3p - 117}{p^2(p-3)(p^2 - 4p + 13)}\right]$

Soln. Let $\dfrac{4p^4 - 14p^3 + 30p^2 - 3p - 117}{p^2(p-3)(p^2 - 4p + 13)} = \dfrac{A}{p-3} + \dfrac{B_1}{p} + \dfrac{B_2}{p^2} + \dfrac{Cp + D}{p^2 - 4p + 13}$

$$4p^4 - 14p^3 + 30p^2 - 3p - 117 = Ap^2 (p^2 - 4p + 13) + B_1 p(p^2 - 4p + 13)$$

$$(p - 3) + B_2 (p - 3)(p^2 - 4p + 13) + (Cp + D) p^2(p - 3)$$

Setting p = 0, $-117 = -39B_2 \quad \therefore B_2 = 3$

Setting p = 3, $90 = 90A \quad \therefore A = 1$

Equating coeff. of p^4, $4 = A + B_1 + C$

Equating coeff. of p^3, $-14 = -4A - 7B_1 + B_2 + D - 3C$

Equating coeff. of p^2, $30 = 13A + 25B_1 - 7B_2 - 3D$

Solving the above equations we get,

$$A_1 = 1, B_1 = 2, B_2 = 3, C = 1, D = 4$$

Therefore, $L^{-1}\left[\dfrac{4p^4 - 14p^3 + 30p^2 - 3p - 117}{p^2(p-3)(p^2 - 4p + 13)}\right]$

$$= L^{-1}\left[\frac{1}{p-3}\right] + 2L^{-1}\left[\frac{1}{p}\right] + 3L^{-1}\left[\frac{1}{p^2}\right] + L^{-1}\left[\frac{p+4}{p^2 - 4p + 13}\right]$$

$$= e^{3t} + 2 + 3t + L^{-1}\left[\frac{p-2}{(p-2)^2 + 3^2}\right] + 2L^{-1}\left[\frac{3}{(p-2)^2 + 3^2}\right]$$

$$= e^{3t} + 2 + 3t + e^{2t}\cos 3t + 2e^{2t}\sin 3t$$

V. INTERGRAL FORMULAS FOR LAPLACE INVERSION

If it is known that $F(p)$ is the Laplace transform of a function $f(t)$, which vanishes for $t < 0$, piecewise continuous in any finite interval of t and is of exponential order, then we have the following theorem for construction of the function $f(t)$ from $F(p)$.

Theorem-7. Let it be known that a given function $F(p)$ of complex variable p in the domain $\mathrm{Re}(p) > a$ is the Laplace transform of a function $f(t)$ of real variable t, which is such that (i) $f(t) = 0$, for $t < 0$, (ii) in any finite interval of t, the function $f(t)$ is piecewise continuous and (iii) $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$. Then

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}dp, \text{ where } r > a \tag{3.27}$$

Proof. Let us consider the function

$$\varphi(t) = e^{-rt}f(t), r > a$$

f(t) being of exponential order, there exist two positive constants a and M such that

$$|f(t)| \le Me^{at}$$

Therefore, $\int_{-\infty}^{\infty} |\varphi(t)dt = \int_{0}^{\infty} e^{-rt}|f(t)|dt \le M\int_{0}^{\infty} e^{-t(r-a)}dt$

which is convergent since $r > a$.

In any finite invertval of t, the function $\varphi(t)$ is piecewise continuous, since $f(t)$ in so. Moreover the integral $\int_{-\infty}^{\infty}|\varphi(t)|dt$ is convergent. So in any finite interval of t the function $\varphi(t)$ satisfies Dirichlet conditions. Thus we can use Fourier inversion theorem to get $\varphi(t)$ from its Fourier transform $\overline{\varphi}(\xi)$ which is

$$\overline{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(t)e^{i\xi t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(-t)e^{i\xi t}dt$$

Therefore by Fourier inversion theorem we have at places of continuity,

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\varphi}(\xi) e^{-i\xi t} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\varphi}(-\xi) e^{i\xi t} d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi e^{i\xi t} \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-r\eta} f(\eta) e^{-i\xi\eta} d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\xi t} \int_{0}^{\infty} f(\eta) e^{-\eta(r+i\xi)} d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\xi t} \int_{0}^{\infty} f(\eta) e^{-\eta p} d\eta, \; p = r + i\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\xi t} F(p), \; dp = i d\xi$$

$$= \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} F(p) e^{(p-r)t} dp$$

Hence, $f(t) = \dfrac{1}{2\pi i} \displaystyle\int_{r-i\infty}^{r+i\infty} F(p) e^{pt} dt$

This completes the proof of the theorem.

Now the question that naturally arises is whether any function $F(p)$ of complex variable p is the Laplace transform of a function $f(t)$ of real variable t, and if so, how to construct $f(t)$ from $F(p)$. The following theorem gives certain sufficient conditions under which a given function $F(p)$ of complex variable p is the Laplace transform of a function $f(t)$ of real variable t and also gives the method of constructing this function $f(t)$ from the function $F(p)$.

Theorem-8. Let $F(p)$ be any function of complex variable p, which is analytic and is of order $O(p^{-k})$*, $(k > 1)$ for $\text{Re}(p) > a$; also $F(x)$ is real for real $x > a$. Then the integral.

---

*By saying that $F(p)$ is of order $O(p - k)$, we mean that there exists a constant M (> 0) such that $|F(p)| < M/|p|k$ for sufficient large $|p|$.

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp$$

is independent of r whenever $r > a$, and the function $f(t)$ defined by

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp,\ r > a \tag{3.28}$$

has the following properties :

(i) $f(t)$ is continuous for all t, (ii) $f(t)$ is of exponential order $O(e^{at})$ at $t \to \infty$, (iii) $f(t) = 0$ for $t < 0$ and (iv) $F(p)$ is the Laplace transform of $f(t)$.

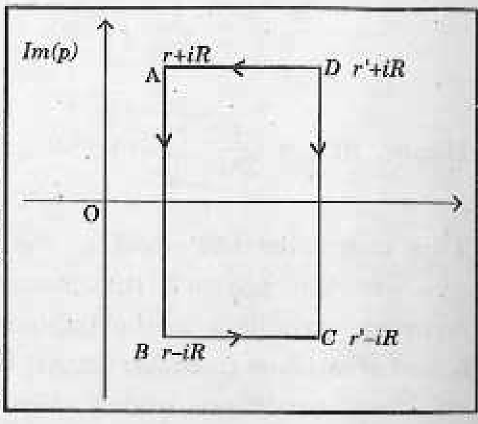
Proof. To prove that the integral

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp$$

is independent of r, we consider the integral

$$\int_{\Gamma} e^{pt}F(p)dp$$

round the closed contour Γ consisting of straight line segments AB, BC, CD, DA joining the points $r + iR$ to $r - iR$, $r - iR$ to $r' - iR$, $r' - iR$ to $r' + iR$, $r' + iR$ to $r + iR$ respectively, where $R > 0$, $r' > r$, $r > a$.



Since the function e^{pt} $F(p)$ is analytic in the region $\text{Re}(p) > a$ and since $r > a$, by Cauchys integral theorem we have

$$\int_{\Gamma} e^{pt}F(p)dp = 0$$

$$\text{or}, -\int_{r-iR}^{r+iR} e^{pt}F(p)dp + \int_{r'-iR}^{r'+iR} e^{pt}F(p)dp + \int_{BC} e^{pt}F(p)dp + \int_{DA} e^{pt}F(p)dp = 0 \tag{3.28a}$$

On BC, $p = x - iR$, therefore we have

$$\left|\int_{\overline{BC}} e^{pt}F(p)dp\right| \le \int_r^{r'} e^{xt}|F(p)|\,dx < M\int_r^{r'} \frac{e^{xt}}{|p|^k}dx$$

$$= M\int_r^{r'} \frac{e^{xt}dx}{\left(x^2+R^2\right)^{k/2}} < \frac{M}{R^k}\int_r^{r'} e^{xt}dx = \frac{M}{R^k}\cdot\frac{e^{r't}-e^{rt}}{t} \to 0 \text{ as } R \to \infty$$

since $k > 1$.

Therefore, $\int_{\overline{BC}} e^{pt}F(p)dp \to 0$ as $R \to \infty$

Similarly we can show that

$$\int_{\overline{DA}} e^{pt}F(p)dp \to 0 \text{ as } R \to \infty$$

So proceeding to the limit $R \to \infty$ in (3.28a) we get

$$\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp = \int_{r'-i\infty}^{r'+i\infty} e^{pt}F(p)dp$$

whenever $r, r' > a$, provided the integrals exists, *i.e.* the integral on the left hand side is independent of r, if $r > a$.

We now prove the existence of this integral and continuity of $f(t)$ given by (3.28).

Let $F(r + iy) = u(r, y) + iv(r, y)$

Since $F(x)$ is real for real x,

$$F(r - iy) = u(r, y) - iv(r, y)$$

So $|u|, |v| < \left(u^2+v^2\right)^{\frac{1}{2}} = |F(r+iy)| < \dfrac{M}{\left(r^2+y^2\right)^{\frac{k}{2}}} < \dfrac{M}{|y|^k}$

Whenever $|y| > y_0$.

Now, $f(t) = \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp = \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} e^{(r+iy)t}F(r+iy)i\,dy$

$$= \frac{1}{2\pi} e^{rt} \left[\int_0^\infty e^{iyt} F(r+iy) + \int_0^\infty e^{-iyt} F(r-iy) dy \right]$$

$$= \frac{1}{2\pi} e^{rt} \int_0^\infty \left[e^{iyt} \{u(r,y) + iv(r,y)\} + e^{-iyt} \{u(r,y) - iv(r,y)\} \right] dy$$

$$= \frac{1}{\pi} e^{rt} \int_0^\infty [u(r,y) \cos yt - v(r,y) \sin yt] dy$$

$$= \frac{1}{\pi} e^{rt} \left[\int_0^{y_0} g(y,t) dy + \int_{y_0}^\infty g(y,t) dy \right]$$

where $g(y, t) = u\,(r, y)\cos yt - v\,(r, y)\sin yt$

$$\left| \int_{y_0}^\infty g(y,t) dy \right| \leq \int_{y_0}^\infty |g(y,t)| dy \leq \int_{y_0}^\infty [|u(r,y)| + |v(r,y)|] dy$$

$$< 2M \int_{y_0}^\infty \frac{dy}{y^k} = \frac{2k}{k-1} \frac{1}{y_0^{k-1}}, \; k-1 > 0$$

Hence the integral $\int_{y_0}^\infty g(y,t) dy$ is uniformly convergent with respect to t. Further, since $F(p)$ is analytic in the domain $\text{Re}(p) > a$, its real and imaginary parts are continuous in this domain and consequently the functions $u(r, y)$ and $v(r, y)$ are continuous functions of y. Therefore the integrands of (3.28b) being continuous functions of y and t and the second integral being uniformly convergent with respect to t, both the integrals in (3.28b) represent continuous functions of t. Hence the function $f(t)$ is a continuous function to t.

We now show that $f(t)$ is of exponential order. From (3.28b) we get

$$|f(t)| \leq \frac{e^{rt}}{\pi} \left[\int_0^{y_0} |g(y,t)| dy + \int_{y_0}^\infty |g(y,t)| g(y,t)| dy \right]$$

$$\leq \frac{e^{rt}}{\pi}\left[A + \frac{2M}{k-1}\cdot {y_0}^{k+1}\right],$$

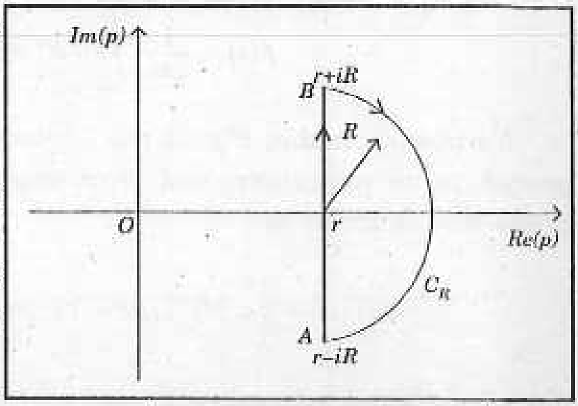
since the first integral being bounded is less than A.

$= Be^{rt}$ where B is a positive constant

Hence $f(t)$ is of exponential order $O(e^{rt})$ at $t \to \infty$, where $r > a$.

Next we prove that $f(t) = 0$ for $t < 0$. To prove this we consider the integral

$$\int_c e^{pt}F(p)dp$$



taken round the closed contour C consisting of the straight line segment AB joining point $r - iR$ to $r + iR$ and a semicircular are $C_R : |p - r| = R, -\frac{\pi}{2} < \arg(p - r) \leq \frac{\pi}{g}$, where $r > a$.

Since C lies in the region in which e^{pt} $F(p)$ in analytic, we have by Cauchy's integral theorem

$$\int_c e^{pt}F(p)dp = 0$$

$$\text{or,} \quad \int_{r-iR}^{r+iR} e^{pt}F(p)dp + \int_{C_R} e^{pt}F(p)dp = 0 \qquad (3.28c)$$

Since on C_R, $p = r + Re^{i0}$, we have

$$\left|\int_{C_R} e^{pt}F(p)dp\right| \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|e^{(r+Re^{i0})t}\right|\left|F(r + Re^{i\theta})\right|\left|Rie^{i\theta}\right|d\theta$$

$$\leq e^{rt}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{Rt\cos 0}\frac{MR}{|r + \mathrm{Re}^{i0}|k}d0 \leq \frac{Me^{rt}}{R^{k-1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d0$$

$$= \frac{M\pi e^{rt}}{R^{k-1}}\left\{\text{since } t < 0 \text{ and } \cos\theta \geq 0 \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ we have } e^{Rt\cos\theta} < 1\right\}$$

$\to 0$ as $R \to \infty$, since $k - 1 > 0$.

Therefore, proceeding to the limit $R \to \infty$ in (3.28c) we get

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp = 0, \text{for } t < 0 \tag{3.28d}$$

We now show that $F(p)$ is the Laplace transform of $f(t)$. Since f(t) has been proved to be continuous and is of exponential order, its Laplace transform exists and is given by

$$L[f(t)] = \int_0^\infty e^{-pt}f(t)\,dt = \int_0^\infty e^{-pt}dt\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{qt}F(q)dq$$

Since the second integral is independent of r, provided $r > a$, we can take $\text{Re}(p) > r > a$ and $r = \text{Re}(q)$. We have proved that the second integral is uniformly convergent with respect to t. Hence we can change the order of integration *in* the above and get

$$L[f(t)] = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} dqF(q)\int_0^\infty e^{-(p-q)t}dt = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} dqF.\frac{1}{p-q} \tag{3.28e}$$

since, $\text{Re}(p) > r = \text{Re}(q)$, *i.e,* $\text{Re}(p) - \text{Re}(q) > 0$.

The function $\frac{F(q)}{p-q}$ is analytic in the domain $\text{Re}(q) > a$ except at the point $q = p$, where the function has a simple pole. We consider the integral

$$\frac{1}{2\pi i}\int_{c'}\frac{F(q)}{p-q}dq$$

taken round the contour C' consisting of straight line segment AB joining the points $r + iR$ to $r - iR$ and a semicircular arc $C_R'; |q - r| = R, -\frac{\pi}{2} \leq \arg\,(q - r) \leq \frac{\pi}{2}$.

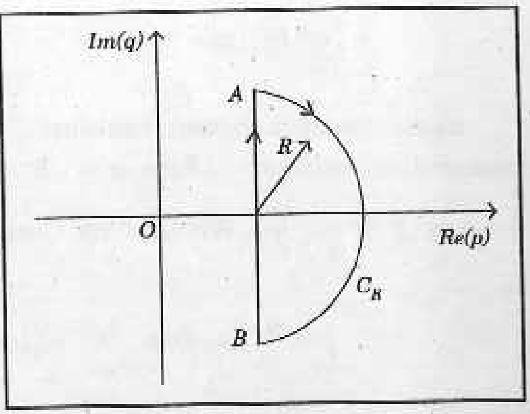
Since, $\text{Re}(p) > r > a$, the pole $q = p$ lies inside the contour C', if R is sufficiently large. Therefore by Cauchy's residue theorem we have

$$\frac{1}{2\pi i}\int_{r+iR}^{r-iR}\frac{F(q)}{p-q}dq+\frac{1}{2\pi i}\int_{C_R}\frac{F(q)}{p-q}dq=-F(p) \tag{3.28f}$$

Now, $\left|\int_{C_R}\frac{F(q)dq}{p-q}\right|\le\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{|F(r+\mathrm{Re}^{i\theta})||Rie^{i\theta}|d\theta}{|p-r-\mathrm{Re}^{i\theta}|}$

$$\le\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{MRd\theta}{|p-r-\mathrm{Re}^{i\theta}||r+\mathrm{Re}^{i\theta}|^k}\simeq\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{MRd\theta}{|\mathrm{Re}^{i\theta}||\mathrm{Re}^{i\theta}|^k} \text{ for sufficiently large } R$$

$$=\frac{M\pi}{R^k}\to 0 \text{ as } R\to\infty.$$



Therefore, $\int_{C_R}\frac{F(q)dq}{p-q}\to 0$ as $R\to\infty$.

So proceeding to the limit $R\to\infty$ in (3.28f) we get

$$\frac{1}{2\pi i}\int_{r+i\infty}^{r-i\infty}\frac{F(q)}{p-q}dq=-F(p)$$

or, $$L[f(t)]=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{F(q)}{p-q}dq=F(b), \text{ by (3.28e)}$$

This completes the proof of part (iv) of the theorem.

Evaluation of inversion integrals : In many cases of applications, the inversion integrals (3.27) or (3.28), which expresses the original function in terms of a given function $F(p)$ of complex variable p, can be evaluated by the help of the theory of residues. Let $F(p)$, which is defined and analytic in the domain $\mathrm{Re}(p) > a$, can be continued analytically in the domain $\mathrm{Re}(p) < a$ with exception of a finite number of isolated singularities and let its analytic continuation $F_1(p)$ tends uniformly to zero in arg $(p-r), \left(\frac{\pi}{2} \le \arg(p-r) \le \frac{3\pi}{2}\right)$ as $|p-r| \to \infty$, then for $t > 0$ (according to Jordan's lemma)

$$\int_{C_R} e^{pt} F_1(p)dp \to 0 \text{ as } R \to \infty,$$

where C_R is the arc of the semicircle $|p-r| = R$, $r > a$ in the left hand half plane. In this case the integral in (3.27) or (3.28) can be evaluated with the aid of the theory of residues.

For example, if $F(p)$ itself be its analytic continuation in the domain $\text{Re}(p) < a$, with exception of a finite number of isolated poles, then consider the integral

$$\int_C e^{pt} F(p)dp$$

taken round a closed contour C consisting of straight line segment AB joining the points $r - iR$ to $r + iR$ and a semicircular arc $C_R : |p-r| = R$, $\frac{\pi}{2} \le \arg(p-r) \le \frac{3\pi}{2}$. We find by Cauchy's residue theorem that

$$\int_{r-iR}^{r+iR} e^{pt} F(p)dp + \int_{C_R} e^{pt} F(p)dp = 2\pi i \times \text{(sum of}$$

the residues of $F(p)\, e^{pt}$ at its poles lying within (C).

If $F(p)$ tends uniformly to zero in arg $(p-r)$, $\left(\frac{\pi}{2} \le \arg(p-r) \le \frac{3\pi}{2}\right)$ as $|p-r| = R \to \infty$, then according to Jordan's lemma the second integral in (3.28g) vanishes as $R \to \infty$. Therefore proceeding to the limit $R \to \infty$, in (3.28g) we get

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp = \sum_{k=1}^{n} R_k, \text{ by (3.28)} \tag{3.28h}$$

where R_k is the residue of $e^{pt} F(p)$ at its k-th pole $p = p_k$. If there are infinite number of poles of $F(p)$, then also (3.28h) holds good provided in that case the sum $\sum_{k=1}^{\infty} R_k$ is convergent.

The condition imposed on $F(p)$ that it is of order $O(p^{-k})$, $k > 1$, stated in Theorem 8 is only a sufficient condition and not necessary. The Laplace inversion of $F(p)$ may exist even if it does not satisfy this conditin we shall see this in an exmaple of this section.

Ex. 3.6.8. Find by evaluating the inversion integral the function whose Laplace transform is

$$F(p) = \frac{2\omega p}{\left(p^2 + \omega^2\right)^2}.$$

Soln. The singularities of the function $F(p)$ are at $p = \pm\, i\omega$, the real parts of which are zero. So we can take $r > 0$ in the inversion integral. $F(p)$ is analytic in the domain $\text{Re}(p) > r$ and the function itself is its analytic continuation in the domain $\text{Re}(p) < r$, except at the singularities mentioned above. Hence the function $f(t)$, whose Laplace transform is $F(p)$, is given by the inversion integral.

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp$$

$$= R_1 + R_2, \text{ by (3.28h)}$$

since $F(p)$ tends uniformly to zero in arg $(p - r)$, $\frac{\pi}{2} \le \arg(p-r) \le \frac{3\pi}{2}$, as $|p - r| \to \infty$. Here R_1 and R_2 are the residues of $e^{pt}\, F(p)$ at $p_1 = i\omega$ and $p_2 = -i\omega$. As both of the two singularies are poles of order 2, we have

$$R_1 = \lim_{p\to i\omega} \frac{d}{dp}\left\{(p-i\omega)^2 \frac{e^{pt}2\omega p}{\left(p^2+\omega^2\right)^2}\right\} = \lim_{p\to i\omega} \frac{d}{dp}\frac{2\omega p e^{pt}}{(p+i\omega)^2}$$

$$= \frac{2\omega(e^{i\omega t} + i\omega t e^{i\omega t})}{(2i\omega)^2} - \frac{4\omega i\omega e^{i\omega t}}{(2i\omega)^3} = \frac{t}{2i}e^{i\omega t}$$

Similarly, $R_2 = -\frac{t}{2i}e^{-i\omega t}$

Therefore $f(t) = \frac{t}{2i}(e^{i\omega t} - e^{-i\omega t}) = t\sin\omega t$

Ex. 3.6.9. Find the function whose Laplace transform is, $\frac{\sinh x\sqrt{p}}{p\sinh a\sqrt{p}}$, where both x and a are positive and $a - x > 0$.

Soln. Let $F(p) = \frac{\sinh x\sqrt{p}}{p\sinh a\sqrt{p}}$, From this expression it appears that $p = 0$ is a branch point pf $F(p)$, but this is not so, as can be seen by substituting the expansions of the hyperbolic functions, which is shown below.

$$\text{F(p)} = \frac{x\sqrt{p} + \frac{1}{3!}(x\sqrt{p})^3 + \frac{1}{5!}(x\sqrt{p})^5 + \ldots}{p\left[a\sqrt{p} + \frac{1}{3!}(a\sqrt{p})^3 + \frac{1}{5!}(a\sqrt{p})^5 + \ldots\right]}$$

$$= \frac{x\left(1 + \frac{1}{3!}x^2 p + \frac{1}{5!}x^4 p^2 + \ldots\right)}{ap\left[1 + \frac{1}{3!}a^2 p + \frac{1}{5!}a^4 p^2 + \ldots\right]}$$

This expression shows that $p = 0$ is not a branch point of $F(p)$ but is a simple pole and other singularities of $F(p)$ are at the zeros of $\sinh a\sqrt{p}$ except at $p = 0$, which are all simple poles. Now $\sinh a\sqrt{p} = 0$ when $a\sqrt{p} = \pm in\pi$, $n =$ 0, 1, 2, ..., i.e. when $p = -\frac{n^2\pi^2}{a^2}$, $n = 0, 1, 2, \ldots$. Therefore the singularities of $F(p)$ are at

$$p = 0, p = -\frac{n^2\pi^2}{a^2},\ n = 1,2,\ldots.,$$

Now which are all simple poles.

$$R_0 = [\text{residue of } e^{pt} F(p) \text{ at } p = 0] = \lim_{p\to 0}\frac{\sinh x\sqrt{p}}{\sinh a\sqrt{p}} = \frac{x}{a}$$

$$R_n = [\text{ residue of } e^{pt} F(p) \text{ at } p = \frac{n^2\pi^2}{a^2}\ ]\cdot$$

$$= \lim_{p\to -\frac{n^2\pi^2}{a^2}}\left(p + \frac{n^2\pi^2}{a}\right)\frac{\sinh(x\sqrt{p})e^{pt}}{p\sinh(a\sqrt{p})} = \lim_{p\to -\frac{n^2\pi^2}{a^2}}\frac{\sinh(x\sqrt{p})e^{pt}}{p\cosh(a\sqrt{p})\frac{1}{2}ap^{-\frac{1}{2}}}$$

$$= \frac{2\sinh\left(\pm ix\frac{n\pi}{a}\right)e^{-\frac{n^2\pi^2}{a^2}t}.(\pm in\pi)}{\left(-\frac{n^2\pi^2}{a^2}\right)\cosh\left(\pm ia\frac{n\pi}{a}\right)a^2} = \frac{2}{n\pi}(-1)^n e^{-\frac{n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a}$$

(1) for $\frac{\pi}{2} \le \arg(p-r) = \theta \le \pi$, we have

$$F(p) = \frac{\sinh\left(x\sqrt{r+\mathrm{Re}^{i\theta}}\right)}{(r+\mathrm{Re}^{i\theta})\sinh\left(a\sqrt{r+\mathrm{Re}^{i\theta}}\right)} \simeq \frac{\sinh\left(xR^{\frac{1}{2}}e^{i\frac{\theta}{2}}\right)}{\mathrm{Re}^{i\theta}\sinh\left(aR^{\frac{1}{2}}e^{i\frac{\theta}{2}}\right)}, \text{ for large } R$$

$$= \frac{\exp\left\{xR^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}}{\mathrm{Re}\,xp\left\{aR^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}} = \frac{1}{R}\exp\left\{-(a-x)R^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\},$$

$$\left[\because \cos\frac{\theta}{2} > 0\right]$$

Therefore, $|F(p)| = \frac{1}{R}e^{-(a-x)R^{\frac{1}{2}}\cos\frac{\theta}{2}} \to 0$ as $R = |p-r| \to \infty$

since $a - x > 0$ and $\cos\frac{\theta}{2} > 0$.

(ii) For $\pi \le \arg(p-r) = \theta \le \frac{3\pi}{2}$, we have

$$F(p) \approx \frac{\exp\left\{-xR^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}}{R\exp\left\{-aR^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}}, \text{ since } \cos\frac{\theta}{2} < 0$$

$$= \frac{1}{R}\exp\left\{(a-x)R^{\frac{1}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right\}$$

Therefore $|F(p)| = e^{(a-x)R^{\frac{1}{2}}\cos\frac{\theta}{2}} \to 0$ as $R \to \infty$,

since $a - x > 0$ and $\cos\dfrac{\theta}{2} < 0$.

Hence, $F(p) \to 0$ uniformly in arg $(p - r)$, $\left(\frac{\pi}{2} \le \arg(p-r) \le \frac{3\pi}{2}\right)$ as $|p - r| \to \infty$. Consequently the function $f(t)$, whose Laplace transform in $F(p)$, is given by

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp,\ r > 0$$

(since real part of all the singularities are zero)

$$= \sum_{n=0}^{\infty} R_n, \text{ by (2.9)}$$

$$= \frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-\frac{n^2\pi^2 t}{a^2}} \sin\frac{n\pi x}{a}$$

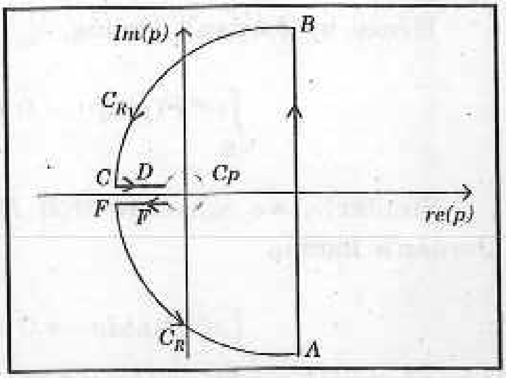
Ex.3.6.10. By evaluating the inversion integral find the function whose Laplace transform is $\dfrac{1}{p^{\alpha+1}}, -1 < \alpha < 0$.

Soln. Let $F(p) = \dfrac{1}{p^{\alpha+1}}$. This function has only one singularity at $p = 0$, which is a branch point. In the whole complex plane with a branch cut along negative real axis, the function $F(p)$, which is that branch of the multiple-valued function that assumes positive values on positive real axis, is analytic.

The function $f(t)$, whose Laplace transform is $F(p)$ is given by the following inversion integral

$$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp,\ r > 0 \tag{3.29a}$$

We consider a closed contour C as shown in figure consisting of (i) straight line segment AB joining points $r - iR$ to $r + iR$ and straight line segments CD and EF drawn just above and below negative real axis, (ii) circular arcs $C_R : |p - r| = R, \frac{\pi}{2} \le \arg(p - r) < \pi$, $C'_R : |p - r| = R, -\pi < \arg(p - r) \le -\frac{\pi}{2}$ and $C_\rho : |p| = \rho, -\pi < \arg(p) < \pi$.



As C is drawn in the domain in which $e^{pt} F(p)$ is analytic and single-valued by Cauchy's integral theorem we have

$$\int_C e^{pt} F(p)\,dp = 0$$

$$\text{or, } \int_{r-iR}^{r+iR} e^{pt}F(p)dp + \int_{\overline{CD}+\overline{EF}} e^{pt}F(p)dp + \int_{C_R+C'_R} e^{pt}F(p)dp + \int_{C_\rho} e^{pt}F(p)dp = 0 \tag{3.29b}$$

(i) on $C\rho$, $p = \rho e^{i\theta}$

$$\text{Therefore, } \lim_{\rho\to 0}\int_{C_\rho} e^{pt}F(p)dp = \lim_{p\to 0}\int_{\pi}^{-\pi} e^{\rho t e^{i\theta}} \frac{i\rho e^{i\theta}d\theta}{\rho^{\alpha+1}e^{i(\alpha+1)\theta}}$$

$$= -\int_{-\pi}^{\pi} \lim_{p\to 0} e^{\rho t e^{i\theta}} \frac{\rho^{-\alpha} i d\theta}{e^{i\alpha\theta}} = 0, \text{ since } -\alpha > 0.$$

(2) On C_R, $p = r + Re^{i\theta}$.

$$\text{Therefore, } |F(p)| = \frac{1}{|r + \text{R}e^{i\theta}|^{\alpha+1}} \simeq \frac{1}{R^{\alpha+1}} \text{ for large R}$$

$$\to 0 \text{ as } R \to \infty, \text{ since } \alpha + 1 > 0,$$

and consequently $F(p) \to 0$ uniformly in C_R

Hence by Jordan's lemma,

$$\int_{C_R} e^{pt} F(p)dp \to 0 \text{ as } R \to \infty.$$

Similarly, we can show that $F(p) \to 0$ uniformly in C'_R and therefore by Jordan's lemma

$$\int_{C_R} e^{pt} F(p)dp \to 0 \text{ as R} \to \infty.$$

Therefore proceeding to the limit $R \to \infty$ and $\rho \to 0$ in (3.29b) we get (since on CD, $p = xe^{i\pi}$ and on EF, $p = xe^{-i\pi}$)

$$\int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp - \int_0^\infty \frac{e^{-xt e^{i\pi}} dx}{x^{\alpha+1} e^{i(\alpha+1)\pi}} + \int_0^\infty \frac{e^{-xt e^{-i\pi}} dx}{x^{\alpha+1} e^{-i(\alpha+1)\pi}} = 0$$

$$\text{or, } f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp = \frac{\sin(-\alpha\pi)}{\pi} \int_0^\infty \frac{e^{-xt}}{x^{\alpha+1}} dx, \text{ by (3.29a)}$$

$$= \frac{\sin(-\alpha\pi)}{\pi} t^\alpha \int_0^\infty e^{-u} u^{-\alpha-1} du, xt = u$$

$$= \frac{t^\alpha}{\pi} \sin(-\alpha\pi)\Gamma(-\alpha)$$

$$= \frac{t^\alpha}{\Gamma(\alpha+1)}$$

where we have used the relation $\Gamma(q)\,\Gamma(1-q) = \frac{\pi}{\sin \pi q}$ and set $q = -\alpha$.

Ex. 3.6.11. Find by evaluating the inversion integral the function whose Laplace transform $\frac{1}{p} e^{-\alpha\sqrt{p}}$, $\alpha > 0$.

Soln. Let $F(p)=\frac{1}{p}e^{-a\sqrt{p}}$. The only singularity of this function is at $p = 0$, which is a branch point. In the whole complex plane with a branch cut along negative real axix the function $F(p)$ with that branch of $\sqrt{p}$ which assumes positive values of positive real axis is analytic and single-valued.

The function $f(t)$, whose Laplace transform is $F(p)$, is given by the inversion integral

$$f(t)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}e^{pt}F(p)\,dp \tag{3.30a}$$

We take the same closed contour C as in Ex. 3.6.10. As C is drawn in the domain in which $e^{pt}F(p)$ is analytic and single-valued, we have by Cauchy's integral theorem

$$\int_C e^{pt}F(p)dp=0$$

$$\text{or,}\quad \int_{r-iR}^{r+iR}e^{pt}F(p)dp+\int_{\overline{CD}+\overline{EF}}e^{pt}F(p)dp+\int_{C_R+C'_R}e^{pt}F(p)dp+\int_{C_p}e^{pt}F(p)dp=0 \tag{3.30b}$$

(1) On C_ρ, $p = \rho e^{i\theta}$

$$\text{Therefore,}\quad \lim_{\rho\to 0}\int_{C_p}e^{pt}F(p)dp=\lim_{\rho\to 0}\int_{\pi}^{-\pi}e^{t\rho e^{i\theta}}\frac{e^{-\alpha\sqrt{\rho e^{i\frac{\theta}{2}}}}}{\rho e^{i\theta}}.i\rho e^{i\theta}d\theta$$

$$=-i\int_{-\pi}^{\pi}\lim_{\rho\to 0}e^{t\rho e^{i\theta}-\alpha\sqrt{\rho e^{i\frac{\theta}{2}}}}d\theta=-i\int_{-\pi}^{\pi}d\theta=-2\pi i$$

(2) On C_R, $p=r+\mathrm{Re}^{i\theta},\frac{\pi}{2}\le\theta<\pi$

$$\text{Therefore,}\quad |F(p)|\approx\left|\frac{e^{-\alpha\sqrt{R}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}}{\mathrm{Re}^{i\theta}}\right|,\ \text{for large } R$$

$$=\frac{e^{-\alpha\sqrt{R}\cos\frac{\theta}{2}}}{R}<\frac{1}{R}\to 0\ \text{as}\ R\to\infty,\ \text{since}\ \cos\frac{\theta}{2}>0$$

for $\frac{\pi}{2}\le\theta<\pi$ and $\alpha>0$.

Consequently $F(p) \to 0$ uniformly in C_R and so by Jordon's lemma we get

$$\int_{C_R} e^{pt} F(p)dp \to 0 \text{ as } R \to \infty$$

(3) On C'_R, $p = r + \text{Re}^{i\theta}$, $-\pi < \theta \le -\frac{\pi}{2}$

Therefore, $|F(p)| \approx \left| \frac{e^{-\alpha\sqrt{R}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}}{\text{Re}^{i\theta}} \right|$, for large R

$$= \frac{e^{-\alpha\sqrt{R}\cos\frac{\theta}{2}}}{R} < \frac{1}{R} \to 0 \text{ as } R \to \infty, \text{ since } \cos\frac{\theta}{2} > 0 \text{ for } -\pi < \theta \le -\frac{\pi}{2} \text{ and } \alpha > 0.$$

Consequently it follows as stated for C_R that

$$\int_{C'_R} e^{pt} F(p)dp \to 0 \text{ as } R \to \infty$$

Therefore proceeding to the limit $R \to \infty$ and $\rho \to 0$ is (3.30b) we get

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp = 1 + \frac{1}{2\pi i} \int_0^\infty \frac{e^{-xt} . e^{-\alpha\sqrt{xe}^{i\frac{\pi}{2}}}}{xe^{i\pi}} e^{i\pi}\, dx$$

$$-\frac{1}{2\pi i} \int_0^\infty \frac{e^{-xt} . e^{-a\sqrt{xe}^{-i\frac{\pi}{2}}}}{xe^{-i\pi}} e^{-i\pi} dx,$$

(since as CD, $p = xe^{i\pi}$ and on EF, $p = xe^{-i\pi}$

$$= 1 - \frac{1}{\pi} \int_0^\infty \frac{e^{-xt}}{x} \sin(a\sqrt{x})dx$$

$$= 1 - \frac{2}{\pi} \int_0^\infty e^{-ty^2} \frac{\sin \alpha y}{y} dy, \text{ (by the substitution } x = y^2) \tag{3.30c}$$

We now express this in terms of error function as follows.

We first evaluate the integral $\int_{-\infty}^{\infty} e^{-tx^2 + i\alpha x} dx.$

$$\int_{-\infty}^{\infty} e^{-tx^2 + i\alpha x} dx = e^{-\frac{a^2}{4t}} \int_{-\infty}^{\infty} e^{-t\left(x - \frac{i\alpha}{2t}\right)} dx$$

$= e^{-\frac{\alpha^2}{4t}} \int_C e^{-tz^2} dz$, where the integration is along a line C through

$z = -\frac{i\alpha}{2t}$ parallel to real axis in the complex z-plane.

$= e^{-\frac{\alpha^2}{4t}} \int\limits_{Re(z)=0} e^{-tz^2} dz$, since the integrand is analytic in the whole

complex plane

$$= e^{-\frac{\alpha^2}{4t}} \int_{-\infty}^{\infty} e^{-tx^2} dx = 2e^{-\frac{\alpha^2}{4t}} \int_0^{\infty} e^{-tx^2} dx$$

$$= e^{-\frac{\alpha^2}{4t}} \frac{1}{\sqrt{t}} \int_0^{\infty} e^{-u} u^{\frac{1}{2}-1} du = \sqrt{\frac{\pi}{t}} e^{-\frac{\alpha^2}{4t}}$$

Taking real parts on both sides,

$$\int_0^{\infty} e^{-tx^2} \cos \alpha x \, dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} e^{-\frac{\alpha^2}{4t}}$$

Let $I(\alpha) = \int_0^{\infty} e^{-tx^2} \frac{\sin \alpha x}{x} dx$. Differentiating under sign of integration we get,

$$I'(\alpha) = \int_0^{\infty} e^{-tx^2} \cos \alpha x \, dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} e^{-\frac{\alpha^2}{4t}}$$

Since I (0) = 0, integrating this we get

$$I(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{t}} \int_0^{\alpha} e^{-\frac{\xi^2}{4t}} d\xi = \sqrt{\pi} \int_0^{\alpha/2\sqrt{t}} e^{-\tau^2} d\tau$$

or, $$\int_0^{\infty} e^{-tx^2} \frac{\sin \alpha x}{x} dx = \sqrt{\pi} \int_0^{\alpha/2\sqrt{t}} e^{-\tau^2} d\tau$$

Therefore from (3.30c) we get

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\alpha/2/\sqrt{t}} e^{-\tau^2} d\tau = 1 - \phi\left(\frac{\alpha}{2\sqrt{t}}\right)$$

where $\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2} d\tau$ is the so-called error function.

VI. HEAVISIDE'S SERIES EXPANSION

If $F(p)$, the Laplace transform of a function $f(t)$, can be expanded in a series of powers of $\frac{1}{p}$, then from this $f(t)$ can be obtained as a power series in t. This method of Laplace inversion due to Heaviside is stated in the following theorem.

Theorem-9. Let $F(p)$, the Laplace transform of a function $f(t)$, be analytic in the domain $|p| > a$ including the point at infinity, $F(\infty) = 0$ and let it can be expanded in this domain in negative integral powers of p as

$$F(p) = \sum_{n=1}^{\infty} a_n p^{-n}.$$

Then $f(t) = \sum_{n=1}^{\infty} a_n \frac{t^{n-1}}{(n-1)!}$ (3.31)

Proof. From Laurent's expansion theorem we find that the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{|P|=R} p^{n-1} F(p) dp, \ R > a \tag{3.31a}$$

Since $F(\infty) = 0$, *i.e.*, $p = \infty$ is a zero of $F(p)$, there exists a positive number M such that

$$|F(p)| \le \frac{M}{\rho} \text{ for } |p| = \rho \ge a$$

Therefore from (3.31a)

$$|a_n| \le \frac{1}{2\pi} \int_{|p|=R} |p|^{n-1} |F(p) ds = \frac{1}{2\pi} . R^{n-1} \frac{M}{R} .2\pi R = MR^{n-1}$$

where ds is an element of arc of the circle $|p| = R$.

Now we consider the series $\sum_{n=1}^{\infty}\frac{a_n}{p^{n-2}}$. For the n-th term $u_n(p) = \frac{a_n}{p^{n-2}}$ of this series we have

$$|u_n(p)| = \frac{|a_n|}{|p|^{n-2}} \le \frac{MR^{n-1}}{R_1^{n-2}}, \text{ for } |p| > R_1 > R > a.$$

The series $\sum_{n=1}^{\infty}\frac{MR^{n-1}}{R_1^{n-2}} = MR_1\sum_{n=1}^{\infty}\left(\frac{R}{R_1}\right)^{n-1}$ being a geometric series of constant terms of common ratio $\frac{R}{R_1} < 1$, is convergent.

Hence by Weirstrass's M-test, the series $\sum_{n=1}^{\infty}\frac{a_n}{p^{n-2}}$ is uniformly convergent for $|p| > R_1$. Therefore, corresponding to an arbitrary positve $\in$ there exists an integer N (> 0) such that

$$|R_n(p)| < \in \text{ for } n \ge N \tag{3.31b}$$

where $R_n(p) = \sum_{k=n+1}^{\infty}\frac{a_k}{p^{k-2}}$ is the remainder after n terms of the series $\sum_{n=1}^{\infty}\frac{a_n}{p^{n-2}}$.

Now, $$\left|\int_{r-i\infty}^{r+i\infty} e^{pt}F(p)dp - \sum_{k=1}^{n}\int_{r-i\infty}^{r+i\infty} e^{pt}\frac{a_k}{p^k}dp\right|$$

$$= \left|\int_{r-i\infty}^{r+i\infty} e^{pt}\left[F(p) - \sum_{k=1}^{n}\frac{a_k}{p^k}\right]dp\right| = \left|\int_{r-i\infty}^{r+i\infty} e^{pt}\left(\sum_{k=n+1}^{\infty}\frac{a_k}{p^k}\right)dp\right|$$

$$= \left|\int_{r-i\infty}^{r+i\infty} e^{pt}\frac{1}{p^2}\left(\sum_{k=n+1}^{\infty}\frac{a_k}{p^{k-2}}\right)dp\right| = \left|\int_{r-i\infty}^{r+i\infty} e^{pt}\frac{1}{p^2}R_n(p)dp\right|$$

$$\le \int_{-\infty}^{\infty}|e^{(r+iy)t}|.\frac{1}{|r+iy|}|R_n(p)|dy \le \in e^{rt}\int_{-\infty}^{\infty}\frac{dy}{r^2+y^2}, \text{ by (3.31b)}$$

$= \pi e^{rt} \in = k \in$, for $n \geq N$, where $k = \pi e^{rt}$ is a constant.

This inequality implies that

$$\int_{r-i\infty}^{r+i\infty} e^{pt} F(p)dp = \lim_{n\to\infty} \sum_{n=1}^{n} \int_{r-i\infty}^{r+i\infty} e^{pt} \frac{a_k}{p^k} dp = \sum_{n=1}^{\infty} \int_{r-i\infty}^{r+i\infty} e^{pt} \frac{a_n}{p^n}$$

or, $f(t) = \sum_{n=1}^{\infty} a_n \frac{t^{n-1}}{(n-1)!}$, by the inversion formula.

Ex. 3.6.12. Obtain the Laplace inversion of

$$F(p) = \frac{1}{\sqrt{1+p^2}}$$

in a series of powers of t. Hence show that

$$\sin t = \int_0^t J_0(\tau) J_0(t-\tau) d\tau$$

where $J_0(t)$ is Bessel's function of order zero.

Soln. Expanding $F(p)$ in negative integral power of p, we get

$$F(p) = \frac{1}{p}\left(1+\frac{1}{p^2}\right)^{-\frac{1}{2}} = \frac{1}{p}\left[1+\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\left\{-\frac{1}{2}-(n-1)\right\}}{n!}\cdot\left(\frac{1}{p^2}\right)^n\right]$$

$$= \frac{1}{p} + \sum_{n=1}^{\infty} \frac{(-1)^n 1.3.5\ldots.(2n-1)}{2^n n!} \cdot \frac{1}{p^{2n+1}} = \frac{1}{p} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(2.4\ldots 2n)2^n n!} \cdot \frac{1}{p^{2n+1}}$$

$$= \frac{1}{p} + \sum_{n=1}^{\infty} \frac{(-1)^n (2n!)}{(2^n n!)^2} \cdot \frac{1}{p^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2^n n!)^2} \cdot \frac{1}{p^{2n+1}}$$

Therefore, $f(t) = L^{-1}[F(p)] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2^n n!)^2} L^{-1}\left[\frac{1}{p^{2n+1}}\right]$

[by Heaviside's series expansion]

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{\left(2^n n!\right)^2} \cdot \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n!2^n\right)^n} \left(\frac{t}{2}\right)^{2n} = J_0(t)$$

Since, series expansion for $J_\lambda(t)$, the Bessel's function or order λ, is

$$J_\lambda(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\lambda+1)} \left(\frac{t}{2}\right)^{\lambda+2n}$$

We can write $\dfrac{1}{p^2+1} = \dfrac{1}{\sqrt{p^2+1}} \cdot \dfrac{1}{\sqrt{p^2+1}}$

Therefore $L^{-1}\left[\dfrac{1}{p^2+1}\right] = L^{-1}\left[\dfrac{1}{\sqrt{p^2+1}} \cdot \dfrac{1}{\sqrt{p^2+1}}\right]$

or, $\sin t = \int_0^t J_0(\tau) J_0(t-\tau) d\tau$, by the use of convolution theorem.

VII. ASYMPTOTIC EVALUATION OF LAPLACE INVERSION INTEGRAL

By evaluating the Laplace inversion integral, the original function $f(t)$ can be determined from its Laplace transform $F(p)$. But in many problems of applications it remains sufficient to know the value to $f(t)$ for large t instead of its exact analytical form. For such cases a method of asymptotic evaluation of the inversion integral is very much needed. The following theorem, which is stated without proof, gives a method of asymptotic evaluation of the inversion integral for large t.

Theorem-10. Evaluation of the inversion integral

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} F(p) dp$$

for large t can be obtained from the behaviour of $F(p)$ near its singularity with largest real part. Let this singularity by $p = p_0 = a + ib$, $a < r$. We now make the following assupmtions, Let $p = x + iy$.

(i) $F(p)$ is analytic in the domain $\text{Re}(p) \geq a - \delta$, $\delta > 0$ except at $p = p_0$.

(ii) $F(p) \to 0$ uniformly with x for $a - \delta \leq x < r$ as $y \to \pm\infty$ in the region $a - \delta \leq x < r$.

(iii) The integral $\int^{y} |F(p)| dy$ converges for $y \to \pm\infty$ in the region $a - \delta \leq x < r$.

If $F(p)$ can be expanded near $p = p_0$ in the form

$$F(p) = \sum_{n=-m}^{\infty} a_n (p - p_0)^n + (p - p_0)^{-\beta} \sum_{n=0}^{\infty} b_n (p - p_0)^n, 0 < \beta < 1,$$

where the two series converge for $0 < |p - p_0| \leq l$. then $f(t)$ has the following expansion for large t.

$$f(t) \sim e^{p_0 t}\left[a_{-1} + \frac{\sin \pi\beta}{\pi} \sum_{n=0}^{\infty} (-1)^n b_n \Gamma(1 - \beta + n) \frac{1}{t^{1-\beta+n}}\right] \tag{3.32}$$

Ex. 3.6.13. If Laplace transform of a function $f(t)$ be $\dfrac{e^{-x\sqrt{p/k}}}{1+\sqrt{\dfrac{p}{k}}}$, find asymptotic expansion of $f(t)$ for large t.

Soln. We are to consider two cases according as $x = 0$ and $x \neq 0$.

(i) for $x = 0$

In this case the expansion for $F(p) = \dfrac{1}{p}\left(1 + \sqrt{\dfrac{p}{k}}\right)^{-1}$ becomes

$$F(p) = \frac{1}{p}\left[1 - \left(\frac{p}{k}\right)^{\frac{1}{2}} + \left(\frac{p}{k}\right) - \left(\frac{p}{k}\right)^{\frac{3}{2}} + \ldots\right]$$

$$= \left[\frac{1}{p} + \frac{1}{k} + \frac{p}{k^2} + \ldots\right] - p^{-\frac{1}{2}}\left[\frac{1}{k^{1/2}} + \frac{p}{k^{3/2}} + \frac{p^2}{k^{5/2}} + \ldots\right]$$

So far large t we have

$$f(t) \sim 1 - \frac{\sin\frac{\pi}{2}}{\pi}\left[\frac{1}{k^{1/2}}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{t^{1/2}} - \frac{1}{k^{3/2}}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{t^{3/2}} + \frac{1}{k^{5/2}}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{t^{5/2}} - \ldots\right], \text{ by} \qquad (3.32)$$

$$= 1 - \frac{1}{\sqrt{\pi}}\left[\frac{1}{(kt)^{1/2}} - \frac{1}{2(kt)^{3/2}} + \frac{1.3}{2^2(kt)^{5/2}} - \ldots\right]$$

(ii) For $x \neq 0$.

In this case the expansion for $F(p) = \frac{1}{p^{\left(1+\sqrt{\frac{p}{k}}\right)}} e^{-x\sqrt{\frac{p}{k}}}$ becomes

$$F(p) = \frac{1}{p}\left[1 + \sqrt{\frac{p}{k}}\right]^{-1} e^{-x\sqrt{\frac{p}{k}}}$$

$$= \frac{1}{p}\left[1 - \left(\frac{p}{k}\right)^{\frac{1}{2}} + \left(\frac{p}{k}\right) - \left(\frac{p}{k}\right)^{\frac{3}{2}} + \ldots\right]\left[1 - x\left(\frac{p}{k}\right)^{\frac{1}{2}} + \frac{x^2}{2!}\left(\frac{p}{k}\right) - \frac{x^3}{3!}\left(\frac{p}{k}\right)^{\frac{3}{2}} + \ldots\right]$$

$$= \frac{1}{p}\left[1 - \left(\frac{p}{k}\right)^{\frac{1}{2}}(1+x) + \frac{p}{k}\left(\frac{x^2}{2} + x + 1\right) - \left(\frac{p}{k}\right)^{\frac{3}{2}}\left(\frac{x^3}{6} + \frac{x^2}{2} + x + 1\right) + \ldots\right]$$

$$= \left[\frac{1}{p} + \frac{1}{k}\left(\frac{x^2}{2!} + x + 1\right) + \ldots\right] - p^{-\frac{1}{2}}\left[\frac{1}{k^{1/2}}(1+x) + \frac{p}{k^{3/2}}\left(\frac{x^3}{3!} + \frac{x^2}{2!} + x + 1\right) + \ldots\right]$$

Therefore for large t we have

$$f(t) \sim 1 - \frac{\sin\frac{\pi}{2}}{\pi}\left[\frac{1}{k^{1/2}}\frac{\Gamma\left(\frac{1}{2}\right)}{t^{1/2}}(x+1) - \frac{1}{k^{3/2}}\frac{\Gamma\left(\frac{3}{2}\right)}{t^{3/2}}\left(\frac{x^3}{3!} + \frac{x^2}{2!} + x + 1\right) + \ldots\right]$$

$$= 1 - \frac{1}{\sqrt{\pi}}\left[(x+1)\frac{1}{(kt)^{1/2}} - \left(\frac{x^3}{3!} + \frac{x^2}{2!} + x + 1\right)\frac{1}{2(kt)^{3/2}} + \ldots\right]$$

Ex. 3.6.14. Assuming that the Laplace transform of $J_0(t)$ is $1/\sqrt{1+p^2}$, show that the asymptotic expansion of $J_0(t)$ for large t is

$$J_0(t) \sim \sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi}{4}\right)\left[1-\frac{9}{128t^2}+\ldots\right]+\sqrt{\frac{2}{\pi t}}\sin\left(t-\frac{\pi}{4}\right)\left[\frac{1}{8t}-\ldots\right]$$

Soln. The Laplace transform $F(p)=\dfrac{1}{\sqrt{1+p^2}}$ to $J_0(t)$ has two singularities at $p = \pm i$ and the real parts of both are same. So we are to take contributions from both these singularities. If f_+ (t) and f_- (t) are contributions from the singularities $p = i$ and $p = -i$ respectively in the asymptotic expansion of $J_0(t)$, then

$$J_0(t) \sim f_+(t) + f_-(t)$$

(i) To find f_+ (t) we expand $F(p)$ about $p = i$.

$$F(p)=\frac{1}{(p-i)^{1/2}(p+i)^{1/2}}=(p-i)^{-\frac{1}{2}}\left[2i+(p-i)\right]^{-\frac{1}{2}}$$

$$=(p-i)^{-\frac{1}{2}}\frac{1}{\sqrt{2i}}\left[1+\frac{p-i}{2i}\right]^{-\frac{1}{2}}$$

$$=(p-i)^{-\frac{1}{2}}\frac{1}{\sqrt{2i}}\left[1-\frac{1}{2}\left(\frac{p-i}{2i}\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{p-i}{2i}\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{p-i}{2i}\right)^3+\ldots\right]$$

$$=(p-i)^{-\frac{1}{2}}\frac{e^{-\frac{\pi i}{4}}}{\sqrt{2}}\left[1-\frac{2}{2^2}\left(\frac{p-i}{2i}\right)+\frac{4!}{2.4.2!2^2}\left(\frac{p-i}{2i}\right)^2-\frac{6!}{2.4.6.3!2^3}\left(\frac{p-i}{2i}\right)^3+\ldots\right]$$

$$=(p-i)^{-\frac{1}{2}}\frac{1}{2}(1-i)\left[1-\frac{2!}{(1!2)^2}\left(\frac{p-i}{2!}\right)+\frac{4!}{(2!2^2)^2}\left(\frac{p-i}{2i}\right)^2-\frac{6!}{(3!2^3)^2}\left(\frac{p-i}{2i}\right)^3+\ldots\right]$$

$$\text{Therefore, } f_+(t) = e^{it}\ \frac{\sin\frac{\pi}{2}}{\pi}\,\frac{1}{2}(1-i)\left[\frac{\Gamma\left(\frac{1}{2}\right)}{t^{1/2}}+\frac{2!}{(1!2)^2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{2it^{3/2}}+\frac{4!}{(2!2^2)^2}\cdot\frac{\Gamma\left(\frac{5}{2}\right)}{(2i)^2t^{5/2}}+\ldots\right]$$

$$= \frac{e^{it}}{2\sqrt{\pi}}(1-i)\left[\frac{1}{t^{1/2}}+\frac{1}{8it^{3/2}}-\frac{9}{128t^{5/2}}+\ldots\right]$$

$$\text{Similarly, } f_-(t) = \frac{e^{-it}}{2\sqrt{\pi}}(1+i)\left[\frac{1}{t^{1/2}}-\frac{1}{8it^{3/2}}-\frac{9}{128t^{5/2}}+\ldots\right]$$

Therefore $f(t) \sim f_+(t) + f_-(t)$

$$= \frac{1}{2\sqrt{\pi}}\left[\left\{(e^{it}+e^{-it})\frac{1}{t^{1/2}}+\frac{1}{i}(e^{it}-e^{-it})\frac{1}{t^{1/2}}\right\}\right.$$

$$+\left\{\frac{1}{i}(e^{it}-e^{it})\frac{1}{8t^{3/2}}-(e^{it}+e^{-it})\frac{1}{8t^{3/2}}\right\}$$

$$\left.+\left\{-(e^{it}+e^{-it})\frac{1}{128t^{5/2}}-\frac{1}{i}(e^{it}-e^{-it})\frac{1}{128t^{5/2}}\right\}+\ldots\right]$$

$$\text{or, }\quad f(t) \sim \frac{\sqrt{2}}{\sqrt{\pi}}\left[\left(\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t\right)\frac{1}{t^{1/2}}+\left(\frac{1}{\sqrt{2}}\sin t-\frac{1}{\sqrt{2}}\cos t\right)\frac{1}{8t^{3/2}}\right.$$

$$\left.-\left(\frac{1}{\sqrt{2}}\cos t+\frac{1}{\sqrt{2}}\sin t\right)\frac{9}{128t^{5/2}}+\ldots\right]$$

$$= \sqrt{\frac{2}{\pi t}}\left[\cos\left(t-\frac{\pi}{2}\right)\left(1-\frac{9}{128t^2}+\ldots\right)+\sin\left(t-\frac{\pi}{2}\right)\left(\frac{1}{8t}+\ldots\right)\right]$$

3.7. SOLUTION OF LINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

By the help of Laplace transform the problem of solving linear differential equations, both ordinary and partial, can be reduced to problems, which are easier to solve than solving the original ones. Ordinary and partial differential equations with constant coefficients, for instance, reduce respectively to algebraic and ordinary differential equations, which are rather simpler to solve. These reductions of ordinary and partial differential equations and finally finding their solutions are made in the next two subsections. In the last subsection it is shown how in some cases ordinary differential equations with variable coefficients can be solved by the use of Laplace transform.

I. ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

We consider the following oridinary differential equation with constant coefficients together with initial conditions.

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) = f(t) \tag{3.33}$$

$$y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}, \tag{3.34}$$

where $y_0, y_1, \ldots, y_{n-1}$ are some given constants.

The coefficients $a_0, a_1, \ldots, a_n$ of the equation are constants, $f(t)$ is a given function and $y^{(j)}(t) = \dfrac{d^j y}{dt^j}$.

Taking Laplace transform of (3.33) we get

$$\begin{aligned}
&a_0\left[p^n \bar{y}(p) - p^{n-1} y_0 - p^{n-2} y_1 - \ldots y_{n-1}\right] \\
&+a_1\left[p^{n-1}\bar{y}(p) - p^{n-2} y_0 - p^{n-3} y_1 - \ldots - y_{n-2}\right] \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&+a_{n-1}[p\,\bar{y}(p) - y_0] \\
&+a_n \bar{y}(p) = \bar{f}(p),
\end{aligned}$$

where $\bar{y}(p)$ and $\bar{f}(p)$ are the Laplace transforms of $y(t)$ and $f(t)$ respectively.

$$\text{or,}\quad \bar{y}(p)\left[a_0p^n + a_1p^{n-1}+\ldots+a_n\right]$$

$$-y_0\left[a_0p^{n-1} + a_1p^{n-2}+\ldots a_{n-1}\right]$$

$$-y_1\left[a_0p^{n-2} + a_1p^{n-3}+\ldots a_{n-2}\right]$$

$$-\quad \ldots\quad \ldots\quad \ldots$$

$$-y_{n-2}[a_0p + a_1]$$

$$-y_{n-1}a_0$$

$$= \bar{f}(p)$$

$$\text{or,}\quad \bar{y}(p)p_n(p) = \sum_{k=0}^{n-1} y_kQ_k(p) + \bar{f}(p) \tag{3.35a}$$

where $P_n(p) = a_0p^n + a_1p^{n-1}+\ldots+a_n$

$$Q_k(p) = a_0p^{n-(k+1)}a_1p^{n-(k+2)}+\ldots a_{n-(k+1)} \tag{3.35b}$$

$$\text{Therefore}\quad \bar{y}(p) = \sum_{k=0}^{n-1} y_k \frac{Q_k(p)}{P_n(p)} + \frac{\bar{f}(p)}{a_0}\cdot\frac{a_0}{P_n(p)}$$

Since, $Q_{n-1}(p) = a_0$, we can write $\bar{y}(p)$ as follows.

$$\bar{y}(p) = \sum_{k=0}^{n-1} y_k \frac{Q_k(p)}{P_n(p)} + \frac{\bar{f}(p)}{a_0}\frac{Q_{n-1}(p)}{P_n(p)} \tag{3.35c}$$

$$\text{Now let}\quad \psi_k(t) = L^{-1}\left[\frac{Q_k(p)}{P_n(p)}\right] \tag{3.35d}$$

If the zeros of $P_n(p)$ are at $p = p_j$, $j = 1, 2, \ldots m \le n$, then the singularities of $e^{pt}\frac{Q_k(p)}{P_n(p)}$ are at $p = p_j, j = 1, 2, \ldots m \le n$ which are poles. Let R_{kj} be the residue of $e^{pt}\frac{Q_k(p)}{P_n(p)}$ at $p = p_j$. Since $\frac{Q_k(p)}{P_n(p)}$ is the ratio of two ploynomials and the degree of $Q_k(p)$ is less than that of $P_n(p)$, $\frac{Q_k(p)}{P_n(p)} \to 0$ uniformly in arg $(p - r)$ as $|P - r| \to \infty$, where r is greater than all the real parts of p_j's. Therefore according to (3.28h) we have

$$\psi_k(t) = \sum_{j=1}^{m} R_{kj} \tag{3.35e}$$

$\psi_k(t)$ can also be obtained from (3.35d) by resolving $\frac{Q_k(p)}{P_n(p)}$ into partial fractions as explained in Sec. 3.6(iv).

Since, $L[f(t)] = \bar{f}(p)$ and $L[\psi_{n-1}(t)] = \frac{Q_{n-1}(p)}{P_n(p)}$ we have by convolution theorem

$$L^{-1}\left[\frac{\bar{f}(p)}{a_0}.\frac{Q_{n-1}(p)}{P_n(p)}\right] = \frac{1}{a_0}\int_0^t f(\tau)\psi_{n-1}(t-\tau)d\tau \tag{3.35f}$$

Hence taking inversion of (3.35c) we finally get

$$y(t) = \sum_{k=0}^{n-1} y_k\psi_k(t) + \frac{1}{a_0}\int_0^t t(\tau)\psi_{n-1}(t-\tau)d\tau \tag{3.35g}$$

by the use of (3.35d) and (3.35f), where $\psi_k(t)$ is given by (3.35e)

If $\frac{\bar{f}(p)}{a_0}.\frac{Q_{n-1}(p)}{P_n(p)} \to 0$ uniformly in arg $(p - r)$ as $|r - p| \to \infty$, then without using convolution theorem we have the following expression for the Laplace inversion given in (3.35f).

$$L^{-1}\left[\frac{\bar{f}(p)}{a_0}.\frac{Q_{n-1}(p)}{P_n(p)}\right] = \sum_j S_j \tag{3.35h}$$

where S_j's are the residues of the function $e^{pt}.\frac{\bar{f}(p)}{a_0}.\frac{Q_{n-1}(p)}{P_n(p)}$ at its singularities $p = q_j$, which are supposed to be ploes.

Ex. 3.7.1. Find the solution of the equation

$$x'''(t) + x'(t) = \sin t,$$

satisfying the initial condition

$$x(0) = 0,\ x'(0) = -2, x''(0) = 0.$$

Soln. Taking Laplace transform of the equation we get

$$\left[p^3\bar{x}(p) - p^2x(0) - px'(0) - x''(0)\right] + p\bar{x}(p) - x(0) = \frac{1}{p^2+1}$$

$$\text{or,}\quad (p^3 + p)\bar{x}(p) + 2p = \frac{1}{p^2+1}$$

$$\text{or,}\quad \bar{x}(p) = -\frac{2}{p^2+1} + \frac{1}{p(p^2+1)^2} \tag{3.36}$$

since $\frac{1}{p(p^2+1)^2} \to 0$ uniformly in arg $(p-r)$, as $|p-r| \to \infty$, when $r > 0$,

$$L^{-1}\left[\frac{1}{p(p^2+1)^2}\right] = R_0 + R_+ + R_-$$

where R_0, R_+, R_- are the residues of the function $e^{pt}.\frac{1}{p(p^2+1)^2}$ at the poles $p = 0$, $p = i$, $p = -i$ respectively.

$p = 0$ is a simple pole and $p = \pm\, i$ are ples of order 2.

$$R_0 = \lim_{p\to 0} e^{pt}\frac{1}{(p^2+1)^2} = 1$$

$$R_1 = \lim_{p\to i}\frac{d}{dp}\left\{\frac{e^{pt}}{p(p+i)^2}\right\} = -\frac{1}{4i}te^{it} - \frac{1}{2}e^{it}$$

$$R_2 = \lim_{p\to -i}\frac{d}{dp}\left\{\frac{e^{pt}}{p(p-i)^2}\right\} = \frac{1}{4i}te^{it} - \frac{1}{2}e^{it}$$

Therefore taking inversion of (3.36) we get

$$x(t) = 1 - \cos t - \frac{1}{2}t\sin t - 2\sin t$$

Ex. 3.7.2. Find the solution of the equations

$$x''(t) = \lambda + \Omega y'(t)$$

and $$y''(t) = -\Omega x'(t)$$

satisfying the initial conditions $x(0) = 0$, $y(0) = 0$, $x'(0) = u$, $y'(0) = v$, where λ and Ω are two constants.

Slon. Taking Laplace transform of the two equations we get

$$p^2\bar{x}(p) - u = \frac{\lambda}{p} + \Omega p\bar{y}(p) \text{ and } p^2\bar{y}(p) - v = -\Omega p\bar{x}(p)$$

or, $$\left.\begin{aligned} p^2\bar{x}(p) - \Omega p\bar{y}(p) &= u + \frac{\lambda}{p} \\ p^2\bar{y}(p) + \Omega p\bar{x}(p) &= v \end{aligned}\right\}$$

where $\bar{x}(p)$ and $\bar{y}(p)$ are Laplace transform of $x(t)$ and $y(t)$ respectively.

Solving the above two equations we have

$$\bar{x}(p) = \frac{\lambda + v\Omega}{p(p^2 + \Omega^2)} + \frac{u}{p^2 + \Omega^2}$$

$$\bar{y}(p) = \frac{-\lambda\Omega}{p^2(p^2 + \Omega^2)} - \frac{u\Omega}{p(p^2 + \Omega^2)} + \frac{v}{p^2 + \Omega^2}$$

Breaking up into partial fraction, these two equations can be written as

$$\bar{x}(p) = \frac{\lambda + v\Omega}{\Omega^2}\left[\frac{1}{p} - \frac{p}{p^2 + \Omega^2}\right] + \frac{u}{\Omega}\cdot\frac{\Omega}{p^2 + \Omega^2}$$

$$\bar{y}(p) = -\frac{\lambda}{\Omega}\left[\frac{1}{p^2} - \frac{1}{p^2 + \Omega^2}\right] - \frac{u}{\Omega}\left[\frac{1}{p} - \frac{1}{p^2 + \Omega^2}\right] + \frac{v}{\Omega}, \frac{\Omega}{p^2 + \Omega^2}$$

Taking Laplace inversion of these two equations we get the following solutions of the given equations by the use of the formulas given in Sec. 3.4.

$$x(t) = \frac{(\lambda + v\Omega)}{\Omega^2}(1 - \cos\Omega t) + \frac{v}{\Omega}\sin\Omega t$$

$$y(t) = -\frac{\lambda}{\Omega}(t - \sin\Omega t) - \frac{u}{\Omega}(1 - \cos\Omega t) + \frac{v}{\Omega}\sin\Omega t$$

II. PARTIAL DIFFERENTIAL EQUATIONS

We consider the particular case of a function depending only on one space co-ordinate and time. Let a function $u(x, t)$, $(t > 0)$ satisfies a partial differential equation together with some initial and boundary conditions. If the co-efficients of this equation do not depened on t, then taking Laplace transform of this equaiton and the boundary conditions with respect to t, we get an ordinary differential equation for $\overline{u}(x,p)$, the Laplace transform of $u(x, t)$, together with some boundary conditions for $\overline{u}(x,p)$. Solving this ordinary differential equation we get $\overline{u}(x,p)$, the Laplace inversion of which gives the solution of the partial differential equation.

Ex. 3.7.3. Find the solution of the equation,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0,$$

which remains bounded for $x \geq 0$ and obeys the following initial and boundary conditions.

$$u\ (x,\ 0) = 0, \quad u\ (0,\ t) = f(t).$$

Soln. Taking Laplace transform of the equation and the boundary condition with respect to t, we get

$$p\overline{u} = a^2 \frac{d^2\overline{u}}{dx^2}$$

$$\overline{u}(o,p) = \bar{f}(p)$$

where $\overline{u} = \overline{u}(x,p)$ and $\bar{f}(p)$ are the Laplace transforms of $u(x, t)$ and $f(t)$ respectively.

The solution of the above ordinary differential equation, which remains bounded for $x > 0$, is

$$\overline{u}(x,p) = Ae^{-\frac{\sqrt{p}}{a}x}$$

Setting $x = 0$, we get

$$\overline{u}(0,p) = A \quad \text{or, } \bar{f}(p) = A$$

Therefore the above solution becomes

$$\overline{u}(x,p) = \bar{f}(p)e^{-\frac{\sqrt{p}}{a}x} = \bar{f}(p)\overline{g}(p) \tag{3.37}$$

where $\overline{g}(p) = e^{-\frac{\sqrt{p}}{a}x}$. We can write

$$\bar{g}(p) = p.\frac{1}{p}e^{-\frac{\sqrt{p}}{a}x} = p\bar{h}(p)$$

where $\bar{h}(p) = \frac{1}{p}e^{-\frac{\sqrt{p}}{a}x}$

Now if $\bar{h}(p)$ is the Laplace transform of a function $h(t)$, then

$$h(t) = L^{-1}\left[\frac{1}{p}e^{-\frac{\sqrt{p}}{a}x}\right] = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}} e^{-\tau^2}d\tau, \text{ by Ex. 2.6.11}$$

$$L[h'(t)] = p\bar{h}(p) - h(0) = p\bar{h}(p)$$

Since, $h(0) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\tau^2}d\tau = 0$

Therefore, $L^{-1}[p\bar{h}(p)] = h'(t) = \frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4a^2t}}\left(\frac{x}{4at^{3/2}}\right)$

or, $L^{-1}[\bar{g}(p)] = \frac{x}{2a\sqrt{\pi}t^{3/2}}e^{-\frac{x^2}{4a^2t}} = g(t)$, whose Laplace transform is $\bar{g}(p)$.

Hence by (3.37) using convolution theorem we get

$$u(x,t) = \int_0^t f(\xi)g(t-\xi)d\xi$$

$$= \frac{x}{2a\sqrt{\pi}}\int_0^t \frac{f(\tau)}{(t-\tau)^{3/2}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau$$

Ex. 3.7.4. Find the solution of the equation,

$$\frac{\partial v}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right),\ t > 0,\ 0 \le r \le a$$

satisfying the initial and boundary conditions :

$v(r, 0) = 0$, for $0 \le r \le a$, $v(a, t) = v_0$, $t > 0$.

Soln. Taking Laplace transform of the given equation and the boundary condition with respect to t we get

$$p\bar{v}(r,p) = \frac{k}{r^2}\frac{d}{dr}\left(r^2\frac{d\bar{v}}{dr}\right) \tag{3.37a}$$

and $$\bar{v}(a,p) = \frac{v_0}{p} \tag{3.37b}$$

where $\bar{v}(r,p)$ is the Laplace transform of $v(r, t)$ with respect to t. In deriving (3.37a) the initial condition $v(r, 0) = 0$ has been used.

To solve the ordinary differential equation (3.37a) we set $\bar{v} = \frac{f(r)}{r}$. Then the equation becomes

$$\frac{d^2f}{dr^2} = \frac{p}{k}f$$

the solution of which is

$$f = Ae^{\sqrt{\frac{p}{k}}r} + Be^{-\sqrt{\frac{p}{k}}r}$$

Therefore, $\bar{v} = \frac{1}{r}\left(Ae^{\sqrt{\frac{p}{k}}r} + Be^{-\sqrt{\frac{p}{k}}r}\right)$

For small r, $\bar{v} = \frac{1}{r}(A+B) + 0(1)$

For $\bar{v}$ to remain finite we must have $A + B = 0$ and consequently the solution for $\bar{v}$ becomes

$$\bar{v} = 2A\frac{\sinh\alpha\sqrt{p}}{r}, \text{where } \alpha = \frac{r}{\sqrt{k}} \tag{3.37c}$$

By the use of the condition (3.37b) we have

$$\frac{v_0}{p} = 2A\frac{\sinh\beta\sqrt{p}}{a}, \text{ where } \beta = \frac{a}{\sqrt{k}} \tag{3.37d}$$

Eliminating A between (3.37c) and (3.37d), we get

$$\bar{v} = \frac{v_0}{p}.\frac{a}{r}\frac{\sinh\alpha\sqrt{p}}{\sinh\beta\sqrt{p}} \tag{3.37e}$$

Following Ex. 3.6.9. we find that

$$L^{-1}\left[\frac{1}{p}\cdot\frac{\sinh a\sqrt{p}}{\sinh\beta\sqrt{p}}\right]=\frac{\alpha}{\beta}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2\pi^2 t}{\beta^2}}\sin\frac{n\pi\alpha}{\beta}$$

$$=\frac{r}{a}+\frac{2}{\pi}\sum_{n=1}^{2}\frac{(-1)^n}{n}e^{-\frac{n^2k\pi^2 t}{a^2}}\sin\frac{n\pi r}{a}$$

where x and a are repalced respectively by α and β.

Therefore from (3.37e) we get

$$v(x,\ t)\quad=v_0\frac{a}{r}\left[\frac{r}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2k\pi^2 t}{a^2}}\sin\frac{n\pi r}{a}\right]$$

$$=v_0\left[1+\frac{2a}{r}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2k\pi^2 t}{a^2}}\sin\frac{n\pi r}{a}\right]$$

Note. This example solves the problem of conduction of heat in a sphere which is initially at zero temperature and whose surface is maintained at a constant temperature. Here $v(r,\ t)$ is the temperature at a distance r from the centre of the sphere and at time t.

Ex. 3.7.4. Find the solution of the equation

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\ 0\le x\le l,\ t>0$$

satisfying the following initial and boundary conditions

$$u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0$$

$$u(0,t)=a\sin\omega t, u(l,t)=0$$

Soln. Taking Laplace transform of the given equation with respect to t, we get

$$c^2\frac{d^2\overline{u}}{dx^2}=p^2\overline{u}-pu(x,0)-\frac{\partial}{\partial t}u(x,0)$$

which by the use of the initial conditions become

$$\frac{d^2\overline{u}}{dx^2}=\frac{p^2}{c^2}\overline{u}\qquad(3.38a)$$

where $\overline{u}=\overline{u}(x,p)$ is the Laplace transform of $u(x,\ t)$.

The Laplace transforms of the two boundary conditions become

$$\overline{u}(0,p)=\frac{a\omega}{p^2+\omega^2},\ \overline{u}(l,p)=0 \quad (3.38b)$$

Solution of the equation (3.38a) is

$$\overline{u}(x,p)=Ae^{\frac{p}{c}x}+Be^{-\frac{p}{c}x}$$

The conditions (3.38b) therefore give

$$A+B=\frac{a\omega}{p^2+\omega^2},\ Ae^{\frac{pl}{c}}+Be^{-\frac{pl}{c}}=0$$

Solution of these two equations for A and B give

$$A=-\frac{a\omega e^{-\frac{pl}{c}}}{2\left(p^2+\omega^2\right)\sinh\frac{pl}{c}},\ B=\frac{a\omega e^{\frac{pl}{c}}}{2\left(p^2+\omega^2\right)\sinh\frac{pl}{c}}$$

With these values of A and B, $\overline{u}$ is given by

$$\overline{u}(x,p)=\frac{a\omega}{p^2+w^2}\cdot\frac{\sinh\frac{p}{c}(l-x)}{\sinh\frac{pl}{c}} \quad (3.38c)$$

The singularities of this function are at

(i) $p=\pm i\omega$, which are simple poles

(ii) $p=\pm\frac{\pi inc}{l}$, $(n=1, 2, \ldots)$, which are also simple poles.

The real part of all these singularities is zero.

It can be shown as in Ex. 3.6.9 that $\overline{u}(x,p)\to 0$ uniformly in arg $(p-r)$, $\frac{\pi}{2}<\arg(p-r)<\frac{3\pi}{2}$, where $r>0$ (since the real part of all the singularities is zero).

Hence,

$$u(x,t)=\sum_{n=0}^{\infty}\left(R_n^{+}+R_n^{-}\right) \quad (3.38c)$$

where R_0^{+}, R_0^{-} are the residues of $e^{pt}\overline{u}(x,p)$ at $p=+i\omega$, $p=-i\omega$ respectively and R_n^+, R_n^- $(n \neq 0)$ are the residues of the same function at $p = \frac{\pi inc}{l}, p = -\frac{\pi inc}{l}$ respectively.

$$\text{Now (i) } R_0^+ + R_0^- = \frac{a\omega e^{i\omega t}\sinh\left\{\frac{i\omega}{c}\right\}(l-x)}{2i\omega\sinh\left\{\frac{i\omega l}{c}\right\}} + \frac{a\omega e^{-i\omega t}\sinh\left\{-\frac{i\omega}{c}(l-c)\right\}}{-2i\omega\sinh\left\{-\frac{i\omega l}{c}\right\}}$$

$$= \frac{a\sin\frac{\omega}{c}(l-x)}{\sin\frac{\omega l}{c}}\sin\omega t$$

(ii) $R_n^+ + R_n^-$, for $n \neq 0$

$$= e^{\frac{\pi nic}{l}t}\cdot\frac{a\omega}{-\frac{\pi^2n^2c^2}{l^2}+\omega^2}\cdot\frac{\sinh\left\{\frac{\pi in}{l}(l-x)\right\}}{\frac{l}{c}\cosh(\pi in)}$$

$$+e^{-\frac{\pi inc}{l}t}\cdot\frac{a\omega}{-\frac{\pi^2n^2c^2}{l^2}+\omega^2}\cdot\frac{\sinh\left\{-\frac{\pi in}{l}(l-x)\right\}}{\frac{l}{c}\cosh(-\pi in)}$$

$$= -\frac{a\omega cl}{\omega^2l^2-n^2\pi^2c^2}\cdot\frac{1}{i\cos n\pi}\sin\left(n\pi-\frac{n\pi x}{l}\right)\left[e^{\frac{\pi inct}{l}}-e^{-\frac{\pi inct}{l}}\right]$$

$$= -\frac{2a\omega cl}{\omega^2l^2-n^2\pi^2c^2}\,\frac{(-1)^{n+1}}{(-1)^n}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

$$= \frac{2a\omega cl}{\omega^2l^2-n^2\pi^2c^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

With these values for R_n^+, R_n^- $(n = 0, 1, 2, ...)$ the expression (3.38d) gives for $u(x, t)$

$$u(x,t) = \frac{a\sin\frac{\omega}{c}(l-x)}{\sin\frac{\omega l}{c}}\sin\omega t + \sum_{n=1}^{\infty}\frac{2a\omega cl}{\omega^2l^2-n^2\pi^2c^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

Note. The example solves the problem of vibration of a stretched string of finite length whose one end is fixed and the other and vibrates periodically.

III. VARIABLE COEFFICIENT ORDINARY DIFFERENTIAL EQUATIONS

By the use of the formula given by (3.19), i.e. the formula

$$F^{(n)}(p) = (-1)^n L[t^n f(t)]$$

where $F(p)$ is the Laplace transfrom of $f(t)$, we can express terms like $t^n \frac{d^m y(t)}{dt^m}$, m, n are positive integers) in terms of derivatives of $\bar{y}(p)$, which is the Laplace transform of $y(t)$. So a variable coefficient ordinary differential equation having terms of the type $t^n \frac{d^m y(t)}{dt^m}$ can sometimes be transformed by Laplace transform into such an equation which can be solved easily.

Ex. 3.7.5. Use Laplace transform to solve Bessel equation of order zero.

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0$$

satisfying the condition $y(0) = 1$.

Soln. By the use of the formula (3.19) we have

(i) $L\left[t\frac{d^2y}{dt^2}\right] = (-1)\frac{d}{dp}\left(L\left[\frac{d^2y}{dt^2}\right]\right) = -\frac{d}{dp}[p^2\bar{y}(p) - py(0) - y'(0)]$

$= -p^2\frac{d\bar{y}}{dp} - 2p\bar{y} + 1$, since $y(0) = 1$

(ii) $L[ty] = (-1)\frac{d}{dp}(L[y]) = -\frac{d\bar{y}}{dp}$

Here $\bar{y}(p)$ is the Laplace transform of $y(t)$.

Therefore taking Laplace transform of the given equation we get

$$-p^2\frac{d\bar{y}}{dp} - 2p\bar{y} + 1 + p\bar{y} - 1 - \frac{d\bar{y}}{dp} = 0$$

or, $\quad (1+p^2)\frac{d\bar{y}}{dp} + p\bar{y} = 0$, or $\frac{d\bar{y}}{\bar{y}} = -\frac{p}{1+p^2}dp$

Integrating this we get

$$ln\bar{y} = -\frac{1}{2}ln(1+p^2) + lnc, \text{ c is a constant.}$$

or, $$\bar{y} = \frac{c}{\sqrt{1+p^2}} = \frac{c}{p}\left(1+\frac{1}{p^2}\right)^{-\frac{1}{2}}$$

Expanding this in negative integral powers of p, $\bar{y}$ can be written as (similar to Ex. 3.6.12.)

$$\bar{y} = c\sum_{n=0}^{\infty}\frac{(-1)^n}{(2^n n!)^2}\cdot\frac{(2n)!}{p^{2n+1}}$$

Term by term Laplace inversion of this, which is possible according to the theorem-9 of Sec. 3.7.(vi)., gives

$$y(t) = c\sum_{n=0}^{\infty}\frac{(-1)^n}{(2^n n!)^2}.t^{2n}$$

Since $y(0) = 1$, we have $c = 1$. Therefore the solution of the given equation becomes

$$y(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2^n n!)^2}t^{2n} = J_0(t)$$

Ex. 3.7.5. By the use of Laplace transform find the solution of the equation

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = \sin t$$

satisfying the initial condition $y(0) = 1$.

Soln. By the same procedure as in the previous example the following equation is obtained after taking the Laplace transform of the given equation.

$$-p^2\frac{d\bar{y}}{dp} - 2p\bar{y} + 1 + 2p\bar{y} - 2 - \frac{d\bar{y}}{dp} = \frac{1}{1+p^2}$$

or, $$\frac{d\bar{y}}{dp} = -\frac{1}{p^2+1} - \frac{1}{(p^2+1)^2}$$

Therefore $$L^{-1}\left[\frac{d\bar{y}}{dp}\right] = -L^{-1}\left[\frac{1}{p^2+1}\right] - L^{-1}\left[\frac{1}{p^2+1}.\frac{1}{p^2+1}\right]$$

$$= -\sin t - \int_0^t \sin\tau\sin(t-\tau)d\tau \text{ (by convolution theorem.)}$$

$$= -\sin t - \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos t]d\tau$$

$$= -\frac{3}{2}\sin t - \frac{1}{2}t\cos t$$

By the use of the formula $L[ty(t)] = -\frac{d\bar{y}}{dp}$, we get $L^{-1}\left[\frac{d\bar{y}}{dp}\right] = -ty(t)$.

Hence the above relation gives $-ty(t) = -\frac{3}{2}\sin t - \frac{t}{2}\cos t$

or, $y(t) = \frac{3}{2t}\sin t + \frac{1}{2}\cos t$

which is the solution of the given equation.

MODEL QUESTIONS

I. Short Questions

1. Define Laplace transform of a given function.

2. State the conditions for the existence of Laplace transform of a function.

3. Deduce the (i) linear property, (ii) similarity theorem, (iii) shifting theorem and (iv) translation property for Laplace transform.

4. Find (i) L $(e^{\lambda t})$, λ being a complex constant, (ii) L (sin ωt) and L (cos ωt), ω being a real conrtaut, (iii) $L(t^n e^{\lambda t})$, (iv) $L(\sin \omega t . e^{\lambda t})$ and $L(\cos \omega t . e^{\lambda t})$, (v) L (t cos ωt) and L (t sin ωt)

5. If $F(p)$ be the Laplace transform of a function $f(t)$, then assuming the necessary conditions, show that $\lim_{p \to \infty} F(p) = 0$.

6. State the prove Initial and Final value theorems.

II. Broad Questins

1. Prove the Theorems 1 – 10 for Laplace transform.

2. State and prove the convolution theorem.

3. Assuming the necessary conditions, find (i) $L\left(\int_0^t f(\tau)d\tau\right)$ (ii) $L(t^\gamma)$, $\gamma > -1$

4. Describe various methods for the inversion of Laplace transform.

5. Describe hwo Laplace transform solving (i) linear ordinary and partial differential equations with constant coefficients and (ii) ordinary differential equations with variable coefficients.

III. Problems

1. By evaluating the inversion integral find the function whose Laplace transform is (i) $\frac{p+3}{(p^2+6p+13)^2}$ (ii) $\frac{1}{(p^2+a^2)^2}$

Ans. (i) $3e^{-3t}t\sin t$ (ii) $\frac{1}{2a^3}(\sin at - at\cos at)$

2. Show that $L[J_0(t)] = \frac{1}{\sqrt{p^2+1}}$, Hence find $L[J_1(t)]$ and $L[e^{-at}J_0(pt)]$

Ans. $1-\frac{p}{\sqrt{p^2+1}}, \frac{1}{\sqrt{(p+a)^2+1}}$

3. Show that $\int_0^\infty J_0(t)dt = 1$

4. By the use of Laplace transform above that

$$\int_0^\infty \frac{\cos tx}{x^2+1}dx = \frac{\pi}{2}e^{-t}, t>0$$

5. By the use of Laplace transform find the solution of the equation.

(i) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y = e^t\sin t,\ y(0)=0,\ y'(0)=1$

(ii) $\frac{d^2x}{dt^2}+x = t\cos 2t,\ x(0)=0,\ x'(0)=0$

(iii) $\frac{d^4x}{dt^4}+2\frac{d^2x}{dt^2}+x=\sin t,\ x(0)=x'(0)=x''(0)=x'''(0)=0.$

(iv) $\frac{d^3x}{dt^3}+\frac{dx}{dt}=\cos t,\ x(0)=0,\ x'(0)=-2, x'''(0)=0.$

(v) $\frac{dx}{dt}-y=e^t,\ \frac{dy}{dt}+x=\sin t,\ x(0)=1,\ y(0)=0.$

Ans. (i) $y(t) = \frac{11}{3}e^{-t}(\sin t + \sin 2t)$

(ii) $x(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t$

(iii) $x(t) = \frac{1}{8}(3-t^2)\sin t - \frac{3}{8}t\cos t$

(iv) $x(t) = -\frac{1}{2}t\cos t - \frac{3}{2}\sin t$

(v) $x(t) = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$, $y(t) = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t)$

6. By the use of Laplace transform find the solution of the equation,

$$ty'' + (1-2t)y' - 2y = 0$$

satisfying the initial conditions, $y(0) = 1,\ y'(0) = 2$

Ans. $y(t) = e^{2t}$

7. By the use of Laplace transform find the solution of the equation

$$\frac{\partial u}{\partial t} = \lambda\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),\ 0 \le r \le a,\ t > 0$$

satisfying the initial-boundary conditions : (i) $u(r,0) = 0,\ 0 \le r \le a$, (ii) $u(a, t) = u_0,\ t > 0$

Ans. $u(r,t) = u_0\left[1 - 2\sum_{s=1}^{\infty}\frac{1}{\alpha_s}e^{-\frac{\lambda\alpha_s^2 t}{a^2}}\cdot\frac{J_0\left(\frac{r}{a}\alpha_s\right)}{J_1(\alpha_s)}\right]$

where, α_s is the s-th root of the equation $J_0(x) = 0$ in x and consequently $J_0(\alpha_s) = 0$.

8. For the following problem find $v(\infty)$ by the use of final value theorem.

Equation : $\frac{\partial v}{\partial t} = \lambda\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right),\ 0 \le r \le a,\ t > 0$

Initial-boundary conditions : $v(r, 0) = 0,\ 0 < r \le a,\ v(a, t) = v_0,\ t > 0.$

Summary : In the present unit, the concept of Laplace transform, conditions for existence and inversions have been introduced. Laplace transforms of some particular functions have been obtained. It has also been shown how this transform can be applied to solve differential equations arising in practice.

CHAPTER 4 ☐ HANKEL TRANSFORMS

Structure

4.0 INTRODUCTION

German mathematician Hermann Hankel introduced another type of integral transform, known as Hankel transform, which is very different from Fourier and Laplace transforms. Hankel transform occurs in the study of functions and is widly used to transform partial differential equations into ordinary differential equations, specially for problems arising with cylindrical polar coordinate system. Here the integral Kernel is not in the form of exponential function, but Bessel function.

We in the present unit, introduce the concept of infinite and finite Hankel transforms, their inversion formulae, derivatives etc.

4.1 HANKEL TRANSFORMS : DEFINITION AND INVERSION FORMULA

Hankel transform of order υ of a function $f(r)$, $0 \le r < \infty$, denoted by $H_\upsilon[f(r)]$ is defined by the integral,

$$H_\upsilon[f(r)] = \int_0^\infty r f(r) J_\upsilon(\xi r) dr, \text{ for } \upsilon > -\frac{1}{2} \quad (4.1)$$

where $J_\upsilon(\xi r)$ is the Bessel function of order ν and argument ξr.

The corresponding inversion formula is stated in the following theorem.

Theorem 1. If the integral $\int_0^\infty f(r)dr$ is absolutely convergent and $f(r)$ is continuous in the neighbourhood of r, then

$$f(r) = \int_0^\infty \xi \bar{f}_v(\xi) J_v(\xi r) d\xi \tag{4.2}$$

where $\bar{f}_v(\xi)$ is the Hankel transform of order v of the function $f(r)$.

We give below the proof of this theorem for the particular case of $v = 0$, which can be stated as follows :

Theorem 2. If the integral $\int_0^\infty f(r)dr$ is absolutely convergent and $f(r)$ is continuous in the neighbourhood of r, then

$$f(r) = \int_0^\infty \xi \bar{f}_0(\xi) J_0(\xi r) d\xi \tag{4.3}$$

where $\bar{f}_0(\xi)$ is the Hankel transform of order 0 of the function $f(r)$.

Proof. We can write

$$f(r) = f\left(\sqrt{x^2 + y^2}\right) = g(x, y), \text{ say,} \tag{4.4a}$$

where $x = r \cos \theta$, $y = r \sin \theta$,

$0 \le r < \infty \quad 0 \le \theta < 2\pi$

The two-dimensional Fourier transform $\bar{g}(k, l)$ of the function $g(x, y)$ is

$$\bar{g}(k,l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) e^{i(kx+ly)} dx\, dy$$

$$= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} f(r) e^{i\xi r \cos\theta} r\, d\theta\, dr, \text{ where } \xi = \sqrt{k^2 + l^2}$$ and where we have taken the line joining (0, 0) and (k, l) in xy-plane as the initial line.

$$= \int_0^\infty r f(r) J_0(\xi r)\, dr, \text{ since } \int_0^{2\pi} e^{i\lambda \cos\theta} d\theta = 2\pi J_0(\lambda)$$

Therefore, $\bar{g}(k,l) = \int_0^\infty r\, f(r)\, J_0(\xi r)\,dr = \bar{f}_0(\xi)$ (4.4b)

By Fourier inversion theorem we get,

$$g(x,y) = \frac{1}{2\pi}\iint \bar{g}(h,l)e^{-i(kx+ly)}dk\,dl$$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \bar{f}_0(\xi)e^{-i\xi r\cos\varphi}\xi d\varphi d\xi, \text{ by (4.4b)}$$

$$= \int_0^\infty J_0(-\xi r)\bar{f}_0(r)\xi\, d\xi \left[\because \int_0^{2\pi} e^{-i\lambda\cos\theta}d\theta = 2\pi J_0(-\lambda)\right]$$

where the line joining (0, 0) and the point (x, y) in kl-plane has been taken as initial line and $k = \xi\cos\varphi$, $l = \xi\sin\varphi$

Since by (4.4a) $f(r) = g(x, y)$ and $J_0(-x) = J_0(x)$ the above relation gives

$$f(r) = \int_0^\infty \xi\,\bar{f}_0(\xi)J_0(\xi r)\,d\xi$$

This completes the proof of the theorem.

4.2 HANKEL TRANSFORMS OF DERIVATIVES

Instead of deriving the formulas for the Hankel transform of the derivatives of a function, we derive in this section the Hankel transform of a function when it is operated on by the derivative operator $\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\upsilon^2}{r^2}$, as this is needed in solving boundary value problems by means of Hankel transform.

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\upsilon^2}{r^2}f,$$

where f is a function of r, $0 \le r < \infty$, is given by

$$H_\upsilon\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\upsilon^2}{r^2}f\right]$$

$$= \int_0^\infty r\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{v^2}{r^2}f\right]J_v(\xi r)\,dr$$

Now, (a) $\displaystyle\int_0^\infty \frac{d^2f}{dr^2}\,rJ_v(\xi r)\,dr$

$$=\left[rJ_v(\xi r)\frac{df}{dr}\right]_0^\infty-\int_0^\infty \frac{d}{dr}\{rJ_v(\xi r)\}\frac{df}{dr}dr$$

$= -\displaystyle\int_0^\infty \frac{d}{dr}\{rJ_v(\xi r)\}\frac{df}{dr}dr$, assuming that the expression within the square bracket vanishes at $r = 0, \infty$

$$=\left[-\frac{d}{dr}\{rJ_v(\xi r)\}f\right]_0^\infty+\int_0^\infty \frac{d^2}{dr^2}\{rJ_v(\xi r)\}f\,dr$$

$$=\int_0^\infty \frac{d^2}{dr^2}\{rJ_v(\xi r)\}f\,dr \tag{4.5a}$$

assuming that the expression within the square bracket vanishes at $r = 0, \infty$.

Also (b) $\displaystyle\int_0^\infty \frac{df}{dr}J_v(\xi r)dr=[fJ_v(\xi r)]_0^\infty-\int_0^\infty f\frac{d}{dr}J_v(\xi r)dr$

$$=-\int_0^\infty f\frac{d}{dr}J_v(\xi r)\,dr \tag{4.5c}$$

assuming that the expression within the square bracket vanishes ar $r = 0, \infty$.

Therefore by the use of (4.5a) and (4.5b) we get

$$H_v\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{v^2}{r^2}f\right]$$

$$=\int_0^\infty f(r)\left[\frac{d^2}{dr^2}\{rJ_v(\xi r)\}-\frac{d}{dr}\{J_v(\xi r)\}-\frac{v^2}{r}J_v(\xi r)\right]dr$$

$$= \int_0^\infty f(r)\left[r\frac{d^2}{dr^2}J_\upsilon(\xi r) + 2\frac{d}{dr}J_\upsilon(\xi r) - \frac{d}{dr}J_\upsilon(\xi r) - \frac{\upsilon^2}{r}J_\upsilon(\xi r)\right]dr$$

$$= \int_0^\infty rf(r)\left[\frac{d^2}{dr^2}J_\upsilon(\xi r) + \frac{1}{r}\frac{d}{dr}J_\upsilon(\xi r) - \frac{\upsilon^2}{r^2}J_\upsilon(\xi r)\right]dr$$

$$= \int_0^\infty rf(r)\xi^2\left[\frac{d^2}{dp^2}J_\upsilon(p) + \frac{1}{p}\frac{d}{dp}J_\upsilon(p) - \frac{\gamma^2}{p^2}J_\upsilon(p)\right]dr,\ p = \xi r$$

$$= -\xi^2\int_0^\infty rf(r)J_\upsilon(\xi r)\,dr = -\xi^2\bar{f}_\upsilon(\xi)$$

Since, Bessel function of order υ satisfies the equation

$$\frac{d^2J_\upsilon(p)}{dp^2} + \frac{1}{p}\frac{dJ_\upsilon(p)}{dp} + \left(1 - \frac{\upsilon^2}{p^2}\right)J_\upsilon(p) = 0$$

Therefore we have obtained the relations

$$H_\upsilon\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\upsilon^2}{r^2}f\right] = -\xi^2\bar{f}_\upsilon(\xi) \tag{4.6}$$

which is valid provided the function $f(r)$ is such that the expression inside the square brackets in the above paragraphs (a) and (b) vanish at $r = 0, \infty$.

In the particular case when $\upsilon = 0$ the above relation becomes

$$H_0\left[\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right] = -\xi^2\bar{f}_0(\xi) \tag{4.7}$$

provided the function $f(r)$ is such that $r\,f'(r)$ tends to zero as $r \to 0$ and $r \to \infty$. It can be shown that under these conditions the expression inside the square brackets of the pragraphs (a) and (b) vanishes in the particular case of $\upsilon = 0$.

Ex. 4.2.1 Find the solution of the following problem of free symmetric vibration of a stretched membrane of infinite extent.

(i) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,\ 0 \le r < \infty,$

(ii) $u(r, 0) = f(r)$

(iii) $u_t(r, 0) = g(r)$

where $u(r, t)$ is the transverse displacement of the membrane.

Soln. Taking Hankel transform of order 0 of equation (i), with respect to r *i.e.*, multiplying both sides of equation (i) by $rJ_0(\xi r)$ and then integrating with respect to r between the limits 0 to ∞, we get

$$\int_0^\infty \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) rJ_0(\xi r)dr - \frac{1}{c^2}\frac{d^2}{dt^2}\int_0^\infty urJ_0(\xi r)\,dr = 0$$

or, $$\xi^2 \overline{u}_0(\xi) + \frac{1}{c^2}\frac{d^2\overline{u}_0(\xi)}{dt^2} = 0, \text{ by (4.7)} \qquad (4.8a)$$

where $\overline{u}_0(\xi, t)$ is the Hankel transform of $u(r, t)$. Here we assume that the function $u(r, t)$ is such that $r\frac{\partial u}{\partial r} \to 0$ as $r \to 0$ and $r \to \infty$.

Taking Hankel transform of the initial conditions (ii) and (iii) we get,

$$\overline{u}_0(\xi, 0) = \bar{f}_0(\xi) \qquad (4.8b)$$

$$\frac{\partial \overline{u}_0}{\partial t}(\xi, 0) = \overline{g}_0(\xi)$$

where $\bar{f}_0(\xi)$ and $\overline{g}_0(\xi)$ are Hankel transform of $f(x)$ and $g(x)$ respectively

Solution of equation (4.8a) is

$$\overline{u}_0(\xi, t) = A\cos(c\xi t) + B\sin(c\xi t) \qquad (4.8c)$$

where A and B are two constants, *i.e.* they are independent of t.

Noting that $$\frac{\partial \overline{u}_0(\xi, t)}{\partial t} = c\xi\,[-A\sin(c\xi t) + B\cos(c\xi t)] \qquad (4.8d)$$

Setting $t = 0$ in (4.8c) and using (4.8d) we get

$$A = \bar{f}_0(\xi),\ B = \frac{\overline{g}_0(\xi)}{c\xi}$$

Therefore, the solution for $\overline{u}_0(\xi, t)$ given by (4.8c) becomes

$$\overline{u}_0(\xi, t) = \bar{f}_0(\xi)\cos(c\xi t) + \frac{\overline{g}_0(\xi)}{c\xi}\sin(c\xi t)$$

Taking inversion of this we get the desired solution of the problem.

$$u(r, t) = \int_0^\infty \xi\,\bar{f}(\xi)\cos(c\xi t)J_0(\xi r)d\xi + \frac{1}{c}\int_0^\infty \overline{g}(\xi)\sin(c\xi t)J_0(\xi r)\,d\xi$$

$$= \int_0^\infty d\xi \cos(c\xi t) J_0(\xi r) \int_0^\infty \alpha f(\alpha) J_0(\xi \alpha) d\alpha$$

$$+ \frac{1}{c} \int_0^\infty \sin(c\xi t) J_0(\xi r) \int_0^\infty \alpha g(\alpha) J_0(\xi \alpha) d\alpha$$

4.3 FINITE HANKEL TRANSFORM : DEFINTION AND INVERSION FORMULA

The finite Hankel transform of order v of a function $f(r)$, $0 \le r \le a$, denoted by $H_{v,i}[f(x)]$ or, $\bar{f}_v(\xi_i)$, is defined by

$$H_{v,i}[f(x)] = \bar{f}_v(\xi_i) = \int_0^a r f(r) J_v(r\xi_i) dr \qquad (4.9)$$

where ξ_i is the root of the transcendental equation

$$J_v(a\xi_i) = 0 \qquad (4.10)$$

$J_v(x)$ being the Bessel function of order v and argument x.

The inversion formula for finite Hankel transform is stated in the following theorem, the proof of which is not given.

Theorem : If $f(r)$ satisfies Dirichlet's conditions in $(0, a)$ and if its finite Hankel transform of order v is given by

$$\bar{f}_v(\xi_i) = \int_0^a r f(r) J_v(r\xi_i) dr$$

where ξ_i is a root of the transcendental equation $Jv(a\xi_i) = 0$, then at any point of the interval $(0, a)$ at which the function $f(r)$ is continuous

$$f(r) = \frac{2}{a^2} \sum_i \bar{f}_v(\xi_i) \frac{J_v(r\xi_i)}{\left[J_v'(a\xi_i)\right]^2}, \qquad (4.11)$$

where the sum is taken over all positive roots of the equation $J_v(a\xi_i) = 0$.

4.4 FINITE HANKEL TRANSFORM OF DERIVATIVES

Due to the reason stated in Sec. 4.2. we derive only the finite Hankel transform of

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\nu^2}{r^2} f,$$

where $f(r)$ is a function of r defined in the interval $(0, a)$, restricting υ to the case $\upsilon \geq 0$.

$$H_{\upsilon,i}\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\upsilon^2}{r^2} f\right]$$

$$= \int_0^a r\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\nu^2}{r^2} f\right] J_\upsilon(r, \xi i)\, dr$$

Now (a) $\int_0^a \frac{d^2 f}{dr^2} rJ_\upsilon(r, \xi_i)\, dr$

$$= \left[rJ_\upsilon(r\xi_i)\frac{df}{dr}\right]_0^a - \int_0^a \frac{d}{dr}\{rJ_\upsilon(\xi_i r)\}\frac{df}{dr} dr$$

$$= -\int_0^a \frac{d}{dr}\{rJ_\upsilon(\xi_i r)\}\frac{df}{dr} dr, \text{ since } J_\upsilon(a\xi_i) = 0$$

$$= -\left[\frac{d}{dr}\{rJ_\upsilon(\xi_i r)\} f(r)\right]_0^a + \int_0^a \frac{d^2}{dr^2}\{rJ_\upsilon(\xi_i r)\} f\, dr$$

$$= -[\{J_\upsilon(\xi_i r) + r\xi_i J'_\upsilon(\xi_i r\} f(r)]_0^a + \int_0^a \frac{d^2}{dr^2}\{rJ_\upsilon(\xi_i r)\} f\, dr$$

$$= -a\xi_i J'_\upsilon(a\xi_i) f(a) + \int_0^a \frac{d^2}{dr^2}\{rJ_\upsilon(\xi_i r)\} f\, dr \qquad (4.12a)$$

since $J_\upsilon(\xi ia) = 0$ and $\upsilon \geq 0$.

(b) $\int_0^a \frac{df}{dr} J_\upsilon(r\xi_i)\, dr = [f(r) J_\upsilon(r\xi_i)]_0^a - \int_0^a f \frac{d}{dr} J_\upsilon(r\xi_i) dr$

$$= -\int_0^a f \frac{dJ_\upsilon}{dr}(r\xi_i)\, dr \qquad (4.12b)$$

since $J_\upsilon\ (a\xi i) = 0$ and $\upsilon \geq 0$.
Therefore by the use of (4.12a) and (4.12b) we get

$$H_{v,i}\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{v^2}{r^2}f\right]$$

$$=\int_0^a f(r)\left[\frac{d^2}{dr^2}\{rJ_v(\xi_i r)\}-\frac{d}{dr}\{J_v(\xi_i r)\}-\frac{v^2}{r^2}J_v(\xi_i r)\right]dr-a\xi_i J_v'(a\xi_i)f(a)$$

$$=\int_0^a rf(r)\left[\frac{d^2}{dr^2}J_v(\xi_i r)+\frac{1}{r}\frac{d}{dr}J_v(\xi_i r)-\frac{v^2}{r^2}J_v(\xi_i r)\right]dr-a\xi_i J_v'(a\xi_i)f(a)$$

$$=\int_0^a rf(r)\xi_i^2\left[\frac{d^2}{dp^2}J_v(p)+\frac{1}{p}\frac{d}{dp}J_v(p)-\frac{v^2}{p^2}J_v(p)\right]dr-a\xi_i J_v'(a\xi_i)f(a)$$

where $p = r\xi_i$

$$=-a\xi_i J_v'(a\xi_i)f(a)-\xi_i^2\int_0^a rf(r)J_v(r\xi_i)\,dr$$

$$=-a\xi_i J_v'(a\xi_i)f(a)-\xi_i^2\bar{f}_v(\xi_i).$$

since Bessel function of order v satisfies the equation,

$$\frac{d^2}{dp^2}J_v(p)+\frac{1}{p}\frac{d}{dp}J_v(p)+\left(1-\frac{v^2}{p^2}\right)J_v(p)=0$$

Hence we have derived the relation

$$H_{v,i}\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{v^2}{r^2}f\right]$$

$$=-a\xi_i J_v'(a\xi_i)f(a)-\xi_i^2\bar{f}_v(\xi_i), \tag{4.12c}$$

which is valid for $v \geq 0$

For the particular case $v = 0$, the relation (4.12c) becomes

$$H_{0,i}\left[\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}\right]=a\xi_i J_1(a\xi_i)f(a)-\xi_i^2\bar{f}_v(\xi_i) \tag{4.12d}$$

since $J_0'(a\xi_i)=-J_1(a\xi_i)$.

Ex. 4.4.1. Solve the following problem of free symmetric vibration of a stretched circular membrane

(i) $\quad \dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}-\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}=0,\ 0 \leq r < a.$

(ii) $\quad u(a, t) = 0$

(iii) $\quad u(r, 0) = f(r)$

(iv) $\frac{\partial}{\partial t}u(r,0) = g(r)$

where $u(r, t)$ is the transverse displacement of the membrane.

Soln. Taking finite Hankel transform of order 0 of the given equation (i), *i.e.* multiplying both sides of equation (i) by $rJ_0(r\xi_i)$ and then integrating with respect to r between the limits 0 to a, we get the following by the use of the relation (4.12d) and the boundary condition (ii)

$$H_{0,i}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]-\frac{1}{c^2}\frac{d^2}{dt^2}\int_0^a ru(r,t)\,J_0(r\xi_i)\,dr = 0$$

$$\text{or,}\quad \xi_i^2\overline{u}_0(\xi_i,t)+\frac{1}{c^2}\frac{d^2\overline{u}_0}{dt^2}(\xi_i t) = 0, \tag{4.13a}$$

where $\overline{u}_0(\xi_i,t)$ is the finite Hankel transform of order 0 with respect to r defined by

$$\overline{u}_0(\xi_i,t) = \int_0^a ru(r,t)J_0(r\xi_i)dr$$

ξ_i being the root of the equation $J_0(ax) = 0$ in x.

Also taking finite Hankel transform of order 0 of the initial conditions (iii) and (iv) we get

$$\overline{u}_0(\xi_i,0) = \bar{f}_0(\xi_i)$$

$$\frac{d\overline{u}_0}{dt}(\xi_i,0) = \overline{g}_0(\xi_i) \tag{4.13b}$$

where $\bar{f}_0(\xi_i)$ and $\overline{g}_0(\xi_i)$ are the finite Hankel transforms of order 0 of $f(r)$ and $g(r)$ respectively.

The solution of equation (4.13a) for $\overline{u}_0(\xi_i,t)$ is

$$\overline{u}_0(\xi_i,t) = A\cos(c\xi_i t)+B\sin(c\xi_i t)$$

From this we get

$$\frac{d}{dt}\overline{u}_0(\xi_i,t) = -c\xi_i A\sin(c\xi_i t)+c\xi_i B\cos(c\xi_i t)$$

Here A and B are two constants, *i.e.* they are independent of time.

Setting $t = 0$ in the above and using (4.13b) we get

$$A = \overline{u}_0(\xi_i,0) = \bar{f}_0(\xi_i),$$

$$c\xi_i B = \frac{d\overline{u}_0}{dt}(\xi_i, t) = \overline{g}_0(\xi_i),$$

Therefore the solution for $\overline{u}_0(\xi_i, t)$ becomes

$$\overline{u}_0(\xi_i, t) = \bar{f}_0(\xi_i)\cos(c\xi_i t) + \frac{1}{c\xi_i}\overline{g}_0(\xi_i)\sin(c\xi_i t)$$

Taking inversion of this according to the formula (4.11) we get

$$u(r,t) = \frac{2}{a^2}\sum_i\left[\bar{f}_0(\xi_i)\cos(c\xi_i t) + \frac{1}{c\xi_i}\overline{g}(\xi_i)\sin(c\xi_i t)\right]\times\frac{J_0(r\xi_i)}{[J_0'(a\xi_i)]^2}$$

$$= \frac{2}{a^2}\sum_i\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}\cos(c\xi_i t)\int_0^a \alpha f(\alpha)J_0(\alpha\xi_i)dx$$

$$+\frac{2}{ca^2}\sum_i\frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}\frac{\sin(c\xi_i t)}{\xi_i}\int_0^a \alpha g(\alpha)J_0(\alpha\xi_i)d\alpha$$

where we have substituted for $\bar{f}_0(\xi_i)$ and $\overline{g}_0(\xi_i)$ according to the definition of finite Hankel transform.

Ex. 4.4.2. Slove the following problem of conduction of heat in an infinite circular cyliner

(i) $\frac{\partial u}{\partial t} = \lambda\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),\ 0 \le r \le a,$

(ii) $u(a, t) = 0$

(iii) $u(r, 0) = f(r),\ 0 \le r \le a$

Taking finite Hankel transform of order 0 of the given equation (i) with respect to r, *i.e.* multiplying both sides of the equation by $rJ_0(\xi_i r)$ and then integrating with respect to r between the limits 0 to a, we get the following by the use of the relation (4.12d) and the boundary condition (ii).

$$\lambda\xi_i^2\overline{u}_0(\xi_i, t) + \frac{d}{dt}\overline{u}_0(\xi_i, t) = 0 \tag{4.14}$$

Here $\overline{u}_0(\xi_i, t)$ denotes the finite Hankel transform of order 0 with respect to r defined by

$$\overline{u}_0(\xi_i, t) = \int_0^a ru(r,t)J_0(r\xi_i)dr$$

ξ_i being the root of the equation $J_0(ax) = 0$ in x.

Also taking finite Hankel transform of order 0 of the initial condition (iii) we get

$$\overline{u}_0(\xi_i, 0) = \bar{f}_0(\xi_i)$$

where $\bar{f}_0(\xi_i)$ is the finite Hankel transform of order zero of the functions $f(r)$.

Solution of equation (4.14) is

$$\overline{u}_0(\xi_i, t) = Ae^{-\lambda\xi_i^2 t}$$

where A is a constant, *i.e.* intependent of t. Setting $t = 0$ we get

$$A = \overline{u}_0(\xi_i, 0) = \bar{f}_0(\xi_i),$$

Therefore the solution for $\overline{u}_0(\xi_i, t)$ becomes

$$\overline{u}_0(\xi_i, t) = \bar{f}_0(\xi_i)e^{-\lambda\xi_i^2 t}$$

Taking inversion of this according to the fornula (4.11) we get

$$u(r, t) = \frac{2}{a^2}\sum_i \overline{f}_0(\xi_i)e^{-\lambda\xi_i^2 t}\frac{J_0(r\xi_i)}{[J_0'(a\xi_i)]^2}$$

$$= \frac{2}{a^2}\sum_i \frac{J_0(r\xi_i)}{[J_1(a\xi_i)]^2}e^{-\lambda\xi_i^2 t}\int_0^a \alpha f(\alpha)J_0(\alpha\xi_i)d\alpha$$

subsituting for $\bar{f}_0(\xi_i)$ according to the definition and using the realtion $J_0'(x) = -J_1(x)$.

MODEL QUESTIONS

I. Short Questions

1. Define Hankel transform and finite Hankel transform of a function of order γ.

2. State the inversion formlae for Hankel and finite Hankel transforms.

II. Broad Questions

1. Assuming the integral $\int_0^\infty f(r)dr$ to be absolutely convergent and $f(r)$ is continous in the neighbourhood of r, show that $f(r) = \int_0^\infty \xi \bar{f}_0(\xi) J_0(\xi r) d\xi$ where $\bar{f}_0(\xi)$ is the Hankel transform of order 0 of the function $f(r)$.

2. Assuming the necessary conditions, show that

$$H_\gamma\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\gamma^2}{r^2} f\right] = -\xi^2 \bar{f}_\gamma(\xi)$$

3. Assuming the necessary conditions, show that for finite Hankel transform

$$H_{\gamma,i}\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{dr}{dr} - \frac{\gamma^2}{r^2} f\right] = -a\xi_i J'_\gamma(a\xi_i) f(a) - \xi_i^2 \bar{f}_\gamma(\xi_i)$$

for a function $f(r)$, $0 \le r \le a$ and ξ_i are the roots of the equation $J_\gamma(a\xi_i) = 0$.

III. Problems

1. Show that $H_0\left\{\frac{1}{r}\right\} = \frac{1}{\xi}$ by using the fact that the Hankel transform is its own inverse.

2. Find $H_0\left(\frac{\sin r}{r^2}\right)$

3. Applying Hankel transform, show that the solutions of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

with initial conditions $u(r, 0) = \frac{1}{\sqrt{1+r^2}}, \frac{\partial u}{\partial t}(r,0) = 0$ is

$$u(r,t) = \text{Re}\left\{\frac{1}{\sqrt{(1-it)^2 + r^2}}\right\}$$

[Given that $\int\limits_0^{\infty} e^{-z}J_0(\xi r)d\xi = \dfrac{1}{\sqrt{z^2+r^2}}$]

4. Solve the axisymmetric diffusion equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), 0 < r < \infty, t > 0$$

$u(r, 0) = u_0$, $0 < r < \infty$, where k is diffusion constant.

5. Find a function $u(r, z)$ harmonic in the half space $z \geq 0$ and satisfies the mixed boundary conditions.

$$\frac{\partial u}{\partial z}(r,0) = f(r),\ 0 \leq r \leq a$$

$$u(r, 0) = 0,\ r > a$$

and the limiting condition $u(r, z) \to 0$ as $p \to \infty$

Summary. A sketch of Hankel transform has been introduced without going into details. Some problems in cylindrical coordinate system have been solved by using the transform.

Group - B

INTEGRAL EQUATIONS

UNIT 1 ❒ PRELIMINARY CONCEPTS

Vito Volterra
(1860–1940)

Vito Volterra was born in a extremely poor family in Italy. He showed interest in mathematics at the age of 11 when he began to study Legendre's Geometry. His doctoral work on hydrodynamics included some results of Stokes, discovered later but independently by Volterra.

He became the Professor of Mechanics at Pisa in 1883. At that time he conceived the idea of theory of functions which depend on a continuous set of values of another function, which at present termed as 'functional' introduced by Hadamard.

During the years 1892 to 1894, Volterra published several papers on partial differential equations. He began this study in 1884 and in 1896 he published his most famous work—'On integral equation of Volterra type.'

During the First World War Volterra joined the Air Force and made many journeys to France and England to promote scientific collaboration.

In 1938 he was offered an honorary degree by the University of St. Andrews.

Erik Ivar Fredholm
(1866–1927)

Ivar Fredholm, a Swedish mathematician, came from a merchant family. He proved his brilliancy since studying at school and was awarded his baccalaureate in 1885. He took Master of Science degree from the university of Uppsala in the year 1888. Fredholm studied for the doctorate degree under Mittag-Leffler and was awarded Ph. D. from the university of Uppsala in the year 1893 and then in the year 1898 he received the degree of Doctor of Science from the same university. His doctoral dissertation involved study of partial differential equations, the study of which was motivated by an equilibrium problem in elasticity.

Fredholm is best remembered for his work on integral equations and spectral theory. In 1900 a preliminary report on his theory of Fredholm integral equations was published as *Sur une nouvelle méthode parr la résolution du probléme de Dirichlet*. Volterra had earlier studied some aspects of integral equations but before Fredholm little had been done.

Fredholm's contributions quickly became well known to the world of mathematics. Hilbert immediately saw the importance of Fredholm theory and extended Fredholm's work to include a complete eigenvalue theory for the Fredholm integral equation. This work led directly to the theory of Hilbert spaces.

Fredholm's fuller version of work entitled *sur une classe d'equations fonctionelle* appeared in Acta Mathematica in 1903.

After the award of his Doctor of Science degree in 1898 he was appointed as a lecturer in mathematical physics at the University of Stockholm, later he attained the chair of Dean in the same university.

Fredholm received many honours for his mathematical contributions, including the V.A. Wallmarks Prize for the theory of differential equations in 1903, the Poncelet Prize from the French Academy of Sciences in 1908, and an honourary doctorate from the university of Leipzig in 1909.

1.1 Integral equation : An integral equation is an equation in which an unknown function appears under an integral sign. There is a close relationship between differential and integral equations and some problems may be formulated either way.

The most basic type of integral equation is Fredholm integral equation of the first kind given by :

$$f(x)=\int_a^b K(x,t)\,u(t)\,dt\,,\quad a\le x\le b. \qquad (1.1)$$

Here $u(t)$ is an unknown function, $f(x)$ is a known function and $K(x,t)$ is another known function of two variables, often called the kernel function. Note that the limits of integration are constant, this is what characterizes a Fredholm integral equation.

If the unknown function occurs both inside and outside the integral sign, it is known as a Fredholm integral equation of the second kind :

$$u(x)=f(x)+\lambda\int_a^b K(x,t)\,u(t)\,dt,\quad a\le x\le b \qquad (1.2)$$

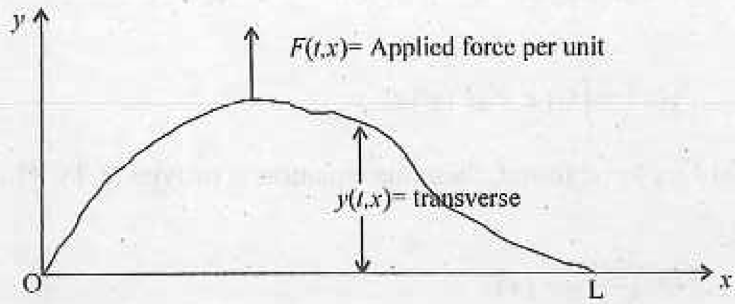
in which λ is a real or complex parameter ($\neq 0$), having the same role as the eigenvalue in linear algebra.

If one of the limits of integration is a variable, it is called Volterra integration equation. Thus Volterra integral equation of the first and the second kind are respectively as :

$$f(x)=\int_a^x K(x,t)\,u(t)\,dt \text{ and } u(x)=f(x)+\lambda\int_a^x K(x,t)\,u(t)\,dt,\ a\le x\le b \quad (1.3)\text{–}(1.4)$$

The equations (1.1.)–(1.4) are termed as homogeneous or inhomogeneous according as f is identically zero or not.

Integral equations are important in many applications. A problem initially formulated as a PDE or ODE can be converted into an equivalent integral equation. As an example, consider the problem of a stretched string, vibrating transversely subject to applied forces :



The parameters in the motion are

- mass per unit length $\rho(x)$,
- tension T

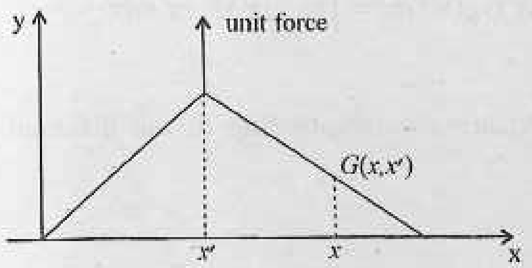
and the assumptions that we make are

- the nonlinear effects are negligible (*e.g.*, y^2 and y_x^2 etc.), T is unaffected by the motion,
- the ends of the string are fixed at O and L,
- there are no other energy losses.

Here the motion is governed by the PDE

$$-T\frac{\partial^2 y}{\partial x^2}+\rho(x)\frac{\partial^2 y}{\partial t^2}=F(t,x),\ 0<x<L$$

We will now convert this equation into an integral equation of the type described previously. Define $G(x,x')$ as the displacement at x per unit point force applied at x',



Clearly,

$$G(x,x')=\begin{cases}\dfrac{x(L-x')}{TL}, & x\le x'\\ \dfrac{(L-x)x'}{TL}, & x'\le x\end{cases}$$

If point forces, F_i act at point x_i' then the displacement at x is

$$\sum_i G(x,x_i')F_i .$$

Extending this, if a force distribution $F(x)$ per unit length acts then the displacement y is

$$y(x)=\int_0^L G(x,x')F(x')dx'$$

If y is given and F is to be found, then this equation is of type (1.1). This equation is equivalent to

$$-T\frac{\partial^2 y}{\partial x^2}=F(x)$$

subject to the usual boundary conditions.

The PDE which we had before is

$$-T\frac{\partial^2 y}{\partial x^2}=F(x)-\rho(x)\frac{\partial^2 y}{\partial t^2}$$

Then we must have

$$y(t,x)=\int_0^L G(x,x')\left\{F(x')-\rho(x')\frac{\partial^2 y}{\partial t^2}\right\}dx'$$

We will look for solutions of the form $y(t,x)=y_0(x)\cos\omega t$ where we are given $F(t,x)=F_0(x)\cos\omega t$. Substituting we find that

$$y_0(x)=\int_0^L G(x,x')\{F_0x')+\rho(x')\omega^2 y_0(x')\}\,dx'$$

which on re-arranging takes the form

$$y_0(x)-\omega^2\int_0^L G(x,x')\,\rho(x')y_0(x')\,dx'=\int_0^L G(x,x')F_0(x')\,dx'$$

which is of same type as (1.2).

Example 1.1. Form the integral equation corresponding to the differential equation

$$\frac{d^3y}{dx^3}-2xy=0$$

with initial conditions $y(0)=\frac{1}{2},\ y'(0)=y''(0)=1$

Solution : Let $\frac{d^3y}{dx^3}=u(x)$. Then we have by using given initial conditions

$$\frac{d^2y}{dx^2}=\int_0^x u(t)+1$$

$$\frac{dy}{dx}=\int_0^x (x-t)u(t)dt+x+1$$

$$y=\frac{1}{2}\int_0^x (x-t)^2u(t)dt+\frac{1}{2}x^2+x$$

Substituting the values of $\frac{d^3y}{dx^3}$ and y in the given differntial equation we get

$$u(x)-2x\left[\frac{1}{2}\int_0^x (x-t)^2u(t)dt+\frac{1}{2}x^2+x\right]=0$$

or, $$u(x)=x(x+1)^2+\int_0^x x(x-t)^2u(t)dt$$

which is Volterra's integral equation of second kind.

Example 1.2. Obtain Fredholm integral equation of second kind corresponding to the boundary value problem

$$\frac{d^2y}{dx^2}+xy=1,\ y(0)=0 \text{ and } y(1)=1$$

Solution : Integrating both sides of the given differential equation with respect to x, we get

$$\frac{dy}{dx}-\left(\frac{dy}{dx}\right)_0+\int_0^x t\,y(t)dt=x$$

or $$\frac{dy}{dx}=c+x-\int_0^x t\,y(t)dt,\ \text{where } c=\left(\frac{dy}{dx}\right)_0=y'(0).$$

Integrating both sides again with respect to x we get

$$y(x)-y(0)=cx+\frac{1}{2}x^2-\int_0^1 t(x-t)y(t)dt$$

Using the bounding conditions $y(0)=0, y(1)=1$, it follows that

$$1=c+\frac{1}{2}-\int_0^1 t(1-t)y(t)dt$$

so that $c=\frac{1}{2}+\int_0^1 t(1-t)y(t)dt$ and hence

$$y(x)=\frac{1}{2}x(1+x)+\int_0^1 xt(1-t)y(t)dt-\int_0^x t(x-t)y(t)dt \tag{1}$$

or, $$y(x)=\frac{1}{2}x(1+x)+\int_0^x xt(1-t)y(t)dt-\int_0^x t(x-t)y(t)dt+\int_x^1 xt(1-t)y(t)dt$$

or, $$y(x)=\frac{1}{2}x(1+x)+\int_0^x (1-x)t^2y(t)+\int_x^1 x(1-t)ty(t)dt \tag{2}$$

Thus the required Fredholm integral equation of the second kind is

$$y(x)=\frac{1}{2}x(1+x)+\int_0^x K(x,t)\, y(t)dt$$

where $K(x,t)=\begin{cases}(1-x)t^2, t<x\\(1-t)tx, t>x\end{cases}$

1.2 SPECIAL TYPES OF KERNEL

(a) Symmetric kernels–Integral equations with kernels satisfying

$$K(x,t)=K(t,x)$$

have certain advantage in determining solutions. For this reason it is to be noted that the integral equation

$$u(x) = f(x) + \lambda \int_a^b H(x,t) p(t) u(t) dt \tag{1.5}$$

with the unsymmetric kernel $H(x,t)p(t)$, where however $H(x,t) = H(t,x)$, can be transformed into the integral equation (1.2) with symmetric kernel

$$K(x,t) = \sqrt{p(x)} H(x,t) \sqrt{p(t)}$$

by multiplying (1.5) by $\sqrt{p(x)}$ and setting

$$\phi(x) = \sqrt{p(x)}\, u(x)$$

and

$$g(x) = \sqrt{p(x)}\, f(x)$$

A complex-valued function $K(x,t)$ satisfying $K(x,t) = K^*(t,x)$, where the star denotes the complex conjugate, is called Hermitian. For a real kernel, this coincides with definition $K(x,t) = K(t,x)$.

(b) Kernels producing convolution integrals—A class of integral equations which is of particular interest has a kernel of the form

$$K(x,t) = k(x-t).$$

The Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x k(x-t) u(t)\, dt$$

and the corresponding Fredholm integral equation are called integral equations of the convolution type. Integral equations of this type can be solved by using integral transforms such as Laplace and Fourier transforms.

Volterra integral equations are closely related to initial value problems. Given the ordinary differential equation

$$u''(x) = \lambda u(x) + g(x), \;\; u(0) = 1, \;\; u'(0) = 0.$$

Integrating with respect to x and taking $\lambda u + g$ as $F(x)$, we find

$$u'(x) = \int_0^x F(t)\, dt + c_1$$

and that

$$u(x) = \int_0^x \int_0^t F(\xi)\, d\xi\, dt + c_1 x + c_2 .$$

Using the identity

$$\int_0^x \int_0^t F(\xi)\, d\xi\, dt = \int_0^x (x-t)\, F(t)\, dt .$$

We can write

$$u(x) = \int_0^x (x-t)\, \{\lambda u(t) + g(t)\}\, dt + c_1 x + c_2 .$$

Applying the initial conditions

$$u(0) = 1 \text{ and } u'(0) = 0$$

we get $c_2 = 1$ and $c_1 = 0$.

Therefore, $u(x) = \lambda \int_0^x (x-t)\, u(t)\, dt + \int_0^x (x-t)\, g(t)\, dt .$

This is Volterra integral equation of the second kind.

Similarly, the boundary value problems in ordinary differential equation lead to Fredholm integral equations.

Solution by differentiation

It may be observed that Volterra integral equation can be reduced to initial value problem while Fredholm integral equation reduces to boundary value problem. To show this we consider the following examples.

Example 1.3. Deduce the initial value problem corresponding to Volterra integral equation

$$u(x) = \cos x + x - 1 - \int_0^x (x-t) u(t) dt$$

Solution. Differentiating both sides of the given integral equation with respect to *x*, we get

$$u'(x) = -\sin x + 1 - \int_0^x u(t) dt$$

Differentiating again, $u''(x) = -\cos x - u(x)$. Also we have $u(0) = 0$, $u'(0) = 1$

Thus the required initial value problem is :

To solve the differential equation $\dfrac{d^2u}{dx^2} + u = -\cos x$ with initial conditions $u(0) = 0$ and $u'(0) = 1$.

Example 1.4. Obtain the boundary value problem from the Fredholm integral equation

$$u(x) = \frac{1}{6}(x^3 - 3x) + \lambda \int_0^1 K(x,t)u(t)\,dt$$

where $K(x,t) = \begin{cases} x, & x > t \\ t, & x < t \end{cases}$.

Solution : The integral equation can be written as

$$u(x) = -\frac{1}{2}x + \frac{1}{6}x^3 + \lambda\left\{\int_0^x t(t)\,dt + \int_x^1 x\,u(t)\,dt\right\}$$

or, $$u(x) = -\frac{1}{2}x + \frac{1}{6}x^3 + \lambda\left\{\int_0^x xu(t)\,dt + \int_x^1 x\,u(t)\,dt - \int_0^x (x-t)u(t)\,dt\right\}$$

or, $$u(x) = -\frac{1}{2}x + \frac{1}{6}x^3 + \lambda\int_0^1 xu(t)\,dt - \lambda\int_0^x (x-t)u(t)\,dt \qquad (1)$$

Differening both sides w.r.t., x, we get

$$u'(x) = -\frac{1}{2} + \frac{1}{2}x^2 + \lambda\int_0^1 u(t)\,dt - \lambda\int_0^x u(t)\,dt$$

i.e. $$u'(x) = -\frac{1}{2} + \frac{1}{2}x^2 + \lambda\int_x^1 u(t)\,dt \qquad (2)$$

Differentiating again w.r.t., x we have

$$u''(x) = x - \lambda u(x)$$

i.e. $$\frac{d^2u}{dx^2}+\lambda u = x$$

Now putting $x = 0$ in (1) and $x = 1$ in (2) we have the boundary conditions $u(0) = 0$, $u'(1) = 0$

Thus the required boundary value problem is :

To solve the differential equation $\frac{d^2u}{dx^2}+\lambda u = x$ subject to boundary conditions $u(0) = 0$, $u'(1) = 0$.

Solution by Laplace transform

If the kernel of the Volterra integral equation is of the form $K(x-t)$, the equation is said to be of convolution type and may be solved by use of the Laplace transform.

Consider the Volterra type integral equation of the first kind

$$f(x)=\int_0^x K(x-t)\phi(t)dt$$

where the kernel $K(x-t)$ depends only on the difference $x-t$. Taking Laplace transform to both sides of the above integral equation with s as transform parameter, we get

$$F(s)=K(s)\phi(s)\Rightarrow \phi(s)=\frac{F(s)}{K(s)}$$

where $F(s)=\int_0^\infty e^{-sx}f(x)$ etc. Laplace inversion of the above equation gives the value of $\phi(x)$.

Similarly we may apply the transform method to the Volterra integral equation of the second kind with a convolution type kernel. Consider a non-homogeneous integral equation of second knd as

$$\phi(x)=f(x)+\int_0^x K(x-t)\phi(t)dt$$

Taking Laplace transform to both sides and applying the convolution formula, we have

$$\Phi(s) = F(s) + K(s)\Phi(s) \Rightarrow \Phi(s) = \frac{F(s)}{1-K(s)},$$

Laplace inversion of which leads to the disired result.

Example 1.5. Solve the integral equation

$$\sin\beta x = \int_0^x \cos\alpha(x-t)\phi(t)dt$$

Solution. Applying Laplace transform on both sides and using Laplace transform of convolution formula, we find

$$\frac{\beta}{s^2+\beta^2} = \frac{s}{s^2+\alpha^2}\Phi(s)$$

whence $\Phi(s) = \frac{\beta}{s}\cdot\frac{s^2+\alpha^2}{s^2+\beta^2} = \frac{\beta}{s} + \frac{\alpha^2-\beta^2}{\beta}\left(\frac{1}{s} - \frac{s}{s^2+\beta^2}\right) = \frac{\alpha^2}{\beta}\cdot\frac{1}{s} + \frac{\beta^2-\alpha^2}{\beta}\cdot\frac{s}{s^2+\beta^2}$,

Laplace inversion of which gives

$$\phi(x) = \frac{\alpha^2}{\beta} + \frac{\beta^2-\alpha^2}{\beta}\cos\beta x$$

Example 1.6. Solve the integral equation

$$\phi(x) = e^{-x} - 2\int_0^x \cos(x-t)\phi(t)dt$$

Solution. Taking Laplace transform we get

where $\Phi(s) = \int_0^\infty e^{-sx}\phi(x)dx$. Then $\Phi(s) = \frac{s^2+1}{(s+1)^3} = \frac{\{(s+1)-1\}^2+1}{(s+1)^3}$, the inversion of which leads to

$$\phi(x) = L^{-1}\left[\frac{\{(s+1)-1\}^2+1}{(s+1)^3}\right]$$

Using first shifting theorem, we get

$$\phi(x) = e^{-x}L^{-1}\left[\frac{(s-1)^2+1}{s^3}\right] = e^{-x}L^{-1}\left[\frac{1}{s}-\frac{2}{s^2}+\frac{2}{s^3}\right]$$

i.e. $$\phi(x) = e^{-x}\left[1-2x+2.\frac{x^2}{2!}\right] = e^{-x}(1-2x+x^2) = e^{-x}(1-x)^2$$

Abel's integral equation

In 1825 Abel solved the integral equation named after him having the form

$$f(x) = \int_0^x (x-t)^{-\alpha}\phi(t)\,dt \tag{1.6}$$

where $f(x)$ is a continuous function satisfying $f(0)=0$ and $0<\alpha<1$. For $\alpha=1/2$ Abel's integral equation corresponds to the famous tautochrone problem—*To determine the shape of the curve with a given initial point along which a particle slides without friction in a constant time independent of its initial position.* Abel's equation may be solved by the use of the Laplace transform. However we solve it directly in the following manner.

If $f(x) = \int_0^x (x-t)^{-\alpha}\phi(t)\,dt, \quad 0<\alpha<1$

then,
$$\int_0^u \frac{f(x)}{(u-x)^{1-\alpha}}dx = \int_0^x \frac{1}{(u-x)^{1-\alpha}}\left\{\int_0^x \frac{\phi(t)}{(x-t)^\alpha}dt\right\}dx$$

$$= \int_0^u \phi(t)\left\{\int_0^u \frac{dx}{(u-t)^{1-\alpha}(x-t)^\alpha}\right\}dt. \tag{1.7}$$

In fact the inner integral in the right hand side of Eq. (1.7) is equal to $\pi\operatorname{cosec}\pi\alpha$. Thus

$$\int_0^u \phi(t)\,dt = \frac{\sin\alpha\pi}{\pi}\int_0^u \frac{f(x)}{(u-x)^{1-\alpha}}dx$$

and we see that the solution in (1.6) is

$$\phi(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \tag{1.8}$$

Integrating by parts on the R.H.S. and assuming f is differentiable we obtain

$$\phi(x) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dx}\left[\frac{1}{\alpha}x^{\alpha}f(0)+\frac{1}{\alpha}\int_0^x (x-t)^{\alpha}f'(t)\,dt\right]$$

$$= \frac{\sin\alpha\pi}{\pi}\left[\frac{f(0)}{x^{1-\alpha}}+\int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}}\,dt\right] \tag{1.9}$$

and in particular for $\alpha = 1/2$,

$$\phi(x) = \frac{f(0)}{\pi\sqrt{x}}+\frac{1}{\pi}\int_0^x \frac{f'(t)}{\sqrt{(x-t)}}\,dt \tag{1.10}$$

(c) Separable or Degenerate kernel : A kernel $K(x,t)$ is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which in its turn is the product of two factors, one depends only on x and the other only on t,

$$K(x,t) = \sum_{i=1}^{n} p_i(x)\overline{q_i(t)} \tag{1.11}$$

where the functions p_i and q_i are two sets of linearly independent [otherwise the no. of terms in (1.11) can be reduced] functions. The number n is called the rank of the kernel.

1.3 NOTATION

Let $C[a,b]$ be the set of all sectionally continuous functions $\phi:[a,b]\to C$. If ϕ and ψ are functions in $C[a,b]$ we define their inner product as

$$<\phi,\psi> = \int_a^b \overline{\phi(t)}\,\psi(t)\,dt \tag{1.12}$$

We say that ϕ and ψ are orthogonal if $<\phi,\psi> = 0$. The norm of ϕ is

$$\|\phi\| = <\phi,\phi>^{1/2} = \left(\int_a^b |\phi(t)|^2\,dt\right)^{1/2}$$

We recall that

$$\|\phi+\psi\| \le \|\phi\|+\|\psi\|$$

and Schwartz inequality

$$|<\phi,\psi>| \le \|\phi\|.\|\psi\| \tag{1.13}$$

Functions $\phi(t)$ which are square-integrable in the interval $a \le t \le b$ satisfy the condition

$$\int_a^b |\phi(t)|^2 \, dt < \infty \tag{1.14}$$

where the integral is taken to be Riemann or Lebesgue sense for greater generality. In the former case it is said that $\phi(t)$ is an R^2 function and in the latter case that it is an L^2-function. Continuous functions are square-integrable over a finite interval since they are bounded. However the converse is not necessarily true, that is square-integrable functions need not be continuous or bounded.

Kernels $K(x,t)$ defined in $a \le x \le b$, $a \le t \le b$ are said to be square-integrable if they satisfy

$$\| ¼ \|^2 = \int_a^b \int_a^b |K(x,t)|^2 \, dx\, dt < \infty \tag{1.15}$$

together with

$$\int_a^b |K(x,t)|^2 \, dt < \infty, \; a \le x \le b \text{ and } \int_a^b |K(x,t)|^2 \, dx < \infty, \; a \le t \le b.$$

The integral operator

With the kernel K of the integral equation, we define the function $K\phi$ to be

$$(K\phi)(x) = \int_a^b K(x,t)\,\phi(t)\, dt \tag{1.16}$$

The transformation $\phi \to K\phi$ is called the integral operator with the kernel K.

Remarks :

1. We will show later that $K\phi$ is a continuous function of x on $[a,b]$. This depends upon the continuity of K.
2. $K(c_1\phi_1 + c_2\phi_2) = c_1(K\phi_1) + c_2(K\phi_2)$ *i.e.* the map $\phi \to K\phi$ is a linear transformation from $C[a,b]$ to itself.
3. The standard Fredholm integral equation of the second kind is written as

$$\phi(x) - \lambda K\phi(x) = f(x) \text{ on } [a,b]$$

or, more briefly,

$$\phi = \lambda K\phi + f$$

(where the two sides are equal as functions on $[a,b]$).

If K is a kernel, the adjoint kernel K^* is defined by

$$K^*(x,t) = \overline{K(t,x)} \tag{1.17}$$

Proposition 1. $< K^*\psi, \phi) > = < \psi, K\phi >$ for all $\psi, \phi \in C[a,b]$.

Proof. First we note that

$$K^*\psi(t) = \int_a^b K^*(t,x)\,\psi(x)\,dx = \int_a^b \overline{K(x,t)}\,\psi(x)\,dx$$

and then we consider the double integral

$$I = \int_a^b \int_a^b \overline{\psi(x)}\,K(x,t)\,\phi(t)\,dx\,dt$$

A double integral of a continuous function over a closed rectangle in $\tilde{N}^2$ may be carried out in either order. So we integrate with respect to x first

$$I = \int_a^b \overline{K^*\psi(t)}\,\phi(t)\,dt = < K^*\psi, \phi >$$

and then with respect to t to get

$$I = \int_a^b K\phi(x)\,\overline{\psi(x)}\,dx = < \psi, K\phi >$$

Note : $\psi \to K^*\psi$ is the adjoint linear transformation of $\phi \to K\phi$ on $C[a,b]$.

Definition. λ is a regular value of K, if $\phi - \lambda K\phi = f$ has a unique solution ϕ for each continuous f. λ is an eigenvalue of K, if $\phi = \lambda K\phi$ for some non-zero function ϕ.

Notes :

1. If λ is regular value of K then $\phi = 0$ is the unique solution of $\phi - \lambda K\phi = 0$ and so λ is not an eigenvalue of K.
2. If λ is an eigenvalue then the functions satisfying $\phi = \lambda K\phi$ are called eigen-functions for λ.

The Resolvent :

We may express the Fredholm integral equation of the second kind as

$$\phi = f + \lambda K\phi$$

or, $(I-\lambda K)\phi = f$, I = identity operator (1.18)

We now seek an integral opeator R given by

$$R = \int_a^b R(x,t;\lambda)\,dt \tag{1.19}$$

such that $\phi = f + \lambda Rf$

or, $\phi = (I+\lambda R)f$ (1.20)

The operator R depends on the parameter λ and since it provides the solution to the integral equation (1.2), R is called the resolvent and $R(x,t;\lambda)$ the resolvent kernel.

If R exists, (1.20) will satisfy (1.18) *i.e.*,

$$(I-\lambda K)(I+\lambda R)f = f$$

$$\Rightarrow R - K = \lambda KR$$

Substituting (1.18) into (1.20) we find that

$$\phi = (I+\lambda R)(I-\lambda K)\,\phi$$

$$\Rightarrow R - K = \lambda RK$$

Thus, $R - K = \lambda KR = \lambda RK$. (1.21)

This equation is called the **resolvent equation.** If there exists an operator R with a square-integrable kernel $R(x,t;\lambda)$ satisfying the resolvent equation (1.16) for a given value of λ, then this value of λ is said to be a regular value (alternate definition) of the kernel $K(x,t)$. The set of all regular values of an operator K is known as the resolvent set Λ.

Proposition 2. If there exists a resolvent kernel $R(x,t;\lambda)$ of the kernel $K(x,t)$ for a given value of the parameter λ, then it is unique.

Proof. Suppose that there are two such kernels $R_1(x,t;\lambda)$ and $R_2(x,t;\lambda)$, then

$$R_1 - K = \lambda R_1 K = \lambda KR_1$$

and $R_2 - K = \lambda R_2 K = \lambda KR_2$

Subtracting we find that $\Gamma = \lambda\Gamma K = \lambda K\Gamma$, where $\Gamma = R_1 - R_2$ (1.22)

Hence from the equation $\Gamma = \lambda K\Gamma$, we have

$$R_1\Gamma = \lambda R_1 K\Gamma$$
$$= (R_1 - K)\Gamma$$
$$= R_1\Gamma - K\Gamma$$

i.e., $K\Gamma = 0$ and from (1.22) we then obtain $\Gamma = 0$. Thus $R_1 = R_2$.

Proposition 3. Given $f(x)$ be a square-integrable function and λ be a regular value of the square-integrable kernel $K(x,t)$ possessing the square-integrable resolvent kernel $R(x,t;\lambda)$, then the equation (1.2) has the unique square-integrable solution (1.20).

Proof. Suppose that the function $\phi(x)$ is given by (1.20). Then we have

$$f + \lambda K\phi = f + \lambda K(f + \lambda Rf)$$
$$= f + \lambda Kf + \lambda^2 KRf$$
$$= f + \lambda Kf + \lambda(R - K)f, \text{ using (1.21)}$$
$$= f + \lambda Rf$$
$$= \phi$$

Conversely, if the square-integrable function ϕ satisfies (1.2) we have

$$f = \phi - \lambda K\phi$$

so that
$$f + \lambda Rf = \phi - \lambda K\phi + \lambda R(\phi - \lambda K\phi)$$
$$= \phi + \lambda(R - K - \lambda RK)\phi$$
$$= \phi, \text{ using resolvent equation (1.21),}$$

which proves the uniqueness of the solution. There may exist other solutions of (1.2) which are not square-integrable.

Proposition 4. If $\phi(x)$ be a square-integrable eigen-function of a continuous kernel $K(x,t)$ then $\phi(x)$ is continuous.

Proof. We have

$$|\phi(x) - \phi(x')| = |\lambda| \left| \int_a^b \{K(x,t) - K(x',t)\}\, \phi(t)\, dt \right|$$

$$\leq |\lambda| \left\{ \int_a^b |\phi(t)|\, dt \right\}^{1/2} \left\{ |K(x,t)-K(x',t)|^2 \right\}^{1/2}$$, using Schwarz' inequality (1.13)

$$= |\lambda| \|\phi\| \varepsilon\sqrt{b-a}$$, using continuity of $K(x,t;\lambda)$ for $|x-x'|<\delta$.

Thus $\phi(x)$ is continuous.

Function space :

Function space is composed of all sectionally continuous complex functions of a real variable x, defined in the interval $a \leq x \leq b$, which are square-integrable *i.e.*,

$$\int_a^b |f(x)|^2\, dx < \infty$$

Introducing the inner product of two such functions *f* and *g* as

$$(f,g) = \int_a^b f(x)\overline{g(x)}\, dx$$

we define the norm of the function *f*(*x*) as

$$\|f\| = \sqrt{(f,f)}$$

Let us establish the important inequality named after Cauchy and Schwarz. We have

$$\int_a^b \left| f(x) - \frac{(f,g)}{(g,g)} g(x) \right|^2 dx \geq 0$$

so that $$(f,f) - 2\frac{|(f,g)|^2}{(g,g)} + \frac{|(f,g)|^2}{(g,g)} \geq 0$$

i.e. $(f,f)(g,g) \geq |(f,g)|^2$

Hence $\|f\|\|g\| \geq |(f,g)|$ (1.23)

which is known as Cauchy-Schwarz inequality for square-integrable functions.

Again,

$$(\|f\|+\|g\|)^2 = \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|$$
$$> (f,f) + (g,g) + 2|(f,g)|$$

by Cauchy-Schwarz inequality and since

$$2|(f,g)| \geq (f,g) + \overline{(f,g)}$$

so, $$(\|f\| + \|g\|)^2 \geq (f,f) + (g,g) + (f,g) + (g,f)$$

$$= (f+g, f+g)$$

$$= \|f+g\| \quad (1.24)$$

which is known as triangle inequality or Minkowski's inequality for functions.

Orthonormal system of functions

Two functions $f(x)$ and $g(x)$ belonging to the function space are said to be orthogonal if

$$(f,g) = \int_a^b f(x)\overline{g(x)}\, dx = 0$$

Let $\phi_1(x), \phi_2(x)$, ... be a system of orthogonal functions each of which is an L^2-function which does not vanish almost everywhere *i.e.*, that

$$\|\phi_j\|^2 = \int_a^b |\phi_j|^2 dx > 0$$

If they are also normalized *i.e.*,

$$(\phi_i, \phi_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (1.25)$$

then such a system is called an orthonormal system.

Gram-Schmidt Orthogonalisation

It is clear that the functions of any orthogonal system are linearly independent. If not, there exist constants $c_1, c_2, ..., c_n$ not all zero such that

$$c_1\phi_1(x) + c_2\phi_2(x) + ... + c_n\phi_n(x) = 0$$

almost everywhere in the interval (a,b). Then taking inner product of $\phi_j(x)$ $(j = 1, 2, ..., n)$ with L.H.S. we find that

$$c_j \int_a^b |\phi_j|^2\, dx = 0$$

By (1.25), $c_j = 0\ (j = 1, 2, ..., n)$

Now we prove that given any finite or denumerable system of linearly independent L^2-functions $\psi_1(x), \psi_2(x), ...,$ it is always possible to construct a system of orthogonal functions $\phi_1(x), \phi_2(x), ...$ satisfying

$$\phi_1(x) = \psi_1(x)$$
$$\phi_2(x) = c_{21}\psi_1(x) + \psi_2(x)$$
$$\phi_3(x) = c_{31}\psi_1(x) + c_{32}\psi_2(x) + \psi_3(x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi_n(x) = c_{n1}\psi_1(x) + c_{n2}\psi_2(x) + ... + c_{nn-1}\psi_{n-1}(x) + \psi_n(x)$$

in the interval.

Clearly, applying the orthogonality condition *i.e.*, $(\phi_i, \phi_j) = 0$ for $i \neq j$ we find $c_{21} = -(\psi_2, \phi_1)$. All the c_{ij} can be found out by mathematical induction. Say, we know c_{ij} for

$$1 \leq j < i \leq n-1$$

i.e., the expression for $\phi_{n-1}(x)$ is known. To determine the expression for $\phi_n(x)$, we calculate for

$$0 = (\phi_n, \phi_j) = c_{n1}(\psi_1, \phi_j) + c_{n2}(\psi_2, \phi_j) + ... + c_{nn-1}(\psi_{n-1}, \phi_j) + (\psi_n, \phi_j)$$
$$= c_{nj}(\phi_j, \phi_j) + (\psi_n, \phi_j)$$

and find that

$$c_{nj} = -\frac{(\psi_n, \phi_j)}{(\phi_j, \phi_j)},\ j = 1, 2, ..., n-1$$

exist finitely since $(\phi_j, \phi_j) \neq 0$ because ϕ_j is a linear combination of the linearly independent functions $\psi_1, \psi_2, ..., \psi_j$ each of which cannot be zero almost everywhere.

Approximation and convergence in the mean

Here we wish to represent an arbitrary function $f(x)$ in terms of the orthonormal functions $\phi_1(x), \phi_2(x), ...$ belonging to the function space

$$c_1\phi_1(x) + c_2\phi_2(x) + ...$$

We know that if

$$\lim_{n\to\infty}\left\{f(x)-\sum_{k=1}^{n}c_k\phi_k(x)\right\}=0 \tag{1.26}$$

uniformly, then

$$\lim_{n\to\infty}\int_a^b|f(x)-\sum_{k=1}^{n}c_k\phi_k(x)|^2\,dx=0 \tag{1.27}$$

holds, but not conversely. Condition (1.27) states that to any positive number ε there exists a positive number N such that

$$\int_a^b|f(x)-\sum_{k=1}^{n}c_k\phi_k(x)|^2\,dx<\varepsilon, \qquad (n>N).$$

The infinite series, therefore, converges in the mean to the function $f(x)$.

Let us consider the integral

$$I_n=\int_a^b|f(x)-\sum_{k=1}^{n}c_k\phi_k(x)|^2\,dx$$

Using orthonormality condition, we have

$$I_n=(f,f)-\sum_{k=1}^{n}\left\{\overline{c_k}(f,\phi_k)+c_k\overline{(f,\phi_k)}\right\}+\sum_{k=1}^{n}|c_k|^2$$

$$=(f,f)+\sum_{k=1}^{n}|c_k-\alpha_k|^2-\sum_{k=1}^{n}|\alpha_k|^2 \tag{1.28}$$

where α_k is the Fourier coefficient of the function $f(x)$.

Thus I_n attains its minimal value if and only if the coefficients c_k coincide with the corresponding Fourier coefficients α_k of the function $f(x)$. In that case I_n will take the form

$$I_n=(f,f)-\sum_{k=1}^{n}|\alpha_k|^2$$

Now since $I_n\geq 0$, it follows that

$$(f,f)\geq\sum_{k=1}^{n}|\alpha_k|^2 \tag{1.29}$$

for any positive integer n. This is known as Bessel's inequality.

The Riesz-Fischer Theorem (Riesz form)

Given an orthonormal system of L^2-functions $\{\phi_k\}$ and a sequence of constants $\{c_k\}$, then

$$\sum_{k=1}^{n} c_k \phi_k$$

converges in the mean to a L^2-function $f(x)$ whose Fourier coefficients are $\{c_k\}$ if and only if

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty$$

The proof of the theorem rests on a preliminary proposition termed as Weyl's lemma :

A necessary and sufficient condition for convergence in the mean of a sequence of L^2-functions $\{f_k\}$ to a certain function $f(x)$ of the same class L^2, is that to any positive ε there corresponds a positive integer N such that for integers $m > N$ and $n > N$ we have

$$\int_a^b |f_m(x) - f_n(x)|^2 \, dx < \varepsilon$$

To prove the necessary part of the theorem we assume this lemma and the convergence of series $\sum_{k=1}^{\infty} |c_k|^2$. We take

$$f_n(x) = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x)$$

If $m > n$, it simply follows that

$$\int_a^b |f_m(x) - f_n(x)|^2 \, dx = \int_a^b \Big| \sum_{k=n+1}^{m} c_k\phi_k(x) \Big|^2 \, dx = \sum_{k=n+1}^{m} |c_k|^2$$

and since the series $\sum_{k=1}^{\infty} |c_k|^2$ is convergent for any $\varepsilon > 0$ there exists a positive integer N such that for $n > N$ $\sum_{k=n+1}^{\infty} |c_k|^2 < \varepsilon$. Thus for $m > n > N$

$$\int_a^b |f_m(x) - f_n(x)|^2 \, dx < \varepsilon$$

holds and by virtue of Weyl's lemma the result follows.

Sufficiency : If $\alpha_1, \alpha_2, ...$ be taken as the Fourier coefficients of $f(x)$, then as in (1.28),

$$\int_a^b | f(x) - f_n(x) |^2 = (f, f,) + \sum_{k=1}^{n} | c_k - \alpha_k |^2 - \sum_{k=1}^{n} | \alpha_k |^2 \geq \sum_{k=1}^{n} | c_k - \alpha_k |^2.$$

Clearly, if one α_k were different from the corresponding c_k, say $\alpha_1 \neq c_1$, it would lead us

$$\int_a^b | f(x) - f_n(x) |^2 \, dx \geq | c_1 - \alpha_1 |^2$$

contradicting the given hypothesis. The theorem is thus completely proved.

EXERCISES

1. Show that the function $u(x) = (1 + x^2)^{-\frac{3}{2}}$ is a solution of the Volterra integral equation

$$u(x) = \frac{1}{1+x^2} - \int_o^x \frac{t}{1+x^2} u(t) dt$$

2. Verify that the function $u(x) = \sin \frac{\pi x}{2}$ is a solution of the integral equation

$$u(x) = \frac{1}{2} x + \frac{\pi^2}{4} \int_0^1 K(x,t) \, u(t) dt$$

where $K(x,t) = \begin{cases} \frac{1}{2} x(2-t), 0 \leq x \leq t \\ \frac{1}{2} t(2-x), t \leq x \leq 1 \end{cases}$

3. Show that $u(x)$ = cos2x is a solution of the equation

$$u(x) = \cos x + 3 \int_0^{\pi} K(x,t) \, u(t) dt$$

where $K(x,t) = \begin{cases} \sin x \cos t, 0 \leq x \leq t \\ \cos x \sin t, t \leq x \leq \pi \end{cases}$

4. Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + e^x y = x$$

with the initial conditions $y(0) = 1,\ y'(0) = -1$

[Ans. $y(x) = x - \sin x + e^x(x-1) + \int_0^x [\sin x - e^x(x-t)]y(t)dt$]

5. Form an integral equation corresponding to the differential equation

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + (x^2 - x)y = xe^x + 1$$

with initial conditions $y(0) = 1, y'(0) = 1,\ y''(0) = 0$

[Ans. $y(x) = xe^x + 1 - x(x^2 - 1) - \int_0^x [x + \frac{1}{2}(x^2 - x)(x-t)^2]y(t)dt$]

6. Reduce the inital value problem $u''(x) + \lambda u(x) = f(x)$ with $u(0) = 1, u''(0) = 0$ to an integral equation.

[Ans. $u(x) = 1 + \int_0^x (x-t)\{f(t) - \lambda u(t)\}dt$]

7. Convert the differential equation

$$\frac{d^2y}{dx^2} + \lambda u = 0$$

with boundary conditions $u(0) = 0, u(l) = 0$ into Fredholm integral equation

[Ans. $u(x) = \lambda \int_0^l K(x,t)\, u(t)dt$ where $K(x,t) = \begin{cases} \frac{t(l-x)}{l}, 0 \le t \le x \\ \frac{x(l-t)}{l}, x \le t \le l \end{cases}$]

8. Obtain the integral correspending to the boundary value problem

$$\frac{d^2u}{dx^2} + \lambda u = x; u(0) = 0,\ u'(1) = 0$$

[Ans. $u(x)=\frac{1}{6}(x^3-3x)+\lambda\int_0^1 K(x,t)u(t)dt$ where $K(x,t)=\begin{cases} x, x>t \\ t, x<t \end{cases}$]

9. Derive the differential equation and the initial conditions from the integral equation

$$u(x)=1-x+\frac{1}{6}x^3+\int_0^x[\sin t-(x-t)(e^t+\cos t)]u(t)dt$$

[Ans. $u''-\sin x\, u'+e^x u=x;\ u(0)=1,\ u'(0)=-1$]

10. Derive the differential equation with the initial conditions from the integral equation

$$u(x)=1-x-4\sin x+\int_0^x[3-2(x-t)]u(t)dt$$

[Ans. $u''(x)-3u'+2x=4\sin x\,;\ u(0)=1,\ u'(0)=-2$]

11. Deduce the boundary value problem from the integral equation

$$u(x)=\frac{1}{2}x(1+x)+\int_0^x K(x,t)u(t)dt\text{, where } K(x,t)=\begin{cases}(1-x)t^2, t<x \\ (1-t)xt, t>x\end{cases}$$

[Ans. $u''+xu=1;\ u(0)=0, u(1)=1$]

12. Solve the following integral equation by using Laplace transform method :

(i) $x=\int_0^x e^{x-t}\phi(t)dt$ [Ans. $\phi(x)=1-x$]

(ii) $\sin x=\int_0^x J_0(x-t)\phi(t)dt$ [Ans. $\phi(x)=J_0(x)$]

(iii) $\phi(x)=x^2+\int_0^x \sin(x-t)\phi(t)dt$ [Ans. $\phi(x)=x^2+\frac{1}{12}x^4$]

(iv) $\phi(x)=x+2\int_0^x \cos(x-t)\phi(t)dt$ [Ans. $\phi(x)=2e^x(x-1)+x+2$]

(v) $\phi(x) = c\sin x - 2\int_0^x \cos(x-t)\phi(t)dt$ [Ans. $\phi(x) = cxe^{-x}$]

(vi) $\phi(x) = f(x) + \int_0^x (x-t)\phi(t)dt$ [Ans. $\phi(x) = f(x) + \int_0^x \sin(x-t) f(t)dt$]

(vii) $\int_0^x \phi(t)\,\phi(x-t)dt = 16\sin 4x$, [Ans. $\phi(x) = \pm 8 J_0(4x)$]

(viii) $f(x) = \int_0^x \frac{\phi(t)}{(x-t)^\alpha} dt, 0 < \alpha < 1,$ [Ans. $\phi(x) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dr} \{\int_0^x (x-t) f(t) dt\}$

UNIT 2 ☐ METHOD OF SUCCESSIVE APPROXIMATIONS

2.1 NEUMANN SERIES

In the theory of ordinary differential equations we have seen how the first-order differential equations can be solved by Picard's method of successive approximations. An iterative procedure of the same method applied to integral equations of the second kind yields a sequence of approximations leading to an infinite series solution associated with the name of Neumann, called Neumann series.

(i) **Fredholm integral equation of the second kind.**

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t)\, dt, \quad a \le x \le b. \tag{2.1}$$

Let $\phi_0(x) = f(x)$ be the zero-order appoximation (say) for the function $\phi(x)$ in the interval $a \le x \le b$. Substituting this in place of $\phi(t)$ in the right hand side of (2.1) gives the first approximation $\phi_1(x)$ for $\phi(x)$ *i.e.*,

$$\phi_1(x) = \phi_0(x) + \lambda \int_a^b K(x,t)\, \phi_0(t)\, dt.$$

Continuing in this way we obtain an infinite sequence of approximations

$$\phi_0(x), \phi_1(x), \ldots, \phi_n(x), \ldots$$

satisfying the recurrence relations

$$\phi_n(x) = \phi_0(x) + \lambda \int_a^b K(x,t)\, \phi_{n-1}(t)\, dt, \; n = 1, 2, \ldots \tag{2.2}$$

We introduce

$$\phi_n(x) - \phi_{n-1}(x) = \lambda^n \phi^{(n)}(x), \quad n = 1, 2, \ldots \tag{2.3}$$

Then if $\phi^{(0)}(x) = \phi_0(x) = f(x)$ we have

$$\phi_n(x) = \sum_{k=0}^{n} \lambda^k \phi^{(k)}(x) \tag{2.4}$$

and that

$$\phi^{(k)}(x) = \int_a^b K(x,t)\, \phi^{(k-1)}(t)\, dt, \quad k \ge 1 \tag{2.5}$$

Now suppose that $f(x)$ and $K(x,t)$ are continuous functions in $a \le x \le b$, so that they are bounded and therefore, we have

$$\left.\begin{aligned} |f(x)| &\le m, \quad a \le x \le b \\ |K(x,t)| &\le M, \; a \le x \le b, \; a \le t \le b \end{aligned}\right\} \tag{2.6}$$

Then $\quad |\phi^{(0)}(x)| = |f(x)| \le m,$

$$|\phi^{(1)}(x)| \le \int_a^b |K(x,t)||\phi^{(0)}(t)|\, dt \le mM(b-a),$$

$$|\phi^{(2)}(x)| \le \int_a^b |K(x,t)|\,|\phi^{(1)}(t)|\, dt \le mM^2(b-a)^2$$

and in general

$$|\phi^{(n)}(x)| \le mM^n(b-a)^n.$$

So, $$|\sum_{k=0}^{n} \lambda^k \phi^{(k)}(x)| \le \sum_{k=0}^{n} |\lambda|^k |\phi^{(k)}(x)| \le \sum_{k=0}^{\infty} |\lambda|^k mM^k(b-a)^k \tag{2.7}$$

and hence the series in R.H.S. of (2.7) converges absolutely and uniformly if $\sum_{k=0}^{\infty} |\lambda|^k mM^k(b-a)^k$ converges i.e. if $|\lambda| mM(b-a) < 1$, that is if

$$|\lambda| < \frac{1}{mM(b-a)} \tag{2.8}$$

Thus passing to the limit $n \to \infty$ in (2.2), we obtain the so-called Neumann series

$$\phi(x) = f(x) + \sum_{k=1}^{\infty} \lambda^k \phi^{(k)}(x) \tag{2.9}$$

as the continuous solution of (2.1) provided (2.6) holds.

(ii) Volterra integral equation of the second kind

$$\phi(x) = f(x) + \lambda \int_a^x K(x,t)\, \phi(t)\, dt \tag{2.10}$$

Here the Volterra kernel $K(x,t) = 0$ for $t > x$. The previous method holds in this case too. We start by setting $\phi_0(x) = \phi^{(0)}(x) = f(x)$, $\phi_k(x)$ and $\phi^{(k)}(x)$ by (2.2) and

(2.5) respectively as earlier except that the integral from a to b is replaced by the integral from a to x. We have further that

$$|\phi^{(0)}(x)| = |f(x)| \leq m,$$

$$|\phi^{(1)}(x)| \leq \int_a^x |K(x,t)|\,|\phi^{(0)}(t)|\,dt \leq mM\int_a^x dt = mM(x-a),$$

$$|\phi^{(2)}(x)| \leq \int_a^x |K(x,t)|\,|\phi^{(1)}(t)|\,dt \leq mM^2\int_a^x (t-a)\,dt = mM^2\frac{(x-a)^2}{2!},$$

$$|\phi^{(3)}(x)| \leq \int_a^x |K(x,t)|\,|\phi^{(2)}(t)|\,dt \leq \frac{mM^3}{2!}\int_a^x (t-a)^2\,dt = mM^3\frac{(x-a)^3}{3!}$$

and in general, assuming that

$$|\phi^{(n-1)}(x)| \leq mM^{n-1}\frac{(x-a)^{n-1}}{(n-1)!}$$

and by the principle of induction, we get

$$|\phi^{(n)}(x)| \leq \frac{mM^n}{(n-1)!}\int_a^x (t-a)^{n-1}\,dt = mM^n\frac{(x-a)^n}{n!} \tag{2.11}$$

Hence $$\left|\sum_{n=0}^{N}\lambda^n\phi^{(n)}(x)\right| \leq m\sum_{n=0}^{N}|\lambda|^n M^n\frac{(b-a)^n}{n!} \tag{2.12}$$

which shows that the series $\sum_{n=0}^{\infty}\lambda^n\phi^{(n)}(x)$ converges absolutely and uniformly in $a \leq x \leq b$ for all values of λ as it is dominated by

$$m\sum_{n=0}^{\infty}|\lambda|^n\frac{M^n(b-a)^n}{n!} = m\exp\{|\lambda|M(b-a)\} \tag{2.13}$$

Thus $$\phi(x) = \sum_{n=0}^{\infty}\lambda^n\phi^{(n)}(x), \tag{2.14}$$

in the form of the Neumann series, stands as the solution of (2.8).

Note : The Neumann series for Volterra equation converges for all values of λ whereas it converges only for small values of λ in the case of Fredholm equation.

Example 2.1. Solve for $\phi(x)$ by the method of successive approximations

(i) $\phi(x) = x^2 + \frac{3}{4}\int_0^1 xt\,\phi(t)\,dt$

(ii) $\phi(x) = e^x + \lambda \int_0^1 \phi(t)\,dt$

Solution. (i) Here we take the zero-order approximation or the initial approximation $\phi_0(x) = \phi^{(0)}(x)$ as x^2, then

$$\phi^{(1)}(x) = \int_0^1 xt.t^2\,dt = \frac{x}{4}$$

$$\phi^{(2)}(x) = \int_0^1 xt.\frac{t}{4}\,dt = \frac{x}{12}$$

Likewise, $\phi^{(3)}(x) = \frac{x}{36}$, so on

Thus the Neumann series (2.9) for the solution takes the form

$$\phi(x) = x^2 + \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k \int_0^1 xt\,\phi^{(k-1)}(t)\,dt$$

$$= x^2 + \frac{3}{4}\int_0^1 xt.t^2 dt + \left(\frac{3}{4}\right)^2 \int_0^1 xt.\frac{t}{4}\,dt + \left(\frac{3}{4}\right)^3 \int_0^1 xt.\frac{t}{12}\,dt + \,...$$

$$= x^2 + \frac{3}{4}x.\frac{1}{4} + \left(\frac{3}{4}\right)^2 \frac{x}{12} + \left(\frac{3}{4}\right)^3 \frac{x}{36} + ...$$

$$= x^2 + \frac{3x}{16}\left[1 + \frac{1}{4} + \frac{1}{16} + ...\right] = x^2 + \frac{x}{4}$$

Alternately in terms of iterated kernels

$$K_1(x,t) = xt, \quad K_2(x,t) = \int_0^1 xt_1.t_1 t\,dt_1 = xt\int_0^1 t_1^2\,dt_1 = \frac{xt}{3}$$

$$K_3(x,t) = \int_0^1 xt_1 \int_0^1 t_1 t_2 . t_2 t \, dt_2 \, dt_1$$

$$= \int_0^1 xt . \frac{t_1^2}{3} dt_1 = \frac{1}{9} xt$$

$K_n(x,t) = \frac{1}{3^{n-1}} xt$ by induction.

Following (2.12).

$$\phi(x) = x^2 + \frac{3}{4} \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \int_0^1 \frac{xt}{3^k} \cdot t^2 \, dt$$

$$= x^2 + \frac{3x}{4} \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} = x^2 + \frac{x}{4}$$

(ii) As usual we take the initial approximation

$\phi_0(x) = e^x$, then

$$\phi_1(x) = e^x + \lambda \int_0^1 e^t dt = e^x + \lambda(e-1)$$

$$\phi_2(x) = e^x + \lambda \int_0^1 \{e^t + \lambda(e-1)\} \, dt = e^x + \lambda(e-1) + \lambda^2(e-1)$$

....

$$\phi_n(x) = e^x + \lambda(e-1) + ... + \lambda^n(e-1)$$

Thus,

$$\phi(x) = e^x + \lambda(e-1) + \lambda^2(e-1) + ...$$

$$= e^x + \lambda(e-1)\{1 + \lambda + \lambda^2 + ...\}$$

$$= e^x + \frac{\lambda(e-1)}{1-\lambda}, \; |\lambda| < 1$$

Example 2.2. Use the method of successive approximations to solve the Volterra integral equation of the second kind

$$\phi(x) = x - \int_0^x (x-t)\,\phi(t) \, dt$$

Solution. In this case we start with $\phi_0(x) = 0$ and obtain $\phi_1(x) = x$.

So if we let $\phi(x) = \phi_1(x)$ in the R.H.S. of the given equation, we obtain

$$\phi_2(x) = x - \int_0^x (x-t)\,\phi_1(t)\,dt$$

$$= x - \int_0^x (x-t)t\,dt = x - \left(\frac{xt^2}{2} - \frac{t^3}{3}\right)_0^x$$

$$= x - \frac{x^3}{3!}$$

Now, $$\phi_3(x) = x - \int_0^x (x-t)\,\phi_2(t)\,dt$$

$$= x - \int_0^x (x-t)\left(t - \frac{t^3}{6}\right)dt$$

$$= x - x\left(\frac{t^2}{2} - \frac{t^4}{24}\right)_0^x + \left(\frac{t^3}{3} - \frac{t^5}{30}\right)_0^x$$

$$= x - x\left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^3}{3} - \frac{x^5}{30}\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

If we continue the process, we obtain the nth approximation $\phi_n(x)$ as

$$\phi_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!}$$

which is the nth partial sum of the Maclaurin's series of $\sin x$. Hence the solution of the given equation is

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) = \sin x$$

2.2 ITERATED KERNELS

In the preceding section we found solutions to the Fredholm and Volterra equations of the second kind in the form of the infinite series. It is advantageous to express these solutions in terms of iterated kernels defined by

$$K_1(x,t) = K(x,t) \tag{2.15}$$

and
$$K_n(x,t) = \int_a^b K(x,s)\, K_{n-1}(s,t)\, ds \tag{2.16}$$

so that

$$K_2(x,t) = \int_a^b K(x,s_1)\, K(s_1,t)\, ds_1$$

$$K_3(x,t) = \int_a^b K(x,s_1)\, K_2(s_1,t)\, ds_1$$

$$= \int_a^b \int_a^b K(x,s_1)\, K(s_1,s_2)\, K(s_2,t)\, ds_1\, ds_2$$

and in general

$$K_n(x,t) = \int_a^b \cdots \int_a^b K(x,s_1)\, K(s_1,s_2) \cdots K(s_{n-2},s_{n-1})\, K(s_{n-1},t)\, ds_1 \cdots ds_{n-1} \tag{2.17}$$

Moreover, it follows at once that

$$K_n(x,t) = \int_a^b K_r(x,s)\, K_p(s,t)\, ds \tag{2.18}$$

for any r and p with $r+p=n$.

Now from (2.5)

$$\phi^{(n)}(x) = \int_a^b K_n(x,t)\, f(t)\, dt \tag{2.19}$$

and hence the solution (2.9) of the Fredholm equation (2.1) can be expressed in the form

$$\phi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_a^b K_n(x,t)\, f(t)\, dt \tag{2.20}$$

or, equivalently

$$\phi(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \tag{2.21}$$

where $$R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t) \tag{2.22}$$

is the resolvent kernel, which is absolutely and uniformly convergent in $a \le x \le b$, $a \le t \le b$, since

$$|\sum_{k=0}^{N} \lambda^k K_{k+1}(x,t)| \le \sum_{k=0}^{N} |\lambda|^k |K_{k+1}(x,t)|$$

$$\le \sum_{k=0}^{\infty} |\lambda|^k mM^{k+1}(b-a)^k \text{, using (2.6) and (2.11)}$$

$$= mM \sum_{k=0}^{\infty} \rho^k = \frac{mM}{1-\rho} \text{, using (2.8), where } \rho = |\lambda| M(b-a)$$

The solution of the Volterra equation (2.10) can also be expressed in terms of iterated kernels as

$$\phi(x) = f(x) + \int_a^x \sum_{n=1}^{\infty} \lambda^n K_n(x,t) f(t)\, dt \tag{2.23}$$

and the resolvent kernel $R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$ (2.24)

for every λ, where the iterated kernels $K_n(x,t)$ satisfy the relation

$$K_n(x,t) = \int_t^x K(x,s)\, K_{n-1}(s,t)\, ds \tag{2.25}$$

Here the uniqueness follows as shown earlier. To check the convergence of the series (2.23) we consider

$$K_1(x,t) = K(x,t)$$

$$|K_2(x,t)| = |\int_a^x K(x,t_1)\, K(t_1,t)\, dt_1| = |\int_t^x K(x,t_1)\, K(t_1,t)\, dt_1|,$$ [since the integrand vanishes except for $x \ge t_1 \ge t, a \le t \le b$]

$$\le M^2(x-t) \le M^2(x-a).$$

$$|K_3(x,t)| = |\int_a^x K(x,t_1)\, K_2(t_1,t)\, dt_1|$$

$$= |\int_t^x K(x,t_1)\, K_2(t_1,t)\, dt_1|, \quad \text{[under the same reasons as in (2.25)]}$$

$$\leq \int_t^x |K(x,t_1)\, M^2(t_1-t)\, dt_1 \leq \frac{M^3}{2}(x-a)^2$$

By induction,

$$|K_n(x,t)| \leq \frac{M^n}{(n-1)!}(x-a)^{n-1}$$

Thus, the Neumann series $f(x) + \sum_{n=1}^{\infty} \lambda^n K_n f(x)$ is dominated by the series

$$f(x) + \sum_{n=1}^{\infty} \frac{\lambda^n M^n m(x-a)^{n-1}}{(n-1)!} \tag{2.26}$$

Since the series (2.26) converges uniformly on $[a,b]$ for every λ, it follows that the Neumann series (2.23) is convergent.

Example 2.3. Solve the Fredholm equation

$$\phi(x) = 1 + \lambda \int_0^1 xt\, \phi(t)\, dt \quad (0 \leq x \leq 1)$$

Solution. Here $K_1(x,t) = K(x,t) = xt$

$$K_2(x,t) = \int_0^1 xt_1\, t_1 t\, dt_1 = \frac{xt}{3}$$

$$K_3(x,t) = \int_0^1 xt_1 \frac{t_1 t}{3}\, dt_1 = \frac{xt}{3^2}$$

and in general

$$K_{n+1}(x,t) = \frac{xt}{3^n}$$

Hence the resolvent kernel is

$$R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_{n+1}(x,t)$$

$$= xt\sum_{n=0}^{\infty}\left(\frac{\lambda}{3}\right)^n$$

$$= xt\frac{1}{1-\frac{\lambda}{3}}, \quad |\lambda|<3$$

and thus $$\phi(x) = 1+\frac{\lambda x}{1-\frac{\lambda}{3}}\int_0^1 t\,dt$$

$$= 1+\frac{3\lambda x}{2(3-\lambda)}$$

Example 2.4. Find the resolvent kernel to solve the Volterra integral equation of the second kind

$$\phi(x) = f(x)+\int_0^x e^{x-t}\phi(t)\,dt$$

Solution. Here we have

$$K_1(x,t) = K(x,t) = e^{x-t}$$

and $$K_2(x,t) = \int_t^x e^{x-t_1}e^{t_1-t}\,dt_1 = e^{x-t}\int_t^x dt_1 = (x-t)\,e^{x-t}$$

$$K_3(x,t) = \int_t^x e^{x-t_1}(t_1-t)\,e^{t_1-t}\,dt_1 = e^{x-t}\int_t^x (t_1-t)\,dt_1 = \frac{(x-t)^2}{2!}e^{x-t}$$

$$K_4(x,t) = \int_t^x e^{x-t_1}\frac{(t_1-t)^2}{2!}e^{t_1-t}dt_1 = \frac{e^{x-t}}{2!}\int_t^x (t_1-t)^2\,dt = \frac{(x-t)^3}{3!}e^{x-t}$$

and in general

$$K_{n+1}(x,t) = \frac{(x-t)^n}{n!}e^{x-t}$$

and consequently,

$$R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n \frac{(x-t)^n}{n!} e^{x-t} = e^{x-t} \sum_{n=0}^{\infty} \frac{\{(x-t)\}^n}{n!}$$

$$= e^{2(x-t)}$$

for all λ. Therefore we obtain the solution of the given equation as

$$\phi(x) = f(x) + \int_0^x e^{2(x-t)} f(t)\,dt.$$

Example 2.5. Using resolvent kernel, solve the integral equation

$$\phi(x) = 1 + \lambda \int_0^{\pi} \sin(x+t)\phi(t)dt$$

Solution. Here $K_1(x,t) = K(x,t) = \sin(x+t)$. Then

$$K_2(x,t) = \int_0^{\pi} \sin(x+t_1)\sin(t+t_1)dt_1 = \frac{1}{2}\pi\cos(x-t)$$

$$K_3(x,t) = \frac{1}{2}\pi \int_0^{\pi} \sin(x+t_1)\sin(t_1-t)dt_1 = \left(\frac{1}{2}\pi\right)^2 \sin(x+t)$$

Proceeding in this way, we have

$$K_4(x,t) = \left(\frac{1}{2}\pi\right)^3 \cos(x-t), \quad K_5(x,t) = \left(\frac{1}{2}\pi\right)^4 \sin(x+t)$$

$$K_6(x,t) = \left(\frac{1}{2}\pi\right)^5 \cos(x-t), \quad K_6(x,t) = \left(\frac{1}{2}\pi\right)^6 \sin(x+t)$$

Thus the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_n(x,t)$$

$$= K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \lambda^3 K_4(x,t) + \lambda^4 K_5(x,t) + \lambda^5 K_6(x,t) + \dots$$

$$= \sin(x+t)\left\{1 + \left(\frac{\lambda\pi}{2}\right)^2 (\lambda\pi)^4 + \dots\right\}$$

$$+ \frac{\lambda\pi}{2}\cos(x-t)\left\{1 + \left(\frac{\lambda\pi}{2}\right)^2 + \left(\frac{\lambda\pi}{2}\right)4 + \dots\right\}$$

$$=\left\{\sin(x+t)+\frac{\lambda\pi}{2}\cos(x-t)\right\}\left\{1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\dots\right\}$$

$$=\left\{\sin(x+t)+\frac{\lambda\pi}{2}\cos(x-t)\right\}.\frac{1}{1-\left(\frac{\lambda\pi}{2}\right)^2}$$

provided $\left|\frac{\lambda\pi}{2}\right|<1$, i.e. $|\lambda|<\frac{2}{\pi}$, Hence

$$R(x,t;\lambda)=\frac{2}{4-\lambda^2\pi^2}\{2\sin(x+t)+2\pi\cos(x-t)\}$$

Therefore the solution of the given integral equation is

$$\phi(x)=1+\frac{2\lambda}{4-\lambda^2\pi^2}\int_0^\pi\{2\sin(x+t)+\lambda\pi\cos(x-t)\}dt$$

i.e. $\phi(x)=1+\frac{2\lambda}{4-\lambda^2\pi^2}(2\cos x+\lambda\pi\sin x)$, provided $|x|<\frac{2}{\pi}$

Volterra integral equation of the first kind

In the Volterra integral equation of the first kind

$$f(x) = \int_a^x K(x,t)\,\phi(t)\,dt,\ a\le x\le b \tag{2.27}$$

we assume

(i) $K(x,x)\neq 0$ for all x belonging to (a,b)

(ii) $\frac{df(x)}{dx}\equiv f'(x),\ \frac{\partial K}{\partial x}\equiv K_x(x,y),\ \frac{\partial K}{\partial y}\equiv K_y(x,y)$

exist and are continuous. Then differentiating both sides of (2.27) with respect to x, we obtain

$$f'(x)=K(x,x)\,\phi(x)+\int_a^x K_x(x,t)\,\phi(t)\,dt$$

so that $$\phi(x) = \frac{f'(x)}{K(x,x)}-\frac{1}{K(x,x)}\int_a^x K_x(x,t)\,\phi(t)\,dt \tag{2.28}$$

which is of the form of the Volterra integral equation of the second kind and can be solved accordingly.

EXERCISES

1. Find the iterated kernal of the following kernels :

(i) $K(x,t)=\sin(x-t);\ a=0,\ b=\frac{\pi}{2}$ for, $n=2,3$

[Ans. $k_2(x,t)=\frac{1}{2}\sin(x+t)-\frac{\pi}{4}\cos(x-t),\ k_3(x,t)=\frac{4-\pi^2}{16}\sin(x-t)$]

(ii) $K(x,t)=e^{|x-t|};\ a=0,\ b=1$ for $n=1$

[Ans. $K_2(x,t)=\begin{cases}\frac{1}{2}[e^{x+t}+e^{2-x-t}]+(t-x-1)e^{t-x}, & 0\le x\le t\\ \frac{1}{2}[e^{x+t}+e^{2-x-t}]+(x-t-1)e^{x-t}, & t\le x\le 1\end{cases}$]

(iii) $K(x,t)=x+\sin t;\ a=-\pi,\ b=\pi$

[Ans. $K_{2n-1}(x,t)=(2\pi)^{2n-2}(x+\sin t)$, $K_{2n}(x,t)=(2\pi)^{2n-1}(1+x\sin t)$, where $n=$ 1, 2, 3, ...]

(iv) $K(x,t)=e^x\cos t;\ a=b,\ b=\pi$

[Ans. $K_n(x,t)=(-1)^{n-1}\left(\frac{1+e^{\pi}}{2}\right)^{n-1}e^x\cos t;\ n=1,2,...$]

2. Find the resolvent kernels for the following kernels :

(i) $K(x,t)=(1+x)(1-t);\ a=-1,\ b=0$

[Ans. $R(x,t;\lambda)=\frac{3(1+x)(1-t)}{3-2\lambda}$ where $|\lambda|<\frac{3}{2}$]

(ii) $K(x,t)=\sin x\cos t;\ a=0,\ b=\frac{\pi}{2}$

[Ans. $R(x,t;\lambda)=\frac{2\sin x\cos t}{2-\pi},\ |\lambda|<2$]

(iii) $K(x,t)=x^2t^2;\ a=-1,\ b=1$

[Ans. $R(x,t;\lambda)=\frac{5-x^2t^2}{5-2\lambda},\ |\lambda|<\frac{5}{2}$

3. Solve the integral equation

$$\phi(x) = 3 + \lambda\int_0^{\pi} K(x,t)\phi(t)dt,\ 0 \le x \le \pi$$

where $K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t \le \pi \\ \sin t \cos x, & 0 \le t \le x \le \pi \end{cases}$

4. Show that the solution of the integral equation

$$\phi(x) = f(x) + \lambda\int_0^1 xe^t\phi(t)dt,\ \lambda \ne 1$$

is $$\phi(x) = f(x) + \frac{\lambda}{\lambda-1}\int_0^1 xe^t f(t)dt$$

and distinguish this with the case $\lambda = 1$

5. Prove that the solutions of the integral equation

$$x(e^{-ax} - 1) = \int_0^x \phi(x-t)(\phi(t)-2)dt,\ x \ge 0$$

are $1 \pm e^{-ax}$

6. Solve the integral equation

$$\sin x - 3\cos x + 3 = \int_0^x (3x - 3t + 4)\,\phi(t)dt$$

7. Solve the following integral equations by using resolvent kernel :

(i) $\phi(x) = x + \int_0^{\frac{1}{2}} \phi(x)dt$, [Ans. $\phi(x) = x + \frac{1}{4}$]

(ii) $\phi(x) = \sin x - \frac{x}{4} + \frac{1}{4}\int_0^{\pi/2} tx\phi(t)dt$, [Ans. $\phi(x) = \sin x$]

(iii) $\phi(x) = \left(\frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{2}\right) + \frac{1}{2}\int_0^1 t\phi(t)dt$, [Ans. $\phi(x) = \frac{3}{2}e^x - \frac{1}{2}xe^x - \frac{1}{3}e + 1$]

(iv) $\phi(x) = 1 + \lambda\int_0^1 (1-3xt)\phi(t)dt$ [Ans. $\phi(x) = \frac{4+2\lambda(2-3x)}{4-\lambda^2}, |\lambda| < 2$]

8. Solve the following integral equations by the method of successive approximateion to the third order

(i) $\phi(x) = 2x + \lambda\int_0^1 (x+t)\phi(t)dt,\ \phi_0(x) = 1$

[Ans. $\phi_3(x) = 2x + \lambda\left(x+\frac{2}{3}\right) + \lambda^2\left(\frac{7}{6}x+\frac{2}{3}\right) + \lambda^3\left(\frac{13}{12}x+\frac{5}{8}\right)$]

(ii) $\phi(x) = 1 + \lambda\int_0^1 (x+t)\phi(t)dt,\ \phi_0(x) = 1$

[Ans. $\phi_3(x) = 1 + \lambda\left(x+\frac{1}{2}\right) + \lambda^2\left(x+\frac{7}{12}\right) + \lambda^3\left(\frac{13}{12}x+\frac{5}{8}\right)$]

9. Show that the resolvent kernel of the equation

$$\phi(x) = f(x+\mu) + \int_0^a R(x,t;\lambda)\phi(t)dt$$

is equal to $R(x,t;\lambda+\mu)$

UNIT 3 □ FREDHOLM THEORY

In the last chapter we studied the solutions of the Fredholm integral equation and expressed these in the form of a uniformly convergent series for small values of λ. We are now in a position to discuss the theory originally developed by Fredholm in 1903 for all values of the parameter λ. We shall approach this by considering different types of kernels namely, degenerate, square-integrable or when it is only continuous, etc.

3.1 THE FREDHOLM THEOREMS

I. [The Fredholm Alternative] Each $\lambda \in \mathbb{C}$ is either a regular value of K or an eigenvalue of K. [consult in this respect eq. (1.16) and definition, § 1.3]

II. For each real positive number A there are only finitely many eigen values λ such that $|\lambda| < A$.

III. If λ is an eigenvalue of K, then

(a) $\bar{\lambda}$ is an eigenvalue of K^*

(b) the spaces $\{\phi : \phi = \lambda K\phi\}$ and $\{\psi : \psi = \bar{\lambda} K^* \psi\}$ have the same finite dimension.

(c) $\phi - \lambda K\phi = f$ has a solution if and only if $< \psi, f > = 0$ for every ψ satisfying $\psi = \bar{\lambda} K^* \psi$.

IV. If λ is a regular value of K then

(a) the solution of $\phi - \lambda K\phi = f$ has the form

$$\phi(x) = f(x) + \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

where $R(x,t;\lambda)$ is the resolvent kernel of K at λ.

(b) For fixed x and t, $R(x,t;\lambda)$ is an analytic function of λ for $\lambda \in$ {regular values of K}.

We prove Fredholm theorems in three special cases :

(i) Degenerate kernels
(ii) Continuous kernels
(iii) Hilbert-Schmidt kernels

3.2 DEGENERATE KERNELS

Here we consider the kernel $K(x,t)=\sum_{k=1}^{n} u_k(x)\overline{v_k(t)}$ and then the corresponding Fredholm integral equation of the second kind takes the form

$$\phi(x)=f(x)+\lambda\sum_{k=1}^{n} u_k(x)\int_a^b \overline{v_k(t)}\,\phi(t)\,dt. \tag{3.1}$$

Let $c_k=\int_a^b \overline{v_k(t)}\phi(t)dt$

then the equation (3.1) reduces to

$$\phi(x)=f(x)+\lambda\sum_{k=1}^{n} c_k\,u_k(x) \tag{3.2}$$

In order to determine the constants c_k we substitute the expression (3.2) for $\phi(x)$ in (3.1) and obtain

$$\sum_{k=1}^{n} c_k u_k(x) = \sum_{k=1}^{n} u_k(x)\int_a^b \overline{v_k(t)}\,\{f(t)+\lambda\sum_{j=1}^{n} c_j\,u_j(t)\}\,dt$$

$$i.e.,\ \sum_{k=1}^{n} u_k(x)\,[c_k-\int_a^b \overline{v_k(t)}\,\{f(t)+\lambda\sum_{j=1}^{n} c_j\,u_j(t)\}\,dt] = 0$$

and since the $u_k(x)$ are linearly independent functions [see (1.11)] it follows that

$$c_k-\int_a^b \overline{v_k(t)}\,\{f(t)+\lambda\sum_{j=1}^{n} c_j u_j(t)\}\,dt = 0. \tag{3.3}$$

Substituting

$$f_k=\int_a^b \overline{v_k(t)}\,f(t)\,dt \text{ and } a_{kj}=\int_a^b \overline{v_k(t)}u_j(t)\,dt, \tag{3.4-5}$$

equation (3.3.) reduces to a system of linear algebraic system of n number of non-homogeneous equations in n number of unknowns $c_1,c_2,...,c_n$ given by

$$c_k - \lambda \sum_{j=1}^{n} a_{kj} c_j = f_k, \quad k = 1, 2, \ldots, n \tag{3.6}$$

which will have a unique solution if and only if the determinant

$$D(\lambda) = \det(I - \lambda A)$$

$$= \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \neq 0 \tag{3.7}$$

Clearly, $D(\lambda)$ is a polynomial in λ of degree not greater than n. The two possible cases are :

Case (i) $\boldsymbol{D(\lambda) \neq 0}$

The system is satisfied by one and only one set of values of $c_1, c_2, \ldots, c_n$ given by (following Cramer's rule)

$$c_k = \frac{1}{D(\lambda)} \sum_{j=1}^{n} d_{kj}(\lambda) f_j, \quad k = 1, 2, \ldots, n \tag{3.8}$$

where (d_{kj}) are the elements of the adjugate of the matrix $I - \lambda A$. Now from (3.2) we get

$$\phi(x) = f(x) + \frac{\lambda}{D(\lambda)} \sum_{k=1}^{n} \sum_{j=1}^{n} u_k(x)\, d_{kj}(\lambda) f_j$$

$$= f(x) + \lambda \int_a^b \left[\frac{1}{D(\lambda)} \sum_{k=1}^{n} \sum_{j=1}^{n} u_k(x)\, d_{kj}(\lambda)\, \overline{v_j(t)} \right] f(t)\, dt \tag{3.9}$$

which shows that the resolvent kernel is

$$R(x, t; \lambda) = \frac{1}{D(\lambda)} \sum_{k,j=1}^{n} u_k(x)\, d_{kj}(\lambda)\, \overline{v_j(t)} \tag{3.10}$$

where $d_{kj}(\lambda) / D(\lambda)$ are rational functions of λ and are analytic except at points where $D(\lambda) = 0$.

Further it is evident that the homogeneous equation (*i.e.*, the Fredholm equation of the first kind) has the unique solution $\phi(x) = 0$, if $D(\lambda) \neq 0$ that is when λ is a regular value of K.

Case (ii) $D(\lambda)=0$

It gives rise to non-trivial solutions of the corresponding homogeneous equation

$$c_k-\lambda\sum_{j=1}^{n}a_{kj}c_j = 0,\quad k=1,2,\dots,n;\ \text{that is, of the equation } \phi(x)=\lambda K\phi(x) \qquad (3.11)$$

where λ belongs to the set of eigenvalues (λ : roots of the equation $D(\lambda)=0$) of the equation (3.11) and the system (3.6) is either nonsolvable or else has an infinite number of solutions.

Here $D(\lambda)=0$

i.e. $\det(I-\lambda A)=0$,

equivalently, $\det(I-\bar{\lambda}A^*)=0$, which implies $\bar{\lambda}$ is an eigenvalue of K^*.

Let r be the rank of the matrix $I-\lambda A$. Then the system of n homogeneous equations

$$c_k-\lambda\sum_{j=1}^{n}a_{kj}c_j = 0,\quad k=1,2,\dots,n \qquad (3.6')$$

has $p=n-r$ number of linearly independent solutions $\{c_k^{(i)}\}$, $i=1,2,\dots,p$, and the general solution of the homogeneous equation (3.11) can be expressed as

$$\phi(x)=\sum_{i=1}^{p}\alpha_i\phi_i(x),$$

by using (3.2) with $f(x)=0$, $c_k=\sum_{i=1}^{p}\alpha_i c_k^{(i)}$, $k=1,2,\dots,n$ and taking $\phi_i(x)=\lambda\sum_{k=1}^{n}c_k^{(i)}u_k(x), i=1,2,\dots,p$. This dimension p is called the rank (sometimes the index to avoid confusion) of the eigenvalue λ of K.

Now since $I-\bar{\lambda}A^*=(I-\lambda A)^*$ it follows that the eigenvalue $\bar{\lambda}$ of the adjoint homogeneous equation

$$\psi(x)=\bar{\lambda}K^*\psi(x) \qquad (3.12)$$

will have the same rank p and therefore the same number of linearly independent solutions as the homogeneous equation (3.11).

Lastly we show that the inhomogeneous equation (3.1) has a solution for a given value of λ if and only if $f(x)$ is orthogonal to every solution of the adjoint homogeneous equation (3.12).

If λ is a regular value of K then $\psi = 0$ and then the result is obvious. But when λ is an eigenvalue of K and ϕ is a solution of (3.1) *i.e.*, of $\phi = f + \lambda K\phi$ we see that

$\langle f, \psi \rangle = \langle \phi - \lambda K\phi, \psi \rangle$

$= \langle \phi, \psi \rangle - \langle \lambda K\phi, \psi \rangle = \langle \phi, \psi \rangle - \langle \phi, \bar{\lambda} K^* \psi \rangle$, using Proposition–1, §1.3.

$$= \langle \phi, \psi - \bar{\lambda} K^* \psi \rangle = \langle \phi, 0 \rangle = 0 \tag{3.13}$$

To prove the sufficiency of the condition we write the equation (3.12) explicitly as

$$\psi(x) = \bar{\lambda} \sum_{k=1}^{n} v_k(x) \int_a^b \overline{u_k(t)} \psi(t)\, dt$$

which on introducing

$$\bar{c}_k = \int_a^b \overline{u_k(t)} \psi(t)\, dt \quad \text{and} \quad \bar{a}_{kj} = \int_a^b \overline{u_k(t)} v_j(t)\, dt$$

takes the form

$$\psi(x) = \bar{\lambda} \sum_{k=1}^{n} \bar{c}_k v_k(x) \tag{3.14}$$

where $\bar{c}_k$ are to be determined from

$$\bar{c}_k - \bar{\lambda} \sum_{j=1}^{n} \bar{a}_{kj} \bar{c}_j = 0, \; k = 1, 2, ..., n \tag{3.15}$$

Now we recall that the system (3.6) is solvable, when $D(\lambda) = 0$, if and only if the vector $\bar{f} = (f_1, f_2, ..., f_n)$ is orthogonal to the solution $\bar{c} = (\bar{c}_1, \bar{c}_2, ..., \bar{c}_n)$ of the system (3.15) *i.e.* $\langle \bar{c}, \bar{f} \rangle = \sum_{k=1}^{n} \bar{c}_k f_k = 0$

By hypothesis,

$\langle \psi, f \rangle = 0$

But $< \psi, f > = \int_a^b \lambda \sum_{k=1}^{n} \tilde{c}_k \overline{v_k(t)}\, f(t)\, dt$

$$= \lambda \sum_{k=1}^{n} \tilde{c}_k \int_a^b \overline{v_k(t)}\, f(t)\, dt$$

$$= \lambda \sum_{k=1}^{n} \tilde{c}_k f_k = \lambda\, (\tilde{c}, \tilde{f})$$

Thus we have proved the Fredholm theorems completely in the case of degenerate kernels.

Example 3.1. Considering the kernel

$$K(x,t) = \sin(x+t),\ 0 \le x \le 2\pi,\ 0 \le t \le 2\pi,$$

deduce the general form of solutions of the Fredholm integral equations of the second kind.

Solution. Here $K(x,t) = \sin(x+t)$

$$= \sin x \cos t + \cos x \sin t$$

$$= \sum_{i=1}^{2} u_i(x) v_i(t), \text{ say}$$

where

$$\left.\begin{aligned} u_1(x) &= \sin x \\ v_1(t) &= \cos t \end{aligned}\right\} \text{ and } \left.\begin{aligned} u_2(x) &= \cos x \\ v_2(t) &= \sin t \end{aligned}\right\}$$

Introducing

$$a_{kj} = \int_0^{2\pi} v_k(t)\, u_j(t)\, dt$$

we find that

$$a_{11} = \int_0^{2\pi} \sin t \cos t\, dt = a_{22}\,; \text{ clearly } \quad a_{11} = a_{22} = 0,$$

$$a_{12} = \int_0^{2\pi} \cos^2 t\, dt = \pi \text{ and } a_{21} = \int_0^{2\pi} \sin^2 t\, dt = \pi$$

and thus

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} \\ -\lambda a_{21} & 1-\lambda a_{22} \end{vmatrix} = 1 - \lambda^2 \pi^2$$

Now, when $D(\lambda)$ is not zero, $\lambda \neq \pm 1/\pi$

Here $d_{11} = d_{22} = 1$ and $d_{12} = d_{21} = \lambda\pi$ so that using the equation (3.9) for $\phi(x)$ we find that

$$\phi(x) = f(x) + \frac{\lambda}{1-\lambda^2\pi^2}\sum_{k=1}^{2} u_k(x)\,(d_{k1}f_1 + d_{k2}f_2)$$

$$= f(x) + \frac{\lambda}{1-\lambda^2\pi^2}\;(\sin x + \lambda\pi\cos x\,)f_1 + (\,\lambda\pi\sin x + \cos x\,)f_2$$

$$= f(x) + \frac{\lambda}{1-x^2\pi^2}\int_0^{2\pi}[(\sin x + \lambda\pi\cos x)\cos t + (\lambda\pi\sin x + \cos x)\sin t]f(t)\,dt$$

$$= f(x) + \frac{\lambda}{1-\lambda^2\pi^2}\int_0^{2\pi}\{\sin(x+t) + \lambda\pi\cos(x-t)\}f(t)\,dt\,. \tag{3.16}$$

Thus the resolvent kernel is

$$R(x,t;\lambda) = \frac{\sin(x+t) + \lambda\pi\cos(x-t)}{1-\lambda^2\pi^2}.$$

Now we show that the solution $\phi(x)$ in (3.16) is orthogonal to the solution of the adjoint homogeneous equation for $\lambda = \pm 1/\pi$, which is the same as the corresponding homogeneous equation since the given kernel is real and symmetric. To find the eigenfunction corresponding to the eigenvalues $\lambda = 1/\pi$ or $-1/\pi$, we first calculate for c_1, c_2 in equation (3.6) *i.e.*, $c_1 - \lambda\pi c_2 = 0$, $-\lambda\pi c_1 + c_2 = 0$ and each of these equations is reduced to $c_1 - c_2 = 0$ for $\lambda = 1/\pi$ and to $c_1 + c_2 = 0$ for $\lambda = -1/\pi$ respectively. We see that the solutions in the form of orthonormalized eigenfunctions [see (3.2)] are

$$\psi_1(x) = \frac{1}{\pi}\sum_{k=1}^{2} c_k u_k(x),\ \text{ for } \lambda = \frac{1}{\pi} \text{ and } c_1 = c_2 = \sqrt{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}(\sin x + \cos x)$$

and $\psi_2(x) = -\dfrac{1}{\sqrt{2\pi}}(\sin x - \cos x)$ for $\lambda = -\dfrac{1}{\pi}$ and $c_1 = -c_2 = \sqrt{\dfrac{\pi}{2}}$.

Suppose $f(x) = 1$; clearly

$$< \psi_j, f > \; = \; 0, \; j = 1,2.$$

Thus f is orthogonal to the solutions of the adjoint homogeneous equation. Hence the given inhomogeneous equation has the general solution

$$\phi(x) = 1 + k_1\psi_1(x) \text{ or } \phi(x) = 1 + k_2\psi_2(x)$$

according as $\lambda = 1/\pi$ or $-1/\pi$ respectively when $f(x) = 1$.

But when $f(x) = x$,

$$<\psi_1, f> = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} t(\sin t + \cos t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[(-t\cos t + t\sin t)\Big|_0^{2\pi} - \int_0^{2\pi} (\sin t - \cos t)\, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-2\pi + (\sin t + \cos t)\Big|_0^{2\pi} \right] = -\sqrt{2\pi}$$

and $\quad <\psi_2, f> = -\sqrt{2\pi}$

Here we see that the orthogonal condition is not satisfied and hence the inhomogeneous equation has no solution for $\lambda = \pm 1/\pi$.

Example 3.2. Given, f is continuous and $1 + \frac{\lambda}{2} - \frac{\lambda^2}{240} \neq 0$, show that

$$y(x) = f(x) + \lambda \int_0^1 (x^2 t + x t^2) y(t)\, dt$$

has a solution.

Solution. Here, $K(x,t) = x^2 t + x t^2$ and that

$$\left.\begin{matrix} u_1(x) = x^2 \\ v_1(t) = t \end{matrix}\right\} \text{ and } \left\{\begin{matrix} u_2(x) = x \\ v_2(t) = t^2 \end{matrix}\right.$$

Letting $\quad a_{kj} = \int_0^1 v_k(t) u_j(t) dt$

we get $\quad a_{11} = \int_0^1 t \cdot t^2 dt = a_{22} = \frac{1}{4}$

$$a_{12} = \int_0^1 t^2 dt = \frac{1}{3}, \; a_{21} = \int_0^1 t^4 dt = \frac{1}{5}$$

and thus $D(\lambda) = \begin{vmatrix} 1+\lambda a_{11} & +\lambda a_{12} \\ +\lambda a_{21} & 1+\lambda a_{22} \end{vmatrix} = \begin{vmatrix} 1+\frac{\lambda}{4} & \frac{\lambda}{3} \\ \frac{\lambda}{5} & 1+\frac{\lambda}{4} \end{vmatrix} = 1+\frac{\lambda}{2}-\frac{\lambda^2}{240}$

As $1+\frac{\lambda}{2}-\frac{\lambda^2}{240} \neq 0$ given, the solution $y(x)$ turns out to be

$$y(x) = f(x) - \lambda \sum_{k=1}^{n} c_k u_k(x)$$

where c_k's are determined from

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1+\frac{\lambda}{4} & -\frac{\lambda}{3} \\ -\frac{\lambda}{5} & 1+\frac{\lambda}{4} \end{pmatrix} \begin{pmatrix} < v_1, f > \\ < v_2, f > \end{pmatrix}$$

Example 3.3. For what functions f does the equation

$$y(x) = f(x) + \sqrt{3}\lambda \int_0^1 (xt^2 - \frac{1}{4}) y(t)\, dt$$

have a solution?

Solution. Here $K(x,t) = xt^2 - \frac{1}{4}$, we take

$$\left.\begin{matrix} u_1(x) = x \\ v_1(t) = t^2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} u_2(x) = -1 \\ v_2(t) = \frac{1}{4} \end{matrix}\right.$$

Introducing,

$$a_{kj} = \int_0^1 v_k(t) u_j(t)\, dt$$

we find that

$$a_{11} = \frac{1}{4}, \; a_{22} = -\frac{1}{4}, a_{12} = -\frac{1}{3} \text{ and } a_{21} = \frac{1}{8}$$

so that

$$D(\lambda) = \det D = \begin{vmatrix} 1-\sqrt{3}\lambda/4 & \sqrt{3}\lambda/3 \\ -\sqrt{3}\lambda/8 & 1+\sqrt{3}\lambda/4 \end{vmatrix} = 1-\frac{\lambda^2}{16}$$

Therefore the given equation will have a solution if and only if $\lambda \neq \pm 4$ (then the corresponding homogeneous equation has only the trivial solution) and the form of a solution is given by

$$y(x) = f(x) + \sqrt{3}\lambda \sum_{k=1}^{n} c_k u_k(x) \tag{3.17}$$

where c_k's are determined from

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1+\dfrac{\sqrt{3}\lambda}{4} & -\dfrac{\lambda}{\sqrt{3}} \\ +\dfrac{\sqrt{3}\lambda}{8} & 1-\dfrac{\sqrt{3}\lambda}{4} \end{pmatrix} \begin{pmatrix} < v_1, f > \\ < v_2, f > \end{pmatrix}$$

for any continuous function $f(x)$ on [0.1].

If λ is equal to the one of the eigenvalues (4 or -4) then the given equation will have solutions if f is orthogonal to the solution of the transposed homogeneous equation

$$y(x) = \sqrt{3}\lambda \int_0^1 \left(xt^2 - \frac{1}{4}\right) y(t)\, dt .$$

For $\lambda = 4$, the algebraic equations (3.6') give

$$(1-\sqrt{3})\, c_1 + \frac{4}{\sqrt{3}} c_2 = 0 .$$

We choose $c_1 = 2(\sqrt{3}+1)$ and $c_2 = \sqrt{3}$ and get the corresponding eigenfunction from (3.17) as

$$y(x) = c\,\{2(3+\sqrt{3})x - 3\},\ c \equiv \text{ arbitrary constant.}$$

Similarly for $\lambda = -4$, the corresponding eigenfunction is

$$y(x) = \alpha\,\{4x - (3+\sqrt{3})\},\ \alpha \equiv \text{ arbitrary constant.}$$

It follows from the Fredholm theorem III(c) that the integral equation

$$y(x) = f(x) + 4\sqrt{3} \int_0^1 (xt^2 - \frac{1}{4}) y(t)\, dt$$

will have a solution if $f(x)$ satisfies the condition

$$\int_0^1 \{2(3+\sqrt{3})x-3\} f(x)\, dx = 0$$

while the integral equation

$$y(x) = f(x) - 4\sqrt{3}\int_0^1 (xt^2 - \frac{1}{4}) y(t)\, dt$$

will have a solution if the following condition holds :

$$\int_0^1 \{4x-(3+\sqrt{3}\} f(x)\, dx = 0.$$

3.2.1. Method of approximation by Degenerate Kernels

In this section it will be shown that we can approximate a square-integrable kernel as close as we please, in the mean square sense, to a degenerate kernel of sufficiently high rank.

Let us suppose that the given kernel $K(x,t)$ is square-integrable [see notation §1.3], we shall prove that given any $\varepsilon > 0$ there exists a degenerate kernel $Q(x,t)$ such that

$$\| ¼ - Ð \|_2 < \varepsilon \tag{3.18}$$

Let $v_1(t), v_2(t), \ldots, v_k(t), \ldots$ be a complete orthonormal system of square-integrable functions and let

$$\int_a^b K(x,t) v_i(t)\, dt = u_i(x) \tag{3.19}$$

Using the Parseval's formula [see Th. 2 pp. 177, Goffman & Pedrick] we have

$$\sum_{j=1}^{\infty} | u_j(x) |^2 = \int_a^b | K(x,t) |^2\, dt \tag{3.20}$$

and so $$\sum_{j=1}^{\infty} \int_a^b | u_j(x) |^2\, dx = \| ¼ \|_2^2 \tag{3.21}$$

from which it follows that there exists a number N such that for $n > N$,

$$\sum_{j=n+1}^{\infty} \int_a^b |u_j(x)|^2 \, dx < \varepsilon^2 \tag{3.22}$$

Again, $\int_a^b |K(x,t) - \sum_{j=1}^{n} u_j(x)\overline{v_j(t)}|^2 \, dt$

$$= \int_a^b \{K(x,t) - \sum_{j=1}^{n} u_j(x)\overline{v_j(t)}\} \{\overline{K(x,t)} - \sum_{j=1}^{n} \overline{u_j(x)} v_j(t)\} \, dt$$

$$= \int_a^b |K(x,t)|^2 \, dt - \sum_{j=1}^{n} u_j(x) \int_a^b \overline{K(x,t) v_j(t)} \, dt - \sum_{j=1}^{n} \overline{u_j(x)} \int_a^b K(x,t) v_j(t) \, dt$$

$$+ \sum_{j=1}^{n} |u_j(x)|^2$$

$$= \sum_{j=1}^{\infty} |u_j(x)|^2 - 2\sum_{j=1}^{n} |u_j(x)|^2 + \sum_{j=1}^{n} |u_j(x)|^2 \quad \text{[using (3.19) and (3.20)]}$$

$$= \sum_{j=n+1}^{\infty} |u_j(x)|^2 \tag{3.23}$$

and so $\int_a^b \int_a^b |K(x,t) - \sum_{j=1}^{n} u_j(x)\overline{v_j(t)}|^2 \, dx \, dt < \varepsilon^2$, [using (3.22)] (3.24)

Finally taking $Q(x,t) = \sum_{j=1}^{n} u_j(x)\overline{v_j(t)}, \; n > N,$ (3.25)

we have thus proved the following result :

Result 3.2 If $\int_a^b \int_a^b |K(x,t)|^2 \, dx \, dt < \infty$

then there are kernels Q and S such that

(i) $K = Q + S$ (3.26)

(ii) Q is degenerate and square-integrable and

(iii) $\int_a^b \int_a^b |S(x,t)|^2 \, dx \, dt < 1.$

3.2.2. Fredholm theorems

We consider the Fredholm integral equation of the second kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt \tag{3.27}$$

when $K(x,t)$ is square-integrable and λ is so chosen that $|\lambda| < 1$.

The expression of $K(x,t)$ in (3.26) was introduced by Schmidt in 1907 and it enables us to write the given equation (3.27) as

$$\phi(x) = f(x) + \lambda \int_a^b Q(x,t)\,\phi(t)\,dt + \lambda \int_a^b S(x,t)\,\phi(t)\,dt$$

$$= f_1(x,\lambda) + \lambda \int_a^b S(x,t)\,\phi(t)\,dt \tag{3.28}$$

Now since $|\lambda|\ \|S\|_2 < 1$ it follows that the Neumann series for the resolvent kernel $R_s(x,t;\lambda)$ of $S(x,t)$ is convergent and the solution of the integral equation (3.28) will be of the form

$$\phi(x) = f_1(x,\lambda) + \lambda \int_a^b R_s(x,t;\lambda) f_1(t;\lambda)\,dt \tag{3.29}$$

This may be written as

$$\phi(x) = f_2(x,\lambda) + \lambda \int_a^b T(x,t;\lambda)\,\phi(t)\,dt \tag{3.30}$$

where $\quad f_2(x,\lambda) = f(x) + \lambda \int_a^b R_s x,t;\lambda) f(t)\,dt$

and $\quad T(x,t;\lambda) = Q(x,t) + \lambda \int_a^b R_s(x,s;\lambda)\,Q(s,t)\,ds$

$$= \sum_{j=1}^{n} u_j(x)\overline{v_j(t)} + \lambda \int_a^b R_s(x,s;\lambda)\left(\sum_{j=1}^{n} u_j(s)\overline{v_j(t)}\right) ds$$

$$= \sum_{j=1}^{n} \left\{ u_j(x) + \lambda \int_a^b R_s(x,s;\lambda)\,u_j(s)\,ds \right\} \overline{v_j(t)}$$

$$= \sum_{1}^{n} y_j(x)\overline{v_j(t)}. \tag{3.31}$$

Clearly $T(x,t;\lambda)$ is degenerate.

Thus the original equation (3.27) with square-integrable kernel $K(x,t)$ took the form (3.30) possessing a degenerate kernel $T(x,t;\lambda)$. Here we note that when the equation (3.27) is homogeneous (*i.e.*, when $f = 0$) the equation (3.30) is also homogeneous i.e., $f_2(x,\lambda) = 0$. These indicate that the results obtained in §3.2 for degenerate kernel equally hold in the case of general square-integrable kernel.

Using the expression (3.9) for degenerate kernel we obtain the solution for (3.27) as

$$\phi(x) = f_2(x,\lambda) + \lambda \int_a^b \emptyset(x)[\tilde{A}(\lambda)]^{-1}[\overline{\tilde{O}(t)}]^T f_2(t;\lambda)\, dt \tag{3.32}$$

where $\tilde{A}(\lambda) = I - \lambda A$ and the matrix A has elements

$$a_{ij} = \int_a^b \overline{v_i(t)}\, y_j(t)\, dt$$

and $\emptyset = (y_1, y_2, \ldots, y_n)$, $\tilde{O} = (v_1, v_2, \ldots, v_n)$

Hence the resolvent kernel of $K(x,t)$ will be of the form

$$R(x,t;\lambda) = R_s(x,t;\lambda) + \emptyset(x)\tilde{A}^{-1}(\lambda)\,\overline{\dot{U}^T(t)} \tag{3.33}$$

where $$\dot{U}(t) = \tilde{O}(t) + \bar{\lambda}\int_a^b \tilde{O}(t)\,\overline{R_s(s,t;\lambda)}\, dt \tag{3.34}$$

3.3 CONTINUOUS KERNELS

Here we derive the Fredholm solutions of the integral equation

$$\phi(x) = f(x) + \lambda\int_a^b K(x,t)\,\phi(t)\, dt \tag{3.35}$$

when the kernel $K(x,t)$ is continuous in the square $a \le x \le b$, $a \le t \le b$ and $f(x)$ is continuous in the interval $a \le x \le b$. In determining the solution Fredholm treats the integral equation (3.35) as the limiting case of a finite system of linear algebraic equations and obtained this in some forms of determinants.

To introduce the Fredholm's method we divide the interval (a,b) into n equal sub-intervals of length $h=(b-a)/n$ in a form

$$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$$

where $x=a+h$. We approximate the Riemann integral on the R.H.S. of (3.35) by the finite sum

$$\int_a^b K(x,t)\,\phi(t)\,dt \approx h\sum_{j=1}^{n} K(x,x_j)\,\phi(x_j) \tag{3.36}$$

Inserting this in the equation (3.35) we get

$$\phi(x) \approx f(x) + \lambda h\sum_{j=1}^{n} K(x,x_j)\,\phi(x_j) \tag{3.37}$$

for all $x\in[a,b]$ and in particular for $x=x_k,\ k=1,\ldots,n,$ which give rise to a system of n algebraic equations

$$\phi(x_k)=f(x_k)+\lambda h\sum_{j=1}^{n} K(x_k,x_j)\,\phi(x_j),\ k=1,2,\ldots,n \tag{3.38}$$

Writing $\phi(x_k)=\phi_k$, $f(x_k)=f_k$ and $K(x_k,x_j)=K_{kj}$, eq. (3.38) takes simpler form

$$\phi_k-\lambda h\sum_{j=1}^{n} K_{kj}\,\phi_j=f_k,\ \ k=1,2,\ldots,n$$

The solutions of this system of equations or, equivalently solution of the vector-matrix equation

$$\tilde{A}_n(\lambda)\,\phi = f$$

give the values of the approximate solution for the equation (3.35) at $x=x_1,x_2\ldots,x_n$;

where $\phi=(\phi_1,\phi_2,\ldots,\phi_n)^T$, $f=(f_1,f_2,\ldots,f_n)^T$ and

$$\tilde{A}_n(\lambda)=\begin{pmatrix} 1-\lambda hK_{11} & -\lambda hK_{12} & \ldots & -\lambda hK_{1n} \\ -\lambda hK_{21} & 1-\lambda hK_{22} & \ldots & -\lambda hK_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ -\lambda hK_{n1} & -\lambda hK_{n2} & \ldots & 1-\lambda hK_{nn} \end{pmatrix} \tag{3.39}$$

provided $d_n(\lambda) = \det \tilde{A}_n(\lambda) \neq 0$. Then the system of equations (3.38) has the solution

$$\phi(x_n) = \frac{1}{d_n(\lambda)} \sum_{j=1}^{n} C_n(x_n, x_t) f(x_t) \tag{3.40}$$

where $C_n(x_r, x_t)$ is the (r, t)th element of the adjugate matrix of $\tilde{A}_n(\lambda)$ that is $C_n(x_r, x_t)$ is the cofactor of the (t, r)th element in $d_n(\lambda)$. Expanding $d_n(\lambda)$ from its determinantal form, we get in ascending powers of λ

$$d_n(\lambda) = 1 - \frac{\lambda h}{1!}\mathsf{K}_1 + \frac{\lambda^2 h^2}{2!}\mathsf{K}_2 + \ldots + \frac{(-\lambda h)^n}{n!}\mathsf{K}_n \tag{3.41}$$

where $\mathsf{K}_1 = \sum_{\upsilon=1}^{n} K_{\upsilon\upsilon}$, $\mathsf{K}_2 = \sum_{p,q=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$

and in general

$$\mathsf{K}_n = \sum_{p_1, p_2, \ldots, p_m = 1}^{n} \begin{vmatrix} K_{p_1p_1} & K_{p_1p_2} & \cdots & K_{p_1p_m} \\ K_{p_2p_1} & K_{p_2p_2} & \cdots & K_{p_2p_m} \\ \cdots & \cdots & \cdots & \cdots \\ K_{p_mp_1} & K_{p_mp_2} & \cdots & K_{p_mp_m} \end{vmatrix} \tag{3.42}$$

Now if we allow $n \to \infty$ then $h \to 0$ and the terms in (3.41) starting from second, third, fourth etc. tend to single, double, triple integral etc and that $d_n(\lambda) \to d(\lambda)$ where

$$d(\lambda) = 1 - \lambda \int_a^b K(x,x)\, dx + \frac{\lambda^2}{2!} \int_a^b \int_a^b \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} dx_1\, dx_2$$

$$-\frac{\lambda^3}{3!} \int_a^b \int_a^b \int_a^b \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} dx_1\, dx_2\, dx_3 + \cdots$$

$$= \sum_{m=0}^{\infty} d_m \lambda^m \text{, say.} \tag{3.43}$$

$d(\lambda)$ is called the **Fredholm's determinant.** Now the cofactor $C_n(x_r, x_t)$ of the (t, s) element in $d_n(\lambda)$ can be expressed, for $r \neq t$, in ascending powers of λ, as

$$C_n(x_r,x_t) = \lambda h\,[K(x_r,x_t) - \frac{\lambda h}{1!}\sum_{t_1=1}^{n}\begin{vmatrix} K(x_r,x) & K(x_r,x_{t_1}) \\ K(x_{t_1},x) & K(x_{t_1},x_{t_1}) \end{vmatrix}$$

$$+\frac{(\lambda h)^2}{2!}\sum_{t_1,t_2=1}^{n}\begin{vmatrix} K(x_r,x_t) & K(x_r,x_{t_1}) & K(x_r,x_{t_2}) \\ K(x_{t_1},x_t) & K(x_{t_1},x_{t_1}) & K(x_{t_1},x_{t_2}) \\ K(x_{t_2},x_t) & K(x_{t_2},x_{t_1}) & K(x_{t_2},x_{t_2}) \end{vmatrix} - \cdots]$$

and allowing $n \to \infty$ and $x_r \to x, x_t \to t$ we find that $h^{-1}C_n(x_r,x_t) \to \lambda\, D(x,t;\lambda)$ where

$$D(x,t;\lambda) = K(x,t) - \frac{\lambda}{1!}\int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) \\ K(t_1,t) & K(t_1,t_1) \end{vmatrix} dt_1$$

$$+\frac{\lambda^2}{2!}\int_a^b\int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & K(x,t_2) \\ K(t_1,t) & K(t_1,t_1) & K(t_1,t_2) \\ K(t_2,t) & K(t_2,t_1) & K(t_2,t_2) \end{vmatrix} dt_1\,dt_2 - \cdots$$

$$= \sum_{m=0}^{\infty} D_m(x,t)\,\lambda^m \tag{3.44}$$

For $r = t, C_n$ will have similar expansion like $d_n(\lambda)$ and can be shown easily that $C_n(x_r,x_r) \to d(\lambda)$ as $n \to \infty$. The series in (3.44) is called the first **Fredholm's minor**. The series for $d(\lambda)$ and $D(x,t;\lambda)$ converge for all values of the parameter λ by **Hadamard's inequality** :

Given the elements a_{ij} of the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

be real and satisfy $|a_{ij}| \le M$ then $\det A \le \sqrt{n^n}\,M^n$ holds.

For the $(m+1)$th term of the series in (3.43) satisfies

$$|t_m| \le \frac{\sqrt{m^m}\, M^m (b-a)^m}{m!} = \alpha_m \text{, say}$$

[since the kernel $K(x,t)$ is continuous, it is also bounded]

and $$|d(\lambda)| \le \sum_{m=0}^{\infty} |t_m|\,|\lambda|^m \le \sum_{m=0}^{\infty} \alpha_m |\lambda|^m$$

and the series on the extreme right converges for all sufficiently small λ and hence the series for $d(\lambda)$ in (3.43) converges absolutely for all λ sufficiently small.

Again the coefficient of λ^m in the $(m+1)$th term of the series (3.44) for $d(x,t;\lambda)$ satisfies

$$|T_m(x,t)| \le \frac{\sqrt{(m+1)^{m+1}}\, M^{m+1} (b-a)^m}{m!} = \beta_m, \text{ say}$$

and thus

$$|D(x,t;\lambda)| \le \sum_{m=0}^{\infty} |T_m(x,t)|\,|\lambda|^m \le \sum_{m=0}^{\infty} \beta_m |\lambda|^m.$$

Again the series $\sum_{m=0}^{\infty} \beta_m |\lambda|^m$ converges for all λ and therefore the series for $D(x,t;\lambda)$ in (3.44) converges absolutely and uniformly for all values of λ.

The results obtained so far are expressed in the following theorem :

Theorem 3.1. Given $K(x,t)$ is a continuous kernel and d_m, $D_m(x,t)$ are defined by (3.43) and (3.44) respectively, then

(a) the series $d(\lambda) = \sum_{m=0}^{\infty} d_m \lambda^m$ is convergent for all complex λ, and $d(\lambda)$ is an integral function of λ.

(b) for given (x, t) the series $D(x,t;\lambda) = \sum_{m=0}^{\infty} D_m(x,t)\, \lambda^m$ is convergent for all complex λ, and $D(x,t;\lambda)$ is an integral function of λ.

(c) the series $D(x,t;\lambda) = \sum_{m=0}^{\infty} D_m(x,t)\, \lambda^m$ is uniformly and absolutely convergent in (x,t,λ) for λ in the entire complex plane.

The Resolvent kernel

We now show that, when $d(\lambda) \neq 0$, λ is a regular value of $K(x, t)$; the resolvent kernel can then be expressed in terms of $d(\lambda)$ and $D(x,t;\lambda)$.

Result. Let (i) $K(x, t)$ be a continuous kernel and (ii) $d(\lambda)$ and $D(x,t;\lambda)$ be its Fredholm determinant and first Fredholm minor respectively. If $d(\lambda) \neq 0$ and λ is a regular value of $K(x, t)$, then the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{d(\lambda)} \tag{3.45}$$

Proof. We have from (3.40)

$$\phi(x_r) = \frac{1}{d_n(\lambda)} \sum_{t=1}^{n} C_n(x_n, x_t) f(x_t)$$

$$= \frac{h}{d_n(\lambda)} \sum_{t=1}^{n} \frac{C_n(x_r, x_t)}{h} f(x_t)$$

$$= \frac{1}{d_n(\lambda)} [C_n(x_r, x_r) f(x_r) + h \sum_{t=1}^{n} \frac{C_n(x_r, x_t)}{h} f(x_t)]$$

Now as $n \to \infty$ and $x_r \to x$, $x_t \to t$

$$\phi(x) = \frac{1}{d(\lambda)} \int_a^b \lambda\, D(x,t;\lambda)\, f(t)\, dt + f(x)$$

$$= f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt, \tag{3.46}$$

where $R(x,t;\lambda)$ is given in (3.45).

The resolvent kernel (3.45) satisfies the resolvent equation.

In this respect we observe that

$$D_m(x,t) = \frac{(-1)^m}{m!} \int_a^b \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & \ldots & K(x,t_m) \\ K(t_1,t) & K(t_1,t_1) & \ldots & K(t_1,t_m) \\ \ldots & \ldots & \ldots & \ldots \\ K(t_m,t) & K(t_m,t_1) & \ldots & K(t_m,t_m) \end{vmatrix} dt_1 dt_2 \ldots dt_m$$

$$= \frac{(-1)^m}{m!}\int_a^b\int_a^b\ldots\int_a^b K\begin{pmatrix} x, t_1, \ldots, t_m \\ t, t_1, \ldots, t_m \end{pmatrix} dt_1\, dt_2 \ldots dt_m \text{, say} \tag{3.47}$$

$$= K(x,t)\,\frac{(-1)^m}{m!}\int_a^b\ldots\int_a^b K\begin{pmatrix} t_1, t_2, \ldots, t_m \\ t_1, t_2, \ldots, t_m \end{pmatrix} dt_1\, dt_2 \ldots dt_m$$

$$+\frac{(-1)^m}{m!}\sum_{j=1}^{m}\int_a^b\ldots\int_a^b K\begin{pmatrix} x, t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m \\ t_1, t_2, \ldots, \ \ t_j, \ \ t_{j+1}, \ldots, t_m \end{pmatrix} K(t_j, t)\, dt_1 \ldots dt_m$$

[Expanding the determinant in terms of the minors of the first column]

$$= d_m\, K(x,t) + \frac{(-1)^m}{m!}\left\{-1 + 1.-1 + \ldots + (-1)^m.(-1)^{m-1}\right\}.$$

$$\int_a^b\ldots\int_a^b K\begin{pmatrix} x, t_1, t_2, \ldots, t_{m-1} \\ s, t_1, t_2, \ldots, t_{m-1} \end{pmatrix} K(s,t)\, ds\, dt_1 \ldots dt_{m-1}$$

$$= \frac{(-1)^{m-1}}{(m-1)!}\int_a^b\ldots\int_a^b K\begin{pmatrix} x, t_1, \ldots, t_{m-1} \\ s, t_1, \ldots, t_{m-1} \end{pmatrix} K(s,t)\, dt_1 \ldots dt_{m-1}\, ds + d_m\, K(x,t)$$

$$= \int_a^b D_{m-1}(x,s)\, K(s,t)\, ds + d_m\, K(x,t) \tag{3.48}$$

Multiplying both sides of (3.48) by λ^m and summing over 0 to ∞, we find

$$D(x,t;\lambda) = K(x,t)\ d(\lambda) + \lambda\int_a^b D(x,s;\lambda)\ K(s,t)\, ds \tag{3.49}$$

Similarly expanding the determinant in (3.47) in terms of the minors of the first row we can have

$$D_m(x,t) = K(x,t)\, d_m + \int_a^b K(x,s)\, D_{m-1}(s,t)\, ds \tag{3.48'}$$

and as earlier

$$D(x,t;\lambda) = K(x,t)\, d(\lambda) + \lambda\int_a^b K(x,s)\, D(s,t;\lambda)\, ds \tag{3.49'}$$

Dividing both sides of (3.49) and (3.49') by $d(\lambda)$ we obtain the resolvent equation [see (1.21)]

$$R(x,t;\lambda) - K(x,t) = \lambda \int_a^b R(x,s;\lambda)\, K(s,t)\, ds$$

$$= \lambda \int_a^b K(x,s)\, R(s,t;\lambda)\, ds$$

The resolvent kernel $R(x,t;\lambda)$ is the quotient of two entire functions of the parameter λ and hence is a meromorphic function of λ *i.e.*, an analytic function whose only singularities are poles (here the zeros of $d(\lambda)$).

Following Proposition 2, §1.3. the results obained so far can be unified in the next theorem.

Theorem 3.2 Given $K(x,t)$ is a continuous kernel and $f(x)$ is a continuous function. Let $d(\lambda)$ and $D(x,t;\lambda)$ be the Fredholm determinant and the first Fredholm minor of $K(x,t)$ respectively. If λ is not a zero of $d(\lambda)$, then the integral equation

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t)\, dt$$

has the unique continuous solution

$$\phi(x) = f(x) + \frac{\lambda}{d(\lambda)} \int_a^b D(x,t;\lambda) f(t)\, dt$$

In particular, the homogeneous equation

$$\phi(x) = \lambda \int_a^b K(x,t)\, \phi(t)\, dt$$

has the unique continuous solution $\phi(x) = 0$; every eigenvalue of $K(x,t)$ must be a zero of $d(\lambda)$.

We now establish a simple interesting expression for logarithmic derivative of $d(\lambda)$.

From (3.47), we have

$$D_{m-1}(x,t) = \frac{(-1)^{m-1}}{(m-1)!} \int_a^b \cdots \int_a^b K\begin{pmatrix} x, t_1, \ldots, t_{m-1} \\ t, t_1, \ldots, t_{m-1} \end{pmatrix} dt_1\, dt_2 \ldots dt_{m-1}$$

and consequently for $x = t$, it follows that

$$\int_a^b D_{m-1}(t,t)\,dt = \frac{(-1)^{m-1}}{(m-1)!}\int_a^b\int_a^b\ldots\int_a^b K\begin{pmatrix} t,t_1,\ldots,t_{m-1} \\ t,t_1,\ldots,t_{m-1}\end{pmatrix} dt\,dt_1\ldots d_{m-1}$$

and so we obtain the relation

$$d_m = -\frac{1}{m}\int_a^b D_{m-1}(t,t)\,dt, \text{ [see (3.43)]} \tag{3.50}$$

Also

$$d'(\lambda) = \sum_{m=1}^{\infty} m d_m \lambda^{m-1}$$

$$= -\sum_{m=0}^{\infty} \lambda^m \int_a^b D_m(t,t)\,dt$$

$$= -\int_a^b D(t,t,\lambda)\,dt \tag{3.51}$$

and so using (3.45)

$$\frac{d'(\lambda)}{d(\lambda)} = -\int_a^b R(t,t;\lambda)\,dt$$

$$= -\sum_{n=0}^{\infty} \lambda^n \int_a^b K_{n+1}(t,t)\,dt$$

by employing the Neumann expansion (2.22) of the resolvent kernel. Now if we put

$$\int_a^b K_n(t,t)\,dt = a_n \quad (n = 1,2,3,\ldots) \tag{3.52}$$

[$a_n \equiv$ trace of the iterated kernel $K_n(x,t)$ which is defined in (2.17)].

We obtain the logarithmic derivative of $d(\lambda)$ as

$$\frac{d'(\lambda)}{d(\lambda)} = -\sum_{n=0}^{\infty} a_{n+1}\lambda^n \tag{3.52'}$$

whose radius of convergence obviously equals the modulus of the eigenvalue of the kernel $K(x,t)$ having the least value.

The homogeneous equation

We shall now show that every zero of the Fredholm determinant $d(\lambda)$ of a continuous kernel $K(x,t)$ is an eigen value of the kernel $K(x,t)$.

Theorem 3.3 Given $K(x,t)$ be a continuous kernel and $d(\lambda)$ be its Fredholm determinant. If $\lambda = \lambda_0$ be a zero of $d(\lambda)$, then the homogeneous equation

$$\phi(x) = \lambda_0 \int_a^b K(x,t)\,\phi(t)\,dt \tag{3.53}$$

has a continuous solution $\phi(x)$, not identically zero; in other words, $\lambda = \lambda_0$ is an eigenvalue of $K(x,t)$.

Proof. From (3.49'), we get

$$D(x,t;\lambda) = K(x,t)\,d(\lambda) + \lambda \int_a^b K(x,u)\,D(u,t;\lambda)\,du \tag{3.54}$$

if $\lambda = \lambda_0$, this equation reduces to

$$D(x,t;\lambda_0) = \lambda_0 \int_a^b K(x,u)\,D(u,t;\lambda_0)\,du \tag{3.55}$$

If $D(x,t;\lambda_0)$ does not vanish identically as a function of (x,t), we can choose $t = t_0$ so that $D(x,t_0;\lambda_0)$ as a function of x does not vanish identically. So, taking $\phi(x) = D(x,t_0;\lambda_0)$ we find that

$$\phi(x) = \lambda_0 \int_a^b K(x,u)\,\phi(u)\,du$$

and $\phi(x)$ is the required eigenfunction. It is to be noted that $D(x,t;\lambda)$ is not identically zero for $\lambda = \lambda_0$ when $\lambda = \lambda_0$ is a simple zero of $d(\lambda)$ as follows from (3.51) *i.e.*,

$$\int_a^b D(x,x;\lambda_0)\,dx = -\,d'(\lambda_0) \neq 0.$$

But when $D(x,t;\lambda) = 0$ for all (x,t) we proceed as follows :

Since $d(x,t;\lambda)$ is an entire function of λ, we have

$$D(x,t;\lambda) = (\lambda-\lambda_0)^m \alpha_m(x,t) + (\lambda-\lambda_0)^{m+1} \alpha_{m+1}(x,t) + \cdots \tag{3.56}$$

by Taylor's theorem for some $m \geq 1$, where the coefficients $\alpha_m(x,t)$ are given by [see **8**, pp. 86, 88]

$$\alpha_j(x,t) = \frac{\{D(x,t;\lambda)\}^{(j)}(\lambda_0)}{j!}$$

Now let $|D(x,t;\lambda)| \leq M$ for all λ lying in the circle $|\lambda-\lambda_0| < R_0$ and arbitrary (x,t). We get using Cauchy's estimate [**8**, pp. 62]

$$|\alpha_j(x,t)| \leq \frac{M}{R_0^j}$$

So the series in the R.H.S. of (3.56) converges uniformly and absolutely in (x,t,λ) for $|\lambda-\lambda_0| \leq R < R_0$. Since R_0 is arbitrary, the series converges uniformly in any bounded subset of the complex λ-plane.

Again by (3.51)

$$d'(\lambda) = -\int D(x,x;\lambda)\, dx. \tag{3.57}$$

Substituting (3.56) in the R.H.S. of (3.57), we see that $d'(\lambda)$ has a zero of order at least m at $\lambda = \lambda_0$; so that the zero of $d(\lambda)$ at $\lambda = \lambda_0$ must be of order $m + 1$. Thus we can write

$$d(\lambda) = \beta_{m+1}(\lambda-\lambda_0)^{m+1} + \beta_{m+2}(\lambda-\lambda_0)^{m+2} + \cdots \tag{3.58}$$

Finally substituting the series for $D(x,t;\lambda)$ and $d(\lambda)$ in (3.54) and equating the coefficients of $(\lambda-\lambda_0)^m$, we obtain

$$\alpha_m(x,t) = \lambda_0 \int_a^b K(x,u)\, \alpha_m(u,t)\, du \tag{3.59}$$

Now let us choose $t = t_0$ so that $\alpha_m(x,t_0)$ does not vanish identically as a function of x, and writing $\phi(x) = \alpha_m(x,t_0)$, we see that $\phi(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda = \lambda_0$.

Note : Theorems 3.2 and 3.3 together show that the zeros of the Fredholm determinant are precisely the eigenvalues of the kernel; in other words, λ is a regular value of the kernel if and only if it is not a zero of the Fredholm determinant.

Example 3.4. Solve the Fredholm integral equation of the second kind having the kernel $\sin(x+t)$, $0 \le x \le 2\pi$, $0 \le t \le 2\pi$.

Solution. From Theorem 3.2, if $f(x)$ is given as inhomogeneous term, the form of the unique solution is

$$\phi(x) = f(x) + \frac{\lambda}{d(\lambda)} \int_0^{2\pi} D(x,t;\lambda) f(t) dt$$

To determine this we use formulae (3.48') and (3.50) *i.e.*

$$d_m = -\frac{1}{m} \int_0^{2\pi} D_{m-1}(x,x)\, dx$$

and $D_m(x,t) = d_m K(x,t) + \int_0^{2\pi} D_{m-1}(x,u)\, K(u,t)\, du$

Here, $\quad d_0 = 1,\ D_0(x,t) = K(x,t) = \sin(x+t)$

$$d_1 = -\int_0^{2\pi} D_0(x,x)\, dx = -\int_0^{2\pi} \sin 2x\, dx = 0$$

$$D_1(x,t) = \int_0^{2\pi} D_0(x,u)\, K(u,t)\, du = \int_0^{2\pi} \sin(x+u) \sin(u+t)\, du$$

$$= \frac{1}{2} \int_0^{2\pi} [\cos(x-t) - \cos(x+2u+t)]\, du = \pi \cos(x-t)$$

$$d_2 = -\frac{1}{2} \int_0^{2\pi} D_1(x,x)\, dx = -\frac{1}{2} \int_0^{2\pi} \pi\, dx = -\pi^2$$

$$D_2(x,t) = -\pi^2 K(x,t) + \int_0^{2\pi} D_1(x,u)\, K(u,t)\, du$$

$$= -\pi^2 K(x,t) + \int_0^{2\pi} \pi \cos(x-u) \sin(u+t)\, du$$

$$= -\pi^2 K(x,t) + \frac{\pi}{2} \int_0^{2\pi} [\sin(x+t) + \sin(2u+t-x)]\, du$$

$$= -\pi^2 K(x,t) + \frac{\pi}{2}\{2\pi \sin(x+t)\} = -\pi^2 K(x,t) + \pi^2 K(x,t)$$

$$= 0.$$

and so, $d_j = D_j(x,t) = 0, \; j \geq 3$.

Therefore,

$$d(\lambda) = 1 - \lambda^2\pi^2 \quad \text{and} \quad D(x,t;\lambda) = \sin(x+t) + \lambda\pi\cos(x-t)$$

Thus, if $\lambda \neq \pm\frac{1}{\pi}$

$$\phi(x) = f(x) + \frac{\lambda}{1-\lambda^2\pi^2}\int_0^{2\pi}\{\sin(x+t) + \lambda\pi\cos(x-t)\}f(t)\,dt$$

If $\lambda = \pm\frac{1}{\pi}$ *i.e.*, $d(\lambda) = 0$, we seek for the solution of the homogeneous equation from Theorem 3.3.

Here $D(x,t;\frac{1}{\pi}) = \sin(x+t) + \cos(x-t)$ and $D(x,t;-\frac{1}{\pi}) = \sin(x+t) + \cos(x-t)$ do not vanish identically as a function of (x,t). So we can choose $t = t_0 = 0$ and take $\phi_1(x) = \sin x + \cos x$ and $\phi_2(x) = \sin x - \cos x$ as the solution of the homogeneous equation corresponding to the eigenvalues $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$ respectively.

[Compare the procedure of this example with that given in example 3]

EXERCISES

1. Using Fredholm determinant, find the resolvent kernel of the following kernels :

   (i) $K(x,t) = xe^t$; $a = 0$, $b = 1$ [Ans. $R(x,t;\lambda) = \frac{xe^t}{1-\lambda}$]

   (ii) $K(x,t) = x^2t - xt^2$; $0 \leq x \leq$, $0 \leq t \leq 1$

   [Ans. $R(x,t;\lambda) = \dfrac{x^2t - xt^2 + xt\left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)\lambda}{1 + \frac{\lambda^2}{240}}$]

2. Find the resolvent kernel of the following kernels :

   (i) $K(x,t) = x - 2t$; $0 \leq x \leq 1$, $0 \leq t \leq 1$

[Ans. $R(x,t;\lambda)=\dfrac{x-2t+\lambda(x+t)-2xt-\frac{2}{3}}{1+\frac{1}{2}\lambda+\frac{1}{6}\lambda^2}$]

(ii) $K(x,t)=4xt-x^2;\ 0\le x\le 1,\ 0\le t\le 1$

[Ans. $R(x,t;\lambda)=\dfrac{4xt-t^2-(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt)\lambda}{1-x+\frac{\lambda^2}{18}}$]

3. Using the method of degenerate kernels, determine the solution of the following integral equations :

(i) $\phi(x)=\tan x+\int_{-1}^{1} e^{\sin^{-1}x}\phi(t)dt$ [Ans. $\phi(x)=\tan x$]

(ii) $\phi(x)=e^x+\lambda\int_0^{20} xt\phi(t)dt$ [Ans. $\phi(x)=e^x+\dfrac{\lambda x(9e^{10}+1)}{1-(10^3/3)\lambda}$

(iii) $\phi(x)=\sec^2 x+\lambda\int_0^1 \phi(t)dt$ [Ans. $\phi(x)=\sec^2 x+\dfrac{\lambda}{1-\lambda}\tan 1,\ \lambda\ne 1$]

(iv) $\phi(x)=x+\lambda\int_0^1 (xt^2+x^2t)\,\phi(t)dt$ [Ans. $\phi(x)=\dfrac{e^x}{1-\lambda(e^2-1)}$]

(v) $\phi(x)=f(x)+\lambda\int_0^1 xt\phi(t)dt$ [Ans. $\phi(x)=f(x)+\dfrac{3\lambda x}{3-\lambda}\int_0^1 tf(t)dt,\ \lambda\ne 3$]

(vi) $\phi(x)=\cos x+\int_0^1 \sin(x-t)\phi(t)dt$ [Ans. $\phi(x)=\dfrac{2\cos x+\pi\sin x}{2\pi^2}$]

4. Solve the following homogeneous integral equations with degenerate kernels :

(i) $\phi(x)=-\int_0^1 \phi(t)dt$ [Ans. $\phi(x)=0$]

(ii) $\phi(x)=\dfrac{1}{50}\int_0^{10} tp(t)dt$ [Ans. $\phi(x)=0$]

5. Find the eigenvalues and eigen functions for the following homogeneous integral equations with degenerate kernels :

(i) $\phi(x) = \lambda \int_0^1 (45x^2 \log t - 9t^2 \log x)\phi(t)dt$

[Ans. no real eigenvalues and eigen functions]

(ii) $\phi(x) = \lambda \int_0^{\pi/4} \sin^2 x \, \phi(t)dt$ [Ans. $\lambda_1 = \frac{8}{\pi - 2}, \phi_1(x) = \sin^2 x$]

(ii) $\phi(x) = \lambda \int_0^{\pi} \cos(x+t)\phi(t)dt$ [Ans. $\lambda_1 = \frac{2}{\pi}, \lambda_2 = \frac{2}{\pi}, \ \phi_1 = \sin x, \ \phi_2 = \cos x$]

6. Prove that the integral equation

$$\phi(x) = \lambda \int_0^1 [\sqrt{x}t - \sqrt{t}x] \, \phi(t)dt$$

does not have real eigenvalues and eigenfunctions.

UNIT 4 ❐ HILBERT-SCHMIDT KERNEL

Finally we focus our attention to Hermitian integral operators $\mathcal{K}$ satisfying

$$(\mathcal{K}\phi, \psi) = (\phi, \mathcal{K}\psi) \tag{4.1}$$

for all ϕ, ψ belonging to the Hilbert space of L^2 functions. The present objective is to establish the theory initiated by Hilbert and Schmidt where it was shown that the resolvent kernel $R(x,t;\lambda)$ can be expanded in terms of the eigenfunctions and eigen values of the Hermitian kernel $K(x,t)$.

Hermitian Kernel. A kernel $K(x,t)$ is said to be **Hermitian** (or symmetric) for which $K(x,t) = \overline{K(t,x)}$ holds where $\overline{K(t,x)}$ denotes complex conjugate. (4.2)

The integral operators defined by

$$\mathcal{K} = \int_a^b K(x,t)dt \text{ and } \mathcal{K}^* = \int_a^b K(t,x)dx \tag{4.3}$$

are said to be Hermitian or self-adjoint if $\mathcal{K} = \mathcal{K}^*$.

Definition. A Hermitian kernel which is square-integrable [see (1.17)] is called a Hilbert-Schmidt kernel.

Theorem 4.1 Every non-null operator $\mathcal{K}$ possessing a Hilbert-Schmidt kernel has at least one eigenvalue.

Proof. Here $\mathcal{K}$ is Hermitian and so $\mathcal{K}^n$ is also Hermitian following (4.1). Applying trace of the integral operator $\mathcal{K}^n$ using (3.52) we find that

$$a_{2n} = \text{trace } (\mathcal{K}^n \mathcal{K}^n)$$

$$= \int_a^b \int_a^b K_n(x,t)\, K_n(t,x)\, dt\, dx$$

$$= \int_a^b \int_a^b K_n(x,t)\, \overline{K_n(x,t)}\, dt\, dx = \int_a^b \int_a^b |K_n(x,t)|^2\, dx\, dt$$

$$= \| \mathcal{K}^n \|^2 \tag{4.4}$$

from which it follows that $a_{2n} \geq 0$.

We also have

$$a_{2n}^2 = \{\text{trace}\,(\mathcal{K}^{n-1}\mathcal{K}^{n+1})\}^2$$

$$= \left\{\int_a^b\int_a^b K_{n-1}(x,t)\,K_{n+1}(t,x)\,dt\,dx\right\}^2$$

$$\leq \left\{\int_a^b\int_a^b |K_{n-1}(x,t)|^2\,dt\,dx\right\}\left\{\int_a^b\int_a^b |K_{n+1}(x,t)|^2\,dt\,dx\right\},$$

[using Cauchy-Schwarz's inequality]

$$= \|\mathcal{K}^{n-1}\|^2\,\|\mathcal{K}^{n+1}\|^2 = a_{2n-2}a_{2n+2} \qquad (n \geq 2) \tag{4.5}$$

Now since $\mathcal{K}$ is non-null $\|\mathcal{K}\| > 0$ and so $a_2 > 0$. Again $a_4 > 0$; if not, $a_4 = 0$ implies that $K_2(x,t) = 0$ almost everywhere and in particular $K_2(t,t) = 0$ so that $a_2 = 0$, which is not the actual case. Thus from (4.5) $a_{2n} > 0$ for all n and

$$\frac{a_{2n+2}}{a_{2n}} \geq \frac{a_{2n}}{a_{2n-2}} \geq \cdots \geq \frac{a_4}{a_2} \tag{4.6}$$

Let us suppose that $\mathcal{K}$ has no eigenvalue. Then $d(\lambda)$ has no zeros and then (3.52') must hold for all λ. We shall show that this leads to a contradiction. The series in the R.H.S. of (3.52') is absolutely convergent for all sufficiently small λ, whence the series

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} |a_{2n}|\,|\lambda|^{2n-1} \tag{4.7}$$

is convergent for all sufficiently small λ. On the other hand, we observe that

$$\frac{u_{n+1}}{u_n} = |\lambda|^2\,\frac{a_{2n+2}}{a_{2n}} \geq |\lambda|^2\,\frac{a_4}{a_2}$$

which shows that the series (4.7) is divergent when $|\lambda| > \sqrt{(a_2/a_4)}$, a contradiction. So the integral operator $\mathcal{K}$ possessing Hilbert-Schmidt kernel must have at least one eigenvalue whose absolute value does not exceed $\sqrt{(a_2/a_4)}$.

Theorem 4.2. Every eigenvalue of $\mathcal{K}$ is real.

Proof. Let $\phi = \lambda ¼ \phi$, where ϕ is a non-null eigenfunction associated with the eigenvalue λ. Then

$$\lambda(¼\phi,\phi) = (\lambda ¼\phi,\phi) = (\phi,\phi) \neq 0$$

so that $\lambda(¼\phi,\phi)$ is real and non-zero. Again since ¼ is Hermitian,

$$(¼\phi,\phi) = (\phi, ¼\phi) = \overline{(¼\phi,\phi)}$$

that is $(¼\phi,\phi)$ is real. The eigenvalue λ is therefore real.

Theorem 4.3. Eigenfunctions of ¼ corresponding to distinct eigenvalues are orthogonal to one another.

Proof. Let ϕ_1, ϕ_2 be two eigenfunctions of ¼ corresponding to eigenvalues λ_1 and λ_2 respectively. Then

$$(\phi_1,\phi_2) = (\lambda_1 ¼\phi_1,\phi_2) = \lambda_1(¼\phi_1,\phi_2)$$

and $$(\phi_1,\phi_2) = (\phi_1,\lambda_2 ¼\phi_2) = \lambda_2(¼\phi_1,\phi_2)$$

since ¼ is Hermitian and λ_2 is real. Therefore

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)(\phi_1,\phi_2) = 0$$

and as $\lambda_1 \neq \lambda_2$ it follows that $(\phi_1,\phi_2) = 0$.

Theorem 4.4. The eigenvalues of ¼ form a finite or enumerable sequence $\{\lambda_n\}$ with no finite limit point. If the eigenvalues are counted according to its rank in the sequence, then

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \leq \| ¼ \|^2 < \infty \tag{4.8}$$

Proof. We have proved in Theorem 4.1 that every non-null Hermitian kernel $K(x,t)$ possesses at least one eigenvalue λ_1 and its corresponding normalized eigen-function is ϕ_1, say.

Let us consider the Hermitian kernel

$$K^{(2)}(x,t) = K(x,t) - \frac{\phi_1(x)\overline{\phi_1(t)}}{\lambda_1}$$

If $K^{(2)}(x,t)$ is non-null, it will also possess at least one eigenvalue λ_2 with a corresponding normalized eigenfunction ϕ_2. Now one thing might happen— λ_1 equals to λ_2, but it is certain that ϕ_1 can not be an eigenfunction of the kernel $K^{(2)}(x,t)$, since

$$\int_a^b K^{(2)}(x,t)\,\phi_1(t)\,dt = \int_a^b K(x,t)\,\phi_1(t)\,dt - \frac{\phi_1(x)}{\lambda_1}\int_a^b |\phi_1(t)|^2\,dt = 0$$

Likewise, we consider the third kernel $K^{(3)}(x,t)$,

$$K^{(3)}(x,t) = K^{(2)}(x,t) - \frac{\phi_2(x)\,\overline{\phi_2(t)}}{\lambda_2}$$

$$= K(x,t) - \sum_{k=1}^{2}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}$$

and continuing in this way either we may arrive after n steps,

$$K^{(n+1)}(x,t) = K(x,t) - \sum_{k=1}^{n}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k} = 0$$

and obtain the *Bilinear formula*, $K(x,t) = \sum_{k=1}^{n}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}$. (4.9)

Otherwise the process will continue indefinitely and as a result, an infinite number of eigen-values λ_k and the associated eigenfunctions ϕ_k will occur.

Applying Bessel's inequality (1.29) to $K(x,t)$, we find that

$$\int_a^b |K(x,t)|^2\,dt \geq \sum_{k=1}^{n} \left|\int_a^b K(x,t)\,\phi_k(t)\,dt\right|^2$$

$$= \sum_{k=1}^{n}\frac{|\phi_k(x)|^2}{\lambda_k^2} \tag{4.10}$$

Integrating over (a,b) with respect to x it follows that

$$\| ¼ \|^2 \geq \sum_{k=1}^{n}\frac{1}{\lambda_k^2}$$

Proof. Let $\phi = \lambda \mathcal{K}\phi$, where ϕ is a non-null eigenfunction associated with the eigenvalue λ. Then

$$\lambda(\mathcal{K}\phi,\phi) = (\lambda\mathcal{K}\phi,\phi) = (\phi,\phi) \neq 0$$

so that $\lambda(\mathcal{K}\phi,\phi)$ is real and non-zero. Again since $\mathcal{K}$ is Hermitian,

$$(\mathcal{K}\phi,\phi) = (\phi,\mathcal{K}\phi) = \overline{(\mathcal{K}\phi,\phi)}$$

that is $(\mathcal{K}\phi,\phi)$ is real. The eigenvalue λ is therefore real.

Theorem 4.3. Eigenfunctions of $\mathcal{K}$ corresponding to distinct eigenvalues are orthogonal to one another.

Proof. Let ϕ_1, ϕ_2 be two eigenfunctions of $\mathcal{K}$ corresponding to eigenvalues λ_1 and λ_2 respectively. Then

$$(\phi_1,\phi_2) = (\lambda_1\mathcal{K}\phi_1,\phi_2) = \lambda_1(\mathcal{K}\phi_1,\phi_2)$$

and $$(\phi_1,\phi_2) = (\phi_1,\lambda_2\mathcal{K}\phi_2) = \lambda_2(\mathcal{K}\phi_1,\phi_2)$$

since $\mathcal{K}$ is Hermitian and λ_2 is real. Therefore

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)(\phi_1,\phi_2) = 0$$

and as $\lambda_1 \neq \lambda_2$ it follows that $(\phi_1,\phi_2) = 0$.

Theorem 4.4. The eigenvalues of $\mathcal{K}$ form a finite or enumerable sequence $\{\lambda_n\}$ with no finite limit point. If the eigenvalues are counted according to its rank in the sequence, then

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} \leq \|\mathcal{K}\|^2 < \infty \tag{4.8}$$

Proof. We have proved in Theorem 4.1 that every non-null Hermitian kernel $K(x,t)$ possesses at least one eigenvalue λ_1 and its corresponding normalized eigenfunction is ϕ_1, say.

Let us consider the Hermitian kernel

$$K^{(2)}(x,t) = K(x,t) - \frac{\phi_1(x)\overline{\phi_1(t)}}{\lambda_1}$$

If $K^{(2)}(x,t)$ is non-null, it will also possess at least one eigenvalue λ_2 with a corresponding normalized eigenfunction ϕ_2. Now one thing might happen — λ_1 equals to λ_2, but it is certain that ϕ_1 can not be an eigenfunction of the kernel $K^{(2)}(x,t)$, since

$$\int_a^b K^{(2)}(x,t)\,\phi_1(t)\,dt = \int_a^b K(x,t)\,\phi_1(t)\,dt - \frac{\phi_1(x)}{\lambda_1}\int_a^b |\phi_1(t)|^2\,dt = 0$$

Likewise, we consider the third kernel $K^{(3)}(x,t)$,

$$K^{(3)}(x,t) = K^{(2)}(x,t) - \frac{\phi_2(x)\,\overline{\phi_2(t)}}{\lambda_2}$$

$$= K(x,t) - \sum_{k=1}^{2}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}$$

and continuing in this way either we may arrive after n steps,

$$K^{(n+1)}(x,t) = K(x,t) - \sum_{k=1}^{n}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k} = 0$$

and obtain the *Bilinear formula*, $K(x,t) = \sum_{k=1}^{n}\frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}$. (4.9)

Otherwise the process will continue indefinitely and as a result, an infinite number of eigen-values λ_k and the associated eigenfunctions ϕ_k will occur.

Applying Bessel's inequality (1.29) to $K(x,t)$, we find that

$$\int_a^b |K(x,t)|^2\,dt \geq \sum_{k=1}^{n} \left|\int_a^b K(x,t)\,\phi_k(t)\,dt\right|^2$$

$$= \sum_{k=1}^{n}\frac{|\phi_k(x)|^2}{\lambda_k^2} \tag{4.10}$$

Integrating over (a,b) with respect to x it follows that

$$\| ¼ \|^2 \geq \sum_{k=1}^{n}\frac{1}{\lambda_k^2}$$

Hence, if $|\lambda_k| \leq \gamma (1 \leq k \leq n)$, we shall have

$$\| ¼ \|^2 \geq \sum_{k=1}^{n} \frac{1}{\gamma^2} = \frac{n}{\gamma^2}$$

i.e., $n \leq \gamma^2 \| ¼ \|^2$ and thus there are only a finite number of eigenvalues in the interval $(-\gamma, \gamma)$. Therefore the system possesses at most enumerable sequence $\{\lambda_n\}$ and the inequality (4.8) follows at once.

In the discussions we shall suppose that the sequence of eigenvalues $\{\lambda_n\}$ is such that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots$$

and each eigenvalue appears in the sequence a number of times equal to its rank. We call $\phi_1(x), \phi_2(x), \ldots,$ the eigen functions corresponding to $\lambda_1, \lambda_2, \ldots$ respectively and they form a complete orthonormal system $\{\phi_n\}$, which is not necessarily complete.

Expansion Theorems

The *Bilinear formula* given in (4.9) which is valid for a finite number of eigenvalues cannot be extended to the general case *i.e.*, for infinite number of eigen-values as the corresponding infinite series for the sum in R.H.S. of (4.9) may not be convergent. However, we can have the following theorem.

Theorem 4.5. Given $\{\phi_k\}$ an orthonormal system of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$ of the Hilbert-Schmidt kernel $K(x,t)$, then the series $\sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k}$ converges in the mean to the kernel $K(x,t)$ in the sense that

$$\lim_{n\to\infty} \int_a^b \int_a^b \left| K(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k} \right|^2 dx\,dt = 0 \tag{4.11}$$

holds.

Proof. Applying Riesz form of the Riesz-Fishcher theorem we can have a L^2 Hermitian kernel (Hilbert-Schmidt kernel) $Q(x,t)$ such that

$$\lim_{n\to\infty} \int_a^b \left| Q(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k} \right|^2 dt = 0 \tag{4.12}$$

and $$\int_a^b Q(x,t)\,\phi_k(t)\,dt = \frac{\phi_k(x)}{\lambda_k} \quad (k=1,2,\ldots) \tag{4.13}$$

provided $$\sum_{k=1}^{\infty} \left|\frac{\phi_k(x)}{\lambda_k}\right|^2 \leq \infty \tag{4.14}$$

holds.

We see that Q exists satisfying (4.12) and (4.13) since (4.14) holds following (4.10).

We set $H(x, t) = K(x, t) - Q(x, t)$ (4.15)
and it is clear that

$$\int_a^b H(x,t)\,\phi_k(t)\,dt = 0 \quad (k=1,2,\ldots) \tag{4.16}$$

since $Q(x, t)$ and $K(x, t)$ have the same Fourier coefficient.

To establish (4.11) we prove that $H(x,t)$ is null. Let us suppose that, on the contrary, $H(x,t)$ is non-null and hence by Theorem. 4.1. $Ç$ possesses an eigenvalue λ_0 and a corresponding eigenfunction ϕ_0. So we have

$$\phi_0 = \lambda_0\, Ç\phi_0$$

which gives

$$(\phi_0, \phi_k) = \lambda_0\,(Ç\phi_0, \phi_k) = \lambda_0\,(\phi_0, Ç\phi_k) = 0 \quad (k=1,2,\ldots) \tag{4.17}$$

using (4.16). Now we can express

$$\int_a^b Q(x,t)\,\phi_0'(t)\,dt = \int_a^b \{Q(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}\}\,\phi_0(t)\,dt$$

by (4.17). So given any $\varepsilon > 0$ there corresponds an integer N such that for $n > N$,

$$|Đ\phi_0|^2 \leq \int_a^b |\,Q(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k}|^2\,dt < \varepsilon$$

by using Cauchy-Schwarz inequality and (4.12). Now since $Đ\phi_0$ is independent of n it follows that $Đ\phi_0 = 0$ and thus

$$\lambda_0\, ¼\phi_0 = \lambda_0\, Ç\phi_0 = \phi_0$$

i.e., ϕ_0 is equivalent to an eigenfunction of ¼ and as a result following (4.17) it has to present in the full orthonormal system of eigenfunctions of ¼ which is not possible. Hence $H(x, t)$ must be a null kernel and that

$$Q(x,t) = K(x,t)$$

and (4.11) is established.

The Hilbert-Schmidt theorem

If $$f(x) = \int_a^b K(x,t) g(t)\, dt \tag{4.18}$$

where $K(x,t)$ is a Hilbert-Schmidt kernel and $g(t)$ is an L^2 function then $f(x)$ can be represented in terms of the orthonormal system $\{\phi_k\}$ of eigenfunctions of $K(x,t)$ as

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x) \tag{4.19}$$

where $$a_k = (f, \phi_k). \tag{4.20}$$

Moreover if $\int_a^b |K(x,t)|^2\, dt = A^2(x) \leq N^2$ ($N \equiv$ constant) $\forall\, x \in [a,b]$ (4.21)

then the infinite series (4.19) converges absolutely and uniformly for every $f(x)$ given by (4.18).

Proof. We observe that

$$a_k = (f, \phi_k) = \int_a^b \overline{\phi_k(x)} \{\int_a^b K(x,t)\, g(t)\, dt\}\, dx$$

$$= \int_a^b g(t) \{\int_a^b \overline{K(t,x)}\ \overline{\phi_k(x)}\, dx\}\, dt$$

$$= g(g, ¼\phi_k) = \frac{1}{\lambda_k}(g, \phi_k) \tag{4.22}$$

To prove the first part of the theorem, we express

$$f(x) = \int_a^b K(x,t) g(t)\, dt$$

$$= \int_a^b \{K(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k}\} g(t)\,dt + \sum_{k=1}^{n} \frac{\phi_k(x)}{\lambda_k} \int_a^b \overline{\phi_k(t)}\, g(t)\,dt$$

$$= \int_a^b \{K(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k}\} g(t)\,dt + \sum_{k=1}^{n} a_k \phi_k(x), \text{ [using (4.22)]}$$

and hence applying Cauchy-Schwarz's inequality, we get

$$| f(x) - \sum_{k=1}^{n} a_k \phi_k(x) |^2 \le \int_a^b | K(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k} |^2 \,dt \int_a^b | g(t) |^2 \,dt$$

and using Theorem 4.5, we find that

$$\lim_{n\to\infty} | f(x) - \sum_{k=1}^{n} a_k \phi_k(x) |^2 \le \int_a^b |g(t)|^2 \,dt \lim_{n\to\infty} \int_a^b | K(x,t) - \sum_{k=1}^{n} \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k} |^2 \,dt$$

$$= 0 \qquad (4.23)$$

Now setting $b_k = (g, \phi_k)$ and using Cauchy-Schwarz's inequality for sums, we have

$$\{ \sum_{k=n+1}^{\infty} | a_k \phi_k(x) | \}^2 = \{ \sum_{k=n+1}^{\infty} \left| b_k \frac{\phi_k(x)}{\lambda_k} \right| \}^2$$

$$\le \{ \sum_{k=n+1}^{\infty} | b_k |^2 \} \{ \sum_{k=n+1}^{\infty} \frac{| \phi_k(x) |^2}{\lambda_k} \} \qquad (4.24)$$

Again since $g(x)$ is square-integrable, the series $\sum_{k=1}^{\infty} | b_k |^2$ is convergent and hence, given any $\varepsilon > 0$ there exists an integer N such that

$$\sum_{k=n+1}^{\infty} | b_k |^2 < \varepsilon^2 \quad (n > N)$$

Also (4.10) with the condition (4.21) gives

$$\sum_{k=n+1}^{\infty} \frac{| \phi_k(x) |^2}{\lambda_k} \le \int_a^b | K(x,t) |^2 \,dt = A^2(x) \le N^2$$

Thus from (4.24).

$$\sum_{k=n+1}^{\infty} |a_k\phi_k(x)| < \varepsilon N$$

This shows that the series (4.19) converges absolutely and uniformly.

Hilbert's formula :

If $g(x)$ and $h(x)$ are both square-integrable then

$$(\not{k}g, h) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k}(g, \phi_k)\,(\phi_k, h) \tag{4.25}$$

Proof. This formula follows from the Hilbert-Schmidt theorem as a corollary replacing f by $\not{k}g$ in the inner product (f, h) and the fact that the series on the R.H.S. of (4.19) is uniformly convergent.

For $h = g$ *Hilbert's formula* takes the form

$$(\not{k}g, g) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} | g, \phi_k |^2 \tag{4.26}$$

Applications of Hilbert-Schmidt Theorem :

(i) *Expansion of iterated kernels*

Let us consider iterated kernels. By definition (2.25)

$$K_m(x,t) = \int_a^b K(x,s) K_{m-1}(s,t)\, ds \quad (m \geq 2)$$

and this equation will be of the form (4.18) if g is replaced by K_{m-1} and hence it follows from the Hilbert-Schmidt theorem that if $K(x,t)$ is Hilbert-Schmidt kernel then

$$K_m(x,t) = \sum_{k=1}^{\infty} a_k(t)\, \phi_k(x)$$

where $\quad a_k(t) = \int_a^b K_m(x,t)\, \overline{\phi_k(x)}\, dx \; = \; \frac{1}{\lambda_k^m} \overline{\phi_k(t)} \;\; (k = 1, 2, \ldots)$

that is, $\quad K_m(x,t) \; = \; \sum_{k=1}^{\infty} \frac{\phi_k(x)\, \overline{\phi_k(t)}}{\lambda_k^m} \; (m \geq 2) \qquad (4.27)$

where the series is absolutely and uniformly convergent if the kernel $K(x,t)$ obeys the condition (4.21).

In particular if $x = t$ in (4.27), we obtain

$$a_m = \text{trace}(\mathcal{K}^m) = \int_a^b K_n(x,x)\,dx$$

$$= \sum_{k=1}^{\infty} \frac{1}{\lambda_k^m} \quad (m \geq 2) \tag{4.28}$$

(ii) *Expansion of the resolvent kernel*

We consider the Fredholm equation of the second kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt \tag{4.29}$$

where

(i) $K(x,t)$ is a Hilbert-Schmidt kernel satisfying (4.21).

(ii) λ is a regular value.

Here we observe that $\phi(x) - f(x)$ has an integral representation like (4.18) and hence by Hilbert-Schmidt theorem it can be expressed as an absolutely and uniformly convergent series

$$\phi(x) - f(x) = \sum_{k=1}^{\infty} C_k\,\phi_k(x)$$

where $\quad C_k = \phi - (f,\phi_k) = (\phi,\phi_k) - (f,\phi_k)$

Again using (4.29)

$$(\phi,\phi_k) - (f,\phi_k) = \lambda(K\phi,\,\phi_k) = \frac{\lambda}{\lambda_k}(\phi,\,\phi_k)$$

i.e., $(\phi,\phi_k) = \dfrac{\lambda_k}{\lambda_k - \lambda}(f,\phi_k)$. So, $C_k = \dfrac{\lambda}{\lambda_k - \lambda}(f,\phi_k)$

and thus

$$\phi(x) = f(x) + \lambda \sum_{k=1}^{\infty} \frac{(f,\phi_k)\,\phi_k(x)}{\lambda_k - \lambda}$$

$$= f(x) + \lambda \sum_{k=1}^{\infty} \int_a^b \frac{\phi_k(x)\overline{\phi_k(t)}}{\lambda_k - \lambda} f(t)\,dt \tag{4.30}$$

From this we see that the series

$$\sum_{k=1}^{\infty} \frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k - \lambda} \tag{4.31}$$

may be taken as the resolvent kernel $R(x,t;\lambda)$ provided the series is uniformly convergent in (a,b).

Again since $K(x,t)$ is a Hilbert-Schmidt kernel we have from (4.10)

$$\sum_{k=1}^{\infty} \frac{|\phi_k(x)|^2}{\lambda_k^2} < \infty$$

and also $\dfrac{\lambda_k}{\lambda_k - \lambda} \to 1$ as $k \to \infty$. Thus it follows that

$$\sum_{k=1}^{\infty} \frac{|\phi_k(x)|^2}{(\lambda_k - \lambda)^2} < \infty$$

Hence by the Riesz-Fischer theorem

$$\sum_{k=1}^{n} \frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k - \lambda}$$

converges in the mean to a square-integrable kernel $R(x,t;\lambda)$. Now considering the resolvent equation [see Eq. (1.21)]

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,s)\, R(t,s;\lambda)\, ds\,,$$

we obtain, using Hilbert-Schmidt theorem,

$$R(x,t;\lambda) = K(x,t) \;+\; \lambda \sum_{k=1}^{\infty} \frac{\phi_k(x)\,\overline{\phi_k(t)}}{\lambda_k(\lambda_k - \lambda)} \tag{4.32}$$

which is absolutely and uniformly convergent.

Example 4.1. Solve equation

$$\phi(x) = x \;+\; \lambda \int_0^1 K(x,t)\;\phi(t)\,dt$$

where $\lambda \neq n^2\pi^2$, $n = 1, 2, \ldots$ and

$$K(x,t)=\begin{cases} x(1-t), & 0\le x\le t\le 1 \\ t(1-x), & 0\le t\le x\le 1 \end{cases}$$

Solution. We consider the eigenvalue problem

$$\phi = \lambda \text{¼}\phi$$

where $\text{¼}\phi = \int_0^1 K(x,t)\,\phi(t)\,dt$.

It is clear that $K(x,t)$ is zero at $x = 0$ and $x = 1$ and so $\phi(0) = \phi(1) = 0$ for any solution to the eigenvalue problem. Now, we need to solve the eigenvalue problem– in this sake we turn the integral equation into ordinary differential equation by repeatedly differentiating the equation $\phi = \lambda\text{¼}\phi$ with respect to x.

$$\phi(x) = \lambda \int_0^1 K(x,t)\,\phi(t)\,dt$$

$$= \lambda \{\int_0^x (1-x)t\,\phi(t)\,dt + \int_x^1 x(1-t)\,\phi(t)\,dt\}$$

$$\frac{d\phi}{dx} = \lambda\{x(1-x)\,\phi(x) + \int_0^x (-t)\,\phi(t)\,dt - x(1-x)\,\phi(x) + \int_x^1 (1-t)\,\phi(t)\,dt\}$$

$$\frac{d^2\phi}{dx^2} = \lambda\{-x\,\phi(x) - (1-x)\,\phi(x)\}$$

which simplifies to

$$\phi'' + \lambda\phi = 0$$

with the boundary condition

$$\phi(0) = \phi(1) = 0$$

For negative λ, we only get the trivial solution $\phi(x) \equiv 0$

So write $\lambda = c^2 \ge 0$. This gives

$$\phi(x) = A\cos cx + B\sin cx$$

As $\phi(0)=0$, we get $A = 0$ and $\phi(1)=0$ gives us $B\sin c = 0$, so, $c = n\pi$.

Therefore the eigenvalues of the ordinary differential equation are $\lambda_n = n^2\pi^2$ and the corresponding eigenfunctions are $\phi_n(x) = B_n \sin n\pi x$. We choose B_n by requiring that $\| \phi_n \| = 1$ which gives us $B_n = \sqrt{2}$.

So for $\lambda \neq \lambda_n = n^2\pi^2$ we have a solution for any $f(x)$, which is x here. Thus using Hilbert-Schmidt theorem we find from (4.30) that

$$\phi(x) = x + \lambda \sum_{n=1}^{\infty} \frac{(t, \sqrt{2}\sin n\pi t)\sqrt{2}\sin n\pi x}{n^2\pi^2 - \lambda}$$

$$= x + \frac{2\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n(n^2\pi^2 - \lambda)}$$

The resolvent kernel according to (4.31) is

$$R(x,t;\lambda) = 2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi t}{n^2\pi^2 - \lambda}, \quad \lambda \neq n^2\pi^2$$

Example 4.2. Solve $\phi(x) = \cos 2x + \lambda \int_0^{\pi/2} K(x,t)\, u(t)\, dt$

with $$K(x,t) = \begin{cases} \sin x \cos t, & 0 \leq x \leq t \\ \sin t \cos x, & t \leq x \leq \frac{\pi}{2} \end{cases}$$

Proof. It is easy to check that $K(x,t)$ is symmetric and square integrable on $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$.

We need to solve the eigenvalue problem $\phi = \lambda ¼\phi$. So, as before we reduce $\phi = \lambda ¼\phi$ to an ordinary differential equation

$$\phi(x) = \lambda\left[\int_0^x \cos x \sin t\, \phi(t)\, dt + \int_x^{\pi/2} \sin x \cos t\, \phi(t)\, dt\right]$$

$$\frac{d\phi}{dx} = \lambda \left[\cos x \sin x \phi(x) + \int_0^x -\sin x \sin t\, \phi(t)\, dt - \sin x \cos x\, \phi(x) + \right.$$

$$\left. + \int_x^{\pi/2} \cos x \cos t\, \phi(t)\, dt \right]$$

$$\frac{d^2\phi}{dx^2} = \lambda\{-\sin^2 x\,\phi(x) - \int_0^x \cos x \sin t\,\phi(t)\,dt - \cos^2 x\,\phi(x) - \int_x^{\pi/2} \sin x \cos t\,\phi(t)\,dt\}$$

$$= -\ (\lambda+1)\,\phi$$

So our eigenvalue problem reduces to $\phi'' = (\lambda+1)\,\phi$. It is not hard to check that the definition of the kernel implies that $\phi(0) = \phi\left(\frac{\pi}{2}\right) = 0$. As usual there is no non-trivial solution for $\lambda + 1 < 0$ or equivalently for $\lambda < -1$. So we take $1 + \lambda = c^2 \geq 0$ and $\phi(x) = A\cos cx + B\sin cx$. Applying the boundary conditions, we get $A = 0$ and $B\sin c\,\pi/2 = 0$, so $c = 2n$.

We therefore have $\lambda_n = 4n^2 - 1$ and $\phi_n = B_n \sin 2nx$. If we take $\|\phi_n\|^2 = 1$ we find that $B_n = \frac{2}{\sqrt{\pi}}$ and so $\phi_n = \frac{2}{\sqrt{\pi}}\sin 2nx$.

Here $\lambda_n = 4n^2 - 1$ is always odd. So if for example we take $\lambda = 4$ we expect a unique solution for any $f(x)$. Here $f(x) = \cos 2x$.

So for $\lambda = 4$,

$$\phi(x) = \cos 2x + 4\sum_{k=1}^{\infty} \frac{f_n \phi_n(x)}{\lambda_k - \lambda}$$

$$= \cos 2x + 4.\frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{f_n \sin 2nx}{4k^2 - 5}$$

where $$f_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \sin 2nx \cos 2x\,dx = \begin{cases} \frac{4}{\sqrt{\pi}} \frac{n}{n^2 - 1} & n \equiv \text{even} \\ 0 & n \equiv \text{odd} \end{cases}$$

Fredholm integral equation of the first kind

Here we shall see how the equation of the first kind

$$f(x) = \int_a^b K(x,t)\,\phi(t)\,dt \quad (a \leq x \leq b) \tag{4.33}$$

can be solved by using Hilbert-Schmidt theorem.

Let $\{\phi_k\}$ be the orthonormal system of eigenfunctions of $K(x,t)$, then following as in (4.22) we obtain

$$a_k = (f,\phi_k) = \frac{1}{\lambda_k}(\phi,\phi_k) \tag{4.34}$$

where λ_k 's are the eigenvalues of the kernel $K(x,t)$, (ϕ,ϕ_k) are the unknown Fourier coefficients of $\phi(x)$ with respect to the system $\{\phi_k(x)\}$ and (f,ϕ_k) are the corresponding coefficients of the function $f(x)$. From (4.34) it follows that

$$(\phi,\phi_k) = a_k\lambda_k$$

Then by the Riesz Fischer theorem [see Chapter I] there will exist at least one L^2-function $\phi_0(x)$ which satisfies the equation (4.33) and that

$$\sum_{k=1}^{n} a_k\ \lambda_k\ \phi_k(x)$$

converges in the mean to the function $\phi_0(x)$ provided the series

$$\sum_{k=1}^{\infty}|a_k\ \lambda_k|^2$$

converges. If this hold then $\phi_0(x)$ is unique when $K(x,t)$ is closed.

Example 4.3. Consider the equation

$$f(x) = \int_0^1 K(x,t)\,\phi(t)\,dt$$

where $K(x,t) = \begin{cases} x(1-t), & 0 \le x \le t \le 1 \\ t(1-x), & 0 \le t \le x \le 1 \end{cases}$

Solution. Here we can find as in example. 4.1 that the eigenvalues and eigenfunctions of

$$\phi = \lambda K\phi$$

are $\lambda_k = k^2\pi^2$, $\phi_k(x) = \sqrt{2}\sin k\pi x$ $(k = 1,2,...)$ and that

$$a_k = \sqrt{2}\int_0^1 f(x)\sin k\pi x\, dx$$

Thus the given equation has one and only one L^2-solution if the infinite series

$$\sum_{k=1}^{\infty} a_k^2 \lambda_k^2 = \pi^2 \sum_{k=1}^{\infty} k^2 a_k^2$$

converges.

EXERCISES

1. Solve the integral equation

$$\phi(x) = e^x + \lambda \int_0^1 k(x,t)\phi(t)dt$$

where $k(x,t)\begin{cases} \dfrac{\sinh x.\sinh(x-t)}{\sinh 1}, 0 \le x \le 1 \\ \dfrac{\sinh t.\sinh(x-1)}{\sinh 1}, x \le t \le 1 \end{cases}$

[Ans. $\phi(x) = e^x - 2\lambda\pi \sum \dfrac{n\{1-e(-1)^n\}\sin n\pi x}{(1+n^2\pi^2)(1+n^2\pi^2+\lambda)}$, $(\lambda < -1$

2. Find the eigenvalues and eigenfunctions of the homogeneons integral equations :

(i) $\phi(x) = \lambda \int_1^2 (xt + \dfrac{1}{xt})\,\phi(t)dt$

[Ans. λ_1 □ 16.639, λ_2 □ 0.3606 , eigenfunctions $x - 2.2732\dfrac{1}{x}$, $x + 0.4399.\dfrac{1}{x}$]

(ii) $\phi(x) = \lambda \int_0^{\pi} k(x)\phi(t)dt$

[Ans. $\lambda = \lambda_n = 1 - \left(n + \dfrac{1}{2}\right)^2$ $\phi(x) = A\cos\left(n + \dfrac{1}{2}\right)x$, A being arbitrary constant]

3. Using Hilbert-Schritt theorem, Solve the symmetric integral euqations :

(i) $\phi(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\phi(t)dt$

[Ans. $\phi(x) = 5x^2 + Ax + 1$, A being arbitrary constant.]

(ii) $\phi(x) = (x+1)^2 + \int_{-1}^{1}(xt + x^2t^2)\phi(t)dt$

[Ans. $\phi(x) = \frac{25}{9}x^2 + 6x + 1$, A being arbitrary constant.]

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