NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS

POST GRADUATE

PG (MT) : IX A (II)
(Applied Mathematics)

Operations Research

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in Subjects introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that it may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great part of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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NETAJI SUBHAS OPEN UNIVERSITY

PG (MT): IX A (II)

Operations Research

Unit 1 ❑ Classical Optimization Techniques

Structure

1.1 (Introduction)

The methods of determining relative extrema of functions of several variables using differential calculus are so old and well-known that they are referred to as classical. The classical methods of optimization are used in finding the optimum of continuous and differentiable functions. Since practical problems involve objective functions that are not continuous and/or differentiable, the classical optimization techniques have limited people of applications. But these classical techniques forms a basis for developing most of the numerical techniques of optimization.

In this unit we consider three types of problems viz

(i) Multivariable optimization with no constraints.

(ii) Multivariable optimization with equality constraints and

(iii) Multivariable optimization with inequality constraints

1.2 Multivariable optimization with no constraints

We develop the necessary and sufficient conditions for an n-variable functions f(x) to have extremt. It is assumed that the first and second partial derivatives of f(x) are continuous at every x.

Theorem 1.2.1 A necessary condition for x_0 to be an extreme point of $f(x)$ is that ∇ $f(x_0) = 0$ i.e. $\left[\frac{\partial t}{\partial x_i}\right]_{x0} = 0$ for i = 1, 2, ...n.

Proof : By Taylor's theorem we have

$$f(X_0 + h) = f(X_0) + \sum_{i=1}^{n} h_i \left[\frac{\partial t}{\partial x_i}\right]_{x0} + \frac{1}{\lfloor 2} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x0oh} ......(1)$$

where $0 < \theta < 1$.

Since the last term is of order h_j^2, the terms of order h will dominate the higher order terms for small h. Thus the sign of $f(X_0 + h) - f(X_0)$ is decided by the sign of $\sum_{i=1}^{n} h_i \left[\frac{\partial t}{\partial x_i}\right]_{x0}$. Let X_0 be are extreme point, say miximum point. Then $f(X_0 + h)$ $- f(X_0) > 0$ for all sufficiently small h. We are to show that $\left[\frac{\partial f}{\partial x_i}\right]_{x0} = 0$ $\forall$i = 1, 2, ..., n. If possible, let $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq 0$.

Let us choose $h_i = 0$ for all i $\neq$ k, and h_k sufficiently small. Then the sign of $f(X_0 + h) - f(X_0)$ is decided by the sign of $h_k\left[\frac{\partial f}{\partial x_k}\right]_{x0}$. Since $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq = 0$, let $\left[\frac{\partial f}{\partial x_k}\right]_{x0} > 0$. Then $f(X_0 + h) - f(X_0)$ will be positive for $h_k > 0$ and negative for $h_k < 0$. This is a contradiction as x_0 is a minimum point. Similar contradiction occurs for $\left[\frac{\partial f}{\partial x_k}\right]_{x0} < 0$. Hence $\left[\frac{\partial f}{\partial x_k}\right]_{x0} \neq 0$ is not possible. $\therefore$ $\left[\frac{\partial f}{\partial x_k}\right]_{x0} = 0$. This is true for any k = 1, 2,... n. Hence the theorem.

Theorem 1.2.2 A sufficient condition for a stationery point x_0 to be an extremum is that

(i) $\nabla f(X_0) = 0$ and the Hessian matrix $[H]_{x0}$ is positive definite when x_0 is a minumum point.

(ii) $\nabla f(X_0) = 0$ and the Hessian matrix $[H]_{x_0}$ is negative definite when x_0 is a maximum point.

Prob : By Tayloris theorem we have

$$f(X_0 + h) = f(X_0) + \sum_{i=1}^{n} h_i \left[\frac{\partial f}{\partial x_i}\right]_{x_0} + \frac{1}{\underline{|2}} \sum_{j=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0 + oh}$$

Where $0 < \theta < 1$.

$$f(X_0 + h) - f(X_0) = Q(x_0 + oh)$$

$$\text{Where } Q\ (x_0 + oh) = \frac{1}{\underline{|2}} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0 + oh}$$

Now we have assumed that the second order partial derivative $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is continuous in the neighbourhord of x_0. So far sufficiently small h, the signs of $Q(x_0 + oh)$ and $Q(x_0)$ are some. Hence $f(X_0 + h) - f(X_0)$ and $Q\ (x_0)$ have the same sign.

Let $J(X_0)$ be the Hessian matrix $\left[\left.\frac{\partial f}{\partial x_i \partial x_j}\right|_{x_0}\right]$. From matrix algebra we know that

$Q(X_0) = \frac{1}{\underline{|2}} \sum_{j=1}^{n} \sum_{j=1}^{n} h_i h_j \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{x_0}$ will be positive (negative) for al h if and only if the Hessian matrix $J(X_0)$ is positive definite (negative definite) at $X = X_0$.

Thus for sufficiently small h, the sign of $f(X_0 + h) - f(X_0)$ is positive (negative) if $J(X_0)$ is positive definite (negative definite) i.e., X_0 is a relative minimum (maximum) ifJ (X_0) is positive definite (negative definite). Hence the theorem.

Result : Let $A = [a_{ij}]_{n \times n}$ and

$$A_1 = a_{11},\ A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{vmatrix},\ A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then the matrix A is

(i) positive definite iff $A_i > 0$ for all i = 1,2, ...n

(ii) negative definite iff the sign of Ai is (–i)i for i = 1,2, ...n.

(iii) positive semidefinite iff $A_i \geq 0$ for all i = 1, 2, ...n wih equality holding for at least one i

(iv) negative semidefinite iff Ai ≤ o for all i = 1,2, ... n with equality holding for at least one i

(v) indefinite if it is neither definite nor semidefinite.

Example 1.2.1 Determine the extreme points of the function

$$f(x_1, x_2) = x_1^3 + x_2^3 + 4x_x^2 + 2x_2^2 + 12$$

Solution :

$$\text{Here } \frac{\partial f}{\partial x_1} = 3x_1^2 + 8x_1, \ \frac{\partial f}{\partial x_2} = 3x_2^2 + 4x_2$$

The necessary condition for the existence of an extreme points gives

$$x_1 \ (3x_1 + 8) = 0 \text{ and } x_2 \ (3x_2 + 4) = 0$$

The solutions are (0, 0) (0, –4/3), (–8/3, 0< (–8/3, –4/3). The Hessian matrix of $f(x_1, x_2)$ is given by

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial^2 f}{dx_1^2} & \frac{\partial^2 f}{dx_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 6x_1 + 8 & 0 \\ 0 & 6x_2 + 4 \end{vmatrix}$$

$$\therefore \ J_1 = 6x_1 + 8$$

$$\text{and } J_2 = \begin{vmatrix} 6x_1 + 8 & 0 \\ 0 & 6x_2 + 4 \end{vmatrix} = (6x_1 + 8)\ (6x_2 + 4)$$

For the point, (0, 0) we have

$J_1 = 6.0 + 8 = 8 > 0$ and $J_2 = (6.0 + 8)\ (6.0 + 4) = 32 > 0$

$\therefore$ J is positive definite. Hence (0, 0) is a relative minimum point of $f(x_1, x_2)$

For the point (0, –4/3) we have

$J_1 = 6.0 + 8 = 8 > 0$ and $J_2 = (6.0 + 8)\ (-6.4/3 + 4) = -32 < 0$

$\therefore$ J is indefinite. Hence (0, –4.3) is a saddle point of $f(x_1, x_2)$.

For the point (–8/3, 0) we have

$J_1 = -\ 6.8/3 + 8 = -8 < 0$ and $J_2 = (-6.8/3 + 8)\ (6.0 + 4) = -32 < 0.$

$\therefore$ J is indefinite. Hence (–8/3, 0) is a paddle point of $f(x_1, x_2)$.

For the point (–8/3, –4/3) we have

$J_1 = -6.8/3 + 8 = -8 < 0$ and $J_2 = (-6.8/3 + 8)(-6.4/3 + 4) = 32 > 0$

$\therefore$ J is negative definite. Hence (–8/3, –4/3) is a relative maximum point of $f(x_1, x_2)$.

1.3 Multivariable optimization with equality constraints

We shall consider two methods viz

(i) Method of constrained variation and

(ii) Method of Lagrange multipliers.

The general multivariable optimization problem with equality constraints is

Minimize $f = f(X)$

subject to $g_i\ (X) = 0,\ i = 1, 2, ..., m$

Where $X = [x_1, x_2, ...xn]^T$, $(m < n)$

1.3.1 Method of constrained variation

To understand the salient features of the method we consider the simple problem

Minimize $f(x_1, x_2)$

subject to $g(x_1, x_2) = 0$

Let us assume that $g\ (x_1, x_2) = 0$ can be solved to obtain x_2 as $x_2 = h\ (x_1)$. Then the problem reduces to the unconstrained minimization problem

Minimize $f\ (x_1, h(x_1))$

The necessary condition gives

$$\frac{df}{dx_1} = 0$$

or, $$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial h}\,\frac{dh}{dx_1}=0$$

or, $$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}\,\frac{dx_2}{dx_1}=0$$

or, $$\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2=0\text{.............(1)}$$

Let (x_1^*, x_2^*) be the minimum point. Then (x_1^*, x_2^*) must satisfy the given constraint.

$$\therefore\quad (x_1^*, x_2^*) = 0 \text{ .... (2)}$$

For admissible variations dx_1, dx_2 we have $g\,(x_1^*+dx_1, x_2^*+dx_2)=0$

Using Taylor's theorem we get

$$g(x_1^*, x_2^*)+\left[\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)}dx_1+\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_2 = 0$$

or, $$\left[\frac{\partial g}{\partial x_1}\right]_{(x_1^*, x_2^*)}dx_1+\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_2=0 \quad \text{[by (2)]}$$

Assuming $\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)} \neq 0$ we get,

$$dx_2 = -\frac{\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}}{\left[\frac{\partial g}{\partial x_2}\right]_{(x_1^*, x_2^*)}}dx_1 \text{ ....(3)}$$

Thus the admissible variation dx_2 depends on dx1and dx_1 can be chosen arlitravity.

Using (3) in (1) we have for admissible unviations

$$\left[\frac{\partial f}{\partial x_1}-\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}}\cdot\frac{\partial f}{\partial x_2}\right]_{(x_1^*, x_2^*)}dx_1 = 0$$

Since dx_1 is albitrary we have

$$\left[\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2}-\frac{\partial g}{\partial x_1}\cdot\frac{\partial f}{\partial x_2}\right]_{(x_1^*,x_2^*)} = 0$$

This is the necessary condition for (x_1^*, x_2^*) to be an extreme point.

Result : The solution of the problem

Minimize $f(x_1, x_2)$

subject to $g(x_1, x_2) = 0$

is obtained by sloving

$$\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2}-\frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}=0$$

and $g(x_1, x_2) = 0$

The above result can be generalized for general problem in the following theorem.

Theorem 1.3.1. Necessary conditions for $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ to be an extreme point of the function $f(x_1, x_2, \ldots\ldots, x_n)$ to exist under the m equality constraints $g_j(x_1, x_2, \ldots\ldots x_n) = 0$, $j = 1, 2 \ldots\ldots, m$ $(m < n)$ are the following $(n - m)$ equations ar satisfied at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$.

$$J\left(\frac{f, g_1, g_2, \ldots\ldots, g_m}{x_1, x_2, \ldots\ldots, x_m}\right)=\begin{vmatrix}\frac{\partial f}{\partial x_k} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_m}\\ \frac{\partial g_1}{\partial x_k} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \frac{\partial g_m}{\partial x_k} & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m}\end{vmatrix} = 0$$

$k = m + 1, m + 2, \ldots\ldots n$

$$\text{Where } J\left(\frac{g_1, g_2, \ldots\ldots, g_m}{x_1, x_2, \ldots\ldots, x_m}\right)=\begin{vmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_m}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m}\end{vmatrix} \neq 0$$

Note : In the above theorem $x_{m+1}, x_{m+2}, \ldots\ldots, x_n$ are independent variable. Also we note that the dependent variable, $x_1, x_2, \ldots\ldots x_m$ must satisfy J $\left(\frac{g_1, g_2, \ldots\ldots g_m}{x_1, x_2, \ldots, x_m}\right) \neq 0$

Example 1.3.1 Using method of constrained variation

Minimize $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2$

subject to $2x_1 + 4x_2 + 3x_3 + 9$

$4x_1 + 8x_2 + 5x_3 + 17$

Solution.

We are b minimize

$$f = x_1^2 + 2x_2^2 + x_3^2$$

subject to $g_1 = 2x_1 + 4x_2 + 3x_3 - 9 = 0$ ....(1)

$g_2 = 4x_1 + 8x_2 + 5x_3 - 17 = 0$ ....(2)

We are first to select independent and dependent variable.

Let us consider

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$$

Thus x_3 cannot be chosen as independent variables.

Let us now conisder

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0$$

Thus x_2 cannot be chosen as independent variables.

The necessary condition is

$$J\left(\frac{f, g_1, g_2}{x_2, x_1, x_3}\right) = 0$$

or, $$\begin{vmatrix} \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 4x_2 & 2x_1 & 2x_3 \\ 4 & 2 & 3 \\ 8 & 4 & 5 \end{vmatrix} = 0$$

or, $4x_2\ (10 - 12) + 2x_1\ (24 - 20) + 2x_3\ (16 - 16) = 0$

or, $-8x_2 + 8x_1 + 0 = 0$

or, $x_2 = x_1$..........(3)

Using (3) in (1) & (2) we get respectively

$6x_1 + 3x_3 - 9 = 0$

and $12x_1 + 5x_3 - 17 = 0$

$\therefore \quad x_1 = \frac{-15+45}{30-36} = 1$

$x_3 = \frac{-108+102}{30-36} = 1$

From (3) we have $x_2 = 1$

Hence the required solution is $x_1 = 1,\ x_2 = 1,\ x_3 = 1$.

1.3.2 Method of Lagrange multipliers

In the Lagrange miltiplier method are additional variable is introduced to the problem for each constraint. If the original problem has n variables and m equality constraints then we are to add m additional variables to the problem so that the final number of unknowns becomes n + m.

We now state the famous theorems of Lagrange.

Theorem 1.3.2 A necessary condition for a function $f(x_1, x_2, ..., x_n)$ subject to the constraints $g_j\ (x_1, x_2, ..., x_n) = 0$, j = 1, 2, ..., m to have a relative minimum at a point $(x_1^*, x_2^*, ... x_n^*)$ is that the first partial derivatives of the Lagrange function

$L = (x_1, x_2, \ldots\ldots, x_n, \lambda_1, \lambda_2, \ldots\ldots \lambda_n) = f + \sum_{j=1} \lambda_j g_j$ with respect to each of its arguments must be zero.

The sufficient condition for a function subject to equality constranits is given in the following theorem.

Theorem 1.3.3 A sufficient condition for a function $f(x_1, x_2, \ldots, x_n)$ subject to the constraints $gj(x, x_2, \ldots\ldots, x_n) = 0$, $j = 1, 2, \ldots\ldots, m$ to have a relative minimum (maximum) at a point $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ is that the quadratic Q, defined by

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j \ \ldots\ldots\ldots\ldots (\ )$$ evalualed at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ must be positive (negative) definite for all choice of admissible variations dx_i.

Theorem (Hanock) 1.3.4 A necessary condition for the quadratic form $Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 L}{\partial x_i \partial x_j} dx_i dx_j$; evaluated at $(x_1^*, x_2^*, \ldots\ldots x_n^*)$ to be positive (negative) definite for all admissible variations dx_i is that each root of the polynomial defined by the following determinantal equation, be positive (negative) :

$$\begin{vmatrix} (L_{11} - Z) & L_{12} & \cdots & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & \cdots & g_{m1} \\ L_{21} & (L_{22} - Z) & \cdots & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & \cdots & g_{m2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{n1} & L_{n2} & \cdots & \cdots & (L_{nn} - Z) & g_{1m} & g_{2m} & \cdots & \cdots & g_{mm} \\ g_{11} & g_{12} & \cdots & \cdots & g_{1m} & 0 & 0 & \cdots & \cdots & 0 \\ g_{21} & g_{22} & \cdots & \cdots & g_{2m} & 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{m1} & g_{m2} & \cdots & \cdots & g_{nm} & 0 & 0 & \cdots & \cdots & 0 \end{vmatrix}$$

Where $L_{ij} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{X^*}$

and $g_{ij} = \left[\frac{\partial g_i}{\partial x_j}\right]_{X^*}$, $X^* = (x_1^*, x_2^*, \ldots\ldots x_n^*)$

Result : If some of the roots of the above determinantal equation are positive and some are negative then the point x^* is not an extreme point.

Example 1.3.2 : Using Lagrange multiplier method minimize the function

$f(x_1, x_2, x_3) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$ subjectto the constrain $x_1 + x_2 + 2x_3 = 3$

Solution. Hope $f = 9 - 8x_1 - 6x_2 + 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$

$$g = x_1 - x_2 + 2x_3 - 3 = 0$$

The Lagrange functionis given by

$L(x_1, x_2, x_3, \lambda) = f + \lambda g$

$= (9 - 8x_1 - 6x_2 + 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3) + \lambda\ (x_1, x_2, 2x_3 - 3)$

The necessary condition are

$$\frac{\partial L}{\partial x_1} = 0 \qquad \text{or.} \quad -8 + 4x_1 + 2x_2 + 2x_3 + \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \qquad \text{or,} \quad -6 + 4x_2 + 2x_1 + \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \qquad \text{or,} \quad -4 + 2x_3 + 2x_1 + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \qquad \text{or,} \quad x_1 + x_2 + 2x_3 - 3 = 0$$

Solving these four equations we have

$$x_1^* = 4/3,\ x_2^* = 7/9,\ x_3^* = 4/9 \text{ and } \lambda^* = 2/9$$

We now use sufficient condition to identify this extreme point.

We evaluate L_{ij} and g_{ij} at the point $(4/3, 7/9, 4/9) = X^*$

$$L_{11} = \left[\frac{\partial^2 L}{\partial x_1^2}\right]_{X^*} = 4$$

$$L_{12} = L_{21} = \left[\frac{\partial^2 L}{\partial x_1 x_2}\right]_{X^*} = 2$$

$$L_{13} = L_{31} = \left[\frac{\partial^2 L}{\partial x_1 x_3}\right]_{X^*} = 2$$

$$L_{22} = \left[\frac{\partial^2 L}{\partial x_2^2}\right]_{X^*} = 4$$

$$L_{23} = L_{32} = \left[\frac{\partial^2 L}{\partial x_2 x_3}\right]_{X^*} = 0$$

$$L_{33} = \left[\frac{\partial^2 L}{\partial x_3^2}\right]_{X^*} = 2$$

$$g_{11} = \left[\frac{\partial g}{\partial x_1}\right]_{X^*} = 1$$

$$g_{12} = \left[\frac{\partial g}{\partial x_2}\right]_{X^*} = 1$$

$$g_{13} = \left[\frac{\partial g}{\partial x_3}\right]_{X^*} = 2$$

We now consider the determinautor equation

$$\begin{vmatrix} L_{11} - z & L_{12} & L_{13} & g_{11} \\ L_{21} & L_{22} - z & L_{23} & g_{12} \\ L_{31} & L_{32} & L_{33} - z & g_{13} \\ g_{11} & g_{12} & g_{13} & 0 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 4-z & 2 & 2 & 1 \\ 2 & 4-z & 0 & 1 \\ 2 & 0 & 2-z & 2 \\ 1 & 1 & 2 & 0 \end{vmatrix} = 0$$

or, $$-1\begin{vmatrix} 2 & 2 & 1 \\ 4-z & 0 & 1 \\ 0 & 2-z & 2 \end{vmatrix} + 1\begin{vmatrix} 4-z & 2 & 2 \\ 2 & 0 & 2-z \\ 1 & 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4-z & 2 & 2 \\ 2 & 4-z & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

or, $z^2 - 6z + 9 = 0$

or, $z = 3, 3$

Since the roots are all positive, (4/3, 7/9, 4/9) is a relative minimum of the function.

1.4 Multivariable optimization with inequality constraints

The general multivariable optimization problem with inequality constraints is

Minimize $f = f(X)$

subject to $g_i(x) \le b_j \quad j = 1, 2, ..., m$

where $X = [x_1, x_2, ... x_n]^T$.

This section is concerned with developing the necessary and sufficient conditions for identifying the stationery points of the above problem. These conditions are called Kuhn-Tucker conditions and the development is mainly based on Lagrangian method.

Theorem 1.4.1 (Kuhn-Tucker Necessary Conditions)

Given the problem to minimize

$$f = f(x) = f(x_1, x_2, ..., x_n)$$

subject to gj $(X) = g_j(x_1, x_2, ..., x_n) \le b_j \quad i = 1, 2, ..., m$

the necessary conditions for X_0 to be a local minimum are that

(i) $\dfrac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \dfrac{\partial g_j}{\partial x_i} = 0, \quad i = 1, 2, ...n$

(ii) $\lambda_j [g_j(X) - b_j] = 0, \quad j = 1, 2, ...m$

(iii) $g_j(X) \le b_j, \quad j = 1, 2, ...m$

(iv) $\lambda_j \ge 0, \quad j = 1, 2, ..., m$

are satisfied at X_0.

Introducing slack variables the inequality constraints bcomes

$$g_j(X) + s_j^2 = b_j, \quad j = 1, 2, ..., m$$

or, $\quad g_j(X) + s_j^2 - b_j = 0, \quad j = 1, 2, ...m. \quad ....(1)$

In order to obtain all stationary points, we form the Lograngian function L given by

$$L\ (X,\ l,\ S) = f(X) + \sum_{j=1}^{m} \lambda_j \{g^j\ (X) + s_j^2 - b^j\}$$

Then the stationary points are obtained by polving the equations

$$\frac{\partial L}{\partial x_i} = 0,\ i = 1,\ 2,\ \ldots\ldots,\ n$$

$$\frac{\partial L}{\partial \lambda_j} = 0,\ j = 1,\ 2,\ \ldots\ldots,\ m$$

and $\frac{\partial L}{\partial s_j} = 0,\ j = 1,\ 2,\ \ldots\ldots,\ m$

i.e., $\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0,\quad i = 1,\ 2,\ \ldots..,n$ ....(2)

$g_j + s_j^2 - b_j = 0$ ....(3)

$2\lambda_j s_j = 0,$ ....(4)

Multiplying (4) by $\frac{1}{2}s$; we get,

$$\lambda_j s_j^2 = 0$$

Using (1) this gives

$$\lambda_j \{b_j - g_j(X)\} = 0$$

or, $\lambda_j \{g_j(X) - b_j\} = 0,\quad j = 1,\ 2,\ \ldots\ldots\ m$ ....(5)

From (5) we have when $\lambda_j \neq 0$ then $g_j\ (X) - b_j = 0$ or, $g_j\ (X) = b_j$

or, $\frac{\partial g_j}{\partial b_j} = 1$

Thus $\frac{\partial g_k}{\partial b_j} = s_{jk}$ where $s_{jk} = \begin{cases} 1\ for\ j = k \\ 0\ for\ j \neq k \end{cases}$

Using chain rule of differential calculus we have

$$s_{jk} = \frac{\partial g_k}{\partial b_j} = \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i}{\partial b_j}$$

Multiplying both sides by λk and summing over all values of k we get

$$\sum_{k=1}^{m} \lambda_k s_{jk} = \sum_{k=1}^{m} \lambda_k \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i}{\partial b_j} \right)$$

or, $$\lambda_j = \sum_{k=1}^{m} \lambda_* \left(\sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i} \frac{\partial x_i^*}{\partial b_j} \right) \qquad \text{............(6)}$$

Again $$\frac{\partial f}{\partial b_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial b_j} \qquad \text{............(7)}$$

Adding (6) and (7) we get

$$\frac{\partial f}{\partial b_j} + \lambda_j = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial x_i} \right] \frac{\partial x_i}{\partial b_j}$$

= 0 [using (2)]

or, $$\frac{\partial f}{\partial b_j} = -\lambda_j \qquad \text{...............(8)}$$

Thus when $\lambda_j \neq 0$ then we have $\lambda_j = -\frac{\partial f}{\partial b_j}$ ....(9)

We now show that $\lambda_j > 0$. If possible let $\lambda_j < 0$. Then from (9) we have $\frac{\partial f}{\partial b_j} > 0$

This implies that as b_j is increased, the objective function increases. Now as b_j optimal value of the objective function clearly cannot increase. This contradicts our assumption $\lambda_j > 0$. Thus at an optimal solution we have $\lambda_j > 0$ when $\lambda_j \neq 0$. Hence at the optimal solution we hour $\lambda j \geq 0$.

Note : For the problem

Maximize f = f (x_1, x_2, ..., x_n)

subject to g_j (x_1, x_2, ..., x_n) b_j, i = 1, 2, ... m

the Kuhn-Tucker necessary conditions for $(x_1^*, x_2^*, \ldots x_n^*)$ to be a local maximum are that

(i) $\frac{\partial f}{dx_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_i}{\partial x_i} = 0,\ i = 1, 2, \ldots, n$

(ii) $\lambda_j\ [g_j - b_j] = 0,\ j = 1, 2, \ldots, m$

(iii) $g_j \leq b_j,\ \ j = 1, 2, \ldots, m$

(iv) $\lambda_j\ 0,\ j = 1, 2, \ldots, m$

are satified at $(x_1^*, x_2^*, \ldots x_n^*)$

Sufficency of the Kuhn-Tucker conditions

The Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy certain conditions regarding convexity anc concairty. For maximization problem the objective function should be concave and solution space should be convex set.

For minimization problem the objective function should be convex and the solution space should be convex set.

Example 1.4.1. Solve using Kuhn-Tucker conditions

$$\text{Maximize } z = 5 + 8x_1 + 12x_2 - 4x_1^2 - 4x_2^2 - 4x_3^2$$

$$\text{subject to } x_1 + x_2 \leq 1$$

$$2x_1 + 3x_2 \leq 6$$

Here the constraints are

$$g_1 = x_1 + x_2 \leq 1$$

and $\quad g_2 = 2x_1 + 3x_2 \leq 6$

The Kuhn-Tucker necessary conditions are

$$\frac{\partial Z}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0,\ i = 1, 2, 3$$

$$\lambda_j\ [g_j - b_j] = 0,\ \ j = 1, 2$$

$$\lambda_j \leq 0,\ \ j = 1, 2$$

i,e., $\quad 8 - 8x_1 + \lambda_1 + 2\lambda_2 = 0 \qquad \ldots\ldots..(1)$

$12 - 8x_2 + \lambda_1 + 3\lambda_2 = 0$ ....(2)

$- 8x_1 = 0$ ....(3)

$\lambda_1 + (x_1 + x_2 - 1) = 0$ ....(4)

$\lambda_2 + (2x_1 + 3x_2 - 6) = 0$ ....(5)

$x_1 + x_2 - 1 = 0$ ....(6)

$2x_1 + 3x_2 - 6 \leq 0$ ....(7)

$\lambda_1 \leq 0$ ....(8)

$\lambda_2 \leq 0$ ....(9)

Four cases may arise.

case 1. $\lambda_1 = 0, \lambda_2 = 0$

case 2. $\lambda_1 = 0, \lambda_2 \neq 0$

case 3. $\lambda_1 \neq 0, \lambda_2 = 0$

case 4. $\lambda_1 \neq 0, \lambda_2 \neq 0$

Case 1. Here $\lambda_1 = 0, \lambda_2 = 0$

From (1) we get $x_1 = 1$

From (2) we get $x_2 = 3/2$

This solution does not satisfy (6). So this solution is discarded

Case 1. Here $\lambda_1 = 0, \lambda_2 \neq 0$

From (5) we get $2x_1 + 3x_1 - 6 = 0$ 10

(1) becomes $8 - 8x_1 + 2\lambda_2 = 0$ or, $x_1 = (\lambda_2 + 4)/4$ ....(11)

(2) becomes $12 - 8x_2 + 3\lambda_2 = 0$ or, $x_2 = (3\lambda_2 + 12)/8$ ....(11)

Using (11) and (12) we get from (10)

$(\lambda_2 + 4)/2 + (g\lambda_2 + 36)/S - 6 = 0$

or, $4\lambda_2 + 16 + g\lambda_2 + 36 - 48 = 0$

or, $13\lambda_2 = - 4$

or, $l2 = -4/13 < 0$

From (11) we have $x_1 = -\frac{1}{13} + 1 = \frac{12}{13}$

From (11) we have $x_1 = -\frac{24}{104} + \frac{12}{18} = \frac{18}{13}$

This solution violets (6) and so is discorded.

Case. 3 Here $\lambda_1 \neq 0$ and $\lambda_2 = 0$

From (4) we have $x_1 + x_2 - 1 = 0$ ....(13)

(1) becomes $8 - 8x_1 + \lambda_1 = 0$ or, $x_1 = (\lambda_1 + 8)/8$ ....(14)

(2) becomes $12 - 8x_2 + \lambda_1 = 0$ or, $x_2 = (\lambda_1 + 12)/8$ ....(14)

Using (14), (15) in (13) we have

or, $\lambda_1 = -6$

From (14) and (15) we get $x_1 = 1/4$, $x_2 = 3/4$

From (3) we get $x_3 = 0$

$\therefore \quad x_1 = 1/4, x_2 = 3/4, x_3 = 0$

This solution satifies (6) and (7).

Hence this is the optimum solution.

1.5 Summary

This unit is devoted with the classical theory of optimization for locating the points of maxima and minima of constrained and unconstrained nonlinear problems. This theory deals with the use of diffrential calculus. The topics introduced includs the development of the necessary and sufficient conditions for locating the extreme points for unconstrained problems, the treatment of the constrained problem with equality constraints using Lagrangian methods, and the development of the Kuhn-Tucker conditions for the general problem with inequality constraints. Though the classical optimization techniques are not suitable for obtaining real life problems, the underlying theory gives the basis for devising most of the non-linear programming algorithms.

1.6 Assesment Questions

1. Determine the extreme points of the function

$$f = 8x_1^3 + 27x_2^3 + 16x_1^2 + 18x_2^2 + 6$$

2. Determine the extreme points of the function

$$Z = 121 + 27x_1^3 + 64x_2^3 + 36x_1^2 + 32x_2^2$$

3. Find the extreme points of the function

$$f = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 20$$

4. The total profits (z) of a firm depend upon the level of of output (Q) and the advertising expenditure (A). Find the profit maximizing values of Q (in thousand units) and A (Rs in thousand) given the following relationship.

$$Z = 800 - 3Q^2 - 4Q + 2QA - 5A^2 + 48A$$

5. Using method of constrained variation and method of Lagrange multiplier

(i) Minimize $f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

Subject to $x_1 = x_2$

$x_1 + x_2 + x_3 = 1$

(ii) Minimize $f = 19 - 16x_1 + 6x_2 - 4x_3 + 8x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 + 4x_1x_3$

subject to $2x_1 - x_2 + 2x_3 = 3$

(iii) Maximize $f = 8x_1\, x_2\, x_3$

subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

(iv) Minimize $f = 4x_1^2 + 2x_2^2 + 9x_3^2$

subject to $4x_1 - 4x_2 + 9x_3 = 9$

$8x_1 - 8x_2 + 15x_3 = 17$

6. Using Kunh-Tucker condition determine the variable values to

Maximize $z = x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to $x^1 + x_2 \leq 2$

$2x_1 + 3x_2 \leq 12$

7. Use Kuhn-Tucker conditions of solve the following non-linear programming problems

(i) Maximize $Z = x_1^2 + 6x_1 + 5x_2$

subject to $x_1 + 2x_2 \leq 10$

$x_1 + 3x_2 \leq 9$

(ii) Maximize $Z = 2x_1 - x_1^2 + x_2$

subject to $2x_1 + 3x_2 \leq 6$

$2x_1 + x_2 \leq 4$

(iii) Maximize $Z = 2x_1^2 + 12x_1x_2 - 7x_2^2$

subject to $2x_1 + 5x_2 \leq 98$

$x_1 + x_2 \geq 0$

(iv) Maximize $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Unit 2 ❑ Revised Simplex Method

Structure

2.1 Introduction :

The revised simplex method proceeds through the same steps as simplex method but keeps all important data in a smaller array. The 'revised' aspect concerns the procedure of changing the simplex tables only. The revised simplex method is thus an efficient computational procedure for solving a linear programming problem with less time and labour. For large size problem this method is found to be ..... useful as it reduces the cost of obtaining the solution.

2.2 Revised Simplex Method :

When a linear programming problem is solved simplex method, successive iterations are obtained by using suitable row operations so that the objective function reduces its value in each step if it is a problem of maximization. Also the net evaluations should remain always non-negative in every step. This method requires storing the entire table in the memory of the computer. For large size problem it may not be feasible. So, it requires to device a new method by modifying simplex method to handle LPP will large number of decision variables and constraints.

In fact, it is found tat it is not necessary to compute the entire simplex table during each iteration. The only informations needed to pass from one table to the next one are seen to be

(i) Net evaluations $z_j - e_j$ to determine the non-basic variable that enters the basis.

(ii) The key column.

(iii) The current basic variables and their values to determine the minimum positive ratio, and thereby to determine the basic variable that will leave the basis.

It is sown that all the above informations can be directly obtained from the original equations of the given LPP by making use of the inverse of the current basis matrix.

If B be the current basis then we have

$$x_B = B^{-1} b, y_j = B^{-1} a_j \text{ for all } j = 1, 2, \ldots\ldots, n$$

$$z_j - c_j = C_B B^{-1} a_j - c_j \text{ for all } j = 1, 2, \ldots\ldots, n$$

and $z = c_B x_B$.

We note that all these necessary informations can be calculated if the current value of B^{-1} is known. Much computational work is needed for transformation of all y_j, $j = 1, 2, \ldots\ldots, n$.

But all y ; are not needed to go to next table. As noted above we need only to know the key column i.e. y_k. This will actively save our much labour. At each iteration x_B, z c_B B^{-1} and B^{-1} are transformed and not all the y_j are transformed, only the key column y_k is transformed in the revised simplex method. The criteria for selecting the entering and departing vectors in the revised simplex method precisely the same as that was in the simplex method. The labour saving point in this method lies in the fact of computing the inverse of the next basis directly from that of the current basis without actually having to invert the next basis.

2.3 Standard Form for Revised Simplex Method :

Let the linear programming problem be

$$\begin{aligned} &\text{Maximize} \quad z = cx \\ &\text{subject to} \quad Ax = b \\ &\qquad\qquad\quad x \geq 0 \end{aligned} \qquad \text{....... (1)}$$

where $c, x^T \in R^n$, $bT \in Rm$ and A is an $m \times n$ real matrix. In the revised simplex method we consider the objective function equation $z = cx$ also one constraint. Thus the new system becomes a $(m + 1)$ simultaneous lines equations in $n + 1$ variables z, $x_1, x_2, \ldots, x_n$. The problem thus becomes to get the solution of this system such that z is as large as possible. The simultaneous linear system thus becomes

$$\begin{aligned} &Ax + oz = b \\ &-cx + z = 0 \\ &\quad x \geq 0, z \text{ is unrestricted.} \end{aligned} \qquad \text{....... (2)}$$

Hence the LPP (1) becomes equivalent to the problem of finding the solution of the system (2) such that z is as large as possible.

In matrix notation (2) becomes

$$\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, x \geq 0 \qquad \text{....... (3)}$$

Let B be the initial basis submatrix of A and $x_B = B^{-1}b$ be the initial basic feasible solution to the original LPP (1).

Since the values of the non-basic variables are always zero (2) becomes

$$\begin{aligned} &Bx_B + 0\,z = b \\ &-C_B\, x_B + z = 0 \end{aligned} \qquad \text{....... (4)}$$

$$\text{or, } \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix} \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\text{or, } \hat{B}\,\hat{x}_B = \hat{b} \qquad \text{....... (5)}$$

where, $\hat{B} = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}, \hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix}$ and $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ .... (6)

From (4) we have

$$\hat{x}_B = \hat{B}^{-1} \hat{b} \quad \text{....... (7)}$$

This is the initial basic feasible solution to the reformulated problem (2).

Computation of Inverse of $\hat{B}$ by partitioning we have $B = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}$.

Let $\hat{B}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ .... (8)

Since $\hat{B}\hat{B}^{-1} = I$, we have

$$\begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I_{m+1}$$

or, $\begin{bmatrix} BP + OR & BQ + OS \\ -C_B P + R & -C_B QS \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $BP = I_m$

$BQ = O$

$-C_B P + R = 0$

$-C_B Q + S = 1$

Since B^{-1} exists, we get from above

$P = B^{-1} I_m = B^{-1}$

$Q = B^{-1} O = O$

$R = C_B B^{-1}$

$S = 1 + C_B O = 1$

Thus from (8) we get

$$B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} \quad \text{....... (9)}$$

We note that all the components of $\hat{B}^{-1}$ are known.

Determination of net evaluations, key column and BFS :

We define $A = \begin{bmatrix} A \\ -C \end{bmatrix}$

and $\hat{y} = \hat{B}^{-1}\hat{A}$

Then $\hat{y} = \begin{bmatrix} B & 0 \\ C_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -C \end{bmatrix}$

$$= \begin{bmatrix} B^{-1}A & -OC \\ C_B B^{-1}A & -C \end{bmatrix}$$

$$= \begin{bmatrix} B^{-1}A \\ C_B(B^{-1}A) - C \end{bmatrix} \quad \text{....... (10)}$$

we have

$A = By$

$\therefore\ y = B^{-1}A$

$\therefore$ From (10) we have $\hat{y} = \begin{bmatrix} y \\ C_B\, y - C \end{bmatrix}$

or, $[\hat{y}_1\ \hat{y}_2 \ldots\ldots \hat{y}_n] = \begin{bmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ z_1 - c_1 & z_2 - c_2 & \ldots\ldots & z_n - c_n \end{bmatrix}$

Thus for $j = 1, 2, \ldots\ldots, n$ we have

$$\hat{y}_j = \begin{bmatrix} y_j \\ z_j - c_j \end{bmatrix} \text{ and } y_j = B^{-1} a_j$$

Hence the net evaluation are the components of $C_B B^{-1}A - C$

i.e. $C_B B^{-1}A - C = [z_1 - c_1\ \ z_2 - c_2 \ldots\ldots z_n - C_n]$

Most negative $z_j - c_j$ will determine the key column. Let $z_k - c_k$ be the most negative $z_j - c_j$. Then the key column is

$$\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \begin{bmatrix} B^{-1} a_k \\ z_k - c_k \end{bmatrix} \quad \text{....... (11)}$$

From (7) and (6) we have

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1} \hat{b} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ C_B B^{-1} b \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ C_B x_B \end{bmatrix}$$

we note the important fact that all necessary informations can be obtained from the products $\hat{B}^{-1}\hat{A}$ and $\hat{B}^{-1}\hat{b}$.

Also we note that $\hat{A}$ and $\hat{b}$ remains same in all steps, only $\hat{B}^{-1}$ changes in each step of simplex table depending on the current basis B.

The above discussion enables us now to state the algorithm of revised simplex method.

2.4 ALogarithm of Revised Simplex Method :

It stepwise procedure of revised simplex method are as follows.

Step 1. Introduce necessary slack and surplus variables. Convert the problem into a problem of maximization if it is in minimization form. Restate the LPP in the standard form of revised simplex method *i.e.* in the form $\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, x \geq 0,\ z$ is unrestricted.

Step 2. Begin with the initial basis $B = I_m$ and form the auxiliary matrix $\hat{B} = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}$ and write down

$$B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix}. \text{ Form } \hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} \text{ and } \hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Also form $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1} \hat{b}$.

Step 3. Compute the net evaluations $z_1 - c_1, z_2 - c_2, \ldots.. z_n - c_n$ as the components of the product

$$[C_B B^{-1} \quad 1]\begin{bmatrix} A \\ -c \end{bmatrix}$$

If all $z_j - c_j$ are non-negative, the current basic solution $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b}$ gives the optimal BFS and maximum value of the objective function.

If at least one $z_j - c_j$ is negative, determine the most negative of them. If $z_k - c_k$ is the most negative $z_j - c_j$ then find $\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \hat{B}^{-1}\hat{a}_k$. Go to step 4. If there is a tie for the most negative $z_j - c_j$, resolve the tie by any standard method.

Take x_k as the new basic variable. Go to step 4.

Step 4. If all $y_{ik} \leq 0$ there exists an unbounded solution to the given problem.

If at least one $y_{ik} > 0$, consider the current x_B and compute the replacement ratios.

$$\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\}$$

If $\frac{x_{Br}}{y_{rk}}$ is the minimum of all these ratios then the basic variable x_{Br} becomes non-basic variable in the next table. *ie.* x_{Br} is replaced by x_k. Go to step 5.

Step 5. Write down the results obtained in steps 2, 3 and 4 in a able. This table is known as revised simplex table. This table is of the form

$\hat{y}_B$	$\hat{x}_B$	$\hat{B}^{-1}$	$\hat{y}_k$	$\frac{x_{Bi}}{y_{ik}}: y_{ik>0}$

Step 6. Convert the key element y_{rk} of $\hat{y}_k$ into unity and all other elements into zero by suitable row operations. Same operations are to be applied in the current $\hat{B}^{-1}$. These operation will change $\hat{B}^{-1}$ to new $\hat{B}^{-1}$ for the next table.

Step 7. Consider new $\hat{B}^{-1}$ obtained in step 6 as $\hat{B}^{-1}$ and go to step 3. Repeat the procedure until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution.

Advantages of revised simplex method :

The advantages of the revised simplex method over the regular simplex method are

(i) fewer calculations are required.

(ii) less storage is needed when computing the problem on a computer.

(iii) the round off errors can be controlled as table entries are not repeatedly recalculated.

2.5 Comparison of Simplex Method and Revised Simplex Method :

Let us consider the LPP

Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

where A is a matrix of order $m \times n$. If initially artificial variables are not needed for obtaining the initial basis matrix, then for solving this problem by the simplex method we have to transfer $(n + 1)$ columns at each iteration. (n columns for A and one column for x_B). Also, at each iteration one variable is introduced into the basis and one is removed from it. Thus, in total we compute for $(n - m + 1)$ columns. Further more, for each of these columns, we have to transform $(m + 1)$ elements. For moving from one iteration to another we also need to calculate minium ratio x_B/y_{ik}. Hence in all we have to perform multiplication $(m + 1)(n - m + 1)$ times and addition $m(n - m + 1)$ times.

In the revised simplex method, there are $(m + 1)$ rows and $(m + 2)$ columns. So, for moving from one iteration to another we have to make $(m + 1)^2$ multiplication operations to get an improved solution in addition to $m(n - m)$ operations for calculating $(z_j - c_j)$'s.

In the revised simplex method we need to make $(m + 1)(m + 2)$ entries in each table while in simplex method there are $(m + 1)(n + 1)$ entries in each table.

If the number of variables n is significantly larger than the number of constraints m, then the coputational efforts of the revised simplex method is smaller than that of the simplex method.

Revised simplex method reduces the cumulative round-off error while calculating $(z_j - c_j)$'s and updated column y_k due to the use of original data.

The inverse of the current basis matrix is obtained automatically.

2.6 Illustrative Examples :

Example 2.6.1. Use revised simplex method to solve the LPP.

$$\text{Maximize} \quad z = 2x_1 - 3x_2 + x_3$$

$$\text{subject to} \quad 3x_1 + 6x_2 + x_3 \le 6$$

$$4x_1 + 2x_2 + x_3 \le 4$$

$$x_1 - x_2 + x_3 \le 3$$

$$x_1, x_2, x_3 \ge 0.$$

Solution : Introducing slack variables $x_4 \ge 0, x_5 \ge 0, x_6 \ge 0$, the given LPP becomes in standard form as

$$\text{Maximize} \quad z = 2x_1 - 3x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\text{subject to} \quad 3x_1 + 6x_2 + x_3 + x_4 = x_6$$

$$4x_1 + 2x_2 + x_3 + x_5 = 4$$

$$x_1 - x_2 + x_3 + x_6 = 3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

or, Maximize $z = cx$

subject to $Ax = b$, $x \ge 0$

where $A = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}, c = 0\begin{bmatrix} 2 & -3 & 1 & 0 & 0 & 0 \end{bmatrix}$

$$b = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \, x_2 x_3 \, x_4 \, x_5 \, x_6 \end{bmatrix}^T$$

$\therefore$ we have $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$

Initially

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}, c_B = \begin{bmatrix} c_4 & c_5 & c_6 \end{bmatrix} = [0 \; 0 \; 0]$$

Now $C_B B^{-1} = [0 \; 0 \; 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \; 0 \; 0]$

$$\therefore \; B^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ z \end{bmatrix} = B^{-1} b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$$

i.e. $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ and $z = 0$

The net evaluation are the components of

$$[C_B B^{-1} \ 1]\begin{bmatrix} A \\ -c \end{bmatrix} = [0\ 0\ 0\ 1]\begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = [-2\ 3\ -1\ 0\ 0\ 0]$$

$$= [z_1 - c_1\ \ z_2 - c_2\ \ z_3 - c_3\ \ z_4 - c_4\ \ z_5 - c_5\ \ z_6 - c_6]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_1 - c_1 = -2$. Therefore x_1 will be the new basic variable.

Now we compute

$$y_1 = B^{-1} a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_4	6	1	0	0	0	3	2
x_5	4	0	1	0	0	4	1
x_6	3	0	0	1	0	1	3
z	0	0		0	1	− 2	

Here the minimum ratio is Min $\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = 1$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_1 in the next table.

Using elementary row operations $\hat{y}_1 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$$

The new BFS is given by

$$\hat{x} = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \\ x \end{bmatrix} = \hat{B}^{-1}.\hat{b} = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } E = 2$$

The net evaluations are given by

$$x_B = [C_B \, B^{-1} \, 1] = \begin{bmatrix} A \\ -C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Since there is negative not evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_3 - c_3 = -\frac{1}{2}$. So, x_3 is the next incoming basic variable.

Now we compute

$$y_3 - B^{-1} \hat{a}_3 = \begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$$

These results are shown in the following simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_4	6	1	$-\frac{3}{4}$	0	0	$\frac{1}{4}$	12
x_1	1	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	4
x_6	2	0	$-\frac{1}{4}$	1	0	$\frac{3}{4}$	$\frac{8}{3}$
z	2	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	

Here the minimum ratio is $\frac{8}{3}$ and is associated with the basic variable x_6. Therefore, the outgoing basic variable is x_6. So x_6 is replaced by x_3 is the next iteration. Using elementary row operations $\hat{y}_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are performed in $\hat{B}^{-1}$. This gives the new $\hat{B}^{-1}$ as follows

$$\text{Now } \hat{B}^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{1}{3} \\ \frac{8}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} \text{ and } z = \frac{10}{3}$$

The net evaluation are given by

$$[C_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Here all net evaluations are non-negative. Hence we have obtained the optimal solution. The optimal solution is $x_1 = \frac{1}{3}, x_2 = 0, \frac{x}{3} = \frac{8}{3}$ and $z = \frac{10}{3}$.

Example 2.6.2. Solve by revised simplex method

Maximize $z = 5x_1 + 3x_2$

subject to $4x_1 + 5x_2 \leq 10$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

$x_1, x_2 \geq 0$

Solution : Introducing surplus variable $x_3 \geq 0$, slack variables $x_4 \geq 0$, $x_5 \geq 0$ and artificial variable $x_6 \geq 0$ the standard form of the given LPP is

$$\text{Maximize} \quad z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$$

$$\text{subject to} \quad \begin{aligned} 4x_1 + 5x_2 - x_3 \quad + x_6 &= 10 \\ 5x_1 + 2x_2 \quad + x_4 &= 10 \\ 3x_1 + 8x_2 \quad + x_5 &= 12 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

or, Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

or, Maxmize $z = cx$

subject to $Ax = b$, $x \geq 0$

$$\text{where } A = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \end{bmatrix}, c = [5\ 3\ 0\ 0\ 0 - M]$$

$$b = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}, x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$$

$$\therefore \text{ We have } A = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}, b = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix}$$

$$\text{Initially, } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}, C_B = [c_6\ c_4\ c_5] = [-M\ 0\ 0]$$

$$\therefore\ B^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now } C_B B^{-1} = [-M\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [-M\ 0\ 0]$$

$$\therefore\ \hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ \hat{x}_{B} = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_6 \\ x_4 \\ x_5 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \\ -10M \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix} \text{ and } z = -10M$$

The net evaluations are the components of

$$[c_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = [-M\ 0\ 0\ 1]\begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= [\,-4M-5\ \ -5M-3\ \ M\ \ 0\ \ 0\ \ 0]$$

$$= [z_1 - c_1\ \ z_2 - c_2\ \ z_3 - c_3\ \ z_4 - c_4\ \ z_5 - c_5\ \ z_6 - c_6]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_2 - c_2 = -5M - 3$. Therefore x_2 will be the new basic variable.

Now we compute

$$\hat{y}_2 = \hat{B}^{-1} a_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_6	10	1	0	0	0	5	2
x_4	10	0	1	0	0	2	5
x_5	10	0	0	1	0	8	$\frac{3}{2}$
z	– 10M	– M	0	0	1	– 5M – 3	

Here the minimum ratio is min $\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = \frac{3}{2}$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_2 in the next table.

Using elementary row operations $\hat{y}_2 - \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{5}{8} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix}$$

The new BFS is given by

$$x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & -\frac{5}{8} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 7 \\ \frac{3}{2} \\ \frac{-5M+9}{8} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 7 \\ \frac{3}{2} \end{bmatrix} \text{ and } z = \frac{-5M+9}{8}$$

The net evaluation are the componentts of

$$[c_B\, B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = \left[-M\ 0\ \frac{5M+3}{8}\right]\begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \left[\frac{-17M-3}{8}\ 0\ M\ 0\ \frac{5M+3}{8}\ 0\right]$$

Since there is negative net evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_1 - c_1$. So x_1 is the next incoming basic variable.

Now we compute.

$$\hat{y}_1 + \hat{B}^{-1} a_1 = \begin{bmatrix} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{17}{8} \\ \frac{17}{4} \\ \frac{3}{8} \\ \frac{-17M-31}{8} \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_6	$\frac{5}{2}$	1	0	$-\frac{5}{8}$	0	$\frac{17}{8}$	2
x_4	7	0	1	$-\frac{1}{4}$	0	$\frac{17}{4}$	5
x_2	$\frac{3}{2}$	0	0	$\frac{1}{8}$	0	$\frac{3}{8}$	$\frac{3}{2}$
z	$\frac{-5M+9}{8}$	$-M$	0	$\frac{5M+3}{8}$	1	$\frac{-17M-31}{8}$	

Here the minimum ratio is $\frac{20}{17}$ and is associated with the basic variable x_6. So x_6 is replaced by x_1 is the next iteration. Using elementary row operation $\hat{y}_1$ is converted to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the same operaions are performed in $\hat{B}^{-1}$ as follows

$$\therefore \text{ Now } \hat{B}^{-1} = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & - & -\frac{13}{17} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & 0 & -\frac{13}{17} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{20}{17} \\ 2 \\ \frac{18}{17} \\ \frac{154}{17} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{20}{17} \\ 2 \\ \frac{18}{17} \end{bmatrix} \text{ and } z = \frac{154}{17}$$

The net evaluation are given by

$$\left[c_B B^{-1}\ 1\right]\left[\frac{A}{-c}\right] = \left[\frac{31}{17}\ 0 - \frac{13}{17}\ 1\right] \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \left[0\ 0 - \frac{31}{17}\ 0 - \frac{13}{17}\ \frac{31}{17} + M\right]$$

Since there are negative net evaluation the BFS obtained is not optimal. The most negative $z_j - c_j$ is $z_3 - c_3 = -\frac{31}{17}$ so x_3 is the next incoming basic variable.

Now we compute

$$\hat{y}_3 = \hat{B}^{-1}\hat{a}_3 = \begin{bmatrix} \frac{8}{17} & 0 & -\frac{5}{17} & 0 \\ -2 & 1 & 1 & 0 \\ -\frac{3}{17} & 0 & \frac{4}{17} & 0 \\ \frac{31}{17} & 0 & -\frac{13}{17} & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{8}{17} \\ 2 \\ \frac{3}{17} \\ -\frac{31}{17} \end{bmatrix}$$

These results are shown in the following revised simplex table.

Basic variables	Values	$\hat{B}^{-1}$				$\hat{y}^{-1}$	min ratio
x_1	$\frac{20}{17}$	$\frac{8}{17}$	0	$-\frac{5}{17}$	0	$-\frac{8}{17}$	...
x_4	2	-2	1	1	0	2	1
x_2	$\frac{18}{17}$	$-\frac{3}{17}$	0	$\frac{4}{17}$	0	$\frac{3}{17}$	6
z	$\frac{154}{17}$	$\frac{31}{17}$	0		1	$-\frac{38}{17}$	

Obviously x_4 will be replaced by x_3.

Using elementary row operations $\hat{y}_3$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are used on $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows.

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 0 & \frac{4}{17} & -\frac{1}{17} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{34} & \frac{5}{34} & 0 \\ 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 0 & \frac{4}{17} & -\frac{1}{17} & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{34} & \frac{5}{34} & 0 \\ 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{28}{17} \\ 1 \\ \frac{15}{17} \\ \frac{185}{17} \end{bmatrix}$$

$$\therefore \; x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{28}{17} \\ 1 \\ \frac{15}{17} \end{bmatrix} \text{ and } z = \frac{185}{17}$$

The net evaluations are given by

$$[c_B B^{-1}]\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & \frac{31}{34} & \frac{5}{34} & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{31}{34} & \frac{5}{34} & M \end{bmatrix}$$

Here all net evaluation are found to be non-negative. Hence wehave obtained the optimal solution. The optimal solutions is given by

$$x_1 = \frac{28}{17}, \; x_2 = \frac{15}{17} \text{ and } z_{max} = \frac{185}{17}.$$

2.7 Summary :

Revised simplex method is an efficient method and is very useful for large problem. Only necessary part of the simplex table is calculated to pass from one table to the next table. Standard form of the revised simplex method is devised and computational procedure of revised simplex method is noted and is compared with simplex method. Finally, the method is used to solve some examples.

2.8 Self Assessment Questions :

Use revised simplex method to solve the following LPP

1. Maximize $z = 3x_1 + 5x_2$

subject to $x_1 \leq 4$

$x_2 \leq 6$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 2$, $x_2 = 6$, $z_{max} = 36$]

2. Maximize $z = 6x_1 - 2x_2 + 3x_3$

subject to $2x_1 - x_2 + 2x_3 \leq 2$

$x_1 + 4x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 4$, $x_2 = 6$, $x_3 = 0$ $z_{max} = 12$]

3. Maximize $z = x_1 + x_2$

subject to $x_1 + 2x_2 \geq 7$

$4x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

$\left[\text{Ans: } x_1 = \frac{5}{7}, x_2 = \frac{22}{7}, z_{min} = \frac{27}{7}\right]$

4. Maxmize $z = 2x_1 + x_2$

subject to $3x_1 + x_2 \leq 3$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

$\left[\text{Ans}: x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, z_{min} = \frac{12}{5}\right]$

5. Minimize $z = 4x_1 + 3x_2$

subject to

$$3x_1 + 4x_2 \leq 12$$

$$3x_1 + 3x_2 \leq 10$$

$$2x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

$\left[\text{Ans}: x_1 = \frac{4}{5}, x_2 = \frac{12}{5}, z_{max} = \frac{52}{5}\right]$.

Unit 3 ☐ Dual Simplex Method

Structure

3.1 Introduction :

The Dual Simplex Method gives an algorithm in which we start with a basic optimal solution of the primal in which all $zj - ej \geq 0$ but not feasible is some basic solution are negative. At each iteration the number of negative basic variables are decreased while maintaining the optimality. An optimal solutionis reached in a finite number of steps. The benifit of this procedure lies in the fact that we need not take the help of any artificial variable and hence it reduces a lot of labour.

3.2 Comparison Between Simplex Method and Dual Simplex Method :

In simplex method the initial solution is basic feasible and non optimal. In subsequent tables the value of the objective function gradually increases and finally reaches to its obtimal value. In each table the solution is basic feasible and non-optimal.

In dual simplex method the initial solution is basic non-feasible and optimal. In subsequent tables the value of the objective function gradually decreases and finally reaches to its optimal value. In each table the solution is basic non-feasible and optimal (or better than optimal).

Basic non-feasible and better than optimal some $x_B < 0$ and all $zj - cj \geq 0$, $z_B \geq z_{max}$.

↓ (Dual simplex method) z gradually decreases

Basic feasible and optimal
$x_B \geq 0$ and all $zj - cj \geq 0$
$z_B = z_{max}$

↓ (Simplex method) z gradually increases

Basic feasible and non optimal
$x_B \geq 0$ and some $zj - cj < 0$, $z_B \leq z_{max}$

3.3 Applications of the Dual Simplex Method :

It the given LPP is optimal and infeasible then only dual simplex method is applicable for many practical problem the initial table does not satisty these conditions and as a consequence dual simplex method can not be applied. Simplex method has no such restnction and is applicable to any LPP. Hence as rule the regular simplexmethod preferred over the dual simplex method for solving the general LPP. However, there are instances when the dual simplex method has a distinct advantage over the regular simplex method. There are problems in which a dual feasible table is readily available to start the dual simplex method and for such problems the optimal BFS is obtained easily in comparison to simplex method. Some of the applications of dual simplex method are :

(i) Sensitivity analysis when the right hand side vector be changed or when new constraints are added.

(ii) Parametric programming.

(iii) Integer programming problem.

(iv) Some non-linear programming problem.

3.4 Criteria for Incoming and Outgoing basic Variable in Dual Simplex Method :

In the transmation formula of simplex method if the basic variable xB_r is replaced by the non-basic variable x_k then we have

$$\hat{z} = z - \frac{{}^{x}B_r}{{}^{y}r_k}(z_k - c_k) \quad \text{... (1)}$$

$$\text{and } (\hat{z}_i - \hat{c}_j) - (z_j - c_i) - \frac{z_{rj}}{y_{rk}}(z_k - c_k) \quad \text{... (2)}$$

As we want to remove negative besic variables, we choose the most negative besic variable xB_r (say) as outgoing basic variable from the list of all basic variables.

We know that if for some basic solution (not necessarily feasible) all components of net evaluations are non-negative then the value of the objective function to this basic solution is optimal or better than optimal. So we intend to lower down the value of the objective function to get z_{max}. For this from (1) we should have $y_{rk} < 0$ as $xB_r < 0$ and $z_k - c_k \geq 0$. This should be one criterion for incoming basic variable.

In the next table we want the solution to be optimal or better than optimal. So we should have $\hat{z}_j - \hat{c}_j \geq 0$ for all j.

∴ From (2) We have

$$z_j - c_j - \frac{y_{rj}}{y_{rk}}(z_k - c_k) \geq 0 \text{ for all } j$$

$$\text{or, } z_j - c_j \geq \frac{y_{rj}}{y_{rk}}(z_k - c_k) \text{ for all } j \quad \text{... (3)}$$

When $yj \geq 0$ then (3) is satisfied as $y_{rk} < 0$ and all $zj - cj \geq 0$.

When $y_r j < 0$ then (3) is satisfied if $\frac{z_j - c_j}{y_{rj}} \leq \frac{z_k - c_k}{y_{rk}}$ for all j

Hence we are b choose k such that

$$\max_{y_{rj} < 0} \left\{ \frac{z_j - c_j}{y_{rj}} \right\} = \frac{z_k - c_k}{y_{rk}}$$

3.5 Dual Simplex Algorithm :

The iterative procedure for dual simplex algorithm are as follows :

Step 1 : Convert the minimizatism LPP into that of maximization if it is in the minimization form.

Step 2 : Convert the ≥ type inequalious, representing the constraints of the given LPP, if any, into those of ≤ type by multiplying the corresponding constraints by –1.

Step 3 : Introduce slack variables in the constraints of the given LPP and obtain an initial basic solution. Put this solution in the starting dual simplex table.

Step 4 : Test the nature of the net evaluations $zj - cj$ in the starting simplex table.

(i) If all $zj - ej$ and xBj are non negative for all i and j, then an optimum basic fesible solution has been obtained.

(ii) It all $zj - cj$ are non negative and at least one basic variable, say xB_r, is negative then go to step 5.

(iii) It at least one $zj - cj$ is negative then dual simplex method is not applicable. In this case we are to apply artificial constraint method.

Step 5 : Select the most negative basic variable, say xB_k, as outgoing basic variable.

Step 6 : Test the nature of all $y_r j$, $j = 1, z, ..., n$.

(i) It all $y_r j$ are non-negative, there does not exist any feasible solution to the given LPP.

(ii) It at least one y_kj is negative, then comute

$$\left\{\frac{zj}{y_{kj}} : y_{kj} < 0\right\}, j = 1, 2, \dots, n,$$

and choose the maximum of these, If the maximum of these be $\frac{z_r - c_r}{y_{kr}}$ then x_r is the incoming basic variable i.e. xB_k is replaced by x_r.

Step 7 : With y_{kr} as the key element form the next table. Using elementary row operation convert the key element to unity and all other elements of the key column to zero to get the improoed solution.

Step 8 : Repeat the steps 4 to 7 until either an optimun basic feasible solution is obtained or there is an indication of no feasible solution.

3.6 Illustrative Examples :

Example 3.6.1. Solve the following LPP by dual Simplex Method.

Maximize $z = 2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0.$

Solution : Converting the given LPP into maximization and changing all ≥ type inequations to ≤ type and finally adding slack variables $x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$, the reformulated LPP in its standard form becomes.

Maximize $z' = -2x_1 - x_2 + ox_3 + ox_4 + ox_5$

Subject to $-3x_1 - x_2 + x_3 = 3$

$-4x_1 - x_2 + x_4 = 6$

$-x_1 - 2x_2 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

Solution : The solution of this LPP by dual simplex method is shown in the following tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	
		c_j	-2	-1	0	0	0	
0	y_3	-3	-3	-1	1	0	0	
0	y_4	-6	$\boxed{-4}$	-1	0	1	0	$\rightarrow$
0	y_5	-3	-1	-2	0	0	1	
$z' = 0$		$z_j - c_j$	2	1	0	0	0	
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			$\frac{2}{-4}$	$\frac{1}{-1}$				
0	$\frac{y}{3}$	$\frac{3}{2}$	0	$-\frac{1}{4}$	1	$-\frac{3}{4}$	0	
-2	y_1	$\frac{3}{2}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	
0	y_5	$-\frac{3}{2}$	0	$\boxed{-\frac{7}{4}}$	0	$-\frac{1}{4}$	0	$\rightarrow$
$z' = -3$		$z_j - c_j$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$				$\frac{1}{2}/\left(-\frac{7}{4}\right) = 2\frac{2}{7}$				
0	y_3	$\frac{12}{7}$	0	0	1	$-\frac{5}{7}$	$-\frac{1}{7}$	
-2	y_1	$\frac{9}{7}$	1	0	0	$-\frac{2}{7}$	$\frac{1}{7}$	
-1	y_2	$\frac{6}{7}$	0	1	0	$\frac{1}{7}$	$\frac{4}{7}$	
$z' = -\frac{24}{7}$	$z_j - c_j$	0	0	0	0	$\frac{3}{7}$	$\frac{2}{7}$	

In the first table max $\left\{-\frac{1}{2}, -1\right\} = -\frac{1}{2}$ and is ascociated with y_1 $\therefore$ y_4 is replaced by y_1 for the second table.

In the second table there is only one ratio $-\frac{2}{7}$ and is associated with y_2.

$\therefore$ y_5 is replaced by y_2 for the third table.

In the third table all xB_i are non negative. So this is optimal table. The optimal solution is $x_1 = \frac{9}{7}$, $x_2 = \frac{6}{7}$ and $z'_{max} = -\frac{24}{7}$ $\therefore$ $z_{min} = -z'_{max} = \frac{24}{7}$.

Example 3.6.2. Solve the following LPP by dual Simplex Method.

Maximize $z = -2x_1 - 2x_2 - 4x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \leq 2$

$3x_1 + x_2 + 2x_3 \geq 3$

$x_1 + 4x_2 + 6x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Solution : Converting the $\geq$ type inequations into $\leq$ type and introducing the slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$ the given LPP can be written in the standard form as

Maximize $z = -2x_1 - 2x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $2x_1 + 3x_2 + 5x_3 + x_4 = 2$

$-3x_1 - x_2 - 2x_3 + x_5 = -3$

$-x_1 - 4x_2 - 6x_3 + x_6 = -5$

$x_1, x_2, x_3, x_4, x_6 \geq 0.$

The following tables are obtained by using dual simplex method to this LPP.

		c_j	-2	-2	-4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	2	2	3	5	1	0	0
0	y_5	-3	-3	-1	-2	0	1	0
0	y_6	-5	-1	$\boxed{-4}$	-6	0	0	1 →
$z = 0$		$z_j - c_j$	2	2	4	0	0	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			$\frac{2}{(-1)} = -2$	$\frac{2}{(-4)} = -\frac{1}{2}$	$\frac{4}{(-6)} = -\frac{2}{3}$			
0	y_4	$-\frac{7}{4}$	$\frac{5}{4}$	0	$\frac{1}{2}$	1	0	$\frac{3}{4}$
0	x_5	$-\frac{7}{4}$	$\boxed{-\frac{11}{4}}$	0	$-\frac{1}{2}$	0	1	$-\frac{1}{4}$
-2	y_2	$\frac{5}{4}$	$\frac{1}{4}$	1	$\frac{3}{2}$	0	0	$-\frac{1}{4}$
$z = -\frac{5}{2}$		$z_j - c_j$	$\frac{3}{2}$	0	1	0	0	$\frac{1}{2}$
$\frac{z_j - c_i}{y_{2j}} : y_{2j} < 0$			$\frac{\left(\frac{3}{2}\right)}{\left(-\frac{11}{4}\right)} = -\frac{6}{11}$		$\frac{1}{\left(-\frac{1}{2}\right)} = -2$			$\frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{4}\right)} = -2$
0	y_4	$-\frac{28}{11}$	0	0	$\frac{3}{11}$	1	$\frac{5}{11}$	$\frac{7}{11}$
-2	y_1	$\frac{7}{11}$	1	0	$\frac{2}{11}$	0	$-\frac{4}{11}$	$\frac{1}{11}$
-2	y_2	$\frac{12}{11}$	0	1	$\frac{16}{11}$	0	$\frac{1}{11}$	$-\frac{3}{11}$
$z = -\frac{38}{11}$		$z_j - c_j$	0	0	$\frac{8}{11}$	0	$\frac{6}{11}$	$\frac{4}{11}$

In the first table $x_6 = -5$ is the most negative basic variable and max $\left\{-2, -\frac{1}{2}, -\frac{2}{3}\right\} = -\frac{1}{2}$ which is associated with this non basic variable x_2. So x_6 is replaced by x_2.

In the second table $x_4 = -\frac{7}{4}$, $x_5 = -\frac{7}{4}$ are the most negative basic variables. We choose x_5 arbitrarily. Here max $\left\{-\frac{6}{11}, -2, -2\right\} = -\frac{6}{11}$ which is associated with the non basic variable x_1. So x_5 is replaced by x_1.

In the third table $xB_1 = x_4 < 0$ and all $y_1 j \geq 0$.

∴ The given LPP has no feasible solution.

3.7 Modification of Dual Simplex Method :

If the initial table of the dual simplex method contains some negative basic variables and some of the net-evaluations are negtive then the dual simplex method is not applicable. In such situation dual simplex method is to be modified to from an equivalent LPP in which some basic variables are negative but all netevaluations are non-negative. Hence standard dual simplex method can be applied to that equivalent LPP.

The artificial constraint is one such method. In this method we consider the variables corresponding to which the net evaluations are negative and the variable corresponding to the most negative component of net evaluations is noted. Lt $z_p - c_p$ be the most negative net evaluation. So we consider the corresponding variable x_p. In this method we have to consider the artificial constraint.

$$\Sigma x_j \leq M$$

Where Σ is extended over all j's for which $zj - cj < 0$ and M is a sufficiently large positive number. Adding slack variable x_M to this constraint we get

$$\Sigma xj + x_M = M$$

From this we find x_p as

$$x_p = M - \left(x_M + \sum_{j \neq p} x_j\right)$$

This x_p is then subptituted in the orginal objective function and in the set of all constraints. This new problem together with the new added artificial constraint is equivalent to the given problem. This equivalent LPP will have all $zj - cj \geq 0$. Thus dual samplex method can be applied.

3.8 Illustrative Examples :

Example 3.8.1 : Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it

Maximize $z = -2x_1 - x_2 - x_3$

Subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$-x_1 + 9x_2 - x_3 \geq 3$

$2x_1 + 3x_2 - 5x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Solution : We first convert the minimization problem to maximization and then change the inequation of $\geq$ type into $\leq$ type. Finally adding slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$ we get the standard form LPP in dual simplex method as

Maximize $z' = 2x_1 + x_2 + x_3 + ox_4 + ox_5 + ox_6$

Subject to $4x_1 + 6x_2 + 3x_3 + x_4 = 8$

$x_1 - 9x_2 + x_3 + x_5 = -3$

$-2x_1 - 3x_2 + 5x_3 + x_6 = -4$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

The initial dual simplex table is

		c_j	2	1	1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	8	4	6	3	1	0	0
0	y_5	–3	1	–9	1	0	1	0
0	y_6	–4	–2	–3	5	0	0	1
$z' = 0$		$z_j - c_j$	–2	–1	–1	0	0	0

Here there are negative net evaluation, so standard dual simplex method is not applicable.

The negative net evaluations are $z_1 - c_1, z_2 - c_2, z_3 - c_3$ & most negative net evaluation is $z_1 - c_1 = -2$.

$\therefore$ The artificial constraint is

$x_1 + x_2 + x_3 \leq M$ where M is a very large positive number. Adding slack variable x_M we have

$$x_1 + x_2 + x_3 + x_M = M$$

From this we have $x_1 = M - x_2 - x_3 - x_M$

Using this in the LPP and adding the artificial constraint we have.

Maximize $z' = 2(M - x_2 - x_3 - x_M) + x_2 + x_3 + ox_4 + ox_5 + ox_6$

Subject to $4(M - x_2 - x_3 - x_M) + 6x_2 + 3x_3 + x_4 = 8$

$(M - x_2 - x_3 - x_M) - 9x_2 + x_3 + x_5 = -3$

$-2(M - x_2 - x_3 - x_M) - 3x_2 + 5x_3 + x_6 = -4$

$x_1 + x_2 + x_3 + x_M = M$

$x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0$

or, **Maximize** $z' = -2x_M - x_2 - x_3 + ox_4 + ox_5 + 2M$

Subject to $-4x_M + 2x_2 - x_3 + x_4 = 8 - 4M$

$-x_M - 10x_2 + x_5 = -3 - M$

$2x_M - x_2 + 7x_3 + x_6 = -4 + 2M$

$x_M + x_1 + x_2 + x_3 = M$

$x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0$

The following tables are obtained using simplex method.

c_B	y_B	c_j / x_B	y_M	y_1	y_2	y_3	y_4	y_5	y_6	
		c_j	-2	0	-1	-1	0	0	0	
0	y_4	$8-4M$	$\boxed{-4}$	0	2	-1	1	0	0	→
0	y_5	$-3-M$	-1	0	-10	0	0	1	0	
0	y_6	$-4+2M$	2	0	-1	7	0	0	1	
0	y_1	M	1	1	1	1	0	0	0	
$z'=0$		z_j-c_j	2	1	1	1	0	0	0	
$\frac{z_j-c_j}{y_{1j}}:y_{1j}<0$			$-\frac{1}{2}$	-1						
-2	y_M	$-2+M$	1	0	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	
0	y_5	-5	0	0	$\boxed{-\frac{21}{2}}$	$\frac{1}{4}$	$-\frac{1}{4}$	1	0	
0	y_6	0	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	0	
0	y_1	2	0	1	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	0	0	
$z'=4-2M$		z_j-c_j	0	0	2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	
$\frac{z_j-c_j}{y_{2j}}:y_{2j}<0$					$-\frac{4}{21}$		-2			
-2	y_M	$M-\frac{37}{21}$	1	0	0	$\frac{5}{21}$	$-\frac{5}{21}$	$-\frac{1}{21}$	0	
-1	y_2	$\frac{10}{21}$	0	0	1	$-\frac{1}{42}$	$\frac{1}{42}$	$-\frac{2}{21}$	0	
0	y_6	0	0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	1	
0	y_1	$\frac{9}{7}$	0	1	0	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{1}{7}$	0	
$z'=-2M+\frac{64}{21}$		z_j-c_j	0	0	0	$\frac{13}{42}$	$\frac{19}{42}$	$\frac{4}{21}$	0	

Here all basic variable are non-negative. So thes is the optimal table. The optimal solution is $x_1 = \frac{9}{7}$ $x_2 = \frac{10}{21}$, $x_3 = 0$ and $z'_{max} = \left(-2M + \frac{64}{21}\right) + 2M = \frac{64}{21}$ Therefore $z_{min} = -z'_{max} = -\frac{64}{21}$.

Example 3.8.2. Use the atificial constrant method to dind the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it :

Maximize $z = 2x_1 - 3x_2 - 2x_3$

Subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 \le 10$

$x_2 - 2x_3 \ge 4$

$x_1, x_2, x_3 \ge 0.$

Solution : We first change the inequation of $\ge$ type into $\le$ type. Adding slack variable $x_4 \ge 0$, $x_5 \ge 0$ we get the standard form of the LPP in dual simplex method as

Maximize $z = 2x_1 - 3x_2 - 2x_3$

Subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 + x_4 = 10$

$-x_2 + 2x_3 + x5 = -4$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

The initial dual simplex table is

		cj	2	−3	−2	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
2	y_1	8	1	−2	−3	0	0
0	y_4	10	0	2	1	1	0
0	y_5	−4	0	−1	2	0	1
$z = 16$		$zj - cj$	0	−1	−4	0	0

Since these are negative net evaluations, standard dual simplex method is not applicable. The negative net evaluations are $z_2 - c_2$ and $z_3 - c_3$ and most negative net evaluation is $z_3 - c_3 = -4$.

$\therefore$ The artificial constraint is $x_2 + x_3 \leq M$ where M is a very large positive number. Adding plack variable x_M we have

$$x_2 + x_3 + x_M = M$$

From this we have $x_3 = M - x_2 - x_M$

Using this in the LPP and adding the artificial constraint we have the equivalent LPP as

Maximize $z = 2x_M + 2x_1 - x_2 - 2M$

Subject to

$$3x_M + x_1 + x_2 = 3M + 8a$$

$$-x_M + x_2 + x_4 = -M + 10$$

$$-2x_M - 3x_2 + x\% = -2M - 4$$

$$x_M + x_2 + x_3 = M$$

$$x_M, x_1, x_2, x_3, x_4, x_5 \geq 0.$$

The dual simplex tables are as follows.

		c_j	2	2	−1	0	0	0
c_B	x_B	x_B	y_M	y_1	y_2	y_3	y_4	y_5
2	y_1	$3M + 8$	3	1	1	0	0	0
0	y_4	$-M + 10$	−1	0	1	0	1	0
0	y_5	$-2M - 4$	−2	0	[−3]	0	0	1
0	z_3	M	1	0	1	1	0	0
$z = 6M + 16$		$z_j - c_j$	4	0	3	0	0	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			−2		−1			
2	y_1	$\frac{7M+20}{3}$	$\frac{7}{3}$	1	0	0	0	$\frac{1}{3}$
0	y_4	$\frac{-5M+26}{3}$	[$-\frac{5}{3}$]	0	0	0	1	$\frac{1}{3}$
−1	y_2	$\frac{2M+4}{3}$	$\frac{2}{3}$	0	1	0	0	$-\frac{1}{3}$
0	y_3	$\frac{M-4}{3}$	$\frac{1}{3}$	0	0	1	0	$\frac{1}{3}$
$z = 4M + 12$		$z_j - c_j$	2	0	0	0	0	1
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			$-\frac{6}{5}$					
2	y_1	$\frac{94}{5}$	0	1	0	0	$\frac{7}{5}$	$\frac{4}{5}$
2	y_M	$\frac{5M-26}{5}$	1	0	0	0	$-\frac{3}{5}$	$-\frac{1}{5}$
−1	y_2	$\frac{24}{5}$	0	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$
0	y_3	$\frac{2}{5}$	0	0	0	1	$\frac{1}{5}$	$\frac{2}{5}$
$z = 10M + 112$		$z_j - c_j$	0	0	0	0	$\frac{6}{5}$	$\frac{7}{5}$

In this table all basic variables are non-negative. So this is the optimal table. The optimal solution is $x_1 = \frac{94}{5}$, $x_2 = \frac{24}{5}$, $x_3 = \frac{2}{5}$ and $z_{max} = \frac{10M+112}{5} - 2M = \frac{112}{5}$.

3.9 Summary :

Dual simplex method is found to be very useful is a large class of LPP. It is simple to handle and sige of the tables are not large as no artificial variables are introduced. the method is illustrated through examples. The method is then modified to handle more LPP.

3.10 Self Assessment Questions :

1. Use dual simplex method to solve the LPP

   Maximize $z = -2x_1 - 3x_2 - x_3$

   Subject to $2x_1 + x_2 + 2x_3 \geq 3$

   $3x_1 + 2x_2 + x_3 \geq 4$

   $x_1, x_2, x_3 \geq 0$

   [Ans. $x_1 = \frac{5}{4}$, $x_2 = 0$, $x_3 = \frac{1}{4}$, $z_{max} = -\frac{11}{4}$]

2. Use dual simplex method to solve the LPP

   Maximize $z = 10x_1 + 6x_2 + 2x_3$

   Subject to $-x_1 + x_2 + x_3 \geq 1$

   $3x_1 + x_2 - x_3 \geq 2$

   $x_1, x_2, x_3 \geq 0$

   [Ans. $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, $x_3 = 0$, $z_{min} = 10$]

3. Solve by dual simplex method the fllowing LPP

   Maximize $z = 6x_1 + x_2$

Subject to $2x_1 + x_2 \geq 3$

$x_1 + x_2 \geq 0$

$x_1, x_2 \geq 0$

[Ans. $x_1 = 1$, $x_2 = 1$, $z_{min} = 7$]

4. Solve the following LPP by dual simplex method

Maximize $z = -3x_1 - 2x_2$

Subject to $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

[Ans. $x_1 = 4$, $x_2 = 3$, $z_{ma}x = -18$]

5. Solve by dual simplex method :

Maximize $z = 2x_1 + 3x_2$

Subject to $2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1 - x_2 \geq 0$

$x_1 \geq 5$

$x_2 \geq 0$

[Ans. $x_1 = 5$, $x_2 = \frac{5}{2}$, $z_{min} = \frac{35}{2}$]

6. Solve the following LPP by daul simplex method

Maximize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

[Ans. No feasible solution]

7. Using artificial constraint procedure, solve the follwing problem by dual simplex method and show that the problem has no feasible solution

Maximize $z = -x_1 + x_2$

Subject to $x_1 - 4x_2 \geq 5$

$x_1 - 3x_2 \leq 1$

$2x_1 - 5x_2 \geq 1$

$x_1, x_2 \geq 0$

8. Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorthm to solve it

Maximize $z = x_1 - 3x_2 - 2x_3$

Subject to $x_2 - 2x_3 \geq 2$

$x_1 - 4x_2 - 6x_3 = 8$

$2x_2 + x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

[Ans. $x_1 = \frac{94}{5}$, $x_3 = \frac{12}{5}$, $x_3 = \frac{1}{5}$, $z_{max} = \frac{56}{5}$]

Unit 4 ❑ Post Optimality Analysis

Structure

4.1 Introduction :

In reality the problem accuring are in general large in size and often an error is discovered in the data after the attainment of an optimal solution to the problem. In such a situation there are two alternatives, either to solve the problem from begining or to device some method to use the optimal table. Undoubtedly the second one willsave time and space and is named as post optimality analysis. Also in practical situation the values of the co-efficient matrix A, the components of the requirement vector and the cost vector or neither known exactly nor they are constant for all time and or all situations. so it is important to know how sensitive the optimal solution is to small changes in these parameters. By sensitiveness we mean fulfilments of the condition of optimality as well as determining the limits of variations of these parameters for the solution to remain optimal.

We shall study the following effects of changes in the

(i) co-efficients c_j of the objective function.

(ii) components of the requirement vector to

(iii) addition of a new variable

(iv) deletion of a variable

(v) addition of a new constraint

4.2 Discrete Changes In The Cost Vector :

Let x_B be the optimal basic solution of the LPP

Maximize $z = cx$

Subject to $Ax = b$

$x \geq 0$

Where $c, x^T \in R^n$, $b^T \in R^m$ and A is mxn an real matrix. Let Δc_k be the amount by which c_k is changed. So the new value of c_k is $c_k^* = c_k + \Delta c_k$.

We know that $x_B = B^{-1} b$ and so it independent of c.

As initially x_B was BFS it will remain so after the change. The optimality condition is $z_j - c_j \geq 0$ for all j i.e. $[c_B \; B^{-1} \; 1] \begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$. It invalues c. So change in c will affect this condition. Thus when c_x is changed to c_k^*, the solution x_B may or may not remain optimal solution though it remains BFS.

Two cases will arise

(i) c_k is not in c_B

(ii) c_k is in c_B

Case (i). Here c_k is not in c_B. The net evaluations are the components of $c_B \; B^{-1}A - c_1$ and as $x_B = B^{-1}b$ was optimal solution we have $c_B B^{-1}A - c \geq 0$. i.e. $z_j - c_j \geq 0 \; \forall \; j$.

We note that when c_k is changed to c_k^* only kth component of net evaluation will change Thus for all $j = 1, 2, \ldots, k-1, k+1, \ldots, n$ i.e. for all $j \neq k$ we have new net evaluations.

$$z_j^* - c_j^* = z_j - c_j \geq 0 \text{ [as } z_j - c_j \geq 0 \text{ for all } j]$$

for $j = k$ we have $z_k^* - c_k^* = z_k - (c_k + \Delta c_k) = (z_k - c_k) - \Delta c_k$

We have $z_k - c_k \geq 0$. Therefor for all Δc_k, $z_k^* - c_k^*$ will not remain non negative.

Thus x_B will remain optimal solution for the changed LPP if $z_k - c_k - \Delta c_k \geq 0$ i.e. if $\Delta c_k \leq z_k - c_k$.

Case (ii). Here c_k is one component of c_B. Let $c_k = c_{B_\lambda}$ and so x_k is a basic variable. Thus y_k is a unit vector with its λth component as 1.

The new value of $z_k - c_k$ is given by

$$z_k^* - c_k^* = \sum_{\substack{i=1 \\ i \neq \lambda}}^{m} c_{B_i} y_{i_k} + c_k^*.1 - c_k^*.1 = 0 \quad [\because y_{ik} = 0 \ \forall\ i \neq \lambda].$$

For $j \neq k$, new value of $z_j - c_j$ is given by

$$z_j^* - c_j^* = \left(\sum_{\substack{i=1 \\ i \neq \lambda}}^{m} c_{B_i} y_{i_j} + c_k^*.y_{\lambda_j} \right) - c_j^*$$

$$= \sum_{i \neq \lambda} c_{Bi} y_{ij} + (c_k + \Delta c_k) y_{\lambda j} - c_j \left[\because c_j^* = c_j \ \forall\ j \neq k\right]$$

$$= \sum_{i \neq \lambda} c_{Bi} y_{ij} + c_{B_\lambda} y_{\lambda j} + \Delta c_k y_{\lambda j} - c_j) \ . \left[\because c_k = y_{B\lambda}\right]$$

$$= \sum_{i=1}^{m} c_{Bj} y_{ij} + \Delta c_k y_{\lambda j} - c_j$$

$$= z_j - c_j + \Delta c_k y_{\lambda i} \left[\because \ z_j = \sum_{i=1}^{m} c_{Bi} y_{ij}\right]$$

$\therefore$ x_B remains optimal solution

if $z_j - c_j + \Delta c_k y_{\lambda j} \geq 0 \ \forall \ i \neq k$

i.e. if $\Delta c_k \ y_{\lambda j} \geq -(z_j - c_j) \ \forall \ j \neq k$

Now for $y_{\lambda j} = 0$ this condition is fulfilled automatically as $z_j - c_j \geq 0$.

For $y_{\lambda j} > 0$ this condition is satisfied if $\Delta c_k \geq -\dfrac{z_j - c_i}{y_{\lambda j}} \ \forall j \neq k$

i.e. if $-\dfrac{z_j - c_i}{y_{\lambda j}} \leq \Delta c_k \forall j \neq k$

$$\therefore \text{ We must have } \max_{\substack{y_{\lambda j} > 0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda i}} \right\} \leq \Delta c_k$$

$y_{\lambda j} < 0$ this condition is satistied if $\Delta c_k \leq -\dfrac{z_j - c_j}{y_{\lambda j}} \forall j \neq k$

$$\therefore \text{ We must have } \Delta c_k \leq \min_{\substack{y_{\lambda j} > 0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda j}} \right\}$$

These two conditions can be combined as

$$\max_{\substack{y_{\lambda j} > 0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda j}} \right\} \leq \Delta c_k \leq \min_{\substack{y_{\lambda j} < 0 \\ j \neq k}} \left\{ -\frac{z_j - c_j}{y_{\lambda j}} \right\}$$

Hence if Δc_k lies in this range then the solution x_B remain optimal and if Δc_k falls outsids this range then at least one $z_j - c_j$ will be negative and the solution will no longer remain optimal.

It no $y_{\lambda j} > 0$, then there is no lower bownd of Δc_k and if no $y_{\lambda j} < 0$, then there is no upper bound of Δc_k.

4.3 Illustrative Example :

4.3.1 The optimal solution of the LPP :

Maximize $z = 6x_1 - 2x_2 + 3x_3$

Subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 \qquad + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0.$

is contained in the table.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
$z_j - c_j$		$z = 12$	0	0	9	2	2

Find the ranges of the cost components when (i) changed one at a time (ii) changed two at a time (iii) changed all three at a time to keep the optimal solution same.

Solution :

(i) When one component is changed at a time :

For change of $c_1 = 6$ to c_1^* we have the corresponding changed table as

		c_j	c_1^*	–2	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
			0	0	$4c_1^* - 12$	2	$c_1^* - 4$

This table becomes optimal table

if $4c_1^* - 12 \ge 0$ and $c_1^* - 4 \ge 0$

i.e. if $c_1^* \ge 3$ and $c_1^* \ge 4$

i.e. if $c_1^* \ge 4$

For change of $c_2 = -2$ to c_2^* the table corresponding to the final table becomes.

			6	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	–1	2
			0	0	$24+6c_2^*$	$-c_2^*$	$6+2c_2^*$

This table becomes the optimal table

$y\ 24+2c_2^* \geq 0$ and $-c_2^* \geq 0$ and $6+2c_2^* \geq 0$

i.e. if $c_2^* \geq -4$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $-3 \leq c_2^* \leq 0$

For change of $c_3 = 3$ to c_3^* the modified table is

			c	–2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	–1	2
			0	0	$12+c_3^*$	2	2

This table remains optimal table

if $12 - c_3^* \geq 0$

i.e. if $c_3^* \leq 12$

(ii) When two components are changed at a time.

For the change of $c_1 = 6$ and $c_2 = -2$ to c_1^* and c_2^* the modified table is

			c_1^*	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	–1	2
			0	0	$4c_1^*+6c_2^*-3$	$-c_2^*$	$c_2^*+2c_2^*$

This table becomes optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* + 6c_2^* - 3 \geq 0$ and $-c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{3-4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\frac{c_1^*}{2}$

i.e. $\max\left\{\frac{3-4c_1^*}{6}, \frac{-c_1^*}{2}\right\} \leq c_2^* \leq 0$ and c_1^* any real number.

For the change of $c_1 = 6$ and $c_3 = 3$ to c_1^* and c_3^* respectively the modified table is

			c_1^*	-2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
-2	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* - 12c_2^* - c_3^*$	2	$c_1^* - 4$

This table remains optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* - 12 - c_3^* \geq 0$ and $c_1^* - 4 \geq 0$

i.e. if $c_1^* \geq \frac{12+c_3^*}{4}$ and $c_1^* \geq 4$

i.e. if $c_1^* \geq \max\left\{4, 3+\frac{c_3^*}{4}\right\}$ and c_3^* any real number.

For the change of $c_2 = -2$ and $c_3 = 3$ to c_2^* and c_3^* respectively the modified table is

			6	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$24 + 6c_2^* - c_3^*$	$-c_2^*$	$6 + 2c_2^*$

This table remains optimal table if all $zj - cj \geq 0$

i.e. if $24 + 6c_2^* - c_3^* \geq 0$ and $- c_2^* \geq 0$ and $6 + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{2c_3^* - 24}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $\max\left\{\frac{c_3^* - 24}{6}, -3\right\} \leq c_2^* \leq 0$ and c_3^* any real number.

When all the three components are changed together :

It $c_1 = 6$, $c = - 2$, $c_3 = 3$ be changed respectively to c_1^*, c_2^*, c_3^*.

The modified table obtained from old optimal table is

			c_1^*	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	−1	2
			0	0	$4c_1^* - 6c_2^* - c_3^*$	$-c_2^*$	$c_1^* + 2c_2^*$

This table becomes an optimal table if all $z_j - c_j \geq 0$.

i.e. if $4c_1^* + 6c_2^* - c_3^* \geq 0$ and $- c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{6c_3^* - 4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\frac{c_1^*}{2}$

i.e. if $\max\left\{\frac{c_3^* - 4c_1^*}{6}, -\frac{c_1^*}{2}\right\} \leq c_3^* \leq 0$ and c_1^* any real number and c_3^* any real number.

4.4 Discrete Change In The Requirement Vector :

Let x_B be the optimal BFS of the LPP

Maximize $\quad z = cx$

subject to $\quad Ax = b, x \geq 0$

where $c, x^T \in R^n$, $b^T \in R^m$ and A is an max real matrix. We have $x_B = B^{-1}b$ and the net evaluations are the components $[c_B B^{-1} \; 1]\begin{bmatrix} A \\ -c \end{bmatrix}$. From these we see that x_B depends on b but net evaluations are independent of b: So change made in b will not affect optimality conditions i.e. optimal solution will remain optimal but it will change the solution x_B and it may become negative i.e. infeasible.

Let the component b_k of b be changed to $b_k^* = b_k + \Delta b_k$.

So the old solution $x_B = B^{-1}b$ becomes

$x_B^* = B^{-1} b^*$ where $b^* = [b_1, b_2, ..., b_{k-1}, b_k + \Delta b_k, b_{k+1}, ..., b_m]^T$

$$\text{Let} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix}$$

$$\therefore x_B^* = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k + \Delta b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ b_{m1} & b_{m2} & & b_{mk} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= B^{-1} + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \dots & \dots \\ b_{mk} & \Delta b_k \end{bmatrix}$$

$$\text{Thus} \begin{bmatrix} x^*_{B_1} \\ x^*_{B_2} \\ \vdots \\ x^*_{B_m} \end{bmatrix} = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{bmatrix} + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \vdots & \\ b_k & \Delta b_k \end{bmatrix}$$

$x^*_{B_i} = x_{B_i} + b_{ik}\Delta b_k$ for all i = 1, 2, ... , M

As we have noted this solution is optimal or better than optimal but may not be feasible shough basic.

Thus x^*_B will be an optimal BFS if $x^*_{B_i} \geq 0$ for all $i = 1, 2, ..., m$

i.e. if $x_{B_i} + b_{ik}\Delta b_k \geq 0$ for all $i = 1, 2, ... , m$

i.e. if $b_{ik}\Delta b_k \geq - x_{B_i}$ for all $i = 1, 2, ... , m$

For all $b_{ik} = 0$ this condition is satisfied.

For all $b_{ik} > 0$ this condition is satisfied if $\Delta b_k \geq -\frac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $-\frac{x_{B_i}}{b_{ik}} \leq \Delta b_k$ for all $b_{ik} > 0$

i.e. we need $\max \left\{ -\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0 \right\} \leq \Delta b_k$

Again for all $b_{ik} < 0$ this condition is satisfied if $\Delta b_k \leq -\frac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $\Delta b_k \leq -\frac{x_{B_i}}{b_{ik}}$ for all $b_{ik} < 0$

i.e. we need $\Delta b_k \le \min\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} < 0\right\}$

Hence x_{B_i} will be optimal basefeasible solution if Δb_k is selected satisfying the condition.

$$\max\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\} \le \Delta b_k \le \min\left\{-\frac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\}$$

4.5 Illustrative Examples :

Example 4.5.1. Given the LPP

Maximize $z = -x_1 + 2x_2 - x_3$

Subject to $3x_1 + x_2 - x_3 \le 10$

$-x_1 + 4x_2 + x_3 \ge 6$

$x_2 + x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

Determine the ranges for discrete changes of the components of b when changed one at a time, so as to maintain the optimality of the current optimum solution for the LPP.

Solution : Introducing slack variables $x_4 \ge 0$, $x_6 \ge 0$, surplus variable $x_5 \ge 0$ and artificial variable $x_7 \ge 0$ we have the standard form as follows

Maximize $z = -x_1 + 2x_2 - x_3 + ox_4 + ox_5 + ox_6 - Mx_7$

Subject to $3x_1 + x_2 - x_3 + x_4 = 10$

$-x_1 + 4x_2 + x_3 - x_5 + x_7 = 6$

$x_2 + x_3 + x_6 = 4$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0$

The tables obtained by simplex method are as follows :

		c_j	-1	2	-1	0	0	0	$-M$
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_4	10	3	1	-1	1	0	0	0
$-M$	y_7	6	-1	$\boxed{4}$	1	0	-1	0	1
0	y_6	4	0	1	1	0	0	1	0
			$M+1$	$-4M-2$	$-M+2$	0	M	0	0
0	y_4	$\frac{17}{2}$	$\frac{13}{4}$	0	$-\frac{5}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$
2	y_2	$\frac{3}{2}$	$-\frac{1}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$
0	y_6	$\frac{5}{2}$	$\frac{1}{4}$	0	$\frac{3}{4}$	0	$\boxed{\frac{1}{4}}$	1	$-\frac{1}{4}$
			$\frac{1}{2}$	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$M+\frac{1}{2}$
0	y_4	6	3	0	-2	1	0	-1	0
2	y_2	4	0	1	1	0	0	1	0
0	y_5	10	1	0	3	0	1	4	-1
			1	0	3	0	0	2	M

In this final table the basis in $B = [a_4\ a_2\ a_5]$ and in the initial table the basis is $I = [a_4\ a_7\ a_6]$

The inverse of the basis in the final table is given by

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

When b_1 is changed to $b_1 + \Delta b_1$ then the range $b\ \Delta b_1$ such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i1}} : b_{ik} > 0\right\} \le \Delta b_1 \le \min\left\{-\frac{x_{Bi}}{b_{i1}} : b_{i1} < 0\right\}$$

$\therefore \max\left\{-\frac{x_{B1}}{b_{11}}\right\} \le \Delta b_1$ [$\because$ only $b_{11} = 1 > 0$ and there are x_0 negative b_{i1}]

or, $-\frac{6}{1} \le \Delta b_1$ or, $\Delta b_1 \ge -6$

$\therefore b_1 + \Delta b_1 \ge b_1 - 6$

or, $b_1^* \ge 10 - 6$

or, $b_1^* \ge 4$

When b_2 is changed to $b_2 + \Delta b_2$ then the range of Δb_2 such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} > 0\right\} \le \Delta b_2 \le \min\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} > 0\right\}$$

$\therefore \Delta b_2 \le \min\left\{-\frac{x_{Bi}}{b_{i2}}\right\}$ [$\therefore$ only $b_{32} = -1 < 0$ and there is no positive b_{i2}]

or, $\Delta b_2 \le -\frac{10}{-1}$

or, $\Delta b_2 \le 10$

$\therefore b_2 + \Delta b_2 \le b_2 + 10$

or, $b_2^* \le 6 = 10$

or, $b_2^* \le 16$

When b_3 is changed to $b_3 + \Delta b_3$ thin the ranges of Δb_3 such that the optimality of the new BFS is not violated are given by

$$\max\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} > 0\right\} \le \Delta b_3 \le \min\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} < 0\right\}$$

or, $\max\left\{-\frac{x_{B2}}{b_{23}}, \frac{x_{B3}}{b_{33}}\right\} \le \Delta b_3 \le \min\left\{-\frac{x_{B1}}{b_{13}}\right\}$

$$\text{or, } \max\left\{-\frac{4}{1}, \frac{-10}{4}\right\} \le \Delta b_3 \le \min\left\{\frac{-6}{-1}\right\}$$

$$\text{or, } -\frac{5}{2} \le \Delta b_3 \le 6$$

$$\therefore\ -\frac{5}{2} + b_3 \le b_3 + \Delta b_3 \le 6 + b_3$$

$$\text{or, } -\frac{5}{2} + 4 \le b_3^* \le 6 + 4$$

$$\text{or, } \frac{3}{2} \le b_3^* \le 10$$

Example 4.5.2 Consider the LPP

Maximize $z = 2x_1 + x_2 + 4x_3 - x_4$

subject to $x_1 + 2x_2 + x_3 - 3x_4 \le 8$

$x_2 + x_3 + 2x_4 \le 0$

$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$

$x_1, x_2, x_3, x_4 \ge 0$

The optimal solution is it is contained in the following table

			2	1	4	–1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_1	1	1	0	3	1	1	2	0
1	y_2	0	0	1	– 1	– 2	0	– 1	0
0	y_7	5	0	0	– 4	2	– 2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0

For each of the parameter change listed below, make the necessary correction in the optimal table and solve the resulting problem.

(a) change c_1 to 1

(b) change c to [1 2 3 4]

(c) change b to $[\ 3\ -2\ 4]^T$

(d) change b_2 to 11

(e) How much c_1 be changed without affecting the optimal solution.

Solution : (a) When c_1 is changed to 1 the modified form of the optimal table becomes

		c_j	1	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	−2	0	1	1	0

From this table we see that changed solution is not optimal as $z_3 - c_3 < 0$. So we are to apply simplex method to get the optimal solution

		c_j	1	1	4	−1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
1	y_1	8	1	0	[3]	1	1	2	0	$\frac{8}{3}$
1	y_2	0	0	1	−1	−2	0	−1	0	
0	y_7	5	0	0	−4	2	−2	3	1	
			0	0	−2	0	1	1	0	
4	y_3	$\frac{8}{3}$	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	
1	y_2	$\frac{8}{3}$	$\frac{1}{3}$	1	0	$-\frac{5}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	
0	y_7	$\frac{95}{3}$	$\frac{4}{3}$	0	0	$\frac{10}{3}$	$-\frac{2}{3}$	$\frac{17}{3}$	1	
$z =$	$\frac{40}{3}$	$z_j - c_j$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{7}{3}$	0	

Since all $z_j - c_j \geq 0$, this optimality conditions are satisfied. The optimal solution is $x_1 = 0$, $x = \frac{8}{3}$, $x_3 = \frac{8}{3}$, $z_{max} = \frac{40}{3}$.

When c is changed from [2 1 4 − 1] to [1 2 3 4] this modified form of the optimal table becomes

		c_j	1	2	3	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
2	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	−2	−7	1	0	0

We see that there are negative $z_j - c_j$ viz $z_3 - c_3 = -2$ and $z_4 - c_4 = -7$. Hence the solution is not optimal. We apply simplex method to get the optimal solution.

		c_j	1	2	3	4	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Min ration
1	y_1	8	1	0	3	1	1	2	0	8
2	y_2	0	0	1	−1	−2	0	−1	0	–
0	y_7	5	0	0	−4	[2]	−2	3	1	$\frac{5}{2}$
		$z_j - c_j$	0	0	−2	−7	1	0	0	
1	y_1	$\frac{11}{2}$	1	0	[5]	0	2	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{11}{10}$
2	y_2	5	0	1	−5	0	−2	2	1	–
4	y_7	$\frac{5}{2}$	0	0	−2	1	−1	$\frac{3}{2}$	$\frac{1}{2}$	–
		$z_j - c_j$	0	0	−16	0	−6	$\frac{21}{2}$	$\frac{7}{2}$	
3	y_3	$\frac{11}{10}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$\frac{1}{10}$	$-\frac{1}{10}$	
2	y_2	$\frac{21}{2}$	1	1	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	
4	y_7	$\frac{47}{10}$	$\frac{2}{5}$	0	0	1	$\frac{1}{5}$	$\frac{17}{10}$	$\frac{3}{10}$	
$z = \frac{431}{10}$		$z_j - c_j$	$\frac{16}{5}$	0	0	0	$\frac{2}{5}$	$\frac{121}{10}$	$\frac{19}{10}$	

Since all $z_j - c_j \geq$ we have obtained this optimal table. The optimal solution is

$$x_1 = 0,$$

$$x_2 = \frac{21}{12},$$

$$x_3 = \frac{11}{10},$$

$$x_4 = \frac{47}{10}$$

and $z_{max} = \frac{431}{10} = 43\frac{1}{10}$.

..... initial table the basis is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_5 & a_6 & a_7 \end{bmatrix}$ and so inverse of the basis

of the final table is given by $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$.

The new solution when b is changed from $[8\ 0\ 21]^T$ to $[3\ -2\ 4]^T$ is given by

$$x_B^* = B^{-1} b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix}$$

This solution is not feasible but optimal. Hence to get the optimal solution we are to apply dual simplex method. The following are the modified optimal table and tables obtained by dual simplex method.

c_B	y_B	x_B (c_j)	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	-1	1	0	3	1	1	2	0
1	y_2	2	0	1	-1	-2	0	-1	0
0	y_7	-8	0	0	$\boxed{-4}$	2	-2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$				$-\frac{1}{4}$		-1			
2	y_1	-7	1	0	0	$\frac{5}{2}$	$\boxed{-\frac{1}{2}}$	$\frac{17}{4}$	$\frac{3}{4}$
1	y_2	4	0	1	0	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{7}{4}$	$-\frac{1}{4}$
4	y_3	2	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{4}$
		$z_j - c_j$	0	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{15}{4}$	$\frac{1}{4}$
$\frac{z_j - c_j}{y_{3j}} : y_{1j} < 0$				0		-3			
0	y_5	-14	-2	0	0	-5	1	$\frac{17}{4}$	$-\frac{3}{2}$
1	y_2	-3	1	1	0	0	0	$\frac{5}{2}$	$-\frac{1}{2}$
4	y_3	-5	1	0	1	2	0	$\frac{7}{2}$	$\frac{1}{2a}$
		$z_j - c_j$	3	0	1	9	0	$\frac{33}{2}$	$\frac{5}{2}$

We note here that $x_{B_3} = -5 < 0$ but all $y_{3j} \geq 0$. Hence this changed problem has no feasible solution.

When b_2 changed to 11, the new solution is given by

$$x_B^* = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix} = B^{-1} b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \\ 21 \end{bmatrix} = \begin{bmatrix} 30 \\ -11 \\ 38 \end{bmatrix}$$

Since $x_{B_2}=-11<0$, the solution is not feasible but optimal. So to get optimal solution we are to apply dual simplex method in the modified optimal table. The dual simplex tables are as follows :

		c_j	2	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	30	1	0	3	1	1	2	0
1	y_2	−11	0	1	−1	−2	0	−1	0
0	y_7	38	0	0	−4	2	−2	3	1
		z_j-c_j	0	0	1	1	2	3	0
$\frac{z_j-c_j}{y_{2j}}:y_{2j}<0$					−1	$-\frac{1}{2}$		3	
2	y_1	$\frac{49}{2}$	1	$\frac{1}{2}$	$\frac{5}{2}$	0	1	$\frac{3}{2}$	0
−1	y_4	$\frac{11}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0
0	y_7	27	0	1	−5	0	−2	2	1
$z=\frac{87}{2}$		z_j-c_j	0	$\frac{1}{2}$	$\frac{1}{2}$	0	2	$\frac{5}{2}$	0

In this tabl all $x_{B_i}>0$ and all $z_j-c_j\geq 0$. So we have reached to the optimal table.

The optimal solution is $x_1=\frac{49}{2}, x_2=0, x_3=0, x_4=\frac{11}{2}$ and $z_{max}=\frac{87}{2}$.

(e) When $c_1=1$ is replaced by c_1^* the modified form of the optimal table is given by

		c_1^*	1	4	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
c_1^*	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		z_j-c_j	0	0	$3c_1^*-5$	c_1^*-1	c_1^*	$2c_1^*-1$	0

This table remains as optimal tablie if all $z_j - c_j \geq 0$

i.e. if $3c_1^* - 5 \geq$ and $c_1^* - 1 \geq 0$ and $c_1^* \geq 0$ and $2c_1^* - 1 \geq 0$

i.e. if $c_1^* \geq \frac{5}{3}$ and $c_1^* \geq$ and c_1^* and $c_1^* \geq \frac{1}{2}$

i.e. if $c_1^* \geq \frac{5}{3}$.

(e) Alternative method using formula :

Since $c_1 \in c_B$, the range of Δc_1 for which the optimality of the solution is maintained is given by

$$\max\left\{\frac{z_j - c_j}{y_{1j}} : y_{1j} > 0\right\} \leq \Delta c_1 \leq \min\left\{\frac{z_j - c_j}{y_{1j}} : y_{1j} < 0\right\}$$

i.e. $\max\left\{-\frac{z_3 - c_3}{y_{13}}, -\frac{z_4 - c_4}{y_{14}}, -\frac{z_5 - c_5}{y_{15}}, -\frac{z_5 - c_5}{y_{16}}\right\} \leq \Delta c_1$

i.e. $\max\left\{-\frac{1}{3}, -\frac{1}{1}, -\frac{2}{1}, -\frac{3}{2}\right\} \leq \Delta c_1$

i.e. $-\frac{1}{3} \leq \Delta c_1 < \infty$

$\therefore \quad c_1 - \frac{1}{3} \leq c_1 + \Delta c_1 < c_1 + \infty$

or, $2 - \frac{1}{3} \leq c_1^* < \infty$

or, $\frac{5}{3} \leq c_1^* < \infty$

$\therefore$ If $c_1^* \geq \frac{5}{3}$ the optimal solution remain optimal.mα

4.6 Addition Of A Single Variable :

Let the optimal solution of the given LPP

Maximize $z = cx$

subject to $Ax = b, x \geq 0$

be known. Let x_{n+1} be added with it and the coefficient vector associated with x_{n+1} be a_{n+1} and the cost coefficient for x_{n+1} be c_{n+1}.

Since b is not changed the old optimal solution will be feasible solution of the new LPP but it may not be optimal. Let B be the optimal basis and C_B be the associated cost vector of the old LPP. Then they are also the same for the new LPP. It is optimum for the new LPP if $z_{n+1} - C_{n+1} \geq 0$.

In case $z_{n+1} - C_{n+1} < 0$, x_{n+1} will enter the solution and simplex method is to be applied to the old optimal table added with $(n + 1)$th column as $y_{n+1} = B^{-1} a_{n+1}$.

4.7 Illustrative Example :

Example 4.7.1: Consider the LPP

Maximize $z = x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 - x_3 \leq 2$

$2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Let a new variable $x_3' \geq 0$ be introduced with cost (i) 3 (ii) 5 and $a_3' = [2 - 1 - 4]$. Discuss the effect.

The solution of the LPP is obtained by simplex method. The following are the tables.

		c_j	1	2	1	−1	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
0	y_4	2	2	[1]	−1	1	0	0	2
0	y_5	6	2	−1	5	0	1	0	–
0	y_6	6	4	1	1	0	0	1	–
		$z_j - c_j$	−1	−2	−1	0	0	0	
2	y_2	2	2	−1	1	0	0	0	
0	y_5	8	4	0	[4]	1	1	0	2
0	y_6	4	2	2	−1	0	0	1	2
		$z_j - c_j$	3	−3	2	0	0	0	
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	
2	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	
$z = 10$		$z_j - c_j$	6	0	0	$\frac{1}{4}$	$\frac{3}{4}$	0	

$\therefore$ The optimal solution is $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

The inverse of the basis in the optimal table is

$$B^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 2 \end{bmatrix}$$

The added column for x_3' is $a_3' = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

The corresponding column in the final table is given by

$$y_3' = B^{-1} a_3' = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix}$$

(i) When $c_3' = 3$, we have

$$z_3' - c_3' - c_B y_3' - c_3' = [2\ 1\ 0] \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} - 3 = \frac{19}{4} - 3 = \frac{7}{4} > 0$$

∴ The optimatity condition is satisfied for the changed problem also. The optimal solution is $x_1 = 0$, $x_2 = 4$, $x_3 = 2$.

(ii) When $z_3' - c_3' = c_B y_3' - c_3' = [2\ 1\ 0] \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix} = \frac{19}{4} - 5 = -\frac{1}{4} < 0$

∴ Optimatity condition is not satisfied here.

We shall modify the optimal table of the old problem with added column $y_3' = \begin{bmatrix} \frac{9}{4} \\ \frac{1}{4} \\ \frac{6}{4} \end{bmatrix}$

and $z_3' - c_3' = -\frac{1}{4}$ and $c_3' = 5$. Then to get the optimal solution we are to apply simplex method. The tables obtained are as follows.

		c_j	1	2	1	5	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_3'	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	$\frac{9}{4}$	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{16}{9}$
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	8
0	y_6	0	0	0	0	$\boxed{\frac{6}{4}}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0 →
		$z_j - c_j$	6	0	0	$-\frac{1}{4}$	$\frac{11}{4}$	$\frac{3}{4}$	0	
2	y_2	4	3	1	0	0	$\frac{7}{2}$	1	$-\frac{3}{2}$	
1	y_3	2	1	0	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$	
5	y_3'	0	0	0	0	1	-1	$-\frac{1}{3}$	$\frac{2}{3}$	
$z = 10$		$z_j - c_j$	6	0	0	0	$\frac{5}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	

In this table all $x_{B_i} \geq 0$ and all $z_j - c_j \geq$. So the optimatily conditions are satisfied. The optimal solution is given by $x_1 = 0$, $x_2 = 4$, x_3 - 2, $x_3' = 0$ and $z_{max} = 10$.

4.8 Deletion of A Variable :

From a LPP if we delete a variable them two cases any arise.

Case 1. If this variable deleted is non basic then the feasibility and optimality conditions are not affected. So the optimal solution of the old problem is the optimal solution of the new problem.

Case 2. If the variable deleted is basic then the conditions of optimality may be affected and so a new solution is to be obtained. For this new optimal solution, we are assign a cost – M coresopnding to the basic variable to be deleted and apply simplex method after modifying the old optimal table.

4.9 Illustrative Example :

Example 4.9.1 For the LPP

Maximize $z = x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 - x_3 \leq 2$

$2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

the optimal table is

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1
$z = 10$		$z_j - c_j$	6	0	0	$\frac{1}{4}$	$\frac{3}{4}$	0

Discuss the effect of deletion of the variable (i) x_1 (ii) x_2 (iii) x_3.

(i) From the optimal table we see that x_1 is a deleted the optimal solution remains unaffected. Hence old optimal solution is also the new optimal solution is

$x_1 = 0, x_2 = 4, x_3 = 2$ & $z_{max} = 10$

(ii) From the optimal table we see that x_2 is a basic variable. Hence we make a new starting table by changing $c_2 = 2$ by $-$ M, where M is a big positive number. As M is very large the optimatity conditions are not affected and once it goes out from the basis it never reappears in the basis in the simplex method.

The modified table is

		c_j	1	−M	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
−M	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{4}{3}$
1	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	2
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
			−3M	0	0	$-\frac{5M}{4}+\frac{1}{4}$	$-\frac{M}{4}+\frac{1}{4}$	0	
1	y_1	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{5}{12}$	$\frac{1}{12}$	0	
1	y_3	$\frac{2}{3}$	0	$-\frac{1}{3}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	
$z = 2$		$z_j - c_j$	0	M	0	$\frac{1}{4}$	$\frac{1}{6}$	0	

In this table all $z_j - c_j \geq$ and all $x_{B_i} \geq 0$, so we have reached to optimal table. The optimal solution is

$$x_1 = \frac{4}{3}, x_2 = 0, x_3 = \frac{2}{3}$$

and $z_{max} = 2$.

(iii) From the optimal table we see that x_3 is a basic variable. Hence we make a new starting table by changing $c_1 = 1$ by − M, where M is a big positive number. As M is very large the optimality conditions are satisfied. Also once it goes out from the basis it never reappears in the basis in the simplex method. The modified table and other simplex tables are as follows :

		c_j	1	$-M$	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{4}{3}$
$-M$	y_3	2	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	2
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
			$-M+5$	0	0	$-\frac{5M}{4}+\frac{5}{2}$	$-\frac{M}{4}+\frac{1}{2}$	0	
1	y_1	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{5}{12}$	$\frac{1}{12}$	0	16
$-M$	y_3	$\frac{2}{3}$	0	$-\frac{1}{3}$	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	4
0	y_6	0	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	...
1	y_1	1	1	$\frac{1}{2}$		$\frac{1}{2}$	0	0	2.
0	y_5	4	0	-2		-11	1	0	...
0	y_6	2	0	-1		-2	0	1	...
2	y_2	2	2	1		1	0	0	
0	y_5	8	2	0		$-\frac{3}{2}$	1	0	
0	y_6	4	0	0		$-\frac{3}{2}$	0	1	
		4	3	0		2	0	0	

The optimal table is obtained and this optimal solution is $x_1 = 0$, x_2, $= 2$, $x_3 = 0$ and $z_{max} = 4.\alpha$

4.10 Addition Of A New Constraint :

Addition of a new constraint may or may not affect the current optimal solution. Two cases will arise.

(i) If the added constraint is satisfied by the old optimal solution then teh old optimal solution is also the new optimal solution.

(ii) If the added constraiant is not satisfied by the old optimal solution, then this old optimal solution becomes an infeasible solution for the new problem.

To obtain the optimal solution for the changed problem we are first to modify the fimal table and then apply dual simplex method.

The following three situations will arise depending on the nature of the solution to the original LPP.

If original LPP has an optimal solution then the modified LPP may have an optimal solution or it will give no F.S.

If the original LPP has unbounded solution then the modified LPP may have optimal solution or it will have no F.S. or it will have unbounded solution.

If the original LPP has no F.S. then the modified LPP will have also no F.S.α

4.11 Illustrative Examples :

Example 4.11.1 Let us consider the final table of a LPP

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
2	y_1	3	1	0	0	– 1	0	·5	·2	– 1
4	y_2	1	0	1	0	2	1	– 1	0	·5
1	y_3	7	0	0	1	1	– 2	5	– ·3	2
		$z_j - c_j$	0	0	0	– 1	0	2	1	2

where y_6, y_7 and y_8 are slack variables.

If the constraint

(i) $2x^1 + 3x^2 - x^3 + 2x^4 + 4x^5 \leq 5$

(ii) $2x^1 + 3x^2 - x^3 + 2x^4 + 4x^5 \leq 1$

is added then find the solution of the changed LPP.

Solution :

From the final table we see that the optimal solution of the old LPP is

$$x_1 = 3,\ x_2 = 1,\ x_3 = 7,\ x_4 = 0,\ x_5 = 0,\ x_6 = 0,\ x_7 = 0,\ x_8 = 0$$

(i) The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we have

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \leq 5$$

or, $6 + 3 - 7 \leq 5$

or, $2 \leq 5$.

This is true. So the solution satisfies the added constraint. Hence the old optimal solution is also optimal solution to the new problem.

The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 1$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we get

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \leq 1$$

or, $6 + 3 - 2 \leq 1$

or, $2 \leq 1$

This is not time *i.e.* the optimal solution to the old problem does not satisfy the added constraint. To get the solution of the new LPP we introduces the new constraint with a new slack variable in the optimal table of the old problem. We then modify this table to have a unit basis and then apply dual simplex method to it. The following are the tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	1	0	1	0	2	1	− 1	0	·5	0
1	y_3	7	0	0	1	− 1	− 2	5	− ·3	2	0
0	y_9	1	2	3	− 1	2	4	0	0	0	1
		$z_j - c_j$	0	0	0	2	0	2	·1	2	0
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	1	0	1	0	2	1	− 1	0	·5	0
1	y_3	7	0	0	1	− 1	− 2	5	− ·3	2	0
0	y_9	− 1	0	0	0	− 3	[− 1]	7	− ·7	2·5	1
			0	0	0	2	0	2	·1	2	0
$\frac{z_j - c_j}{y_{4j}} : y_{4j} < 0$						$-\frac{2}{3}$	0		$-\frac{\cdot 1}{7}$		
2	y_1	3	1	0	0	− 1	0	·5	·2	− 1	0
4	y_2	0	0	1	0	− 1	0	6	− ·7	3	0
1	y_3	9	0	0	1	5	0	− 9	1·1	− 3	− 2
2	y_5	1	0	0	0	3	1	− 7	·7	− 2·5	− 1
			0	0	0	2	0	2	·1	2	0

The second table is obtained by the operation $R_4^1 = R_4 - 2R_1 - 3R_2 + R_3$. The third table is obtained by using dual simplex method to the second table and is the the final table. The optimal solution is $x_1 = 3, x_2 = 0, x_3 = 9, x_4 = 0, x_5 = 1$.

4.12 Summary :

The usefulness of post-optimality analysis is discussed. Then onle by one the different situations viz discrete changes in the cost vector and requirement vector, addition and deletion of a single variable, and addition of a new constraint are discussed. Each situation is illustrated by examples.

4.13 Self Assessment Questions :

1. For the LPP

Maximize $z = 15x_1 + 45x_2$

subject to $5x_1 + 2x_2 \le 162$

$x_1 + 16x_2 \le 240$

$x_2 \le 50$

$x_1, x_2 \ge 0$

find the optimal solution. Find the range of each cost coefficient (changed one at a time) to give same optimal solution.

[Ans : $x_1 = 352/13$, $x_2 = 173/13$, $z_{max} = 1005$]

2. Find how much the 7 in the first constraint of the problem

Minimize $z = x_1 - 3x_2 + 2x_3$

subject to $3x_1 - x_2 + 2x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

be changed before the basis of the optimal table would change.

3. Find the optimal solution of the LPP and the separate ranges of variations of b_2 and b_3 consistent with the optimatity of the solution

Minimize $z = -x_1 + 2x_2 - x_3$

subject to $3x_1 + x_2 - x_3 \le 10$

$-x_1 + 4x_2 + x_3 \ge 6$

$x_2 + x_3 \le 4$

$x_1, x_2, x_3 \ge 0.$

Determine also this efficient discrete changes in the components of the cost vector which correspond to the basic variables.

[Ans : $x_1 = 0$, $x_2 = 4$, $x_3 = 0$; $\Delta b_2 \le 10$, $-5/2 \le \Delta b_3 \le 6$, $-2 \le \Delta c_2$]

4. Following is the optimal table for an LPP

		c_j	2	1	1	2	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
2	a_1	3	1	0	− 1	3	2
1	a_2	4	0	1	4	− 1	− 2
			0	0	1	3	2

(i) Find the limitations of this values of c_3, c_4, c_5 (taking one at a time) for which the current solution will remain optimal.

(ii) Find the optimal solution to the problem, if c_3 is changed to 3.

(iii) Find the limitations of the values of c_1 for which the current solution remains optimal.

(iv) Find the optimal solution to this problem, if c_1 is changed to 5.

[Ans : (i) $-\alpha < c_3 \leq 2, -\alpha < c_4 \leq 5, -\alpha < c_5 \leq 2$

(ii) $x_1 = 4, x_3 = 1, x_2 = 0, x_4 = 0$

(iii) $1 \leq c_1 \leq 3$

(iv) $x_1 = 13/4, x_2 = 0, x_3 = 1, x_4 = 0$]

5. Find the optimal solution of the IPP

Maximize $z = 4x_1 + 3x_2$

subject to $x_1 + x_2 \leq 5$

$3x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

Show how to find the optimal solution of the problem, if

(i) the first component of the original requirement vector be increased by one unit and the third component be decreased by one unit.

(ii) the second component of the original requirement vector be decreased by two units.

[Ans : (i) $x_1 = 1, x_2 = 4, z_{max} = 16$

(ii) $x_1 = 0, x_2 = 5, z_{max} = 15$]

Unit 5 ☐ Quadratic Programming Problem

Structure

5.1 Introduction :

Quadratic programiing problem is the most well behaved nonlinear programming problem. Quadratic programming deals with non-linear programming problem of maximizing (or minimizing) quadratic objective function subject to a set of linear inequality constraints. The solution of this problem is based on the Kuhn-Tucker conditions. The quadratic objective function to be optimized is taken as strictly convex for minimization and strictly concave for maximization. As the solution space is always convex, the optimal the solution obtained is global is nature.

Definition 5.1.1 : Let x^T and $C \in R^n$ and Q be a symmetric $n \times n$ real matrix then, the problem quadratic pogamming problem is

Maximize (or minimize) $f(x) = cx + \frac{1}{2} x^T Q x$

subject to $Ax \le b$

$x \ge 0$

where $x = [x_1, x_2, ..., x_n]^T$

$c = [c_1, c_2, ..., c_n]$

$b = [b_1, b_2, ... b_m]^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{and } Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

The function $x^T Qx$ defines a quadratic form when Q is a symmetric matrix.

The quadratic form $x^T Qx$ is said to be positive-definite if $x^T Qx \geq$ for all $x \neq 0$.

The quadratic form $x^T Qx$ is said to be positive semi definite if $x^T Qx \geq$ for at one $x \neq 0$.

The quadratic form $x^T Qx$ is said to be negative definite and negative semi-definite if $- x^T Qx$ is positive definite and positive semi-definite respectively.

In quadratic programming problem $x^T Qx$ is assmed to be negative definite in the maximization case, and positive definite in the minimization case. These means that $f(x) = cx + \frac{1}{2} x^T Qx$ is assumed to be strictly convex function for minimmzation case and strictly concave fomaximization case.

As the constraints are always assumed to be linear, the solution space of a quadratic programming problem is always convex.

Thus the solution obtained using Kuhn-Tucker conditions given global optimum of the quadatic programming problem.

5.2 Kuhn-Tucker Conditions for Quadratic Programming Problem :

Let the quadratic programming problem be

Maximize $f(x) = \sum_{j=1}^{n} c_j x_j \frac{1}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} c_{jk} x_j x_k$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots\ldots, m$$

and $x_j \geq 0, j = 1, 2, \ldots\ldots, n$

where $c_{jk} = c_{kj}$ for all j and k.

Introducing slack variables q_i^2 and r_j^2 the problem reduces to

$$\text{Maximize } f = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 = 0, \; i = 1, 2, \ldots\ldots m$$

$$-x_j + r_j^2 = 0, \; j = 1, 2, \ldots\ldots n.$$

The Lagangian function is given by

$$L(x_1, x_2, \ldots, x_n, q_1, q_2, \ldots, q_m, r_1, r_2, \ldots, r_n, \lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n)$$

$$= \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k - \sum_{i=1}^{m} \lambda_j \left(\sum_{j=1}^{n} a_{ij} x_j - b_i + q_i^2 \right) - \sum_{j=1}^{n} \mu_j \left(-x_j + r_j^2 \right)$$

The Kuhn-Tucher conditions are given by

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j(-1) = 0, \quad j = 1, 2, \ldots\ldots, n$$

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0, \quad i = 1, 2, \ldots\ldots, m$$

$$\mu_j x_j = 0, \quad j = 1, 2, \ldots\ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j, \quad i = 1, 2, \ldots\ldots, m$$

$$x_j \geq 0, \quad j = 1, 2, \ldots\ldots, m$$

$$\lambda_i \geq 0, \quad i = 1, 2, \ldots\ldots, m$$

$$\mu_j \geq 0, \quad j = 1, 2, \ldots\ldots, n$$

Letting $q_i^2 = s_i \geq 0$ these equations becomes

$$\left.\begin{array}{l} c_j + \sum_{k=1}^{n} c_{jk}\, x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0 \\ \sum_{j=1}^{n} a_{ij}\, x_j - b_i + s_i = 0, i = 1,2,\ldots,m \end{array}\right\} \quad \ldots \quad \ldots \quad (1)$$

$$\left.\begin{array}{l} \lambda_1 s_i = 0, i = 1,2,\ldots,m \\ \mu_j x_j = 0, j = 1,2,\ldots,n \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$$

$$\left.\begin{array}{l} \lambda_i \geq 0, i = 1,2,\ldots,m \\ \mu_j \geq 0, j = 1,2,\ldots,n \\ x_j \geq 0, j = 1,2,\ldots,n \\ s_i \geq 0, i = 1,2,\ldots,n \end{array}\right\} \quad \ldots \quad \ldots \quad (3$$

(1) is a system of $m + n$ linear equations in x_j, λ_i, μ_j and s_i.

The solution of these system which will satisfy also (2) and (3) is the required optimal solution of the quadrative programming problem.

5.3 Wolfe's Modified Simplex Method :

To solve the system (1) satisfying the conditions (2) and (3) Wolfe suggested to introduce the non-negative artificial variables $\beta_1, \beta_2, \ldots\ldots \beta_n$ in the Kuhn-Tucker conditions (1) and to constract an objective function $z = -\beta_1 - \beta_2 - \ldots - \beta_n$ and to consider the following LPP with complementary slackness condition.

Maximize $\quad z = -\beta_1 - \beta_2 - \ldots\ldots - \beta_n$

subject to $\quad \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = -c_j, j = 1, c, \ldots, n$

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, i = 1,2,\ldots\ldots,m$$

$$\lambda_1, s_i, x_j, \mu_i, \mu_j, \beta_j \geq 0, i = 1,2,\ldots\ldots,m, j = 1,2,\ldots\ldots,n$$

and satisfying the complementarhy slackness conditions

$$\lambda_i s_i = 0, i = 1,2,\ldots\ldots,m$$

$$\mu_j x_j = 0, j = 1,2,\ldots\ldots,n$$

The optimum solution of theis LPP gives the optimum solution of the given QPP.

Note : To maintain the condition $\lambda_i s_i = 0 = \mu_j x_j$ all the time we should note that if λ_i is in the basic solution with positive value then s_i can not be basic with positive value. Similarly μ_j and x_j cannot be in the basic solution (i.e. positive) simultaneously.

Example 5.3.1 Using Wolfe's method solve the quadratic programming problem

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$

$$\text{subject to} \quad 2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Solution : First be write all constraints with '≥' sign to get the problem as

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$

$$\text{subject to} \quad 2x_1 + 3x_2 \le 0$$

$$2x_1 + x_2 \le 4$$

$$x_1 \quad \le 0$$

$$-x_2 \le 0$$

Introducing slack variable q_1^2, q_2^2, r_1^2 and r_2^2 we get

$$\text{Maximize } z = 2x_1 + x_2 - x_1^2$$

$$\text{subject to} \quad 2x_1 + 3x_2 + q_1^2 = 6$$

$$2x_1 + x_2 + q_2^2 = 4$$

$$-x_1 + r_1^2 = 0$$

$$-x_2 + r_2^2 = 0$$

We now constuct the Largrange function

$$L\ (x_1, x_2, q_1, q_2, r_1, r_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$$

$$= \left(2x_1 + x_2 - x_1^2\right) - \lambda_1\left(2x_1 + 3x_2 + q_1^2 - 6\right) - \lambda_2\left(2x_1 + x_2 + q_2^2 - 4\right)$$

$$-\mu_1\left(-x_1 + r_1^2\right) - \mu_2\left(-x_2 + r_2^2\right)$$

The Kuhn-Tucker's necessay and sufficient conditions gives

$$\frac{\partial L}{\partial x_1} = 0 \text{ or, } 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \text{ or, } 1 - 3\lambda_1 - \lambda_2 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \text{ or, } 2x_1 3x_2 + q_1^2 - 6 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \text{ or, } 2x_1 + x_2 + q_2^2 - 4 = 0$$

$$\lambda_1 q_1^2 - 0,\ \lambda_2 q_2^2 = 0,\ \mu_1 x_1 = 0,\ \mu_2 x_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$$

Taking $q_1^2 = s_1$ and $q_2^2 = s_2$ we get

$$2x_1 + 2\lambda_1 - \mu_1 = 2$$

$$3\lambda_1 + \lambda_2 - \mu_2 = 1$$

$$2x_1 + 3x_2 + s_i = 6$$

$$2x_1 + x_2 + s_2 = 6$$

$$\lambda_1 s_1 = 0,\ \lambda_2 s_2 = 0,\ \mu_1 x_1 = 0,\ \mu_2 x_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, s_1, s_2 \geq 0$$

With necessary modification we use phase I of two phase method to solve this system Introducing artificial variables β1 and β2 the modified LPP become

Maximize $z' = -\beta_1 - \beta_2$

subject to $2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 + \beta_1 + \quad = 2$

$$3\lambda_1 + \lambda_2 - \mu_2 + \beta_2 = 1$$

$$2x_1 + 3x_2 \quad s_1 = 6$$

$$2x_1 + x_2 \quad s_2 = 4$$

$$\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \lambda_1 s_1 = 0,\ \lambda_2 s_2 = 0$$

$$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \beta_1, \beta_2, s_1, s_2 \geq 0$$

Initiat talle of Phase–I is

		C_j	0	0	0	0	0	0	–1	–1	0	0	
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2	
–1	β_1	2	[2]	0	2	2	–1	0	1	0	0	0	→
–1	β_2	1	0	0	3	1	0	–1	0	1	0	0	
0	s_1	6	2	3	0	0	0	0	0	0	1	0	
0	s_2	4	2	1	0	0	0	0	0	0	0	1	
z' =	–3		–2	0	–5	–3	1	1	0	0	0	0	
			↑										

Accoding to the regular pocedure λ_1 enters and β_2 leave the basis is $\lambda_1 > 0$ & $\beta_2 = 0$. But $s_1 = 6$ $\quad \therefore \lambda_1 s1 \neq 0$.

$\therefore \lambda_1$ cannot enter the basis.

Next negative z; –e, is associated with λ_2. If λ_2 enters the basis then β_1 and β_2 will leave the basis is $\lambda_1 > 0$.

Since $s_2 = 4$ we have $\lambda_2\, s_2 \neq 0$. So λ_2 cannot enter the basis.

Next negative z ; –e ; is associated with x_1. If x_1 eneris the basis then β_1, leaves the basis ie. $x_1 \geq 0$. This is accepted since $\mu_1 = 0$ & $\mu_1 . x_1 = 0$ is satisfied.

The next table is

		C_j	0	0	0	0	0	0	–1	–1	0	0
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2
–1	x_1	1	1	0	1	1	–1/2	0	1/2	0	0	0
–1	β_2	1	0	0	3	1	0	–1	0	1	0	0
0	s_1	4	0	[3]	–2	–2	1	0	–1	0	1	0
0	s_2	2	0	1	–2	–2	1	0	–1	0	0	1
z' =	–3		0	0	–3	–1	0	1	1	0	0	0
				↑								

Here λ_1 enters and β_1 leaves the basis i.e.. $\lambda_1 > 0$, $\beta_2 = 0$

This is not accepted since $s_1 = 4 \quad \therefore \lambda, s \neq 0$.

If λ_2 enters the basis then x_1 or β_2 leaves the basis..

This is not also accepted since $s_2 = 2$ & so $\lambda_2 s_2 \neq 0$

We select x_2 to enter the basis. Then s_1 leaves the basis.

The next table is

		C_j	0	0	0	0	0	0	–1	–1	0	0	
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2	
0	x_1	1	1	0	1	1	–1/2	0	1/2	0	0	0	
–1	β_2	1	0	0	[3]	1	0	–1	0	1	0	0	→
0	x_2	4/3	0	1	–2/3	–2/3	1/3	0	–1/3	0	1/3	0	
0	s_2	2/3	0	0	–4/3	–4/3	2/3	0	–2/3	0	–1/3	1	
$z' =$	–1		0	0	–3	–1	0	1	1	0	0	0	
					↑								

Hare λ_1 enters the basis and β_2 leaves the basis. This is acceptable since $s_1 = 0 \quad \therefore \lambda_1 s_1 = 0$.

The next table is

		C_j	0	0	0	0	0	0	–1	–1	0	0
C_B	B.V	X_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	β_1	β_2	s_1	s_2
0	x_1	2/3	1	0	0	2/3	–1/2	1/3	1/2	–1/3	0	0
0	λ_1	1/3	0	0	1	1/3	0	–1/3	0	1/3	0	0
0	x_2	14/9	0	0	0	–4/9	1/3	–2/9	–1/3	2/9	1/3	0
0	s_2	10/9	0	0	0	–8/9	2/3	–4/9	–2/3	4/9	–1/3	1
$z' =$	0		0	0	0	0	0	0	1	1	0	0

In this table $\beta_1 = 0$ and $\beta_2 = 0$. So this is the final table.

The optimal solution is

$x_1 = 2/3,\ x_2 = 14/9,\ \lambda_1 = 1/3,\ \lambda_2 = 0,\ s_1 = 0,\ s_2 = 10/9,\ \mu_1 = 0,\ \mu_2 = 0$

The complementay stachness conditions

$\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \lambda_1 s_1 = 0$ & $\lambda_2 s_2 = 0$ are satisfied.

∴ The optimal solution of the given quadratic programming problem is

$x_1 = 2/3,\ x_2 = 14/9$

and $z_{man} = 2(2/3) \mid 14/9 - 2/3 = 22/9$

5.4 Beale's Method

Beale suggested another approach to solve quadratic programming problem (QPP)

Let the QPP be of the from

$$\text{Maximize } f(x) = cx + \frac{1}{2} x^T Q x$$

$$\text{subject to } Ax = b,\ x \geq 0$$

Where $x = [x_1, x_2, \ldots\ldots x_n]^T$, $C = [c_1, c_2, \ldots.c_n]$, A is mxn matrix and Q is symmetric matrix.

In This method the variables are partitioned into basic and non-basic variables. At each iteration, the objective function is expressed in terms of te non-basic variables.

The Beale's iteative procedure of solving QPP is stated below :

Step 1. Express the constaints of the given QPP as equations by introducing slack / surplus variables to get $Ax = b$.

Step 2. Select arbitrarily m variables as basic and the remaining n–m variables as non-basic. With this partitioning, the constraint equation $Ax = b$ can be written as

$$[B \quad R] \begin{bmatrix} x_B \\ x_B \end{bmatrix} = b$$

or, $Bx_B + Rx_R = b$

Where x_B and x_R denote the basic and non-basic vectors respectively. Thus we get

$$x_B = B^{-1}b - B^{-1}Rx_R$$

Step 3. Express the basic x_B in terms of non-basic x_R only, using the given and additional constraint equations, if any.

Step 4. Express the objective function $f(x)$ in terms of xR only using the given and additional constraints, if. As $x_B \geq 0$ we have $B^{-1}RxR \leq B^{-1}b$. Thus, any component of xR can increase only until $\delta f/\delta x_R$ becomes zero, or one or more components of x_B are reduced to zero.

Note that we face the possibilit of having moer than m non-zero variables at any step of teration. This stage comes when the new point generated at some step occurs were $\delta f/\delta x_R$ becomes zero. Geometrically, this means that we are no longe at an extreme point of the convese set formed by the constaints, and thus no longer have a basic solution with respect to the original constraint set. When this happens, we simply define a new variables s_i as $s_i = \delta f/\delta x_{Ri}$ and a new constraint $s_i = 0$.

Step 5. At this stage, we have $m + 1$ non-zero varibles and $m + 1$ constraints, which is a basic solution to the extended set of constaints.

Step. Repeat the above procedure until no further improvement of the objective function may be obtained by increasing one of the non-basic variables.

Example 5.4.1. Using Beale's method solve the QPP

Maximize $z = 5 + 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \leq 0$

$x_1, x_2 \geq 0$

Solution :

Introducing slacle variable $x_3 \geq 0$, the given QPP becomes

Maximize $z = 5 + 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 + x_3 = 2$

$x_1, x_2, x_3 \geq 0$

We choose x_1 arbitarily as basic variable and express it in terms of x_2 and x_3. Thus

$$x_1 = 2 - 2x_2 - x_3$$

We now express the objective functions z in terms of x_R $z = 5 + 4\,(2 - 2x_2 - x_3) + 6x_2 - 2\,(2 - 2x_2 - x_3)^2 - 2\,(2 - 2x_2 - x_3)\,x_2 - 2x_2^2$

$$\therefore\ \frac{\partial z}{\partial x_2} = -8 + 6 - 4\,(2 - 2x_2 - x_3)\,(-2) - 2\,(2 - 4x_2 - x_3) - 4x_2$$

At $x_2 = 0$m $x_3 = 0$ We have $\frac{\partial z}{\partial x_2} = -8 + 6 + 14 - 4 = 10$

This means z will increase if x_2 is increased from zero.

Also $\frac{\partial z}{\partial x_3} = -4 + 4\ (2 - 2x_2 - x_3) + 2x_2$

$\therefore$ At $x_2 = 0$, $x_3 = 0$ we have $\frac{\partial z}{\partial x_3} = -4 + 8 = 4$

We see that the rate of increase of z with respect to x_2 is more.

Hence incease in x_2 will give better improvement in the objective function.

To find how much x_2 should or may increase, we check two quantities.

(i) the value of x_2 for which $\delta z/\delta x_2$ vanishes.

(ii) the largest value of x_2 attained without deriving the basic variable x_1 negative.

Then x_2 will be minimum of these two.

Now $\delta z/\delta x_2 = 0$ gives for $x_3 = 0$

$$-2 + 8\ (2 - 2x_2) - 2\ (2 - 4x_2) - 4x_2 = 0$$

or, $-2 + 16 - 16x_2 - 4 + 8x_2 - 4x_2 = 0$

or, $-12x_2 + 10 = 0$

or, $x_2 = 5/6$

And for $x_3 = 0$, $x_1 < 0$ gives $2 - 2x_2 < 0$ or, $x_2 > 1$

We have min $\{5/6, 1\} = 5.6$. Thus the new basic variable is x_2.

Expressing x_2 is terms of x_1 and x_3 we get

$$x_2 = 1 - x_{1/2} - x_{3/2}$$

We now express z in terms of x_1 and x_3 as

$$z = 5 + 4x_1 + 6\ (1 - x_{1/2} - x_{3/2}) - 2x_1^2 - 2x_1\ (1 - x_{1/2} - x_{3/2})$$
$$-2\ (1 - x_{1/2} - x_{3/2})^2$$

Now $\frac{\partial z}{\partial x_1} = 4 - 6\ (-1/2) - 4x_1 - 2x_1\ (-1/2) - 2\ (1 - x_{1/2} - x_{3/2})$

$$- 4\ (1 - x_{1/2} - x_{3/2})\ (-1/2)$$

$$= 1 - 3x_1$$

$$\frac{\partial z}{\partial x_3} = 6\ (1 - 1/2) - 2x_1\ (-1/2)\ \ 4\ (1 - x_{1/2} - x_{3/2})\ (-1/2)$$

$$= -1 - x_3$$

At $x_2 = 0,\ x_3 = 0$ We have $\frac{\partial z}{\partial x_1} = 1$ and $\frac{\partial z}{\partial x_3} = -1$

Ths z increases as x_1 is increases. So x_1 can be introduced to incease z.

To find how much x_1 should or may increase, we check two quantities.

(i) the value of x_1 for which $\delta z/\delta x_1$ vanishes.

(ii) the largest value of x_1 attained without deriving the basic variable x_2 negative.

The x_1 will be minimum of these two.

Fo $x_3 = 0$, $\delta z/\delta x_1 = 0$ gives $1 - 3x_1 = 0$ or $x_1 = 1/3$

For $x_2 = 0$, $x_2 < 0$ gives $1 - x_{1/2} < 0$ or, $x_1 > 2$

We have min $\{1/3, 2\} = 1/3$

Hence we find $x_1 = 1/3$ and the new basic vaiable is x_1.

At $x_1 = \frac{1}{3}$, $x_3 = 0$ we have $\frac{\partial z}{\partial x_1} = 0, \frac{\partial z}{\partial x_3} = -1$. Thus the optimal solution has been attained & the optimal solution is $x_1 = 1/3$, $x_2 = 1 - 1/6 - 0 = 5/6$, $x_3 = 0$ and man $x = 5 + 4/3 + 6 \times 5/6 - 2x\ (1/3)^2 - 2\ (1/3)\ (5/6) - 2x\ (5/6)^2 = 55/6$

5.5 Summary

Quadratic programming problem is concerned with non linear progamming problem of maximizing (or minimizing) the quadratic objective function subject to a set of linear inequality constaints. Wolfe's modified simplex method and Beale's method are discussed here with examples.

5.6 Self Assessment Questions

1. Applying wolfe's method solve the following quadatic pogramming problems

(i) Maximize $f = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \leq 2$

$x_1, x_1 \geq 0$

(ii) Maximize $z = 12x_1 + 12x_2 - 18x_1^2 - 12x_1x_2 - 8x_2^2$

subject to $3x_1 + 4x_2 \leq 2$

$x_1, x_1 \geq 0$

(iii) Maximize $f = 3x_1 + 2x_2 - 2x_2^2$

subject to $4x_1 + x_2 \leq 4$

$2x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 10x_1 + 6x_2 - 50x_1^2$

subject to $5x_1 + 8x_2 \leq 4$

$5x_1 + 4x_2 \leq 2$

$x_1, x_2 \geq 0$

(iv) Maximize $f = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $2x_1 + x_2 \leq 6$

$x_1 - 4x_2 \leq 0$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 2x_1 + 3x_2 - 2x_1^2$

subject to $x_1 + 4x_2 \leq 4$

$x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

2. Use Beale's method of solve the following quadiratic linear programming problems

(i) Maximize $z = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

(ii) Maximize $z = 2x_1 + 3x_2 - x_1^2$

subject to $x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

(iii) Maximize $f = 2x_1 + 3x_2 - 2x_2^2$

subject to $x_1 + 4x_2 \le 4$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

(iv) Maximize $f = 12x_1 + 6x_2 - 18x_1^2 - 6x_1x_2 - 2x_2^2$

subject to $3x_1 + 2x_2 \le 2$

$x_1, x_2 \ge 0$

Unit 6 ❑ Integer Programming Problem

Structure

6.1 Introduction

Integer Programming Problem (IPP) is a special class of Linear Programming Problem where all or some of the variables in the optimal solution are restricted to the integers. If all the variables are restricted to take integral values the IPP is termed as pure IPP. On the other hand, if only some variables are restricted to take only integer values then the problem is called mixed IPP.

In 1956, R. E. Gomory developed a method to solve pure IPP. Later, he extended the method to solve mixed IPP. Another important approach, called the "branch and bound" technique was developed for solving both the all integer and he mixed integer programming problems.

Several algorillms have yet been developed for solving both types of IPP. We shall discuss only.

(i) Gomory's cutting plane method for pure IPP. and

(ii) Branch and bound method.

6.2 Need for Integer Programming

To solve an IPP one may think to get the optimal solution just by rounding down the optimal solution of the corresponding LPP obtained by regular simplex method. But there is no gaurantee for this. It may or may not happen so. The integer solution obtaind by rounding down the optimal solution of the corresponding LPP will not always satisfy all constraints or will not give the actual optimal solution of the IPP. These are explained by following examples.

Example 6.2.1

Maximize $z = 3x_1 - 2x_2$

subject to $12x_1 + 7x_2 \leq 28$

$x_1, x_2 \geq 0$

x_1, x_2 are integers.

Ignoring the integer restriction here the optimal solution is $x_1 = 2\frac{1}{3}$, $x_2 = 0$ with max $z = 7$.

The solution obtained by rounding down this optimal solution is $x_1 = 2$, $x_2 = 0$ this solution is the optimal solution of the given Integer programming problem.

Example 6.2.2

Minimize $z = 2x_1 + 3x_2$

subject to $80x_1 + 31x_2 \geq 248$

$x_1, x_2 \geq 0$, x_1, x_2 are integers.

Here, ignoring the integer restriction, the optimal solution is $x_1 = 3\frac{1}{10}$, $x_2 = 0$ with min $z = 6\frac{1}{5}$

Rounding down the solution we get $x_1 = 3$, $x_2 = 0$

But this point does not lie in the feasible region since $80 \times 3 + 31.0 = 240 < 248$.

Hence just rounding the optimal solution of the corresponding LPP to the given IPP we may not get the optimal solution of the IPP.

Example 6.2.3

Maximize $z = 3x_1 + 4x_2$

subject to $4x_1 + 6x_2 \le 15$

$x_1, x_2 \ge 0$

x_1, x_2 are integers.

Ignoring the integer-valued restriction The optimal solution of the problem is $x_1 = 3\frac{3}{4}$, $x_2 = 0$ with max $z = 11\frac{1}{4}$

Rounding off this solution we get $x_1 = 3$, $x_2 =$ or, $x_1 = 4$, $x_2 = 0$.

For $x_1 = 3$, $x_2 = 0$ we have $z = 3 \times 3 + 4 \times 0 = 9$

$x_1 = 4$, $x_2 = 0$ does no satisfy $4x + 6x_2 \le 15$. Here the actual solution tothis IPP is $x_1 = 2$, $x_2 = 1$ with max $z = 10$.

6.3 Gomory's cutting plane method for all IPP

In this method we first find the optimal solution to the IPP by simplex method ingoring the integer valued restriction. If in the optimal solution all the variables have integer values, then it is also the optimum solution of the given IPP. But if not, then a new constraint, called secondary an Gomory's constraint is introduced to the problem which slice away non-integer optimal solution exhibited by the extreme point of the feasitle region of the associated LPP and at the same time leave all feasible integer solutions untouched. The new related LPP is then solved as usual. If the new optimal solution obtained does not satisfy the integer requirement, then another Gomory's constraint is added and the process is repeated iteratively until the required integer valued optimum solution is obtained. As each introduced Gomory's constraint cut off a portion of the feasible region of the related LPP, the method is called Gomory's cutting plane method.

6.3.1 Construction of Gomory's constraints

Ignoring the integer restriction let the optimal solution of the given IPP using simplex method be x_B. Also let this optimal solution has at least one non-integer component. If more than one basic variable are fractional, we select that non-integral variable which involves the largest fractional part.

As x_{Br} corresponds to the rth now of simplex table we consider the rth now. of the final tables as

$$\sum_{j=1}^{n} y_{rj} x_j = b_r \quad .... \quad .... \quad .... (1)$$

Let $[y_{rj}]$ denote the greatest integer less than y_{rj} and fri denote the positive fractional part of y_{rj}. Similarly, let $[b_r]$ and f_r be resputively the greatest integer less than b_r and the positive fractional part of br.

Then we have $y_{rj} = [y_{rj}] + f_{rj}$

and $b_r = [b_r] + f_r$ where $0 < f_{rj} < 1$ and $o < f_r < 1$.

From (1) we have thus

$$\sum_{j=1}^{n}[y_{rj}]x_j + \sum_{j=1}^{n} f_{rj}\, x_j = [b_r] + f_r$$

and $$f_r - \sum_{j=1}^{n} f_{rj}\, x_j = [b_r] - \sum_{j=1}^{n}[y_{rj}]x_j \quad .... \quad .... \quad .... (2)$$

For integer value of x_j the RHS of (2) is an integer. So LHS of (2) must be an integer. Now f_r is a proper fraction *i.e.* $0 < f_r < 1$ and $\sum_{j=1}^{n} f_{rj} x_j$ is positive thus (2) gives.

(A proper fraction) – (positive number) = (integer)

Hence RHS is either zero or negative integes.

So LHS is also either zero or negative integer

i.e. LHS ≤ 0

or, $$\sum_{j=1}^{n} f_{rj} x_j \leq 0$$

or, $$-\sum_{n=1}^{n} f_{rj} x_j \leq -f_r$$

Introducing plack variable x_s this becomes

$$-\sum_{n=1}^{n} f_{ij}x_j + x_s = -f_t$$

This is the Gomory's constraints which is to be introduced to the given problem to form a new LPP to be solved the dual simplex method.

6.3.2 Gomory's cutting Plane Algorithm

The following are the four steps of solving all integer IPP by Gomory's cutting plane method.

Step 1. Using simplex method find the optimal solution of the IPP ignoringthe integral value restructions.

Step 2. If all the variables have integral values, take this solution as the optimal solution of the given IPP.

If at least one varible in the optimal solution obtained in setp 1 has fractional value then identify the now involving the largest fractional part. Using this row from the Gomory's constraint.

Step 3. Augment the IPP by introducing the Gomory's constraint formed in step 2 and modify the table. Using dual simplex method find the new optimal solution of the augmented LPP.

Step 4. If all variables of the optimal solution obtained in setp 3 are integers, then this is the required optimal solution of the original IPP. Otherwise go to step 2 and again augment the IPP by a new Gomory's constraint.

Example 6.3.1 Use Gomory's cutting plane method to find the optimal solution of the IPP

Maximize $z = x_1 + x_2$

subject to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0$

x_1, x_2 are integers.

Solution : Ignoring the intergal value restriction we solve itby simplex method. Introducing slack variables x_3 and x_4 the LPP becomes

Maximize $z = x_1 + x_2 + 0x_3 + 0x_4$

subject to $2x_1 + 5x_2 + x_3 = 16$

$6x_1 + 5x_2 + \quad + x_4 = 30$

$x_1, x_2, x_3, x_4 \geq 0$

Using simplex method the tables are obtained

	c_j		1	1	0	0	
C_B	y_B	x_B	y_1	y_2	y_3	y_4	min ratio
0	y_3	16	2	5	1	0	8
0	y_4	30	[6]	5	0	1	5 →
z = 0			–1	–1	0	0	
0	y_3	6	0	[10/3]	1	–1/3	9/5 →
1	y_1	5	1	5/6	0	1/6	6
z = 5			0	–1/6	0	1/6	
1	y_2	9/5	0	1	3/10	–1/10	
1	y_1	7/2	1	0	–1/4	1/4	
z = 53/10			0	0	1/20	3/20	

In this object table we see that both the variables are fractional and are 9/5 = 1 + 4/5, 7/2 = 3 + 1/2. The largest fractional part is 4/5 and is associated with the first row. The first row written in the form of equation is

$x_2 + (3/10)\ x_3 - (1/10)\ x_4 = 9/5$

Writing 3/10 = 0 + 3/10, –1/10 = –2 + 9/10 and 9/5 = 1 + 4/5 this becomes

$x_2 + 0x_3 + (3/10)\ x_3 - 2x_4 + (9/10)\ x_4 = 1 + 4/5$

∴ The Gomory's constraint is

$-(3/10)\ x_3 - (9/10)\ x_4 - \leq (4/5)$

Introducing plack variable $x_5 \leq 0$ we get

$-(3/10)\ x_3 - (9/10)\ x_4 + x_5 = -(4/5)$

Adding this Gomory's constraint to the above optimum table, we get modified table as follows :

C_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
	c_j		1	1	0	0	0
1	y_2	9/5	0	1	3/10	–1/10	8
1	y_1	7/2	1	0	–1/4	1/4	0
0	y_5	–4/5	0	0	[–3/10]	–9/10	1
		$z_j - c_j$	0	0	1/20	3/20	1
$\frac{(z_j - c_j)}{y_{3j}}$: $y_{3j} < 0$					$\frac{1/20}{-3/10}$	$\frac{3/20}{-9/10}$	
1	y_2	1	0	1	0	-1	1
1	y_1	25/6	1	0	0	1	–5/6
0	y_3	8/3	0	0	1	3	–10/3
		$z_j - c_j$	0	0	0	0	1/6

In this optimal table the basic variable x_1 is fractional (it is a variable of the original given IPP). It is associated with second row. We consider the second row and write it as equation to form Gomory's second constraint.

$x_1 + x_5 \; - (5/6)\, x_5 = 25/6$

or, $x_1 + x_5 + (-1)\, x_5 + (1/6)\, x_5 = 4 + 1/6$.

The Gomory's constraint is

$-\, (1/6)\, x_5 \leq -\, (1/6)$

or, $-x_5 \leq -\, 1$

Adding slack variable x_6 0 we get

$-\, x_5 + x_6 = -\, 1$

Adding the Gomory's constraint to the above optimum table and modifying the table we get

		c_j	1	1	0	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
1	y_2	1	0	1	0	–1	1	0
1	y_1	25/6	1	0	0	1	–5/6	0
0	y_3	8/3	0	0	1	3	–10/3	0
0	y6	–1	0	0	0	0	[–1]	1
		$z_j - c_j$	0	0	0	0	1/6	0
$\frac{(z_j - c_j)}{y_{3j}}$	$: y_{4j} < 0$						$\frac{1/6}{(-1)}$	
1	y_2	0	0	1	0	–1	0	1
1	y_1	5	1	0	0	1	0	–5/6
0	y_3	6	0	0	1	3	0	–10/3
	y_5	1	0	0	0	0	1	–1
		$z_j - c_j$	0	0	0	0	1/6	

As the original variables are integers this is the final table of the IPP. The optimal solution is $x_1 = 5$, $x_2 = 0$ and max $z = 5$.

6.4 The Branch and Bound Method

The Branch and Bound method is most powerful method and is applicable to both pure as well as mixed integer programming prolbems. This method was developed by Landand Doig. The principal idea underlying the branch and bound method is an follows. First we are to solve the problem ignoring the integer valued restriction. If the optimal solution has non-integral value, say x_j, then there is an integer k such that $k < x_j < k + 1$. As we want x_j to have integer value, the value of x_j must satisfy either $x_j \le k$ or xj $\ge k+1$ but not noth. Adding these constraints individually to the constraints of the given problem two subproblems are obtained. These two subproblems are solved. Repating the branching, the desired optimal solution is obtained.

6.4.1 "ranch and Bound Algorithm

The step by step procedure of branch and bound algorithm is as follows :

Let the IPP be

Maximize z = cx

subject to $Ax = b$

$x \ge 0$

x_j is integer for $j \in I$

Where $c = [c_1, c_2, ..., c_n]$, $x = [x_1, x_2, ..., x_n]^T$, $b = [b_1, b_2, ..., b_m]^T$

$A = [a_{ij}]_{m \times n}$

If $I = \{1, 2, ..., n\}$ then it is a pure (or all) IPP and if I is a proper subset of $\{1, 2, ..., n\}$ then it is a mixed IPP.

Step 1. Ignoringthe integer restriction solve the IPP. If the optimal solution be such that all x_j, $j \in I$ are integers, then this is the required optimal solution. If at least one x_j, $J \in I$ be non-integer then go to next step.

Step 2. Among non-integer x_j, $j \in I$ chope any one, Then there exists integer k such that

$$K < x_j < k + 1$$

As we want x_j to be an integer, the integer solution must satisfy one of the following

$$x_j \le k \text{ or } x_j \ge k + 1$$

Add these constraints indirectly to the constraints of the current problem and get two sub-problems. Solve these two sub-problems.

Step 3. If for any of the subproblem integer solution is obtained then that problem is not further branched.

But if any subproblem involves some non-integer variable, then it is again branched. This process of branching is continued, until each subproblem either admits an integer valued solution or there is eirdence that it cannot yield a better solution or it gives no feasible solution.

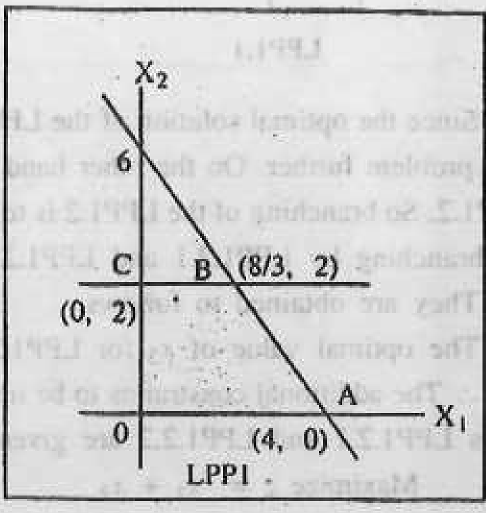
Among all subproblems select that integer valued solution which gives the over all maximum value of the object function.

Note : Main disadvantage of this method is that it requires the optimal solution of each subproblem. For large size problem this become very tedions job. Inspite of this drawback it is most effective method for solving IPP. Also the method is applicable for both all and mixed IPP

Example 6.4.1 Using Branch and Bound technique solve the following IPP

$$\begin{aligned} \text{Maximize } z &= x_1 + x_2 \\ \text{subject to } & 3x_1 + 2x_2 \le 12 \qquad \ldots\ldots \quad \ldots\ldots \text{(LPP1)} \\ & x_1, x_2 \le 0 \\ & x_1, x_2 \ge 0 \\ & x_1, x_2 \text{ are integers.} \end{aligned}$$

Solution : Ignoring the integer valued restriction the solution of the given IPP by graphical method is $x_1 = 8/3, x_2 = 2$, the value of z is $4\frac{2}{3}$. We call the LPP corresponding to this IPP as LPP1.



The value of x_1 is fraction and is 8/3. We note that $2 < 8/3 < 3$.

So we from two subproblems with additional constraints respectively as $x_1 \le 2$ and $x_1 \ge 3$.

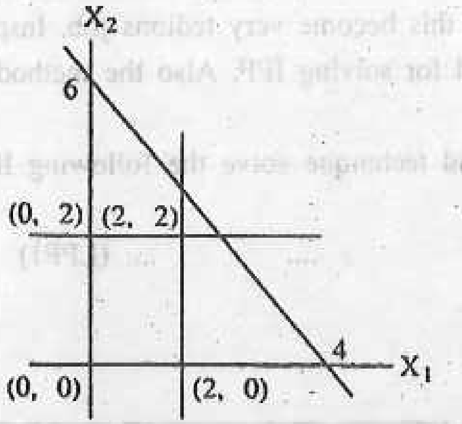
Thus two problems are

$$\begin{aligned} \text{Maximize } z &= x_1 + x_2 \\ \text{subject to } & 3x_1 + 2x_1 \le 12 \\ & x_2 \le 2 \qquad \ldots\ldots \quad \ldots\ldots \text{(LPP1.1)} \\ & x_1 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

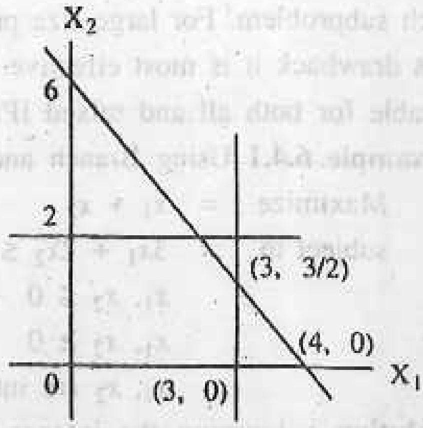
and

$$\begin{aligned} \text{Maximize } z &= x_1 + x_2 \\ \text{subject to } & 3x_1 + 2x_1 \le 12 \\ & x \le 2 \qquad \ldots\ldots \quad \ldots\ldots \text{(LPP1.2)} \\ & x_1 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

By graphical method, the optimal solution of the LPP1.1 is $x_1 = 2$, $x_2 = 2$ with $z = 4$ and that of the LPP1.2 is $x_1 = 3$, $x_2 = 3/2$ with $z = 9/2$



LPP1.1



LPP1.2

Since the optimal solution of the LPP1.1 are integers there is no need to branch this problem further. On the other hand the optimal value of x_2 is fraction for the LPP1.2. So branching of the LPP1.2 is to be done. Let the two subproblems obtained by branching by LPP1.2.1 and LPP1.2.2.

They are obtained to follows.

The optimal value of x_2 for LPP1.2 is 3/2 and $1 < 3/2 < 2$.

∴ The additional constraints to be introduced are $x_2 \le 1$ and $x_2 \ge 2$ respectively.

Thus LPP1.2.1 and LPP1.2.2 are given by

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \le 12$

$x_2 \le 2$

$x_1 \ge 3$ .... (LPP1.2.1)

$x_2 \le 1$

$x_1, x_2 \ge 0$

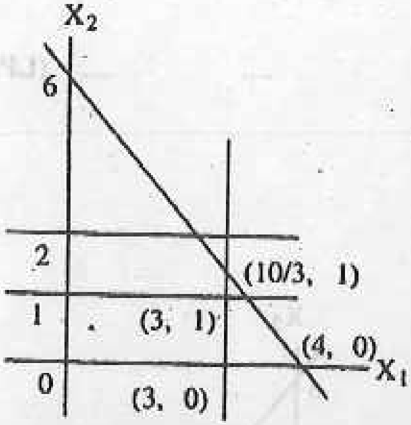
and

Maximize $z = x_1 + x_2$

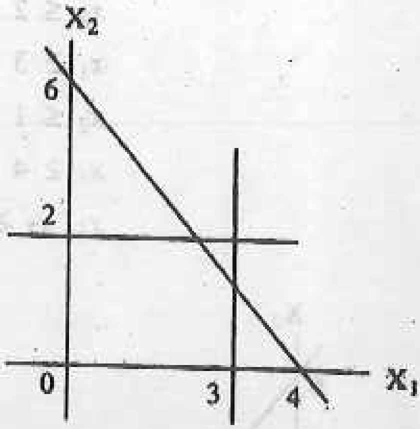
subject to $3x_1 + 2x_1 \le 12$

$x_2 \le 2$

$$x_1 \geq 3$$
$$x_2 \geq 2 \qquad \ldots\ldots \qquad \ldots\ldots \quad \text{(LPP1.2.2)}$$
$$x_1, x_2 \geq 0$$



LPP1.2.1



LPP1.2.2

Using graphical method the optimal solution of the LPP1.2.1 is $x_1 = 10/3$, $x_2 = 1$ with the value of $z = 13/3 = 4\frac{1}{3}$. As x_1 is not an integer and $z = 13/3$ which is greater that the optimal value $z = 4$ of the LPP1.1, we need branching of this LPP toget LPP1.2.1.1. and LPP1.2.1.2. (Here we note that instead of z = 13/3 if the value of z would be less than 4 then no branching is nedded)

The LPP1.2.2. has no feasible, so no question of branching.

To get branching of LPP1.2.1. we note that $3 < 10/3 < 4$. So that additional constraints to the LPP.1.2.1 to get sub problem are respectively $x_1 \leq 3$ and $x_1 \geq 4$.

Thus the subproblems are given by

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \leq 12$

$$x_2 \leq 2$$
$$x_1 \geq 3 \qquad \ldots\ldots \qquad \ldots\ldots \quad \text{(LPP1.2.1.1)}$$
$$x_2 \leq 1$$
$$x_1 \leq 3$$
$$x_1, x_2 \geq 0$$

and

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_1 \leq 12$

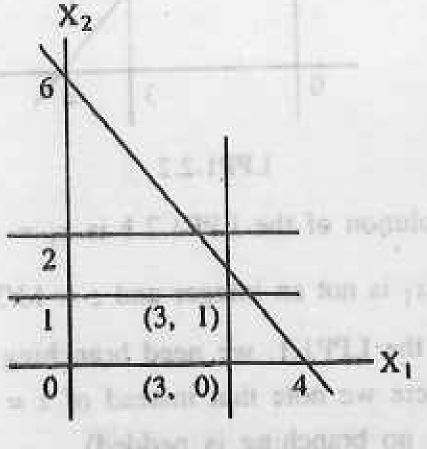
$x_2 \leq 2$

$x_1 \geq 3$ .... (LPP1.2.1.2)

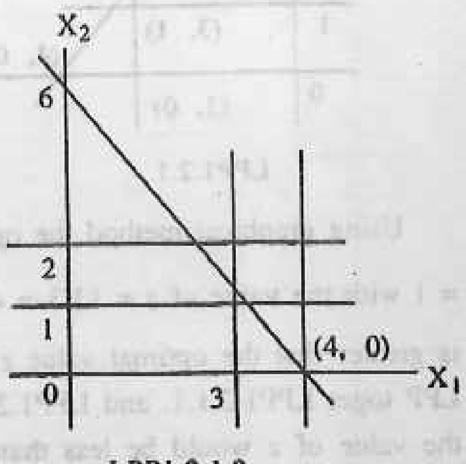
$x_2 \leq 1$

$x_1 \geq 4$

$x_1, x_2 \geq 0$



LPP1.2.1.1



LPP1.2.1.2

Graphical we get the optimal solution of the LPP1.2.11 as $x_1 = 3$, $x_2 = 1$ with $z = 4$ which is some as the optimal value of z of the LPP1.1. The optimal solution of the LPP 1.2.1.2. is $x_1 = 4$, $x_2 = 0$ with $z = 4$. No further branching is necessary.

The over all maximum value of the objective function is $z = 4$ and the integer valued solution are $x_1 = 2$; $x_1 = 3$, $x_2 = 1$; $x_1 = 4$, $x_2 = 0$.

6.5 Summary

Gomory cutting plane method for all IPP and Branch and bound method for general IPP have been considered and explained with examples. Need forIPP has been explained in detail with examples.

6.6 Self Assessment Questions

1. Solve the following IPP using Gomory's cutting plane method.

(i) Maximize $z = 2x_1 + 2x_2$

subject to
$$5x_1 + 3x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$
x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 1$, max $z = 4$]

(ii) Maximize $z = 4x_1 + 3x_2$

subject to
$$3x_1 + 4x_2 \le 12$$
$$4x_1 + 2x_2 \le 9$$
$$x_1, x_2 \ge 0$$
x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 2$, max $z = 10$]

(iii) Maximize $z = x_1 - 2x_2$

subject to
$$4x_1 + 2x_2 \le 15$$
$$x_1, x_2 \ge 0$$
x_1, x_2 are integers

[Ans : $x_1 = 3, x_2 = 0$, max $z = 3$]

2. Using Branch and Bound method solve the following IPP

(i) Maximize $z = 3x_1 + 4x_2$

subject to
$$3x_1 + 2x_2 \le 8$$
$$x_1 + 4x_2 \le 0$$
$$x_1, x_2 \ge 0$$
x_1, x_2 are integers

[Ans : $x_1 = 1, x_2 = 1$, max $z = 11$]

(ii) Maximize $z = 7x_1 + 9x_2$

subject to
$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$0 \le x_1 \le 7$$
$$0 \le x_2 \le 7$$
x_1, x_2 are integers

[Ans : $x_1 = 4, x_2 = 3$, max $z = 55$]

Unit 7 ❑ One dimensional minimization method

Structure

7.1 Introduction

Numerical method ofoptimization are used to solve the problems involving objective function and/or constraints which are two complicated or cannot be expressed as explicit function.

One dimensional minimization method plays an important role to solve the problems using numerical technique. In numerical methods we are to minimize $f(x_i + \lambda_i S_i)$ with respect to li for known values of x_i and s_i.

This is nothing but aone dimensional minimization problem. Among many one-dimensional minimization methods Fibonacci method and golden section method are simple and important. They are discussed in this unit. These two methods are used for unimodal functions.

7.2 Unimodal Function

In the process of finding optimal point often it becomes necessary that the function has only one optimum point in the domain of search. As in many methods we need only the values of the function at various points, the functionmay not be continuous and differentiable. What we need is that it should be unimodal. Unimodality of a function of one variable is defined as follows

7.2.1. Definition

A real valued function f(x) is said to be unimodal (minimum) is $[a, b]$ if there is a point $x^* \in [a, b]$ such that

(i) it $a < x_1 < x_2$ x^* then $f(x_1) > f(x_2)$

(ii) it $a < x_1 < x_2 < b$ then $f(x_2) > f(x_1)$

7.3 Fibonacci Method

Fibonacci method is based on Fibonacci sequence (Fn) defined by

$$F_0 = F_1 = 1$$

$$F_n + F_{n-1} + F_{n-2},\ n = 2, 3, 4, ...$$

Thus

$F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21$, $F_8 = 34$, $F_9 = 55$, $F_{10} = 89$, $F_{11} = 144$, ...

Fibonacci method can be used to find the optimum of a function of one variable. The function must be unimodal, it may ormay not be continuous or differtiable. This method has the following limitations :

(i) The initial interval of uncertainty $[a, b]$, in which the optimum lies, has to be known

(ii) The function to be optimized has to be unimodal in the initial interval of uncertainty.

(iii) The exact optimum point cannot be located by this method. Only an interval, known as the final interval of uncertainty can be obtained.

(iv) The number of function evaluations to be used in the search has to be specified beforehand.

The final interval of uncertainly can be made as small as we desire by making the number of function evaluations more.

Procedure : Let L be the length of the initial interval of uncertainty $[a, b]$ be the initial interval of uncertanity. Therefore $L_0 = b - a$.

Let n be the total number of experiments to be conducted. We define

$$L_2^* = \frac{F_{n-2}}{F_n} L_0$$

The first two experiments are placed at the points x_1 and x_2 which are located at a distance L_2^* from each end of L_0. The values of the function f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ at $f_2 = f(x_2)$. Using unimodality assumption one of the intervals $[a, x_1]$ and $[x_2, b]$ is to be discarded. The remaning interval of uncertainty is denoted by L_2.

Then $L_2 = L_0 - L_2^*$

$$= L_0 - \frac{F_{n-2}}{F_n} . L_0$$

$$= L_0\left(\frac{F_n - F_{n-2}}{F_n}\right)$$

$$= \frac{F_{n-1}}{F_n} L_0$$

Now $L_2 - L_2^*$

$$= \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-2}}{F_n} L_0$$

$$= \frac{L_0}{F_n}(F_{n-1} - F_{n-2})$$

$$= \frac{L_0}{F_n}(F_{n-2} + F_{n-3} - F_{n-2})$$

$$= \frac{F_{n-3}}{F_n} L_0$$

$$\therefore \quad \frac{L_2 - L_2^*}{L_2^*} = \frac{F_{n-3}}{F_{n-2}} < 1$$

i.e. $L_2 - L_2^* < L_2^*$

Thus in the interval of uncertainty L2 there is one point, either x_1 or x_2, whose distance from the two ends of L_2 are L_2^* and $L_2 - L_2^*$. The smaller of the two $L_2 - L_2^*$ & L_2^* is denoted) by. i.e L_3^*. $L_3^* = L2 - L_2^*$

Now, $L_3^* = L_2 - L_2^* = \frac{F_{n-3}}{F_n} L_0$.

We now place the third experiment x_3 and L_2 so that the current two experiment are located at a distance L_3^* from each end of L_2. Again by the unimodal property we can reduce the interval of uncertainty from L_2 to L_3 given by $L_3 = L_2 - L_3^*$ $= \frac{F_{n-2}}{F_n} L_0$.

∴ The interval of uncertainty at the end of 3rd experiment is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

and this obtained by discarding $L_3^* = \frac{F_{n-3}}{F_n} L_0$ continuing in this manner we have the following result in general.

The j th experiment is to be placed at a distance $L_j^* = \frac{F_{n-j}}{F_n} L_0$ from one end of L_{j-1} and the interval of uncertainty at the end of j th experiment is given by $L_j = \frac{F_{n-j+1}}{F_n} L_0$

Taking j = n we see that the nth experiment is to be placed at a distance L_n^*

$= \frac{F_0}{F_n} L_0 = \frac{L_0}{F_n}$ from one end of L_{n-1} and the interval of uncertainty at the end of n th experiment is given by $L_n = \frac{F_1}{F_n} L_0 = \frac{L_0}{F_n}$

$$\text{Now } L_{n-1} = \frac{F_{n-(n-1)+1}}{F_n} L_0 = \frac{F_2}{F_n} L_0 = \frac{2L_0}{F_n}$$

$$\therefore L_n^* = \frac{1}{2} L_{n-1}$$

Therefore, the last two experiments are located at a distance $L_n^* = \frac{1}{2} L_{n-1}$ from each end of L_{n-1}. So they have the same location. To remove this difficulty we place the n th experiment very close to the remaining valid experiment in L_{n-1}. This enables us to obtain the final interest of uncertainty of length $\frac{1}{2} L_{n-1} = L_n = \frac{L_0}{F_n}$

From $L_n = \frac{L_0}{F_n}$ we note that we can determine n for given L_n

7.4 Illustrative Examples

Example 7.4.1 : Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5 - x, & x > 3 \end{cases}$

in the interest [1, 4] by Fibonacci method using $n = 6$

Solution : Here number it experiment to be performed is $n = 6$.

From Fibonacei sequence we have

$F_0 = F_1 = 1$

$F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, F_7 = 21$ etc.

Here $L_0 = 4 - 1 = 3$.

$$\therefore L_2^* = \frac{F_4}{F_6} L_0 = \frac{5}{13} \times 3 = 1 \cdot 1538$$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_1^* = 1 + 1{\cdot}1538 = 2{\cdot}1538$$

& $x_2 = 4 - L_1^* = 4 - 1{\cdot}1538 = 2{\cdot}8462$

Now $f_1 = f(x_1) = \frac{2x_1}{3} = \frac{2 \times 2 \cdot 1538}{3} = 1 \cdot 4359$

and $f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2 \cdot 8462}{3} = 1 \cdot 8975$

Since $f_1 < f_2$, using uniondal property we delete the interval $[1, x_1]$. Thus the reduced interval of uncertainly is $[x_1, 4]$ i.e., $[1{\cdot}4359, 4]$ with x_2 inside it and near to x_1.

The third experiment is placed at the position x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $x_3 = 4 - x_2 + x1$

$= 4 - 2{\cdot}8462 + 2{\cdot}1538$

$= 3{\cdot}3076$

Now $f_3 = f(x_3) = 5 - x_3 = 5 - 3{\cdot}3076 = 1{\cdot}6924$

Here $f_3 < f_2$. So by unimodaily we delete the interval $[x_3, 4]$. The remaining interval of uncertainly becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3.

The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

$\therefore x_4 = x_1 + x_3 - x_2 = 2{\cdot}1538 + 3{\cdot}3076 - 2{\cdot}8462 = 2{\cdot}6152$

Now, $f_4 = (x_4) = \frac{2x_4}{3} = \frac{2 \times 2 \cdot 1652}{3} = 1 \cdot 7435$

Since $f_4 < f_2$ we delite the interval $[x_1, x_4]$. The remaining interval of uncertainly is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed of x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3{\cdot}3076 - 2{\cdot}8462 + 2{\cdot}6152 = 3{\cdot}0766$

Now $f_5 = f(x_5) = 5 - x_5 = 5 - 3{\cdot}0766 = 1{\cdot}9234$

Since $f_5 < f_2$, using unimodal property we delete the interval $[x_4, x_2]$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2$$

or, $x_6 = x_3 - x_5 + x_2 = 3{\cdot}3076 - 3{\cdot}0766 + 2{\cdot}8462 = 3{\cdot}0772$

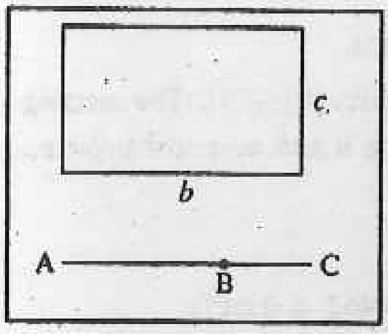
Now $f_6 = f(x_6) = 5 - x_6 = 5 - 3{\cdot}0772 = 1{\cdot}9228$

since $f_6 < f_5$, using unimodality we delete the interval $[x_6, x_3]$. The final interval of uncertainty is $[x_2, x_6] = [2{\cdot}8462, 3{\cdot}0772]$

Here we note that if the exact calculation be carried out then we would get $x_5 = x_6$. In that situation x_6 should be selected very close to x_5. But here we see $x_5 \neq x_6$. This is due to round off error involved in the calcution.

7.5 Golden Section

Ancient Greek architects believed that a building having sides b and c satisfying



the relation $\frac{b+c}{b} = \frac{b}{c} = \gamma$ will be having the most pleasing properties. This ratio is called Golden ration. It is also found in Euclid's geometry that the division of a line segment into unequal parts so that the ration of the whole to the largest part is equal to the ratio of the large part to the smaller part: This section is known as the golden section

Thus the Golden section

$$\frac{AC}{AB} = \frac{AB}{BC} = \gamma \quad \text{i.e.,} \quad \frac{AB+BC}{AB} = \frac{AB}{BC} = \gamma$$

From this we have

$$\frac{AB}{AB} + \frac{BC}{AB} = \frac{AB}{BC} = \gamma$$

or, $1+\frac{1}{\gamma}=\gamma$

or, $\gamma^2 - \gamma - 1 = 0$

$$\therefore \quad \gamma = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.(-1)}}{2.1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since γ is a positive number we have

$$\gamma = \frac{\sqrt{5}+1}{2} = 1 \cdot 618$$

7.6 Golden Section Method

Golden section method is similar to the Fibonacei method except for one difference. The difference is that in Fibonacei method the total number of experiments to be performed has to be specified before beginning the calculation, whereas, this is not required in golden section method. In fact when n is very large then Fibonacei method reduces to golden section method. In Fibonacei method the number of experiments to be performed is decided at the begining but in golden section method the total number of experiments are to be decided during the computations.

In the Fibonacei method, the interval of uncertanity at the end of two experiments is given by $L_2 = \frac{F_{n-1}}{F_n} L_0$

In Golden Section method is n is very large this L_2 becomes

$$L_2 = \lim_{n \to \infty} \frac{F_{n-1}}{F_n} L_0 = L_0 \left(\lim_{n \to \infty} \frac{F_{n-1}}{F_n} \right)$$

Also in Fibona method L_3 is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

$\therefore$ In Golden section method L_3 will be given by

$$L_3 = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} L_0$$

$$= \lim_{n\to\infty} \left(\frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} L_0 \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}} \right) \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right) \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \cdot \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^2$$

Similary, we get $L_4 = L_0 \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^3$

Generalizing these results we have

$$L_k = \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^{k-1} . L_0$$

We have the relation

$$F_n = F_{n-1} + F_{n-2}$$

$$\therefore \quad \frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

$$\text{or, } \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}$$

$$= 1 + \lim_{n\to\infty} \frac{1}{\frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim\limits_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}}$$

Let $\gamma = \lim\limits_{n\to\infty} \frac{F_n}{F_{n-1}}$

$\therefore$ We have $\gamma = 1 + \frac{1}{\gamma}$

or, $\gamma^2 = g + 1$

or, $\gamma^2 - g - 1 = 0$

or, $\gamma = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1(-1)}}{2.1}$

$= \frac{1 \pm \sqrt{5}}{2}$

Since γ is a positive real number, we have $\gamma = \frac{\sqrt{5}+1}{2} = 1{\cdot}618$, which is nothing but golden ratio or golden section.

Hence we have in general,

$$L_k = \left(\frac{1}{\gamma}\right)^{k-1} L_0 = (0{\cdot}618)^{k-1}\ L_0$$

$\therefore$ In the Golden section method the interval of uncertainty at the end of k th experiment is given by

$$L_k = (0{\cdot}618)^{k-1}\ L_0$$

7.7 Procedure of Golden Section Method

In the Fibonacci method, the location of the first two experiments are the points situated at a distance L_2^* from the two ends of the initial interval of uncertainty, where L_2^* is given by

$$L_2^* = \frac{F_{n-2}}{F_n} L_0$$

In Golden section method n is very large. Therefore L_2^* is given by

$$L_2^* = \lim_{n\to\infty} \frac{F_{n-2}}{F_n} L_0$$

$$= \lim_{n\to\infty} \left(\frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} L_0 \right)$$

$$= L_0 . \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}} . \lim_{n\to\infty} \frac{F_{n-1}}{F_n}$$

$$= L_0 . \lim_{n\to\infty} \frac{F_{n-1}}{F_n} . \lim_{n\to\infty} \frac{F_{n-1}}{F_n}$$

$$= L_0 . \left(\lim_{n\to\infty} \frac{F_{n-1}}{F_n} \right)^2$$

$$= L_0 . \left(\frac{1}{\gamma} \right)^2$$

$$= L_0 . (\cdot 613)^2 = 0\cdot 382 \ L_0 .$$

∴ In the Golden section method, the first two experiments are placed at the points x_1 and x_2 which are located at a distance $L_2^* = 0\cdot 382 \ L_0$ from each end of L_0. The values of the functions f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ and $f_2 = f(x_2)$. Using the assumption of unimodality, one of the two intervals $[a, x]$ and $[x_2, b]$ can be discarded. The remaining interval of uncertainty will be $L_2 = 0\cdot 618 \ L_0$. The interval will contain one experiment point. The smaller distance of this experiment point from the ends of L_2 is denoted by L_3^*. The third experiment x_3 is placed in L_2 so that the current two experiments are located at a distance L_3^* from each end of L_2. Again using unimodelity we can discard one of the end intervals and the reduced internal of uncertainty at the end of 3rd experiment becomes $L_3 = (0{\cdot}618)^2 L_0$. This process is continued until the desired length of the interval of uncertainty is obtained.

7.8 Illustrative Examples

Example 7.8.1 Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5 - x, & x > 3 \end{cases}$

in the interval [1, 4] by Golden selection method up to six experiments.

Solution : We have $L_0 = 4 - 1 = 3$

Now $L_2^* = {\cdot}382\, L_0 = {\cdot}382 \times 3 = 1{\cdot}146$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_2^* = 1 + 1{\cdot}164 = 2{\cdot}146$$

$$x_2 = 4 - L_2^* = 1{\cdot}146 = 2{\cdot}854$$

Now $f_1 = f(x_1) = \dfrac{2x_1}{3} = \dfrac{2 \times 2 \cdot 146}{3} = 1 \cdot 43066$

$$f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2 \cdot 854}{3} = 1 \cdot 90266$$

As $f_1 < f_2$ and the problem is of maximization, using unimodal property we delete the interval $[1, x_1]$. Thus the reduced interval of uncertainly is $[x, 4]$ with x_2 inside it and near to the point x_1.

The third experiment is to be placed at x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $x_3 = 4 - x_2 + x_1 = 4 - 2{\cdot}854 + 2{\cdot}146 = 3{\cdot}292$

Now, $f_3 = f(x_3) = 5 - 3{\cdot}292 = 1{\cdot}708$

Here $f_3 < f_2$. So by unimodality we delete the interval $[x_3, 4]$. The remaning interval of uncertainty becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3. The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

or, $x_4 = x_1 + x_3 - x_2 = 2{\cdot}146 + 3{\cdot}292 - 2{\cdot}854 = 2{\cdot}584$

Now $f_4 = f(x_4) = \frac{2x_4}{3} = \frac{2 \times 2 \cdot 584}{3} = 1 \cdot 7226$

Hence, $f_4 < f_2$. Using unimodality we delete the interval $[x_1, x_4]$. The remaning interval of uncertainty is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed at x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3{\cdot}292 - 2{\cdot}854 + 2{\cdot}584 = 3{\cdot}022$

Now $f_5 = f(x_5) = 5 - 3{\cdot}022 = 1{\cdot}978$

Since $f_5 > f_2$, using unimodal properly we delete the interval $[x_4, x_2]$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near to x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2$$

or, $x_6 = x_3 - x_5 + x_2 = 3{\cdot}292 - 3{\cdot}022 + 2{\cdot}854 = 3{\cdot}124$

Now $f_6 = f(x_6) = 5 - 3{\cdot}124 = 1{\cdot}876$

Since $f_6 < f_5$, using unimodality we delete the interval $[x_6, x_3]$. The final interval of uncertainty is given by $[x_2, x_6]$ is $[2{\cdot}854, 3{\cdot}124]$

7.9 Summary

The necessity of numerical methods of optimization is discussed. The inportance of one-dimensional minimization methods is solving multivariable optimization problems in described. The concept of unimodal function and its role in the elimination methods is presented. Fibonacci method and Golden section methods are discussed in detail through examples.

7.10 Self Assesment Questions

1. Minimize $f(x) = \begin{cases} 8-x, & x \le 4 \\ x, & x \le 4 \end{cases}$

   in the interval [1, 7] by Fibonacci method using $n = 6$

2. Minimize $f(x) = |x - 1|$ in the interval [–1, 5] by Fibonacci method using $n = 5$.

3. Manimize $f(x) = \begin{cases} 4x/3, & x \le 3 \\ 7-x, & x \le 3 \end{cases}$

   in the interval [1, 5] by Golden section method upto six experiments.

4. Minimize = $f(x) = \begin{cases} 6-x, & x \le 5 \\ 2x-9, & x \le 5 \end{cases}$

   in the interval [2, 8] by Golden section method upto five experiments.

5. Minimize = $f(x) = \begin{cases} 2\sqrt{x}, & x \le 1 \\ 3-x, & x \ge 1 \end{cases}$

   in the interval [0, 5] by Golden section method upto six experiments.

6. Minimize $f(x) = |x|$ in the interval [–2, 2] by Golden section method upto six experiments.

Unit 8 ❑ Unconstrained Optimization Technique

Structure

8.1 Introduction

The solution ofunconstrained optimization problem need not satisfy any constraints, Unconstrained optimization technique is important because of the following reasons

(i) Some of the most powerful and convenient methods of solving constrained optimization problems involve the transformation of the problem into one of unconstrained optimization.

(ii) The study of the unconstrained optimization methods provides the basic understanding necessary for the study of the constrained optimization methods.

Several methods are available for solving an unconstrained optimization problem. These methods are classified into two broad categories viz direct search methods and descent methods. The different methods of these two categories are shown below.

8.2 General Iterative scheme of optimization

All the unconstrained optimization methods are iterative in nature. Hence they start from an initial trial solution and proceed towards the optimum point in a sequential manner. It is importai tc ote that all the unconstrained optimization methods requires an initial point x_1 to start the iterative procedure. One method differs from another only in the method of generation the new point x_{i+1} from x_i and in testing the point x_{i+1} for optimality.

If the search d tion from x_i be s_i and the step length for movement along the search direction s_i be λ_i^* then the next point to x_i is obtained as $x_{i+1} = x_i + \lambda_i^* s_i$. Thus the terative scheme becomes.

(i) Start with an initial trial point x_1.

(ii) Find a suitable direction s_i (i = 1 to start with) which points is general direction of the minimum.

(iii) Find and appropriate step length λ_i^* for movement along the direction si.

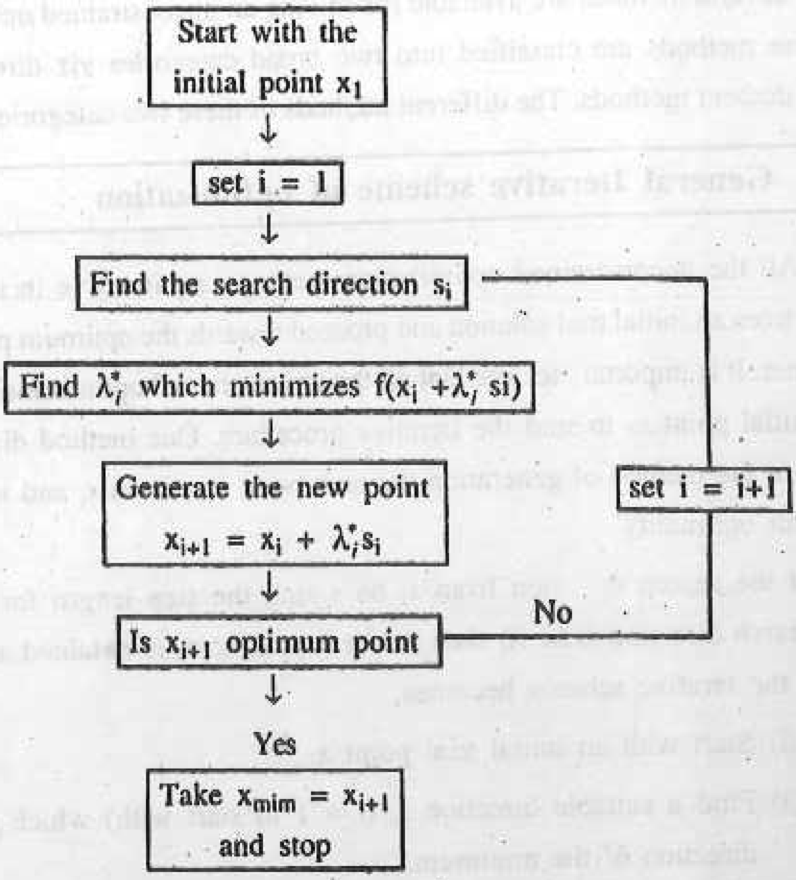
(iv) Obtain the new approximation x_{i+1} as $x_{i+1} = x_i + \lambda_i^* s_i$

(v) Test whether x_{i+1} is optimum. If x_{i+1} is optimum then stop the procedure, otherwise set new $i = i+1$ and repeat step (ii) onward.

Thus as mentioned before, the efficiency of an optimization method depends an the efficiency with which the quantities λ_i^* and si are determined to generate the new point x_{i+1} as $x_i + \lambda_i^* s_i$. To find we are to minimize $f(x_i + \lambda_i s_i)$ regarding it as a function of λ_i only.

$$f(x_i + \lambda_i^* s_i) = \min_{\lambda_i} (f(x_i + \lambda_i s_i))$$

The flow chart for the iterative scheme may thus be shown as follows



8.3 Steepest Descent Method

In the steepest descent method of minimize a function f of n variables x_1, x_2, ..., xn we use the gradient of the the function f defined by

$$\nabla f = \left[\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2} \ldots\ldots \frac{\partial f}{\partial x_n}\right]^T$$

The gradient of f is a n-companent vector and has a very important property viz if we move along the gradient direction from any point in the n-dimensional space, then the function value increases at the fastest rate. To prove this properly we first define directional derivalive.

Definition 8.3.1 Directional Devivative : The directional devivative of f(x) in the direction of the unit vector y is defined as the following limit

$$\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$$

The directional derivative of f(x) in the direction y is thus given by using Taylor's theorem

$$\lim_{t\to 0} \frac{\{f(x)+(ty)\nabla f(x)+\text{terms of higher degree in t}\}-f(x)}{t}$$

$= y'\nabla f(x)$

∴ The directional derivative of f(x) in the direction of unit vector y

$= y'\nabla f(x)$

= rate of change of f(x) in the direction of y.

Theorem 8.3.1 Prove that f(x) increases at the fastest rate in the direction of ∇f.

Proof : We have that the rate of change of f(x) in the direction of the unit vector y is $y'\nabla f(x)$ .... (1)

Now the unit vector in the direction of the gradient vector ∇f is $\nabla f / |\nabla f|$. Therefore, the rate of change of $f(x)$ in the direction of the gradient vector

$$= \left(\frac{\nabla f}{|\nabla f|}\right)' \nabla f = \frac{(\nabla f)'(\nabla f)}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f| \text{ ....(2)}$$

Since $|\nabla f| > 0$, it follows the $f(x)$ increases in the direction of ∇f.

Using cauchy schwarz inequality we have

$$|y'\nabla f| \le |y|\,|\nabla f| = |\nabla f| \; [\because |y| = 1] \text{.......... (3)}$$

From (1), (2) and (3) it follows that the rate of change of $f(x)$ in the direction of ∇f is greater than that in the direction of any unit vector y. In other words $f(x)$ increases at the fastest rate in the direction of ∇f.

Note : Since $f(x)$ increases at the fastest rate in the direction of ∇f, it follows that $f(x)$ decreases at the fastest rate in the direction of $-\nabla f$. Thus the direction of ∇f and $-\nabla f$ are respectively the directions of the steepest ascent and steepest descent.

8.4 Iterative Scheme of Steepest Descent Method

The steepest descent method uses the properly that a function $f(x)$ decreases a the fastest rate in the direction of $-\nabla f$. Thus at x_i the function decreases at the fastes rate along the direction si given by $s_i = [-\nabla f]_{x_i} = -\nabla f_i$.

The iterative scheme of steepest descent method is given below.

(i) Start with an initial point x_1.

(ii) Take the search direction s_i at x_i (i = 1 to start with) as $s_i = [-\nabla f]_{x_i}$ an denote it by $-\nabla f_i$.

(iii) Find the step length λ_i^* for movement along si which minimizes $f(x_i + \lambda_i^* s_i$

(iv) Obtain the new appromimation point x_{i+1} as $x_{i+1} = x_i + \lambda_i^* s_i$

(v) Test whether x_{i+1} is optimum. If x_{i+1} is optimum then stop the procedure Otherwise set new $i = i+1$ and repeat setp (ii) onward.

8.5 Illustrative Examples

Example 8.5.1 Using steepest descent method minimize f = $x_1^2 + x_2^2 + 2gx_1 + 2fy_1 + c$ starting from the point

Solution : Here $f(x_1, x_2) = x_1^2 + x_2^2 + 2gx_1 + 2fy_1 + c$

∴ The gradient of f is given by

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2g \\ 2x_2 + 2f \end{bmatrix}$$

The starting point is $x_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Using steepest desant method the search directio at x_1 is given by

$$s_i = [-\nabla f]_{x_i} = \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix}$$

The step length λ_1^* is obtained by minimising $f(x_1 + \lambda_1 s_1)$ with respect to λ_1.

$$\text{Now } (x_1 + \lambda_1 s_1) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \lambda_1 \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix} = \begin{bmatrix} \alpha - 2\lambda_1\alpha - 2\lambda_1 g \\ \beta - 2\lambda_1\beta - 2\lambda_1 f \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

Where $\gamma = \alpha + \lambda_1 (-2\alpha - 2g)$

and $\delta = \beta + \lambda_1 (-2\beta - 2f)$

$\therefore\ f(x_1 + \lambda_1 s_1) = \gamma^2 + \delta^2 + 2g\gamma + 2f\delta + c$

For minimum value of f we have $\frac{df}{d\lambda_1} = 0$. This gives $\frac{\partial f}{\partial \gamma}\frac{\partial \lambda}{\partial \lambda_1} + \frac{\partial f}{\partial \delta}\cdot\frac{\partial \delta}{\partial \lambda_1} = 0$.

or, $(2\gamma + 2g)(-2\alpha - 2g) + (2\delta + 2f)(-2\beta - 2f) = 0$

or, $(\gamma + g)(\alpha + g) + (\delta + f)(\beta + f) = 0$

or, $\{\alpha + \lambda_1(-2\alpha - 2g)\}(\alpha + g) + \{\beta + \lambda_1(-2\beta - 2f) + f\}(\beta - f) = 0$

or, $(\alpha + g)^2 - 2\lambda_1(\alpha + g)^2 + (\beta + f)^2 - 2\lambda_1(\beta + f)^2 = 0$

or, $(1 - 2\lambda_1)[(\alpha + g)^2 + (\beta + f)^2] = 0$

or, $1 - 2\lambda_1 = 0$

or, $\lambda_1 = \frac{1}{2}$

$\therefore\ \lambda_1^* = \frac{1}{2}$

$$\text{Now, } x_1 \text{ is given by } x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix} = \begin{bmatrix} \alpha - \alpha - g \\ \beta - \beta - f \end{bmatrix} = \begin{bmatrix} -g \\ -f \end{bmatrix}$$

The gradient of f at x_2 is given by

$$[\nabla f]_{x_2} = \begin{bmatrix} 2(-g) + 2g \\ 2(-f) + 2f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that x_2 is the optimum point

$$\therefore \quad x_{opt} = x_2 = \begin{bmatrix} -g \\ -f \end{bmatrix}$$

Example 8.5.2 Using steepest descent method minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution : Here $f = f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ and the starting point is $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The gradient of f is given by

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$$\therefore \quad \nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The search direction at x_1 is given by $s_1 = -\nabla f_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

To find x_2 we are to find the optimal step length λ_1^*. For this we are to minimize $f(x_1 + \lambda_1 s_1)$ with respect to λ_1.

Now $x_1 + \lambda_1 s_1 = -\lambda_1 - \lambda_1 + 2\lambda_1^2 - 2\lambda_1^2 + \lambda_1^2 = \lambda_1^2 - 2\lambda_1$

For minimum value of f we have $\frac{df}{d\lambda_1} = 0$.

From this we have $2\lambda_1 - 2 = 0$

or, $\lambda_1 = 1$

$\therefore \quad \lambda_1^* = 1$

Thus we obtain x_2

$$x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The gradient of f at x_2 is given by

$$\nabla f_2 = [\nabla f]_{x2} = \begin{bmatrix} 1-4+2 \\ -1-1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ x_2 is not an optimum point. So we proceed to the next iteration.

The search direction at x_2 is given by

$$s_2 = -\nabla f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find x_3 we find the step length λ_3^* by minimizing $f(x_2 + \lambda_2 s_2)$ with respect to λ_2.

Now $x_2 + \lambda_2 s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+\lambda_2 \\ 1+\lambda_2 \end{bmatrix}$

$$\therefore \ f(x_2 + \lambda_2 s_2) = (-1 + \lambda_2)(1 + \lambda_2) + 2(-1 + \lambda_2)^2 + 2(-1 + \lambda_2)(1 + \lambda_2) + (1 + \lambda_2)^2$$

$$= -1 + \lambda_2 - 1 - \lambda_2 + 2 - 4\lambda_2 + 2\lambda_2^2 - 2 + 2\lambda_2^2 + 1 + 2\lambda_2 + \lambda_2^2$$

$$= -1 - 2\lambda_2 + 5\lambda_2^2$$

To minimize f we set $\frac{df}{d\lambda_2} = 0$

Form this we have $-2 + 10\lambda 2 = 0$

or, $\lambda_2 = +\frac{1}{5}$

$\therefore \ \lambda_2^* = \frac{1}{5}$

Hence $x_3 = x_2 + \lambda_2^* s_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix}$

The gradient of f at x_3 is given by

$$\nabla f_3 = [\nabla f]_{x_2} = \begin{bmatrix} 1+4(-0\cdot 8)+2(1\cdot 2) \\ -1+2(-0\cdot 8)+2(1\cdot 2) \end{bmatrix} = \begin{bmatrix} 0\cdot 2 \\ -0\cdot 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ x_3 is not optimum and we proceed to the next iteration.

The search direction at x_3 is given by

$$s_3 = -\nabla f_3 = \begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix}$$

To find x_4 we are to find the step length λ_4^* by minimizing $f(x_3 + \lambda_3 s_3)$ with respect to λ_3.

Now $x_3 + \lambda_3 s_3 + \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix} = \begin{bmatrix} -0\cdot 8-\lambda_3 0\cdot 2 \\ 1\cdot 2+\lambda_3 0\cdot 2 \end{bmatrix}$

$\therefore$ $f(x_3 + \lambda_3 s_3) = (-0{\cdot}8 - 0{\cdot}2\ \lambda_3) - (1{\cdot}2 + 0{\cdot}2\ \lambda_3) + 2\ (-0{\cdot}8 - 0{\cdot}2\lambda_3)^2$

$+ 2\ (-\ 0{\cdot}8 - 0{\cdot}2\ \lambda_3)\ (1{\cdot}2 + 0{\cdot}2\ \lambda_3) + (1{\cdot}2 + 0{\cdot}2\ \lambda_3)^2$

$= 0\cdot 04\lambda_3^2 - 0\cdot 08\lambda_3 - 1\cdot 20$

To minimize f we set $\dfrac{df}{d\lambda_3} = 0$

To gives $2 \times 0{\cdot}04\lambda_3 - 0{\cdot}08 = 0$

or, $\lambda_3 = 1$

$\therefore$ $\lambda_3^* = 1$

Hence $x_4 = x_3 + \lambda_3^* s_3 = \begin{bmatrix} -0\cdot 8 \\ 1\cdot 2 \end{bmatrix} + 1\begin{bmatrix} -0\cdot 2 \\ 0\cdot 2 \end{bmatrix} = \begin{bmatrix} -1\cdot 0 \\ 1\cdot 4 \end{bmatrix}$

The gradient of f at x_4 is given by

$$\nabla f_4 = [\nabla f]_{x_4} = \begin{bmatrix} 1+4(-1\cdot 0)+2(1\cdot 4) \\ -1+2(-1\cdot 0)+2(1\cdot 4) \end{bmatrix} = \begin{bmatrix} -0\cdot 20 \\ -0\cdot 20 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So x_4 is also not optinum and we are to continue the iterations until we have $\nabla f_n \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and then x_n is taken has the optimum point.

Convergence Creteria: The following criteria can be used to terminate the iteratice process.

(i) $\left|\dfrac{f(x_{i+1}) - f(x_i)}{f(x_i)}\right| \leq \in$

(ii) $\left|\dfrac{\partial f}{\partial x_i}\right| < \in$ for all $i = 1, 2, \ldots\ldots, n$

(iii) $|x_{i+1} - x_i| \leq \in$

8.6 Quadratically Convergent Method

Example 8.6.1 A minimization method is called quadratically convergent method if it locates the minimum of general function in no more than a pre-determined number of operations and if the limitting number of operations is directly related to the number of variates.

Definition 8.6.2 Let A be an nxn pymmetric matrix. A set of n vectors s_1, s_2, ..., sn is said to be A conjugate directions if $s_j^T A\ s_i = 0$ for all $i \neq j$, i, j = 1,2, 3, ..., n.

Example 8.6.1 Find the conjugate direction for the symmetric matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

Solution : Let A = $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ and A-conjugate direction be $s_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $s_2 = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$

$\therefore\ s_i^T A\ s_2 = 0$

or, $[\alpha\ \beta]\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0$

or, $\gamma\,(2\alpha - 3\beta) + \delta\,(-3\alpha + 2\beta) = 0$

Let, $\alpha = 1, \beta = 2, \gamma = 1$ $\quad \therefore\ -1\,(2.1 - 3.2) + \delta\,(-3.1 + 2.2) = 0$

or, $4 + \delta\,(+1) = 0$

or, $\delta = -4$

Thus the conjugate direction are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$

We note that for a given matrix there are many conjugate directions.

Matrix representation of quadratic expression :

Any quadratic expression can be expressed with the help of matrices as

$$\frac{1}{2}x^T Ax + B^T x + c$$

Where A is asymmetric matrix

eg. $3x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2 - x_2x_3 + 3x_3x_1 + 3x_1 - 2x + x_3 + 7$

can be written as $\frac{1}{2}x^T Ax + B^T x + c$

Where $A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 4 & -1 \\ 3 & -1 & 8 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, C - 7, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

We state the following important theorem.

Theorem 8.6.1 If quadratic function $Q(x) = \frac{1}{2}x^T Ax + B^T x + c$ is minimized sequentially once along each direction of a set of n A-conjugate directions them the global minimum of Q(x) will be located at a before the n th setp regareless of the starting point and the order in which the directions are used.

8.7 Newton's Method

If the function $f(x)$ is continuously differentiable then the local minimum point x^* is given by $[\nabla f]_{x^*} = 0$. Solving the set of n nonlinear equations $\nabla f = 0$ we get the optimal point x^*.

Newton's method : To get the minimum point x^* of the continuously differentiable function $f(x)$ we are to solve the n nontinear equation $\nabla f = 0$. To solve these n nonlinear equations by the Newton's method, we first linearize the set of equation about the i th appromimations x_i to the minimum point x^* of f.

Let $x^* = x_i + s$ and $\nabla f = g$

From $[\nabla f]_{x^*} = 0$ we have $g(x^*) = 0$ or, $g(x_i + s) = 0$

By Taylor's series expansion we get

$g(x_i) + [J]_{x_i} s + \ldots\ldots = 0$ where $[J]_{x_i}$ is the matrix of second partial devivatives of f evaluated at the point Negleeting the higher order terms we get

$$g(x_i) + [J]_{x_i} s = 0$$

or, $g_i + J_i s = 0$ where $g(x_i) = gi$ and $[J]_{x_i} = J_i$. If J_i is non singular, then we have

$$S = -J_i^{-1} g_i$$

But the higher order terms are not negligible in general. Hence an iterative procedure has to be used to find the improved approximations. The iterative scheme is given by

$$x_{i+1} = x_i + s_i = x_i - J_i^{-1} g_i$$

If J is nonsingular then it can be shown that the sequence of points $x_1, x_2, \ldots\ldots, x_i, \ldots\ldots$ converges to the actual solution x^* from any initial point x_1 sufficiently close to the solution x^*.

Theorem 8.7.1 If $f(x)$ is a quadratics then the minimum point can be obtained in a single step by Newton's method.

Proof : Let $f(x) = \frac{1}{2}x^T Ax + B^T x + c$ & the minimum point be x^*. Then $[\nabla f]_{x^*} = 0$

or, $[Ax + B]_{x^*} = 0$

or, $Ax^* + B = 0$

or, $x^* = -A^{-1}B$.

From $f(x) = \frac{1}{2}x^T Ax + B^T x + c$ we have $\nabla f = Ax + B$ and J = matrix of second partial derivatives of f = A. By Newton's method we have

$$\begin{aligned} x_{i+1} &= x_i - J_i^{-1} g_i \\ &= x_i - A^{-1}(Ax_i + B) \\ &= x_i - A^{-1}Ax_i + A^{-1}B) \\ &= x_i - x_i - A^{-1}B \\ &= -A^{-1}B = x^* \end{aligned}$$

$\therefore$ $x_2 = -A^{-1}B = x^*$ for any starting point x_1.

Thus the answer is obtained in a single step.

Example 8.7.1 Using Newton's method

minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$ with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as starting point.

Solution : Here $f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$

$$\therefore \quad \frac{\partial f}{\partial x_1} = 1 + 4x_1 + 2x_2, \quad \frac{\partial f}{\partial x_2} = -1 + 2x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 2$$

The starting point is $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 1+0+0 \\ -1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\& \quad J1 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \quad J_1^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

We have $x_1 = x_2 - J_1^{-1}\nabla f_1$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} - 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$

Now $\nabla f_2 = [\nabla f]_{x_2} = \begin{bmatrix} 1 + 4(-1) + 2(3/2) \\ -1 + 2(-1) + 2(3/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As $\nabla f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ is the optimum point.

8.8 Davidon-Fletcher-Powell Method (Variable Metric Method)

Davidon-Fletcher-Powell method is an important quasi-Newton method. This method is the best general purpose unconstrained optimization technique making use of the derivatioes.

The iterative procedure of this method is as follows :

(i) Start with an initial point x_1 and a nxn positive definite symmetric matrix H_1. Usually H_1 is taken as the identely matrix I. Set iteratio number is i = 1.

(ii) Compute the gradient of the function f at the point x_i i.e., compute $\nabla f_i = [\nabla f]_{x_i}$

Take $s_i = H_i \nabla f_i$ as the search direction at x_i.

(iii) Find the optimal step length λ_i^* in the direction s_i and set $x_{i+1} = x_i + \lambda_i^* s_i$

(iv) Test the new point x_{i+1} for optimality. If x_{i+1} is optimal, terminate the iterative process. Otherwise go to setp (v).

(v) Update H_i to H_{i+1} as

$$H_{i+1} = H_i + M_i + N_i$$

Where $M_i = (\lambda_i^* s_i s_i^T) / (s_i^T Q_i)$

$$N_i = -(H_i Q_i)(H_i Q_i)^T / (Q_i^T H_i Q_i)$$

$$Q_i = \nabla f_{i+1} - \nabla f_i$$

(vi) Set the new iteration number i = i + 1 and go to step (ii).

8.9 Illustrative Examples

Example 8.9.1 Using Davidon Fletcher-Powell method minimize $f(x_1, x_2) = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$ with $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as starting point.

Solution : Here $f = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$

$$\therefore \quad \nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 12 \\ 8x_2 + 16 \end{bmatrix}$$

Thus $\nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 4 - 12 \\ 8 + 16 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$

We take $H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad s_1 = -H_1\nabla f_1 = -\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-8\\24\end{bmatrix} = \begin{bmatrix}-8\\24\end{bmatrix}$$

To find the minimizing step length λ_1^* along s_1, we minimize

$f(x_1 + \lambda_1 s_1) = f(1 + 8\lambda_1,\ 1 - 24\lambda_1)$

$$= 2\,(1 + 8\lambda_1)^2 + 4\,(1 - 24\lambda_1)^2 - 12\,(1 + 8\lambda_1) + 16\,(1 - 24\lambda_1) + 41$$

$$= 2 + 32\lambda_1 + 128\lambda_1^2 + 4 - 192\lambda_1 + 2304\lambda_1^2 - 12 - 96\lambda_1 + 16 - 384\lambda_1 + 41$$

$$= 2432\lambda_1^2 - 640\lambda_1 + 51$$

We set $\frac{df}{d\lambda_1} = 0$

$\therefore \quad 2432 \times 2\lambda 1 - 640 = 0$

or, $\lambda 1 = \frac{640}{2 \times 2432} = \frac{10}{76} = 0{\cdot}1316$

$\therefore \quad \lambda_1^* = 0{\cdot}1316$

$\therefore$ The second approximation is given by

$$x_2 = x_1 + \lambda_1^* s_1 = \begin{bmatrix}1\\1\end{bmatrix} + 0\cdot 1316\begin{bmatrix}8\\-24\end{bmatrix} = \begin{bmatrix}2\cdot 0528\\-2\cdot 1584\end{bmatrix}$$

$$\text{Now } \nabla f_2 = [\nabla f]_{x_2} = \begin{bmatrix}4 \times 2\cdot 0528 - 12\\8 \times (-2\cdot 1584) + 16\end{bmatrix} = \begin{bmatrix}-3\cdot 7888\\-1\cdot 2672\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$$

$\therefore$ x_2 is not optimum point

To update the matrix H_1 we compute

$$Q_1 = \nabla f_2 - \nabla f_1 = \begin{bmatrix}-3\cdot 7888\\-1\cdot 2672\end{bmatrix} - \begin{bmatrix}-8\\24\end{bmatrix} = \begin{bmatrix}4\cdot 2112\\-25\cdot 2672\end{bmatrix}$$

$$\therefore \quad S_1^T Q_1 = [8 - 24]\begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = 640\cdot 1024$$

$$S_1 S_1^T = \begin{bmatrix} 8 \\ -24 \end{bmatrix}[8 \quad -24] = \begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix}$$

$$H_1 Q_1 = Q_1 = \begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix}$$

$$\therefore \quad (H_1Q_1)(H_1Q_1)^T = \begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = [\, 4{\cdot}2112 \;\; -25{\cdot}2672 \,]$$

$$= \begin{bmatrix} 17\cdot 7242 & -106\cdot 4052 \\ -106\cdot 4052 & 638\cdot 4314 \end{bmatrix}$$

Also $Q_1^T(H_1Q_1) = [4\cdot 2112 \quad -25\cdot 2672]\begin{bmatrix} 4\cdot 2112 \\ -25\cdot 2672 \end{bmatrix} = 656\cdot 1656$

$$\therefore \quad N_1 = -\frac{(H_1Q_1)(H_1Q_1)^T}{Q_1^T(H_1Q_1)} = -\frac{1}{656\cdot 1656}\begin{bmatrix} 17\cdot 7242 & -106\cdot 4052 \\ -106\cdot 4052 & 638\cdot 4314 \end{bmatrix}$$

$$= -\begin{bmatrix} 0\cdot 027 & -0\cdot 1625 \\ -0\cdot 1625 & 0\cdot 973 \end{bmatrix}$$

$$M_1 = \frac{\lambda_1^* S_1 S_1^T}{S_1^T Q_1} = \frac{0\cdot 1316}{640\cdot 1024}\begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix} = \begin{bmatrix} 0\cdot 0132 & -0\cdot 0395 \\ -0\cdot 0395 & 0\cdot 1184 \end{bmatrix}$$

$$\therefore \quad H_2 + H_1 + M_1 + N_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0\cdot 0132 & -0\cdot 0395 \\ -0\cdot 0395 & 0\cdot 1184 \end{bmatrix} + \begin{bmatrix} -0\cdot 027 & 0\cdot 1625 \\ 0\cdot 1625 & -0\cdot 973 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot 8062 & 0\cdot 123 \\ 0\cdot 123 & 0\cdot 1454 \end{bmatrix}$$

Hence $S_2 = -H_2 \nabla f_2 = -\begin{bmatrix} 0\cdot 8062 & 0\cdot 123 \\ 0\cdot 123 & 0\cdot 1454 \end{bmatrix}\begin{bmatrix} -3\cdot 7888 \\ -1\cdot 2672 \end{bmatrix} = \begin{bmatrix} 3\cdot 21 \\ 0\cdot 622 \end{bmatrix}$

To find the minimizing step length along S_2 we are to minimize $f(x_2 + \lambda_2 S_2)$ $= f(3{\cdot}21\lambda_2, -0{\cdot}9472, 0{\cdot}622\lambda_2 - 0{\cdot}1584) = 2(3{\cdot}21\lambda_2 - 0{\cdot}9472)^2 + 4(0{\cdot}622\lambda_2 - 0{\cdot}1584)$ $- 12\,(3{\cdot}21\lambda_2 - 0{\cdot}9474) + 16\,(0{\cdot}622\lambda_2 - 0{\cdot}1584) + 41$

We set $\frac{df}{d\lambda_2} = 0$

This gives $\lambda 2 = 0{\cdot}292$

$\therefore\ \lambda_2^* = 0{\cdot}292$

The third approximation is given by

$$x_3 = x_2 + \lambda_2^* s_2 = \begin{bmatrix} 2\cdot 0528 \\ -2\cdot 1584 \end{bmatrix} + 0\cdot 292 \begin{bmatrix} 3\cdot 21 \\ 0\cdot 622 \end{bmatrix} = \begin{bmatrix} 2\cdot 99 \\ -1\cdot 98 \end{bmatrix}$$

Now, $\nabla f_3 = [\nabla f]_{x_3} = \begin{bmatrix} 4\times 2\cdot 99 - 12 \\ 8\times(-1\cdot 98) + 16 \end{bmatrix} = \begin{bmatrix} -0\cdot 04 \\ 0\cdot 16 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ x_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\cdot 99 \\ -1\cdot 98 \end{bmatrix}$ *i.e.*, $x_1 = 2{\cdot}99$, $x_2 = -1{\cdot}98$ is the optimum point.

8.10 Summary

The unit is devoted to some unconstrained method of optimization viz. steepest descent method, Quadralically convergent method, Newton's method and Dairlon-Fletches-Powell method. These methods are explained with examples.

8.11 Self Assessment Questions

1. Using steepest descent method minimize the function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 4x_1 + 3x_3 + 9$ starting from the point (1, 2, 30).

2. Using steepest descent method minimize $f(x_1, x_2) = 2x_1 - x_2 + 8x_1^2 + 4x_1x_2 + x_2^2$ starting from the point (0, 0).

3. Find the conjugate directions for the matrix $\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$

4. Using Davidon Fletcher and Powell method minimize $f(x_1, x_2) = x_1 - 2x_2 + 2x_1^2 + 4x_1x_2 + 4x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

5. Using Davidon-Fletcher Powell method minimize $f(x_1, x_2) = 8x_1^2 + 4x_2^2 - 24x_1 + 16x_2 + 35$ with $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ as the starting point.

6. Using Davidon-Fletcher Powell method minimize $f(x_1, x_2) = 2x_1 + 3x_2 + 8x_1^2 + 12x_1x_2 + 9x_2^2$ with $\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$ as the starting point.

Unit 9 ❑ Constrained Optimization Techniques

Structure

9.1 Introduction

The constrained optimization problem is

Minimize $f(x)$

subject to $g_j, (x) \leq 0, j = 1, 2, ..., m$

There are many techniques to solve a constrained non linear programming problem. All these methods canbe classified as follows.

Constrained optimization techniques

Direct methods

(i) Heuristic search methods

(ii) Methods of feasible directions

(a) Zoutendijlis method

(b) Gradient projection method

(iii) Cutting plane

Indirect methods

(i) By the transformation of variables

(ii) Penalty function methods

(a) Interior penalty function methods

(b) Exterior penalty function methods

In the direct methods, the constraints are handled in an explicit manner whereas in most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems.

In this unit we discuss only cutting plane method.

9.2 Cutting Plane Method

In the cutting plane method, the nonlinear constraints are linearized by using Taylor's series expansion thereby approximating the feasible region by linearized envelopes. Assuming that the objective function is linear, we can solve the approximating LPP by this simplex method. If the solution of the LPP is not sufficiently accurate, we relinearize the binding constraints about the current point and formulate a new approximating LPP as solve it using the simplex method. We repeat this procedure until asufficiently accurate solution is found. We note that the approximating linear constraint cut off a portion of the existing feasible region. Hence the method is called cutting plane method.

To apply cutting plane method it is necessary that the objective function is linear. If the objective function is non-linear then we can formulate an equivalent optimization problem with linear objective function as follows.

Let the given problem be

Find $(x_1, x_2, \ldots\ldots, x_n)$ which minimize $f(x_1, x_2, \ldots\ldots, x_n)$

subject to the constraints gj $(x_1, x_2, \ldots\ldots, x_n) \leq 0$, $j = 1, 2, \ldots\ldots, m$.

We introduced a new variable x_{n+1} and transform this problem into an equivalent problem as follows

Find $(x_1, x_2, \ldots\ldots, x_n, x_{n+1})$ which minimize $0x_1, 0x_2, + \ldots\ldots + 0x_n, + x_{n+1}$ subject to the cosntraints gj $(x_1, x_2, \ldots\ldots, x_n) \leq 0$ $j = 1, 2, \ldots\ldots, m$ and g_{m+1} $(x_1, x_2, \ldots\ldots, x_{n+1}) = f(x_1, x_2, \ldots\ldots, x_n) - x_{n+1} \leq 0$

Thus, without loss of generally, we can assume that the given problem is

Minimize $f(x) = f(x_1, x_2, \ldots\ldots, x_n) = \bar{c}^T x = c_1x_1 + c_2x_2 + \ldots\ldots + c_{nxn}$

subject to the constraints $g_j(x) = g_j(x_1, x_2, \ldots, x_n) \leq 0$ $j = 1, 2, \ldots\ldots, m$

The iterative procedure of cutting plane method can be stated as follows :

9.3 Algorithm of Cutting Plane Method

(i) Start with an initial point x_1 and set the iteration number as i = 1. The point x_1 need not be feasible

(ii) Linearize the nonlinear constraint functions $g_j(x)$ about the point x_i as

$g_j(x) \simeq g_j(x_j) + [\nabla g_j(x_i)]^T (x - x_i), j = 1, 2, ..., m$

(iii) Formulate the approximating linear programming problem as

Minimize $f(x) = c^T x$

subject to $g_j(x_i) + [\nabla g_j(x_j)]^T (x - x_i) \leq 0, j = 1, 2, ..., m$

(iv) Solve the approximating LPP to obtain the solution vector x_{i+1}.

(v) Evaluate the original constraints at x_{i+1} i.e., find gj (x_{i+1}) for all j = 1,2, ..., m.

(vi) If $g_j(x_{i+1}) \leq \in$ for all j = 1, 2, ..., m where $\in$ is a prescribed small positive tolerance then all the original constaints can be assumed to have been satisfied.

Hence stop the procedure and take $x_{opt} = x_{i+1}$

It $g_i(x_{i+1}) >$ for some value of j, find the most violated constranit as

$g_k(x_{i+1}) = \max [g_j(x_{i+1})]$

Relinearize the constrant $g_k(x)$ 0 about the point x_{i+1} as

$g_k(x) \simeq g_k(x_{i+1}) + [g_k(x_{i+1})]^T (x - x_{i+1}) \leq 0$

and add this linear constraint to the previous approximating LPP.

(vii) Set the new iteration number i = i+1 and increase the total number of constraints in thenew approximationg LPP by one and go to step (iv).

Note : To avoid the unbounded solution of the first approximating LPP we may take the first approximating LPP as

Minimize $f(x)$ $c^T x$

subject to l_i x_i ll_i, i = 1, 2, ..., n

Where l_i and ll_i are chosen as lower and upper bounds of x_i take theoptimum solution of this first approximating LPP as x_1 in this first step.

9.4 Illustrative Examples

Example 9.4.1 Using cutting plane method

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 \leq 12$ taking $\in = 0{\cdot}03$

$x_1 + 2x_2 \leq 6$

$1 \leq x_1 \leq 6$

$1 \leq x_2 \leq 6$

Solution : We first consider the LPP

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \leq 6$

$1 \leq x_1 \leq 6$

$1 \leq x_2 \leq 6$

The extreme point of the feasible region are A (1,1), B (4, 1) and C (1, 5/2).

The value of the objective functions are

$(1, 1) = 1, \quad (4, 1) = -5, \quad (1, 5/2) = -5$

$\therefore$ The optimal solution of the LPP is (1, 1)

$\therefore$ The first approximationg point is $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $g(x_1, x_2) = (x_1 - 4)^2 + 2(x_2 - 3)^2 - 12$

$\therefore$ The given non-linear constraint is $g(x_1, x_2) \leq 0$

We gave $g(x) = \begin{bmatrix} 2(x_1 - 4) \\ 4(x_2 - 3) \end{bmatrix}$

Now $g(x_1) = g(1, 1)\ (1 - 4)^2 + 2(1 - 3)^2 - 12 = 5 > \in = 0{\cdot}03$.

Hence we linearize g (x) about x_1 as follows to replace

$$g(x)\ 0 \text{ as } g(x_1) + [\nabla g(x_1)]^T \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $5 + [-6, -8] \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 0$

or, $5 + (-6)(x_1 - 1) + (-8)(x_2 - 1) \le 0$

or, $-6x_1 - 8x_2 + 19 \le 0$

or, $6x_1 + 8x_2 \ge 19$

We now consider the following LPP by adding the constraint $6x_1 + 8x_2 \ge 19$ as

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$6x_1 + 8x_2 \ge 19$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme points of the feasible region are

A_1 $(1, {}^{13}/_8)$, A_2 $({}^{11}/_6, 1)$, B $(4, 1)$ and C $(1, {}^5/_2)$

The values of the objective function are

$f(1, {}^{13}/_8) = -{}^3/_2, f({}^{11}/_6, 1) = -{}^2/_3, f(4, 1) = -5, f(1, {}^5/_2) = -5$

∴ The optimal solution of the LPP is

$x_1 = {}^{11}/_6, x_2 = 1$

∴ We take the next approminality point as $x_2 = \begin{bmatrix} 11/6 \\ 1 \end{bmatrix}$.

Now $g(x_2) = g({}^{11}/_6, 1) = \left(\frac{11}{6} - 4\right)^2 + 2(1 - 3)^2 - 12 = \frac{25}{36} = 0{\cdot}69 > \epsilon = 0{\cdot}03$

We relinearize $g(x)$ about x_2 as follows and consider

$$g(x) \le 0 \text{ as } g(x_2) + [\nabla g(x_2)]^T \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $$\frac{25}{36} + \left[-\frac{13}{3} - 8\right] \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

or, $165x_1 + 288x_2 \ge 599$

We add this constraint to the previous LPP to get the following LPP

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$6x_1 + 8x_2 \ge 19$

$156x_1 + 288x_2 \ge 599$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme points of the feasible region are

A_1 $(1, {}^{13}/_{8})$, B $({}^{17}/_{12}, {}^{21}/_{16})$ and C_1 $({}^{311}/_{156}, 1)$

The values of the objective function are

$f(1, {}^{13}/_{8}) = -{}^{3}/_{2}$, $f({}^{17}/_{12}, {}^{21}/_{16}) = -{}^{13}/_{12}$, $f({}^{311}/_{156}, 1) = -{}^{77}/_{78}$

$\therefore$ The optimum solution is (311/156, 1)

$$\text{We take } x_3 = \begin{bmatrix} 311/156 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 994 \\ 1 \end{bmatrix}$$

Now $g(x_3) = g(1{\cdot}994, 1) = (1{\cdot}994 - 4)^2 + 2(1 - 3)^2 - 12 = 0{\cdot}027 < 0{\cdot}03 \in$

Hence, the optimum solution is given by $x_1 = 1{\cdot}994$, $x_2 = 1$

9.5 Summary

Among all the methods of constrained optimization here we have considered only the cutting plane method. The method is explained with the help of an example.

9.6 Self Assessment Questions

Using cutting plane method

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 - 12 \geq 0$

$x_1 + 2x_2 - 6 \leq 0$

$1 \leq x_1, x_2 \leq 6$

with the tolerance as $\in = 0{\cdot}3$

Using cutting plane method

Maximize $f = 1 - 4x_1 - 2x_2$

subject to $2(x_1 - 2)^2 + (x_2 - 3)^2 - 12 \geq 0$

$2x_1 + x_2 - 3 \leq 0$

$0 \leq x_1, x_2 \leq 5$

with $\in = 0{\cdot}2$

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