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Special Paper :
Pure Mathematics
Advanced Topology

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate Degree Programme, the opportunity to pursue Post-Graduate course in any subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental–in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Group
B
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Unit–I ❑ Compactness

Introduction

In this chapter we mainly deal with the notion of compactness and some of its variants. We start with the idea of nets and filters which was in the Topology course in PG–I, and present some more definitions like cluster points of nets, subnets, ultrafilters etc. Which will help us to establish some more characterizations of compactness in topological spaces. Next, the notions of three more types of compactness, namely countable compactness, Frechet compactness and sequential compactness are introduced which arise naturally from equivalent criteria of compactness in Real line with which you are already aware of. In a topological space all the four types of compactness turn out to be distinct and we establish their interrelationships.

In the remaining part of the chapter, we deal with compactness in stronger structures. First, we consider metric spaces and establish equivalent criteria of compactness by showing that all the four types of compactness are equivalent in metric spaces.

1.1 More on nets and filters

First, recall the following definitions from the earlier course on Topology.

Definition. Let $(D, \geq)$ be a directed set and X be a non-empty set. A mapping $s : D \to X$ is called a net in X. It is denoted by $\{s_n : n \in D\}$ or simply by $\{s_n\}_n$

A net $\{s_n\}_n$ is said to be eventually in $A \subset X$ if $\exists$ $n_0 \in D$ such that $s_n \in A$, $\forall n \in D$ with $n \geq n_0$.

A net $\{s_n\}_n$ is said to be frequently in $A \subset X$ if for each $m \in D$, $\exists$ *an* $n \in D$ with $n \geq m$ such that $s_n \in A$.

Definition. Let X be a topological space. A net $\{s_n\}_n$ is said to converge to $x_0 \in X$ if $\{s_n\}_n$ is eventually in every neighbourhood of x_0 and we write $\lim s_n = x_0$, x_0 is called a limit point or just a limit of $\{s_n\}_n$.

Definition. A point x_0 in a topological space X is said to be a cluster point of the net $\{s_n\}_n$ if it is frequently in every neighbourhood of x_0.

From the definition, it is clear that if a net $\{s_n\}_n$ is convergent then its limit points are the only cluster points of the net. But existence of a cluster point does not necessarily mean that the net is convergent. You have already come across such examples. Recall that taking $D = \mathbb{N}$, we had non-convergent sequences which have convergent subsequences and the limits of those convergent subsequences are in fact cluster points. This takes us to the next definition.

Definition. A net $\{t_\alpha : \alpha \in E\}$ is said to be a subnet of the net $\{s_n : n \in D\}$ if there is a mapping $i : E \to D$ such that

(a) $t = soi$,

(b) for any $m \in D$ there is $\alpha_0 \in E$ with the property that $i(\alpha) \geq m$ for all $\alpha \in E$ with $\alpha \geq \alpha_0$.

Theorem. Let X be a topological space and $\{s_n : n \in D\}$ be a net in X. A point $x_0 \in X$ is a cluster point of $\{s_n : n \in D\}$ iff some subnet of $\{s_n\}_n$ converges to x_0.

Proof. Let x_0 be a cluster point of the net $\{s_n : n \in D\}$. Denote by N_{x_0} the family of all neighbourhoods of x_0 and let $E = \{(U, n) : n \in D \text{ and } U \in N_{x_0}\}$. For (U, n) and (V, p) in E, define $(U, n) \geq (V, p)$ iff $U \subset V$ and $n \geq p$ in $(D, \geq)$. It is easy to verify that $(E, \geq)$ is a directed set.

Let $(U, m) \in E$. Since x_0 is a cluster point of $\{s_n : n \in D\}$ it is frequently in U. So there is an element $p_{(U, m)}$ in D with $p_{(U, m)} \geq m$ such that $s_{p(U,m)} \in U$. Now define the mappings $i. E \to D$ and $t : E \to X$ as follows : $i(U, m) = p_{(U, m)}$ and $t(U, m) = s_{p(U,m)}$. Then $(soi)(U, m) = s(i(U, m)) = s_{p(U,m)}$. so $t = soi$. Finally let $m \in D$. Choose any $U \in Nx_0$ so that $(U, m) \in E$. Now, let $(V, n) \in E$ and $(V, n) \geq (U, m)$. Then $i(V, n) = p_{(V, n)} \geq n \geq m$. This shows that $\{t_{(U, m)} : (U, m) \in E\}$ is a subnet of the net $\{s_n : n \in D\}$.

Now let U be any neighbourhood of x_0. Choose any $m \in D$ so as to get an element $(U, m) \in E$. Now, for any $(V, n) \in E$ with $(V, n) \geq (U, m)$, we have $t_{(V, n)} = s_{p(V,n)} \in V \subset U$ which shows that the net $\{t_{(U, m)} : (U, m) \in E\}$ converges to x_0.

Next suppose that some subnet $\{t_\alpha : \alpha \in E\}$ of the net $\{s_n : n \in D\}$ converges to x_0. Then there is a mapping $i : E \rightarrow D$ satisfying the conditions for a subnet. Let U be any neighbourhood of x_0 and $m \in D$. Since $\{t_\alpha : \alpha \in E\}$ converges to x_0, $\exists\ \alpha_1 \in E$ such that $t_\alpha \in U, \forall\ \alpha \geq \alpha_1, \alpha \in E$. Again by (*b*) of the above definition $\exists\ \alpha_2 \in E$ such that $i(\alpha) \geq m\ \forall\ \alpha \geq \alpha_2, \alpha \in E$. Choose $\alpha_0 \in E$ with $\alpha_0 \geq \alpha_1, \alpha_2$. Take $\alpha \in E$ with $\alpha \geq \alpha_0$. Then $i(\alpha) \geq m$ and $t_\alpha = (s o i)(\alpha) = s_{i(\alpha)} \in U$. So the net $\{s_n : n \in D\}$ is frequently in U. Hence x_0 is a cluster point of $\{s_n : n \in D\}$.

Exercise A net $\{s_n : n \in D\}$ is called a maximal net (or an ultranet) in X if for any $A \subset X$ it is either eventually in A or in $X \backslash A$. Prove that if x_0 is a cluster point of a maximal net $\{s_n : n \in D\}$ then it is convergent to x_0.

Solution : Let U be any neighbourhood of the point x_0. Since $\{s_n : n \in D\}$ is maximal, so either it is eventually in U or eventually in $X \backslash U$. If possible, suppose that it is eventually in $X \backslash U$. Then $\exists\ m \in D$ such that $s_n \in X \backslash U$ for all $n \in D$, with $n \geq m$. But as x_0 is a cluster point of $\{s_n : n \in D\}$, we can find a $p \geq m$ such that $s_p \in U$ which is a contradiction. Therefore $\{s_n : n \in D\}$ is eventually in U. Since this is true for every neighbourhood U of x_0, so $\{s_n : n \in D\}$ converges to x_0.

We now move to the idea of filters. Recall the basic definitions.

Definition. A nonempty family $\mathscr{F}$ of subsets of X is called a filter in X if (i) $\phi \notin \mathscr{F}$, (ii) $A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$, (iii) $A \in \mathscr{F}, A \subset B \Rightarrow B \in \mathscr{F}$.

A filter $\mathscr{F}$ is said to converge to x_0 in a topological space X if every neighbourhood of x_0 belongs to $\mathscr{F}$.

Definition. A point $x_0 \in X$ is called a cluster point of a filter $\mathscr{F}$ if for every neighbourhood U of x_0 and $F \in \mathscr{F}$, $U \cap F \neq \phi$ or equivalently $x_0 \in \overline{F}$, $\forall\ F \in \mathscr{F}$.

Definition. A filter $\mathscr{F}$ in X is said to be an ultrafilter if it is not properly contained in any other filter in X.

We will now prove some interesting results about ultrafilters.

Theorem : Let X be a non-empty set and $\hat{\mathscr{F}}$ be a family of subsets of X with finite intersection property. Then there exists an ultrafilter $\mathscr{F}^*$ in X containing $\hat{\mathscr{F}}$.

Proof : Let $\mathbb{C}$ denote the collection of all families of subsets of X with finite intersection property and containing the family $\hat{\mathscr{F}}$. For $\mathscr{F}_1$, $\mathscr{F}_2$ in $\mathbb{C}$, Let $\mathscr{F}_1 \geq \mathscr{F}_2$, iff $\mathscr{F}_2 \subset \mathscr{F}_1$. It is easy to see that $(\mathbb{C}, \geq)$ is a partially ordered set.

Let $\mathbb{B}$ be any totally ordered subset of $\mathbb{C}$. Write $\mathscr{F}_0 = \cup\{\mathscr{F} : \mathscr{F} \in \mathbb{B}\}$. Clearly $\hat{\mathscr{F}} \subseteq \mathscr{F}_0$. Let $\{A_1, A_2, ..., A_n\}$ be any finite subfamily of $\mathscr{F}_0$. Without loss of generality, suppose $A_i \in \mathscr{F}_i$ (i = 1, 2, ..., n) where $\mathscr{F}_i \in \mathbb{B}$. Since $\mathbb{B}$ is totally ordered, $\exists$ a $p \in N$, $1 \leq p \leq n$, such that $\mathscr{F}_p \geq \mathscr{F}_i$ $\forall$ i = 1, 2, ... , n. Then $A_1, A_2, ... A_n \in \mathscr{F}_p$ and so $A_1 \cap A_2 \cap .. \cap A_n \neq \phi$. Thus $\mathscr{F}_0$ also has the finite intersection property and hence $\mathscr{F}_0 \in \mathbb{C}$. Clearly $\mathscr{F}_0$ is an upper bound of $\mathbb{B}$. Therefore by Zorn's Lemma $\mathbb{C}$ has a maximal element $\mathscr{F}^*$ (say).

Clearly $\phi \notin \mathscr{F}^*$. Let $A, B \in \mathscr{F}^*$. If $\mathscr{F}_0 = \mathscr{F}^* \cup \{A \cap B\}$ then $\mathscr{F}_0$ has fip and contains $\hat{\mathscr{F}}$. So $\mathscr{F}_0 \in \mathbb{C}$. But as $\mathscr{F}^*$ is maximal, we must have $\mathscr{F}_0 = \mathscr{F}^*$ and hence $A \cap B \in \mathscr{F}^*$. Again let $A \in \mathscr{F}^*$ and $A \subset B$. By similar argument we can show that $B \in \mathscr{F}^*$. Therefore $\mathscr{F}^*$ is a filter.

Finally, note that if $\mathscr{F}'$ is any filter containing $\mathscr{F}^*$ then $\hat{\mathscr{F}} \subset \mathscr{F}^* \subset \mathscr{F}'$ and so $\mathscr{F}' \in \mathbb{C}$. Since $\mathscr{F}^*$ is a maximal element of $\mathbb{C}$ so we must have $\mathscr{F}^* = \mathscr{F}'$. This proves that $\mathscr{F}^*$ is an ultrafilter.

Theorem : A filter $\mathscr{F}^*$ is an ultrafilter in X iff any subset A of X which intersects every member of $\mathscr{F}^*$ belongs to $\mathscr{F}^*$.

Proof : First suppose that $\mathscr{F}^*$ is an ultrafilter in X. Let A be a subset of X which intersects every member of $\mathscr{F}^*$.

Let $\mathscr{F}_0 = \{C \subset X : A \cap B \subset C$ for some $B \in \mathscr{F}^*\}$. Clearly $\phi \notin \mathscr{F}_0$, $\mathscr{F}^* \subset \mathscr{F}_0$ and $A \in \mathscr{F}_0$. Let $C_1, C_2 \in \mathscr{F}_0$. Then $A \cap B_1 \subset C_1$ and $A \cap B_2 \subset C_2$ for $B_1, B_2 \in \mathscr{F}^*$. Then $B = B_1 \cap B_2 \in \mathscr{F}^*$ and we have $C_1 \cap C_2 \supset (A \cap B_1) \cap (A \cap B_2) = A \cap (B_1 \cap B_2) = A \cap B$ which implies $C_1 \cap C_2 \in \mathscr{F}_0$. Again if $C \in \mathscr{F}_0$ and $C \subset C'$ $(\subset X)$ then $\exists$ $B \in \mathscr{F}^*$ such that $A \cap B \subset C$ and so $A \cap B \subset C'$ which implies $C' \in \mathscr{F}_0$. Therefore $\mathscr{F}_0$ is a filter in X. Since $\mathscr{F}^*$ is an ultrafilter so $\mathscr{F}^* = \mathscr{F}_0$ and so $A \in \mathscr{F}^*$.

Next suppose that the given condition holds. Let $\mathscr{F}$ be any filter in X containing $\mathscr{F}^*$. Let $A \in \mathscr{F}$. If $B \in \mathscr{F}^*$ then $B \in \mathscr{F}$ and so $A \cap B \neq \phi$. So by our hypothesis $A \in \mathscr{F}^*$. This shows that $\mathscr{F} = \mathscr{F}^*$. Hence $\mathscr{F}^*$ is an ultrafilter.

Exercise : Let $\mathscr{F}^*$ be an ultrafilter in X and A, B be two subsets of X such that $A \cup B \in \mathscr{F}^*$. Then either $A \in \mathscr{F}^*$ or $B \in \mathscr{F}^*$.

Solution : Suppose that $A \notin \mathscr{F}^*$. Consider the family $\mathscr{F}_0 = \{C \subset X : A \cup C \in \mathscr{F}^*\}$. Then $B \in \mathscr{F}_0$. Since $A \notin \mathscr{F}^*$, $\phi \notin \mathscr{F}_0$. Let $C_1, C_2 \in \mathscr{F}_0$. Then $A \cup C_1, A \cup C_2 \in \mathscr{F}^*$ and so

$A \cup (C_1 \cap C_2) = (A \cup C_1) \cap (A \cup C_2) \in \mathscr{F}^*$. This proves that $C_1 \cap C_2 \in \mathscr{F}_0$. Again let $C \in \mathscr{F}_0$ and $C \subset C'$. Then $A \cup C \in \mathscr{F}^*$. But since $A \cup C \subset A \cup C'$, so $A \cup C' \in \mathscr{F}^*$ which then implies $C' \in \mathscr{F}_0$. So $\mathscr{F}_0$ is a filter in X.

Finally as we can see, $C \in \mathscr{F}^* \Rightarrow A \cup C \in \mathscr{F}^*$ and so $C \in \mathscr{F}_0$. Thus $\mathscr{F}^* \subset \mathscr{F}_0$. But as $\mathscr{F}^*$ is an ultrafilter so $\mathscr{F}_0 = \mathscr{F}^*$. Therefore $B \in \mathscr{F}^*$.

Exercise : A filter $\mathscr{F}^*$ in X is an ultrafilter iff for any $A \subset X$ either $A \in \mathscr{F}^*$ or $X \setminus A \in \mathscr{F}^*$.

Solution. First suppose that $\mathscr{F}^*$ is an ultrafilter. Let $A \subset X$. Since $X \in \mathscr{F}^*$ and $X = A \cup (X \setminus A)$ so either $A \in \mathscr{F}^*$ or $X \setminus A \in \mathscr{F}^*$. Conversely, suppose that the given condition holds. Let $\mathscr{F}$ be a filter containing $\mathscr{F}^*$. If $\mathscr{F}^* \subseteq \mathscr{F}$ then we can choose some $A \in \mathscr{F}$ such that $A \notin \mathscr{F}^*$. But then by the given condition $X \setminus A \in \mathscr{F}^*$ which implies $X \setminus A \in \mathscr{F}$. Then $\phi = A \cap (X \setminus A) \in \mathscr{F}$ which is a contradiction. Hence $\mathscr{F}^* = \mathscr{F}$ and so $\mathscr{F}^*$ must be an ultrafilter.

1.2 Compactness

We first recall the following definitions and a result from earlier Topology course.

Definition : A topological space (X, τ) is said to be compact if every open covering of X has a finite subcovering.

Compactness can be characterised in terms of "the finite intersection property" of closed sets.

Definition : (Finite intersection property) : A collection of subsets $\{F_\nu : \nu \in \wedge\}$ of a given set X ($\wedge$ being an indexing set) is said to possess the finite intersection property, if every finite sub-collection of $\{F_\nu\}$ has non-empty intersection.

Theorem : A topological space (X, τ) is compact if and only if for every collection of closed sets $\{F_\nu : \nu \in \wedge\}$ in (X, τ), possessing the finite intersection property, the intersection $\cap\{F_\nu : \nu \in \wedge\}$ of the entire collection is non-empty.

We now prove the following characterizations of compactness.

Theorem : Let (X, τ) be a topological space. Then the following statements are equivalent.

(i) X is compact

(ii) Every filter in X has a cluster point

(iii) Every ultrafilter in X converges.

Proof : (i) ⇒ (ii) : Suppose that X is compact. Let $\mathscr{F}$ be any filter in X. Let $\mathscr{F}^* = \{\overline{A} : A \in \mathscr{F}\}$. Then $\mathscr{F}^*$ is a family of closed sets with finite intersection property. Since X is compact, so

$$\cap\{\overline{A} : A \in \mathscr{F}\} \neq \phi.$$

Choose a point x_0 in $\cap\{\overline{A} : A \in \mathscr{F}\}$. Then $x_0 \in \overline{A}, \forall A \in \mathscr{F}$ and from definition x_0 is a cluster point of $\mathscr{F}$.

(ii) ⇒ (iii) : Let $\hat{\mathscr{F}}$ be an ultrafilter in X. By (ii), $\hat{\mathscr{F}}$ has a cluster point x_0 in X. Let U be any neighbourhood of x_0. Then $U \cap F \neq \phi \;\forall F \in \hat{\mathscr{F}}$. But then we must have $U \in \hat{\mathscr{F}}$. This shows that $\hat{\mathscr{F}}$ converges to x_0.

(iii) ⇒ (i) : Finally suppose that (iii) holds. Let $\mathscr{F}$ be a family of closed sets in X with finite intersection property. Then there exixts an ultrafilter $\hat{\mathscr{F}}$ containing $\mathscr{F}$. By (iii), $\hat{\mathscr{F}}$ converges to a point $x_0 \in X$. Then for any neighbourhood U of x_0, $U \in \hat{\mathscr{F}}$. Take any $F \in \mathscr{F}$. Then $F \in \hat{\mathscr{F}}$ and so $U \cap F \neq \phi$. This shows that x_0 is a limit point of F and so $x_0 \in \overline{F}$. But since each $F \in \mathscr{F}$ is closed, $x_0 \in \overline{F} = F$. This is true for any $F \in \mathscr{F}$ and so

$$\cap\{F : F \in \mathscr{F}\} \neq \phi.$$

This proves that X is compact.

We now use the concept of ultrafilter, developed so far, to prove the following important theorem due to Tychonoff.

Theorem (Tychonoff Product Theorem).

Let $\{X_\alpha : \alpha \in \wedge\}$ be a collection of topological spaces. Then the topological product space X is compact iff each X_α is so.

Proof : If X is compact, then clearly each factor space X_α, being continuous image of X under the projection map $p_\alpha : X \to X_\alpha$, is compact.

Conversely, let each space X_α be compact. By the above theorem it suffices to show that any ultrafilter $\mathscr{F}$ on X converges in X. For each $\alpha \in \Lambda$, $\mathscr{B}_\alpha = \{p_\alpha(F) : F \in \mathscr{F}\}$ is clearly a base for a filter $\mathscr{F}_\alpha$ on X_α. We claim that $\mathscr{F}_\alpha$ is an ultrafilter on X_α. For this, we need to show that for any subset A of X_α, either $A \in \mathscr{F}_\alpha$ or $X_\alpha \backslash A \in \mathscr{F}_\alpha$. Let us write $B = p_\alpha^{-1}(A)$. Since $\mathscr{F}$ is an ultrafilter on X, either $B \in \mathscr{F}$ or $X \backslash B \in \mathscr{F}$. Consequently, either $A = p_\alpha(B) \in \mathscr{B}_\alpha \subseteq \mathscr{F}_\alpha$ or $(X \backslash A) \in \mathscr{F}_\alpha$. Hence $\mathscr{F}_\alpha$ is an ultrafilter in X_α, for each $\alpha \in \Lambda$. As each X_α is compact, $\mathscr{F}_\alpha$ converges to some $x_\alpha \in X_\alpha$, for each $\alpha \in \Lambda$. Then $\mathscr{F}$ converges to the point $x = (x_\alpha)_{\alpha \in \Lambda}$ in X and hence X is compact.

1.3 Countable Compactness

We now look into another type of compactness which is weaker than compactness but is equivalent to compactness in the real line.

Definition : A topological space (X, τ) is said to be countably compact, if every countable open covering of X has a finite subcovering.

We shall obtain several necessary and sufficient conditions for a topological space to be countably compact. One such condition is given in terms of the concept of cluster point of a sequence. A point p is called a cluster point of an infinite sequence $\{x_n : n = 1, 2, \ldots\}$ in a topological space (X, τ) if, for any given open set U, containing p, and any positive integer r, there always exists a positive integer $m > r$, such that $x_m \in U$.

Theorem : For a topological space (X, τ) the following conditions are equivalent:

(a) (X, τ) is countably compact.

(b) Every countable aggregate of closed sets, possessing the finite intersection property, has a non-empty intersection in (X, τ).

(c) Every descending chain of non-empty closed sets, $F_1 \supset F_2 \supset ...$, has a non-empty intersection in (X, τ). (Cantor's intersection theorem)

(d) Every infinite sequence in X has a cluster point in X.

(e) Every infinite set $S \subset X$ has an w-accumulation point in X.

Proof : (a) ⇒ (b) : This is quite similar to the corresponding theorem on compactness.

(b) ⇒ (c) : Clearly $\{F_n : n \in \mathbb{N}\}$ is a countable collection of closed sets with the finite intersection property and hence by (b), $\bigcap_{n=1}^{\infty} F_n \neq \phi$

(c) ⇒ (d) : Let $\{x_n\}$ be a sequence in X and Let $A_n = \{x_m : m > n\}$ for each $n \in \mathbb{N}$. Clearly $\{\overline{A}_n : n \in \mathbb{N}\}$ is a descending sequence of nonempty closed sets in X. *By (c)*, there is a point $a \in \bigcap_{n=1}^{\infty} \overline{A}_n$. We claim that a is a cluster point of the given sequence. Indeed, for any open neighbourhood U of a and any $m \in \mathbb{N}$, we have some $x_n \in A_m \cap U$ as $a \in \overline{A}_m$. Then $n > m$ such that $x_n \in U$.

(d) ⇒ (a). If possible, suppose (a) does not hold. Then there is a countable open covering $\{U_n : n \in \mathbb{N}\}$ of X having no finite subcovering. Let $C_n = X \setminus (U_1 \cup ... \cup U_n)$. Clearly, $\{C_n : n \in \mathbb{N}\}$ is a descending sequence of nonempty closed sets in X. Choose $x_n \in C_n$, for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ has a cluster point x (say) in X (by (d)). Since $\{U_n : n \in \mathbb{N}\}$ is a cover of X, $\exists\, m \in \mathbb{N}$ such that $x \in U_m$. Now, $n > m \Rightarrow C_n \subseteq C_m \Rightarrow x_n \in C_m \Rightarrow x_n \notin U_m$. Thus x cannot be a cluster point of $\{x_n : n \in \mathbb{N}\}$, a contradiction.

To complete the proof, it now suffices to prove '(d) ⇔ (e)' which we do as follows:

(d) ⇒ (e) : Given an infinite set S in X, we can always construct a sequence $\{a_n\}$ in S such that $a_n \neq a_m$ for $n \neq m$ ($n, m \in \mathbb{N}$). By (d), this sequence has a cluster point p (say) in X. Then erery neighbourhood of ρ contains infinitely many terms of the sequence, i.e., contains infinitely many points of S. Hence p is an w-accumulation point of S.

(e) ⇒ (d) : Let $\{a_n\}$ be a sequence in X and let A be the set formed by the values taken by the sequence. If A is a finite set, then there is an element a such that $a_n = a$ for infinitely many natural numbers n. Obviously, then a is a cluster point of the sequence. If A is an infinite set, then by (e), A has an w-accumulation point p (say). Obviously, p is then a cluster point of $\{a_n\}$.

Exercise

(a) A subspace of a countably compact space need not be countably compact.

(b) Every closed subspace of a countably compact space is countably compact.

(c) The union of a finite collection of countably compact subspaces of a topological space is a countably compact subspace.

Solution : (a) The closed unit interval [0, 1] is compact, by Heine-Borel theorem, hence it is also countably compact. The subspace (0, 1) of [0, 1], is, however, not countably compact.

1.4 Sequentially Compact and Frechet Compact spaces

Finally we look into two types of compactness, one of which is defined by using sequences and the other defined by using the idea of limit points of sets.

Definition : (Sequentially Compact) : A topological space (X, τ) is said to be sequentially compact, if every infinite sequence in X contains a convergent subsequence.

Definition : (Frechet Compact) : A topological space (X, τ) is said to be Frechet compact (or B-W compact i.e., Bolzano-Weierstrass compact), if every infinite subset of X has an accumulation point.

Theorem : (a) Every closed subspace of a sequentially compact space is sequentially compact.

(b) Every closed subspace of a Frechet compact space is Frechet compact.

It follows from the following example that :

(i) a subspace of a sequentially compact space need not be sequentially compact, and (ii) a subspace of a Frechet compact space need not be Frechet compact.

Example : Let $\mathbb{R}$ be the set of reals, and v consists of (i) all those subsets of $\mathbb{R}$, which do not contain 0, and (ii) the 4 subsets $\mathbb{R}\setminus\{1, 2\}$, $\mathbb{R}\setminus\{1\}$, $\mathbb{R}\setminus\{2\}$, and $\mathbb{R}$. Then $(\mathbb{R}, v)$ is a first countable, Lindeloff space. Any open covering V of $\mathbb{R}$ must include at least one of the sets in (ii) (in order that 0 may be covered). Let G be such a set for the open covering V of $\mathbb{R}$, then $X\setminus G$ consists of at most two points 1 & 2. Let H_1 & H_2 be two members of V, containing the points 1 & 2 respectively. Then $\{G, H_1, H_2\}$ forms a finite subcovering of V for $\mathbb{R}$. Hence $(\mathbb{R}, v)$ is compact. let $S = \mathbb{R}\setminus\{0\}$, then the subspace $(S, v_s\}$ is not a Lindeloff space.

As v_s is the discrete topology on S, S is an infinite set having no accumulation point in S. Hence the subspace (S, v_s) is not Frechet Compact. The space $(\mathbb{R}, v)$ is also sequentially compact. In fact, any infinite sequence $\{x_i ; i = 1, 2 ...\}$ in $\mathbb{R}$ is of any one of the following two types :

(i) $x_i \neq 1$ and 2 for all i, except for finitely many values of i and the sequence $\{x_i : i = 1, 2 ...\}$ is itself convergent. Converging to the limit 0;

(ii) $x_i = 1$ or 2 for infinitely many values of i, and then there exists an infinite subsequence of $\{x_i : i = 1, 2 ...\}$, which converges to the limit 1 or 2.

1.5 Mutual dependence of different types of compactness

Now we investigate the interrelationships between the four types of compactness we have come across.

Theorem : (a) Every compact space is countably compact and also a Lindeloff space.

(b) A countably compact Lindeloff space is compact.

Proof : (a) Let (X, τ) be a compact space. Since for every open covering of X, there exists a finite sub-covering, the same is true for every countable open covering. Hence (X, τ) is countably compact. Also, since a finite sub-covering is necessarily a countable sub-covering, it follows that (X, τ) is also a Lindeloff space.

(b) Let (X, τ) be a countably compact, Lindeloff space. Let U be any open covering of X. As (X, τ) is a Lindeloff space, there exists a countable subcovering V of U for X. Again, since (X, τ) is countably compact, for the countable open covering V of X, there exists a finite subcovering W. Then W is a finite sub-covering of U for X. Hence the space (X, τ) is compact.

Theorem : (a) A countably compact space is Frechet compact.

(b) Any Frechet compact T_1-space is countably compact.

Proof : (a) Let (X, τ) be a countably compact space. Then every infinite subset S of X has an w-accumulation point in X ; thus S has an accumulation point in X. Hence (X, τ) is a Frechet compact space.

(b) Let (X, τ) be a Frechet compact, T_1-space, and S be an infinite subset of X. As (X, τ) is Frechet compact, S has an accumulation point x (say) in X and since (X, τ) is a T_1–space, the accumulation point x is an ω-accumulation point. Hence (X, τ) is countably compact.

Theorem : (a) A sequentially compact space is countably compact.

(b) Any countably compact, first countable space is sequentially compact.

Proof : (a) Let (X, τ) be a sequentially compact space and let $\{x_i : i = 1, 2, ...\}$ be any infinite sequence in X. Then the sequence $\{x_i : i = 1, 2 ...\}$ contains a convergent subsequence. The limit of the convergent subsequence is a cluster point of the sequence $\{x_i : i = 1, 2, ...\}$. Hence (X, τ) is countably compact.

(b) Let (X, τ) be a countably compact, first countable space. Let $\{x_i : i = 1, 2, ...\}$ be an infinite sequence in X. Since (X, τ) is countably compact, it is also Frechet compact; hence the infinite sequence $\{x_i : i = 1, 2 ...\}$ has an accumulation point x (say) in X. Again, since the space (X, τ) is first countable, it follows that there exists a sub-sequence $\left\{x_{k_i} : i = 1,2,...\right\}$ of the sequence $\{x_i : i = 1, 2, ...\}$, such that lim $x_{k_i} = x$. Thus the sequence $\{x_i ; i = 1, 2, ...\}$ contains a convergent subsequence $\left\{x_{k_i} : i = 1,2,...\right\}$. Hence the space (X, τ) is sequentially compact.

Note : In proving the part (b) of the above theorem, we have merely used the property that (X, τ) is Frechet compact (in place of its countable compactness). Hence, every Frechet compact, first countable space is sequentially compact.

In view of the fact that a second countable space is first countable and also a Lindeloff space, it follows that :

Theorem : For a second countable T_1-space, any one of the four properties (i) compactness, (ii) countable compactness, (iii) sequential compactness and (iv) Frechet compactness, implies the other three.

If can be shown by constructing suitable counter-examples that no other direct implication exists between the Lindeloff property and the four compactness properties.

Exercise : Give an example of a second countable (and hence Lindeloff), Frechet compact space that is not countably compact.

Solution : Consider the topological space $(\mathbb{N}, \tau)$. Here $\mathbb{N}$ is the set of natural numbers and τ is the odd-even topology on $\mathbb{N}$. The topology τ is generated by the base $B = \{\phi\} \cup \{(2n - 1, 2n) : n = 1, 2, ...\}$. The space $(\mathbb{N}, \tau)$ is second countable, since the open base B of τ is countable. Also B forms a countable open covering of $\mathbb{N}$, for which there is no finite sub covering, hence $(\mathbb{N}, \tau)$ is not countably compact.

Let P be an infinite subset of $\mathbb{N}$ and let $p \in P$. Let now, $x = p + 1$ if p is odd, and $x = p - 1$ if p is even. Then every open set, containing x, also contains p ; hence x is an accumulation point of P in $\mathbb{N}$. Consequently, the space $(\mathbb{N}, \tau)$ is Frechet compact.

Exercise : Give an example of a compact Hausdorff space that is not sequentially compact.

Solution : Let I denote the closed unit interval $[0, 1]$ with the subspace topology σ_1, induced by the usual topology σ of the real number space $(\mathbb{R}, \sigma)$. Let $(X, \tau) = \Pi \{I_r : I_r = I, r \in \mathbb{R}\}$. Thus, $X = I^I$ is the uncountable product of I. Hence X is compact and T_2, since I is so. Again X is not sequentially compact, since the sequence of functions $f_n \in X$ defined by $f_n(x)$ = the nth digit in the binary expansion of x, has no convergent sub-sequence.

For, suppose $\{f_{n_k}\}$ is a subsequence which converges to a point $f \in X$. Then, for each $x \in I$, $f_{n_k}(x)$ converges in I to $f(x)$. Let $p \in$ I have the property that $\alpha_{n_k}(p) = 0$ or 1 according as whether k is odd or even. Then $\{\alpha_{n_k}(p)\}$ is 0, 1, 0, 1, ... which cannot converge.

1.6 Compactness in Metric spaces :

In the final section of the chapter we consider the four types of compactness in metric spaces.

Theorem : In a metric space (X, d), the concepts of second countability, separability and Lindeloffness are equivalent.

Proof : Suppose (X, d) is separable. Let $A = \{x_i\}$ be a countable dense subset of X. Let $B = \{B(x_i, r)$; r-rational, $i = 1, 2, ...\}$. Then B is countable. Also if U is any open set and $x \in U$, $\exists$ an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Since A in dense in X, we can choose x_k s.t. $d(x_k, x) < \varepsilon/2$. Then it is easy to show that $x \in B(x_k, \varepsilon/2) \subset U$ and so B is also a base in X. Hence X is second countable.

It is known that a second countable space is Lindeloff. Now let (X, d) be Lindeloff. Let $\varepsilon = \frac{1}{n}$. From the open cover $\left\{B\left(x, \frac{1}{n}\right); x \in X\right\}$ of X, we can find a countable covering $\left\{B\left(x_i, \frac{1}{n}\right) : i \in \wedge\right\}$. Let $A_{\frac{1}{n}} = \{x_i ; i \in \mathbb{N}\}$. Then $A = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ is a countable dense subset of X and so X is separable.

We now introduce the following definition.

Definition. A finite subset F of a metric space X is called an ε-net for X if $X \subset \cup\{B(x, \varepsilon) : x \in F\}$ X is called totally bounded if it has an ε-net for every $\varepsilon > 0$.

Theorem : A countably compact metric space is totally bounded.

Proof : If possible let (X, d) be not totally bounded. Then $\exists$ an $\varepsilon > 0$ such that there is no ε-net for X. Let $p_1 \in X$. Clearly $X \not\subset B(p_1, \varepsilon)$. Choose $p_2 \in X \setminus B(p_1, \varepsilon)$. Since $\{p_1, p_2\}$ is not an ε-net, we can find $p_3 \in X$ such that $d(p_1, p_3) \geq \varepsilon$, $d(p_2, p_3) \geq \varepsilon$. Proceeding in this way we get a sequence $\{p_n\}$ of distinct points in X such that $d(p_i, p_j) \geq \varepsilon$, $i \neq j$. Since a metric space is first countable, so countable compactness of X implies sequential compactness of X. But evidently the sequence $\{p_n\}$ has no convergent subsequence in X. This contradicts the fact that X is countably compact.

Exercise : Prove that a totally bounded metric space X is separable.

Solution : For each $n \in \mathbb{N}$, X has an $\frac{1}{n}$-net F_n. Let $F = \bigcup_{n=1}^{\infty} F_n$. Then F is clearly countable. Let $x \in X$ and let $\delta > 0$ be given. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$. Now since F_m is a $\frac{1}{m}$-net, $\exists$ *a* $w \in F_m$ such that $d(x, w) < \frac{1}{m} < \delta$ i.e. $w \in B(x, \delta)$. This proves that F is also dense in X and so X is separable.

Theorem. For a metric space, compactness, countable compactness, Frechet compactness and sequential compactness are all equivalent.

Note. A totally bounded metric space is not necessarily compact. If $A = \{x \in Q : 0 \leq x \leq 1\}$ and $d^* = d_{A \times A}$ where d is the usual metric, then (A, d^*) is totally bounded but not compact.

Exercise : A set $A \subset (X, d)$ is called relatively compact if $\overline{A}$ is compact in X. Show that a relatively compact set A in a metric space (X, d) is totally bounded.

Recall that in a metric space (X, d), a sequence $\{x_n\}$ is called a Cauchy sequence if for any $\varepsilon > 0$ $\exists$ a $k \in \mathbb{N}$ s.t. $m, n \geq k \Rightarrow d(x_m, x_n) < \varepsilon$. Every convergent sequence is Cauchy. Also a Cauchy sequence having a convergent subsequence is also convergent.

Definition : (X, d) is said to be complete if every Cauchy sequence in X converges in X.

Theorem : A compact metric space is complete.

Theorem : If (X, d) is complete and totally bounded then it is compact.

Proof : We show that (X, d) is sequentially compact. Let $\{x_n\}$ be a sequence in X. Since X is totally bounded, X is contained in the union of a finite number of open balls of radius 1. At least one of them must contains a sub-sequence of $\{x_n\}$, say $\{x_{11}, x_{12}, ...\}$. Again from the property of total boundedness of X, we can find an open ball of radius $\frac{1}{2}$ which contains a subsequence of $\{x_{1n} ; n\in\mathbb{N}\}$, say $(x_{2n}, n = 1, 2, ...\}$. Proceeding in this way, by induction we obtain sequences $\{x_{ki} ; i = 1, 2, ...\}$ $(k = 1, 2...)$ each sequence is a subsequence of the predecessor and the k^{th} sequence is contained in a ball of radius $\frac{1}{k}$. It is easy to see that $\{x_{kk}; k = 1, 2, ...\}$ is a subsequence of $\{x_n\}$ which is Cauchy in X and so is convergent in X, since X is complete. Hence (X, d) is compact.

Note. The space l_2 is complete but not totally bounded and so is not compact.

Theorem : A metric space is compact iff it is complete and totally bounded.

Exercise : Prove that Lindeloff'ness is not a hereditary property.

Solution : Let X be an uncountable set and let $x_0\in X$ be chosen. Define a topology τ on X as follows : (i) $\phi, X \in\tau$. (ii) $A\ (\subset X) \in\tau$ iff $x_0\notin A$. First we will show that (X, τ) is a Lindeloff space. Let v be an open cover of X. Since the only open set containing x_0 is X itself, so $X\in v$ and $\{X\}$ is the required subcover. Now if $Y = X\backslash\{x_0\}$ then (Y, τ_y) is a discrete topological space. Taking v_l as the collectioin of all singletons from τ_Y, we see that v_1 cannot have a countable family which also covers Y.

Group–A
(Short questions)

1. Show that the collection $\mathscr{F}(p)$ of all subsets of a set X which contain a given element $p\in X$ is an ultrafilter on X.
2. Show that in a discrete topological space every neighbourhood filter is an ultrafilter.
3. Prove that the net associated with an ultrafilter is a maximal net.
4. Show that a filter $\mathscr{F}$ converges to a point $x\in X$ iff every ultrafilter containing $\mathscr{F}$ converges to x.
5. Give an example of a totally bounded space which is not compact.
6. Prove that continuous image of a sequentially compact set is sequentially compact.

7. Examine whether $\{\{a\}, \{b\}, \{a, b\}, X\}$ where $X = \{a, b, c\}$ is a filter.

8. Give an example to show that sequential compactness is not a hereditary property.

9. Prove that a finite union of compact subspaces of a topological space is compact.

Group–B
(Long questions)

1. Prove that every filter $\mathscr{F}$ on X is the intersection of all the ultrafilters finer than $\mathscr{F}$.

2. For an ultrafilter $\mathscr{F}$ on a set X prove that $\cap \{F : F \in \mathscr{F}\}$ is either empty or a singleton subset of X.

3. For a topological space (X, τ), prove that following are equivalent.

   (i) X is compact.

   (ii) Every net in X has a convergent subnet.

   (iii) Every maximal net in X converges in X.

4. Let (X, τ) be a topological space and $A \subset X$. Prove that A is τ-open iff A belongs to every filter which converges to a point of A.

5. If $\{f_n\}_n$ is a sequence of real valued continuous functions on a compact topological space X and $f_n \to$ f on X then prove that $f_n \to f$ uniformly on X.

6. Show that a subspace of R^n is bounded iff it is totally bounded.

7. If (X, d) is a complete metric space and $A \subset X$ is totally bounded then prove that A is relatively compact (i.e. $\overline{A}$ is compact).

8. Prove that a subnet of a subnet of a net $\{x_n : n \in D\}$ is a subnet of $\{x_n : n \in D\}$.

9. Give an example to show that the continuous image of a Frechet compact space need not be Frechet compact. If $f : X \to Y$ is a continuous bijection and X is Frechet compact, is it true that Y is also Frechet compact? Justify.

Unit–II ❑ Compactification

Introduction

In this chapter we start with the idea of local compactness. Though the definition was mentioned in the basic Topology course, here, it is dealt with in full detail as locally compact spaces are more common than compact spaces and they have many interesting properties. We then consider the notion of compactification. Though one-point compactification has been already included in the earlier course, it is again described in detail for the sake of completeness. We then consider two more ideas of compactification, which are much deeper and which have played important role in the advancement of the subject. The first type of compactification, that is dealt with, is Stone-Cech compactification which happens to be the largest Hausdorff compactification among all possible Hausdorff compactifications of a given Tychonoff space and there-in lies its importance. Another very strong result is the Stone-Cech theorem showing that any continuous function on a Tychonoff space can be extended to its Stone-Cech compactification. Finally we consider Wallman's compactification which is different from the other two compactifications in view of the use of ultrafilters in its construction.

2.1 Locally Compact spaces

A topological space is said to be locally compact if each point of the space has at least one compact neighbourhood.

Clearly every compact space is locally compact but the converse is not true, as R with usual or discrete topology is locally compact but not compact.

Theorem : 1. Let X be a locally compact space. The family of all closed compact neighbourhoods of each point x of X forms a neighbourhoods basis at x if in addition X is regular or Hausdorff.

Proof : Let $x \in X$. Denote by $\mathscr{V}$ the family of all closed and compact neighbourhoods of x. Since X is locally compact, there is a compact neighbourhood C of x.

(I) First suppose that X is a regular space. Let U be any neighbourhood of x. Then $(U \cap C)^o$ is an open neighbourhood of x. So there is an open neighbourhood V of x suct that $x \in V \subset \bar{V} \subset (U \cap C)^o$. Since $\bar{V} \subset C$ and C is compact, so $\bar{V}$ is also compact. Clearly $\bar{V}$ is then a closed compact neighbourhood of x and so $\bar{V} \in \mathscr{V}$. Also $\bar{V} \subset U$ and hence $\mathscr{V}$ is a neighbourhood basis at x.

(II) Next let X be T_2 and U be any neighbourhood of x. Let us take $W = (U \cap C)^o$. Then W is an open neighbourhood of x. Since C is compact and X is T_2, C is closed. Then $W \subset C \Rightarrow \bar{W} \subset C$ which implies $\bar{W}$ is compact.

Write $F = \bar{W} \setminus W$. Then F is a closed compact set and $x \notin F$. Since X is T_2, $\exists$ two open sets V_1, G_1 such that

$$x \in V_1,\ F \subset G_1 \text{ and } V_1 \cap G_1 = \phi.$$

Let $V = W \cap V_1$ and $G = \bar{W} \cap G_1$. Then V is an open neighbourhood of x, $F \subset G$ and $V \cap G = \phi$. Now

$$V \subset W \setminus G \subset \bar{W} \setminus G \subset \bar{W} \setminus F = W \subset U$$

and $\bar{V} \subset \bar{W} \setminus G \subset W \subset \bar{W} \subset C$

$= (\bar{W} \cap X \setminus G_1)$ ($\because$ $V \subset V_1 \subset X \setminus G_1$ closed), which implies that $\bar{V}$ is compact. Hence $\bar{V} \in \mathscr{V}$. Since $\bar{V} \subset U$, hence $\mathscr{V}$ forms a neighbourhood basis at x.

Exercise : Every locally compact T_2 space is regular.

Solution : Let X be a locally compact T_2 space.

Let $x \in X$ and U be a neighbourhood of x. Then proceeding exactly in the same as in the last part of the proof of theorem 1, we get two open neighbourhood V and W, and an open set G_1 such that.

$\bar{V} \subset \bar{W} \setminus G \subset W \subset U$, where $G = \bar{W} \cap G_1$. Thus $x \in V \subset \bar{V} \subset U$ which proves that X is regular.

Theorem 3 : A locally compact regular space is completely regular.

Proof : We prove the theorem by the following steps.

(I) Let A be a compact subset of X and U be an open set with $A \subset U$. Let $x \in A$. Since X is locally compact, there is a closed compact neighbourhood W_x of x (which is also closed by from theorem 1) such that $W_x \subset U$. Now the collection $\{W_x^\circ ; x \in A\}$ from an open cover A. Since A is compact, there exists a finite number of points x_1, x_2, .., x_n such that $A \subset W_{x_1}^\circ \cup W_{x_2}^\circ \cup .. \cup W_{x_n}^\circ$. Write $V = \bigcup_{i=1}^{n} W_{x_i}^\circ$. Then V is an open set containing A. Also $\bar{V} = \bigcup_{i=1}^{n} Wx_i$. So $\bar{V}$ is compact and $A \subset V \subset \bar{V} \subset U$.

(II) Let F be any closed subset of X and let $x_0 \in X \setminus F$. Since $X \setminus F$ is open, there is a closed compact neighbourhood A of x_0 with $A \subset X \setminus F$. By step (I), there is an open set V such that $\bar{V}$ is compact and

$$A \subset V \subset \bar{V} \subset X \setminus F.$$

Write $B = \bar{V} \setminus V$. Then B is also a closed compact set with $A \cap B = \phi$.

Clearly $\bar{V}$ with the relative topology, is a compact Hausdorff space and so is normal. Hence there is a continuous mapping $g : \bar{V} \to [0, 1]$

such that $g(x) = 0 \quad \forall x \in A$

$= 1 \quad \forall x \in B$.

We now define the mapping $f : X \to [0, 1]$ by

$f(x) = g(x) \quad \forall x \in \bar{V}$

$= 1 \quad \forall x \in X \setminus \bar{V}$.

We now show that f is continuous.

(a) Let $z \in V$. Choose any $\in > 0$. Since g is continuous, there is an open neighbourhood U of z in the space $\bar{V}$ such that $|g(x) - g(z)| < \in$, $\forall x \in U$. we can write $U = \bar{V} \cap G$, where G is an open set in X. Let $W = V \cap G$. Then W is an open neighbourhood of z in X and we have

$|f(x) - f(z)| = |g(x) - g(z)| < \in$, $\forall x \in W$ (since $W \subset U$) and so f is continuous at z.

(b) Let $z \in X \setminus \bar{V} = W$ (say). Let $\in > 0$ be given. Clearly W is an open neighbourhood of z in X and we have

$|f(x) - f(z)| = |1 - 1| = 0 < \in, \ \forall x \in W.$

So f is continuous at z.

(c) Finally let $z \in \bar{V} \setminus V$. Then $f(z) = g(z) = 1$. Let $\in > 0$ be given. Since g is continuous, there is an open neighbourhood U of z in the space $\bar{V}$ such that

$$|g(x) - g(z)| < \in, \quad \forall x \in U.$$

We can write $U = \bar{V} \cap G$, where G is an open set in X. We have $G = (G \cap \bar{V}) \cup (G \setminus \bar{V})$. Now if $x \in G \cap \bar{V}$, then $f(x) = g(x)$ and so

$$|f(x) - f(z)| = |g(x) - g(z)| < \in.$$

If $x \in G \setminus \bar{V} \subset X \setminus \bar{V}$, $f(x) = 1$ and so

$$|f(x) - f(z)| = |1 - 1| = 0 < \in.$$

Thus $|f(x) - f(z)| < \in, \ \forall x \in G$. Hence f is continuous at z.

Since $x_0 \in A$(which is contained in $\bar{V}$) and $F \subset X \setminus \bar{V}$, we have $f(x_0) = 0$ and $f(x) = 1, \ \forall x \in F$. This proves that the space X is completely regular.

Theorem : 4. A locally compact Hausdorff space is completely regular.

Proof. Follows from Theorems 2 and 3.

Exercise : Let (X, τ) be a T_2 space. Then the following statements are equivalent.

(i) X is locally compact.

(ii) For each $x \in X$ and each neighbourhood U of x there is a relatively compact open set V such that $x \in V \subset \bar{V} \subset U$.

(iii) For each compact set C and each open set U with $C \subset U$, there is a relatively compact open set V such that $C \subset V \subset \bar{V} \subset U$.

(iv) τ has a basis consisting of relatively compact open sets.

Solution : (i) ⇒ (ii) :

As in Theorem 2 we can show that ∃ an open set V such that $\bar{V}$ is compact (i.e. V is relatively compact) and $x \in V \subset \bar{V} \subset U$.

(ii) ⇒ (iii) : Suppose (ii) holds good. Let C be a compact set and U be an open set with $C \subset U$. Take any $x \in C$. Then $x \in U$ and by (ii) there is a relatively compact open set V_x suct that.

$$x \in V_x \subset \bar{V}_x \subset U.$$

Now $\{V_x \,;\, x \in C\}$ forms an open cover of C. Since C is compact, ∃ a finite number.

of open sets $V_{x_1}, V_{x_2}, ..., V_{x_n}$ s.t. $C \subset \bigcup_{i=1}^{n} V_{x_i}$. Write $V = \bigcup_{i=1}^{n} V_{x_i}$. Then V is an open set. Also

$\bar{V} = \bigcup_{i=1}^{n} \bar{V}_{x_i}$, being finite union of compact sets is also compact. Hence V is a relatively

compact set such that

$$C \subset V \subset \bar{V} \subset U.$$

(iii) ⇒ (iv) : Suppose (iii) holds good. Let $\mathscr{B}$ be the family of all relatively compact open sets. Let G be any open set and $x \in G$. Since $\{x\}$ is compact, by (iii) ∃ a relatively compact open set V (i.e., $V \in \mathscr{B}$) such that $x \in V \subset \bar{V} \subset G$. Hence $\mathscr{B}$ forms a basis of τ.

(iv) ⇒ (i) : Let $x \in X$. Since X is open by (iv) there is a relatively compact open set V such that $x \in V \subset \bar{V} \subset X$. Clearly $\bar{V}$ is a compact neighbourhood of x and so X is locally compact.

Exercise : Let (X, τ) be locally compact and let $f : (X, \tau), \to (Y, \tau')$ be open, continuous and onto. Then show that Y is also locally compact.

Solution : Let $y \in Y$ and let V be a τ'-open set containing y. Let $f(x) = y$, $x \in X$. Since f is continuous at x, we can find an open set U containing x such that $f(U) \subset V$. By local compactness of (X, τ), there is a compact set A such that

$$x \in A^{\circ} \subset A \subset U.$$

Then $\quad y = f(x) \in f(A^\circ) \subset f(A) \subset f(U) \subset V.$

Write $f(A) = B$. Since f is continuous and A is compact, so B is also compact. Again as f is open so $f(A^\circ)$ is an open set contained in $f(A)$ and so

$$f(A^\circ) \subset (f(A))^\circ = B^\circ.$$

Thus we have $y \in B^\circ \subset B \subset V$; which shows that Y is locally compact.

Exercise : Prove that a closed subspace of a locally compact space is locally compact.

Solution : Let (X, τ) be locally compact and let $Y \subset X$. To show that (Y, τ_Y) is so, choose $y \in Y$ and let V be a τ_Y-neighbourhood of y. Then $V = U \cap Y$ for some τ-neighbourhood U of y. Since X is locally compact, so there is a compact set A such that

$$y \in A^\circ \subset A \subset U.$$

Then $y \in A^\circ \cap Y \subset A \cap Y \subset U \cap Y = V.$

Write $B = A \cap Y$. Choose an open cover $\mathscr{V}$ of τ_Y-open sets covering B. Note that every $W \in \mathscr{V}$ is of the form $W = W' \cap Y$, $W' \in \tau$. Then $\{W' : W = W' \cap Y \in \mathscr{V}\} \cup (X \setminus Y)$ forms an open cover of A. By compactness of A, this cover has a finite subcover and consequently $\mathscr{V}$ also has a finite subcover of B. Hence B is τ_Y-compact. Clearly

$$y \in B^\circ \subset B \subset V$$

and this proves the result.

Exercise : The cartesian product $\prod_{\alpha \in \Lambda} X_\alpha$ (provided non-empty) is locally compact iff each X_α is locally compact $\forall\, \alpha \in \Lambda$ and all X_α, except for finite number of spaces, are compact.

Solution : First let X be locally compact. Since each projection map $p_\alpha : X \to X_\alpha$ is a continuous, open surjection, (by previous exercise) X_α is locally compact, $\forall\, \alpha \in \Lambda$.

Now let $x \in X$. By local compactness of X, x has a compact neighbourhood $U = \prod U_\alpha$ (say). Then $U_\alpha = X_\alpha$, $\forall \alpha \in \Lambda \setminus F$, where F is a finite subset of $\wedge$. Thus $p_\alpha(U) = X_\alpha$ which is compact (being continuous image of the compact set U), for all $\alpha \in \Lambda \setminus F$. Hence all spaces X_α, except for finite number of $\alpha's$, are compact.

Conversely, to prove X to be locally compact (under the stated conditions), let $x = (x_\alpha)_{\alpha \in \Lambda} \in X$. By hypothesis, there is a finite subset $F = \{\alpha_1, ..., \alpha_m\}$ (say) of Λ such that X_α is compact, $\forall \alpha \in \Lambda \setminus F$. For $\alpha_i \in F$ ($i = 1, 2, ..., m$), there exists a compact neighbourhood U_{α_i} of X_{α_i} in X_{α_i} (by local compactness of each X_α). Then $V = \prod_{\alpha \in \Lambda} V_\alpha$, where $V_\alpha = U_\alpha$ for $\alpha = \alpha_1, ..., \alpha_m$ and $V_\alpha = X_\alpha$ for $\alpha \notin F$ is a neighbourhood of x, and V is compact by Tychonoff product theorem. Hence each point of X has a compact neighbourhood and so X is locally compact.

2.2 Compactification

Let X be a topological space. A pair (f, Y) is said to be a compactification of X if the following conditions hold.

(i) Y is a compact space.

(ii) There is a subspace Y_0 of Y such that Y_0 is dense in Y and f is a homeomorphism of X onto Y_0.

Let (f, Y) be a compactification of the topological space X. If $Y \setminus f(X)$ consists of one point only, (f, Y) is called a one point compactification of the space X.

Exercise : 1. If X is a compact space, (i_X, X) is a compactification of X where i_X is the identity mapping.

Exercise : 2. If $X = (0, 1)$, then (f, Y) is a compactification of X where $Y = [0, 1]$ and $f : (0, 1) \to (0, 1) \subset [0, 1]$ is the inclusion.

Exercise 3. Take $Y = [a, b]$, $Y_0 = (a, b)$. Then also (f, Y) is a compactification of $(0, 1) = X$, where $f : X \to Y_0$ is defined by $f(x) = a + (b - a)x$, $\forall x \in X$.

Theorem : 1. Let (X, τ) be a non-compact topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an element not in X. Denote by τ^* the family consisting of the void set ϕ, the set X^*, the members of τ and all those subsets U of X^* such that $X^* \setminus U$ is a closed compact subset of X. Then τ^* is a topology on X^* and (X^*, τ^*) is a compactification of X.

Proof : We prove the theorem in the following steps.

(I) We first verify that τ^* is a topology on X^*. Let G_1, G_2 be two members of τ^* and let $G = G_1 \cap G_2$. If $G = \phi$ then clearly $G \in \tau^*$. Suppose that $G \neq \phi$. If $G_1, G_2 \in \tau$ then $G \in \tau \subset \tau^*$. Let $G_1, G_2 \notin \tau$. Then $X^* \setminus G_1$ and $X^* \setminus G_2$ are closed compact subsets of X. So $X^* - G = (X^* - G_1) \cup (X^* - G_2)$ is also a closed compact subset of X and so $G \in \tau^*$. Again let $G_1 \in \tau$ but $G_2 \notin \tau$. Then $X^* \setminus G_2 = F$(say) is a closed compact subset of X.

Since $F \subset X$, $\infty \notin F$. So $\infty \in G_2$ and we may write

$G_2 = X^* \setminus F = \{\infty\} \cup (X \setminus F) = \{\infty\} \cup W$ (say) where $W \in \tau$. Then $G = G_1 \cap G_2 = G_1 \cap [\{\infty\} \cup W] = G_1 \cap W \in \tau \subset \tau^*$. If $G_1 \notin \tau$ and $G_2 \in \tau$ then one can similarly show that $G \in \tau^*$.

Now let $\{G_a \,;\, a \in A\}$ be any nonempty subfamily of τ^* (where A is an index set) and let $G = \cup \{G_a \,;\, a \in A\}$. If $G_a \in \tau \;\forall\, a \in A$, then clearly $G \in \tau \subset \tau^*$. Suppose $G_a \notin \tau$ for some $a \in A$. Let $A_1 = \{a \,;\, G_a \in \tau\}$ and $A_2 = A \setminus A_1$. Write $U_1 = \cup\{G_a \,;\, a \in A_1\}$ and $U_2 = \cup\{G_a \,;\, a \in A_2\}$. For $a \in A_2$, we may write $G_a = \{\infty\}\ U\ W_a$ where $W_a \in \tau$. Also write $W_a = G_a$ if $a \in A_1$. Then

$$
\begin{aligned}
G &= \cup\{G_a \,;\, a \in A\} \\
&= \{\infty\} \cup [\cup \{W_a \,;\, a \in A\}] \\
&= \{\infty\} \cup W \text{ (say)}.
\end{aligned}
$$

clearly $W = \cup\{W_a \,;\, a \in A\} \in \tau$.

Take $a_0 \in A_2$. Then $W_{a_0} \subset W$. Now we have $X^* \setminus G = X^* \setminus [\{\infty\} \cup W] = X \setminus W$ which is closed is X. Also $X \setminus W \subset X \setminus W_{a_0} = X^* \setminus G_{a_0}$ where $X^* \setminus Ga_0$ is compact in X. Hence it follows that $X^* \setminus G$ is also compact in X (being a closed subset of a compact set). Therefore $G \in \tau^*$. Obviously ϕ, $X \in \tau^*$ and τ^* is a topology on X^*.

(II) Let $W \in \tau$. Then $\infty \notin W$. So we may write $W = X \cap W$ where $W \in \tau^*$. Again if $G \in \tau^*$ but $G \notin \tau$ then G is of the form $G = \{\infty\} \cup W$, where $W \in \tau$ and so $G \cap X = W \in \tau$. This shows that τ consists of exactly all those sets of the form $X \cap G$ where $G \in \tau^*$. Hence (X, τ) is a subspace of (X^*, τ^*). Also since $G \cap X \neq \phi$ for every open set G containing ∞, so X is dense in X^*.

(III) Let $G = \{G_a \ ; \ a \in A\}$ be any open cover of X^*. Then $\infty \in G_a$ for some $a \in A$. Let $A_1 = \{a \ ; \ \infty \notin G_a\}$ and $A_2 = A \setminus A_1$. For $a \in A_2$ we may write $G_a = \{\infty\} \cup W_a$ where $W_a \in \tau$. Also write $G_a = W_a$ for $a \in A_1$. Take any $a_0 \in A_2$. Then $X^* \setminus G_{a_0} = X \setminus W_{a_0}$ is a compact subset of X. Clearly $\{W_a \ ; \ a \in A\}$ is an open cover of $X \setminus W_{a_0}$. So there is a finite number of sets $W_{a_1}, W_{a_2}, .., W_{a_n}$ from the family such that

$$X^* \setminus G_{a_0} = X \setminus W_{a_0} \subset \bigcup_{i=1}^{n} W_{a_i} \subset \bigcup_{i=1}^{n} G_{a_i}.$$

So $X^* = \bigcup_{i=0}^{n} G_{a_i}$ and X^* is compact.

(IV) Take $Y = X^*$ and $Y_0 = X$. Define the mapping $f : X \to Y_0$ by $f(x) = x$ for $x \in X$. Then f is a homeomorphism of X onto Y_0. Therefore (f, Y) is a compactification of X.

Note : In the above theorem clearly f is the identity mapping i_X on X. The compactification (i_X, X^*) is called the Alexandroff's one point compactification.

Definition : If (f, Y) is a compactification of a topological space X where Y is T_2 then (f, Y) is called a T_2-compactification of X.

Theorem 2 : Let X be a topological space which is not compact. Then Alexandroff's one point compactification (i_X, X^*) of X is a T_2-compactification iff X is a locally compact T_2 space.

Proof : First suppose that (i_X, X^*) is a T_2-compactification. Then X^* is a T_2 space and so X is also a T_2-space.

Let $x \in X$. Then x and ∞ are two distinct points of the space X^*. So there are open sets G_1, G_2 in X^* such that $x \in G_1$, $\infty \in G_2$ and $G_1 \cap G_2 = \phi$. Since $\infty \notin G_1$, $G_1 \subset X$ and is open in X. Again $X^* \setminus G_2$ is closed and compact in X. Since $G_1 \subset X^* \setminus G_2$, so $X^* \setminus G_2$ itself is a compact neighbourhood of x in X. Thus every point of X has a compact neighbourhood and so X is locally compact.

Next suppose that X is locally compact and T_2. Let x, y be two distinct points of X^*. If $x, y \in X$ then Hausdorffness of X implies that there are two disjoint open sets G_1, G_2 in X containing x and y respectively. Since open sets of X are also open in X^* so the result follows. Now let $x \in X$ and $y = \infty$. Since X is locally compact, there is a compact neighbourhood U of x in the space X. Since X is a T_2-space, U is also closed in X. There is an open set G_1 in X with $x \in G_1 \subset U$. Take $G_2 = X^* \setminus U$. Then G_2 is open in X^* and $\infty \in G_2$. Also $G_1 \cap G_2 = \phi$. Hence the space X^* is a T_2-space.

Definition : Let (X, τ) be a topological space. If there exists a topological space $(Y, \hat{\tau})$ s.t. X is homeomorphic to a subspace Y_0 of Y, then we say that X can be embedded in the space Y.

Let (X, τ) be a topological space and $\mathscr{F}$ be a family of functions s.t. each function f in $\mathscr{F}$ is a mapping from X to a topological space Y_f. Denote by Y the product of the spaces Y_f i.e., $Y = \prod_{f \in \mathscr{F}} Y_f$.

Definition : Define the mapping $e : X \to Y$ as follows. For $x \in X$, $e(x)_f = f(x)$, where $e(x)_f$ denotes the f th component of $e(x)$, i.e., $p_f(e(x)) = f(x)$, $\forall x \in X$ so that $p_f \circ e = f$, $p_f : \Pi Y_f \to Y_f$ being the f th projection map. e is called the evaluation map. We say that the family $\mathscr{F}$ distinguishes points iff for any two distinct points $x, y \in X$, $\exists f \in \mathscr{F}$ s.t. $f(x) \neq f(y)$. We say that the family $\mathscr{F}$ distinguishes points from closed sets if for any closed set A in X and each point x in $X \setminus A$, there is a $f \in \mathscr{F}$ s.t. $f(x) \notin \overline{f(A)}$.

Embedding Lemma

Let (X, τ) be a topological space and $\mathcal{F}$ be a family of functions on X such that each f in $\mathcal{F}$ is a continuous mapping of X into a topological space Y_f. Then followings hold.

(a) $e : X \rightarrow Y = \prod_{f\in\mathcal{F}} Y_f$ is continuous.

(b) e is an open mapping onto $e(X)$ if $\mathcal{F}$ distinguishes points from closed sets.

(c) e is one-to-one iff $\mathcal{F}$ distinguishes points.

(d) e is a homeomorphism of X onto $e(X)$ if $\mathcal{F}$ distinguishes points as also points from closed sets.

Proof : (a) Let P_f denote the projection of the product space Y to the f th co-ordinate space Y_f. For $x\in X$, $(P_f o e)(x) = P_f(e(x)) = e(x)_f = f(x)$. Since each f is continuous, so each $P_f o e$ is continuous for each $f\in\mathcal{F}$. Therefore e is continuous.

(b) Suppose that $\mathcal{F}$ distinguishes points from closed sets. Let G be any open set in X and let $y\in e(G)$. Then $\exists x\in G$ such that $y = e(X)$. Since $\mathcal{F}$ distinguishes points from closed sets, there is a function f in $\mathcal{F}$ such that $f(x) \notin \overline{f(A)}$ where $A = X \backslash G$. Write $U_f = Y_f \setminus \overline{f(A)}$. Then U_f is open in Y_f and $P_f^{-1}(U_f)$ is open in the product space $\prod Y_f$. Therefore $W_y = P_f^{-1}(U_f) \cap e(X)$ is an open set in the subspace $e(X)$ of Y. We now show that

$$y\in W_y \subset e(G).$$

Since $x\notin A$, by our hypothesis $f(x) \notin \overline{f(A)}$ and so $f(x) \in Y_f \setminus \overline{f(A)} = U_f$. Since $P_f(e(x)) = f(x)$, so $y = e(x) \in P_f^{-1}(U_f)$. Hence $y\in W_y$. Next let $z\in W_y$. Then $z\in P_f^{-1}(U_f)$ and $z\in e(X)$. Clearly $z = e(u)$ for some $u\in X$. Now $e(u) \in P_f^{-1}(U_f) \Rightarrow P_f(e(u)) = e(u)_f = f(u) \in U_f \Rightarrow f(u) \notin \overline{f(A)} \Rightarrow u \notin A \Rightarrow u\in G \Rightarrow z = e(u) \in e(G)$. So $W_y \subset e(G)$. Thus $e(G)$ is a neighbourhood of y in the space $e(X)$. Since y is an arbitary point of $e(G)$, it follows that $e(G)$ is an open set in the space $e(X)$. Hence $e : X \rightarrow e(X)$ is an open mapping.

(c) Suppose that $\mathcal{F}$ distinguishes points of X. Take any two points x, y $(x \neq y)$ in

X. Then there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$, i.e., $e(x)_f \neq e(y)_f$. This gives that $e(x) \neq e(y)$. Hence e is one-to-one.

Next, suppose that e is one-to-one. Let x and y be two distinct points of X. Then $e(x) \neq e(y)$. So there is a function f in $\mathcal{F}$ suct that $e(x)_f \neq e(y)_f$ i.e. $f(x) \neq f(y)$. Hence the family $\mathcal{F}$ distinguishes points of X.

(d) If the family $\mathcal{F}$ distinguishes points as well as distinguishes points from closed sets, then by (a), (b) and (c), e is a bijective mapping from X onto $e(X)$ which is both open and continuous. Hence e is a homeomorphism from X onto $e(X)$.

Definition : Let X be a topological space. Denote by $C^*(X)$ the family of all continuous mappings of X into the unit closed interval $[0, 1] = Q$. Now by Tychonoff's theorem $Q^{C^*(X)}$ [the product of the unit interval Q taken $C^*(X)$ times] is compact. As before let $e : X \to Q^{C^*(X)}$ be the evaluation map defined by $e(x)_f = f(x)$ for $x \in X$. Then e is continuous. Now suppose that X is a Tychonoff space (completely regular T_1 space). Then from definition it follows that the family $C^*(X)$ distinguishes points of X as well as points from closed sets. Then by Embedding Lemma, e is a homeomorphism of X onto the subspace $e(X)$ of $Q^{C^*(X)}$. We write $\beta(X) = \overline{e(X)}$. Then $\beta(X)$ is compact and the pair $(e, \beta(X))$ is a compactification of X which is called the Stone-Cech compactification of X.

Theorem : 3. (Stone-Cech Theorem)

Let X be a Tychonoff space and f be a continuous mapping of X into a compact T_2 space Y. Then there is a continuous extension of f which carries $\beta(X)$ into Y.

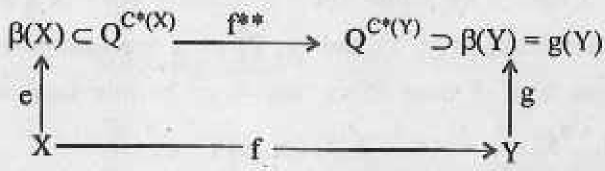
Proof : Let e denote the evaluation map of X into $Q^{C^*(X)}$ and g be the evaluation map of Y into $Q^{C^*(Y)}$ where $Q = [0, 1]$. If $a \in C^*(Y)$, then aof is a continuous mapping of X into Q and so aof $\in C^*(X)$. Denote the mapping $f^* : C^*(Y) \to C^*(X)$ by $f^*(a) = a$of, for all $a \in C^*(Y)$. Then for any $q \in C^*(X) \to Q$, $q$of* is a mapping of $C^*(Y)$ into Q. Define the mapping $f^{**} : Q^{C^*(X)} \to Q^{C^*(Y)}$ by

$$f^{**}(q) = q\text{of}^* \text{ for all } q \in Q^{C^*(X)}.$$

Let $a \in C^*(Y)$ and $q \in Q^{C^*(X)}$. We have

$$P_a o f^{**}(q) = P_a(f^{**}(q)) = P_a(q o f^*)$$
$$= (q o f^*)\ (a) = q\ (f^*(a)).$$

But $q(f^*(a))$ is simply the projection of q into the $f^*(a)$-th co-ordinate space of $Q^{C^*(X)}$ and this is a continuous mapping. Hence the mapping f^{**} is continuous. By embedding Lemma, e is a homeomorphism of X onto $e(X)$ and g is a homeomorphism of Y onto $g(Y) = \beta(Y)$, because Y is a compact T_2 space.



Let $x \in X$ and $a \in C^*(Y)$. Write $q = e(x)$ and $y = f(x)$.

Then $q \in Q^{C^*(X)}$ and $y \in Y$. We have

$$[(f^{**} o e)\ (x)]\ (a) = [f^{**}(e(x))]\ (a) = [f^{**}(q)]\ (a)$$
$$= (q o f^*)\ (a) = q(f^*\ (a)) = q(a o f)$$
$$= e(x)\ (a o f) = (a o f)\ (x) = a(f(x)) = a(y)$$
$$= g(y)_a = g(y)\ (a) = [g(f(x))]\ (a)$$

This gives that

$$(f^{**} o e)\ (x) = g(f(x)) \in g(Y).$$

Let $q \in e(X)$. Then there is a point x in X such that $e(x) = q$. So

$$f^{**}(q) = f^{**}(e(x)) = (f^{**} o e)\ (x) = g(f\ (x)) \in g(Y).$$

Or, $(g^{-1} o f^{**})\ (q) = f(x) \in Y$..............(1)

Now let $q \in \beta(X) \setminus e(X)$. Since $e(X)$ is dense in $\beta(X)$, there is a net $\{q_n\ ;\ n \in D\}$ in $e(X)$ such that $\{q_n\ ;\ n \in D\}$ converges to q. Clearly $g^{-1} o f^{**}$ is continuous. So the net

$\{(g^{-1}of^{**})(q_n) ; n\in D\}$ converges to $(g^{-1}of^{**})(q)$. Since $(g^{-1}of^{**})(q_n) \in Y$, for every $n\in D$ and Y is a compact T_2-space, $(g^{-1}of^{**})(q) \in Y$. Hence $g^{-1}of^{**}$ carries $\beta(X)$ into Y.

Write $h = foe^{-1}$. Let $q\in e(X)$. Then $q = e(x)$ for some $x\in X$. We have

$$h(q) = (foe^{-1})(q) = f(e^{-1}(q)) = f(x) = (g^{-1}of^{**})(q). \qquad \text{(by (1))}$$

Hence $g^{-1}of^{**}$ is the required extension of h.

Definition : Let X be a topological space. Let us denote by $\mathscr{C}$ the collection of all compactifications of X. For (f, Y) and (g, Z) in $\mathscr{C}$ we define $(f, Y) \geq (g, Z)$ if there is a continuous mapping h of Y onto Z such that hof = g. Clearly gof^{-1} is a homeomorphism of Y_0 onto Z_0. The compactifications (f, Y) and (g, Z) of X are said to be equivalent if there is a homeomorphism h of Y onto Z s.t. hof = g. In this case we write $(f, Y) = (g, Z)$.

Theorem : 4. Let X be a topological space. Denote by $\mathscr{C}$ the collection of all T_2-compactifications of X. Then $\mathscr{C}$ is partially ordered by $\geq$.

Proof : Clearly $\geq$ is reflexive. Let (f, Y), (g, Z) and (h, W) be three elements of $\mathscr{C}$ and let $(f, Y) \geq (g, Z)$ and $(g, Z) \geq (h, W)$. There are continuous functions $j : Y \to Z$ (onto) and $k : Z \to W$ (onto) suct that $g = j$of and h = kog. Then clearly $h = k$o(jof) = (koj)of, where koj is a constant mapping of Y onto W. So $(f, Y) \geq (h, W)$. Hence $\geq$ is transitive.

Next let (f, Y) and $(g, Z) \in \mathscr{C}$ such that $(f, Y) \geq (g, Z)$ and $(g, Z) \geq (f, Y)$. There are continuous mappings $j : Y \to Z$ (onto) and $k : Z \to Y$ (onto) such that $j$of = g and kog = f.

So $f = ko(jof) = (koj)$of

and $g = (jof)$ = (jok)og.

Let $y\in Y_0$. Since f is a homeomorphism of X onto Y_0, there is a point x in X suct that $y = f(x)$. We have

$y = f(x) = [(ko\,j)o\,f](x)$ = (koj) $(f(x))$ = (koj) (y). Again let $y\in Y\setminus Y_0$. Since Y_0 is dense in Y, there is a net $\{y_n ; n\in D\}$ in Y_0 converging to y. For each n in D, there is a point x_n in X such that $y_n = f(x_n)$. We have

$y = \lim y_n = \lim f(x_n) = \lim [(ko\,j)of]\ (x_n)$

$= \lim (koj)\ (f(x_n)) = \lim (koj)\ (y_n) = (koj)\ (y)$ (koj is a continuous mapping of Y onto inself). Thus $(koj)\ (y) = y,\ \forall y \in Y,$ which shows that koj is the identity mapping of Y. Similarly, we can show that jok is the identity mapping of Z. Let $u, v\ (u \neq v)$ be two elements of Y. Then

$$(koj)\ (u) \neq (koj)\ (v)$$

i.e., $k(j(u)) \neq k(j(v))$

which implies that $j(u) \neq j(v)$. So j is one-to-one, similarly, k is one-to-one. Hence j is a homeomorphism of Y onto Z. This gives that (f, Y) and (g, Z) are equivalent i.e., $(f, Y) = (g, Z)$. Hence $(\mathscr{C}, \geq)$ is partially ordered.

Theorem : 5. Let X be a Tychonoff space which is not compact. Let $\mathscr{C}$ denote the collection of all T_2-compactifications of X. Then Alexandroff's one point compactification X^* is the minimal element and Stone-Cech compactification is the maximal element of $(\mathscr{C}, \geq)$.

Proof : Let (f, Y) be any T_2-compactification of X. Then Y is a compact T_2-space and f is a homeomorphism of X onto a dense subspace Y_0 of Y.

(I) We first show that $(e, \beta(X)) \geq (f, Y)$. Let $Q = [0, 1]$ and e denote the evaluation map of X into the space $Q^{C^*(X)}$. By Stone-Cech theorem there is a continuous extension h(say) of the map foe^{-1} such that h carries $\beta(X)$ into the space Y.

Let $y \in Y$. If $y \in Y_0$, then $y = f(x)$, for some x in X. Write $q = e(x)$. Then

$h(q) = (foe^{-1})\ (q) = f(e^{-1}\ (q)) = f(x) = y.$

Suppose that $y \in Y \setminus Y_0$. Since Y_0 is dense in Y, there is a net $\{y_n\ ;\ n \in D\}$ in Y_0 which converges to y. For each n in D, there is a point x_n in X such that $y_n = f(x_n)$. Write $q_n = e(x_n)$. Then $q_n \in e(X) \subset \beta(X),\ \forall n \in D$. Since $\beta(X)$ is compact, the net $\{q_n\ ;\ n \in D\}$ has a convergent subnet. So we may assume that the net $\{q_n\ ;\ n \in D\}$ is convergent. Let $q = \lim q_n$. Since h is continuous, the net $\{h(q_n)\ :\ n \in D\}$ converges to $h(q)$.

We have

$$h(q_n) = (f \circ e^{-1})\ (q_n) = f(e^{-1}(q_n)) = f(x_n) = y_n.$$

So $y = \lim y_n = \lim h(q_n) = h(q)$.

Hence h maps $\beta(X)$ onto the space Y which proves that $(e, \beta(X)) \geq (f, Y)$.

(II) We now assume X to be locally compact also. Then (i_X, X^*) is a T_2-compactification. Further Y_0 is an open subset of the Hausdorff compact space Y as it is homeomorphic with X. We now show that $(f, Y) \geq (i_X, X^*)$. Define the mapping h of Y onto $X^* = X \cup \{\infty\}$ as follows. Let $y \in Y_0$. Then $y = f(x)$, for some x is X. Since f is one-to-one, x is uniquely determined by y. We define $h(y) = x$. Also we define $h(y) = \infty$ if $y \in Y \backslash Y_0$.

Let G be any open set in X^*. First suppose that $G \subset X$. Then $h^{-1}(G) = f(G)$. Since f is a homeomorphism, $h^{-1}(G)$ is open in Y_0 and so is open in Y. Next let $\infty \in G$. Then we can write $G = W \cup \{\infty\}$ where W is open in X and $X^* \backslash G = X \backslash W$ is a closed compact subset of X. We have

$$h^{-1}(X^* \backslash G) = h^{-1}(X^*) \backslash h^{-1}(G) = Y \backslash h^{-1}(G).$$

So $h^{-1}(G) = Y \backslash h^{-1}(X^* \backslash G) = Y \backslash h^{-1}(X \backslash W) = Y \backslash f(X \backslash W)$

Since $X \backslash W$ is compact in X and f continuous, $f(X \backslash W)$ is compact in Y_0 and so is compact in Y. Since Y is T_2, so $f(X \backslash W)$ is closed in Y. Thus $h^{-1}(G)$ is open in Y and so h is continuous. Hence $(f, Y) \geq (i_X, X^*)$. This completes the proof.

2.3 Wallman Compactification

We will now describe another type of compactification.

Let (X, τ) be a T_1-topological space. Denote by $\mathscr{C}$, the collection of all closed subsets of X. Let Y denote the collection of all ultrafilters of $\mathscr{C}$.

(I) For any $G \in \tau$, let

$$G^* = \{\mathscr{F} : \mathscr{F} \in Y \text{ and } A \subset G \text{ for some } A \in \mathscr{F}\} \text{ and } \mathscr{B} = \{G^* : G \in \tau\}.$$

If G_1 and G_2 are two members of τ, then we can verify that

(a) $(G_1 \cap G_2)^* = G_1^* \cap G_2^*$

(b) $(G_1 \cup G_2)^* = G_1^* \cup G_2^*$

The relation (a) gives that $G_1^* \cap G_1^* \in \mathscr{B}$ whenever $G_1^*, G_2^* \in \mathscr{B}$. Let $\mathscr{F} \in Y$. Since $X \in \tau$ and $A \subset X$ for any $A \in \mathscr{F}$, so $\mathscr{F} \in X^*$. Hence $Y = X^*$. Therefore $\mathscr{B}$ forms a basis for a topology τ^* on Y.

(II) For any $x \in X$, let $\mathscr{F}_x = \{A : x \in A \in \mathscr{C}\}$. It is easy to see that each $\mathscr{F}_x$ is an ultrafilter in $\mathscr{C}$ and so $\mathscr{F}_x \in Y$. Let

$$Y_0 = \{\mathscr{F}_x : x \in X\}.$$

(III) Now define the mapping $f : X \to Y_0$ by $f(x) = \mathscr{F}_x$ for $x \in X$. Obviously f is an one-to-one mapping from X onto Y_0.

Let $x_0 \in X$ and let S be any neighbourhood of $f(x_0) = \mathscr{F}_{x_0}$ in the space Y_0. Since $\mathscr{B}$ is a base for τ^*, there is a member $G^* \in \mathscr{B}$ with

$$\mathscr{F}_{x_0} \in G^* \cap Y_0 \subset S.$$

So there is a member A in $\mathscr{F}_{x_0}$ with $A \subset G$. This gives that $x_0 \in G$. Take any $x \in G$ and let $B = \{x\}$. Then $B \in \mathscr{F}_x$. Since $B \in G$, so $\mathscr{F}_x \in G^*$ which shows that $f(x) \in S$. Hence f is continuous.

Write $g = f^{-1}$. Take any $\mathscr{F}_0 \in Y_0$. Then $\mathscr{F}_0 = \mathscr{F}_{x_0}$, for some point $x_0 \in X$. We have

$f(x_0) = \mathscr{F}_{x_0} = \mathscr{F}_0$ and so $g(\mathscr{F}_0) = x_0$. Let G be any open neighbourhood of x_0 in (X, τ). Then G^* is an open neighbourhood of $\mathscr{F}_0$ in the space Y and $S = Y_0 \cap G^*$ is an open neighbourhood of $\mathscr{F}_0$ in the space Y_0. Take any $\mathscr{F} \in S$. Now $\mathscr{F} = \mathscr{F}_x$, for some $x \in X$. So $A \subset G$ for some $A \in \mathscr{F}_x$ which implies that $x \in G$. Since $g(\mathscr{F}) = g(\mathscr{F}_x) = x$, we have $g(\mathscr{F}) \in G$ for all $\mathscr{F} \in S$. Hence g is also continuous. Therefore f is a homeomorphism from X onto Y_0.

(IV) Let $\mathscr{F} \in Y$ and let S be any $\mathscr{F}^*$-neighbourhood of π in the space Y. Since $\mathscr{B}$ is a base for τ^*, $\mathscr{F} \in G^* \subset S$ for some G^* in $\mathscr{B}$. Take any $x \in G$. Then $\mathscr{F}_x \in G^*$ and $\mathscr{F}_x \in Y_0$.

Thus $S \cap Y_0 \neq \phi$, which proves that Y_0 is dense in Y.

(V) Finally we show that (Y, τ^*) is compact. Let $\mathscr{Y}$ be any ultrafilter in Y. We define a subfamily β of $\mathscr{C}$ as follows :

$$\beta = \{A : A \in \mathscr{C} \text{ and } F \subset Y \setminus U_A^*, \text{ for some } F \in \mathscr{Y}\}$$

where $U_A = X \setminus A$ and so $U_A^* \in \tau^*$.

It is easy to see that $\phi \notin \beta$.

Let A_1, A_2 be two members of β. There are members F_1 and F_2 in $\mathscr{Y}$ such that

$$F_1 \subset Y \setminus U_{A_1}^* \text{ and } F_2 \subset Y \setminus U_{A_2}^*.$$

Write $F = F_1 \cap F_2$. Then $F \in \mathscr{Y}$ and we have

$$F \subset (Y \setminus U_{A_1}^*) \cap (Y \setminus U_{A_2}^*) = Y \setminus (U_{A_1}^* \cup U_{A_2}^*)$$

$$= Y \setminus (U_{A_1} \cup U_{A_2})^* = Y \setminus U_{(A_1 \cap A_2)}^*.$$

Hence $A_1 \cap A_2 \in \beta$.

Again let $A \in \beta$, $B \in \mathscr{C}$ and $A \subset B$. Then $F \subset Y \setminus U_A^*$, for some $F \in \mathscr{Y}$. Since $A \subset B$, $U_B \subset U_A$, which implies $U_B^* \subset U_A^* \Rightarrow Y \setminus U_A^* \subset Y \setminus U_B^*$. So $F \subset Y \setminus U_B^*$. This shows that $B \in \beta$. Hence β is a filter on $\mathscr{C}$. Now there is an ultrafilter β^* in $\mathscr{C}$ containing β. Clearly $\beta^* \in Y$.

We will show that β is a cluster point of $\mathscr{Y}$. If not then there is an $F \in \mathscr{Y}$ such that $\beta^* \notin \bar{F}$, where $\bar{F}$ denotes the τ^*-closure of F. Then $\beta^* \in Y \setminus \bar{F}$. Since $\mathscr{B}$ is a basis of τ^*, there is a member G^* in $\mathscr{B}$ with

$$\beta^* \subset G^* \subset Y \setminus \bar{F}.$$

Write $A = X \setminus G$. Then $A \in \mathscr{C}$ and $U_A = X \setminus A = G$. So $U_A^* = G^*$. From above we have

$$U_A^* \subset Y \setminus \bar{F} \text{ or } \bar{F} \subset Y \setminus U_A^*.$$

This gives that $A \in \beta$ and so $A \in \beta^*$. Since $\beta^* \in G^*$, there is a member B in β^* with

$B \subset G = X \setminus A$. But then $A \cap B = \phi \in \beta^*$ which contradicts that β^* is a filter. Hence $\beta^* \in \bar{F}$, for every $F \in \mathscr{Y}$ and so $\mathscr{Y}$ converges to β^*. This shows that (Y, τ^*) is compact. We have then the following

Theorem : (f, Y) is a compactification of (X, τ). This compactification (f, Y) of X is known as Wallman compactification of X and is denoted by $\omega(x)$.

Exercise : Let X be a Tychonoff space. If X is locally compact then prove that for every compactification (f, Y) of X, $Y \setminus f(X)$ is closed.

Solution : Since local compactness remains invariant under a homeomorphism, $f(X)$ is also locally compact. Let $y \in f(X)$. We will show that y is an interior point of $f(X)$. Since $f(X)$ is locally compact, $\exists$ an open neighbourhood U of x in $f(X)$ such that $\tilde{U}$ is compact ($\tilde{A}$ denotes the closure in $f(X)$). Hence there is an open set V in Y such that $U = V \cap f(X)$. Now we have $\bar{V} = \overline{(V \cap f(X))}$, where $\bar{A}$ denotes the closure of A in Y [as we know that if D is dense in a topological space (X, τ) then for any open set W, $\bar{W} = \overline{(W \cap D)}$]. Thus

$$x \in V \subset \bar{V} = \overline{(V \cap f(X))} \subset \bar{U} \subset \tilde{U} \subset f(X)$$

because $\tilde{U}$ is a closed set in $f(X)$ containing U. This proves the assertion.

Group–A
(Short questions)

1. Give an example of a topological space which is locally compact but not compact.
2. Show that the space of rationals with the induced topology from the usual topology of reals in not locally compact.
3. Give examples to justify that two compactifications of a given topological space may not be homeomorphic.
4. Show that any open subspace of a locally compact space is locally compact.

5. Prove that R^{ω} with product topology is not locally compact.

6. Show that if X is connected then $\beta(X)$ is also connected.

7. Show that product of locally compact spaces may not be locally compact.

8. For two compactifications (f, Y) and (g, Z) if $(f, Y) \leq (g, Z)$ and $(g, Z) \leq (f, Y)$, then show that (f, Y) and (g, Z) are equivalent.

9. Show that the Sorgenfrey line $(\mathbb{R}, \tau_l)$ in not locally compact at any point.

Group–B

(Long questions)

1. In order that two compactifications (f, Y) and (g, Z) of a topological space X be equivalent prove that it is necessary and sufficient that for every pair of closed subsets A, B of X,

$$\overline{f(A)} \cap \overline{f(B)} = \phi \Leftrightarrow \overline{g(A)} \cap \overline{g(B)} = \phi .$$

2. Show that if (f_α, Y_α) is a compactification of the space X_α for every $\alpha \in \Delta$ then $\prod_{\alpha\in\Delta} (f_\alpha, Y_\alpha)$ is a compactification of $\prod_{\alpha\in\Delta} X_\alpha$.

3. Let X be a Tychonoff space. Then prove that every pair of sets which can be separated by a real valued continuous function have disjoint closures in βX.

4. Let X be a Tychonoff space. Prove that X is locally compact iff the remainder $\beta X \backslash \beta(X)$ is closed.

5. Prove that in a locally compact space the intersection of a closed subset with an open subset in also locally compact.

6. With reasons give an example of a topological space which has only one compactification.

7. Let X be completely regular and T_1. Show that X is connected if and only if βX is connected.

8. Let X be discrete. Show that if U is open in βX then $\overline{U}$ is also open in βX. Then show that βX is totally disconnected.

9. Let A be any subset of $\mathbb{R}^n$ $(n \in \mathbb{N})$ such that A and $\mathbb{R}^n \backslash A$ are both dense in $\mathbb{R}^n$. Prove that no point of A has a compact neighbourhood.

Unit–III ❑ Paracompactness

Introduction

In this chapter we will study a weaker notion of compactness, which is called para-compactness ; this notion is actually more recent as it was introduced in 1944. The notion uses the idea of locally finite family which is easier to find. The importance of the idea of paracompactness also lies in the fact that many results, especially involving separation axioms on which we will concentrate, which were originally proved using compactness can be found valid by using the weaker notion of paracompactness. Apart from establishing several basic results we will show that a stronger notion of normality, called fully normal spaces can be obtained from paracompactness. We will finally introduce the very important notion of partition of unity and give an equivalent criteria of paracompactness in respect of partition of unity.

Definition : A family of subsets $\{B_i : i \in \Delta\}$ of a topological space X is said to be locally finite if each point x of X has a neighbourhood U which intersects at most finite number. of members of the family i.e., there is a finite subset Δ_1 of Δ suct that $U \cap B_i = \phi$ for all i in $\Delta \backslash \Delta_1$. Thus $\{B_i : i \in \Delta\}$ is locally finite iff there is an open cover α of X such that every member of α meets at most finite number of members of $\{B_i : i \in \Delta\}$.

Remark Every finite family is clearly locally finite. Every subfamily of a locally finite family is so. If $\{B_i : i \in \Delta\}$ is locally finite and $\{C_i : i \in \Delta\}$ is such that $C_i \subset B_i$, $\forall\, i$, then $\{C_i : i \in \Delta\}$ is locally finite.

Lemma 1. Let $\{B_i : i \in \Delta\}$ be a locally finite family of subsets in a topological space X. Then $\{\overline{B_i} : i \in \Delta\}$ is also locally finite and $\cup \{\overline{B_i} : i \in \Delta\} = \overline{\cup\{B_i : i \in \Delta\}}$.

Proof : Let $x \in X$. Then there is an open neighbourhood U of x and a finite subset Δ_1 of Δ such that $U \cap B_i = \phi$, $\forall\, i \in \Delta \backslash \Delta_1$. Then $U \cap \overline{B_i} = \phi$ $\forall\, i \in \Delta \backslash \Delta_1$ and so $\{\overline{B_i} : i \in \Delta\}$

is also locally finite. Now for each i, since $B_i \subset \cup \{B_i : i \in \Delta\}$, so $\overline{B_i} \subset \overline{\cup\{B_i : i \in \Delta\}}$ and hence $\cup\{\overline{B_i} : i \in \Delta\} \subset \overline{\cup\{B_i : i \in \Delta\}}$. Again let $x \in \overline{\cup\{B_i : i \in \Delta\}}$. Now there are a neighbourhood V of x and a finite subset Δ_1 of Δ such that $V \cap B_i = \phi$, $\forall\, i\ \Delta \backslash \Delta_1$. Let W be any neighbourhood of x. Then $V \cap W$ is also a neighbourhood of x and $(V \cap W) \cap (\cup \{B_i : i \in \Delta \backslash \Delta_1\}) = \phi$. Since $(V \cap W) \cap (\cup \{B_i ; i \in \Delta\}) \neq \phi$, so $(V \cap W) \cap (\cup\{B_i : i \in \Delta_1\}) \neq \phi$. Thus $x \in \overline{\bigcup_{i \in \Delta_1} B_i} = \cup\{\overline{B_i} : i \in \Delta_1\} \subset \cup\{\overline{B_i} : i \in \Delta\}$. This proves the result.

Definition 1. A topological space (X, τ) is said to be paracompact if every open cover of X has a locally finite refinement which is also an open cover of X.

Exercise 1. A compact space is paracompact.

Exercise 2. A discrete space is paracompact.

Theorem 1. Every closed subset of a paracompact space is paracompact.

Proof : Let X be paracompact and $F \subset X$ be closed. Let $\{ U_i : i \in \Delta\}$ be an open cover of F. Then for each $i \in \Delta$, $\exists$ an open set V_i in X such that $U_i = V_i \cap F$. Now $\{V_i : i \in \Delta\} \cup \{X \backslash F\}$ is an open cover of X. Since X is paracompact, this open cover has a locally finite refinement α which is also an open cover of X. Then $\beta = \{U \cap F : U \in \alpha\}$ is clearly of locally finite cover of F. Evidently every member of β is open in F and is contained in some U_i. Hence F is paracompact.

Theorem : 2. If every open cover of a topological space X has a closed locally finite refinement then X is paracompact.

Proof : Let α_1 be any open cover of X and let α_2 be a closed locally finite refinement of α_1 covering X. Then for each $x \in X$, there is an open neighbourhood P_x of x which meets at most finite number of members of α_2. Now $\{P_x : x \in X\} = \alpha_3$ (say) is an open cover of X. So $\exists$ a closed locally finite refinement α_4 of α_3 covering X. For each B in α_2, let U_B be a member of α_1 containing B and let V_B be the union of all those members of α_4 which are disjoint from B. By Lemma 1, V_B is a closed subset of X. Put $B^* = U_B \cap (X \backslash V_B)$ and denote by α_5 the class of all sets of the form B^*. Clearly each B^* is open in X. Since $B \subset B^*$, $\forall B \in \alpha_2$ and α_2 covers X, so α_5 also covers X. Evidently from definition, α_5 is a refinement of α_1.

We shall now prove that α_5 is locally finite. For each x in X, there is a neighbourhood Q_x of x which intersects at most a finite number of members of α_4 say $D_1, .., D_m$, because α_4 is locally finite. For $i = 1, 2, .., m$, D_i is contained in a member, say, P_{x_i} of α_3 ($\because$ α_4 is a refinement of α_3). For $i = 1, 2, .. m$, P_{x_i} meets at most a finite number of members of α_2, say, $Bi_1, .., Bi,n_i$. So D_i meets, if at all, these n_i members alone of α_2. If B is a member of α_2 which is different from all the sets B_{ij} ($i = 1, 2, .., m, j = 1, .., n_i$), then B is disjoint from all D_i ($i = 1, .., m$) and so B^* is also disjoint from all D_i ($i = 1, .., m$) ($\because \cup D_i \subset V_B$). Since Q_x meets at most the sets D_i ($i = 1, .., m$) and α_4 covers X, $Q_x \subset \cup\{D_i : i = 1 \text{ to } m)$. Consequently Q_x does not meet B^*. Thus x has a neighbourhood Q_x which meets at most a finite number of numbers of α_5. Hence α_5 is an open locally finite refinement of α_1 covering X and so X is paracompact.

Theorem : 3. If for each open cover of a regular space X there is a locally finite refinement covering X, then for each open cover of X there is a closed locally finite refinement covering X.

Proof : Let α be any open cover of X. For each x in X $\exists$ a $A_x \in \alpha$ such that $x \in A_x$. Since X is regular, $\exists$ an open set B_x such that $x \in B_x \subset \overline{B_x} \subset A_x$. Now $\beta = \{B_x ; x \in X\}$ is an open refinement of α covering X. By our assumption, $\exists$ a locally finite refinement ν of β covering X. Let $\delta = \{\overline{B} : B \in \nu\}$. By Lemma 1, δ is locally finite. Since each $B \in \nu$ is contained in some B_x and $\overline{B_x} \subset A_x \in \alpha$, so δ is a refinement of α. Thus δ is a closed locally finite refinement of α covering X.

Definition : A family of subsets of a topological space is said to be σ-locally finite if it is the union of a countable number of locally finite families.

Theorem : 4. Every open σ-locally finite cover of a topological space has a locally finite refinement.

Proof : Let α be a σ-locally finite cover of a topological space X. Now α is the union of the countable family $\{\alpha_n : n \in N\}$ of locally finite open classes α_n in X. Put B_1

$= \phi$, $B_n = \bigcup_{1 \le m < n} \bigcup_{A \in \alpha_m} (\cup A)$ for $1 < n \in N$ and denote by β the class of all subsets of A of the form $V \setminus B_n$; where $n \in N$ and $V \in \alpha_n$. β is evidently a refinement of α. Let $x \in X$. Let n be the least positive integer such that x belong to some W in α_n. Then $x \in W \setminus B_n$ and so β covers X. Moreover W is an open neighbourhood of x which is disjoint from all members of β of the form $V \setminus B_p$ for all $p > n$. Since for each $q \in N$, α_q is locally finite, for each positive integer $m \le n$, there is a neighbourhood U_m of x which intersects at most a finite number of members of α_m. Consequently the neighbourhood $\cap\{U_m : 1 \le m \le n\} \cap W$ of x meets at most a finite number of members of β. This proves the theorem.

Lemma 2. If every open cover of a regular space X has an open σ-locally finite refinement covering X then X is paracompact.

Proof : Follows from The 2, 3, 4.

Corollary 1. Every regular Lindeloff space is paracompact.

Theorem : 5. Every Hausdorff paracompact space is regular.

Proof : Let X be a T_2 paracompact space. Let F be a closed subset of X and $a \in X \setminus F$. For each x in F, there is an open neighbourhood N_x of x such that $a \notin \overline{N}_x$. Since X is paracompact, the open cover $\{N_x : x \in F\} \cup \{X \setminus F\}$ of X has an open locally finite refinement α covering X. Let β be the class of all those members of α which meet F. Then β, as a subclass of α, is locally finite. By Lemma 1, $\{\overline{B} : B \in \beta\}$ is locally finite. Also $\cup\{\overline{B} : B \in \beta\}$ is a closed set in X. Put $U = \cup\{B : B \in \beta\}$ and $V = X \setminus \cup\{\overline{B} : B \in \beta\}$. Then U, V are disjoint open subsets of X and $F \subset U$. Since β consists of all those members of α which meet F, for each B in β there is an x in F such that $B \subset N_x$. Now $\overline{B} \subset \overline{N}_x$ and $a \notin \overline{N}_x$. Hence for each $B \in \beta$, $a \in X \setminus \overline{B}$. Thus $a \in \cap\{X \setminus \overline{B} : B \in \beta\} = V$. Hence X is regular.

Exercise : Let X be a paracompact space and let A, B be two disjoint closed subsets. If for every $x \in B$ there exist open sets U_x, V_x such that $A \subset U_x$, $x \in V_x$, $U_x \cap V_x = \phi$ then there are open sets U, V such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.

Solution : Clearly the family of open sets $\{X \setminus B\} \cup \{V_x\}_{x \in B}$ forms an open covering of X. Since X is paracompact, this open cover has an open locally finite refinement $\{W_s\}_{s \in \Delta}$. Let

$\Delta_1 = \{s \in \Delta : W_s \subset V_x$ for some $x \in B\}$.

Then $A \cap \bar{W}_s = \phi$ for $s \in \Delta_1$ and $B \subset \underset{s\in\Delta_1}{\cup} W_s$. We know that $\underset{s\in\Delta_1}{\cup} \bar{W}_s = \overline{\underset{s\in\Delta_1}{\cup} W_s}$ which is closed. Consequently $U = X \setminus \underset{s\in\Delta_1}{\cup} \bar{W}_s$ is open. Evidently $A \subset U$, $B \subset V = \underset{s\in\Delta_1}{\cup} W_s$ and $U \cap V = \phi$.

Theorem : 6. Every paracompact Hausdorff space is normal.

Definition : Let $x \in X$, $B \subset X$ and α be a class of subsets of X. Then $\cup \{A \in \alpha : x \in A\}$ is called the star of x over α denoted by st(x, α). $\cup\{A \in \alpha : A \cap B \neq \phi\}$ is called the star of B over α and is denoted by st(B, α). The class of sets $\{st(x, \alpha) : x \in X\}$ is called the star of α.

Exercise : For $B \subset X$ and a class of subsets α of X,

(i) st $(B, \alpha) = \cup$ {st $(x, \alpha) : x \in B\}$

(ii) $B \subset \cup \{A : A \in \alpha\} \Rightarrow B \subset$ st (B, α).

Solution : (i) For every $x \in B$ clearly st $(x, \alpha) \subset$ st(B, α) and so $\cup$\{st$(x, \alpha) : x \in B\} \subset$ st(B, α). Conversely, let $y \in$ st(B, α). Then from definition there is some $A \in \alpha$ such that $y \in A$ and $A \cap B \neq \phi$. Choose $x \in A \cap B$. Then $y \in A \subset$ st(x, α) and so st$(B, \alpha) \subset \cup$ {st$(x, \alpha) : x \in B\}$. This proves the result.

(ii) Let $y \in B$. Since $B \subset \cup\{A : A \in \alpha\}$ so $\exists$ a $A_1 \in \alpha$ such that $y \in A_1$. Then $A_1 \cap B \neq \phi$ and this imlies $y \in A_1 \subset \cup \{A \in \alpha : A \cap B \neq \phi\} =$ st(B, α). This is true for every $y \in B$ and so $B \subset$ st(B, α).

Exercise : For B, $C \subset X$ and a collection α of subsets of X.

(i) $B \subset C \Rightarrow$ st$(B, \alpha) \subset$ st(C, α).

(ii) $B \cap$ st$(C, \alpha) = \phi \Leftrightarrow C \cap$ st$(B, \alpha) = \phi$.

Solution : (i) is obvious.

(ii) First suppose that $B \cap \text{st}(C, \alpha) \neq \phi$. Then $\exists\, y \in B \cap \text{st}(C, \alpha)$. Since $\text{st}(C, \alpha) = \cup\{A \in \alpha : A \cap C \neq \phi\}$, we can find a $A_1 \in \alpha$ such that $y \in A_1$ where $A_1 \cap C \neq \phi$. But then $y \in A_1 \cap B$ which implies $A_1 \subset \text{st}(B, \alpha)$. Then $\text{st}(B, \alpha) \cap C \neq \phi$. Similarly we can show that $C \cap \text{st}(B, \alpha) \neq \phi \Rightarrow B \cap \text{st}(C, \alpha) \neq \phi$. This completes the proof.

Definition : Let α and β be two classes of subsets, of X. Then α is called a star refinement of β if the star of α is a refinement of β.

Lemma : 3. Let α and β be covers of X such that α is a star refinement of β. Then $\{\text{st}(A, \alpha) : A \in \alpha\}$ is a refinement of the star of β.

Proof : Let $A \in \alpha$. Since α is a star refinement of β, for each $x \in A$, there is a member $B_x \in \beta$ such that $\text{st}(x, \alpha) \subset Bx$. Let $x_0 \in A$ be fixed. For each $x \in A$, $B_x \subset \text{st}(x_0, \beta)$, because $x_0 \in A \subset \text{st}(x, \alpha) \subset B_x$. Hence

$$\text{st}(A, \alpha) = \cup\, \{\text{st}(x, \alpha) : x \in A\} \subset \cup\{B_x : x \in A\} \subset \text{st}(x_0, \beta).$$

This completes the proof.

Definition : If for every $x \in X$ there exists $t_x \in \Delta_0$ such that $\text{st}(x, \alpha) \subset B_{t_x} \in \beta = \{B_t\}_{t \in \Delta_0}$, then α is called a pointwise star refinement of β.

Lemma : 4. If an open covering $\alpha = \{U_s\}_{s \in \Delta}$ of X has a closed locally finite refinement then it also has an open pointwise star refinement.

Proof : Let $\alpha = \{U_s\}_{s \in \Delta}$ be an open covering of X. By our assumption α has a closed locally finite refinement $\{F_t\}_{t \in \Delta_0}$. For every $t \in \Delta_0$ let us denote by $s(t)$, a fixed index in Δ such that $F_t \subset U_{s(t)}$. Since $\{F_t\}_{t \in \Delta_0}$ is locally finite, evidently $\Delta_0(x) = \{t \in \Delta_0 : x \in F_t\}$ is finite for every $x \in X$. Then it follows that the set

$$V_x = \bigcap_{t \in \Delta_0(x)} U_{s(t)} \cap \Big(X \setminus \bigcup_{t \notin \Delta_0(x)} F_t\Big)$$

is an open set, for every $x \in X$. Clearly $x \in V_x$ and so $\beta = \{V_x\}_{x \in X}$ in as open cover of X. Now let us consider a point $x_0 \in X$ and choose an index $t_0 \in \Delta_0(x_0)$. From the above construction it follows that if $x_0 \in V_x$ then $t_0 \in \Delta_0(x)$ and $V_x \subset U_{s(t_0)}$. Hence $\mathrm{st}(x_0, \beta) \subset U_{s(t_0)}$ which proves that β is a pointwise star refinement of α.

Lemma : 5. If a covering $\alpha = \{A_s\}_{s \in \Delta}$ of an arbitrary set is a pointwise star-refinement of a covering $\beta = \{B_t\}_{t \in \Delta_0}$ which is a pointwise star refinement of a covering $\nu = \{C_w\}_{w \in \Delta_1}$, then α is a star refinement of ν.

Proof : Let us take a fixed $s_0 \in \Delta$ and for every $x \in A_{s_0}$ choose $t(x) \in \Delta_0$ such that

$$A_{s_0} \subset st(x, \alpha) \subset B_{t(x)}.$$

Then we have

$st(A_{s_0}, \alpha) = \cup\{st(x, \alpha) : x \in A_{s_0}\} \subset \bigcup_{x \in A_{s_0}} B_{t(x)}$. From above we can get that whenever we choose $x_0 \in A_{s_0}$ then $x_0 \in B_{t(x)}$ for every $x \in A_{s_0}$.

Hence $\bigcup_{x \in A_{s_0}} B_{t(x)} \subset \mathrm{st}(x_0, \beta)$. But then

$$\mathrm{st}\,(A_{s_0}, \alpha) \subset \mathrm{st}(x_0, \beta) \subset C_w$$

for some $w \in \Delta_1$. This completes the proof.

Lemma : 6. If every open covering of a topological space X has an open star refinement then every open cover of X has an open σ-locally finite refinement.

We omit the proof as it is too technical.

From the above three Lemmas we can find the following equivalent conditions for paracompactness using the star operation.

Theorem : 7. For a regular Hausdorff space X the following are equivalent.

(i) X is paracompact.

(ii) Every open cover of X has an open pointwise star refinement.

(iii) Every open cover of X has an open star refinement.

(iv) Every open cover of X has an open σ-locally finite refinement.

Definition : A topological space X is said to be fully normal if every open cover of X has an open star refinement.

Exercise : A fully normal space X is normal.

Solution : Let X be a fully normal space. Let B and C be two pairwise disjoint closed subsets of X. Now $\{X\setminus B, X\setminus C\}$ is an open cover of X and so it has an open star refinement α covering X. Let $B^* = \text{st}(B, \alpha)$ and $C^* = \text{st}(C, \alpha)$. Clearly B^* and C^* are open in X and $B \subset B^*$, $C \subset C^*$. We claim that $B^* \cap C^* = \phi$. On the contrary if $x \in B^* \cap C^*$ then $\exists\, M, N \in \alpha$ such that $x \in M \cap N$ where $M \cap B \neq \phi \neq N \cap C$. Then $\text{st}(x, \alpha)$ intersects both B and C and as such can't be contained in either $X\setminus B$ or $X\setminus C$. This contradicts that α is an open star refinement of $\{X\setminus B, X\setminus C\}$. Hence X is normal.

We will end the discussions with giving the idea of partition of unity without going into the full details of the proofs.

Definition : A family $\{f_s\}_{s\in\Delta}$ of continuous functions defined on a space X with values in $[0, 1]$ is called a partition of unity if $\sum_{s\in\Delta} f_s(x) = 1$ for every $x \in X$. This actually means that for a fixed point $y \in X$, at most countably many functional values $f_s(y)$ can be non-zero and clearly the infinite series $\sum_{i=1}^{\infty} f_{s_i}(y)$ is convergent with its sum 1 where $\{s_1, .., , ...\} = \{s \in \Delta : f_s(y) \neq 0\}$. A partition of unity $\{f_s\}_{s\in\Delta}$ is said to be locally finite if the covering $\{f_s^{-1}((0, 1])\}_{s\in\Delta}$ is locally finite. In this case for every point $y \in X$, there is a neighbourhood U_y and a finite set $\Delta_0 = \{S_1, S_2, .., S_n\} \subset \Delta$ such that $f_s(x) = 0$ for $x \in U_y$, $s \in \Delta\setminus\Delta_0$. Clearly $\sum_{i=1}^{n} f_{s_i}(x) = 1$ for $x \in U_y$. We say that the partition of unity $\{f_s\}_{s\in\Delta}$ is subordinate to the covering $\{A_t\}_{t\in\Delta_1}$ if the covering $\{f_s^{-1}((0, 1])\}_{s\in\Delta}$ is a refinement of $\{A_t\}_{t\in\Delta_1}$.

Lemma : 7. For each point finite open cover $\{U_s\}_{s\in\Delta}$ (a cover is called point finite if every point belong to only a finite number of members of the cover) of a normal space

X, there exists an open cover $\{V_s\}_{s\in\Delta}$ such that $\overline{V_s} \subset U_s \forall s \in \Delta$.

Proof is omitted.

Lemma : 8. If an open cover $\mathscr{V}$ of a T_2 topological space X has a partition of unity $\{f_s\}_{s\in\Delta}$ subordinated to it, then $\mathscr{V}$ has a locally finite open refinement.

Proof left as an exercise in Group–B.

Theorem : 8. For a T_1 topological space X, following are equivalent.

(i) X is paracompact.

(ii) Every open cover of X has a locally finite partition of unity subordinated to it.

(iii) Every open cover of X has a partition of unity subordinated to it.

Proof : First suppose X is paracompact. Let $\mathscr{U} = \{U_s\}_{s\in\Delta_1}$ be an open cover of X. Let $\mathscr{V} = \{V_s\}_{s\in\Delta}$ be an open and locally finite refinement of $\mathscr{U}$. Then by previous Lemmas, we can get a cover $\{W_s\}_{s\in\Delta}$ of X such that $\bar{W}_s \subset V_s \quad \forall s \in \Delta$. By Uryshon's Lemma, there exists a continuous function $g_s : X \to [0, 1]$ such that $g_s(x) = 1$ for $x \in \bar{W}_s$, $g_s(x) = 0$ for $x \in X \setminus V_s$. Since $\mathscr{V}$ is locally finite, the function $g = \sum_{s\in\Delta} g_s$ is well-defined. It is easy to verify that defining $f_s = g_s/g \ \forall s \in \Delta$, the family $\{f_s\}_{s\in\Delta}$ is a locally finite partition of unity subordinated to $\mathscr{U}$. This proves (ii).

The implication (ii) $\Rightarrow$ (iii) is obvious. Let (iii) hold. In view of preceding Lemma we only need to prove that X is T_2. We will show that X is Tychonoff. Let $x \in X$, F–a closed set such that $x \notin F$. Now $\mathscr{V} = \{X \setminus F, X \setminus \{x\}\}$ is an open cover of X and so it has a partition of unity $\{f_s\}_{s\in\Delta}$ subordinated to it. So $\exists\ s_0 \in \Delta$ such that $f_{s_0}(x) = a > 0$ and $f_{s_0}^{-1}((0, 1]) \not\subset X \setminus \{x\}$ i.e., it is contained in $X \setminus F$. So $f_{s_0}(F) = 0$. Then $f : X \to [0, 1]$ where $f(x) = \min \left\{\frac{1}{a} f_{s_0}(x), 1\right\}$ is a continuous function with $f(x) = 1$, $f(F) = 0$.

Group–A
(Short questions)

1. Give an example of an open cover which is point finite but not locally finite.
2. Let $\mathscr{V} = \{(-n, n)\}_{n\in N}$. Is $\mathscr{V}$ locally finite? Answer with reasons.
3. Give an example of a paracompact space that is not compact.
4. Prove that a discrete topological space is paracompact.
5. Show that every locally finite family of non-empty subsets of a countably compact space is finite.
6. Prove that a T_1 space is normal if each finite open cover has an open star refinement.
7. Give an example of a paracompact space which is not Lindeloff.
8. Give an example of a paracompact space which is not countably compact.

Group–B
(Long questions)

1. Show that an F_σ-subset of a paracompact space is paracompact.
2. Prove that the cartesian product $X \times Y$ of a paracompact space X and a compact space Y is paracompact.
3. Prove that a paracompact countably compact space is compact.
4. Prove that a Lindeloff space is paracompact.
5. If an open cover v of a T_2 topological space X has a partition of unity $\{f_s\}_{s\in\Lambda}$ subordinated to it then show that v has a locally finite open refinement.
6. Let X be a T_2-space. If $\exists$ a countable open cover $\{U_n\}_n$ of X such that $\overline{U}_n \subset U_{n+1} \forall n$ and $\overline{U}_n$ is compact $\forall n$ then prove that X is paracompact.

7. A topological space X is called (a) metacompact if every open cover of X has an open point finite refinement, (b) countably paracompact if every countable open cover of X has an open, locally finite refinement.

   Prove that a paracompact space is metacompact as well as countably paracompact.

8. Prove that a countable paracompact space is countably metacompact [a space X is countabley metacompact if every countable open cover of X has an open, point finite refinement]

9. Prove that any closed subspace of a countably paracompact (resp. metacompact, countably metacompact) space X is respectively so.

10. Let τ_l be the lower limit topology on $\mathbb{R}$. Assuming that $(\mathbb{R}, \tau_l)$ is Lindeloff, prove that $(\mathbb{R}, \tau_l)$ is a paracompact space.

Unit–IV ❑ Metrization

Introduction

After coming across the two structures, namely, metric spaces and topological spaces it is evident that a metric space is a much stronger structure than a topological space with many more additional properties arising due to the presence of the distance function. We have already seen for example that while four types of compactness are different in a topological space (in general with no additional assumption), they become equivalent in a metric topology. So the natural question is that whether it is possible to get a given topology on a set X as the topology induced by some metric on that set. Metrization deals with this problem. Here we will see that there are metrizable topologies as also topologies which are not metrizable. Uryshon's metrization theorem is recalled here. Nagata-Smirnov theorem; though quite long and tricky with a very deep proof, is the milestone of metrization problems as it gives necessary and sufficient conditions for a space to be metrizable. Further, in the last section we include two very important theorems, namely, Arzela–Ascoli's theorem and Stone Weirstrass theorem.

3.1 Metrization of topological spaces

Definition : A topological space X is said to be metrizable if there exists a metric d on the set X that induces the given topology of X.

Since a metric space is inherently Hausdorff, normal, and it satisfies first axiom of countability, say the least, metrizability is a highly desirable property for a topological space. Before we prove our main results, we recall the following.

Definition : Let (X, d) be a metric space. A subset A of X is called bounded if there is a number M such that $d(x, y) \leq M \ \forall x, y \in A$. If A is bounded, the diameter of A is defined to be the number diam A = lub $\{d(a_1, a_2) : a_1, a_2 \in A\}$.

Boundedness is not a topological property and it only depends on the particular metric d.

Theorem A. Let (X, d) be a metric space. Define $\overline{d} : X \times X \to R$ by the equation $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Then $\overline{d}$ is a bounded metric that induces the topology of (X, d), irrespective of whether d is bounded or unbounded. $\overline{d}$ is called the standard bounded metric on X.

Lemma A. Let d and d' are two metrics on the set X and τ and τ' be their induced topologies. Then τ is finer than τ' iff for each x in X and each $\in > 0$, there is a $\delta > 0$ s.t.

$$B_d(x, \delta) \subset B_{d'}(x, \in)$$

Exercise : For any (+) ve integer n, R^n with the product topology is metrizable.

Solution : We shall prove that the euclidean metric d on R^n defined by

$$d(x, y) = [(x_1 - y_1)^2 + ... + (x_n - y_n)^2]^{1/2}$$

and the square metric defined by

$$\rho(x, y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}$$

for any $x = (x_1, .., x_n)$ and $y = (y_1, .., y_n)$

induce the product topology on R^n.

First note that for $x = (x_1, .., x_n)$, $y = (y_1, .., y_n) \in R^n$, we have

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\,\rho(x, y)$$

which implies that for $x \in R^n$, and any $\in > 0$,

$$B_d(x, \in) \subset B_\rho(x, \in)$$

$$B_\rho(x, \in/\sqrt{n}) \subset B_d(x, \in).$$

Hence the two metric topologies induced by d and ρ are same.

Now we show that the product topology is the same as that induced by the metric ρ. Let $B = (a_1, b_1) \times .. \times (a_n, b_n)$ be a basis element of the product topology and let $x = (x_1, .. , x_n) \in B$. For each i, $\exists$ an $\epsilon_i > 0$ s.t.

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$$

Choose $\epsilon = \min \{\epsilon_1, .. , \epsilon_n\}$. Then $B_\rho(x, \epsilon) \subset B$. Hence the ρ-topology is finer than the product topology. That the product topology is also finer that the ρ-topology follows from the fact that for any $x = (x_1, .. x_n) \in R^n$ and $\epsilon > 0$,

$B_\rho(x, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times .. \times (x_n - \epsilon, x_n + \epsilon)$ is itself a member of the product topology. This completes the proof.

Definition : Given an index set J and given two points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of R^J, the uniform metric on R^J is defined by

$$\bar{\rho}(x, y) = \text{lub}\{\bar{d}(x_\alpha, y_\alpha) / \alpha \in J\}$$

where $\bar{d}$ is the standard bounded metric on R. The topology induced by the metric $\bar{\rho}$ is called the uniform topology.

Lemma : The uniform topology on R^J is finer than the product topology.

Proof left as an Exercise.

Theorem : The countable product of R, R^ω with the product topology is metrizable.

Proof : If $x = (x_n)_{n \in N}$ and $y = (y_n)_{n \in N}$ are two points of R^ω, define

$$D(x, y) = \underset{i}{\text{lub}}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

where $\bar{d}$ is the standard bounded metric on R. Evidently $D(x, y) \geq 0$ and $= 0$ iff $x = y$. Also $D(x, y) = D(y, x)$. To prove the triangle inequality, we note that for $x, y, z \in R^\omega$

$$\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}$$

$$\leq D(x, y) + D(y, z) \quad \forall i \in N$$

and so $D(x, z) = \text{lub}\left\{\frac{\bar{d}(x_i, z_i)}{i}\right\} \leq D(x, y) + D(y, z)$. Thus D is a metric on R^ω. We shall now show that D induces the product topology on R^ω.

First let U be open in the metric topology and let $x \in U$. $\exists$ an $\in > 0$ such that $B_D(x, \in) \subset U$. Choose a (+) ve integer M s.t. $1/M < \in$. Let

$V = (x_1 - \in, x_1 + \in) \times .. \times (x_M - \in, x_M + \in) \times R \times R \times \ldots$. Then V is open in the product topology. Note that for any $y = (y_i) \in R^\omega$,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{M} \quad \forall (i \geq M)$$

Hence $D(x, y) \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, .., \frac{\bar{d}(x_M, y_M)}{M}, \frac{1}{M}\right\}$. Clearly if $y \in V$ then $D(x, y) < \in$ and so $V \subset B_D(x, \in) \subset U$.

Conversely let $U = \prod_{i \in N} U_i$ be a basis element of the product topology, where U_i is open in R for $i = \alpha_1, .., \alpha_n$ and $U_i = R$ for all other indices i. Let $x \in U$. Choose an interval $(x_i - \in_i, x_i + \in_i) \subset U_i$ for $i = \alpha_1, .., \alpha_n$. We can choose each $\in_i \leq 1$. Now take

$$\in = \min\{\in_i \setminus i \,;\, i = \alpha_1, .., \alpha_n\}.$$

If $y \in B_D(x, \in)$, then $(\forall i)$

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \in.$$

Clearly if $i = \alpha_1, .., \alpha_n$, then $\in \leq \in_i \setminus i$, so that $\bar{d}(x_i, y_i) < \in_i \leq 1$ which in turn implies $|x_i - y_i| < \in_i$. Therefore $y \in \Pi U_i$ and so

$$x \in B_D(x, \in) \subset U.$$

This completes the proof of the fact that D induces the product topology on R^ω.

Lemma : Let X be a toplogical space. Let $A \subset X$. If there is a sequence of points of A converging to x then $x \in \bar{A}$. The converse holds if X is metrizable.

This result is sometimes called the sequence Lemma. We shall use this result to conclude the following.

Exercise : An uncountable product of R with itself endowed with the product topology is not metrizable.

Proof : Let J be an uncountable index set. We show that R^J does not satisfy the sequence Lemma.

Let A be the subset of R^J consisting of all points (x_α) such that $x_\alpha = 0$ for finitely many values of α and $x_\alpha = 1$ for all other values of α. Let 0 be the point of R^J each of whose coordinates is 0.

Let ΠU_α be a basis open set in R^J containing 0. Then $U_\alpha \neq R$ for only finitely many values of α, say for $\alpha = \alpha_1, \ldots, \alpha_n$. Let (x_α) be the point defined by $x_\alpha = 0$ for $\alpha = \alpha_1, \ldots, \alpha_n$ and $x_\alpha = 1$ for all other values of α. Then clearly $x = (x_\alpha) \in A \cap \Pi U_\alpha$. This shows that $0 \in \overline{A}$.

But there is no sequence of points of A converging 0. For let $\{a_n\}$ be a sequence of points of A. Each point a_n is a point of the product space having only finitely many coordinates equal to 0. For $n \in N$, let J_n denote the subset of J consisting of those indices α for which the αth coordinate of a_n is zero. Then $\bigcup_{n \in N} J_n$ is a countable set. Since J is uncountable, there is at least one index, say, β s.t. $\beta \in J \setminus \cup J_n$. This means that βth coordinate of each a_n is equal to 1.

Now let $U_\beta = (-1, 1)$ and $U = \Pi_\beta^{-1}(U_\beta)$. Then U is an open nbd of 0 in R^J but $a_n \notin U$ for any $n \in N$. Thus the sequence $\{a_n\}$ cannot converge to 0.

The above example confirms that not every topological space is metrizable. We now give a few necessary and sufficient conditions for metrizability. First we recall the following well known theorem which provides some sufficient conditions for a topological space to be metrizable, for the proof of which we refer to your earlier course on set-topology.

Uryshon Metrization Theorem

Every regular T_1 and 2nd countable space X is metrizable.

Exercise : Let X be a regular space with a basis $\mathscr{B}$ that is σ-locally finite. Then X is normal.

Solution : Step 1.

Since $\mathscr{B}$ is σ-locally finite, we can write $\mathscr{B} = \bigcup_n \mathscr{B}_n$ where each $\mathscr{B}_n$ is locally finite.

Let W be any open set in X. Let C_n be the collection of those basis elements B such that $B \in B_n$ and $\overline{B} \subset W$. Then C_n, being a subcollection of $\mathscr{B}_n$ is locally finite. Define

$$U_n = \bigcup_{B \in C_n} B$$

Then U_n is open and $\overline{U}_n = \bigcup_{B \in C_n} \overline{B}$ ($\because$ C_n is locally finite),

consequently $\cup U_n \subset \cup \overline{U}_n \subset W$.

We now show that $W = \cup U_n = \cup \overline{U}_n$. Let $x \in W$. By the regularity of X, $\exists$ a $B \in \mathscr{B}$ such that $x \in B \subset \overline{B} \subset W$. Now $B \in \mathscr{B}_n$ for some n. Then $B \in C_n$ by definition and so $x \in \overline{U}_n$.

Step 2. Now let C and D be two disjoint closed sets in X. By step 1 we can construct two countable collections of open sets $\{U_n\}$ and $\{V_n\}$ such that

$$\cup U_n = \cup \overline{U}_n = X \setminus D$$

$$\cup V_n = \cup \overline{V}_n = X \setminus C$$

For each n∈N, define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i.$$

and let $U' = \bigcup_{n=1}^{\infty} U'_n$ and $V' = \bigcup_{n=1}^{\infty} V'_n$

Then $U' \supset C$, $D \subset V'$ and U', V' are open sets with $U' \cap V' = \phi$.

Nagata Smirnov Metrization Theorem.

A topological space X is metrizable iff it is regular T_1 and has a σ-locally finite basis.

Proof : (sufficiency)

Let X be a regular T_1 space with a σ-locally finite basis $\mathcal{B}$.

Step 1. Let W be open in X. We have already shown that W is a countable union of closed sets $\{A_n\}$ of X. Using normality of X, for each n, choose a continuous function $f_n : X \to [0, 1]$ such that $f_n(A_n) = \{1\}$ and $f_n(X \setminus W) = \{0\}$. Let

$$f(x) = \Sigma f_n(x)/2^n.$$

Since the series converges uniformly, so the limit function f is continuous. Clearly $f(x) > 0\ \ \forall x \in W$ and $f(X \setminus W) = \{0\}$.

Step 2. We can write $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each collection $\mathcal{B}_n$ is locally finite. For each $n \in N$ and $B \in \mathcal{B}_n$, choose a continuous function

$$f_{n,B} : X \to [0,\ 1/n]$$

such that $f_{n,B}(x) > 0$ for $x \in B$ and $f_{n,B}(X \setminus B) = \{0\}$. Now given any point $x_0 \in X$ and open set U containing x_0, $\exists$ a basis element B such that $x_0 \in B \subset U$. Then $B \in \mathcal{B}_n$ for some n and hence $f_{n,B}(x_0) > 0$ and $f_{n,B}(X \setminus U) = \{0\}$. In other words the collection of functions $\{f_{n,B}\}$ separates points from closed sets. Since X is T_1, so it also separates points.

Let J be the subset of $N \times \mathcal{B}$ consisting of all pairs (n, B) such that $B \in \mathcal{B}_n$. Define

$$F : X \to [0,\ 1]^J$$

by the equation

$$F(x) = (f_{n,B}(x))_{(n,B) \in J}.$$

By Imbedding theorem F is an imbedding of X into $[0,\ 1]^J$ with the product topology.

Step 3. It should be noted that $[0,\ 1]^J$ with the product topology is not always metrizable (if J is uncountable). So instead of taking the product topology, we take the uniform topology induced by the uniform metric $\bar{\rho}$ on $[0,\ 1]^J$.

Since the uniform topology is finer than the product topology, so F is still an open map. Evidently $F : X \to F(X)$ is bijective. We have only to show that F is continuous. Let $x_0 \in X$ and $\epsilon > 0$ be given.

First let $n \in N$ be fixed. Since $\mathcal{B}_n$ is locally finite, $\exists$ a neighbourhood U_n of x_0 which meets only a finite number of members of $\mathcal{B}_n$, say, $B_1, \ldots, B_k$. Then

$$f_{n,B}(U_n) = \{0\}, \quad \forall B \in \mathcal{B}_n \setminus \{B_1 \ldots, B_k\}$$

Again by continuity of f_{n,B_i}, $\exists$ a neighbourhood V'_i of x_0 such that

$$y \in V'_i \Rightarrow |f_{n,B_i}(y) - f_{n,B_i}(x_0)| < \in/4 \text{ for } i = 1 \text{ to } k.$$

Let $V_n = V'_1 \cap .. \cap V'_k \cap U_n$. Then V_n is an open neighbourhood of x_0 and for any $x, y \in V_n$, any $i = 1$ to k.

$$\left|f_{n,B_i}(x) - f_{n,B_i}(y)\right| < \frac{\in}{2}$$

as V_n is a neighbourhood of x_0 on which all but finite functions, $f_{n,B}$ vanishes identically and the remaining functions $f_{n,B}$ vary atmost by $\frac{\in}{2}$.

Now choose $M \in N$ such that $1/M < \in/2$. For each of the (+)ve integers 1, 2 .., M, choose open neighbourhoods $V_1, .., V_M$ of x_0 having the above property. Let

$$W = V_1 \cap V_2 \cap .... \cap V_M.$$

Let $x \in W$. If $n \leq M$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{\in}{2}$$

because each $f_{n,B}$ either vanishes identically or varies by at most $\frac{\in}{2}$ on W.

If $n > M$, then

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{1}{n} < \frac{\in}{2}$$

because $f_{n,B}$ maps X into $\left[0, \frac{1}{n}\right]$. Therefore

$$\bar{\rho}(F(x), F(x_0) \leq \frac{\in}{2} < \in.$$

Thus we have an open neighbourhood W of x_0 such that

$$x \in W \Rightarrow \bar{\rho}(F(x), F(x_0)) < \in.$$

This proves that the function F is continuous. This completes the proof of the sufficiency part.

Proof (necessity) :

Step 1. Let X be a metrizable space. First we prove that any open cover $\mathscr{V}$ of X has a σ-locally finite refinement $\mathscr{D}$ covering X.

Let d be a metric on X that induces the topology of X. Since the collection $\mathscr{V}$ is a poset (w.r.t inclusion) it can be well-ordered by well ordering theorem. Let '<' be the well ordering in $\mathscr{V}$. Let $n \in N$ be fixed. For $U \in \mathscr{V}$, define

$$S_n(U) = \left\{x \,;\, B\left(x, \frac{1}{n}\right) \subset U\right\}$$

Then we define

$$S'_n(U) = S_n(U) \setminus \bigcup_{V<U,\, V\in\mathscr{V}} V$$

We shall show that $\{S'_n(U) : U \in \mathscr{V}\}$ consists of pairwise disjoint sets. For this let $V, W \in \mathscr{V}$, $V \neq W$. Without any loss of generality we may assume that $V < W$. Now $x \in S'_n(V)$ $\Rightarrow x \in S_n(V)$. Again $y \in S'_n(W)$ implies by definition, $y \notin V$ ($\because V < W$). Clearly $x \in S_n(V)$, $y \notin V$ $\Rightarrow d(x, y) \geq \frac{1}{n}$. Thus $S'_n(V) \cap S'_n(W) = \phi$ and

$$x \in S'_n(V) \text{ and } y \in S'_n(V) \Rightarrow d(x, y) \geq \frac{1}{n}.$$

Now let us define

$$E_n(U) = \cup\{B(x, 1/3n) : x \in S'_n(U)\}.$$

Evidently $E_n(U)$ is open for each $U \in \mathscr{V}$. For $V, W \in \mathscr{V}$, $V \neq W$, we assert that $E_n(V) \cap E_n(W) = \phi$. For if not, then $\exists$ *a* $z \in E_n(V) \cap E_n(W) \Rightarrow \exists x \in S'_n(V)$ and $y \in S'_n(W)$ such that $z \in B\left(x, \frac{1}{3n}\right)$ and $z \in B\left(y, \frac{1}{3n}\right) \Rightarrow d(x, y) \leq d(x, z) + d(z, y) < \frac{2}{3n} < \frac{1}{n}$, a contradiction.* Further it is easy to show that $E_n(U) \subset U$, $\forall\, U \in \mathscr{V}$. (* Actually it is easy to see that

$$x \in E_n(V),\ y \in E_n(W) \Rightarrow d(x, y) \geq \frac{1}{3n}.)$$

Now let us define

$$\mathcal{D}_n = \{E_n(U) : U \in \mathcal{V}\} \text{ for } n \in N.$$

Evidently $\mathcal{D}_n$ is a refinement of $\mathcal{V}$. Also for each $x \in X$, $B\left(x, \frac{1}{6n}\right)$ is an open neighbourhood of x which can intersect at most one member of $\mathcal{D}_n$. Thus $\mathcal{D}_n$ is locally finite.

Finally let $\mathcal{D} = \bigcup_{n \in N} \mathcal{D}_n$. Then $\mathcal{D}$ is a σ-locally finite refinement of $\mathcal{V}$. We just have to show that $\mathcal{D}$ covers X.

Let $x \in X$. Choose $U \in \mathcal{V}$ to be the first element that contains x (w.r.t. the well-ordering). Since U is open, choose $n \in N$ such that $B\left(x, \frac{1}{n}\right) \subset U$. Then $x \in S_n(U)$. Since there is no $V \in \mathcal{V}$ with $x \in V$, $V < U$. So clearly $x \in S'_n(U) \subset E_n(U)$. This completes the proof of our assertion.

Step 2. As X is metrizable, it is evidently regular and T_1. We have to show that X has a σ-locally finite basis. For this we note that for any $m \in N$,

$$\{B(x, \frac{1}{m}) : x \in X\} = \vartheta_m \text{ (say)}$$

is an open cover of X. By step 1, ϑ_m has a σ-locally finite refinement $\mathcal{D}_m$ covering X. Note that every member of $\mathcal{D}_m$ has diameter at most $2/m$. Let $\mathcal{D} = \bigcup_{m \in N} \mathcal{D}_n$. Evidently $\mathcal{D}$ is also σ-locally finite. Further given any $x \in X$ and $\in > 0$, choose $m \in N$ so that $1/m < \varepsilon/2$. As $\mathcal{D}_m$ covers X, we can choose $D \in \mathcal{D}_m$ such that $x \in D$. But as diam $(D) \leq 2/m$, so $D \subset B(x, \in)$. Hence $\mathcal{D}$ is a basis of X as $\bigcup_{m \in N} \vartheta_m$ is already a basis of X.

Theorem : (stone) Every metrizable space is paracompact.

The results follows from Lemma 2(paracompactness chapter) and the above Theorem.

Exercise : Let (X, d) be a metric space. Let $X \times X$ be endowed with the corresponding product topology. Then $d : X \times X \rightarrow R$ given by $(x, y) \rightarrow d(x, y)$ is a continuous function.

Solution : Let $\in > 0$ be given. Consider the open interval $(d(x, y) - \in, d(x, y) + \in)$. Choose the basic open set $B\left(x, \frac{\in}{2}\right) \times B\left(y, \frac{\in}{2}\right)$ containing (x, y) in the product topology in $X \times X$. Take $(x_1, y_1) \in B\left(x, \frac{\in}{2}\right) \times B\left(y, \frac{\in}{2}\right)$. Then $d(x, x_1) < \frac{\in}{2}$ and $d(y, y_1) < \frac{\in}{2}$.

Note that

$$d(x, y) \leq d(x, x_1) + d(x_1, y_1) + d(y_1, y)$$

i.e., $d(x, y) - d(x_1, y_1) \leq d(x, x_1) + d(y, y_1)$

i.e., $| d(x, y) - d(x_1, y_1)| \leq d(x, x_1) + d(y, y_1) < \in$

i.e., $d(x_1, y_1) \in (d(x, y) - \in, d(x, y) + \in)$.

This shows that d is continuous.

Exercise : Show that in a compact metrizable space X, every metric for X is a B-metric (bounded metric).

Solution : In order to prove the result we show that there exist points $a, b \in X$ such that $d(a, b) = \text{diam}(X)$ where d is the metric on X corresponding to the given topology. Let $X^* = X \times X$ with the product topology and let $f : X^* \rightarrow R$ be defined as before $f((x_1, x_2)) = d(x_1, x_2)$. We have already shown that f is a continuous mapping to R. Since X is compact, $X \times X = X^*$ is also compact. Then $f(X^*)$ being continuous image of a compact set is also a compact subset of R. Consequently $f(X^*)$ is closed and bounded in R. Let $C = \text{lub } f(X^*)$. Then $C \in f(X^*)$ and there exists $p \in X^*$ such that $f(p) = C$. Let $p = (a, b)$. Obviously $d(a, b) = \text{diam}(X)$ and the assertion follows immediately.

We will now show that the cartesian product of countably many metrizable spaces is also metrizable.

Theorem : Let $\{(X_n, d_n)\}_n$ be a countable family of metrizable spaces. Let diam $(X_n) \leq M$ for all large n and diam $(X_n) \to 0$ as $n \to \infty$. Let us define $e(x, y) = \sup_n \{d_n(x_n, y_n)\}$.

Then τ_e (the topology corresponding to e) is the product topology of $\prod_n (X_n, \tau d_n)$.

Proof : Let for $n_0 \in N$, diam $(X_n) \leq M \ \forall n \geq n_0$. Clearly e is well-defined. We only show the triangle inequality to prove that e is a metric. For $x = (x_n)_n$, $y = (y_n)_n$, $z = (z_n)_n \in \Pi Xn$,

$$e(x, z) = \sup_n d_n(x_n, z_n) \leq \sup_n \{d_n(x_n, y_n) + d_n(y_n, z_n)\}$$

$$\leq \sup_n d_n(x_n, y_n) + \sup_n d_n(y_n, z_n)$$

$$= e(x, y) + e(y, z).$$

To show that the product topology is given by the metric e, let $x = (x_n)_n \in \Pi X_n$ and

$$x = (x_n)_n \in S(x_1, \in_1) \times ... \times S(x_n, \in_n) \times \prod_{n+1}^{\infty} X_n = U(say).$$

Choose $\in = \min \{\in_1, \in_2, .., \in_n\}$. Then $\in > 0$ and $y \in B_e(x, \in) \Rightarrow e(x, y) < \in \Rightarrow$

$\sup_n d_n(x_n, y_n) < \in$

$\Rightarrow d_i(x_i, y_i) < \in_i$ for $1 \leq i \leq n$.

Hence $y \in U$ and so $B(x, \in) \subset U$. Thus one side is proved.

To prove the converse take a ball $B_e(x, \in)$. Since diam $(X_n) \to 0$, $\exists\, n_0 \in N$ with diam $(X_n) < \in/2 \ \ \forall n \geq n_0$. Let

$$U = B(x_1, \in/2) \times ... \times B(x_{n_0}, \in/2) \times \prod_{n_0+1}^{\infty} X_n$$

Take $y = (y_n)_n \in U$. Clearly $d_i(x_i, y_i) < \in/2 \ \ \forall\, 1 \leq i \leq n_0$.

Obviously then $y \in U \Rightarrow e(x, y) \leq \in/2 \Rightarrow y \in B_e(x, \in)$.

This completes the proof.

Theorem : Let $\{X_n : n \in N\}$ be a family of metrizable spaces. Then $\prod_{n=1}^{\infty} X_n$ with product topology τ is metrizable.

Proof : Let d_n be the metric for X_n. We know that e_n for each X_n where $e_n(x, y) = \min\left\{\frac{1}{n}, d_n(x,y)\right\}$ for $(x, y) \in X_n \times X_n$ induces the same topology as d_n. Obviously diam $(X_n) \to 0$ as $n \to \infty$ and of course it is bounded uniformly. If $e(x, y) = \sup_n e_n(x_n, y_n)$ then $\tau_e = \tau$ and the result is proved.

Exercise : Let (X_1, d_1) and (X_2, d_2) be two metric spaces, $K \subset X_1$ and $f : K \to X_2$ be uniformly continuous. Then show that the oscillation $w(p)$ of f is zero at every $p \in X_1$.

Solution : For each $p \in K$, $w(p) = 0$ follows from continuity of f at p. If $p \in X_1 \setminus \bar{K}$ then we have

$$0 \leq w(p) \leq \text{diam } (f(X_1 \setminus \bar{K}) \cap K) = 0$$

as $X_1 \setminus \bar{K}$ is an open neighbourhood of p.

Finally let $p \in \bar{K} \setminus K$. Let $\in > 0$ be given. By the uniform continuity of f, $\exists$ a $\delta > 0$ such that

$$d_1(r, q) < \delta \Rightarrow d_2(f(r), f(q)) < \in/2.$$

Choose $r, q \in B(p, \frac{\delta}{2}) \cap K$. Then $d_1(q, r) \leq d_1(p, q) + d_1(p, r) < \frac{\delta}{2} + \frac{\delta}{2} = \delta \Rightarrow d_2(f(q), f(r)) < \in/2$. Hence

$$\text{diam } (f(B(p, \delta/2) \cap K)) \leq \in/2 < \in.$$

Thus $0 \leq w(p) \leq \text{diam } (f(B(p, \delta/2) \cap K)) < \in$. Since this is true for any $\in > 0$, $w(p) = 0$.

3.2 Two important theorems

We first consider compactness in $C[a, b]$ the space of all real continuous functions on $[a, b]$ endowed with sup metric ρ. $C[a, b]$ itself is not compact as it is not bounded.

Definition : Let M be a class of real functions defined on $[a, b]$. M is said to be uniformly bounded if $\exists$ a $k > 0$ such that $|f(t)| \le k \quad \forall t \in [a, b], \quad \forall f \in M$. M is said to be equi-continuous if for every $\in > 0$, $\exists$ a $\delta > 0$ such that $|f(t_1) - f(t_2)| < \in$ if $|t_1 - t_2| < \delta \quad \forall f \in M$.

We now prove the following characterization.

Arzela-Ascoli's Thorem

A set $M \subset C[a, b]$ is relatively compact iff it is uniformly bounded and equi-continuous.

Proof : Suppose that M is relatively compact in $C[a, b]$. Then M is bounded. This is equivalent to saying that for $b_1(t) \in C[a, b]$, there is a $k > 0$ such that

$$\rho(x, b_1) = \sup_{a \le t \le b} |x(t) - b_1(t)| \le K \quad \forall x \in M.$$

So $\forall x \in M$,

$$\sup_{a \le t \le b} |x(t)| \le \sup_{a \le t \le b} |x(t) - b_1(t)| + \sup_{a \le t \le b} |b_1(t)| \le k + k' \quad \text{(say).}$$

This shows that M is uniformly bounded.

To prove equi-continuity, choose $\in > 0$ and construct a finite $\frac{\varepsilon}{3}$- net,

$A = \{x_1(t), x_2(t), \ldots, x_k(t)\}$ for M ($\because$ M is totally bounded). The functions $x_i(t)$ are continuous and so uniformly continuous on $[a, b] \quad \forall i = 1, 2 \ldots, n$. So $\exists$ $\delta_i > 0$ such that $|x_i(t_1) - x_i(t_2)| < \frac{\varepsilon}{3}$ when $|t_1 - t_2| < \delta_i$, $t_1, t_2 \in [a, b]$ for $i = 1, 2, .., k$. Choose $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Then for $i = 1, 2, .., k$ $|x_i(t_1) - x_i(t_2)| < \frac{\varepsilon}{3}$ when $|t_1 - t_2)| < \delta$, $t_1, t_2 \in [a, b]$. Let $x(t) \in M$. There exists a $x_i(t) \in A$ such that $\rho(x, x_i) < \frac{\varepsilon}{3}$.

Then if $|t_1 - t_2| < \delta$, $t_1, t_2 \in [a, b]$,

$$|x(t_1) - x(t_2)| \le |x(t_1) - x_i(t_1)| + |x_i(t_1) - x_i(t_2)| + |x_i(t_2) - x(t_2)|$$

$$< \rho(x, x_i) + \frac{\varepsilon}{3} + \rho(x, x_i) < \varepsilon.$$

This is evidently true for any $x(t) \in M$ and so M is equi-continuous. Conversely suppose that $M \subset C[a, b]$ is uniformly bounded and equi-continuous. To show that M is relatively compact it is sufficient to prove that it is totally bounded. Suppose K is a (+)ve integer such that $|x(t)| \le K \ \forall x \in M$ and $t \in [a, b]$. Let $\varepsilon > 0$ be given. Since M is equicontinuous, choose $\delta > 0$ such that $|x(t_1) - x(t_2)| < \frac{\varepsilon}{4} \ \forall x \in M$ when $|t_1 - t_2| < \delta$. Since $[a, b]$ in compact, it has a δ-net $t_1, \ldots, t_n$. Choose a (+)ve integer m such that $\frac{1}{m} < \frac{\varepsilon}{4}$ and divide $[-K, K]$ into 2km equal parts by the points

$y_0 = -K < y_1 < y_2 < \ldots < y_k = K$. where $K = 2Km$.

Consider those n-tuples $(y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ of the numbers y_i, $i = 0, \ldots, k$ such that some $x \in M$ has the property that

$$\left|x(t_j) - y_{i_j}\right| < \frac{\varepsilon}{4}, j = 1, 2 \ldots, n$$

and choose one such $x \in M$ for each such n-tuple.

We shall show that the resulting finite subset E of M in an ε-net for M. Let $x \in M$; choose $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ so that $\left|x(t_j) - y_{i_j}\right| < \frac{\varepsilon}{4}, j = 1, 2 \ldots n$ and so there is a corresponding $e \in E$. Let $t \in [a, b]$ and choose j so that $|t - t_j| < \delta$. Then $|x(t) - e(t)| \le |x(t) - x(t_j)| + |x(t_j) - y_{i_j}| + |y_{i_j} - e(t_j)| + |e(t_j) - e(t)| < \varepsilon$

Hence $\rho(x, e) = \sup\limits_{a \le t \le b} |x(t) - e(t)| < \varepsilon$

Note : The above theorem can be generalized for any compact metric spa· X in place of $[a, b]$.

We next refer to the Weierstrass approximation theorem and its generalization to a compact space by stone ; recall that Weierstrass approximation theorem states that $P[a, b]$, the space of all polynomials on $[a, b]$ is dense in $C[a, b]$.

Let X be a compact metric space and let $C(X)$ be the space of continuous real functions on X with the usual metric

$$\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

We define algebraic operations in $C(X)$ as follows : If $f, g \in C(X)$ and a is real, then, for $x \in X$.

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)\, g(x)$$

$$(af)(x) = af(x)$$

A set $A \subset C(X)$ is called an algebra if $f, g \in A$ and a real imply $f + g, fg, af \in A$. If A is an algebra then it is easy to show that $\bar{A}$ is also an algebra.

Stone-Weierstrass Theorem

Let A be a closed algebra in $C(X)$, X a compact metric space. Assume that $1 \in A$ and A separates points (i.e. if $x, y \in X$, $x \neq y$, $\exists\, f \in A$ for which $f(x) \neq f(y)$). Then $A = C(X)$.

Proof : We first show that $f \in A \Rightarrow |f| \in A$. First suppose that $\sup\{|f(x)| : x \in X\} \leq 1$. Let $\varepsilon > 0$ and let $p(t) = a_0 + a_1 t + \ldots + a_n t^n$ be a polynomial such that $||(t)| - p(t)| < \varepsilon$ $\forall\, t \in [-1, 1]$.

Then $p(f) = a_0 + a_1 f + \ldots + a_n f^n \in A$($\because$ A is an algebra) and $||f(x)| - p(f(x))| < \varepsilon$ $\forall\, x \in X$.

This shows that $|f|$ is a limit of A and so $|f| \in A$ as A is closed. For any $f \in A$ we can choose a constant $a (\neq 0)$ such that $|af(x)| \leq 1$ $\forall\, x \in X$. Then as above we can show that $|af| \in A$ and so $|f| \in A$.

We next note that if $f, g \in A$ then min (f, g) and $\max(f, g)$ are in A. It follows readily from the facts that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

$$\min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

and $|f - g| \in A$ if $f, g \in A$ and A is an algebra.

Next let $f \in C(X)$. Let $x, y \in X$, $x \neq y$. Let g be the function which takes constant value $f(x)$ at all points. Then $g \in A$. Since A separates points, $\exists\, h \in A$ such that $h(x) \neq h(y)$. Without any loss of generality assume that $h(x) = 0$. There is a constant a such that the function f_{xy} given by $f_{xy} = g + ah$. satisfies $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$ and clearly $f_{xy} \in A$. Let $\varepsilon > 0$. Since $(f_{xy} - f)(y) = f(y) - f(y) = 0 < \varepsilon$.

From the continuity of $f_{xy} - f$ we can find an open ball s_y such that $y \in s_y$ and $f_{xy}(z) < f(z) + \varepsilon \quad \forall z \in s_y$.

Since X is compact the open cover $\{S_y : y \in X\}$ has a finite subcover, say, $Sy_1, Sy_2, \ldots, Sy_n$. Let $f_x = \min(f_{xy_1}, \ldots, f_{xy_n})$. Then $f_x \in A$, $f_x(x) = f(x)$ and for every $z \in X$, $f_x(z) < f(z) + \varepsilon$.

Again using the same argument for each $x \in X$, choose an open ball T_x such that $f_x(z) > f(z) - \varepsilon \quad \forall z \in T_x$.

Since X is compact, a finite number of these balls $T_{x_1}, \ldots, T_{x_m}$ covers X. Let $F = \max\left\{f_{x_1}, \ldots, f_{x_m}\right\}$.

Then $F \in A$ and $\forall z \in X$, $|f(z) - F(z)| < \varepsilon$. This proves the theorem.

Group–A
(Short questions)

1. Is the discrete topology defined on a non-empty set X metrizable? If so explain with reasons.
2. Is the real number space endowed with the cofinite topology metrizable? Answer with reasons.

3. Show that a metrizable space is normal.

4. In a metric space (X, d) prove that $x \in \overline{A}$ iff $d(x, A) = 0$ where $x \in X$, $A \subset X$.

5. Let (X, d) be a metric space and $A \subset X$. If p is a limit point of M then show that A contains an infinite sequence of distinct points converging to p.

6. Let $X = N \cup \{b\}$ where N is the set of natural numbers and $b \notin N$. Define

$$d(x, y) = 1 \text{ if } x, y \in N, x \neq y$$

$$d(b, x) = d(x, b) = 1 + \frac{1}{x} \text{ if } x \in N$$

$$d(x, y) = 0 \text{ if } x = y$$

Show that d is a metric on X. Find dist $(N, \{b\})$.

7. Let $f : (X, d) \to (Y, e)$ be uniformly continuous.

If $A, B \subset X$ be such that $d(A, B) = 0$, show that $e(f(A), f(B)) = 0$.

8. Is the real number space endowed with lower limit topology metrizable. Answer with reasons.

Group–B
(Long questions)

1. Prove that metrization is invariant under homeomorphism.
2. Show that the derived set of a countably compact set in a metric space is countably compact.

3. For a pseudo-metric space (X, d) if $Y = \{\{\overline{x}\} : x \in X\}$, define $e(\{\overline{x}\}, \{\overline{y}\}) = d(x, y)$.

First show that $\{\overline{x}\} = \{y : d(x,y) = 0\}$. Then prove that e is a metric on Y.

4. Let K be a subset of a metric space (X, d) and let $r > 0$. Define $S_r(K) = \{x \in X : d(x, y) < r$ for at least one point $y \in K\}$. Prove that $S_r(K)$ is open.

5. Let $\{x_n\}_n$ and $\{y_n\}_n$ be Cauchy sequences in a metric space (X, d). Define a relation '$\sim$' as follows : $\{x_n\}_n \sim \{y_n\}_n$ iff. $\lim_{n \to \infty} d(x_n, y_n) = 0$. Then '$\sim$' is an equivalence

relation and let X^* be the collection of equivalence classes. For $x^*, y^* \in X^*$, define $d^*(x^*, y^*) = \lim_{n\to\infty} d(x_n, y_n)$ where $\{x_n\}_n \in x^*$, $\{y_n\}_n \in y^*$. Prove that the limit exists and the limit does not depend on the members closen from the equivalence classes.

6. Prove that (X^*, d^*) (described in (5) above) is a metric space.
7. (a) If a separable space is also metrizable then prove that the space has a countable basis.

   (b) Show that any finite subspace of a metrizable space is always discrete.
8. Prove that a topological space (X, τ) is metrizable iff there is a homeomorphism of X onto a subspace of some metric space.

Unit–V ❑ Uniform spaces and proximity spaces

Introduction

We are already familiar with the notions of topological spaces and metric spaces. Due to the presence of a distance function, the topology induced by a metric is much stronger and also we can define notions like Cauchy condition, completeness, uniform continuity in metric spaces which cannot be defined in general topological spaces. The theory of uniformity was developed to bridge this gap and it is a tool which can be seen as a structure which is stronger than a topological space but weaker than a metric space. The theory of uniform spaces is somewhat analogas to the theory of metric spaces but can be applied to a large number of spaces; in particularl to those spaces, not necessarily satisfying the axiom of countability (i.e., which cannot be metrizable). We will see that every uniformity induces a topology on a set, whereas every metric or more generally, every family of pseudo-metrics induces a uniformity on a set. We will study the conditions under which a given topology can be induced by a uniformity (i.e., when the topology is uniformizable) and when a given uniformity can be induced by a metric (i.e., when the uniform space is metrizable). We will also study many more properties of these spaces. Finally we will study another related structure, called proximity structure.

5.1. Basic definitions and properties

Let X be a nonempty set. A nonempty subsat U of $X \times X$ is called a relation on X. If U is a relation on X, its inverse relation U^{-1} is defined by $U^{-1} = \{(x, y) : (y, x) \in U\}$. Clearly $(U^{-1})^{-1} = U$. If $U^{-1} = U$, the relation U is said to be symmetric. If U and V are two relation on X, then their composition is defined by,

$UoV = \{(x, y) : (x, z) \in V$ and $(z, y) \in U$ for some z in $X\}$. It is easy to verify that for any three relations U, V, W on X, $(UoV)o\, W = Uo\,(VoW)$ and $(UoV)^{-1} = V^{-1}oU^{-1}$, We write $\Delta = \{(x, x) : x \in X\}$. For any relation U on X, $\Delta \, o \, U = U \, o \, \Delta = U$. So Δ is the identity relation on X.

Let A be any nonempty subset of X, U any relation on X and $x_o \in X$. We define $U[A] = \{y : (x, y) \in U$ for some x in $A\}$ and $U[x_0] = \{y : (x_0, y) \in U\}$.

If A is a subset of X and U and V are two relations on X, then we can verify that $(U \, o \, V)[A] = U[V[A]]$.

Uniformity and Uniform space.

Let X be a nonempty set. A nonempty family $\mathscr{U}$ of subsets of $X \times X$ is said to be a uniformity on X if the following hold :

(i) $\Delta \subset U$ for every $U \in \mathscr{U}$.

(ii) If $U \in \mathscr{U}$, then $U^{-1} \in \mathscr{U}$.

(iii) If $U \in \mathscr{U}$, there is a member V of $\mathscr{U}$ with $V \circ V \subset U$.

(iv) If U_1, U_2 are in $\mathscr{U}$, then $U_1 \cap U_2 \in \mathscr{U}$.

(v) If $W \subset X \times X$ and $U \subset W$ for some U in $\mathscr{U}$, then $W \in \mathscr{U}$.

The pair $(X, \mathscr{U})$ is called a uniform space.

Base and subbase of a uniformity.

Let $(X, \mathscr{U})$ be a uniform space. A nonempty subfamily $\mathscr{B}$ of $\mathscr{U}$ is said to be a base for the uniformity $\mathscr{U}$ if for every U in $\mathscr{U}$ there is a member V of $\mathscr{B}$ with $V \subset U$.

A nonempty subfamily $\mathscr{S}$ of $\mathscr{U}$ is said to be a subbase for the uniformity $\mathscr{U}$ if the family ($\mathscr{B}$ of all finite intersections of the members of $\mathscr{S}$) is base for the uniformity $\mathscr{U}$.

Theorem 1 : Let X be a nonempty set. A nonempty family β of subsets of $X \times X$ is base for some uniformity on X if the following hold.

(i) $\Delta \subset U$ for every $U \in \beta$.

(ii) If $U \in \beta$, there is a member V of β with $V \subset U^{-1}$.

(iii) If $U \in \beta$, there is a member V of β with $VoV \subset U$.

(iv) If U_1, U_2 are in β, then $U_1 \cap U_2$ contains a member of β.

Proof : First suppose that β is a base for the uniformity $\mathscr{U}$ on X. Then $\beta \subset \mathscr{U}$ and each member of $\mathscr{U}$ contains a member of β.

(i) Let $U \in \beta$. Then $U \in \mathscr{U}$ and so $\Delta \subset U$.

(ii) Let $U \in \beta$. Then $U \in \mathscr{U}$ and so $U^{-1} \in \mathscr{U}$.

So there is a member V of β with $V \subset U^{-1}$.

(iii) Let $U \in \beta$. Then $U \in \mathscr{U}$. So there is a member W of $\mathscr{U}$ with $W \circ W \subset U$. Again, W contains a member V of β. So $V \circ V \subset W \circ W \subset U$.

(iv) Let U_1, U_2 belong to β. Then U_1, U_2 are in $\mathscr{U}$ and so $U_1 \cap U_2 \in \mathscr{U}$. Hence there is a member V of β with $V \subset U_1 \cap U_2$.

Thus the conditions are necessary.

Next, let β be a nonempty family of subsets of $X \times X$ satisfying the given conditions. Denote by $\mathscr{U}$ the family of all those subsets $\cup$ of $X \times X$ such that $U \in \mathscr{U}$, iff $V \subset U$ for some V in β. Clearly $\beta \subset \mathscr{U}$.

(a) Let $U \in \mathscr{U}$. Then there is a member V of β with $V \subset U$. Since $\Delta \subset V$, $\Delta \subset U$.

(b) Let $U \in \mathscr{U}$. There is a member V of β with $V \subset U$. Also, there is a member W of β with $W \subset V^{-1}$. Since $V^{-1} \subset U^{-1}$, $W \subset U^{-1}$. So $U^{-1} \in \mathscr{U}$.

(c) Let $U \in \mathscr{U}$. Then $V \subset U$ for some V in β. So there is a member W of β such that $W \circ W \subset V$. Since $\beta \subset \mathscr{U}$, $W \in \mathscr{U}$. So, $W \circ W \subset U$.

(d) Let U_1, U_2 be two members of $\mathscr{U}$. Then there are two members V_1, V_2 in β with $V_1 \subset U_1$ and $V_2 \subset U_2$. By condition (iv), there is a member V of β with $V \subset V_1 \cap V_2$. So $V \subset U_1 \cap U_2$. This gives that $U_1 \cap U_2 \in \mathscr{U}$.

(e) Let W be a subset of $X \times X$ such that $U \subset W$ for some U in $\mathscr{U}$. There is a member V in β with $V \subset U$. So $V \subset W$ which gives that $W \in \mathscr{U}$.

Hence, $\mathscr{U}$ is a uniformity on X. From the construction of $\mathscr{U}$ it is obvious that β is a base for the uniformity $\mathscr{U}$.

Exercise : Let X be a nonempty set. A nonempty family s of subsets of $X \times X$ is a subbase for some uniformity on X if the following hold.

(i) $\Delta \subset U$ for every, $U \in s$.

(ii) If $U \in s$, then there exist finitely many members $V_1, ..., V_n$ (say) in s such that U^{-1} contains $V_1 \cap ... \cap V_n$.

(iii) If $U \in s$, then there exist finitely many members $V_1, ..., V_m$ (say) in s with $VoV \subset U$. Where $V = V_1 \cap ..... \cap V_n$.

Solution : Let the family s of subsets of $X \times X$ satisfy the given conditions.

Denote by β the collection of all finite intersections of the members of s. Clearly $s \subset \beta$.

(a) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. Since $\Delta \subset U_i$ for each i, $\Delta \subset \cap_{i=1}^{n} U_i = U$.

(b) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. We have $U^{-1} = \cap_{i=1}^{n} = U_i^{-1}$. For each i, there is a member V_i in s with $V_i \subset U_i^{-1}$. Write $V = \cap_{i=1}^{n} V_i$. Then $V \in \beta$. We have $V \subset \cap_{i=1}^{n} U_i^{-1} = U^{-1}$.

(c) Let $U \in \beta$. Then $U = \cap_{i=1}^{n} U_i$, where $U_i \in s$. For each i, there is a member W_i of s with $W_i o W_i \subset U_i$. Let $W = \cap_{i=1}^{n} W_i$. Then $W \in \beta$ and $WoW \subset \cap_{i=1}^{n} (W_i o W_i) \subset \cap_{i=1}^{n} U_i = U$.

(d) Let U and V be any two members of β. Then $U = \cap_{i=1}^{m} U_i$ and $V = \cap_{i=1}^{n} V_i$, where $U_i, V_i \in s$.

We have $U \cap V = U_1 \cap U_2 \cap .. \cap U_m \cap V_1 \cap V_2 \cap .. \cap V_n$. This gives that $U \cap V \in \beta$.

Thus β is a base for some uniformity $\mathcal{U}$ on X. From the construction of β it is clear that s is a subbase for the uniformity $\mathcal{U}$.

Theorem 3. Let $(X, \mathcal{U})$ be a uniform space and let τ denote the family consisting of the void set ϕ and all those subsets S of X such that if $x \in S$, then $U[x] \subset S$ for some U in $\mathcal{U}$. Then τ is a topology on X.

Let $x_0 \in X$ and $\mathcal{U} = \{U[x_0] : U \in \mathcal{U}\}$. Then $\mathcal{U}$ is a neighbourhood base at x_0.

Proof : Let $x \in X$ and $U \in \mathcal{U}$. Then $U[x] \subset X$ which gives that $X \in \tau$. Let S_1 and S_2 be any two members of τ. Write $S = S_1 \cap S_2$. If $S = \phi$, then $S \in \tau$. Suppose that $S \neq \phi$. Let $x \in S$.

Then $x \in S_1$ and $x \in S_2$. So there are members U_1, U_2 in $\mathcal{U}$ such that $U_1[x] \subset S_1$ and $U_2[x] \subset S_2$. Write $U = U_1 \cap U_2$. Then $U \in \mathcal{U}$ and $U[x] \subset U_1[x] \subset S_1$. $U[x] \subset U_2[x] \subset S_2$. So $U[x] \subset S_1 \cap S_2$. This gives that $S \in \tau$.

Let $\mathcal{F} = \{S_\alpha : \alpha \in \Delta\}$ be a nonempty subfamily of τ and let $S = \cup\{S_\alpha : \alpha \in \Delta\}$.

Let $x \in S$. Then $x \in S_\alpha$ for some α in Δ. There is a member U of $\mathscr{U}$ with $U[x] \subset S_\alpha$. Since $S_\alpha \subset S$, $U[x] \subset S$. So $S \in \tau$.

Therefore τ is a topology on X.

For the second part we proceed as follows. Let $x_0 \in X$ and U be a member of $\mathscr{U}$. Consider the set $U[x_0]$.

Let $A = \{x : V[x] \subset U[x_0]$ for some V in $\mathscr{U}\}$. Taking $V = U$, we have $V[x_0] \subset U[x_0]$ which gives that $x_0 \in A$. Also from definition we get $A \subset U[x_0]$. We now show that A is open.

Let $x \in A$. Then there is a member V in $\mathscr{U}$ with $V[x] \subset U[x_0]$. Choose a member W in $\mathscr{U}$ with $WoW \subset V$.

Take any $y \in W[x]$. Then $(x, y) \in W$. If $z \in W[y]$, then $(y, z) \in W$. So $(x, z) \in WoW \subset V$. This gives that $z \in V[x] \Rightarrow W[y] \subset U[x] \Rightarrow y \in A$.

Thus $W[x] \subset A$. So A is open.

Since $x_0 \in A \subset U[x_0]$, it follows that $U[x_0]$ is a neighbourhood of x_0.

Let W be any neighbourhood of x_0. Then there is an open set G with $x_0 \in G \subset W$. So there is a member U of $\mathscr{U}$ with $U[x_0] \subset G \subset W$.

This gives that the family.

$\mathscr{V} = \{U[x_0] : U \in \mathscr{U}\}$ is a neighbourhood base at x_0.

Note : We say that the topology τ on X in Theorem 3, generated by the uniformity $\mathscr{U}$, is the uniform topology.

Definition : A topological space (X, τ) is said to be uniformisable if there is a uniformity $\mathscr{U}$ on X such that the topology generated by the uniformity $\mathscr{U}$ is identical with the topology τ.

Example 1 : Let (X, d) be a pseudometric space. For positive number r, let

$W_r = \{(x, y) : x, y \in X$ and $d(x, y) < r\}$. and $\beta = \{W_r : r > 0\}$.

We verify that β is a base for some uniformity on X.

(i) Since $d(x, x) = 0$ for all $x \in X$, it follows that $\Delta \subset W_r$ for every $r > 0$.

(ii) Since $d(y, x) = d(x, y)$ for all x, y in X, we get $W_r^{-1} = W_r$ for $r > 0$. So $W_r^{-1} \in \beta$.

(iii) Let r be any positive number and let $p=\frac{1}{2}r$. Let $(x, y) \in W_p \circ W_p$. Then (z, z), $(x, y) \in W_p$ for some z in X. We have

$d(x, y) \leq d(x, z) + d(z, y) < 2p = r.$

This gives that $(x, y) \in W_r$ and so

$$W_p \circ W_p \subset W_r.$$

(iv) Let $Wr_1, Wr_2 \in \beta$. Let $r = \min \{r_1, r_2\}$. If $(x, y) \in W_r$, then $d(x, y) < r \leq r_i$ $(i = 1, 2)$ which gives that $W_r \subset W_{r_1}$ and $W_r \subset W_{r_2}$. So, $W_r \subset W_{r_1} \cap W_{r_2}$.

Hence β is a base for some uniformity $\mathcal{U}$ on X. A subset U of $X \times X$ belongs to $\mathcal{U}$ if $W_r \subset U$ for some $r > 0$.

We now show that the topology τ_1 generated by the pseudometric d is identical with the topology τ_2 generated by the uniformity $\mathcal{U}$.

Let $G \in \tau_1$ and $x \in G$. Then there is a positive number r such that

$$S(x, r ; d) \subset G,$$

where $S(x, r ; d) = \{y : y \in X \text{ and } d(x, y) < r\}$

$= W_r [x].$

Thus $W_r [x] \subset G$. Since $W_r \in \mathcal{U}$, $G \in \tau_2$.

Again, let $G \in \tau_2$ and $x \in G$. Then there is a member U in $\mathcal{U}$ with $U[x] \subset G$. Since β is a base for $\mathcal{U}$, there is a positive number r, such that $W_r \subset U$. So $W_r [x] \subset U[x] \subset G$.

Since $W_r [x] = S(x, r ; d)$, $S(x, r ; d) \subset G$. This gives that $G \in \tau_1$. Hence $\tau_1 = \tau_2$. Therefore the pseudometric space (X, d) is uniformisable.

5.2. Uniformizability and metrizability

Example 2. Let X be a nonempty set and ρ be a family of pseudometrics on X. For $d \in \theta$ and $r > 0$.

Let $W_{(d, r)} = (x, y) : x, y \in X$ and $d (x, y) < r\}$, and

$$s = \{W_{(d, r)} : d \in P \text{ and } r > 0\}.$$

Let $d \in \rho$ and $r > 0$. Since $d(x, x) = 0$ for all $x \in X$, we have $\Delta \subset W_{(d, r)}$. Again, since $d(y, x) = d(x, y)$ for all x, y in X, $W_{(d,r)}^{-1} = W_{(d,r)}$.

Also $W_{\left(d,\frac{1}{2}r\right)} o W_{\left(d,\frac{1}{2}r\right)} \subset W_{(d, r)}$.

Therefore s is a subbase for some uniformity u on X. Let β denote the family of all finite intersections of the members of s. Then β is a base for the uniformity u.

Let τ_1 denote the topology generated by the family ρ of pseudo-metrics and τ_2 the topology generated by the uniformity u.

Let $G \in \tau_1$ and $x \in G$. Then there is a set of the form

$B = \cap_{i=1}^{n} S(x, r_i, d_i)$, where $d_i \in \rho$ and $r_2 > 0$, such that $x \in B \subset G$.

Since $S(x, r_i, d_i) = W_{(d_i,r_i)}[x]$, we have $B = \cap_{i=1}^{n} S(x, r_i, d_i) = W[x]$, where $W = \cap_{i=1}^{n} W_{(d_i,r_i)} \in u$.

This gives that $G \in \tau_2$.

Again, let $G \in \tau_2$ and $x \in G$. Then there is a member U in u with $U[x] \subset G$. Since β is a base for u, there is a set of the form

$W = \cap_{i=1}^{n} W_{(d_i,r_i)}$ where $d_i \in \theta$ and $r_i > 0$ such that $W \subset U$.

So $W[x] \subset U[x] \subset G$.

Since $W[x] = \cap_{i=1}^{n} S(x, r_i, d_i)$, we have

$\cap_{i=1}^{n} S(x, r_i, d_i) \subset G$. So $G \in \tau_1$.

Hence $\tau_1 = \tau_2$.

Theorem 4 : Every completely regular space is uniformisable.

Proof : Let (X, τ) be a completely regular space. We first show that its topology τ can be generated by a family P of pseudometrics on X.

In fact, let us denote the family of all real valued continuous functions defined on

X by $C(X)$ and let $C^*(X)$ denote the subfamily of $C(X)$ consisting of bounded functions. For a finite number of functions $f_1, f_2 .., f_k \in C^*(X)$ define.

$$\rho_{f_1,f_2,..f_k}(x,y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|, ..., |f_k(x) - f_k(y)|\}$$

for $x, y \in X$. It is easy to verify that $\rho_{f_1,f_2,..f_k}$ is a pseudo-metric on X. Let us consider the family ρ of all these pseudo-metrics $\rho_{f_1,f_2,..f_k}$ where $f_1, f_2, .., f_k \in C^*(X)$.

Observe that from the construction it follows that every $\rho : X \times X \to R$ is continuous where $\rho \in \rho$. Let τ_1 be the topology induced by ρ. Let $G \in \tau_1$ and $x \in G$. Then we have

$$\bigcap_{i=1}^{n} S(x, r_i, \rho_i) = \{y \in X : \rho_i(x,y) < r_i\} \subset G \text{ for some } r_i > 0 \text{ and } \rho_i \in \rho,\ i = 1, 2, ..,n.$$

Take a fixed i. If ρ_i is generated by $f_1, f_2, . . , f_k$ (say) $\in C^*(X)$ then clearly

$$x \in \bigcap_{i=1}^{k} f_i^{-1}\ (f_i(x) - r_i, f_i(x) + r_i) \subset S(x, r_i, \rho_i).$$

As each f_i is continuous so $\bigcap_{i=1}^{k} f_i^{-1}(f_i(x) - r_i, f_i(x) + r_i) = v_i$ (say) $\in \tau$ and this shows that

$$x \in \bigcap_{i=1}^{n} v_i = v \text{ (say) } \subset G,$$

where $v \in \tau$. Hence $G \in \tau$.

On the other hand is $U \in \tau$ and $x \in U$, by complete regularity of X, $\exists$ a function $f \in C^*(X)$ such that $f(x) = 0$ and $f(y) = 1\ \forall y \in X \backslash U$. Then clearly $\rho_f \in P$ and

$$x \in S(x, \tfrac{1}{2}, \rho_f) \subset U.$$

This shows that $U \in \tau_1$. Therefore $\tau = \tau_1$.

For d in P and $r > 0$, let

$$W_{(d,r)} = \{(x,y) : x, y \in X \text{ and } d(x, y) < r\}$$

and $s = \{W_{(d,r)} : d \in p \text{ and } r > o\}$.

Let $d \in \theta$ and $r > 0$. Since $d(x, x) = 0$ for all $x \in X$, $\Delta \subset W_{(d, r)}$. Again, since $d(y, x) = d(x, y)$ for all x, y in X, we get $W^{-1}_{(d,r)} = W_{(d,r)}$. Also $W_{(d,\frac{1}{2}r)} o W_{(d,\frac{1}{2}r)} \subset W_{(d,r)}$.

Therefore s is a base for some uniformity u on X. Denote by β the family of all finite intersections of the members of s. Thus β is a base for the uniformity u. Let τ_0 denote the topology generated by the uniformity u.

Let $G \in \tau$ and $x \in G$. Then as before $d_i \in P$ and $r_i > 0$ there is a such that $x \in B = S(x, r_i, d_i) \subset G$.

Since $S(x, r_i, d_i) = W_{(d_i,r_i)}[x] \in u$,

Thus $W[x] \subset G$ which gives that $G \in \tau_0$. Again, let $G \in \tau_0$ and $x \in G$. Then there is a member U of u with $U[x] \subset G$. Since β is a base for u, there is a set of the form

$$W = \cap_{i=1}^{n} W_{(d_i,r_i)} \quad (d_i \in \theta \text{ and } r_i)$$

such that $W \subset U$. Thus $W[x] \subset U[x] \subset G$.

Since $W[x] = \cap_{i=1}^{n} S(x, r_i, d_i)$, it follows that $G \in \tau$. Hence $\tau = \tau_0$.

Therefore the space (X, τ) is uniformisable.

Definition : Let X be a nonempty set. A mapping $q : X \times X \to R$ is said to be a quasimetric on X if the following hold.

(i) $q(x, y) \geq 0$ and $q(x, x) = 0$.

(ii) $q(x, y) \leq q(x, z) + q(z, y)$

for x, y, z in X.

Theorem 5. (Metrization Lemma) :

Let X be a nonempty set and let $\{U_n\}_{n=0}^{\infty}$ be a sequence of subsets of $X \times X$ such that (i) $U_0 = X \times X$ (ii) $\Delta \subset U_n$ for each n and

(iii) $U_n \circ U_n \circ U_n \subset U_{n-1}$ $(n = 1, 2, 3, \ldots)$.

Then there exists a quasimetric q on X such that

(a) $U_n \subset \{(x, y) : q(x, y) < 2^{-n+2}\} \subset U_{n-1}$ $(n = 1, 2, 3, \ldots)$.

If each U_n is symmetric, q becomes a pseudometric.

Proof : It is easy to see that $U_n \subset U_{n-1}$ for $n = 1, 2, 3, \ldots$ We define a mapping $f : X \times X \to [0, \infty)$ as follows.

$$f(x, y) = 2^{-n+2}, \text{ if } (x, y) \in U_{n-1} \setminus U_n.$$
$$= 0, \text{ if } (x, y) \in U_n \text{ for all } n.$$

Now we define the mapping $q : X \times X \to [0, \infty]$ as follows : Let $(x, y) \in X \times X$. Then

$$q(x, y) = \inf \left\{ \sum_{i=1}^{k} f(x_{i-1}, x_i) \right\},$$

where the infimium is taken over all finite sequences $\{x_0, x_1, x_2, \ldots, x_k\} \subset X$ with $x_0 = x$, $x_k = y$.

It is obvious that $q(x, y) \geq 0$ and $q(x, x) = 0$.

Let x, y, z be three points of X. Choose any $\epsilon > 0$. Thus there are finite sequences $\{x_0, x_1, x_2, \ldots, x_k\}$ and $\{z_0, z_1, z_2, \ldots, z_r\}$ in X with $x_0 = x$, $x_k = z$, $z_0 = z$ and $z_r = y$. Such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < q(x, z) + \frac{1}{2}\varepsilon$$

$$\sum_{i=1}^{r} f(z_{i-1}, z_i) < q(z, y) + \frac{1}{2}\varepsilon$$

Now, $\{x_0, x_1, x_2, \ldots, x_k, z_1, z_2, \ldots, z_r\}$ is a finite sequence in X with $x_0 = x$, $z_r = y$. So.

$$q(x, y) \leq \sum_{i=1}^{k} f(x_{i-1}, x_i) + \sum_{i=1}^{r} f(z_{i-1}, z_i).$$

Or, $q(x, y) < q(x, z) + q(z, y) + \varepsilon$.

This gives that

$$q(x, y) \leq q(x, z) + q(z, y).$$

Thus q is a quasimetric on X.

To prove the relation (a) we first prove the inequality

(b) $f(x_0, x_k) \le 2\sum_{i=1}^{k} f(x_{i-1}, x_i)$

for any finite sequence $\{x_0, x_1, x_2, .., x_k\} \subset X$.

Clearly the inequality (b) holds for $k = 1$. Take any positive integer $m > 1$. Suppose that the inequality (b) holds for all positive integers $K < m$.

Let $S = \sum_{i=1}^{m} f(x_{i-1}, x_i)$ and $S > 0$.

We consider the following cases :

(i) $f(x_0, x_1) \le \frac{1}{2}S$ and (ii) $f(x_0, x_1) > \frac{1}{2}S$.

Case (i) : Denote by k the largest positive integer such that

$\sum_{i=1}^{k} f(x_{i-1}, x_i) \le \frac{1}{2}S$. Thus $k < m$

Clearly $\sum_{i=k+2}^{m} f(x_{i-1}, x_i) \le \frac{1}{2}S$.

Also $f(x_k, x_{k+1}) \le S$.

By induction hypothesis

$$f(x_0, x_k) \le 2\sum_{i=1}^{k} f(x_{i-1}, x_i) \le S$$

and $f(x_{k+1}, x_m) \le 2\sum_{i=k+2}^{m} f(x_{i-1}, x_i) \le S$.

Let n be the least positive integer such that $2^{-n+1} \le S$. Then clearly (x_0, x_k), (x_k, x_{k+1}) and (x_{k+1}, x_m) all belong to U_n. Again since $U_n \circ U_n \circ U_n \subset U_{n-1}$, $(x_0, x_m) \in U_{n-1}$.

So $f(x_0, x_m) \le 2^{-n+2} \le 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_i)$.

Case (ii) : We have

$$f(x_0, x_1) \le S \text{ and } \sum_{i=2}^{m} f(x_{i-1}, x_i) < \frac{1}{2}S.$$

By induction hypothesis, $f(x_1, x_m) \le 2\sum_{i=2}^{m} f(x_{i-1}, x_i) < S$.

Let n denote the least positive integer such that $2^{-n+1} \le S$. Thus $(x_0, x_1), (x_1, x_m) \in U_n$. So $(x_0, x_n) \in U_n o\ U_n = \Delta\ oU_n\ o\ U_n \subset U_{n-1}$.
This gives that

$$f(x_0, x_m) \le 2^{-n+2} \le 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_i)$$

Suppose that $S = 0$. Thus $f(x_{i-1}, x_i) = 0$ for i = 1, 2, .., m. Let n be any positive integer. Then $(x_0, x_1), (x_1, x_2) \in U_{n+1}$. So $(x_0, x_2) \in U_{n+1}\ o\ U_{n+1} \subset U_n$. This gives that $(x_0, x_2) \in \cap_{n=0}^{\infty} U_n = W$ (say).

Similarly $(x_0, x_3) \in W$, $(x_0, x_4) \in W$. At $(m-1)$ th step we get $(x_0, x_m) \in W$.

So $f(x_0, x_m) = 0$, and

$$f(x_0, x_m) = 2S = 2\sum_{i=1}^{m} f(x_{i-1}, x_m).$$

Thus in any case the inequality (b) also holds for $k = m$. Hence by the principle of finite induction (b) holds for every positive integer k.

Take any positive integer n and let $(x, y) \in U_n$.

Then $q(x, y) \le f(x, y) \le 2^{-n+1} < 2^{-n+2}$.

This gives that

(c) $U_n \subset \{(x, y) : q(x, y) < 2^{-n+2}\}$.

Now let $q(x, y) < 2^{-n+2}$. Thus there exists a finite sequence $\{x_0, x_1, x_2, ..., x_k\}$ in X with $x_0 = x$, $x_k = y$ such that

$$\sum_{i=1}^{k} f(x_{i-1}, x_i) < 2^{-n+2}.$$

By inequality (b) we have

$$f(x, y) = f(x_0, x_k) \leq 2\sum_{i=1}^{k} f(x_{i-1}, x_i) < 2^{-n+3}.$$

Since $f(x, y)$ takes values of the form $0, 2^{-p+2}$ $(p = 1, 2, 3, ..)$ it follows that $f(x, y) \leq 2^{-n+2}$.

So $(x, y) \in U_{n-1}$ which gives that

(d) $\{(x, y) : q(x, y) < 2^{-n+2}\} \subset U_{n-1}$

Combining (c) and (d) we obtain (a).

If each U_n is symmetric, then $f(x, y) = f(y, x)$ for all x, y in X. This implies that $q(x, y) = q(y, x)$.

So q is a pseudometric.

Theorem 6. Every uniformity on a set X can be generated by a family of pseudometrics on X.

Proof : Let u be a uniformity on the set X. Let β be a base for the uniformity u such that each member of β is symmetric and is different from $X \times X$. For each V in β we choose a sequence $\{U_n(v)\}_{n=0}^{\infty}$ of symmetric sets in u such that

$$U_{n+1}^{(v)} \circ U_{n+1}^{(v)} \circ U_{n+1}^{(v)} \subset U_n^{(v)},$$

where $U_0^{(v)} = X \times X$ and $U_1^{(v)} = V$.

By metrization Lemma there exists a pseudometric d_v on X such that

(1) $U_n^{(v)} \subset \left\{(x,y) : d_v(x,y) < 2^{-n+2}\right\} \subset U_{n-1}^{(v)}$.

Let $P = \{d_v : V \in \beta\}$: Denote by γ the uniformity on X generated by the family P of pseudometrics on X. For $V \in \beta$ and $r > 0$, let

$$W_{(v, r)} = \{(x, y) : d_v (x, y) < r\}.$$

and $\beta_0 = \{W_{(v, r)} : V \in \beta \text{ and } r > 0\}$.

Thus β_0 is a subbase for the uniformity γ.

Let U be any member of u. Since β is a base for u, there is a member V in β with $V \subset U$. From (1) we have $W_{(v, 1)} \subset U_1^{(v)} = V$.

So $W_{(v, 1)} \subset U$. This gives that $U \in \gamma$.

Next let $W \in \gamma$. Then there is a set of the form,

$$\hat{W} = \cap_{i=1}^{k} W_{(v_i r_i)} \qquad (V_i \in \beta \text{ and } r_i > 0)$$

such that $\hat{W} \subset W$.

Choose positive integers $n_1, n_2, \ldots, n_k$ such that $2^{-n_i+2} < r_i (i = 1,2,\ldots k)$. From (1) we have

$$U_{n_i}^{(v_i)} \subset \left\{(x,y) : dv_i(x,y) < 2^{-n_i+2}\right\} \subset W_{(v_i, r_i)}.$$

Write $U = \cap_{i=1}^{k} U_{n_i}^{(v_i)}$. Then $U \in u$ and

$$U \subset \cap_{i=1}^{k} W_{(v_i r_i)} = \hat{W} \subset W.$$

This gives that $W \in u$.

Therefore $\mathscr{V} = u$. Which proves the theorem.

Theorem 7 : As in theorem 6, for every $V \in \mathscr{V}$ We can construct a sequence $\left\{U_n^v\right\}_{n=0}^{\infty}$ of symmetric sets in U such that

$U_0^v = X \times X,\ V_1^v = V$ and

$U_{n+1}^v \, o \, U_{n+1}^v \, o \, U_{n+1}^v \subset U_n^v \qquad \forall n \in N.$

By metrization Lemma there is a pseudometric d_v on X such that

$$U_n^v \subset \left\{(x,y) : d_v(x,y) < 2^{-n+2}\right\} \subset V_{n-1}^v.$$

clearly $\{(x, y) : d_v(x, y) < 1\} \subset U_1^v = v$.

We will first show that the pseudometric d_v thus constructed is a continuous function from $X \times X$ to R.

Take any point $(x_0, y_0) \in X \times X$ and choose $\in > 0$. We can find $n \in N$ so that $2^{-n+2} < \frac{\in}{2}$. Then taking $U_n^v = W$ (say) we have a $W \in \mathscr{V}$ such that

$$(x, y) \in W \Rightarrow d_v(x, y) < \frac{\in}{2}.$$

Consider the open neighborhood $W(x_0) \times W(y_0)$ of the point (x_0, y_0) in the product topology of $X \times X$. Clearly for $(x, y) \in W(x_0) \times W(y_0)$,

$$\left|d_v(x_0,y_0) - d_v(x,y)\right| \le d_v(x_0,x) + d_v(y,y_0) < \frac{\in}{2} + \frac{\in}{2} = \in.$$

This proves our assertion.

Finally to prove that (x, τ) is completely regular, choose $x_0 \in X$ and a closed set F, $x_0 \notin F$. Since τ is induced by the uniformity $\mathscr{V}$ So $\exists$ a $V \in \mathscr{V}$ such that $V[x_0] \cap F = \phi$. Define $f : X \to [0, 1]$ by $f(x) = \min \{1, d_v(x_0, x)\}$. Then f is continuous (by above assertion) and $f(x_0) = 0$ and $f(y) = 1 \, \forall y \in F$. This completes the proof.

Theorem 8 : A uniform space is pseudometrizable if its uniformity has a countable base.

Proof : Let (X, u) be a uniform space with a countable base $\{V_n\}_{n=0}^{\infty}$ for its uniformity u, where $V_0 = X \times X$.

First observe that $\beta' = \{V_n'\}_{n=0}^{\infty}$ where $V_n' = V_n \cap V_n^{-1}$ against forms a countable basis of $\mathscr{V}$ consisting of symmetric sets. Let $U_0 = V_0 = X \times X$. First choose $U'_1 \in V$ such that $U'_1 \text{ o } U'_1 \subset U_0$ and then choose $U''_1 \in U$ such that $U_1'' \text{o} U_1'' \subset U'_1$. Then

$U''_1 \text{ o } U''_1 \text{ o } U''_1 \subset U''_1 \text{ o } U''_1 \text{ o } U''_1 \text{ o } \Delta \subset U''_1 \text{ o } U''_1 \text{ o } U''_1 \text{ o } U''_1 \subset U'_1 \text{ o } U'_1 \subset U_0$.

Take $U'''_1 = U''_1 \cap V_1$ and choose a member U_1 (say) from B' such that $U_1 \subset U'''_1$. Then U_1 is symmetric,

$U_1 \subset V_1$ and $U_1 \text{ o } U_1 \text{ o } U_1 \subset U_0$.

For each positive integer n we proceed in this way.

Hence we obtain a sequence of symmetric sets $\{U_n\}_{n=0}^{\infty}$ in u which forms a base for u and possesses the properties.

(i) $U_0 = V_0$ (ii) $U_n \subset V_n$ and (iii) $U_n \text{ o } U_n \text{ o } U_n \subset U_{n-1}$.

By Metrization Lemma there exists a pseudometric d on X such that

(1) $U_n \subset \{(x, y) : d(x, y) < 2^{-n+2}\} \subset U_{n-1}$ $(n = 1, 2, 3, ...)$.

For any positive number r, let

$W_r = \{(x, y) : x, y \in X \text{ and } d(x, y) < r\}$

and $\beta = \{W_r : r > 0\}$.

Then β is a base for some uniformity $\mathscr{V}$ on X. We verify that $\mathscr{V} = u$.

Let $W \in u$. Since $\{U_n\}_{n=0}^{\infty}$ is a base for the uniformity u, $U_{n-1} \subset W$ for some positive integer n. Choose a positive number r with $r < 2^{-n+2}$. Then from (1) we have

$W_r \subset U_{n-1} \subset W$.

This gives that $W \in \mathscr{V}$.

Next, let $W \in \mathscr{V}$. Then $W_r \subset W$ for some $r > 0$. Choose a positive integer n with $2^{-n+2} < r$. Then from (1) we have $U_n \subset W_r \subset W$ which gives that $W \in u$.

Hence $\mathscr{V} = u$. Therefore the uniform space (X, u) is pseudometrizable.

Definition : Let (X, u) be a uniform space. A subset E of X is said to be totally bounded if for every U in u there are finite number of points x_1 x_2, ...x_n in X such that

$$E \subset U_{i=1}^{n} \; U[x_i].$$

Example 3 : Every compact subset of a uniform space is totally bounded.

Solution : Let (X, u) be a uniform space and let E be a compact subset of X. Take any U in u. For $x \in E$, $U[x]$ is a neighbourhood of x. So there is an open set G_x with $x \subset G_x \subset U[x]$. Let $\mathscr{F} = \{G_x : x \in E\}$. Then $\mathscr{F}$ is an open cover of the set E. Since E is compact, there are finite number of open sets G_{x_1}, G_{x_2}, ..., G_{x_n} in $\mathscr{F}$ such that

$$E \subset U_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U[x_i]$. $(i = 1, 2, ..., n)$ we get

$$E \subset \cup_{i=1}^{n} U[x_i].$$

Hence E is totally bounded.

5.3 Cauchy nets and Cauchy filters : Completeness.

Let (X, u) be a uniform space. A net $\{S_n : n \in (D, \geq)\}$ in X is said to be a Cauchy net if for every U in u, there is an element n_0 in D such that

$$(X_m, x_n) \in U \text{ for all } m, n \text{ in } D \text{ with } m \geq n_0, n \geq n_0.$$

A filter $\mathscr{F}$ in X is said to be a Cauchy filter if for every U in u, there is a point p in X such that $U[p] \in \mathscr{F}$.

Completeness : A uniform space (X, u) is said to be complete if every Cauchy net in X is convergent.

Theorem 8 : A uniform space (X, u) is complete iff every Cauchy filter in X is convergent.

Proof : First suppose that the uniform space (X, u) is complete.

Take any Cauchy filter $\mathscr{F}$ in X. Let $\{s_A : A \in \mathscr{F}\}$ be a derived net of the filter $\mathscr{F}$.

Let U be any member of u. Choose a symmetric member V in u with $VoV \subset U$. Then there is a point x_0 in X such that $V[x_0] \in \mathscr{F}$.

Write $A_0 = V[x_0]$. Take any A, B in $\mathscr{F}$ with $A \subset A_0$, $B \subset A_0$. Then $s_A, s_B, \in A_0 = V[x_0]$.

So $(s_A, x_0) \in V$, and $(s_B, x_0) \in V$ which gives that $(s_A, s_B) \in V \circ V \subset U$. Thus $\{s_A : A \in \mathscr{F}\}$ is a Cauchy net in X. Since (X, u) is complete, $\{s_A : A \in \mathscr{F}\}$ is convergent.

Let $p = \lim_{\mathscr{F}} s_A$. Let $A \in \mathscr{F}$. Choose any B in $\mathscr{F}$ with $B \subset A$. Then $s_B \in B \subset A$ which gives that $p \in \overline{A}$. So p is a cluster point of $\mathscr{F}$. Since $\mathscr{F}$ is a Cauchy filter, $\mathscr{F}$ converges to p. [see Ex.4 page 18].

Next suppose that every Cauchy filter in X converges. Let $\{s_n : n \in (D, \geq)\}$ be a Cauchy net in X. Denote by $\mathscr{F}$ the derived filter of the net $\{s_n : n \in D\}$. Take any U in u. Then there is an element n_0 in D such that $(x_m, x_n) \in U$ for all m, n in D with $m \geq n_0$, $n \geq n_0$. In particular

$$\left(x_m, x_{n_0}\right) \in U \text{ for all } n \text{ in } D \text{ with } n \geq n_0.$$

$$\text{Or, } x_n \in U\left[x_{n_0}\right] \text{ for all } n \text{ in } D \text{ with } n \geq n_0.$$

This gives that $U\left[x_{n_0}\right] \in \mathscr{F}$. Thus $\mathscr{F}$ is a Cauchy filter in X. By our hypothesis $\mathscr{F}$ converges to a point x_0 in X.

This completes the proof of the theorem.

Exercise : Let (X, u) be a uniform space and let $\mathscr{F}$ be a Cauchy filter in X. If x_0 is a cluster point of $\mathscr{F}$, then $\mathscr{F}$ converges to x_0.

Solution : Let U be any member of $\mathscr{U}$. Choose a symmetric member V of u with $V o V o V \subset U$. Since $\mathscr{F}$ is a Cauchy filter in X, there is a point p in X such that $V[p] \in \mathscr{F}$.

Again, since x_0 is a cluster point of $\mathscr{F}$, $x_0 \in \overline{V[p]}$. Then $V[x_0] \cap V[p] \neq \phi$. Let $z \in V[x_0] \cap V[p]$. Then $(z, x_0) \in V$ and $(z, p) \in V$. Let $u \in V[p]$.

Then $(u, p) \in V$. From above three we see that $(u, x_0) \in V o V o V \subset U$; so $u \in U[x_0]$ which gives that $V[p] \subset U[x_0]$. Thus $U[x_0] \in \mathscr{F}$.

Therefore $\mathscr{F}$ converges to x_0.

Theorem 9 : A uniform space is compact iff it is totally bounded and complete.

Proof : Let (X, u) be a uniform space.

First suppose that X is compact. Let U be any member of u. Take any point x in X. Then $U[x]$ is a neighbourhood of x. So there is an open set G_x such that $x \in G_x \subset U[x]$. Let

$$G = \{G_x : x \in X\}.$$

Then G is an open cover of X. Since X is compact there are finite number of open sets $G_{x_1}, G_{x_2}, \ldots G_{x_n}$ in G such that.

$$X = \cup_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U[x_i]$, we have

$$X = \cup_{i=1}^{n} U[x_i].$$

So X is totally bounded.

Let $\mathscr{F}$ be a Cauchy filter in X. Since X is compact, $\mathscr{F}$ has a cluster point x_0 (say). So $\mathscr{F}$ converges to x_0. Hence the space X is complete.

Next, suppose that the space (X, u) is totally bounded and complete.

Let $\mathscr{F}$ be an ultrafilter in X. Take any member U in u. Since X is totally bounded, there are finite number of points $x_1, x_2, \ldots\ldots x_n$ in X such that

(1) $X = \cup_{i=1}^{n} U[x_i]$.

Since $\mathscr{F}$ is an ultrafilter and $X \in \mathscr{F}$, (1) implies that $U[x_i] \in \mathscr{F}$ for some i $(1 \le i \le n)$. So $\mathscr{F}$ is a Cauchy filter in X. Since X is complete, $\mathscr{F}$ is convergent. Hence the space X is compact.

This completes the proof of the theorem.

Definition : Let (X, u) and $(Y, \mathscr{V})$ be two uniform spaces and let $f : X \to Y$. The function f is said to be uniformly continuous if for every member V of $\mathscr{V}$, there is a member U in $\mathscr{V}$ such that

$(x', x'') \in U \Rightarrow (f(x'), f(x'')) \in V.$

Note : For any $f : X \to Y$, let us define $f_2 : X \times X \to Y \times Y$ as follows. For $(x', x'') \in X \times X, f_2 (x', x'') = (f(x^1), f(x''))$. Then the uniform continuity may be defined as follows : The function $f : X \to Y$ is said to be uniformly continuous if

$f_2^{-1} (V) \in u$ for every V in $\mathscr{V}$.

Theorem 10 : Let (X, u) and $(Y, \mathscr{V})$ be uniform spaces and let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof : Let V be any member of $\mathscr{V}$. Choose a symmetric member V_0 of $\mathscr{V}$ with $V_0 o V_0 \subset V$. Let $x \in X$. Since f is continuous, there is a symmetric member $W^{(x)}$ of u such that

(1) $u \in W^{(x)}[X] \Rightarrow f(u) \in v_0 \, [f(x)].$

Choose a symmetric member $U^{(x)}$ in u with $U^{(x)}$ o $U^{(x)} \subset W^{(x)}$. Since $U^{(x)}[X]$ is a neighbourhood of x, there is an open set G_x with $x \in G_x \subset U^{(x)} [x]$.

Let $G = \{G_x : x \in X\}$.

Then G is an open cover of X. Since X is compact, we can select finite number of open sets $G_{x_1}, G_{x_2}, .., G_{x_n}$ from the family G such that

$$X = \cup_{i=1}^{n} G_{x_i}$$

Since $G_{x_i} \subset U^{(x_i)}[x_i]$, we have

(2) $X = U_{i=1}^{n} \; U^{(x_i)}[x_i]$

Let $U = \cap_{i=1}^{n} U^{(x_i)}$. Then $U \in u$.

Take any two points x', x'' in X with $(x', x'') \in U$. From (2) we see that $x' \in U^{(x_i)}[x_i]$ for some i $(1 \leq i \leq n)$. Thus $(x', x_i) \in U^{(x_i)}$; also $(x', x'') \in U \subset U^{(x_i)}$.

So $(x'', x_i) \in U^{(x_i)}$ o $U^{(x_i)} \subset W^{(x_i)}$, that is, $x'' \in W^{(x_i)} [x_i]$. Therefore by (1)

$(f(x''), f(x_i) \in V_0$ and $(f(x'), f(x_i)) \in V_0$.

So $(f(x'), f(x'') \in V_0 \text{ o } V_0 \subset V$.

Hence f is uniformly continuous.

Theorem 11 : Let X be a nonempty set and let $\{(Y_a, \mathscr{V}_a) : a \in A\}$ be a family of uniform spaces and for each $a \in A$, $f_a : X \to Y_a$. Then there exists a smallest uniformity u on X relative to which the functions f_a are uniformly continuous.

Proof : We prove the theorem by the following steps.

(I) Let $s = \{(f_a)_2^{-1}(U_a) : U_a \in \mathscr{V}_a \text{ and } a \in A\}$.

We first verify that s is a subbase for some uniformity U on X. Let $\Delta = \{(x, x) : x \in X\}$ and $\Delta_a = \{(y_a, y_a) : y_a \in Y_a\}$.

(i) Let $U \in s$. Then $U = (f_a)_2^{-1} (U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. Take any $x \in X$. Then $(f_a(x), f_a(x)) \in \Delta_a$ i.e. $(f_a)_2 (x, x) \in \Delta_a \subset U_a$.

So $(x, x) \in (f_a)_2^{-1} (U_a)$ and hence $\Delta \subset U$.

(ii) Let $U \in s$. Then $U = (f_a)_2^{-1} (U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. There is a member V_a in $\mathscr{V}_a$ with $V_a \circ V_a \subset U_a$.

Write $V = (f_a)_2^{-1} (V_a)$.

Let $(x, y) \in V \circ V$. Then there is an element z in X such that $(z, y) \in V$ and $(x, z) \in V$. This gives that $(f_a)_2 (z, y) \in V_a$ and $(f_a)_2 (x, z) \in V_a$.

i.e $(f_a(z), f_a(y)) \in V_a$ and $(f_a(x), f_a(z)) \in V_a$.

$\Rightarrow (f_a(x), f_a(y)) \in V_a \circ V_a \subset U_a$

i.e. $(f_a)_2 (x, y) \in U_a$

$\Rightarrow (x, y) \in (f_a)_2^{-1} (U_a) = U$.

$\Rightarrow V \circ V \subset U$

(iii) Let $U \in s$. Then $U = (f_a)_2^{-1} (U_a)$ for some $a \in A$ and $U_a \in \mathscr{V}_a$. Since $U_a^{-1} \in \mathscr{V}_a$ we have

$$W = (f_a)_2^{-1} (U_a^{-1}) \in s.$$

Let $(x, y) \in W$. Then $(f_a)_2 (x, y) \in U_a^{-1}$.

i.e. $(f_a(x), f_a(y)) \in U_a^{-1}$

$\Rightarrow (f_a(y), f_a(x)) \in U_a$

$\Rightarrow (y, x) \in (f_a)_2^{-1} (U_a) = U$

$\Rightarrow (x, y) \in U^{-1}$

$\Rightarrow W \subset U^{-1}$.

From (i), (ii) and (iii) we see that s is a subbase for some uniformity u on X.

(II) Let $a \in A$, consider the function f_a . Take any member V_a in $\mathscr{V}_a$. Then $U = (f_a)_2^{-1} (V_a) \in u$.

Let $(x, y) \in U$. Then $(f_a)_2 (x, y) \in V_a$ i.e. $(f_a(x), f_a(y)) \in V_a$.

Hence f_a is uniformly countinuous.

(III) Let $\tilde{u}$ be any uniformity on X relative to which the function f_a $(a \in A)$ are uniformly continuous.

Let $U \in u$. Then there is a set of the form

$W = \cap_{i=1}^{n} (f_{a_i})_2^{-1} (V_{a_i}) (a_i \in A, V_{a_i} \in \mathscr{V}_{a_i})$ such that $W \subset U$. Since the functions $fa_1, fa_2, \ldots f_{a_n}$ are uniformly countinous relative to the uniformity $\tilde{u}$, there are members $w_1, w_2, \ldots w_n$ in $\tilde{u}$ such that

$(x, y) \in W_i \Rightarrow (f_{a_i}(x), f_{a_i}(y)) \in V_{a_i}$ $(i = 1, 2, \ldots, n)$. Write $\tilde{W} = \cap_{i=1}^{n} W_i$. Let $(x, y) \in \tilde{W}$. Then $(x, y) \in W_i$ and so $(f_{a_i}(x), f_{a_i}(y)) \in V_{a_i}$

$$\Rightarrow (f_{a_i})_2 (x, y) \in V_{a_i} \Rightarrow (x, y) \in \bigcap_{i=1}^{n} (f_{a_i})_2^{-1} (V_{a_i}) = W \subset U.$$

So $\tilde{W} \subset U$. which gives that $U \in \tilde{u}$.

Hence $u \subset \tilde{u}$ and the proof is complete.

5.4. Proximity Spaces :

Definition : Let X be a nonempty set and let δ be a relation on the power set $P(X)$ of the set X. Suppose that δ satisfies the following axioms.

(1) For A, B in $P(X)$, $A\ \delta\ B \Rightarrow B\ \delta\ A$.

(2) Let A, B, C be in $P(X)$. Then $(A \cup B)\delta\ C$ iff $A\ \delta\ C$ or $B\ \delta\ C$.

(3) For A, B in $P(X)$, $A\ \delta\ B \Rightarrow A \neq \phi$ and $B \neq \phi$.

(4) Let A, B be in $P(X)$. Then $A \cap B \neq \phi \Rightarrow A\ \delta\ B$.

(5) Let A, B be in $P(X)$. If $A \bar{\delta} B$, there exists a subset E of X such that $A \bar{\delta} E$ and $\bar{E} \bar{\delta} B$, where $\bar{E}$ denotes the complement of E and $\bar{\delta}$ denotes the negation of δ.

Then δ is called a proximity on X and the pair (X, δ) is called a proximity space.

Example 1. Let (X, d) be a metric space. For two subsets A, B of X, let $d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}$.

Now define the relation δ on the power set $P(X)$ of the set X as follows : For A, B in $P(X)$, let

$$A \delta B \text{ hold iff } d(A, B) = 0$$

Then δ is a proximity on the set X.

Solution : (1) Since $d(y, x) = d(x, y)$ for all x, y in X, it follows that $d(B, A) = d(A, B)$ for A, B in $P(X)$. Let A, B be in $P(X)$ and $A\ \delta\ B$. Then $d(A, B) = 0$. So $d(B, A) = 0$ which gives that $B\ \delta\ A$.

(2) Let A, B, C be in $P(X)$. Suppose that $(A \cup B)\delta C$. Then $d(A \cup B, C) = 0$. Let $A \bar{\delta} C$. Then $d(A, C) = r > 0$. Choose any ε with $0 < \varepsilon < r$.

Since $d(A \cup B, C) = 0$, there is a point x in $A \cup B$ and a point Z in C such that

$$d(x, Z) < \varepsilon \ \dots \quad \dots \quad \dots \quad (*)$$

If $x \in A$, then $d(x, Z) \geq d(A, C) = r > \varepsilon$. This contradicts (*). So $x \in B$ and $d(B, C) \leq d(x, Z) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary it follows that $d(B, C) = 0$. So $B\ \delta\ C$.

Next, let $A\ \delta\ C$. Then $d(A, C) = 0$. Choose any $\varepsilon > 0$. Then there is a point x in A and a point Z in C such that $d(x, Z) < \varepsilon$. Since $x \in A \cup B$, we have $d(A \cup B, C) \leq d(x, Z) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $d(A \cup B, C) = 0$. This gives that $(A \cup B)\ \delta\ C$.

If $B \delta C$, as above we can show that $(A \cup B)\ \delta\ C$. Thus $(A \cup B)\ \delta\ C$ iff $A\ \delta\ C$ or $B \delta C$.

(3) Let A, B be in $P(X)$ and $A \delta B$. Choose any $\varepsilon > 0$. Then there is a point x in A and a point y in B with $d(x, y) < \varepsilon$. This gives that $A \neq \phi$ and $B \neq \phi$.

(4) Let A, B be in $P(X)$ and $A \cap B \neq \phi$. Take any $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $d(A, B) \leq d(x, x)$, we have $d(A, B) = 0$. So $A \ \delta \ B$.

(5) Let A, B be in $P(X)$ and let $A \bar{\delta} B$. Then

$$d(A, B) = r > 0 \quad \ldots \quad \ldots \quad \ldots (**)$$

Let $E = \left\{ y : y \in X \text{ and } d(y, B) < \frac{1}{3} r \right\}$

Assume that $A \ \delta \ E$. Then $d(A, E) = 0$. So there is a point x in A and a point y in E with $d(x, y) < \frac{1}{3} r$. Since $d(y, B) < \frac{1}{3} r$, there is a point Z in B such that $d(y, Z) < \frac{1}{3} r$.

We have $d(x, Z) \leq d(x, y) + d(y, Z) < \frac{1}{3} r + \frac{1}{3} r = \frac{2}{3} r$. Since $d(A, B) \leq d(x, z)$, $d(A, B) \leq \frac{2}{3} r$.

This contradicts (**). Hence $A \bar{\delta} E$. Next, let $\tilde{E} \delta B$. Then $d(\tilde{E}, B) = 0$. So there is a point x in $\tilde{E}$ and a point y in B such that $d(x, y) < \frac{1}{3} r$. Since $d(x, B) \leq d(x, y)$, we have $d(x, B) < \frac{1}{3} r$ which gives that $x \in E$. This contradicts the fact that $x \in \tilde{E}$.

Hence $\tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X.

Example 2 : Let $(X, \mathscr{U})$ be a uniform space and let the relation δ be defined on $P(X)$ as follows : For A, B in $P(X)$, let $A \ \delta \ B$ if $(A \times B) \cap U \neq \phi$ for every U in $\mathscr{U}$. Then δ is a proximity on X.

Solution :

(i) Let A, B be in $P(X)$ and let $A \ \delta \ B$. Take any U in $\mathscr{U}$. Then $U^{-1} \in \mathscr{U}$ and so $(A \times B) \cap U^{-1} \neq \phi$. This gives that there is a point x in A and *a* point y in B such that $(x, y) \in U^{-1}$ or $(y, x) \in U$. So $(B \times A) \cap U \neq \phi$. Hence $B \delta A$.

(ii) Let A, B, C be in $\rho(X)$. Suppose that $(A \cup B) \ \delta \ C$. Assume that $A \bar{\delta} C$. Then there is a member V in $\mathscr{U}$ with $(A \times C) \cap V = \phi$. Take any U in $\mathscr{U}$. Write $W = U \cap V$. Then $W \in \mathscr{U}$. Since $(A \cup B) \delta C$, $[(A \cup B) \times C] \cap W \neq \phi$. This gives that there is a point x in $A \cup B$ and a point z in C such that $(x, Z) \in W$. If $x \in A$, then $(x, Z) \in A \times C$ and so $(x, Z) \in (A \times C) \cap W$

$\subset (A \times C) \cap V$ which condradicts the fact that $(A \times C) \cap V = \phi$. So $x \in B$. Hence $(x, Z) \in B \times C$; and $(x, Z) \in (B \times C) \cap U$. This gives that $(B \times C) \cap U \neq \phi$ and $B\ \delta\ C$.

Thus $(A \cup B) \delta C \Rightarrow$ either $A \delta C$ or $B\ \delta\ C$. Let $A\ \delta\ C$. Take any $U \in \mathcal{U}$. Thus $(A \times C) \cap U \neq \phi$. This gives that $[(A \cup B) \times C] \cap U \neq \phi$. So $(A \cup B) \delta C$. Similarly $B\ \delta\ C \Rightarrow (A \cup B) \delta C$.

(iii) Let A, B be in $P(X)$ and let $A\ \delta\ B$. Take any U is $\mathcal{U}$. Then $(A \times B) \cap U \neq \phi$. So there is a point x in A and a point y in B such that $(x, y) \in U$. This gives that $A \neq \phi$ and $B \neq \phi$.

(iv) Let A, $B \in \rho(X)$ and $A \cap B \neq \phi$. Take any point $x \in A \cup B$. Then $x \in A$ and $x \in B$. Let $U \in \mathcal{U}$. Since $(x, x) \in U$, we have $(A \times B) \cap U \neq \phi$. So $A\ \delta\ B$.

(v) Let A, $B \in P(X)$ and let $A \bar{\delta} B$. Thus there is a member U_0 in $\mathcal{U}$ such that

$(A \times B) \cap U_0 = \phi$ ... ... (1)

Choose a symmetric member V in $\mathcal{U}$ with $V_0 V \subset U_0$. Let

$E = \{y : y \in X$ and $(y, z) \in V$ for some Z in $B\}$. Assume that $A \delta E$. Then there is a point x in A and *a* point y in E with $(x, y) \in V$. Aslo from the definition of E, there is a point Z in B such that $(y, Z) \in V$. This gives that $(x, Z) \in V_0 V \subset U_0 \Rightarrow (A \times B) \cap U_0 \neq \phi$ which contradicts (1). Hence $A \bar{\delta} E$.

Again, assume that $\tilde{E} \delta B$. Then $(\tilde{E} \times B) \cap V \neq \phi$. So there is a point y in $\tilde{E}$ and a point Z in B such that $(y, Z) \in V$ which implies that $y \in E$. This contradicts the fact that $y \in \tilde{E}$. Hence $\tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X.

Lemma 1 : Let (X, δ) be a proximity space and let A, B, C, D be subsets of X.

(i) If $A\ \delta\ B$ and $A \subset C$ & $B \subset D$, then $C\ \delta\ D$.

(ii) If $x \in X$ and $A \delta x$ & $x \delta B$, then $A \delta B$.

(iii) If $A \bar{\delta} B$, then $\bar{A} \bar{\delta} B$, $A \bar{\delta} \bar{B}$ and $\bar{A} \bar{\delta} \bar{B}$, where $\bar{A} = \{y : y \in X$ and $y\ \delta\ A\}$.

(iv) $A \delta B$ if $\bar{A} \delta \bar{B}$, where $\bar{A}$ is defined as in (iii).

(v) If $A \bar{\delta} B$, then $\bar{B} \subset X \setminus A$ and $\bar{A} \subset X \setminus B$.

Proof : (i) Suppose that $A\ \delta\ B$ and $A \subset C$ & $B \subset D$. We have $C = A \cup (C \setminus A) =$

$A \cup E$, where $E = C \setminus A$. Since $A\delta B$, we get $(A \cup E)\ \delta\ B$ i.e., $C\delta B$. Again, since $D = B \cup (D \setminus B)$, we get $C\delta D$.

(ii) Suppose that there is a point x is X with

$A\delta x$ and $x\delta B$ ... ... (1)

Assume that $A\bar{\delta}B$. Then there is a subset E of X such that

$A\bar{\delta}E$ and $\tilde{E}\,\bar{\delta}B$ ... ... (2)

where $\tilde{E} = X \setminus E$.

If $x \in E$, then by (1) and (i) we have $A\delta E$ which contradicts (2). Again, if $x \in \tilde{E}$, then (1) and (i) imply that $\tilde{E}\delta B$ which also contradicts (2). Hence $A\delta B$.

(iii) Suppose that $A\bar{\delta}B$. Thus there is a subset E of X such that

$A\bar{\delta}E$ and $\tilde{E}\bar{\delta}B$ ... ... (3)

(a) Let $y \in \bar{B}$. Then $y\delta B$. If $y \in \tilde{E}$, then $\tilde{E}\delta y$ and so by (ii) $\tilde{E}\delta B$ which contradicts (3). So $y \in E$; this gives that $\bar{B} \subset E$. This with $A\bar{\delta}E$ and (i) imply that $A\bar{\delta}\bar{B}$.

Let $y \in \bar{A}$. Then $y\delta A$ and so $A\delta y$.

(b) If $y \in E$, then $y\delta E$ and by (ii) $A\delta E$ which contradicts (3). So $y \in \tilde{E}$; this gives that $\bar{A} \subset \tilde{E}$. If $\bar{A}\delta B$, then by (i) we get $\tilde{E}\delta B$ which contradicts (3). Hence $\bar{A}\bar{\delta}B$. Step (iiia) implies that $\bar{A}\bar{\delta}\bar{B}$.

(iv) Suppose that $A\delta B$. Clearly $A \subset \bar{A}$ and $B \subset \bar{B}$. So by (i) $\bar{A}\delta\bar{B}$.

Next, suppose that $\bar{A}\delta\bar{B}$. ... ... (3a)

Assume that $A\bar{\delta}B$. Then by (iii) $\bar{A}\bar{\delta}\bar{B}$.

This contradicts (3a). Hence $A\delta B$.

(v) Suppose that $A\bar{\delta}B$. ... ... (4)

Let $y \in \bar{A}$. Then $y\delta A$ and so $A\delta y$. If $y \in B$, then by (ii) $A\delta B$ which contradicts (4). So $y \in X \setminus B$. Hence $\bar{A} \subset X \setminus B$.

Let $y \in \overline{B}$. Then $y\delta B$. If $y \in A$, then $A\delta y$ and by (ii) $A\delta B$ which contadits (4). So $y \in X\backslash A$. Hence $\overline{B} \subset X \setminus A$.

Theorem 1 : Let (X, δ) be a proximity space and for any subset A of X let

$$C(A) = \{y : y \in X \text{ and } y\delta A\}.$$

Then C is the Kuratowski closure operator on X.

Proof : (i) Let A be any subset of X. Take any x in A. There $\{x\} \cap A \neq \phi$; so $x \ \delta \ A$ which gives that $x \in C(A)$. Hence $A \subset C(\text{A})$.

Clearly $C(\phi) = \phi$ and $C(X) = X$

(ii) Let A, B be two subsets of X and $A \subset B$. Take any $x \in C(A)$. Then $x \ \delta \ A$; so $x \ \delta \ B$. This gives that $x \in C(B)$.

Hence $C(A) \subset C(B)$.

(iii) Let A, B be any two subsets of X.

Take any $x \in C(A) \cup C(B)$.

Then $x \in C(A)$ or $x \in C(\text{B})$

This gives that $x \ \delta \ A$ or $x \ \delta \ B$ and so

$x\delta(A \cup B) \Rightarrow x \in C(A \cup B)$.

Hence $C(A) \cup C(B) \subset C(A \cup B)$. ... ... (1)

Next, let $x \in C(A \cup B)$. Then $x \ \delta(A \cup B) \Rightarrow$

either $x\delta A$ or $x\delta B$. This gives that $x \in C(\text{A})$ or $x \in C(B) \Rightarrow x \in C(A) \cup C(B)$.

So, $C(A \cup B) \subset C(A) \cup C(B)$ ... (2)

From (1) and (2) we have

$C(A \cup B) = C(A) \cup C(B)$.

(iv) Let A be any subset of X.

If $A = \phi$, then $C(A) = C(\phi) = \phi$ and so

$C(C(A)) = C(\phi) = \phi = A$

Suppose that $A \neq \phi$. By (i) we have

$$C(A) \subset C(C(A)). \quad \dots \dots \quad (3)$$

Let $y \in C(C(A))$. Then

$$y \delta C(A) \quad \dots \quad \dots (4)$$

Assume that $y \notin C(A)$. By the definition of $C(A)$ we get $y \bar{\delta} A$ and so $y \bar{\delta} C(A)$ which contradicts (4). So $y \in C(A)$ and

$$C(C(A)) = C(A).$$

Note : Let (X, δ) be a proximity space. For any subset A of X let $C(A) = \{y : y \in X$ and $y \ \delta \ A\}$. Then by theorem 1, C is the kuratowski closure operator on X. This closure operator C induces a topology $\tau_{(\delta)}$ on X. We say that the proximity δ induces the topology $\tau_{(\delta)}$ on X, and the topology $\tau_{(\delta)}$ is compatible with the proximity δ.

Theorem 2 : Let (X, τ) be a completely regular space. Then there exists a proximity δ on X compatible with the topology τ.

Proof : Since (X, τ) is completely regular, it is uniformisable. So there is a uniformity $\mathscr{U}$ on X such that the topology induced by the uniformity $\mathscr{U}$ is identical with the topology τ.

Now define the relation δ on $P(X)$, the power set of X, as follows :

For A, B in $P(X)$, $A \ \delta \ B$ if $(A \times B) \cap \mathscr{U} \neq \phi$ for every U in $\mathscr{U}$. Then δ is a proximity on X. The proximity δ induces a topology $\tau_{(\delta)}$ on X. Let A be any subset of X. Denote by $\bar{A}$ and $C(A)$ respectively the τ-closure and $\tau_{(\delta)}$-closure of the set A.

Let $x \in \bar{A}$. Take any $U \in \mathscr{U}$. Thus $U[x]$ is a neighd of x. So $A \cap U[x] \neq \phi$. This gives that there is a point y in A such that $y \in U[x]$ i.e. $(x, y) \in U$.

So, $(x, y) \in (\{x\} \times A) \cap U$

i.e. $(\{x\} \times A) \cap U \neq \phi \Rightarrow x \ \delta \ A$.

Hence $x \in C(A) \Rightarrow \bar{A} \subset C(A)$.

Next, let $x \in C(A)$.

Then $x \,\delta\, A \Rightarrow (\{x\} \times A) \cap U \neq \phi$ for any $U \in \mathscr{U}$

Let $U \in \mathscr{U}$. Then there is a point y in A such that $(x, y) \in U \Rightarrow y \in U[x]$

$\Rightarrow A \cap U[x] \neq \phi \Rightarrow x \in \overline{A}$

So, $C(A) \subset \overline{A}$

Thus $C(A) = \overline{A}$

This gives that $\tau_{(\delta)} = \tau$

Hence δ is complatible with τ.

Theorem 3 : Let (X, τ) be a T_4 space and let δ be a relation on the power set $P(X)$ defined as follows. For A, B in $\rho(X)$, $A \,\delta\, B$ iff $\overline{A} \cap \overline{B} \neq \phi$. Then δ is a proximity on X compatible with τ.

Proof : We first verify that δ is a proximity on X. (i) Let $A, B \in P(X)$ and $A \,\delta\, B$. Then $\overline{A} \cap \overline{B} \neq \phi$. Since $\overline{B} \cap \overline{A} = \overline{A} \cap \overline{B} \neq \phi$, we get $B \,\delta\, A$.

(ii) Let A, B, C be in $P(X)$.

Suppose that $(A \cup B) \,\delta\, c$.

Then $\overline{A \cup B} \cap \overline{C} \neq \phi$ i.e. $(\overline{A} \cup \overline{B}) \cap \overline{C} \neq \phi$

i.e. $(\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \neq \phi$

This gives that either $\overline{A} \cap \overline{C} \neq \phi$ or $\overline{B} \cap \overline{C} \neq \phi$.

so either $A \,\delta\, C$ or $B \,\delta\, C$.

Next, suppose that $A \,\delta\, C$. Then $\overline{A} \cap \overline{C} \neq \phi$.

This gives that $(\overline{A} \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \neq \phi$.

ie. $(\overline{A} \cup \overline{B}) \cap \overline{C} \neq \phi$

or, $\overline{(A \cup B)} \cap \overline{C} \neq \phi$

So, $(A \cup B) \,\delta\, C$.

If $B \,\delta\, C$, we can show that $(A \cup B) \,\delta\, C$.

(iii) Let $A, B \in P(X)$ and let $A \,\delta\, B$. Then $\overline{A} \cap \overline{B} \neq \phi$. If A = ϕ, then $\overline{A} = \phi$ and so $\overline{A} \cap \overline{B} = \phi$ which is a contradiction. Hence $A \neq \phi$. Similarly $B \neq \phi$.

(iv) Let $A, B \in P(X)$ and $A \cap B \neq \phi$. This gives that $\overline{A} \cap \overline{B} \neq \phi$ and so $A \,\delta\, B$.

(v) Let $A, B \in P(X)$ and $A \bar{\delta} B$. then $\overline{A} \cap \overline{B} = \phi$.

Since (X, τ) is a T_4 space it is normal. So there are open sets G_1, G_2 such that $\overline{A} \subset G_1$, $\overline{B} \subset G_2$ and $G_1 \cap G_2 = \phi$.

Write $E = X \backslash G_1$. Then E is a closed set. We have $\overline{A} \cap \overline{E} = \overline{A} \cap E = \phi$.

So $A \bar{\delta} E$.

Again, since $E \supset G_2 \supset \overline{B}$, $\tilde{E} \subset X | G_2 \subset X | \overline{B}$ and so $\mathrm{cl}(\tilde{E}) \subset X | G_2 \subset X | \overline{B}$, where $\tilde{E} = X | E$.

This gives that cl $(\tilde{E}) \cap \overline{B} = \phi \Rightarrow \tilde{E} \bar{\delta} B$.

Therefore δ is a proximity on X. So it induces a topology τ_δ on X.

Let A be any subset of X. Denote by $\overline{A}$ and $C(A)$ respectively the τ-closure and τ_δ-closure of A.

$x \in C(A) \Rightarrow x \,\delta\, A \Rightarrow \overline{\{x\}} \cap \overline{A} \neq \phi \Rightarrow \{x\} \cap \overline{A} \neq \phi$. $\left[\because \overline{\{x\}} = \{x\}\right]$

$\Rightarrow x \in \overline{A}$.

Again, $x \in \overline{A} \Rightarrow \{x\} \cap \overline{A} \neq \phi \Rightarrow \overline{\{x\}} \cap \overline{A} \neq \phi$

$\Rightarrow x \,\delta\, A \Rightarrow x \in C(A)$.

Hence $C(A) = \overline{A}$. $\Rightarrow \tau_\delta = \tau$.

Therefore δ is compatible with τ.

Lemma 2 : Let (X, δ) be a proximity space and let A, B be any two subsets of X.

(i) $A \bar{\delta} X \backslash B \Rightarrow A \bar{\delta} X \backslash \mathrm{int}(B)$

(ii) $A \bar{\delta} X \backslash B \Rightarrow \overline{A} \subset \mathrm{int}(B)$.

(iii) Let $A\bar{\delta}X\backslash B$. Then there is a subset C of X such that $A\bar{\delta}X\setminus C$ and $C\bar{\delta}X\backslash B$, where closure and interior are taken with respect to $\tau_{(\delta)}$.

Proof : (i) Let $A\bar{\delta}X\backslash B$. Then there is a subset E of X such that

$$A\bar{\delta}E \text{ and } \tilde{E}\bar{\delta}(X\setminus B) \;\dots\;\dots (1)$$

where $\tilde{E}=X\setminus E$.

Let $x \in \mathrm{cl}(X\backslash B)$. Then $x\delta(X\backslash B)$. If $x\in\tilde{E}$, then $\tilde{E}\delta x$ and so $\tilde{E}\delta(X\setminus B)$ which contradicts (1).

Hence $x\notin\tilde{E}\Rightarrow x\in E$; so $\mathrm{cl}(X\backslash B)\subset E$.

This with (1) implies that $A\bar{\delta}\overline{(X\setminus B)}$.

Now $\overline{X\setminus B} = X\backslash\,\mathrm{int(B)}$

So $A\bar{\delta}[X\setminus \mathrm{int}(B)]$ ... ... (2)

(ii) Let $A\bar{\delta}(X\setminus B)$. Then (2) holds. Take any x in $\bar{A}$. This with (2) implies that

$x\notin[X\setminus\mathrm{int}(B)]$; so $x\in\mathrm{int}(B)$.

Hence $\bar{A}\subset \mathrm{int}(B)$.

(iii) Let $A\bar{\delta}(X\setminus B)$. Then there is a subset E of X such that (1) holds. Write $C=X\backslash E$.

Then $C=\tilde{E}$ and $E = X\backslash C$. From (1) we have $A\bar{\delta}(X\setminus C)$ and $C\bar{\delta}(X\setminus B)$.

Theorem 4 : Let (X, δ) be a proximity space. Then the topology τ_δ is completely regular.

Proof : Let A be a closed set in X with respect to the topology τ_δ and $x\in X\backslash A$. Write $U_0 = X\backslash A$. Then U_0 is an open set and $x\in U_0$.

Since $x\notin A=\bar{A}$, $x\bar{\delta}(X\setminus U_0)$. $[\because A = X\backslash U_0]$

So there is a set $E\subset X$ such that

$$x\bar{\delta}(X\setminus E) \text{ and } E\bar{\delta}(X\setminus U_0) \;\dots\quad\dots\quad (1).$$

Write $U_{1/2} = \text{int}(E)$. Then $U_{1/2}$ is an open set. By (1) and Lemma 2,

$$x\bar{\delta}\ (X \backslash U_{1/2}) \text{ and } U_{1/2}\ \bar{\delta}\ (X \backslash U_0)$$

There are subsets E_1 and E_2 of X such that $x\bar{\delta} X\backslash E_1$, $E_1 \bar{\delta}\ X\backslash U_{1/2}$ and $U_{1/2}\ \bar{\delta}\ X\backslash E_2$, $E_2 \bar{\delta}\ X\backslash U_0$ ... (2)

Write $U_{1/4} = \text{int}(E_1)$ and $U_{3/4} = \text{int}(E_2)$.

Then $U_{1/4}$ and $U_{3/4}$ are open sets. By (2) and Lemma 2 we get.

$x\bar{\delta}\ (X\backslash U_{1/4})$, $U_{1/4}\ \bar{\delta} (X\backslash U_{1/2})$ and

$$U_{1/2}\ \bar{\delta} (X\backslash U_{3/4}),\ U_{3/4}\ \bar{\delta}\ (X\backslash U_0) \ \ldots \quad \ldots \ (3)$$

So, $x \in U_{1/4} \subset \bar{U}_{1/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{3/4} \subset \bar{U}_{3/4} \subset U_0.$

Denote by D the set of numbers of the form

$$\frac{m}{2^n} : m = 1,3,5,\ldots,2^{n-1} \text{ and } n = 1,\ 2,\ 3,\ \ldots..$$

Then D is dense in $[0,1]$.

Proceeding as above we can select a family of open sets $\{U_t : t \in D\}$ such that if t, $s \in D$ and $t < s$, then

$$x \in U_t \subset \bar{U}_t \subset U_s \subset \bar{U}_s \subset U_0.$$

We now define the functions f on X as follows :

Let $z \in X$. $f(z) = 0$ if $z \in \cap\{U_t : t \in D\}$

$= 1$ if $z \in X\backslash U_0$

$= \inf\ \{t : z \in U_t\}$ otherwise.

Clearly values of f lie in the closed interval $[0, 1]$ and $f(x) = 0$ and $f(z) = 1$ for all $z \in A$.

Now we prove that f is continuous.

The family of all intervals $[0, a)$, $(b, 1]$ $(0 < a, b < 1)$ forms a subbase for the topology on $[0, 1]$. It is easy to see that

$$f(z) < a \Leftrightarrow z \in U_t \text{ for some } t \text{ in } D \text{ with } t < a.$$

This gives that $f^{-1}([0, a)) = \cup\{U_t : t \in D \text{ and } t < a\}$.

So $f^{-1}([0, a))$ is open. Again,

$f(z) > a \Leftrightarrow z$ lies outside of $\bar{U}_t$ for some t in D with $t > a$.

So $f^{-1}((a, 1]) = \cup \{X \setminus \bar{U}_t : t \in D \text{ and } t > a\}$.

Which is open.

This gives that f is continuous.

Hence the space (X, τ_δ) is completely regular. This proves the theorem.

Theorem 5 : Let (X, τ) be a completely regular space and let it have a compatible proximity δ defined as follows : For any two subsets A, B of X.

$A \,\delta\, B$ iff $\bar{A} \cap \bar{B} \neq \phi$. Then (X, τ) is normal.

Proof : Let A and B be any two disjoint closed subsets of X. Then $A \bar{\delta} B$. So there is a subset E of X such that

$A \bar{\delta} E$ and $\tilde{E} \bar{\delta} B$ ... ... (1)

Where $\tilde{E} = X \setminus E$.

By Lemma 2, we have

$A \subset \text{int}(X \setminus E)$ and $B \subset \text{int}(X \setminus \tilde{E}) = \text{int } E$. .... (2)

Write $G_1 = \text{int}(X \setminus E)$ and $G_2 = \text{int}(E)$.

Then G_1 and G_2 are open. Clearly $G_1 \cap G_2 = \phi$.

By (2) $A \subset G_1$ and $B \subset G_2$.

Hence the space (X, τ) is normal.

Exercise : Prove that the interior of a set. $A \subset X$ endowed with a uniformity $\mathscr{V}$ is

$\mathring{A} = B = \{x \in X : V(x) \subset A \text{ for some } V \in \mathscr{V}\}$

Solution : Since every open set $G \subset A$ is contained in B, so it is sufficient to prove that the set B is open. Take any $x \in B$. Then there is a $V \in \mathscr{V}$ such that $V(x) \subset A$. Choose $W \in \mathscr{V}$ such that $W \circ W \subset V$. Note that for any $y \in W(x)$, if $z \in W(y)$ then $(x, y) \in W$ and $(y, z) \in W$ and so $(x, z) \in W \circ W \subset V \Rightarrow Z \in V(x)$. Thus

$W(y) \subset V(x) \subset A$.

Since this is true for every $y \in W(x)$ this shows that $W(x) \subset B$. So x is an interior point of B. As x is arbitrary, this proves that B is open.

Exercise : Let X be a Tychonoff space. Let $C(X)$ and $C^*(X)$ denote the family of all real valued continuous functions and real valued continuous bounded functions on X respectively. For every finite number of functions $f_1, f_2, ..., f_k \in C(X)$ (or $C^*(X)$)

$$d_{f_1, f_2, ..., f_k}(x,y) = \max_{1 \le i \le k} \{| f_i(x) - f_i(y)|\}$$

define two pseudometrics on X.

Solution : It is easy to note that

$$d_{f_1, f_2 ..., f_k}(x,x) = \max\{|f_1(x) - f_1(x)|, ..., | f_k(x) - f_k(x)|\}$$

$$= 0 \quad \forall x \in X.$$

If $x, y \in X$ then since $|f_i(x) - f_i(y)| = |f_i(y) - f_i(x)|$

for $i = 1, 2, .., k$ so it follows that

$$d_{f_1, f_2 ... f_k}(x,y) = d_{f_1, f_2 ... f_k}(y,x)$$

Finally if $x, y, z \in X$ then we have

$$|f_i(x) - f_i(z)| \le | f_i(x) - f_i(y)| + | f_i(y) - f_i(z)|$$

for $i = 1, 2, ...k$. Hence

$$d_{f_1, f_2 ... f_k}(x,z) \le d_{f_1, f_2, ..., f_k}(x,y) + d_{f_1 ... f_k}(y,z).$$

Exercise : Let P and P^* denote the families of pseudometrics on X defined as in the preceeding exercise. Consequently they generate uniformities $\mathscr{V}$ and $\mathscr{V}^*$ on X. Prove that $\mathscr{V}$ and $\mathscr{V}^*$ induce the same topology identical with the initial topology.

Solution : Since any $f_1, f_2, .., f_k \in C(X)$ (or $C^*(X_i)$) are continuous and the modulus function is continuous so every generated pseudometric

$d_{f_1, f_2 ... f_k}(x,y) = \max \{|f_1(x) - f_1(y)|, .., |f_k(x) - f_k(y)|\}$ is a continuous function from $X \times X \to R$. Thus every element of P (or P^*) is a continuous function from $X \times X \to R$. Hence the sets $\{(x, y) : d(x, y) < z^{-i}\}$ where $d \in P$ and $i \in N$ are open in $X \times X$ and every open set in the topology induced by $\mathscr{V}$ or $\mathscr{V}^*$ is open in the initial topology on X.

Now let us suppose that U is open in the initial topology in X. Let $x_0 \in U$. Since

X is a Tychonoff space, there is a continuous function f, i.e. $f \in C^*(X) \subset C(X)$. Such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in X \setminus U$. Let

$$V = \left\{(x,y) : d_f(x,y) < \frac{1}{2}\right\}.$$

Then we have $V(x_0) \subset U$ and this implies that U is open in the topology induced by $\mathscr{V}$ or $\mathscr{V}^*$.

Exercise : Let X be a Tychonoff space, $C^*(X)$ denote the family of real valued bounded continuous functions on X, p^* the family of generated pseudometrics. Then $(X, \mathscr{V}^*)$ is totally bounded where $\mathscr{V}^*$ is the uniformity generated by p^*.

Solution : It is sufficient to prove that for every system of functions $f_1, f_2, \ldots, f_k, \ldots \in C^*(X)$ and $\in > 0$, there exists a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that for every $x \in X$, there exists an $i \leq n$ with the property.

$$d_{f_1 f_2 \ldots f_k}(x, x'_k) = \max\{|f_1(x) - f_1(x_i)|, \ldots, |f_k(x) - f_k(x'_i)|\} < \in.$$

Since $f_1, f_2 \ldots, f_k \in C^*(X)$ so $f_1(X), f_2(X), \ldots, f_k(X)$ are all bounded sets in R and so we can find a bounded closed interval $J \subset R$ which contains $f_1(X), \ldots, f_k(X)$. Note that J is totally bounded and so we can find a finite number of open intervals $\{A_j\}_{j=1}^m$ of diameter less than $\in$ which cover J. Subsequently the family of sets of the form

$$f_1^{-1}(A_{j_1}) \cap f_2^{-1}(A_{j_2}) \cap \ldots \cap f_k^{-1}(A_{j_k}), \quad \ldots\ldots\ldots(1)$$

where $1 \leq j_i \leq m$ for every $i \leq k$ is a covering of the space X. The diameter of each of these sets with respect to the pseudometric $d_{f_1 f_2 \ldots f_k}$ is less than $\in$. Choosing a point x_i from each of the non-empty sets of the form (1) we get the finite sequence of points $x_1, x_2, \ldots, x_n$ which has the required property.

Group–A
(Short questions)

1. Describe the uniformity on the real number space which induces the usual topology on R and the uniformity which induces the discrete topology on R.
2. If the uniformity $\mathscr{V}$ on a set X has a countable base then show that the induced topology is first countable.

3. If the intersection of all members of the uniformity consists of the diagonal only then prove that the induced topology is T_1.
4. In an uniform space give an example of a set which is totally bounded but not compact.
5. Give an example of a continuous mapping from a uniform space $(X, \mathscr{V})$ to another uniform space $(Y, \mathscr{V})$ that is not uniformly continuous.
6. Give an example of a topological space which is not uniformizable.
7. If a uniform space $(X, \mathscr{V})$ is complete then prove that the space $(M, \mathscr{V}_M)$ is complete for each closed set $M \subset X$.
8. If v_1 and v_2 are two uniformities on X and $v_1 \supset v_2$ then show that v_1 induces a stronger topology than the topology induced by v_2.

Group—B

(Long Questions)

1. Let $(X, \mathscr{V})$ and $(Y, \mathscr{V})$ be uniform spaces. Prove that $f : (X, \mathscr{V}) \to (Y, \mathscr{V})$ is uniformly continuous iff for every $V \in B'$ there is a $U \in B$ such that $U \subset f^{-1}(V)$ where B and B' are bases of $\mathscr{V}$ and V respectively.
2. Show that every family $\{V_s\}_{s \in \Delta}$ of uniformities in a set X has a least upper bound i.e. in the set X there exists a uniformity V which is weakest in the set of all uniformities stronger than V_s for every $s \in \Delta$.
3. Let (X, V) be a uniform space, verify that the product of the topology induced by V on $X \times X$ is identical with the topology induced by $V \times V$ in $X \times X$.
4. If a uniformity V in a set X is induced by a metric ρ then prove that (X, V) is complete if the metric space (X, ρ) is complete.
5. Let (X, V) be a uniform space and let (Y, V) be a complete uniform space. Show that every uniformly continuous funciton f defined on (A, V_A) where A is a dense subset of x, with values in (Y, V) can be extended to a uniformly continuous function from (X, V) to (Y, V).
6. Let X be a compact space. Show that there exists exactly one uniformity V in X which induces the topology on X. The base for the uniformity V consists of all neighbourhoods of the diagonal which are open in the space $X \times X$.
7. Prove that the filter associated with a Cauchy net is a Cauchy filter and conversely every net associated with a Cauchy filter is cauchy.

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Notes

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

—রবীন্দ্রনাথ ঠাকুর

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

—সুভাষচন্দ্র বসু

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

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