# PREFACE

In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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PG MT : XB (II)
Magnetohydrodynamics
(Applied Mathematics)

# Unit 0 ❑ Prerequisites : Electromagnetic Equations

**Structure**

## 0.1 LAW OF FORCE

The study of electrostatics is based upon the experiments of Cavendish and of Coulomb. These show that if two charges $q_1$ and $q_2$ are at a distant r apart, then the force between them is proportional to $q_1q_2/r^2$. If the charges are of like sign, then the force is repulsive ; if unlike, then it is attrative. This is known as the **inverse square law** of electrostatics.

Coulomb's law may be expressed as

$$\mathbf{F} = C \cdot \frac{q_1 q_2}{r^3}\mathbf{r}, \tag{0.1.1}$$

where **F** is the force exerted by $q_1$ on $q_2$ and **r** is the vector displacement from $q_1$ to $q_2$. The constant of proportionality C depends on the system of units chosen. In SI system, which we shall use, the unit of force is newton, the charge—the coulomb and the length—the metre. Thus the dimension of C is newton-metre$^2$/-columb$^2$ or farad/metre. For electrodynamics, we take

$$C = \frac{1}{4\pi\varepsilon_0} \text{newton - metre}^2 \text{ / coulomb}^2$$

where the constant $\varepsilon_0$ is called the **permittivity of free space** and has a value 8·854 × $10^{-2}$ coulomb$^2$/newton-metre$^2$. Thus, the vacuum, we have

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^3}\mathbf{r}. \tag{0.1.2}$$

If the charge $q_1$ is placed at **r** in space while $q_2$ is at $\mathbf{r}_1$, then the force exerted by $q_1$ on $q_2$ is

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2 (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}. \tag{0.1.3}$$

## 0.2 ELECTROSTATIC POTENTIAL AND FIELD

Equation (0.1.1) may be generalized to describe the force on a charge q due to a number of other charges $q_i$, (i = 1, 2, ..., n).

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\mathbf{r}_i \tag{0.2.1}$$

Now suppose that a unit test charge is placed at some point P in presence of a number of fixed charges $q_i$, (i = 1, 2, ...., n). Then, it will, in general, experience a force which we define as the electrostatic field or simply the electric field at P and is denoted by **E**. Thus

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\mathbf{r}_i \tag{0.2.2}$$

We may also deine the electric field **E** for test charge q as the limit

$$\mathbf{E} = \lim_{q\to 0}\frac{F}{q} = \lim_{q\to 0}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\mathbf{r}_i = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\mathbf{r}_i \tag{0.2.3}$$

Noting that $\nabla\left(\frac{1}{r_i}\right) = -\frac{1}{r_i^3}\mathbf{r}_i$ and defining $\phi = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}$, it is easy to see from (0.2.2) or (0.2.3) that

$$\mathbf{E} = -\nabla\phi \tag{0.2.4}$$

The units of **E** and $\phi$ are volt/metre and volt respectively,

Since

$$\nabla \times \mathbf{E} = -\nabla \times \nabla\phi = \mathbf{0}, \tag{0.2.5}$$

the *electrostatic field is irrotational*.

Again, we see that

$$\nabla.\mathbf{E} = \nabla.\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^3}\mathbf{r}_i\right] = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \nabla\cdot\left(\frac{\mathbf{r}_i}{r_i^3}\right) = 0, \qquad (0.2.6)$$

so the *electric field* ***E*** *is a solenoidal vector in a region not occupied by the charge.*

Combining (0.2.4) and (0.2.6), we find that $\phi$ **Satisfies Laplace's equation**

$$\nabla^2\phi = 0. \qquad (0.2.7)$$

*The function* $\phi$ *is called electrostatic potential.*

The flux of the electric field **E** across a surface S, closed or open, is defined by :

$$\text{flux of } \mathbf{E} \text{ across } S = \iint_s \mathbf{E}\cdot\mathbf{dS} \qquad (0.2.8)$$

## 0.3 GAUSS' LAW (CHARGE CONTINUITY EQUATION)

Gauss' law states that *the total flux of the electrostatic field* **E** *out of any closed surface S is equal to* $\frac{1}{\varepsilon_0}$ *times the total charge enclosed by S.*

The above result may easily be extended to a continuous charge distribution characterized by a charge density $\rho$. Regarding the volume distribution as a set of discrete charges $\rho d\tau$, the result is equally applicable is this case also. Here the enclosed charge is $\int_\tau \rho d\tau$ where $\tau$ is the volume of the space enclosed by S, so that by Gauss' theorem.

$$\int_s \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\varepsilon_0}\int \rho d\tau, \quad \text{i.e.,} \int_\tau\left(\nabla, E - \frac{\rho}{\varepsilon_0}\right)d\tau = 0$$

which holds for arbitrary volume $\tau$ and, therefore,

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \qquad (0.3.1)$$

*The equation (0.3.1) is the differential form of Gauss' law.*

Since **E** = – f, it follows from (0.3.1) that f *satisfies Poisson's equation*

$$\nabla^2\phi = -\rho/\varepsilon_0 \qquad (0.3.2)$$

## 0.4 POLARIZATION AND ELECTRIC DISPLACEMENT

**Polarization :**

If a good conductor is subject to an electric field, the conduction electrons respond very quickly to this field and a current flows. On the other hand, a dielectric (insulator) material has no free electron. When a field **E** is applied, the bound electrons in the molecules of the dielectric material move in a direction opposite to **E** but the nuclei are displaced in the direction of **E**. Such displacements are usually very small on a molecular scale. We say that *the medium is polarized by the field*. As a result of this polarization, each molecule of the medium becomes a tiny dipole whose strength depends on the electric field **E**. Suppose that the charges in each molecule are separated by a distance $\delta$. Then for N molecules in unit volume, we define dipole moment per unit volume, **P** by

$$\mathbf{P} = Nq\delta. \qquad (0.4.1)$$

If the field is not too large, then the strength of each microscopic dipole is proportional to **E** and we write $\alpha\mathbf{E}$, where $\alpha$ is called the *polarizability*. The induced dipole $\alpha\mathbf{E}$ is, for many substances, parallel to **E**.

There is also another type of polarizability $\alpha$ where the dielectric molecules are permanent dipoles, such as water. Here the dipoles are randomly oriented in the absence of a field, whereas a couple is established if a field is applied. Thus there will be a resultant moment in the direction of the field. We suppose that this has also been included in the value of α. The total moment induced by an electric field **E** is N$\alpha$**E** per unit volume. Thus for isotropic materials, the polarization **P** is given by

$$\mathbf{P} = N\alpha\mathbf{E} = \varepsilon_0 x\mathbf{E}. \tag{0.4.2}$$

The scalar χ is called the *electric susceptibility* which depends on the density of the material and on the polarizability. Thus the effect of an electric field **E** on a dielectric is to create the polarization **P** with its dipole moment **P**dτ in each volume element dτ. Hence to calculate the potential for such a system we may now set aside any consideration of the dielectric itself, provided that we imagine it to be replaced by the volume polarization **P**dτ.

Now, for a single dipole **p** centered at the origin, the potential at any point is $\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p.r}}{r^3}$, or more generally for a dipole centered at some point **r**'.

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{p\cdot(r-r')}{|r-r'|^3} = \frac{1}{4\pi\varepsilon_0}\,p\cdot\nabla'\left(\frac{1}{r-r'}\right),$$

where prime denotes that the gradient operation is carried out on the primed coordinates. If the dipole be replaced by a volume polarization **P** so that **p** → **P**dτ, then

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\int P(r')\cdot\nabla'\left(\frac{1}{|r-r'|}\right)d\tau' \tag{0.4.3}$$

Since

$$\nabla'\cdot\left(\frac{P(r')}{|r-r'|}\right) = P(r')\cdot\nabla'\left(\frac{1}{|r-r'|}\right) + \frac{\nabla'\cdot P(r')}{|r-r'|}$$

We have from (0.4.3) by using Ganss' theorem

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int \frac{\mathbf{P}(\mathbf{r}')\cdot dS'}{|\mathbf{r}-\mathbf{r}'|} - \int \frac{\nabla'\cdot p(r')}{|r-r'|}\, d\tau'\right] \tag{0.4.4}$$

in which the surface integral is to be evaluated over the dielectric boundary.

It follows from (0.4.4) that for a polarization **P**, the resulting potential is the same as if we have a volume distribution of charge $\rho_p = -\nabla.\mathbf{P}$ throughbout the dielectric, together with a surface distribution $\sigma_p = \mathbf{P}\cdot\hat{\mathbf{n}}$ on the boundary of the dielectric. The charges $\rho_p$ and $\sigma_p$ are known as *polarization* or *bound charge densities*.

**Electric Displacement :**

Suppose that in a dielectric medium, the electric field is **E** with resulting polarization **P**. Then Gauss' Law takes the form

$$\nabla.E = \frac{1}{\varepsilon_0}(\rho - \nabla.P)$$

or, $\nabla . (\varepsilon_0 E + \mathbf{P}) = \rho$

Defining a new macroscopic field vector **D** by

$$\mathbf{D} = \varepsilon_0 E + \mathbf{P}, \tag{0.4.5}$$

the above Gauss' Law becomes

$$\nabla . \mathbf{D} = \rho \tag{0.4.6}$$

The vector **D**, introduced by Maxwell, is called *electric displacement*,

For isotropic materials

$$\mathbf{D} = \varepsilon_0 E + \mathbf{P} = \varepsilon_0(1 + \chi)E \tag{0.4.7}$$

If we write

$$\varepsilon = \varepsilon_0(1 + \chi) \tag{0.4.8}$$

then the equation (0.4.7) gives

$$\mathbf{D} = \varepsilon\mathbf{E}, \tag{0.4.9}$$

where $\varepsilon$ is called the *permittivity of the dielectric medium* $\varepsilon$ has same units as $\varepsilon_0$ (farad/metre). In free space $\varepsilon = \varepsilon_0$, is $\mathbf{D} = \varepsilon_0 E$. Sometimes it is convenient to introduce a dimensionless quantity K, called the *dielectric constant*, by $\varepsilon = K\varepsilon_0$, i.e.,

$$K = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi \tag{0.4.10}$$

The electric displacement **D** is measured in units of coulomb/metre$^2$.

## 0.5 MAXWELL EQUATIONS FOR ELECTROSTATICS

Maxwell equations for electrostatics are given by (0.2.5) and (0.3.1) which may be rewritten as

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{0.5.1}$$

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{0.5.2}$$

where in terms of the potential we have

$$E = -\nabla\phi \tag{0.5.3}$$

$$\nabla^2\phi = -\rho/\varepsilon_0 \tag{0.5.4}$$

which have already been obtained in (0.2.4) and (0.3.2) respectively.

## 0.6 ENERGY OF THE ELECTROSTATIC FIELD

We now proceed to find the mutual potential energy (P.E.) between a given set of charges $\{q_j\}$, $(j = 1, 2, ..., n)$. Denoting $|r_i - r_j|$ by $r_{ij}$, the mutual P.E'. is

$$W_e = \frac{1}{2}\sum_{i\neq j}\sum \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

$$= \frac{1}{2} q_1 \cdot \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_2}{r_{12}} + \frac{q_3}{r_{13}} + \cdots + \frac{q_n}{r_{1n}}\right\}$$

$$+ \frac{1}{2} q_2 \cdot \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_1}{r_{21}} + \frac{q_3}{r_{23}} + \cdots + \frac{q_n}{r_{2n}}\right\} + \cdots$$

$$= \frac{1}{2} q_1 V_1 + \frac{1}{2} q_2 V_2 + \cdots + \frac{1}{2} q_n V_n \tag{0.6.1}$$

where $V_j$, (j = 1, 2, ..., n), are potentials at the positions of the charges $q_j$ (j = 1, 2, ..., n) due to the others.

If the charges are not discrete, but are distributed with volume density $\rho$ and surface density $\sigma$, then the mutual P.E. is

$$W_e = \frac{1}{2}\int \rho V d\tau + \frac{1}{2}\int \sigma V d\tau. \tag{0.6.2}$$

Using the equation (0.5.2) to replace $\rho$ by $\rho = \varepsilon_0 \nabla . \mathbf{E}$**.** the first integral on the right hand side of (0.6.2) gives,

$$\frac{1}{2}\int \rho V d\tau = \frac{1}{2}\ \varepsilon_0 \int V(\nabla . \mathbf{E}) d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int [\nabla . (\nabla \mathbf{E}) - \mathbf{E} . \nabla \nabla] d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int \nabla \mathbf{E} \cdot \mathbf{dS} + \frac{1}{2}\varepsilon_0 \int \mathbf{E} \cdot \mathbf{E} d\tau$$

$$= \frac{1}{2}\varepsilon_0 \int \nabla \mathbf{E} \cdot \mathbf{dS} + \frac{1}{2}\varepsilon_0 \int E^2 d\tau$$

where the surface integral is taken over the sphere at infinity and over the surface of each conductor in which

$$\int V\mathbf{E} \ . \ \mathbf{dS} = \frac{1}{2}\varepsilon_0 \int V E_n ds,$$

$E_n$ being the normal component of **E** out of S into each conductor. Noting that the normal component $E_n$ in the case of surface distribution is $E_n = -\sigma/\varepsilon_0$, we have

$$\frac{1}{2}\rho V d\tau + \frac{1}{2}\int \sigma V dS = \frac{1}{2}\varepsilon_0 \int E^2 d\tau$$

and, therefore, the total electrostatic energy is

$$W_e = \frac{1}{2}\varepsilon_0 \int E^2 d\tau \tag{0.6.3}$$

The quantity $\frac{1}{2}\varepsilon_0 E^2$ is known as the *energy density* of the field. The unit of $W_e$ is coulomb-volt or Joule.

Since $\mathbf{D} = \varepsilon_0 \mathbf{E}$ for free space, we can write (0.6.3) alternatively in the form

$$W_e = \frac{1}{2}\int \mathbf{E} \cdot \mathbf{D} d\tau \tag{0.6.4}$$

## 0.7 THE CURRENT VECTOR

If charges are in motion which results in a net drift of charge, we say that there is a *conduction current* or simply an *electric current*. When the current flowing is independent of time, but may depend spatially, we are concerned with *steady current.*

Current (I) is defined as the rate at which charge Q crosses an area A, i.e.,

$$I = \frac{dQ}{dt} \tag{0.7.1}$$

The unit of current is coulomb/second, or ampere in SI system.

Suppose that the charges are moving at a point P with mean velocity *v*. Then if there are n per unit volume, each carrying a charge q, the *current density* **j** at ρ is defined as

$$\mathbf{j} = nq\mathbf{v}, \tag{0.7.2}$$

the direction of **j** is that in which the current flows.

Consider a cylindrical element with dS as base and generators of length **v**. Then all the charges within this element will cross **ds** in unit time. The total charge contained in it is nq **v. dS = j. dS** so that the flux of j across S is $\int_s \mathbf{j} . \mathbf{dS}$. Thus

$$I = \frac{dQ}{dt} = \frac{d}{dt}\int_s nq\,vt.\,dS = \int_s \mathbf{j}.\,d\mathbf{S} \qquad (0.7.3)$$

since the motion is steady. The unit of **j** is ampere/metre$^2$.

Next suppose that the charge inside a volume $\tau$ is enclosed by a surface S. If $\rho$ be the volume density of charge, the total included charge is $\int_\tau \rho d\tau$ and the rate at which this is decreasing is $-\frac{d}{dt}\int_\tau \rho d\tau$. If the volume $\tau$ remains constant, then $\frac{d}{dt}$ operates solely on $\rho$ which is a function of space and time, i.e.

$$I = -\int_\tau \frac{\partial \rho}{\partial y} d\tau$$

This decrease is due to the outward flow of charge from S and, therefore,

$$-\int_\tau \frac{\partial p}{\partial t}\, d\tau = \int_s \mathbf{j}\cdot \mathbf{dS} = \int_\tau \nabla.\,\mathbf{j}d\tau$$

$$\text{i.e.,}\ \int_\tau \left(\nabla\cdot \mathbf{j} + \frac{\partial \rho}{\partial t}\right) d\tau = 0$$

which is true for arbitrary $\tau$ and so

$$\frac{\partial \rho}{\partial t} + \nabla\cdot \mathbf{j} = 0. \qquad (0.7.4)$$

Equation (0.7.4) is known as *current continuity equation* or *equation of conservation of charge*.

For steady state, $\rho$ is independent of time and then (0.7.4) simplies to

$$\nabla\ .\ \mathbf{j} = 0. \qquad (0.7.5)$$

## 0.8 OHM'S LAW

If the mean velocity of the charges lies in the same direction of the field **E** causing the motion, then **j** and **E** are parallel vectors and, therefore,

$$\mathbf{j} = \sigma \mathbf{E} \tag{0.8.1}$$

where σ is the *electrical conductivity*. Equation (0.8.1) is the *Ohm's Law* which can alternatively be written as

$$\mathbf{E} = \eta \mathbf{j}$$

where η is called the **resistivity** of the conductor. If the conductor be homogeneous and isotropic, then σ and η are constants independent of position. The SI unit of resistivity is volt-metres/ampere, or ohm-metres and that of electrical conductivity is $\text{ohm}^{-1}$-$\text{metre}^{-1}$, or $\text{mho-metre}^{-1}$.

## 0.9 MAGNETIC EFFECTS OF STEADY CURRENTS : LORENTZ FORCE

Magnetic effects are associated with electric currents, which in turn represent charges in motion. We have already defined the electric field **E** by the ratio **E** = **F**/q, where **F** is the force experienced by a test charge q initially at rest in this field. Let us row generalise the expression **F** = q**E** for the electrostatic force by including the effects of the moving charges, i.e. **F** = q**E** + **F'**.

The interaction of currents or charges in motion is described by a magnetic field **B** which we define as a function of the additional force **F'**. From experiments it is observed that a test particle moving in this field experiences a force **F'** which is proportional to the strength of the magnetic field **B** and perpendicular to the velocity **v** of the particle. The magnetic field B is, therefore, defined by the relation.

$$\mathbf{F'} = q\mathbf{v} \times \mathbf{B} \tag{0.9.1}$$

Thus the total electromagnetic force on a particle of charge q moving with velocity **v** is given by

$$\mathbf{F}_{em} = q[\mathbf{E} + v \times \mathbf{B}] \tag{0.9.2}$$

The force $\mathbf{F}_{em}$ defined by (0.9.2) is known as *Lorentz force*. The unit of **B** is gauss. The magnetic field **B** is also termed as *magnetic induction*.

## 0.10 FORCE ON A CONDUCTOR CARRYING CURRENT

Assuming the electric field $\mathbf{E} = \mathbf{0}$, the Lorentz force for n charges is

$$\mathbf{F} = nq\mathbf{v} \times \mathbf{B} = \mathbf{j} \times \mathbf{B},$$

$\mathbf{j}$ being the current density. Thus the total force acting due to the current flowing is

$$\mathbf{F}_{lot} = \int_\tau \mathbf{j} \times \mathbf{B} d\tau, \tag{0.10.1}$$

$\tau$ being the total volume of the plasma. For a plasma column, that is, for a current flowing in a wire, we can rewrite (0.10.1) as

$$\mathbf{F}_{lot} = \iint (\mathbf{j} \times \mathbf{B}) dS \cdot dS \tag{0.10.2}$$

dS being an element of length of the column with its sense along the direction of $\mathbf{j}$ and d$\mathbf{S}$ is an element of area. Since $\mathbf{j}$ and d$\mathbf{S}$ are parallel, we have from (0.10.2)

$$\mathbf{F}_{lot} = \iint (d\mathbf{S} \times \mathbf{B})(\mathbf{j} \cdot \mathbf{dS}) = \int I\, d\mathbf{S} \times \mathbf{B}, \tag{0.10.3}$$

I being the current flowing. Thus the force on an element is

$$d\mathbf{F} = I\, d\mathbf{S} \times \mathbf{B} \tag{0.10.4}$$

where we have dropped the subscript in $\mathbf{F}$. If the current flows in a closed loop of wire, then the force is

$$\mathbf{F} = \oint IdS \times \mathbf{B} = \oint IdS \times \mathbf{B} \tag{0.10.5}$$

## 0.11 THE BIOT-SAVART LAW

Ampére found experimentally that the force between two circuits carrying electric currents is

$$\mathbf{F}_2 = \frac{\mu_e}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{\mathbf{ds}_2 \times [\mathbf{ds}_1 \times \mathbf{r}_{12}]}{r_{12}^3} \tag{0.11.1}$$

This represents the force exerted on the circuit 2 due to the circuit I, $I_1$ and $I_2$ are the currents flowing in the circuits, d$\mathbf{S}_1$ and d$\mathbf{S}_2$ are respective elements of the circuits at a distance $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ apart. The constant $\mu_e$ is known as the *magnetic permeability* of the free space and has unit henry/metre.

Now we can write (0.11.1) as

$$\mathbf{F}_2 = \frac{\mu_e}{4\pi} I_1 I_2 \oint_1 \oint_2 \left[ \frac{(d\mathbf{s}_2 \cdot \mathbf{r}_{12})d\mathbf{s}_1}{r_{12}^3} - \frac{(d\mathbf{s}_1 \cdot d\mathbf{s}_2)\mathbf{r}_{12}}{r_{12}^3} \right].$$

Since

$$\oint_2 \frac{d\mathbf{s}_2 \cdot \mathbf{r}_{12}}{r_{12}^3} = \oint_2 \nabla\left(\frac{1}{r_{12}}\right) \cdot d\mathbf{s}_2 = 0,$$

it follows that

$$\mathbf{F}_2 = -\frac{\mu_e}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{s}_1 \cdot d\mathbf{s}_2)\mathbf{r}_{12}}{r_{12}^3} = -\mathbf{F}_1 \qquad (0.11.2)$$

We may write (0.11.1) in the form

$$\mathbf{F}_2 = I_2 \oint_2 d\mathbf{s}_2 \times \mathbf{B}_2 \qquad (0.11.3)$$

where

$$\mathbf{B}_2 = \frac{\mu_e}{4\pi} I_1 \oint_1 \frac{d\mathbf{s}_1 \times \mathbf{r}_{12}}{r_{12}^3} \qquad (0.11.4)$$

is the magnetic field produced by the current flowing in circuit 1 at the position of circuit 2. The result (0.11.4) is known as *Biot-Savart law* which, in differential form, is given by

$$d\mathbf{B}_2 = \frac{\mu_e}{4\pi} \frac{I_1 d\mathbf{s}_1 \times \mathbf{r}_{12}}{r_{12}^3} \qquad (0.11.5)$$

**Magnetic Intensity Vector :**

The magnetic intensity vector **H**, in the absence of magnetic polarization, is defined by

$$\mathbf{B} = \mu_e \mathbf{H} \qquad (0.11.6)$$

The direction of **H** is the same as that of **B** for an isotropic medium. The unit of **H** is ampere/metre.

## 0.12 THE LAWS OF MAGNETOSTATICS

Suppose that instead of a current flowing in a circuit, we consider a continuous distribution. Then, introducing the current density **j**, we may write analogues to (0.11.3) and (0.11.4) as

$$\mathbf{F} = \int_{\tau} j(\mathbf{r}) \times \mathbf{B}(r) d\tau \qquad (0.12.1)$$

and $$\mathbf{B}(r) = \frac{\mu_e}{4\pi} \int_{\tau} \frac{j(r') \times (r - r')}{|r - r'|} d\tau, \qquad (0.12.2)$$

t being the total volume. Now from (0.12.2) we get

$$\nabla.\mathbf{B} = \frac{\mu_e}{4\pi} \nabla \cdot \int_{\tau} \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d\tau$$

i.e. $$\nabla.\mathbf{B} = \frac{\mu_e}{4\pi} \nabla \cdot \int_{\tau} j(r') \times \nabla \left( \frac{1}{r' - r'} \right) d\tau \qquad (0.12.3)$$

Noting that ∇ . (α × β) = β . ∇ × α – α . ∇ × β, we get by putting α = j and β = $\nabla\left(\frac{1}{r - r'}\right)$

$$\nabla. \left\{ j \times \nabla \left( \frac{1}{|r - r'|} \right) \right\} = \nabla \left( \frac{1}{|r - r'|} \right) \cdot \nabla \times \mathbf{j} - \mathbf{j} \cdot \nabla \times \nabla \left( \frac{1}{|r - r'|} \right) \qquad (0.12.4)$$

Now ∇ × j = σ∇ × E = 0 (by (0.8.1) and (0.5.1)) so that the first term on the right hand side of (0.12.4) vanishes. Also, the second term on the right hand side of

(0.12.4) becomes identically zero, since = $\nabla \times \nabla\left(\frac{1}{|r - r'|}\right) = 0$. Thus we have from (0.12.3).

$$\nabla.B = 0. \tag{0.12.5}$$

Again, from (0.12.2) we get

$$\nabla \times \mathbf{B} = -\frac{\mu_c}{4\pi} \nabla \times \int_\tau \mathbf{j}(\mathbf{r}') \times \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d\tau$$

$$= -\frac{\mu_e}{4\pi} \times \int_\tau \left[\mathbf{j}(\mathbf{r}')\nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) - \{\mathbf{j}(\mathbf{r}')\cdot\nabla\}\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] d\tau \tag{0.12.6}$$

Now, since

$$\nabla \cdot \left[\mathbf{j}(\mathbf{r}')\cdot\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] = \mathbf{j}(\mathbf{r}') \times \nabla \times \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) + \{\mathbf{j}(\mathbf{r}')\cdot\nabla\}\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right),$$

the first term on the right hand side of which is identically zero and noting $\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{\mathbf{r} - \mathbf{r}'}\right)$, we may transform the left hand side of (0.12.6) by writing

$$\int \left[\mathbf{j}(\mathbf{r}')\cdot\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)\right] d\tau = \int \left[\nabla'\cdot\left\{\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right\} - \frac{\nabla'\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - r'|}\right] d\tau.$$

The first term on the right hand side vanishes by Gauss' theorem while the second term is identically zero as $\nabla\ .\ \mathbf{j} = 0$, by (0.7.5). Thus (0.12.6) reduces to

$$\nabla \times \mathbf{B} = \mu_e \int_\tau \mathbf{j}(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')d\tau$$

i.e., $\nabla \times \mathbf{B} = \mu_e\ \mathbf{j}$ (0.12.7)

Equations (0.12.5) and (0.12.7) are the *fundamental laws of magnetostatics.*

**Ampére's Law :**

From (0.12.7) we have

$$\int_s \nabla \times \mathbf{B} \cdot \mathbf{dS} = \mu_e \int_s \mathbf{j} \cdot \mathbf{dS} = \oint_c \mathbf{B} \cdot \mathbf{dS}$$

where S is an open surface bounded by a closed curve C. Since, by definition, $\oint_s \mathbf{j} \cdot \mathbf{dS}$ = I(see (0.7.3)), the current passing through C, it follows that

$$\oint_s \mathbf{B} \cdot d\mathbf{S} = \mu_e I \tag{0.12.8}$$

which is *Ampére's Law* : This result can be stated as : *The circulation of* **B** *around any closed curve C is equal to the product of the total* ***current embraced by the curve*** *and the magnetic permeability of the free space*.

**Magnetic Potentials :**

If we write

$$\mathbf{B} = -\mu_e \nabla\nabla_m, \tag{0.12.9}$$

then $V_m$ is called the *magnetic vector potential*. Since ∇ . B = 0 by (0.12.5), $V_m$ *satisfies Laplace's equation*

$$\nabla^2 V_m = 0 \tag{0.12.10}$$

Again, if we put

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{0.12.11}$$

we can see easily that **A** is not completely prescribed by this equation, since, for a scalar function ψ, $\nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A}$. Thus we must have to add an extra condition on ∇. **A** which in the case of magnetostatics is chosen as $\nabla \cdot \mathbf{A} = 0$. The vector **A** so defined is called the vector *potential*. Since $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_e \mathbf{j}$, so

$$\nabla^2 \mathbf{A} = -\mu_e \mathbf{j}. \tag{0.12.12}$$

Thus the vector potential satisfies Poisson's equation whose solution is given by

$$\mathbf{A} = \frac{\mu_e}{4\pi}\int_{\tau} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau \tag{0.12.13}$$

If we suppose that the current flowing in the loop is I, then we may express (0.12.13) as

$$\mathbf{A(r)} = \frac{\mu_e}{4\pi}\int_{s} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau$$

i.e., $$\mathbf{A(r)} = \frac{\mu_e I}{4\pi}\oint \frac{\mathbf{ds}'}{\mathbf{r}-\mathbf{r}'} \tag{0.12.14}$$

## 0.13 ENERGY OF THE MAGNETOSTATIC FIELD

The energy required or the work done by a current $I_2$ moving in the circuit 2 through the magnetic field produced by the circuit I is

$$d\mathbf{W}m = I_2\, \mathbf{B}(\mathbf{r}_2)\,.\, d\mathbf{S}_2$$

where $d\mathbf{S}_2$ is the surface element of the surface $S_2$ enclosed by the circuit 2. Thus the total magnetic energy is

$$W_m = I_2\int \mathbf{B}(r_2)\cdot d\mathbf{S}_2 = I_2\int_{s_2} (\nabla\times\mathbf{A})\cdot d\mathbf{S}_2 \quad \text{(using (0.12.11))}$$

$$= I_2\oint_2 \mathbf{A}\cdot d\mathbf{s}_2, \quad \text{(by Stokes' theorem),}$$

$d\mathbf{s}_2$ being an element of the circuit 2, Hence

$$W_m = I_2\oint_2\left[\frac{\mu_e}{4\pi}\oint_1 \frac{I_1 d\mathbf{s}_1}{\mathbf{r}}\right]\cdot d\mathbf{s}_2 \quad \text{(by (0.12.14))}$$

i.e., $$W_m = \frac{\mu_e}{4\pi} I_1 I_2 \oint_1\oint_2 \frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{r} \tag{0.13.1}$$

In the case of self-induction, we have

$$W_m = \frac{\mu_e I^2}{8\pi} \oint_c \oint_c \frac{d\mathbf{S} \cdot d\mathbf{S}'}{r} \tag{0.13.2}$$

Again, noting that Id**S** = **j**dτ and Id**s**′ = **j**dτ′, we get from (0.13.2)

$$W_m = \frac{\mu_e I^2}{8\pi} \oint_\tau \oint_\tau \frac{\mathbf{j}d\tau \cdot \mathbf{j}d\tau'}{\mathbf{r}}$$

and, on using (0.12.13), it follows that

$$W_m = \frac{1}{2}\int_\tau \mathbf{j} \cdot \mathbf{A} d\tau \tag{0.13.3}$$

Since $\mathbf{j} = \frac{1}{\mu_e}\nabla \times \mathbf{B} = \nabla \times \mathbf{H}$ (by (0.12.7) and (0.11.6)), **B** = ∇ × **A** (by (0.12.11)) and noting that ∇ · (**H** × **A**) = **A** · (∇ × **H**) – **H** · (∇ × **A**) = **j** · **A** – **H** · **B**, we can write (0.13.3) as

$$W_m = \frac{1}{2}\int_\tau \mathbf{H} \cdot \mathbf{B} d\tau + \frac{1}{2}\int_\tau \nabla.(\mathbf{H} \times \mathrm{A}) d\tau = \frac{1}{2}\int_\tau \mathbf{H} \cdot \mathbf{B} d\tau + \frac{1}{2}\int_S (\mathbf{H} \times \mathbf{A}) \cdot dS$$

If we integrate the second integral over all space for large r, then **H** is of order $\frac{1}{r^2}$ **A** is of order $\frac{1}{r}$ and d**S** is of order $r^2$ and, therefore, the surface integral vanishes, so that

$$W_m = \frac{1}{2}\int_\tau \mathbf{H} \cdot \mathbf{B} d\tau, \tag{0.13.4}$$

where the integration is carried over the whole space. The quantity $\frac{1}{2}$ **H.B.** is known as the *density of the magnetic energy***.** For isotropic medium, the density of magnetic eneergy is $\frac{1}{2}\mu_e H^2$.

It follows, therefore, from (0.6.4) and (0.13.4) that the total electromagnetic energy is given by

$$W_{em} = W_e + W_m = \frac{1}{2}\int_{\tau}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B})d\tau \qquad (0.13.5)$$

## 0.14 GENERALIZED OHM'S LAW

The equation (0.9.2) of Section-0.9 gives the total electromagnetic force on a moving charge and this can be interpreted as the force on a charge due to an electric field moving with the charge velocity **v** through a stationary magnetic field. On the other hand, if an observer moving with the charge will interpret this force due to an electric field **E′** given by

$$\mathbf{E}' = \frac{\mathbf{F}}{q} = \mathbf{E} + v \times \mathbf{B} \qquad (0.14.1)$$

Similarly, if a conducting medium moves with velocity v in a magnetic field, then a moving electric field is produced giving rise to a current called the *conduction current* and is given by

$$\mathbf{j}_e = \sigma E' = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (0.14.2)$$

On the other hand, if there be any free charges moving with the fluid, then it will give rise to a *convection current* given by

$$\mathbf{j}_\mathbf{q} = \rho v, \qquad (0.14.3)$$

ρ being the volume density of charge. Thus the total current density **j** in the medium is

$$\mathbf{j} = \mathbf{j}_e + \mathbf{j}_q = \sigma(E + v \times \mathbf{B}) + \rho v \qquad (0.14.4)$$

which is *generalized ohms law***.**

For neutral conducting fluid ρ = 0 and we have

$$\mathbf{j} = \mathbf{j}_e = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad (0.14.5)$$

## 0.15 GENERALIZED AMPÉRE'S LAW (UNSTEADY CASE) : DISPLACEMENT CURRENT

Suppose a current flows in an open circuit, say in the discharge of condenser for which the current starts from the positively charged plate and flows to the negatively charged one thereby decreasing continuously the charge in the positive plate. Thus we may look upon the condenser plates as sources or sinks of currents.

Now taking divergence of equation (0.12.7) we get

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_e \nabla \cdot \mathbf{j}$$

i.e., $$\nabla \cdot \mathbf{j} = 0$$

leading to the result that the current is always closed and there are no sources or sinks which is a contradiction and, therefore, $\nabla \cdot \mathbf{j} \neq 0$. Thus using the current continuity equation (0.7.4) we have

$$\frac{\partial \rho}{\partial t} = -\nabla . \mathbf{j} \neq 0$$

or, $$\frac{\partial}{\partial t} = (\nabla \cdot \mathbf{D}) = -\nabla . \mathbf{j} \quad \text{(using 0.4.6)}$$

or, $$\nabla . \left( \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \qquad (0.15.1)$$

This equation shows that Ampére's law $\nabla \cdot \mathbf{j} = 0$ is incomplete and the law requires another term (current vector) $\frac{\partial \mathbf{D}}{\partial t}$ which must be added to **j**. The term $\frac{\partial \mathbf{D}}{\partial t}$ is called the *displacement current*.

Hence the generalized Ampére's law is

$$\nabla \times \mathbf{B} = \mu_e \mathbf{j} + \mu_e \frac{\partial \mathbf{D}}{\partial t}$$

i.e., $$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \qquad (0.15.2)$$

## 0.16 ELECTROMAGNETIC INDUCTION : FARADAY'S LAW

Faraday discovered by experiment that if a closed circuit moved across a magnetic field then a current flowed even though no batteries were present there. The same effect was observed if the loop was stationary and the magnetic field varied. This phenomenon was given the name *electromagnetic induction* which may be stated as follows :

**Faraday's Law :**

*The electromagnetic force (e.m.f.) induced in a circuit is equal to the negative of the rate of increase of flux of the magnetic field B through the circuit.*

Mathematically, this can be written as

$$\oint_c \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{S} \qquad (0.16.1)$$

Changing the line integral into a surface one and keeping the surface stationary, we have

$$\int_s \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{S}$$

i.e.,
$$\int_s \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} = 0$$

which is true for all surfaces and, therefore, we have

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad (0.16.2)$$

This is the *differential form of Faraday's Law*.

## 0.17 MAXWELL'S EQUATIONS FOR ELECTROMAGNETISM

According to modern point of view, an electromagnetic field means the domain of five vectors **E, B, D, H** and **j** which satisfy Maxwell's equations already obtained in previous sections in an intertial system* of coordinates when the conductor (medium) is at rest. If the conductor is set in motion, then we reformulate these equations by taking the relativistic effects into consideration. In such cases, Maxwell's equations are unaltered excepting those for **D** and **B** which contain additional terms. These two terms also can be neglected if the velocity of the medium is small in comparison to the velocity of light.

Thus the electromagnetic field equations for non-relativistic motion are :

**Maxwell's equations :**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \text{ (Faraday's law (0.16.2))}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \text{ (Generalized Ampére's Law (0.15.2))}$$

$$\nabla \cdot \mathbf{D} = \rho, \text{ (Gauss' law (0.4.6))}$$

$$\nabla \cdot \mathbf{B} = 0, \text{ (Vide equation (0.12.5))}$$

in which the charge density ρ and the current density **j** are not independent, but are related by the current continuity equation (0.7.4), viz.

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}.$$

In addition to these, we have the following constitutive equations :

$$\mathbf{D} = \varepsilon \mathbf{E}, \text{ (vide equation (0.4.9))}$$

$$\mathbf{B} = \mu_e \mathbf{H}, \text{ (vide equation (0.11.6))}$$

and generalized ohm's law (0.14.4), viz,

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \rho \mathbf{v}.$$

For completion of the set of dynamical principles, we have the Lorentz force (0.9.2)

$$\mathbf{F}_{em} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

* A system of space coordinates in which a free particle subjected to no forces moves in a straight line uniformly is called *inertial system.*

which is electromagnetic force on a moving charge. The corresponding force per unit volume acting on the charge and the current is

$$\mathbf{F}_{em} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}, \tag{0.17.1}$$

since for n charges, we have nq = p and nq**v** = **j** (vide (0.7.2))

## 0.18 ENERGY OF THE ELECTROMAGNETIC FIELD. POYNTING VECTOR

Since

$$\nabla\ (\mathbf{E} \times \mathbf{H}) = \mathbf{H}\ .\ \nabla \times \mathbf{E} - \mathbf{E}\ .\ \nabla \times \mathbf{H}$$

$$= -\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot\mathbf{j} - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} \quad \text{(by (0.16.2) and (0.15.2))}$$

we have by taking volume integral and keeping in view the equations (0.4.9) and (0.11.6)

$$\int_\tau \nabla.(\mathbf{E}\times\mathbf{H})d\tau = \int_\tau \mathbf{E}\cdot\mathbf{j}d\tau - \frac{1}{2}\frac{\partial}{\partial t}\int_\tau (\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B})d\tau$$

or, $$-\frac{\partial \mathbf{W}_{em}}{\partial t} = \int_\tau \mathbf{E}.\,\mathbf{j}_c d\tau + \int_\tau \mathbf{E}.\,\mathbf{j}_q d\tau + \int_\tau \nabla.(\mathbf{E}\times\mathbf{H})d\tau \quad \text{(using (0.13.5))}$$

which with the help of generalized Ohm’s law (0.14.4), gives

$$-\frac{\partial \mathbf{W}_{em}}{\partial t} = \int_\tau \frac{j_e^2}{\sigma}d\tau - \int_\tau (\mathbf{v}\times\mathbf{B}).\,\mathbf{j}_c d\tau + \int_\tau \rho\mathbf{E}.\,\mathrm{v}d\tau + \int_\tau \nabla.(\mathbf{E}\times\mathbf{H})d\tau$$

Use of divergence theorem then leads to

$$-\frac{\partial \mathbf{W}_{em}}{\partial t} = \int_\tau \frac{j_e^2}{\sigma}d\tau + \int_\tau \mathbf{S}.(\mathbf{j}\times\mathbf{B})d\tau + \int_\tau \rho\mathbf{E}.\mathbf{v}d\tau + \int_\tau (\mathbf{E}\times\mathbf{H}).\,d\mathbf{S}, \tag{0.18.1}$$

S being the surface enclosing the volume τ.

Thr equation (0.18.1) states that *the rate of dcrese of the electromagnetic energy is equal to the sum of (i) the energy dissipated due to Joulean heating of the medium at the rate of* $j_c^2/\sigma$ *per unit volume, (ii) the rate of the work done by the medium* *against the Lorentz force* j × **B** *on the currents during the motion, (iii) the rate of work done by the electrical body force* ρ**E** *on the medium and (iv) the instaneous flow of the electromagnetic energy passing through the surface S at the rate of* **E × H.**

The vector **E × H**, usually denoted by **S**, is called Poynting vector, i.e.

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (0.18.2)$$

and *this represents the rate of flow of electromagnetic energy per unit area through the considered surface*. It is obvious that the poynting vector **S** is perpendicular to both **E** and **H**.

If the field extends to infinity, then both **E** and **H** are each of order (distance)$^{-2}$ and, therefore, the surface integral of the normal component of the Poynting vector vanishes, Hence, the energy balance in equation (0.18.1) reduces to

$$-\frac{\partial \mathbf{W}_{em}}{\partial t} = \int_{\tau} \frac{j_e^2}{\sigma} d\tau + \int_{\tau} \mathbf{v}.(\mathbf{j_c} \times \mathbf{B}) d\tau + \int_{\tau} \rho \mathbf{E}.\mathbf{v} d\tau \qquad (0.18.3)$$

where the integration is carried out over the whole space.

## 0.19 ELECTROMAGNETIC WAVE EQUATIONS

In vacuum (i.e. where no charges or current is present), we can write from Maxwell equations.

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$$

or, $$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

or, $$\nabla \times (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_{eo}\varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

$$\therefore \quad \nabla^2 \mathbf{E} = \mu_{e0}\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (0.19.1)$$

where we have used Gauss' Law (0.4.6), viz. $\nabla.\mathbf{D} = 0$ and $\mathbf{D} = \varepsilon_0\mathbf{E}$ for zero charge density.

Similarly, we deduce

$$\nabla^2\mathbf{H} = \mu_{e0}\varepsilon_0 \frac{\partial^2\mathbf{H}}{\partial t^2} \tag{0.19.2}$$

Equations (0.19.1) and (0.19.2) are the *standard equations of wave motion*. It is, therefore, evident that the *electromagnetic waves propagate with velocity* $c = (\mu_{e0}\varepsilon_0)^{-1/2} = 3 \times 10^3$ *metres/sec. which is equal to the velocity of light in free space.*

# Unit 1 ❑ Fundamental Equations of Conducting Liquid

**Structure**



## 1.1 INTRODUCTION

The study of the motion of electrically conducting fluid subjected to a magnetic field is dealt with in the subject *Magnetofluiddynamics (MFD)*. For incompressible fluid, like liquid mercury, if the properties of viscosity, thermal conductivity, electrical conductivity etc. be regarded as constants, then we use the term *Magnetohydrodynamics (MHD)* or *Hydromagnetics*. On the other hand, for compressible fluid such as ionized gas when its other properties specially temperature be variable, then the term *Magnetofluiddynamics* is selected. In general, we consider continuum approach for the subject regarding the conducting fluid to be a continuous medium.

Faraday (1832) observed that when an electrically conducting liquid moves in presence of a magnetic field, then electric currents are induced in the fluid thereby producing their own magnetic field. Moreover, the induced currents also interact with the magnetic field as a result of which electromagnetic force is developed perturbing the original motion. Thus magnetofluid dynamics leads to two important basic results : *(i) the motion of the fluid affects the magnetic field and (ii) the magnetic field affects the motion of the fluid*. In fact, the motion of the fluid slows down due to these electromagnetic forces unless a sufficiently large electrical field is applied along the direction opposite to the induced magnetic field to overcome its effect as a result of which the fluid motion is accelerated by the net electromagnetic force.

It is to be noted that some interesting results of MFD can be achieved in laboratory. However, its importance lies in cosmic problems in geophysics and astrophysics. We now cite some applications of the subject as below :

*(a) MHD power generator :*

Electricity is generated in turbogenerators by the motion of a conductor through a magnetic field (Faraday's Law). Here the conductor is moved by a compressible fluid which expands through a nozzle so that internal energy is transformed into mechanical energy of the conductor and this in turn is transformed into electrical energy.

*(b) MFD flowmeter :*

MFD flowmeter is used in measuring the speed of the ship and is based on the principle that the induced voltage is proportional to the flow rate. This technique is widely applied in oceanography.

*(c) MFD submarines :*

Thrust of MFD submarine is obtained from Lorentz force which is produced by transverse electric and magnetic fields. These pump the electrically conducting sea water through or past the submarine.

*(d) Pinch effect :*

The confinement of hot plasma is of great importance in nuclear fusion devices where a large amount of energy is released. Magnetofluidynamics may be used for magnetically pinching the hot plasma.

Some more applications of the subject are : radio wave propagation in ionosphere, diagnostic techniques, solar flares, space communication system, geomagnetic storms, plasma jets etc.

## 1.2 THE EQUATIONS OF MOTION OF A CONDUCTING FLUID

Navier-Stokes equation of motion of a viscous fluid is given in vector form as

$$\rho \frac{Dv}{Dt} = \mathbf{F} - \nabla p + \frac{1}{3}\rho \nu \nabla(\nabla . \nu) + \rho \upsilon \nabla^2 \nu \tag{1.2.1}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\nu . \Delta)$; $\rho$ is the density of the fluid; **F**, the body force per unit volume; p, the fluid pressure; v, the fluid velocity and $\nu$ is the kinematic coefficient of viscosity.

Now suppose that the fluid is conducting and it moves in presence of an electromagnetic field. Then the body force **F** per unit volume consists of three parts : (i) gravitational force $\rho$**g, g** being the acceleration due to gravity, (ii) electrical force and (iii) magnetic force. Consider an elementary volume $\delta\tau$ of fluid containing a charge of amount $q\delta\tau$ so that the force exerted on it by an electric field of intensity **E** is $(q\delta\tau)$**E** and, therefore, the electrical body force per unit volume is q**E**.

To find the magnetic body force per unit volume, we note that the total current density vector is **j** + q**v** in which **j** represents the electric current density vector and the chosen fluid element moves along with local velocity **v**. Thus the conductive component **j** makes an effective contribution to the magnetic body force but not the convective part q**v**. Let us consider a normal cross-section $\delta$**S** of a fluid element whose length $\delta s$ lies along the direction of **j**. This element moves along with the local fluid velocity **v**, in a magnetic field of intensity **H**. It, therefore, follows that the current flowing through the element is given by **I** = | **j** |ds. Hence, by Biot-Savart law (0.11.3) of Unit-0, the magnetic force $\mathbf{F}_1$ in the element is

$$\delta \mathbf{F}_1 = I\delta \mathbf{S} \times \mathbf{B} = |\mathbf{j}|\delta \mathbf{S}\ \delta s \times \mathbf{B} = (\mathbf{j} \times \mathbf{B})\delta s \delta S$$

which leads the magnetic body force per unit volume as

$$\mathbf{j} \times \mathbf{B} = \mu_e \mathbf{j} \times \mathbf{H}$$

in which **B** is the magnetic induction vector and $\mu_c$ is the magnetic permeability.

Thus the total body force per unit volume is

$$\mathbf{F} = \rho \mathbf{g} + \mu_e \mathbf{j} \times \mathbf{H} + q\mathbf{E} \tag{1.2.2}$$

and, therefore, the equation (1.2.1) can be rewritten as

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho \mathbf{g} + \mu_e \mathbf{j} \times \mathbf{H} + q\mathbf{E} + \frac{1}{3}\rho \nabla(\nabla . \mathbf{v}) + \rho \mathbf{v} \nabla^2 \mathbf{v} \tag{1.2.3}$$

In addition, the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla .(\rho \mathbf{v}) = 0 \tag{1.2.4}$$

## 1.3 RATE OF FLOW OF CHARGE

Since the charge are moving, the generalized Ampére's Law (0.15.2) takes the form

$$\nabla \times \mathbf{H} = \mathbf{j}_e + q\mathbf{v} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{1.3.1}$$

$\mathbf{j}_e$ being the conduction current. Divergence of this equation gives

$$0 = \nabla . \mathbf{j}_e + q\nabla . \mathbf{v} + \mathbf{v} . \nabla q + \varepsilon \frac{\partial}{\partial t}(\nabla . \mathbf{E})$$

where ε is the dielectric constant of the medium. Using equation (0.5.2) (with $\varepsilon_0 = \varepsilon$) and (0.14.5) we get by replacing the charge density ρ by charge q.

$$\frac{Dq}{Dt} + q\nabla . \mathbf{v} + \frac{\sigma}{\varepsilon} q + \mu_e \sigma \nabla .(\mathbf{v} \times \mathbf{H}) = 0 \tag{1.3.2}$$

which represents an equation giving the rate of flow of the charge moving along with the fluid. Two cases arise in this case :

**(i) Fluid at rest :**

Here v = 0 and the equation (1.3.2) reduces to

$$\frac{\partial q}{\partial t} + \frac{q}{\tau_0} = 0 \qquad (1.3.3)$$

where $\tau_0 = \frac{\varepsilon}{\sigma}$, a quantity having dimensions of time. $\tau_0$ is called the *relaxation time* and is usually very small, since $\tau_0 = 1/(\mu_e \sigma c^2)$ where the permeability $\mu_e \sim 1$ and the conductivity σ, the velocity of light c are very large. The solution of (1.3.3) is given by

$$q = q_0 \exp(-t/\tau_0) \qquad (1.3.4)$$

which shows that the charge decays very rapidly in an exponential manner at any point within a conducting liquid.

**(ii) Fluid in motion :**

In general, equation (1.3.2) is not integrable. However, it can be shown that the term $q\nabla.\mathbf{v}$ may by made negligible in some cases. Suppose I is a characteristic, length of the same order of magnitude as distance in which the variables in the equation change by appreciable proportions. Then

$$\{\mu_e \sigma \nabla.(\mathbf{v} \times \mathbf{H})\} = \mu_e \sigma vH/L,$$

If these two are of comparable magnitude, then

$$q \quad \frac{\mu_e \varepsilon vH}{L} = \frac{vH}{Lc^2} \qquad (1.3.5)$$

so that

$$q\nabla.\mathbf{v} \sim \frac{vH}{Lc^2} \cdot \frac{v}{L} = \frac{v^2 H}{L^2 C^2}$$

Hence

$$\frac{0\{q\nabla.\mathbf{v}\}}{0\{\mu_e\sigma\nabla.(\mathbf{v}\times\mathbf{H})\}} = \frac{v^2HL}{L^2c^2\mu_e\sigma vH} = \frac{v}{\sigma L\mu_e c^2} = \frac{\varepsilon v}{\sigma L} \sim \frac{\tau_0}{L/v}$$

The relaxation time $\tau_0$ is very small and in general $\tau_0 << (L/v)$. Thus the term $q\nabla.\mathbf{v}$ can be neglected provided that the condition (1.3.5) holds.

Assuming the result (1.3.5) to hold, we can simplify the equation (1.3.1) as follows :

Noting that

$$O(|\nabla\times\mathbf{H}|) = H/L, \qquad 0(|q\mathbf{v}|) = v^2H/Lc^2$$

so that

$$\frac{O(|q\mathbf{v}|)}{O(|\nabla\times\mathbf{H}|)} = \left(\frac{v}{c}\right)^2 << 1$$

for non-relativistic velocities, we see that the term qv is negligible. Also

$$O\left(\left|\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right|\right) = \frac{\varepsilon E}{L/v} = \frac{vE}{\mu_e c^2 L}$$

and so

$$\frac{O\left(\left|\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right|\right)}{O(|q\mathbf{v}|)} = \frac{vE}{\mu_e c^2L}\times\frac{Lc^2}{v^2H} = \frac{E}{\mu_e Hv}$$

which is of finite size. Hence the term $\varepsilon\frac{\partial\mathbf{E}}{\partial t}$ can also be neglected.

Thus the convective and displacement currents in electromagnetic equation (1.3.1) can be ignored leading to

$$\nabla\times\mathbf{H} = \mathbf{j}_c = \mathbf{j} = \sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad (1.3.6)$$

where the relation (1.3.5) is satisfied and velocities are of non-relativistic order.

## 1.4 THE MAGNETIC REYNOLDS NUMBER

Taking curl on both sides of the equation (1.3.6) we get by using

$$\nabla \times (\nabla \times \mathbf{H}) = \sigma\{\nabla \times \mathbf{E} + \mu_c \nabla \times (\mathbf{v} \times \mathbf{H})\}$$

$$\text{or,} \quad \nabla(\nabla \, . \, \mathbf{H}) - \nabla^2 \mathbf{H} = \sigma\{-\mu_e \frac{\partial H}{\partial t} + \mu_c \nabla \times (\mathbf{v} \times \mathbf{H})\}$$

{by (0.16.2) and (0.11.6}

$$\text{or,} \quad \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}) + \eta \nabla^2 \mathbf{H} \quad \text{\{by (0.12.5) and (0.11.6)\}} \tag{1.4.1}$$

where $\eta = \frac{1}{\mu_e \sigma}$ is called the *magnetic diffusivity* or *magnetic viscosity*.

Now if L denotes a characteristics length and v, a characteristic velocity, then

$$O(|\nabla \times (\mathbf{v} \times \mathbf{B})|) = \frac{vH}{L} \text{ and } 0(|\eta \nabla^2 H|) = \frac{\eta H}{L^2}$$

so that

$$\frac{O(|\nabla \times (\mathbf{v} \times \mathbf{H})|)}{O(|\eta \nabla^2 \mathbf{H}|)} = \frac{vL}{\eta} = R_m \tag{1.4.2}$$

The non-dimensional quantity $R_m = vL/\eta = \mu_e \sigma vL$ is called the *magnetic Reynolds number*.

If $R_m << 1$, then the first term on the right hand side of equation (1.4.1) can be neglected leading to the equation

$$\frac{\partial \mathbf{H}}{\partial t} = \eta \nabla^2 \mathbf{H} \qquad \text{i.e.} \quad \frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}\,. \tag{1.4.3}$$

On the other hand, if $R_m >> 1$, the first term on the right hand side of equation (1.4.1) is predominant and the equation can be approximated as

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}) \qquad \text{i.e.} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \tag{1.4.4}$$

## 1.5 ALFVÉN'S THEOREM

**Statement :** *For a conducting liquid, the flux of the magnetic field through a closed circuit of fluid particles moving along with the fluid is constant for all time.*

**Proof :**

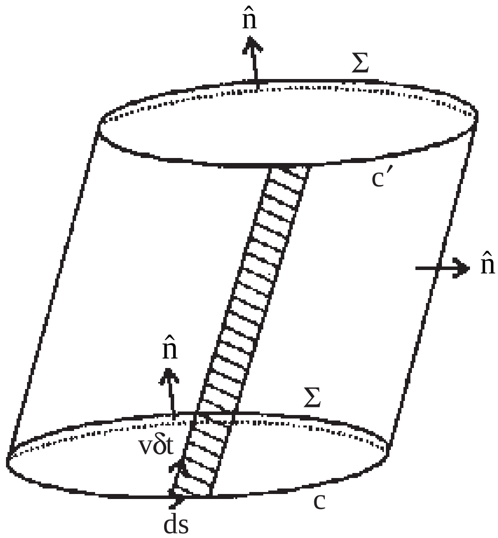



**Fig. 1.** 5(a) : Positions of Σ and Σ′ of a surface of fluid particles bounded by a closed contour at time t and t + δt.

Suppose an open surface Σ of fluid particles be bounded by a closed curve C at any time t. Then the flux of the magnetic field **B**(**r**, t) through the surface Σ at time t is

$$F = \int_{\Sigma} \mathbf{B}(\mathbf{r}, t) . d\mathbf{S} , \qquad (1.5.1)$$

**r** being the position vector of a particle on Σ and d**S** is an element of area of Σ along the normal to the surface associated in the same of description of C. Now let the surface Σ and the curve C move with the fluid to the new positions Σ′ and C′ respectively at time t + δt. Then if **r**′ be the position vector of the same particle on Σ′, the change of flux as Σ moves with the fluid is

$$\delta F = \int_{\Sigma'} \mathbf{B}(\mathbf{r}', t + \delta t) . d\mathbf{S} - \int_{\Sigma} \mathbf{B}(\mathbf{r}, t) . d\mathbf{S}$$

$$= \left[ \int_{\Sigma'} \mathbf{B}(\mathbf{r}', t+\delta t).d\mathbf{S} - \int_{\Sigma} \mathbf{B}(\mathbf{r}, t+\delta t).d\mathbf{S} \right] + \left[ \int_{\Sigma} \mathbf{B}(\mathbf{r}, t+\delta t).d\mathbf{S} - \int_{\Sigma} \mathbf{B}(\mathbf{r}, t+\delta t).d\mathbf{S} \right]$$

$$= I_1 + I_2, \text{ (say)} \qquad (1.5.2)$$

Here $I_1$ refers to the change in flux F at time t + δt due to displacement of Σ and $I_2$ is the change in flux through Σ during the time δt, i.e., it represents the local rate of change of F. Thus, we have

$$I_2 = \int_{\Sigma} [\mathbf{B}(\mathbf{r}, t+\delta t) - \mathbf{B}(\mathbf{r}, t)].d\mathbf{S} = \delta t \int_{\Sigma} \frac{\partial \mathbf{B}}{\partial \mathbf{t}} \cdot d\mathbf{S}$$

to the first order in δt.

Now consider the volume τ enclosed by the surface Σ, Σ′ and the cylindrical surface area S traced out by C as it moves to C′. Then we have at time t + δt.

$$\int_{\Sigma'} \mathbf{B}.d\mathbf{S} - \int_{\Sigma} \mathbf{B}.d\mathbf{S} + \int_{S} \mathbf{B}.d\mathbf{S} = \int_{\tau} \nabla.\mathbf{B} d\tau = 0, \qquad \text{[by (0.12.5)]}$$

In the above d**S** is oriented in each case in the sense of the unit normal vector n as shown in figure 1.5.1. Let us suppose that an element d**s** of the curve C undergoes, to the first order, a displacement vδt in time δt and this displacement traces out a vectorial area d**s** × **v**δt. Hence, to the first order in δt, we have

$$\int_{s} \mathbf{B}.d\mathbf{S} = \oint_{c} \mathbf{B}.(d\mathbf{s} \times \mathbf{v})\, \delta t = \delta t \oint_{c} \mathbf{v} \times (\mathbf{v} \times \mathbf{B}).d\mathbf{s}$$

$$= \delta t \int_{\Sigma} \nabla \times (\mathbf{v} \times \mathbf{B}).d\mathbf{S} \text{ (by Stokes' theorem)}$$

Thus to the first order in δt, we get for time t + δt

$$I_1 = \int_{\Sigma'} \mathbf{B}.d\mathbf{S} - \int_{\Sigma} \mathbf{B}.d\mathbf{S} = -\delta t \int_{\Sigma} \nabla \times (\mathbf{v} \times \mathbf{B}).d\mathbf{S}$$

Then finally, it follows that

$$\delta F = I_1 + I_2 = \delta t \int_{\Sigma} \left[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}).d\mathbf{S} \right]$$

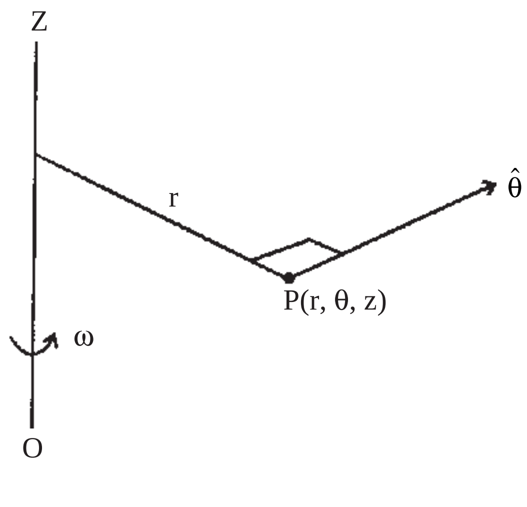
Dividing both sides by $\delta t$ and then making $\delta t \to 0$, we get

$$\frac{dF}{dt} = \int_{\Sigma} \left[ \frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right] \cdot d\mathbf{S} = 0, \text{ (by 1.4.4).}$$

so that F is constant; in other words, we say that *the magnetic lines of force are 'frozen' in the fluid.* Thus the fluid flows freely along the lines of magnetic force, but the motion of the fluid perpendicular to the lines of force carries them with the fluid.

**Ferraro's law of isorotation :**

As a consequence of the Alfvén theorem, we consider the motion of a rotating conducting fluid permeated by a magnetic field. Such type of problems are of great interest in astrophysics. Suppose a star of high electrical conductivity possesses a magnetic field and it rotates non-uniformly about the z-axis with angular velocity $\omega$. Then by Alfvén theorem, the lines of force are frozen in the material and so are carried round by rotation. Thus the magnetic field of the star can be steady provided that it is symmetrical about the axis of rotation and each line of force lies on a surface which is also symmetrical about the same axis and rotates with uniform angular velocity. This is known as *Ferraro's Law of isorotation of the magnetic field.*




**Fig–1.** 5(b) : Non-uniformly rotating star

Let us consider a simple analytical derivation of the above law. Assuming axial symmetry about the z-axis and taking (r, θ, z) as cylindrical coordinates with origin at the centre of the star, all variables are independent of θ and t, i.e., $\frac{\partial}{\partial\theta} \equiv 0, \frac{\partial}{\partial t} \equiv 0.$ Suppose that the star possesses a poloidal magnetic field so that the equation ∇ . B = 0 implies that there exists a scalar function ψ(r, z) such that

$$\mathbf{B} = (B_r, O, B_2) = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \tag{1.5.3}$$

The function ψ is called the *magnetic stream function.*

Now assume that the fluid velocity **v** at a point P(r, θ, z) is

$$\mathbf{v} = r\omega\hat{\theta} \tag{1.5.4}$$

where ω = ω(r, z). Noting that **B** is independent of time, the equation (1.4.4), viz. $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$ gives

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \tag{1.5.5}$$

Now

$$\mathbf{v} \times \mathbf{B} = r\omega\hat{\theta} \times \left(\frac{1}{r}\frac{\partial\psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = -\omega\frac{\partial\psi}{\partial r}\hat{r} - \omega\frac{\partial\psi}{\partial z}\hat{z}$$

and, therefore,

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \frac{1}{1.r.1}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ -\omega\frac{\partial\psi}{\partial r} & 0 & -\omega\frac{\partial\psi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial r}\left(\omega\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\omega\frac{\partial\psi}{\partial r}\right)\right]\hat{\theta}$$

$$= \left[\frac{\partial\omega}{\partial r}\frac{\partial\psi}{\partial z} - \frac{\partial\omega}{\partial z}\frac{\partial\psi}{\partial r}\right]\hat{\theta}$$

$$= \frac{\partial(\omega, \psi)}{\partial(r, z)}\hat{\theta}$$

Hence the equation (1.5.5) gives $\frac{\partial(\omega,\psi)}{\partial(r,z)} = 0$ leading to $\omega = f(\psi)$, or that $\omega$ is constant on the surface $\psi$ = constant, that is the angular velocity is constant over a surface generated by rotation of a line of magnetic force about the axis. Such surfaces are known as *isorotational* or *isotachial* or *magnetic stream surfaces*. Any violation of the law causes the lines of force to be drawn out along the direction of motion as a result of which there arises an azimuthal component of the field.

In a frame of reference rotating with an isorotational surface, the electrical field $\mathbf{E'} = \mathbf{0}$, and therefore, by (0.14.2) we see that in an inertial frame of reference $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$. Thus the lines of force of the electrostatic field E are perpendicular to those of the magnetic field. Hence the *electrostatic potential over an isorotational surface is constant.*

## 1.6 MAGNETIC BODY FORCE

Using equation (0.12.7), viz., $\mu_e \mathbf{j} = \nabla \times \mathbf{B}$, or $\mathbf{j} = \nabla \times \mathbf{H}$, ($\because \mathbf{B} = \mu_e \mathbf{H}$), we can write the equation (1.2.3) as

$$\rho \frac{Dv}{Dt} = -\nabla p + \rho\mathbf{g} + \mu_e(\nabla \times \mathbf{H}) \times \mathbf{H} + q\mathbf{E} + \frac{1}{3}\rho v\nabla(\nabla \cdot \mathbf{v}) + \rho\mathbf{v}\nabla^2\mathbf{v}.$$

Now we have

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = -\mathbf{H} \times \Sigma\left\{\hat{\mathbf{i}} \times \frac{\partial \mathbf{H}}{\partial x}\right\}$$

$$= -\Sigma\left\{\left(\mathbf{H} \times \frac{\partial \mathbf{H}}{\partial x}\right)\hat{\mathbf{i}}\right\} + \Sigma\left\{(\mathbf{H}.\hat{\mathbf{i}})\frac{\partial \mathbf{H}}{\partial x}\right\}$$

$$= -\Sigma\left\{\hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\frac{1}{2}\mathbf{H}^2\right)\right\} + \mathbf{H}.\left(\Sigma\,\hat{\mathbf{i}}\frac{\partial}{\partial x}\right)\mathbf{H}$$

$$= -\nabla\left(\frac{1}{2}\mathbf{H}^2\right) + (\mathbf{H}.\nabla)\mathbf{H} \qquad (1.6.1)$$

To interpret the magnetic body force $\mu_e(\nabla \times \mathbf{H}) \times \mathbf{H})$, we integrate throughout a volume $\nabla\tau$ bounded by a closed surface $\nabla s$ and obtain

$$\mu_e \int_{\Delta\tau} (\nabla \times \mathbf{H}) \times \mathbf{H} d\tau = \mu_e \int_{\Delta\tau} \nabla\left(-\frac{1}{2}\mathbf{H}^2\right) d\tau + \mu_e \int_{\Delta\tau} (\mathbf{H}.\nabla)\mathbf{H} d\tau$$

$$= \mu_e \int_{\Delta s} \left(-\frac{1}{2}\mathbf{H}^2\right) \hat{n} dS + \mu_e \int_{\Delta\tau} (\mathbf{H}.\nabla)\mathbf{H} d\tau \qquad (1.8.2)$$

Now for an arbitrary non-zero constant vector a, we have

$$\mathbf{a}.\int_{\Delta\tau} (\mathbf{H}.\nabla)\mathbf{H} d\tau = \int_{\Delta\tau} \mathbf{a}.\{(\mathbf{H}.\nabla)\mathbf{H}\} d\tau$$

$$= \int_{\Delta\tau} \mathbf{a}.\Sigma\left\{(\mathbf{H}.\hat{\mathbf{i}})\frac{\partial \mathbf{H}}{\partial x}\right\} d\tau$$

$$= \int_{\Delta\tau} \Sigma\left\{(\mathbf{H}.\hat{\mathbf{i}})\frac{\partial}{\partial x}(\mathbf{H}.\mathbf{a})\right\} d\tau$$

$$= \int_{\Delta\tau} \mathbf{H}.\left\{\Sigma\,\hat{\mathbf{i}}\frac{\partial}{\partial x}(\mathbf{H}.\mathbf{a})\right\} d\tau$$

$$= \int_{\Delta\tau} \mathbf{H}.\nabla(\mathbf{H}.\mathbf{a}) d\tau$$

$$= \int_{\Delta\tau} \left[\mathbf{H}.\nabla(\mathbf{H}.\mathbf{a}) + (\nabla.\mathbf{H})(\mathbf{H}.\mathbf{a})\right] d\tau \qquad [\because \nabla.\mathbf{H} = 0 \text{ by } (0.12.5)]$$

$$= \int_{\Delta\tau} \nabla \cdot \left[\mathbf{H}(\mathbf{H}.\mathbf{a})\right] d\tau$$

$$= \int_{\Delta s} \left[\mathbf{H}(\mathbf{H}.\mathbf{a})\right] \cdot \hat{\mathbf{n}} dS \qquad \text{[by divergence theorem]}$$

$$= \int_{\Delta s} (\mathbf{a}.\mathbf{H})(\hat{\mathbf{n}}.\mathbf{H}) dS$$

$$= \mathbf{a}.\int_{\Delta s} \mathbf{H}(\hat{\mathbf{n}}.\mathbf{H}) dS$$

Since **a** is arbitrary, we must have

$$\int_{\Delta\tau} (\mathbf{H} \cdot \nabla)\, \mathbf{H} d\tau = \int_{\Delta s} (\hat{\mathbf{n}} \cdot \mathbf{H})\mathbf{H} dS \tag{1.6.3}$$

The magnetic body force is then defined and given by (1.6.2) and (1.6.3) as

$$\mu_e \int_{\Delta\tau} (\nabla \cdot \mathbf{H}) \times \mathbf{H} d\tau = \int_{\Delta s} \left( -\frac{1}{2}\mu_e H^2 \right) \hat{\mathbf{n}}\, dS + \int_{\Delta s} \mu_e H(\hat{\mathbf{n}}.\mathbf{H})\, dS \tag{1.6.4}$$

Thus the magnetic body force is equivalent to two kinds of surface acting on each surface $\delta S$ given by

$$-\frac{1}{2}\mu_e H^2 \hat{\mathbf{n}} \delta S \text{ and } \mu_e \mathbf{H}(\hat{\mathbf{n}}\,.\,\mathbf{H})\, \delta s.$$

The surface force $-\frac{1}{2}\mu_e H^2 \hat{\mathbf{n}} \delta S$ represents a force $-\frac{1}{2}\mu_e H^2$ per unit area along the direction $-\,\hat{\mathbf{n}}$ and is the hydrostatic pressure $-\frac{1}{2}\mu_e H^2$. To interpret the surface force $\mu_e \mathbf{H}(\hat{\mathbf{n}} \cdot \mathbf{H})\delta S$, we note that $\hat{\mathbf{n}}.H = H \cos\theta$, $\theta$ being the angle between the unit normal vector $\hat{\mathbf{n}}$ to $\delta S$ and the magnetic field intensity vector **H** and suppose that $\delta S'$ is the projection of $\delta S$ normal to $\hat{\mathbf{n}}$ so that $\delta S' = \delta S \cos\theta$; then if $\mathbf{H} = H\hat{\mathbf{H}}$, we have which represents the force $\mu_e H^2$ per unit area in the direction of **H** and may be regarded as a tensile force per unit area amounting to $\mu_e H^2$ in the direction of the magnetic field

We, therefore, conclude that the magnetic body force $\mu_e(\nabla \times \mathbf{H}) \times \mathbf{H}$ per unit volume for a conducting fluid in a magnetic field, is equivalent to a tension $\mu_e H^2$ per unit area along the lines of force, together with a hydrostatic pressure $\frac{1}{2}\mu_e H^2$

## 1.7 MAGNETOHYDRODYNAMIC WAVES

According to Alfvén's theorem, the particles of a fluid of infinite electrical conductivity are tied to the magnetic lines of force. Suppose that $\mathbf{B}_0$ is the undisturbed field intensity in a tube of magnetic force of section $\delta A$ and $\rho$ is the density of the fluid. Then the magnetic forces acting on the tube are equivalent to a tension $B_0^2/\mu_e$ per unit area along the lines of force and a hydrostatic pressure $B_0^2/2\mu_e$. The latter can be balanced by a decrease in fluid pressure leaving the tubes in tension T along the lines of force, where $T = B_0^2\delta A/\mu_e$.

These lines of force, for incompressible fluid, are like stretched strings in tension T and mass m (= ρδA) per unit length. Thus if the liquid is slightly disturbed from its position of rest, the lines of force will execute transverse vibrations, the phase velocity of the waves being given by

$$\left(\frac{\text{tension}}{\text{density}}\right)^{\frac{1}{2}} = \left(\frac{T}{m}\right)^{\frac{1}{2}} = \left(\frac{B_0^2}{\mu_e \rho}\right)^{\frac{1}{2}} = V_A \text{, say} \qquad (1.7.1)$$

*This velocity $V_A$ is called Alfvén's velocity and the waves are known as Alfvén waves.*

In the case of perfectly conducting compressible fluid, longitudinal wave propagation is also fessible, the nature of the wave depending on the direction of the magnetic field $\mathbf{B}_0$ relative to the particle velocity **v**.

If the direction of the particle velocity **v** and the direction of wave propagation are both parallel to $\mathbf{B}_0$, then since the fluid moves along the lines of force, no magnetic effects are called into play. In this case, the waves in the fluid are ordinary acoustic waves which propagate with sound velocity (c), since the motion of the fluid particles parallel to the magnetic field does not give rise to any magnetic perturbation.

Now we suppose that the particle velocity is parallel to the direction of propagation and each is perpendicular to undisturbed magnetic field intensity $\mathbf{B}_0$. We show that a new type of wave is excited in such a case. For a perfectly conducting liquid, let us suppose that **B** is the field intensity at time t and **v** is the particle velocity. From (1.4.4), viz,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (v \times \mathbf{B})$$

we have, by taking $\mathbf{v} = v(x)\,\hat{\mathbf{i}}$, $B = B(x)\,\hat{\mathbf{j}}$ and noting that the motion is steady, i.e., $\frac{\partial \mathbf{B}}{\partial t} = 0$, $-\frac{d}{dx}(vB)\,\hat{\mathbf{j}} = 0$, leading to vB = constant. Also the equation of continuity (1.2.4) gives for steady motion $\nabla\,(\rho \mathbf{v}) = 0$, so that ρv = constant. Thus we get

$$\frac{B}{\rho} = \text{constant} = \frac{B_0}{\rho_0},$$

in which the suffix 0 refers to the undisturbed conditions. Since $B^2/2\mu_e$ is the magnetic pressure, the effective pressure p* is given by $p^* = p + B^2/2\mu_e$ so that

$$\frac{dp^*}{d\rho} = \frac{dp}{d\rho} + \frac{B}{\mu_e}\frac{dB}{d\rho}.$$

But $\frac{dB}{d\rho} = \frac{B_0}{\rho_0} = \frac{B}{\rho}$ and $c^2 = \frac{dp}{d\rho}$ and so

$$\frac{dp^*}{d\rho} = c^2 + \frac{B^2}{2\mu_e} = c^2 + V_A^2,$$

where $V_A$ is the Alfvén wave velocity. Thus the speed of propagation is $\sqrt{(c^2 + V_A^2)}$.

Such type of wave is called *magnetohydrodynamic wave.*

It is to be noted that *Alfvén waves are transverse waves and are propagated in incompressible conducting fluids, but magnetohydrodynamic waves are longitudinal and requires compressible conducting fluid of infinite conductivity for their propagation.*

**A more detailed discussions of Alfvén waves :**

The direction of propagation of Alfvén wave lies along the lines of magnetic force and the fluid particle velocity is at right angles to them. We now consider an undisturbed uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{k}}$ along the direction of z-axis and **b** is the perturbation produced in the field due to a small disturbance so that the resultant field is

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} \tag{1.7.2}$$

Since, for incompressible conducting fluid, there is no charge accumulation at internal points, so the equation of motion (1.2.3) gives

$$\rho\frac{Dv}{Dt} = -\nabla p + \frac{1}{\mu_e}(\nabla \times \mathbf{b}) \times \mathbf{B}, \tag{1.7.3}$$

where we have neglected the viscous effects. Also the magnetic field continuity equation (0.12.5), viz. $\nabla.\mathbf{B} = 0$ gives

$$\nabla.\mathbf{b} = 0 \tag{1.7.4}$$

and the fluid continuity equation (1.2.4) is

$$\nabla.\mathbf{v} = 0 \tag{1.7.5}$$

Assuming that the disturbances to be so small that we can neglect the squares and products of v and b, we have

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t}$$

and

$$\begin{aligned}(\nabla \times \mathbf{b}) \times \mathbf{B} &= (\nabla \times \mathbf{b}) \times (\mathbf{B}_0 + \mathbf{b}) \\ &= (\nabla \times \mathbf{b}) \times \mathbf{B}_0 \\ &= -\mathbf{B}_0 \times \Sigma\left(\hat{\mathbf{i}} \times \frac{\partial \mathbf{b}}{\partial x}\right) \\ &= -\Sigma\left\{\hat{\mathbf{i}}\left(\mathbf{B}_0 \cdot \frac{\partial \mathbf{b}}{\partial x}\right)\right\} + \Sigma\left\{\left(\hat{\mathbf{i}} \cdot \mathbf{B}_0\right)\frac{\partial \mathbf{b}}{\partial x}\right\} \\ &= -\Sigma\left\{\hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\mathbf{B}_0.\mathbf{b}\right)\right\} + B_0\frac{\partial \mathbf{b}}{\partial z} \\ &= -\nabla\left(\mathbf{B}_0.\mathbf{b}\right) + B_0\frac{\partial \mathbf{b}}{\partial z}\end{aligned}$$

Thus (1.7.3) can be approximated as

$$\nabla\left(p + \frac{1}{\mu_e}\mathbf{B}_0.\mathbf{b}\right) = \frac{B_0}{\mu_e}\frac{\partial \mathbf{b}}{\partial z} - \rho\frac{\partial \mathbf{v}}{\partial t} \tag{1.7.6}$$

Taking divergence on both sides of this equation and noting the equations (1.7.4) and (1.7.5) we are led to

$$\nabla^2\left(p + \frac{1}{\mu_e}\mathbf{B}_0.\mathbf{b}\right) = 0 \tag{1.7.7}$$

which shows that $p + \frac{1}{\mu_e}\mathbf{B}_0 \,.\, \mathbf{b}$ is a harmonic function. Here two cases arise :

(i) *the liquid is of infinite extent* and (ii) *the liquid is of finite extent* :

**Case I. Liquid of infinite extent**

The solution of the equation (1.7.7) regular at all points (including at infinity) is given by

$$p + \frac{1}{\mu_e}\mathbf{B}_0 \,.\, \mathbf{b} = \text{constant}$$

and, therefore, the equation (1.7.6) is reduced to

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \frac{B_0}{\mu_e} \frac{\partial \mathbf{v}}{\partial z}. \tag{1.7.8}$$

Now the equation (1.7.1) governing magnetic field variations reduces to the first order

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0)$$

$$= B_0 \nabla \times (\mathbf{v} \times \hat{\mathbf{k}})$$

$$= B_0 \left[ (\nabla . \hat{\mathbf{k}}) \mathbf{v} - (\nabla . \mathbf{v}) \hat{\mathbf{k}} \right]$$

$$\text{i.e., } \frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}_0 \frac{\partial \mathbf{v}}{\partial z}, \text{ [using (1.7.5)].} \tag{1.7.9}$$

Equations (1.7.8) and (1.7.9) then lead to the following wave equation for **b** and **v** :

$$\left( \frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial z^2} \right) (\mathbf{b}, \mathbf{v}) = 0, \tag{1.7.10a, b}$$

where $V_A = \left( B_0^2 / \mu_e \rho \right)^{\frac{1}{2}}$ is the Alfvén wave velocity. Hence the *magnetic field and fluid particles propagate as transverse waves along the lines of force with Alfvén velocity* $V_A$.

If a wave travels along the positive direction of the z-axis, then the solutions of the equations (1.7.10a, b) are

$$\mathbf{b} = \mathbf{b}(z - V_A t), \ \mathbf{v} = \mathbf{v}(z - V_A t)$$

so that $\frac{\partial \mathbf{b}}{\partial t} = -V_A \frac{\partial \mathbf{v}}{\partial z}$ and, therefore, from (1.7.9) it follows that

$$\frac{\partial \mathbf{b}}{\partial z} = -\frac{B_0}{V_A} \cdot \frac{\partial \mathbf{v}}{\partial z} = -(\mu_e \rho)^{\frac{1}{2}} \frac{\partial \mathbf{v}}{\partial z}$$

which is satisfied if

$$\mathbf{b} = -(\mu_e\rho)^{\frac{1}{2}}\mathbf{v}. \tag{1.7.11a}$$

Similarly, for a wave travelling along the negative direction of z-axis, we have

$$\mathbf{b} = (\mu_e\rho)^{\frac{1}{2}}\mathbf{v} \tag{1.7.11b}$$

*The relations (1.7.11a,b) are due to Walén.* It is obvious that $\frac{b^2}{2\mu_e} = \frac{1}{2}\rho v^2$. *Thus the magnetic energy of the perturbed field is equal to the kinetic energy of the motion.*

**Case II. Liquid of finite extent**

If $p + \frac{1}{\mu_e}\mathbf{B}_0.\mathbf{b}$ is not constant, then there occur boundary reflections and transmissions, as a result of which the solution of the equation (1.7.7) becomes much more complicated. However, taking curl on both sides of (1.7.6), it follows that

$$\nabla\times\nabla\left(p + \frac{1}{\mu_e}\mathbf{B}_0\cdot\mathbf{b}\right) = \frac{B_0}{\mu_0}\frac{\partial}{\partial z}(\nabla\times\mathbf{b}) - \rho\frac{\partial}{\partial t}(\nabla\times\mathbf{v})$$

$$0 = \frac{B_0}{\mu_e}\frac{\partial \mathbf{j}}{\partial z} - \rho\frac{\partial \zeta}{\partial t} \tag{1.7.12a}$$

where we have used the relation (0.12.7), viz. $\nabla \times \mathbf{B} = \mu_e\mathbf{j}$ with $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ so that $\nabla \times \mathbf{b} = \mu_e\mathbf{j}$ and $\zeta = \nabla \times \mathbf{v}$ is the vorticity vector. Also taking curl on both sides of (1.7.9) we get

$$\frac{\partial}{\partial t}(\nabla\times\mathbf{b}) = B_0\frac{\partial}{\partial z}(\nabla\times\mathbf{v})$$

$$\text{i.e., } \frac{1}{\mu_e}\frac{\partial \mathbf{j}}{\partial t} = B_0\frac{\partial \zeta}{\partial z} \tag{1.7.12b}$$

Equations (1.7.12a,b) then lead to the wave equations

$$\left(\frac{\partial^2}{\partial t^2} - V_A^2\frac{\partial^2}{\partial z^2}\right)(\zeta, \mathbf{j}) = 0 \tag{1.7.13a,b}$$

*Thus the vorticity ζ and the current density j propagate with Alfvén wave velocity along the lines of force.*

## 1.8 INITIAL AND BOUNDARY CONDITIONS

The solutions of the equations given in sections 1.2 and 1.4 are completely determined if the initial and boundary conditions are specified. However, the appropriate conditions depend on the nature and physical conditions of the problem.

**(i) Initial conditions :**

For steady motion of the conducting fluid, we need not consider initial conditions. However, for unsteady problem, the initial distributions of the velocity **v**, magnetic field **B**, pressure p and density ρ are to be specified to solve the equations (1.2.3) and (1.4.1) and the expressions for the current density **j** and the electric field **E** then follow from (0.12.7) and (0.14.4).

**(ii) Boundary conditions on velocity field :**

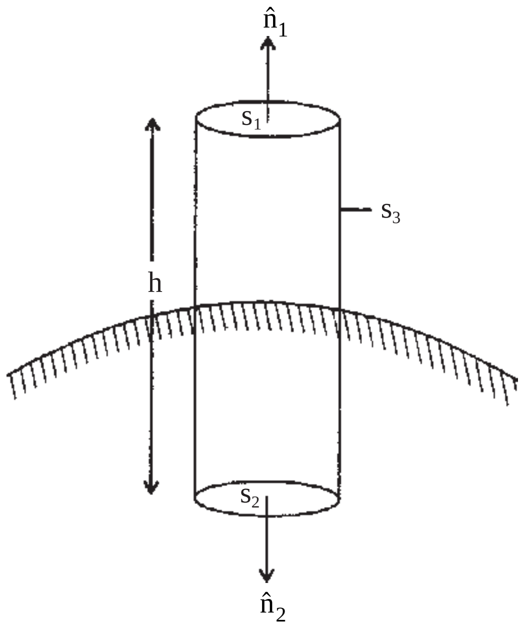
If the flow through the interfaces are not considered, the normal component $v_n$ of the velocity of fluid on either side of the boundary must be equal to its normal velocity. At a fixed wall or at any interface in steady flow $v_n = 0$.

For inviscid fluid, the tangential component $v_t$ of the fluid velocity is discontinuous at the boundary, a situation which we call a *vortex sheet*. On the other hand, on the interface of two viscous liquids or between a viscous fluid and a solid, the tangential component $v_t$ is continuous. At a fixed wall, the no-slip condition $v_t = 0$ is to be specified.

**(iii) Boundary conditions on electromagnetic field :**

The boundary conditions to be satisfied by the magnetic and electric fields at an interface of two media are deduced from Maxwell's equations.

**(a) Normal component of magnetic field**




Fig–1. 8(a) : Thin cylinder on the interface between two media

Construct a thin cylinder of negligibly height h compared to the diameter at the interface of two media (fig–1.8a). Then the equations $\nabla.\mathbf{B} = 0$ shows that the net flux of the magnetic field from a volume is zero, i.e.

$$\int_s \mathbf{B} \cdot \hat{\mathbf{n}} dS = \int_{s_1} \mathbf{B} \cdot \hat{\mathbf{n}} dS + \int_{s_2} \mathbf{B} \cdot \hat{\mathbf{n}} dS + \int_{s_3} \mathbf{B} \cdot \hat{\mathbf{n}} dS = 0 \qquad (1.8.1)$$

If **B** is bounded, then taking limit as $h \to 0$, we see that the last integral or the right hand side of (1.8.1) vanishes and $s_2 \to s_1$. Thus, noting that the unit normal vectors and are in opposite directions, are have

$$B_{n_1} = B_{n_2}, \quad \text{i.e.,} \quad (\mathbf{B}_1 - \mathbf{B}_2) \cdot \hat{\mathbf{n}} = 0 \qquad (1.8.2)$$

which shows that the *normal component of magnetic induction it discontinuous at the interface.*

**(b) Tangential component of magnetic field**

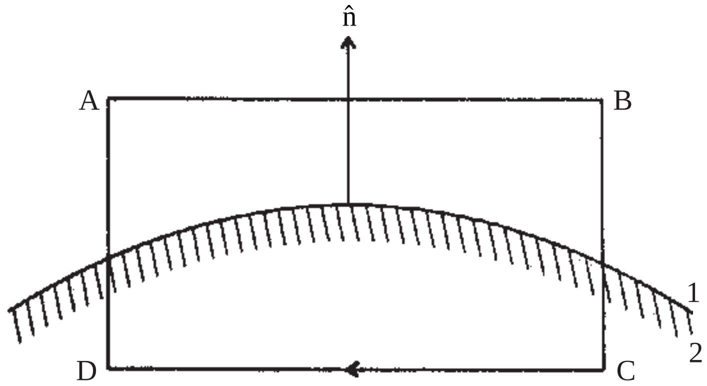



Fig 1.8(b) : Rectangular closed curve intersecting the interface between two media

Consider a rectangular closed curve intersecting the interface as shown in figrue-(1.8b) such that AB = CD = $\Delta$s and the segments AD and BC (= h) are negligibly small. Now from Maxwell equation (0.12.7), viz. $\nabla \times \mathbf{B} = \mu_e \mathbf{j}$, i.e., $\nabla \times \mathbf{H} = \mathbf{j}$ ($\because$ $\mathbf{B} = \mu_e \mathbf{H}$ by (0.11.6)), we have

$$\lim_{h\to 0}\int_s (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{n}} dS = \lim_{h\to 0}\int_s \mathbf{j} \cdot \hat{\mathbf{n}} dS$$

$$\text{or,} \quad \lim_{h\to 0}\int \mathbf{H.dS} = \lim_{h\to 0}\int_s \mathbf{j} \cdot \hat{\mathbf{n}} dS$$

$$\therefore \quad \mathbf{H}_1 \,.\, \Delta S + \mathbf{H}_2 \,.\, (-\Delta S) = |\mathbf{j}_s \times \Delta S|$$

$$\text{i.e. } H_{t_1} = H_{t_2} = |\mathbf{j}_s \times \hat{\mathbf{s}}| \qquad (1.8.3)$$

where $\mathbf{j}_s$ is the surface current density or sheet current and $\hat{\mathbf{s}}$ is the unit vector in the direction of $\Delta$S. Since the equation (1.8.3) holds for any segment $\Delta$s which is parallel to the interface, we can write this as

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{j}_s \qquad (1.8.4)$$

For finite electrical conductivity, $\sigma = \infty$ and so $\mathbf{j}_s = 0$. Thus (1.8.4) gives

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \qquad (1.8.5)$$

which shows that the tangential component of **H** is continuous.

For infinite electrical conductivity, i.e. when $\sigma = \infty$, then $\mathbf{j}_s \neq 0$ and therefore, the equation (1.8.5) gives the jump in the tangential component of the magnetic field in terms of sheet current. Thus *the tangential component of* **H** *is discontinuous across the surface.*

**(c) Normal component of electric field**

From Gauss' Law (0.4.6), viz. $\nabla.\mathbf{D} = \rho$, we get the integral formulation for element of volume $\tau$ (figure—1.8a)

$$\int_\tau \nabla.\mathbf{D}d\tau = \int_\tau \rho d\tau$$

$$\text{or,} \quad \int_s \mathbf{D}.\hat{\mathbf{n}}dS = \int_\tau \rho d\tau$$

$$\text{or,} \quad \int_{s_1} \mathbf{D}.\hat{\mathbf{n}}ds + \int_{s_2} \mathbf{D}.\hat{\mathbf{n}}ds + \int_{s_3} \mathbf{D}.\hat{\mathbf{n}}ds + \int_\tau \rho d\tau$$

Let the volume element tends to zero in such a way that $s_1, s_2 \to s$ as $h \to 0$ and then

$$D_{n_1} - D_{n_2} = \lim_{h\to 0} \frac{1}{s}\int_\tau \rho d\tau = \rho_s$$

where $\rho_s$ is the surface free charge distribution per unit area. Thus we have

$$\hat{\mathbf{n}}.(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s. \qquad (1.8.6)$$

Hence the normal component of the electric displacement vector **D** is discontinuous. However, for MHD, there is usually no surface free charge and in such a case

$$\hat{\mathbf{n}}.(\mathbf{D}_1 - \mathbf{D}_2) = 0$$

i.e., *the normal component of the electric displacement is continuous across the interface.*

**(d) Tangential component of electric field**

From Faraday's law (0.16.2), viz. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, we get by integration over the surface boundary by a rectangular loop (fig.–1.8b)

$$\lim_{h\to 0}\int_s (\nabla \times \mathbf{E}).dS = \lim_{h\to 0} -\frac{\partial}{\partial t}\int_s \mathbf{B}.dS$$

$$\text{or,} \quad \lim_{h\to 0}\int_c \mathbf{E}.dS = \lim_{h\to 0} -\frac{\partial}{\partial t}\int_s \mathbf{B}.dS \qquad (1.8.7)$$

$$\therefore\ \mathbf{E}_1 \,.\, \Delta S + \mathbf{E}_2 \,.\, (-\Delta S) = 0$$

$$\text{i.e., } E_{t_1} - E_{t_2} = 0.$$

the right hand side of (1.8.7) vanishes provided that $\dfrac{\partial \mathbf{B}}{\partial t}$ is bounded. The above equation can be written as

$$\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \qquad (1.8.8)$$

Thus *the tangential component of the electric field* **E** *is continuous.*

**(e) Current density :**

In the absence of charge density ρ, the current continuity equation (0.7.4) gives $\nabla.\mathbf{j} = 0$ which implies that *the normal component of current density is continuous across the interface*, i.e.

$$\hat{\mathbf{n}} \cdot (\mathbf{j}_1 - \mathbf{j}_2) = 0 \qquad (1.8.9)$$

The normal component is zero on the boundary if either region adjoining the boundary is non-conducting or vacuum, or if an insulating layer constitutes the boundary.

Noting Ohm's law $\mathbf{j} = \sigma\mathbf{E}$ and the fact that the tangential component of the electric field **E** is continuous, we see that the *tangential component of current density is discontinuous only if the electrical conductivities of the media are different.*

## 1.9 THERMODYNAMICAL CONSIDERATIONS

A fluid in different thermodynamic states behaves in different ways. Thus to supplement the conservation equations of mass and momentum by the addition of an energy equation, we must take into account thermodynamical environment of the fluid. The energy conservation law is equivalent to the thermodynamic law

$$T\, dS = de + p d\tau \qquad (1.9.1)$$

where T is the temperature, **e** is the internal energy per unit mass of the fluid, S is the entropy per unit mass of the fluid and $\tau = 1/\rho$ is the specific volume, i.e., the volume per unit mass of the fluid. In a *reversible process* TdS can be considered as the heat per unit mass gained by the fluid due to conduction. On the other hand, in an irreversible process, other sources of heat like viscosity are present resulting in TdS exceeding the heat acquired by conduction. The state of a gas is defined by the quantities p, T, $\tau$, S and e in which only two are independent. It is usually most convenient to express all quantities in terms of $\tau$ and S. For a gas, the dependence of e on $\tau$ and S is known and, therefore, the equation (1.9.1) gives $T = \partial e/\partial S$ and $p = -\partial e/\partial \tau$. In our discussions, we shall consider only irreversible process in which the effects of viscosity and of electrical resistane are involved. We shall also interpret the difference between TdS and the heat acquired by conduction as the heat resulting from the effects of viscous forces and of Joule loss.

Now rewriting the equation (1.4.1) with $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_e}$, we have

$$\eta\nabla^2\mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.9.2}$$

Multiplying the equation by the magnetic field vector **B**, we find after some calculations

$$\frac{1}{2}\frac{\partial B^2}{\partial t} = \mathbf{v}.\{\mathbf{B} \times (\nabla \times \mathbf{B})\} - \frac{j^2}{\sigma} - \nabla.\left[\{\mathbf{B} \times (\mathbf{v} \times \mathbf{B})\} - \eta\mathbf{B} \times (\nabla \times \mathbf{B})\right] \tag{1.9.3}$$

Integrating this equation over a fixed volume V and using Gauss divergence theorem, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_v B^2 dv = \int_v \mathbf{v}.(\mathbf{B} \times \mathbf{j})dv - \int_v \frac{j^2}{\sigma} d\mathbf{V} - \int_s \left[\mathbf{B} \times (\mathbf{v} \times \mathbf{B})\right] - \eta\mathbf{B} \times (\nabla \times \mathbf{B}).d\mathbf{S} \tag{1.9.4}$$

where S denotes the surface enclosing the volume V. This equation shows that the heat per unit volume per unit time resulting from Joule loss is $j^2/\sigma$, whereas in ordinary hydrodynamics* the heat due to viscous forces is equal to

$$\mu \sum_{i,k=1}^{3} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \nabla . \mathbf{v} \right) \frac{\partial v_i}{\partial x_k}$$

per unit volume, per unit time, while the heat acquired by conduction per unit volume, per unit time is

$$\nabla \cdot (\chi \nabla T)$$

where $\chi$ is the thermal conductivity. The equation (1.9.1) shows that the increase of heat per unit time of a unit mass of fluid as it moves in space is $T \frac{DS}{Dt}$ and, therefore, we may equate the heat increase per unit volume, $\rho T \frac{DS}{Dt}$, to the heat influx due to viscous dissipation, Joule heating and thermal conduction per unit volume per unit time, to give

$$\rho T \frac{DS}{Dt} = \mu \sum_{i,k=1}^{3} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \nabla . \mathbf{v} \right) \frac{\partial v_i}{\partial x_k} + \frac{j^2}{\sigma} + \nabla . (\chi \nabla T) \qquad (1.9.5)$$

Now, noting that $\tau = 1/\rho$, we can write equation (1.9.1) as

$$\rho T \frac{DS}{Dt} = \rho \frac{De}{Dt} - \frac{p}{\rho} \frac{d\rho}{Dt} \qquad (1.9.6)$$

Also using the continuity equation (1.2.4) in its alternative form

$$\frac{d\rho}{Dt} + \rho \nabla . \mathbf{v} = 0,$$

it follows from (1.9.5) and (1.9.6) that

$$\rho \frac{De}{Dt} = -\rho \nabla . \mathbf{v} + \mu \sum_{i,k=1}^{3} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \nabla . \mathbf{v} \right) \frac{\partial v_i}{\partial x_k} + \frac{j^2}{\sigma} + \nabla . (\chi \nabla T) \qquad (1.9.7)$$

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*See any book on Hydrodynamics, say Fluid Dynamics—Rutherford.

Now the total energy U per unit volume is

$$U = \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu_e} + \rho_e \tag{1.9.8}$$

in which the first term on the right hand side is the contribution due to the kinetic energy, the second term is the energy density of the magnetic field and their term represents the internal energy density of the fluid. Differentiating equation (1.9.8) with resepct to t and making use of equations (0.12.7), (1.9.3), (1.2.4), (1.2.3) (with **g** = 0 and **E** = 0) and (1.9.2) we finally obtain the following expression of the *law of conservation of energy :*

$$\frac{\partial U}{\partial t} + \nabla . \mathbf{G} = 0 \tag{1.9.9}$$

in terms of the energy flow vector **G** given by

$$\mathbf{G} = \rho \mathbf{v}\left(\frac{1}{2}v^2 + i\right) + \frac{1}{\mu_e}\{\mathbf{B} \times (\mathbf{v} \times \mathbf{B}) - \eta \mathbf{B} \times (\nabla \times \mathbf{B})\}$$

$$- \mu \sum_{i,k=1}^{3}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\nabla . \mathbf{v}\right) v_i \hat{\mathbf{e}}_k - \chi \nabla T \tag{1.9.10}$$

where i = e + p$\tau$ is called the *enthalpy per unit mass of the fluid* and $\hat{\mathbf{e}}_i$ is the unit vector parallel to the $x_i$–axis.

When these conservation laws are supplemented by the addition of the constitutive equations of state such that p = p($\rho$, T), e = e($\rho$, T), the resulting set of equations determines completely the magnetohydrodynamic equations.

If there is no energy transport to or from any fluid element and external region of the fluid, then the conditions in the fluid are said to be *adiabatic.* In particular, for an ideal gas, the equation of state is expressed by the law

$$p\tau = RT, \tag{1.9.11}$$

where R is the universal gas constant. The internal energy e of an ideal gas depends only on the temperature T and if the energy is assumed to be proportional to the temperature T, the gas is called *polytropic*.

We define the *specific heat at constant volume*, $C_v$, of a gas to be the limit of the ratio of the energy T $\delta S$ supplied to a unit mass of gas to $\delta T$, the rising in temperature, when the volume is kept constant. Thus from (1.9.1), it follows that $C_v = \frac{\partial e}{\partial T}$. Clearly, for a polytropic gas $e = C_v T$. Similarly, the *specific heat at constant pressure* $C_p$ is defined only when the pressure of the gas instead of volume remains constant during the addition of energy. From (1.9.1), it is obviour that for a polytropic gas

$$C_p = C_v + p\frac{\partial \tau}{\partial T}$$

and, using (1.9.11)

$$C_p - C_v = R \tag{1.9.12}$$

Now writing equation (1.9.1) in the form

$$TdS = c_v dT + pd\tau$$

and using the differential form of equation (1.9.11) in the form

$$pd\tau + \tau dp = RdT = (c_p - c_v)dT$$

we have

$$dS = C_v \frac{dp}{p} + C_p = \frac{d\tau}{\tau}. \tag{1.9.13}$$

Setting $C_p/C_v = \gamma$, the *adiabatic exponent* of the gas, we may integrate (1.9.13) to yield

$$p = A(S)\rho^{\gamma} \tag{1.9.14}$$

where

$$A(S) = p_0 \tau_0^{\gamma} \exp\{(S - S_0)/C_v\} \tag{1.9.15}$$

in which the initial conditions have been denoted by the suffix 0. This shows that *for a polytropic gas the coefficient A(S) in equation (1.9.14) is a function of the entropy only and does not depend on the nature of the gas.*

## SUMMARY

In this unit, the concept of MHD and related fields along with their applications in various fields, e.g. geophysics, astrophysics etc. has been given. The basic equations and their consequences have been derived. Various laws and results in MHD are also given. The waves arising in conducting incompressible viscous fluid have been discussed. Different types of boundary conditions involved in MHD are obtained. An outline of thermodynamical considerations, needed for later Unit, has also been sketched.

## MODEL QUESTIONS

**Short questions :**

1. What is the difference between MHD and MFD?

2. What are the basic results led in MFD?

3. Give some applications of MFD in cosmic problems in geophysics and astrophysics.

4. Show that the change decays very rapidly in an exponential manner at any point within a conducting fluid at rest.

5. Define : Magnetic diffusivity (or magnetic viscosity), magnetic Reynolds number, relaxation time, isorotational (or isotactial or magnetic stream) surfaces, Alfvén velocity, MHD wave, polytropic gas, adiabatic exponent, magnetic stream function, vortex sheet.

6. State Alfvén's theorem, Ferraro's law of isorotation.

7. Show that the electrostatic potential over an isorotational surface is constant.

8. Show that Alfvén waves are transverse waves and are propagated in incompressible conducting fluids.

9. Show that MHD waves are longitudinal and require compressible conducting fluid of infinite conductivity for their propagation.

10. State the initial and boundary conditions involved in MHD flow.

11. Show that the normal component of magnetic induction is discontinuous at the interface.

12. Show that the tangential component of the magnetic field intensity is discontinuous across the surface.

13. Show that the normal component of the electric displacement and tangential component of the electric field are continuous across the interface.

14. Show that the tangential component of the current density is discontinuous only if the electrical conductivities of the media are different.

15. Show that for a polytropic gas, the coefficients A(S) in the pressure density relation $p = A(S)\rho^{\gamma}$, $\gamma$ being the adiabatic exponent, is a function of the entropy S only and not on the nature of the gas.

**Broad questions :**

1. Deduce the basic equations of motion of a conducting fluid.

2. Deduce the equation giving the rate of flow of charge moving along with the fluid and discuss the cases when the fluid is at rest and is in motion.

3. State and prove Alfvén's theorem. When are the magnetic lines of force called 'frozen' the fluid?

4. Show that the magnetic body force per unit volume for a conducting fluid in a magnetic field is equivalent to a tension per unit area along the lines of force, together with a hydrostatic pressure.

5. Discuss Alfvén waves considering its direction along the lines of magnetic force and the fluid velocity is at right angles to them. Hence discuss the cases if the liquid is of infinite extent and of finite extent.

6. Deduce the law of conservation of energy for MHD.

7. State the principle of conservation of charge. Hence or otherwise show that $\nabla \cdot \mathbf{j} = 0$, for the fluid motion, where $\mathbf{j}$ is the current density.

8. Starting from Biot-Savart law, calculate the divergence of magnetic induction vector B.

9. Starting from Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t},$$

derive equation of continuity of charge

10. Write down the boundary conditions to be satisfied by **B**, **H** and **E**, **D** at the interface between two media of different permeabilities.

11. Starting with Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{D}}{\partial t} \text{ and } \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

show that

$$\nabla \cdot \mathbf{B} = 0 \text{ and } \nabla \cdot \mathbf{D} = p,$$

where symbols have their usual meanings.

# Unit 2 ❑ Exact Solutions of MHD Equations

**Structure**



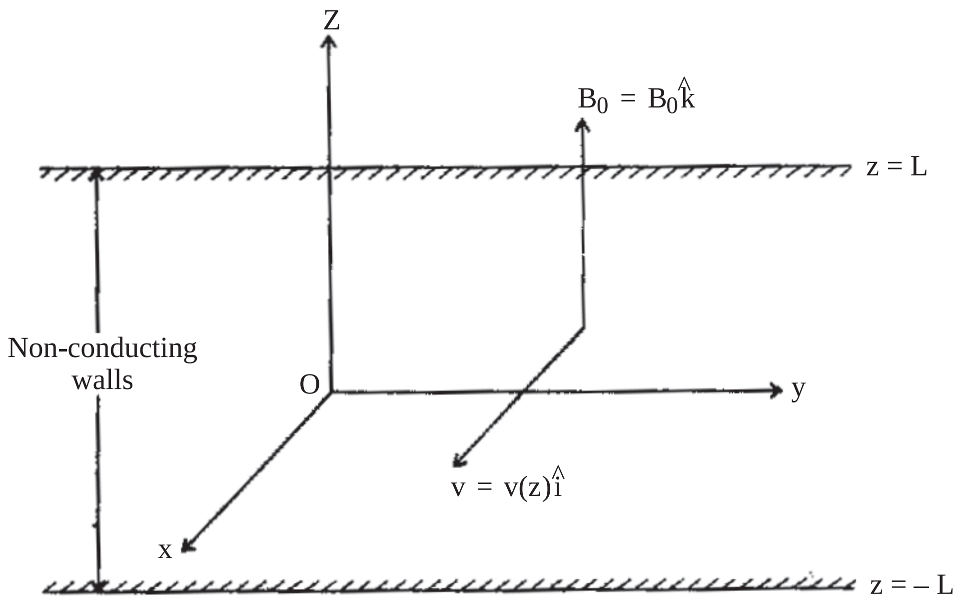
## 2.1 INTRODUCTION

The MHD equations derived in Unit-1 are more complicated and involve more variables than those of ordinary hydrodynamics. However, for a few special cases, we can obtain exact solutions making certain assumptions of the state of the conducting fluid and for simple configuration of the flow pattern. *We assume the fluid to be incompressible and its properties like viscosity, density, electrical conductivity etc. are constants*. Let us now discuss some exact solutions of MHD flow problems.

## 2.2 MHD FLOW BETWEEN PARALLEL PLATES

**Problem 1 :** *Steady laminar flow of a viscous conducting liquid between two horizontal parallel plates in a transverse magnetic field (Hartmann plane Poiseulle flow)*

Suppose a highly conducting viscous liquid, say mercury, flows between two parallel non-conducting horizontal plates and the liquid is ucied on by a uniform transverse magnetic field perpendicular to the plates. As the fluid particles tend to bind themselves to the magnetic field, so the field will inhibit the motion of the liquid in some way. The motion of the liquid then produces tension to the lines of force which can revert to their initial positions because of finite conductivity.




**Fig–2.2(a) :** Hartmann—Plane Poiseuille flow

Let the parallel planes be $z = \pm L$ and the magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{k}}$, acts across them. Then the motion of the liquid across the magnetic field induces electric current at right angles to the liquid velocity $\mathbf{u} = u(z)\hat{\mathbf{i}}$ and applied magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{k}}$. The Lorentz force on the moving stream opposes the motion together with the viscous forces. It is obvious that the equation of continuity $\nabla \cdot \mathbf{v} = 0$ is satisfied identically.

Now the conducting liquid has the tendency to drag the lines of force in the direction of motion and, therefore, the motion of the liquid produces a perturbation field intensity $\mathbf{b} = b(z)\,\hat{\mathbf{i}}$ so that the total magnetic field is

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} \tag{2.2.1}$$

which evidently satisfies the magnetic field continuity equation (0.12.5), viz, $\nabla.\mathbf{B} = 0$.

Let the pressure p(x, z) in the liquid be assumed to be in the form

$$p(x, z) = p_0(x) + p_1(z)$$

in which the first term $p_0(x)$ gives rise to the pressure gradient $-\dfrac{dp_0}{dx}$ in the direction of motion while the second term $p_1(z)$ is ascribable to hydrostatic stress.

Noting that for steady flow $\dfrac{\partial \mathbf{B}}{\partial t} = 0$, the magnetic induction equation (1.4.1) with $\mathbf{H} = \mathbf{B}/\mu_c$ reduces to

$$\nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} = 0 \tag{2.2.2}$$

where $\eta = 1/\mu_e \sigma$ is the magnetic diffusivity.

Also the general equation of motion of conducting liquid for steady condition is given from (1.2.3) by using the continuity equation (1.2.4), viz.

$\nabla.\mathbf{v} = 0$ as

$$\rho(\mathbf{v}.\nabla) = -\nabla(p_0 + p_1) - \rho g \hat{\mathbf{k}} + \frac{1}{\mu_e}(\nabla \times \mathbf{b}) \times \mathbf{B} + \nu\rho \nabla^2 \mathbf{v}. \tag{2.2.3}$$

Noting that $\mathbf{v} \times \mathbf{B} = -u(z)B_0\,\hat{\mathbf{j}}$ and $\nabla \times (\mathbf{v} \times \mathbf{B}) = B_0 u'(z)\,\hat{\mathbf{j}}$, the equation (2.2.2) leads to

$$\frac{d^2 b}{dz^2} + \mu_e \sigma B_0 \frac{du}{dz} = 0 \tag{2.2.4}$$

Again, since $(\nabla \times \mathbf{b}) \times \mathbf{B} = B_0 \dfrac{db}{dz}\,\hat{\mathbf{i}} - b\dfrac{db}{dz}\,\hat{\mathbf{k}}$, we have from (2.2.3)

$$-\frac{dp_0}{dx} + \frac{B_0}{\mu_e}\frac{db}{dz} + \nu\rho \frac{d^2 u}{dz^2} = 0 \tag{2.2.5}$$

$$\rho g + \frac{dp_1}{dz} + \frac{1}{\mu_e} b(z) \frac{db}{dz} = 0 \qquad (2.2.6)$$

From (2.2.5) we have

$$\frac{B_0}{\mu_e} \frac{db}{dz} + v\rho \frac{d^2u}{dz^2} = \frac{dp_0}{dx} \qquad (2.2.7)$$

The left hand side of (2.2.7) is a function of z alone while the right hand side is a function of x only and, therefore, each is equal to the same constant. Thus the pressure gradient, for steady laminar flow, in the idrection of motion remains constant throughout the liquid. Again, integrating (2.2.6) with respect to z, we obtain

$$p_1(z) = c_1 - \rho g z - \frac{1}{2\mu_e} b^2, \qquad (2.2.8)$$

where $c_1$ is integration constant. Also integration of (2.2.4) with respect to z gives

$$\frac{db}{dz} + \sigma\mu_e B_0 L = c_2, \qquad (2.2.9)$$

$c_2$ being integration constant.

Now using equations (0.12.7) and (0.14.5), viz. $\nabla \times \mathbf{B} = \mu_c \mathbf{j}$ and $\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\mathbf{j} = (j_1, j_2, j_3)$, we have

$$j_1 = \sigma E_1 = 0,\ j_2 = \sigma(E_2 - B_0 u) = \frac{1}{\mu_e}\frac{db}{dz},\ j_3 = \sigma E_3 = 0. \qquad (2.2.10)$$

so that $\mathbf{E} = (0, E_2, 0)$ where

$$E_2 = B_0 u + \frac{1}{\sigma\mu_e}\frac{db}{dz} = \frac{c_2}{\sigma\mu_e}, \quad \text{[using (2.2.10) and (2.2.9)]}$$

$$\text{i.e.,}\quad \frac{db}{dz} + \sigma\mu_e B_0 u = \sigma\mu_e E_2 \qquad (2.2.11)$$

The equation (2.2.5) then gives

$$-\frac{dp_0}{dx} + \sigma B_0(E_2 - B_0 u) + v\rho\frac{d^2u}{dz^2} = 0$$

$$\text{or,}\quad v\rho\frac{d^2u}{dz^2} - \sigma B_0^2 u = -(P + \sigma B_0 E_2)$$

$$\text{or,}\quad \frac{d^2u}{dz^2} - \frac{M^2}{L^2}u = -\frac{P + \sigma B_0 E_2}{v\rho} = \text{constant} = -\alpha, \text{ say,} \qquad (2.2.12)$$

where

$$P = -\frac{dp_0}{dx} \text{ and } M = B_0L = \sqrt{\frac{\sigma}{\nu\rho}} = B_0L\sqrt{\frac{\sigma}{\mu}} = \text{ Hartmann number,} \qquad (2.2.13)$$

is a dimensionless quantity, $\mu = \nu\rho$ being the coefficient of viscosity.

The no-slip conditions on the non-conducting boundaries are

$$u = 0, \; b = 0 \text{ on } z = \pm L \qquad (2.2.14)$$

It may be easily seen that the conditions $j_3 = 0$ at $z = \pm L$ are identically satisfied.

The solution of the equation (2.2.12) subject to the conditions $u = 0$ at $z = \pm L$ is

$$u(z) = \frac{\alpha L^2}{M^2 \cosh M}\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\} \qquad (2.2.15)$$

Then, from (2.2.10), it follows that

$$j_2 = \sigma\left[E_2 - \frac{\alpha L^2 B_0}{M^2 \cosh M}\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\}\right] \qquad (2.2.16)$$

If there is no externally applied current, we have

$$\int_{-L}^{L} j_2 dz = 0$$

so that using (2.2.16) we derive)

$$E_2 = \frac{P}{\sigma B_0}(M \coth M - 1) \qquad (2.2.17)$$

whence from (2.2.15) we get by the use of $\alpha = \frac{P + \alpha B_0 E_2}{\nu\rho}$ as given in (2.2.12)

$$u(z) = \frac{PM}{\sigma B_0^2 \sinh M}\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\} \qquad (2.2.18)$$

Again, substituting (2.2.17) and (2.2.18) into (2.2.11) and then integrating with respect to z, we obtain

$$b(z) = \frac{\mu_e PL}{B_0 \sinh M}\left\{\sinh\left(\frac{M}{L}z\right) - \frac{z}{L}\sinh M\right\} \tag{2.2.19}$$

where we have used the conditions b = 0 at z = ±L.

The mean velocity over the section is obtained from (2.2.18) as

$$\bar{u} = \frac{1}{2L}\int_{-L}^{L} u(z)\,dz = \frac{P}{\sigma B_0^2}(M\coth M - 1) \tag{2.2.20}$$

so that (2.2.18) can be written as

$$u(z) = \frac{\bar{u}M\left\{\cosh M - \cosh\left(\frac{M}{L}z\right)\right\}}{M\cosh M - \sinh M} \tag{2.2.21}$$

For weak magnetic field, M = 0 and we have

$$u(z) = \lim_{M\to 0}\frac{\bar{u}M\left\{\left(1 - \frac{M^2}{2!} + \frac{M^4}{3!}\cdots\right)\right\} - \left(1 - \frac{M^2}{2!}\frac{z^2}{L^2} + \frac{M^4}{4!}\frac{z^4}{L^4}\cdots\right)}{M\left(1 - \frac{M^2}{2!} + \frac{M^4}{4!}\cdots\right) - \left(M - \frac{M^3}{3!} + \frac{M^5}{5!}\cdots\right)}$$

i.e., $u(z) = \frac{3}{2}\bar{u}\left(1 - \frac{z^2}{L^2}\right)$ (2.2.22)

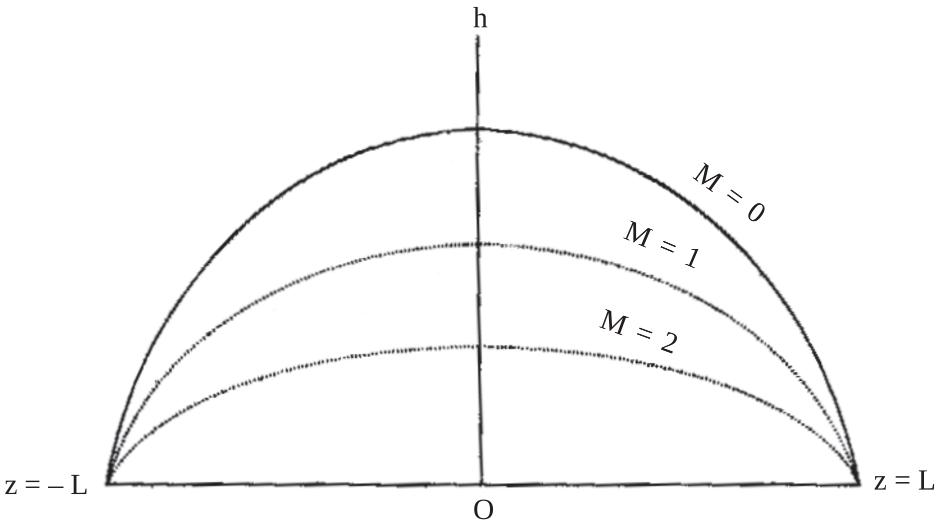



Fig-2.2(b) : Velocity distribution for different values of Hartmann number M which shows that the velocity profile is parabolic for viscous flow in the absence of magnetic field.

The above figure-2.2(b) gives a sketch of the velocity profiles for various values of the Hartmann number M.

**Problem–2 : Magnetohydrodynamic Couette flow :**

Let us consider a viscous incompressile electrically conducting fluid of uniform density ρ flowing steodily between a horizontal conducting plane Z = 0 (lower) and a non-conducting plane Z = L (upper) of which the lower plane is held at rest, but the upper one moves horizontally with uniform velocity $v\hat{\mathbf{j}}$. Suppose a uniform magnetic field $B_0\hat{\mathbf{k}}$ acts vertically upwards and there is no pressure gradient in the liquid. The velocity at any point (x, y, z) of the liquid is $v(z)\hat{\mathbf{j}}$ and the new magnetic

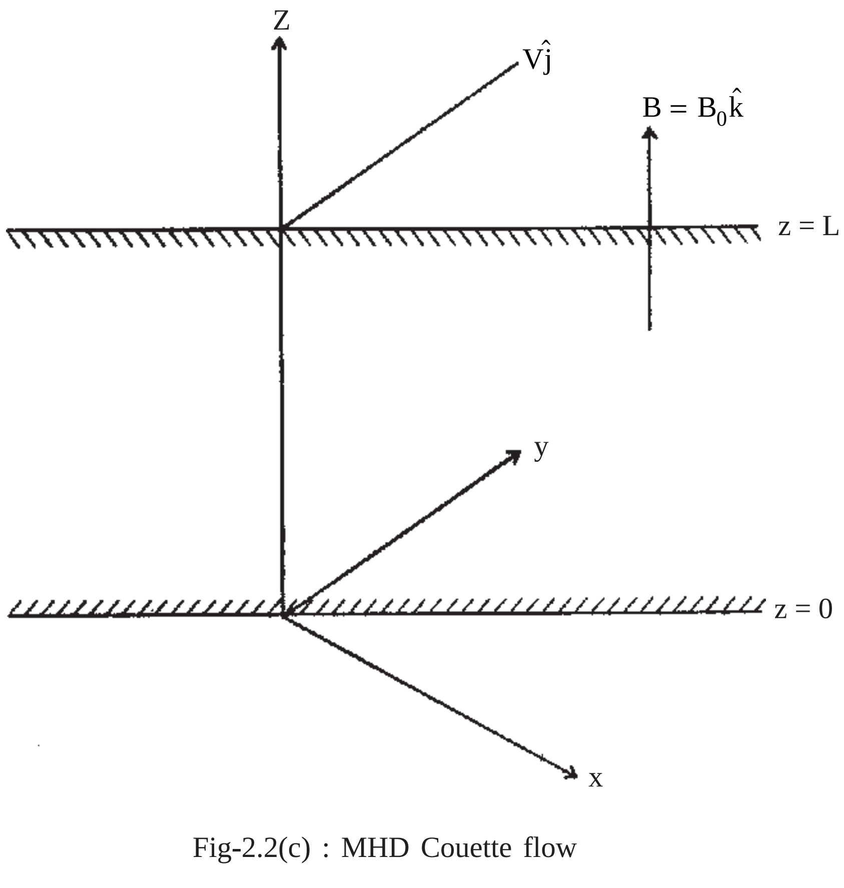



Fig-2.2(c) : MHD Couette flow

field is $\mathbf{B} = B_0\hat{\mathbf{k}} + b(z)\hat{\mathbf{j}}$. It is evident that the magnetic field B satisfies the equation (0.12.5), viz $\nabla.B = 0$.

For steady motion, the magnetic induction equation (1.4.1) with $\mathbf{H} = \mathbf{B}/\mu_e$ gives

$$\nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\nabla^2\mathbf{B} = 0$$

Since $\mathbf{v} = v(z)\hat{\mathbf{j}}$ and $\mathbf{B} = \mathbf{B}_0\hat{\mathbf{k}} + b(z)\hat{\mathbf{j}}$, the above equation leads to

$$B_0\frac{dv}{dz} + \eta\frac{d^2b}{dz^2} = 0 \tag{2.2.23}$$

Also the equation of motion (1.2.3) with the help of continuity equation $\nabla.\mathbf{v} = 0$ reduces for steady condition to

$$\rho(\mathbf{v}.\nabla)\mathbf{v} = -\rho g\hat{\mathbf{k}} - \nabla p + \frac{1}{\mu_e}(\nabla \times \mathbf{b}) \times \mathbf{B} + \nu\rho\nabla^2\mathbf{v}$$

from which we obtain

$$0 = \frac{B_0}{\mu_e}\frac{db}{dz} + \nu\rho\frac{d^2v}{dz^2} \tag{2.2.24}$$

$$\text{and } 0 = -\rho g - \frac{dp}{dz} - \frac{b}{\mu_e}\frac{db}{dz}. \tag{2.2.25}$$

Integrating (2.2.23) and (2.2.25) we have

$$B_0 v + \eta\frac{db}{dz} = \text{ constant } = c_1, \text{ say} \tag{2.2.26}$$

$$\text{and } p(z) + \rho gz + \frac{1}{2\mu_e}b^2(z) = \text{ constant.} \tag{2.2.27}$$

Noting that v = 0 and $\frac{db}{dz} = 0$ (since the plane is perfectly conducting) at z = 0, we have $c_1 = 0$, Thus

$$\frac{db}{dz} = -\frac{B_0}{\eta}v \tag{2.2.28}$$

Substituting this in (2.2.24) we obtain

$$\nu\rho\frac{d^2v}{dz^2} - \frac{B_0^2}{\mu_e\eta}v = 0$$

$$\text{or,}\quad \nu\rho\frac{d^2v}{dz^2} - \sigma B_0^2 v = 0. \qquad \left(\because\ \eta = \frac{1}{\mu_e\sigma}\right)$$

$$\text{or,}\quad \frac{d^2v}{dz^2} - \frac{M^2}{L^2}v = 0 \tag{2.2.29}$$

where $M = B_0 L\sqrt{\dfrac{\sigma}{\nu\rho}}$ = Hartmann number. The solution of the equation (2.2.29) subject to the conditions v = 0 at z = 0 and v = V at z = L is

$$v(z) = V.\frac{\sinh(Mz/L)}{\sinh M} \tag{2.2.30}$$

which gives the velocity of the liquid.

Substitution of (2.2.30) into (2.2.28) and integrating we have

$$b(z) = \frac{B_0 VL}{M\eta \sinh M}\left\{\cosh M - \cosh\left(\frac{Mz}{L}\right)\right\}$$

$$\text{i.e.,}\quad b(z) = \frac{\mu_e\sigma B_0 VL}{M\sinh M}\left\{\cosh M - \cosh\left(\frac{Mz}{L}\right)\right\} \tag{2.2.31}$$

where we have used the condition b = 0 at z = L. The relation (2.2.31) gives the required magnetic field.

## 2.3 MHD FLOW IN A TUBE OF RECTANGULAR CROSS-SECTION

Let us consider the steady laminar flow of a viscous incompressible electrically conducting fluid through a tube of cross-section in the form of a rectangle and the fluid is acted on by a uniform transverse magnetic field $B_0\hat{\mathbf{j}}$. The flow, due to a constant pressure graident, is in the direction of the z-axis. With these assumptions, we have

$$\frac{\partial \mathbf{v}}{\partial t} = 0, \frac{\partial \mathbf{B}}{\partial t} = 0, \mathbf{v} = \{0, 0, w(x, y)\}, \mathbf{B} = \{0, B_0, b(x, y)\}.$$

Then the MHD equations (1.2.3) and (1.4.1) give

$$\frac{1}{\mu_e} b\frac{\partial b}{\partial x} = -\frac{\partial p}{\partial x} \tag{2.3.1}$$

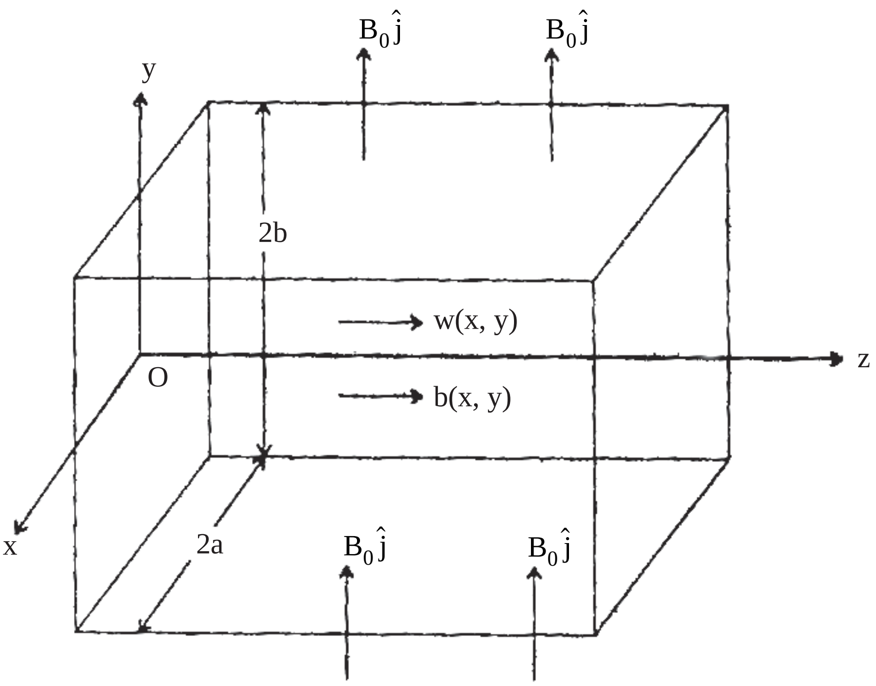



Fig. 2.3 : MHD flow in a rectangular tube

$B_0\hat{j}$

$$\frac{1}{\mu_e} b \frac{\partial b}{\partial x} = -\frac{\partial p}{\partial x} \tag{2.3.2}$$

$$\nu\rho\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{\mu_e} B_0 \frac{\partial b}{\partial y} = \frac{\partial p}{\partial z} \tag{2.3.3}$$

$$\text{and } \eta\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2}\right) + B_0 \frac{\partial w}{\partial y} = 0. \tag{2.3.4}$$

From (2.3.1) and (2.3.2), it follows that

$$p = p_0 - P_0 z - \frac{1}{2\mu} b^2$$

where $P_0$ is constant and

$$\frac{\partial p}{\partial z} = -P_0 \text{ (constant)} \tag{2.3.6}$$

Then equations (2.3.3) and (2.3.4) give

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{B_0}{\mu_e \nu\rho}\frac{\partial b}{\partial y} + \frac{P_0}{\nu\rho} = 0 \tag{2.3.7}$$

and $$\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{B_0}{\eta}\frac{\partial w}{\partial y} = 0 \tag{2.3.8}$$

Now suppose that the cross-section of the tube is bounded by the planes $x = \pm a$ and $y = \pm b$ and the arrangement is an open circuit arrangement so that the induced magnetic field on the boundaries is zero. Thus the boundary conditions are

$$w = 0,\ b = 0 \text{ on } x = a,\ y = \pm b. \tag{2.3.9}$$

The coupled differential equations (2.3.7) and (2.3.8) can be made decoupled by introducing the following variables V and W defined by

$$V = w + \frac{(\sigma\nu\rho)^{-\frac{1}{2}}}{\mu_e} b$$

and $$W = \omega - \frac{(\sigma\nu\rho)^{-\frac{1}{2}}}{\mu_e} b \tag{2.3.10}$$

Then from (2.3.7) and (2.3.8) we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{M}{a}\frac{\partial V}{\partial y} + \frac{P_0}{\nu\rho} = 0 \tag{2.3.11}$$

and $$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} - \frac{M}{a}\frac{\partial W}{\partial y} + \frac{P_0}{\nu\rho} = 0 \tag{2.3.12}$$

where $M = B_0 a\sqrt{\frac{\sigma}{\nu\rho}} =$ Hartmann number. The boundary conditions are then

$$V = W = 0 \text{ at } x = \pm a,\ y = \pm b. \tag{2.3.13}$$

It may easily be seen that it is sufficient to solve only one equation, say V, as their solutions are connected by the relation

$$V(x, y) = W(x, -y) \tag{2.3.14}$$

A particular integral of the equation (2.3.11), satisfying the boundary conditions at $x = \pm a$ is

$$V_1 = \frac{P_0}{2\nu\rho}(a^2 - x^2) \tag{2.3.15}$$

which, however, does not satisfy the boundary conditions at $y = \pm b$. We, therefore, seek the solution $V_2$ satisfying the homogeneous equation

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} + \frac{M}{a}\frac{\partial V_2}{\partial y} = 0 \tag{2.3.16}$$

so that $V = V_1 + V_2$ satisfies the equation (2.3.11) and the boundary conditions $V = 0$ at $x = \pm a$, $y = \pm b$. It is to be noted that $V_2$ should be an even function of x.

To solve the equation (2.3.16) by separation of variables method we put $V_2(x, y) = X(x)Y(y)$ in it and get

$$-\frac{1}{X}\frac{d^2X}{dX^2} = \frac{1}{Y}\frac{d^2Y}{dY^2} + \frac{M}{a}\cdot\frac{1}{Y}\frac{dY}{dY} = k_n^2.\text{(say)},$$

whose solutions are given by

$$X(x) = C\cos(k_n x) + D\sin(k_n x),$$

$$Y(y) = Ee^{m_1 y} + Fe^{m_2 y}, \tag{2.3.17}$$

where $m_1$, $m_2$ are roots of the equation

$$m^2 + \frac{M}{a}m - k_n^2 = 0. \tag{2.3.18}$$

Since $V_2$ is even function of x, so $D = 0$ and then by superposition, we have from

$$V = \frac{P_0}{2\nu\rho}\left(a^2 - x^2\right) + \sum_{n=0}^{\infty}\left\{E_n e^{m_1 y} + F_n e^{m_2 y}\right\}\cos(k_n x) \tag{2.3.19}$$

The boundary condition $V = 0$ at $x = \pm a$ gives $\cos(k_n a) = 0$, i.e., $k_n = \frac{(2n+1)\pi}{2a}$.

Also the boundary condition V = 0 at y = ± b gives

$$-\frac{P_0}{2\nu\rho}(a^2 - x^2) = \sum_{n=0}^{\infty}\left\{E_n e^{m_1 b} + F_n e^{m_2 b}\right\}\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}, \qquad (2.3.20a)$$

$$-\frac{P_0}{2\nu\rho}(a^2 - x^2) = \sum_{n=0}^{\infty}\left\{E_n e^{-m_1 b} + F_n e^{-m_2 b}\right\}\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}, \qquad (2.3.20b)$$

for $-a \le x \le b$. Multiplying both sides of (2.3.20a) by $\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}$ and then integrating between the limits $-a$ to a, we get

$$E_n e^{m_1 b} + F_n e^{m_2 b} = -\frac{16P_0 a^2}{\nu\rho}\cdot\frac{(-1)^n}{(2n+1)^3\pi^3} \qquad (2.3.21a)$$

Similarly, from (2.3.20b) we obtain

$$E_n e^{-m_1 b} + F_n e^{-m_2 b} = -\frac{16P_0 a^2}{\nu\rho}\cdot\frac{(-1)^n}{(2n+1)^3\pi^3} \qquad (2.3.21b)$$

Solving (2.3.21a,b) for $E_n$ and $F_n$ we get

$$E_n = \frac{16P_0 a^2}{\nu\rho}\cdot\frac{(-1)^n}{(2n+1)^3\pi^3}\cdot\frac{\sinh(m_2 b)}{\sinh(m_1 - m_2)b},$$

$$F_n = -\frac{16P_0 a^2}{\nu\rho}\cdot\frac{(-1)^n}{(2n+1)^3\pi^3}\cdot\frac{\sinh(m_1 b)}{\sinh(m_1 - m_2)b}$$

Hence from (2.3.19), we have

$$V = \frac{P_0}{2\nu\rho}\left[\left(a^2 - x^2\right) + \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left\{\frac{e^{m_1 y}\sinh(m_2 b) - e^{m_2 y}\sinh(m_1 b)}{\sinh(m_1 - m_2)b}\right\}\times \cos\left\{\frac{(2n+1)\pi x}{2a}\right\}\right]$$

and, in view of (2.3.14)

$$W = \frac{P_0}{2\nu\rho}\left[(a^2 - x^2) + \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left\{\frac{e^{-m_1 y}\sinh(m_2 b) - e^{-m_2 y}\sinh(m_1 b)}{\sinh(m_1 - m_2)\,b}\right\}\right.$$

$$\left.\times\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}\right].$$

Finally, using (2.3.10) it follows that

$$w = \frac{P_0}{2\mu}\left[(a^2 - x^2) + \frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\times\right.$$

$$\left.\left\{\frac{\cosh(m_1 y)\sinh(m_2 b) - \cosh(m_2 y)\sinh(m_1 b)}{\sinh(m_1 - m_2)b}\right\}\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}\right] \quad (2.3.22)$$

and $$b = \left[\sqrt{\frac{\sigma}{\mu}}\cdot\frac{16P_0 a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\times\right.$$

$$\left.\left\{\frac{\sinh(m_1 y)\sinh(m_2 b) - \sinh(m_2 y)\sinh(m_1 b)}{\sinh(m_1 - m_2)b}\right\}\cos\left\{\frac{(2n+1)\pi x}{2a}\right\}\right], \quad (2.3.23)$$

The volume flow rate Q is given by

$$Q = \int_{-b}^{b}\int_{-a}^{a}\omega\,dx\,dy$$

$$= \frac{4P_0 a^3}{\mu}\left\{\frac{1}{3}b + \frac{128}{\pi^4}\sum_{n=0}^{\infty}\frac{(m_2 - m_1)\sinh(m_1 b)\sinh(m_2 b)}{m_1 m_2 (2n+1)^4\sinh(m_1 - m_2)b}\right\}$$

(2.3.24)

In the above, $\mu(=\nu\rho)$ is the coefficient of viscosity.

## 2.4 MHD FLOW IN A CIRCULAR PIPE

Let us consider the motion of an electrically conducting liquid along a uniform circular pipe under a uniform transform magnetic field $B_0$. The flow is in positive z-direction, the magnetic field is in y-direction and the pipe is electrically non-conducting.

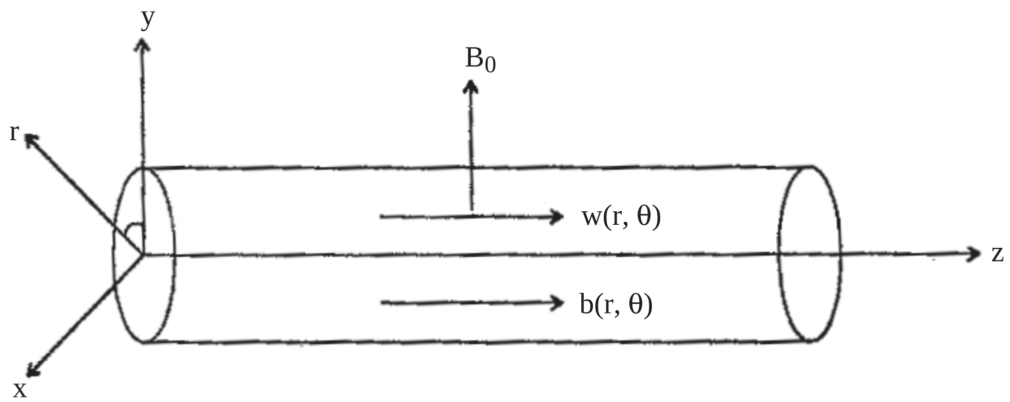



Fig-2.4 : MHD circular pipe flow

In this case, it is convenient to start first with Cartesian coordinates and then convert the equations into polar coordinates. So we have to solve equations (2.3.7) and (2.3.8) subject to the boundary conditions

$$w = 0,\ b = 0 \text{ at } r = a$$

$$\text{and } w, b \text{ are finite at } r = 0, \tag{2.4.1}$$

a being the radius of the pipe,

For convenience, we introduce the following non-dimensional quantities :

$$w^* = \frac{w}{v_0},\ b^* = \frac{b}{v_0\sqrt{\sigma\nu\rho}},\ x^* = \frac{x}{a},\ y^* = \frac{y}{a},\ P = \frac{a^2 P_0}{\nu\rho v_0} \tag{2.4.2}$$

$$M = B_0 a\sqrt{\frac{\sigma}{\nu\rho}} = \text{Hartmann number},$$

$$R_m = \sigma\mu_e v_0 a = \text{ magnetic Reynolds number},$$

where $v_0$ is a characteristic velocity, Using (2.4.2), the equations (2.3.7) and (2.3.8) reduce (omitting asterisks) to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + M\frac{\partial b}{\partial y} + P = 0$$

and $$\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + M\frac{\partial w}{\partial y} = 0$$

respectively and these equations are decoupled to

$$\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} + M\frac{\partial V}{\partial y} + P = 0 \qquad (2.4.3)$$

and $$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + M\frac{\partial W}{\partial y} + p = 0 \qquad (2.4.4)$$

by putting

$$V = w + b, \quad W = w - b \qquad (2.4.5)$$

It is to be noted that for a circular pipe

$$V(x, y, M) = W(x, y, -M) \qquad (2.4.6)$$

To make the equation (2.4.3) homogeneous, we introduce

$$V(x,y,M) = \chi(x,y,M) - \frac{Py}{M} \qquad (2.4.7)$$

in it and get

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + M\frac{\partial \chi}{\partial y} = 0 \qquad (2.4.8)$$

Introducing the transformation

$$\zeta = \chi e^{My/2} \qquad (2.4.9)$$

the equation (2.4.8) reduces to

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{\partial^2 \zeta}{\partial y^2} - \frac{M^2}{4}\zeta = 0 \qquad (2.4.10)$$

which, in cylindrical coordinates x = r cos θ, y = r sin θ, is changed to

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} - \frac{M^2}{4}\zeta = 0 \tag{2.4.11}$$

The boundary condition is

$$\left.\begin{aligned} &\zeta = \frac{P\cos\theta}{M} e^{M\cos\theta/2} \text{ on } r = 1 \\ \text{and} \quad &\zeta \text{ is finite at } r = 0. \end{aligned}\right\} \tag{2.4.12}$$

The solution of (2.4.11) finite at r = 0 is

$$\zeta = \sum_{n=0}^{\infty} A_n I_n \left(\frac{Mr}{2}\right) \cos n\theta \tag{2.4.13}$$

where $A_n$ are constants and $I_n$ are modified Bassel functions of order n of first kind. It is to be noted that since the flow is symmetrical about the magnetic field, we have taken cos θ and not sin θ.

Using the boundary condition (2.4.12) we have

$$\frac{P\cos\theta}{M} e^{M\cos\theta/2} = \sum_{n=0}^{\infty} A_n I_n'(M/2)\cos n\theta, \text{ for all } \theta$$

which leads to

$$A_n = \frac{2P}{M} \cdot \frac{I_n'(M/2)}{I_n(M/2)}$$

Thus

$$V = -\frac{P}{M} r\cos\theta + \sum_{n=0}^{\infty}\left[\frac{2PI_n'(M/2)}{MI_n(M/2)} \cdot I_n(Mr/2)\cos n\theta\right]$$

$$\text{and} \quad W = \frac{P}{M} r\cos\theta + \sum_{n=0}^{\infty}\left[\frac{2P(-1)^n I_n'(M/2)}{MI_n(M/2)} \cdot I_n(Mr/2)\cos n\theta\right]$$

Hence the use of (2.4.5) gives the required velocity and magnetic field as

$$w(r,\theta) = \frac{P}{M}\sum_{n=0}^{\infty}\left\{e^{-(Mr\cos\theta/2)} + (-1)^n e^{(Mr\cos\theta/2)}\right\}\times$$

$$\frac{I_n'(M/2)}{I_n(M/2)} I_n(Mr/2)\cos n\theta \qquad (2.4.14)$$

and $$b(r,\theta) = -\frac{P}{M} r\cos\theta + \frac{P}{M}\sum_{n=0}^{\infty}\left\{e^{-(Mr\cos\theta/2)} - (-1)^n e^{(Mr\cos\theta/2)}\right\}\times$$

$$\times\frac{I_n'(M/2)}{I_n(M/2)} I_n(Mr/2)\cos n\theta \qquad (2.4.15)$$

The mean velocity is obtained, by integrating (2.4.14) over the circular cross-section, i.e.

$$\tilde{w} = \frac{1}{\pi.I^2}\int_{r=0}^{1}\int_{0=0}^{2\pi} w(r,\theta).\, rd\theta\, dr$$

which gives

$$\tilde{w} = -\frac{PI_1^3(M/2)}{MI_0(M/2)} + \frac{P}{M}\sum_{n=0}^{\infty}\left\{(-1)^{n+1}\cdot\frac{I_{n-1}(M/2)+I_{n+1}(M/2)}{I_n(M/2)}\cdot I_{n-1}(M/2)I_{n+1}(M/2)\right\}$$

(2.4.16)

When $M \to 0,\ \tilde{w} \to \frac{P}{8}$.

## 2.5 MHD FLOW IN AN ANNULAR CHANNEL

Consider the steady motion of an electrically conducting incompressible fluid in the annular region of two concentric infinite circular cylinders of radii a and c (c > a) under an applied radial magnetic field $B_0a/r$, where $B_0$ is constant. The flow is in the axial directon which we take as the aixs of z, due to an applied pressure graident. The equation of continuity $\nabla.\mathbf{v} = 0$ then gives $\mathbf{v} = \{0, 0, w(r)\}$. Moreover, the symmetry of the problem states that the induced magnetic field in θ-direction is zero.

In addition, the continuity of the normal component of the magnetic field across the boundary shows that the radial component of the induced magnetic field is also zero. Thus the magnetic field continuity equation $\nabla.\mathbf{B} = 0$ gives $\mathbf{B} = \{B_0a/r,\ 0,\ b(r)\}$

Now from Ohm's law (0.14.5) we have

$$\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B}) = \sigma\left\{0,\ \frac{wB_0a}{r},\ 0\right\}$$

and also

$$\mathbf{j} \times \mathbf{B} = \left\{\frac{\sigma B_0 a}{r} wb, 0, -\frac{\sigma B_0^2 a^2}{r^2} w\right\}.$$

Hence the equation of motion (1.2.3) in cylindrical polar coordinates gives

$$\frac{\sigma B_0 a}{r} wb = \frac{\partial p}{\partial r}, \tag{2.5.1}$$

$$\nu\rho\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) - \frac{\sigma B_0^2 a^2}{r^2} w = \frac{\partial p}{\partial z} \tag{2.5.2}$$

and the Maxwell equation (0.12.7) viz. $\mu_e \mathbf{j} = \nabla \times \mathbf{B}$ gives

$$\frac{\sigma B_0 a}{r} w = -\frac{1}{\mu_e}\frac{db}{dr} \tag{2.5.3}$$

Equations (2.5.1) and (2.5.2) together imply

$$p(r,\ z) = p(r) - zP_0 \tag{2.5.4}$$

where $P_0$ is constant. The equation (2.5.2) can now be written as

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - M^2\frac{w}{r^2} = -\frac{P_0}{\nu\rho}$$

whose solution is

$$w = c_1 r^M + c_2 r^{-M} + f(r) \tag{2.5.5}$$

where

$$f(r) = \begin{cases} -\dfrac{P_0 r^2}{\nu\rho(4 - M^2)}, M \neq 2 \\ -\dfrac{P_0}{\nu\rho} r^2 \log r, M = 2 \end{cases} \tag{2.5.6}$$

and $M = B_0 a\sqrt{\dfrac{\sigma}{\nu\rho}}$ is the Hartmann number. The boundary conditions are

$$w = 0 \text{ at } r = a,\ c. \tag{2.5.7}$$

Hence using these, we have from (2.5.5)

$$w(r) = f(r) - \frac{\{(cr)^M - (a^2c/r)^M\} f(c) - \{(ar)^M - (ac^2/r)^M\} f(a)}{c^{2M} - a^{2M}} \tag{2.5.8}$$

If we take the outer cylinder to be non-conducting so that b = 0 at r = c, then from (2.5.3), the magnetic field is obtained as

$$b(r) = -\mu_e \sigma B_0 a \int_c^r \frac{w}{r}\, dr$$

i.e., $$b(r) = \frac{\mu_e \sigma B_0 a}{M(c^{2M} - a^{2M})}\left[\left\{(cr)^M + \left(\frac{a^2c}{r}\right)^M - c^{2M} - a^{2M}\right\} f(c)\right.$$

$$\left. - \left\{(ar)^M + \left(\frac{ac^2}{r}\right)^M - 2(ac)^M\right\} f(a)\right] - \frac{\sigma B_0 a}{\mu_e}\int_c^r \frac{f(r)}{r}\, dr \tag{2.5.9}$$

where $$\int_c^r \frac{f(r)}{r}\, dr = \begin{cases} -\dfrac{P_0}{\nu\rho(4 - M^2)}(r^2 - c^2), & M \neq 2 \\ -\dfrac{P_0}{8\nu\rho}\left[r^2\left(\log r - \dfrac{1}{2}\right) - c^2\left(\log c - \dfrac{1}{2}\right)\right], & M = 2 \end{cases}$$

## 2.6 MHD FLOW DUE TO A PLANE WALL SUDDENLY SET IN MOTION (RAYLEIGH'S PROBLEM)

Let us now consider the unsteady flow of an electrically conducting liquid due to the impulsive motion of an infinite flat insulated plate with uniform velocity $U_0$ in its own plane in presence of a uniform transverse magnetic field $B_0$ which is otherwise at rest.

Suppose the plate be y = 0 and uniform magnetic field $B_0$ be applied along the direction of y-axis. The nature of flow then suggests the following forms of **v** . **B** and p :

$$\mathbf{v} = \{u(y, t), 0, 0\}, \ \mathbf{B} = \{b(y, t), B_0, 0\}, \ p = \text{constant}.$$

Then the MHD equations (1.2.3) and (1.4.1) with **H** = **B**/$\mu_e$ give

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} + \frac{B_0}{\rho\mu_e}\frac{\partial b}{\partial y}, \tag{2.6.1}$$

$$\frac{\partial b}{\partial t} = B_0\frac{\partial u}{\partial y} + \eta\frac{\partial^2 b}{\partial y^2}, \tag{2.6.2}$$

where $\eta = \dfrac{1}{\mu_e\sigma}$ is the magnetic diffusivity.

The initial conditions are

$$u(y, 0) = 0, \ b(y, 0) = 0 \tag{2.6.3}$$

and the boundary conditions are

$$u(0, t) = U_0, \ b(0, t) = 0$$
$$u, b \to 0 \text{ as } y \to \infty \tag{2.6.4}$$

For convenience, we introduce the following non-dimensional quantities :

$$u^* = \frac{u}{U_0}, \ b^* = \frac{b}{\rho L U_0^2\sqrt{\sigma/\mu_e}}, \ y^* = \frac{y}{L}, \ t^* = \frac{tU_0}{L},$$

$$M = B_0 L\sqrt{\frac{\sigma}{v\rho}} = \text{ Hartmann number}, \tag{2.6.5}$$

$$R_m = U_0 L\sigma\mu_e = \text{magnetic Reynolds number},$$

$$R = U_0 L/v = \text{Reynolds number},$$

where L is a characteristic length.

Then the equations (2.6.1) and (2.6.2) reduce, in non-dimensional form, to (omitting asterisks)

$$\frac{\partial u}{\partial t} = M\frac{\partial b}{\partial y} + \frac{1}{R}\frac{\partial^2 u}{\partial y^2}, \tag{2.6.6}$$

$$\frac{\partial b}{\partial t} = \frac{M}{R.Rm}\frac{\partial u}{\partial y} + \frac{1}{Rm}\frac{\partial^2 b}{\partial y^2}. \tag{2.6.7}$$

The initial and boundary conditions (2.6.3) and (2.6.4), in non-dimensional form, are

$$u(y, 0) = 0, \; b(y, 0) = 0$$
$$u(0, t) = 1, \; b(0, t) = 0 \tag{2.6.8}$$
$$u, b \to 0 \text{ as } y \to \infty.$$

We shall discuss the solutions of (2.6.6) and (2.6.7) for Rm = R and Rm ≪ 1.

**Case I :** $R_m = R$

We decouple equations (2.6.6) and (2.6.7) by substitutions

$$V = u + Rb \text{ and } W = u - Rb \tag{2.6.9}$$

Then these equations give

$$R\frac{\partial V}{\partial t} = M\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial y^2}, \tag{2.6.10}$$

$$R\frac{\partial W}{\partial t} = -M\frac{\partial W}{\partial y} + \frac{\partial^2 W}{\partial y^2}. \tag{2.6.11}$$

It is obvious that

$$W(y, t, M) = V(y, t, -M). \tag{2.6.12}$$

Defining Laplace transform $\bar{\phi}(y,s)$ of the function $\phi(y, t)$ with s as parameter, is

$$\bar{\phi}(y,s) = \int_0^\infty e^{-st}\phi(y,t)\,dt,$$

we see that Laplace-transformed boundary conditions of V(y, t) are

$$\bar{V}(0,s) = \frac{1}{s}, \; V(y,s) \to 0 \text{ as } y \to \infty \tag{2.6.13}$$

and the equation (2.6.10) is transformed to

$$\frac{d^2\bar{V}}{dy^2} + M\frac{d\bar{V}}{dy} - sR\bar{V} = 0$$

whose solution subject to the conditions (2.6.13) is

$$\bar{V}(y,s) = \frac{1}{s}\exp\left[\left\{-\frac{M}{2} - \left(\frac{M^2}{4} + sR\right)^{\frac{1}{2}}\right\}y\right]$$

the inverse transform of which leads to

$$V(y,t) = \frac{1}{2}\left\{\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y + \frac{M}{2}\sqrt{\frac{t}{R}}\right)\right\} + e^{-My}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y - \frac{M}{2}\sqrt{\frac{t}{R}}\right)\Bigg\} \quad (2.6.14)$$

where 'erfcx' is the complementary error function of x defined by

$$\operatorname{erfcx} = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\xi^2} d\xi.$$

Hence, in view of (2.6.12), we get

$$W(y,t) = \frac{1}{2}\left\{\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y - \frac{M}{2}\sqrt{\frac{t}{R}}\right) + e^{My}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y + \frac{M}{2}\sqrt{\frac{t}{R}}\right)\right\}\Bigg\} \quad (2.6.15)$$

Finally, we have by using (2.6.9)

$$u(y,t) = \frac{1}{4}\left\{\left(1+e^{-My}\right)\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y - \frac{M}{2}\sqrt{\frac{t}{R}}\right) + \left(1+e^{My}\right)\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y + \frac{M}{2}\sqrt{\frac{t}{R}}\right)\right\}$$

(2.6.16)

and b(y, t) =

$$\frac{1}{4R}\left\{-\left(1-e^{-My}\right)\right\}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y - \frac{M}{2}\sqrt{\frac{t}{R}}\right) + \left(1-e^{My}\right)\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y + \frac{M}{2}\sqrt{\frac{t}{R}}\right)$$

(2.6.17)

**Case II :** $R_m \ll 1$

In this case the equation (2.6.7) can be written as

$$\frac{\partial^2 b}{\partial y^2} + \frac{M}{R}\frac{\partial u}{\partial y} = 0$$

which, on integration, gives

$$\frac{\partial b}{\partial y} + \frac{M}{R}u = 0, \quad (2.6.18)$$

the constant of integration vanishes by virtue of the boundary conditions at infinity

($\because \nabla \times \mathbf{H} = \mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B})$ and $\mathbf{B} = \mu_e\mathbf{H}$, so $\frac{\partial b}{\partial y} \to 0$ as $y \to \infty$ an du $\to 0$ as $y \to \infty$

by (2.6.8).

Now the equation (2.6.6) with the hep of (2.6.18) becomes

$$\frac{d^2u}{dy^2} - M^2u - R\frac{\partial u}{\partial t} = 0,$$

Laplace transform of which gives

$$\frac{d^2\bar{u}}{dy^2} - M^2\bar{u} - Rs\bar{u} = 0.$$

Its solution satisfying Laplace transformed boundary conditions of (2.6.8) is

$$\bar{u}(y, s) = \frac{1}{s}\exp\left\{-\left(M^2 + Rs\right)^{\frac{1}{2}} y\right\}.$$

Laplace inversion of this leads to the velocity distribution as

$$u(y, t) = \frac{1}{2}\left\{e^{-My}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y - M\sqrt{\frac{t}{R}}\right) + e^{My}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{R}{t}}y + M\sqrt{\frac{t}{R}}\right)\right\}. \quad (2.6.19)$$

The expression for the magnetic field b(y, t) can similarly be obtained.

## SUMMARY

This unit is dealt with the exact solutions of a few MHD flow problems. For this, certain assumptions, like incompressibility and constancy of different properties of the conducting fluid, have been made because of the complicacy of MHD equations.

## MODEL QUESTIONS

**Broad questions :**

1. Solve the problem of steady laminar flow of a viscous conducting incompressible fluid between two horizontal non-conducting parallel plates with no-slip boundary conditions in presence of a uniform transverse magnetic field (Hartmann-plane Poiseuille flow).

2. Solve the problem of MHD Couette flow.

3. Solve the problem of steady laminar flow of a viscous electrically conducting fluid through a tube of rectnagular cross-section with no-slip and non-conducting boundaries and the fluid is acted on by a unifrm transverse magnetic field.

4. Solve the problem of MHD flow in a circular pipe with no-slip and non-conducting boundary conditions in presence of a unform transverse magnetic field.

5. Solve the above problem-4 for an annular channel with the same conditions.

6. Solve the problem of unsteady flow of an electrically conducting liquid due to the impulsive motion of an infinite flat insulated plate moving with uniform velocity in its own plane under the action of a uniform transverse magnetic field which is otherwise at rest (Rayleigh's problem).

7. A viscous incompressible finitely conducting fluid flows steadily under a uniform pressure graident in a channel formed by two infinite parallel plates which are non-conducting. If a uniform magnetic field acts perpendicular to the channel walls, find the velocity and magnetic field in the channel.

8. State and explain Maxwell's electromagnetic field equations governing the motion of conducting fluids. What is Lorentz force?

9. Derive the equation for the magnetic induction in MHD flows and explain the significance of high and low magnetic Reynolds number.

10. A viscous incompressible fluid of uniform density is confined between two horizontal non-conducting planes $z = 0$ (lower) and $z = h$ (upper). The lower plane is held at rest and the upper one is moved horizontally in its own plane with uniform velocity U. A uniform magnetic field $H_0$ acts perpendicular to the planes. Find the velocity and magnetic field between the planes.

11. Show that the magnetic flux linking and loop moving with a perfectly conducting fluid is constant.

12. Show that if a steady axisymmetric motion of a conducting liquid permeated by an axisymmetric magnetic field with no azimuthal component, the liquid at all point of a magnetic field line rotates about the axis of symmetry at a uniform angular velocity.

13. Show that in an infinite mass of an inviscid, perfectly conducting incompressible fluid (of density $\rho$ and magnetic permeability $\mu_e$ permeated by a uniform magnetic field $H_0$, a small disturbance in the magnetic field is propagated in the form of transverse waves along the magnetic lines of force with velocity $H_0(\mu_e/\rho)^{1/2}$.

# Unit 3 ❑ MHD Boundary Layer Flow

**Structure**



## 3.1 INTRODUCTION

In 1904, Prandtl introduced the concept of boundary layer in fluid mechanics on the hypothesis that for fluids with small viscosity, the flow about a solid body can be divided into regions : (i) a very thin layer in the neighbourhood of the body, known as the *velocity boundary layer* or *viscous boundary* layer and (ii) the region outside this layer where the viscous effects can be considered to be negligible and the fluid is regarded as inviscid. According to this hypothesis, Navier-Strokes equations are reduced to mathematically simplified form and it is possible to give physical explanation of the importance of viscosity in the assessment of frictional drag and flow separation.

The MHD boundary layers, which are of recent origin, may be classified into two types by considering the limiting cases of large and small electrical conductivity. If the electrical conductivity is large (i.e. magnetic diffusivity $\eta = 1/\sigma\mu_e$ is small), then the magnetic Reynods number $R_m = \frac{\mu_e \sigma U_0}{L}$ is also large and the diffusion of the magnetic field takes place in a narrow zone, called the *magnetic boundary layer* and is of the same size as the viscous boundary layer. In this case, the axial component of the externally applied magnetic field differentiates the MHD equations from ordinary hydrodynamic equations and for incompressible flow, the MHD equations are solved simultaneously for the velocity and magnetic field. On the other hand, if the electrical conductivity is small, that is, if the magnetic Reynolds number is small, then the thickness of the magnetic boundary layer is very large and the flow direction component of the magnetic interaction is a function of the transverse magneticfield and local velocity in the flow direction. Changes in the transverse magnetic field component and pressure across the boundary layer are negligible. The induced magnetic field is neglected in comparison with the applied magnetic field in the transverse direction.

## 3.2 TWO-DIMENSIONAL MHD BOUNDARY LAYER EQUATIONS FOR FLOW OVER A PLANE SURFACE FOR FLUIDS OF LARGE ELECTRICAL CONDUCTIVITY

The governing equations of motion of viscous incompressible electrically conducting fluid as given in Unit-1 are

$$\nabla.\mathbf{v} = 0 \tag{3.2.1}$$

$$\rho \frac{Dv}{Dt} = -\nabla\left(p + \frac{B^2}{2\mu_e}\right) + \frac{1}{\mu_e}(\mathbf{B}.\nabla)\mathbf{B} + v\rho\nabla^2\mathbf{v} \tag{3.2.2}$$

$$\nabla.\mathbf{B} = 0 \tag{3.2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{B} - (\mathbf{B}.\nabla)\mathbf{v} = \eta\nabla^2\mathbf{B} \tag{3.2.4}$$

For two-dimensional motion, we set

$$\mathbf{v} = \{u(x, y, t), v(x, y, t), 0\} \quad \mathbf{B} = \{B_X(x, y, t), B_y(x, y, t), 0\} \tag{3.2.5}$$

and then the equations (3.2.1) to (3.2.4) reduce respectively to

$$\underset{1}{\frac{\partial u}{dx}} + \underset{1}{\frac{\partial v}{\partial y}} = 0 \tag{3.2.6}$$

$$\underset{1}{\frac{\partial u}{dx}} + \underset{1}{\frac{\partial v}{\partial x}} + \underset{\delta}{v}\,\underset{1/\delta}{\frac{\partial u}{\partial y}} = -\frac{\partial}{\partial x}\left(\underset{1}{\frac{p}{\rho}} + \underset{1}{\frac{B_x^2}{2\mu_e\rho}} + \underset{\delta B^2}{\frac{B_y^2}{2\mu_e\rho}}\right)$$

$$+\frac{1}{\mu_e\rho}\left(B_x \underset{1}{\frac{\partial Bx}{\partial x}} + \underset{\delta_B}{By}\underset{1/\delta_B}{\frac{\partial Bx}{\partial y}}\right) + \underset{\delta^2}{v}\left(\underset{1}{\frac{\partial^2 u}{\delta x^2}} + \underset{1/\delta^2}{\frac{\partial^2 u}{\partial y^2}}\right) \tag{3.2.7}$$

$$\underset{\delta}{\frac{\partial v}{\partial t}} + u\underset{\delta}{\frac{\partial v}{\partial x}} + \underset{\delta}{v}\underset{1}{\frac{\delta v}{\partial y}} = -\frac{\partial}{\partial y}\left(\underset{1}{\frac{p}{\rho}} + \underset{1}{\frac{B_y^2}{2\mu_e\rho}} + \underset{2}{\frac{B_y^2}{2\mu_e\rho}}\right)$$

$$+\underset{\delta}{\frac{1}{\mu_e\rho}}\left(B_x \underset{\delta_B}{\frac{\partial B_x}{\partial x}} + B_y\underset{\delta_B}{\frac{\partial B_y}{\partial y}}\right) + \underset{\delta^2}{v}\left(\underset{\delta}{\frac{\partial^2 v}{\partial x^2}} + \underset{1/\delta}{\frac{\partial^2 v}{\partial y^2}}\right) \tag{3.2.8}$$

$$\underset{1}{\frac{\partial B_x}{\partial x}} + \underset{1}{\frac{\partial B_y}{\partial y}} = 0, \tag{3.2.9}$$

$$\underset{1}{\frac{\partial B_x}{\partial t}} + u\underset{1}{\frac{\partial B_x}{\partial x}} + \underset{\delta}{v}\underset{1/\delta_B}{\frac{\partial Bx}{\partial y}} - \underset{1}{B_x}\frac{\partial u}{\partial x} - \underset{\delta_B}{By}\frac{\partial u}{\partial y} = \underset{\delta_B^2}{\eta}\left(\underset{1}{\frac{\partial^2 Bx}{\partial x^2}} + \underset{1/\delta_B^2}{\frac{\partial Bx}{\partial y^2}}\right) \tag{3.2.10}$$

$$\underset{\delta_B}{\frac{\partial B_y}{\partial t}} + u\underset{1}{\frac{\partial B_y}{\partial x}} + \underset{\delta}{v}\underset{1/\delta_B}{\frac{\partial B_y}{\delta y}} - B_x\underset{\delta}{\frac{\partial v}{\partial x}} - B_y\underset{\delta}{\frac{\partial u}{\partial y}} = \underset{\delta_B^2}{\eta}\left(\underset{\delta_B}{\frac{\partial^2 B_y}{\partial x^2}} + \underset{1/\delta_B}{\frac{\partial^2 B_y}{\partial y^2}}\right) \tag{3.2.11}$$

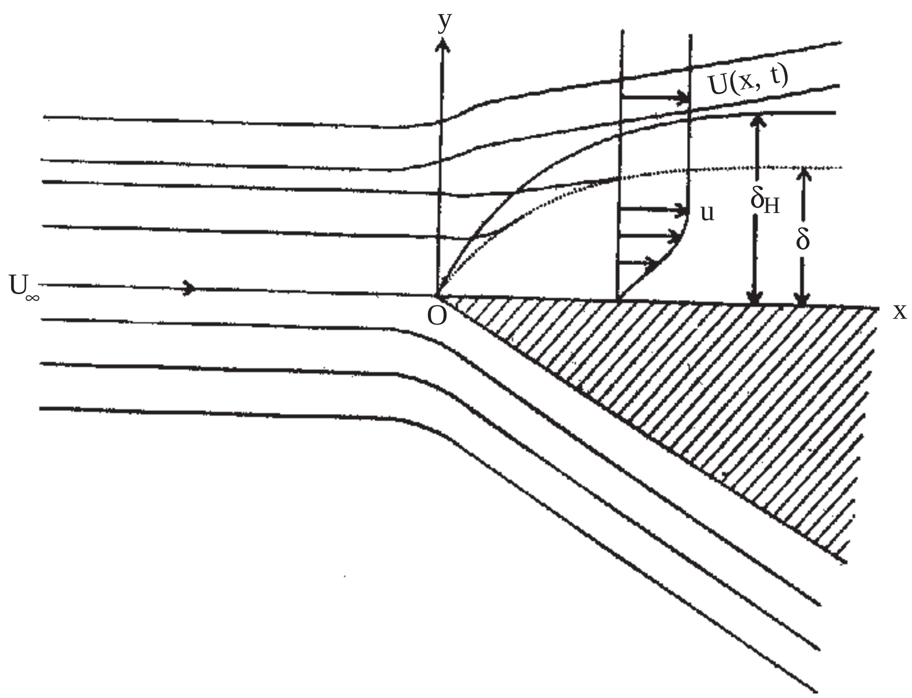
where the plane of motion is the (x, y)-plane, the x-axis is along the wall and y-axis is perpendicular to it. The external magnetic field $B_{X\infty}$ (x, t) is applied in the direction of x-axis.

Assuming no-slip condition and the wall to be solid, we have u = v = 0 at y = 0. Let us now assess the order of magnitude, symbolically 0 ( ), of the terms involved in equations (3.2.6) to (3.2.11). The velocity component u parallel to the wall in the velocity boundary layer rises rapidly from a value zero at the wall to a value U in the main stream within a short distance δ (say), the *thickness of the velocity boundary layer or viscous boundary layer*. In a similar way, the magnetic field component $B_x$ rises rapidly from its nearly zero value to a value $B_{x\infty}$ in the main stream within a short distance $\delta_B$ (say), which is the *thickness of the magnetic boundary layer from the wall*. We take t, x, u and H as quantities of 0(1) and y is of 0(δ) in the velocity variation while y is of $0(\delta_B)$ in magnetic field variation. We have indicated the order of each of magnitude of each term in each equation, such that $\delta << 1$, $\delta_B << 1$, but they themselves are of comparable magnitude ($\delta = \delta_B$). Thus, as in arguments for ordinary fluid dynamics,

$$v = 0(\delta),\ B_y = 0(\delta_B),\ v = 0(\delta^2),\ \eta = 0(\delta_B^2).$$

Hence $$\delta = 0\left(\frac{1}{\sqrt{R}}\right),\ \delta_B = 0\left(\frac{1}{\sqrt{Rm}}\right) \tag{3.2.12}$$

where $R = \dfrac{UL}{v}$ (Reynolds number), $R_m = \dfrac{UL}{\eta}$ (magnetic Reynolds number)




**Fig–3.2** : Coordinate system for two-dimensional MHD boundary layer flow over a plane wall

Neglecting small quantities of order $\delta$, $\delta_B$ and of higher order, the equations (3.2.6) to (3.2.11) reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.2.13}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{B_x^2}{2\mu_e\rho}\right)+\frac{1}{\mu_e\rho}\left(B_x\frac{\partial Bx}{\partial x}+B_y\frac{\partial Bx}{\partial y}\right)+v\frac{\partial^2 u}{\partial y^2}, \tag{3.2.14}$$

$$\frac{\partial}{\partial y}\left(\frac{p}{\rho}+\frac{B_x^2}{2\mu_e\rho}\right)=0, \tag{3.2.15}$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0, \tag{3.2.16}$$

$$\frac{\partial B_x}{\partial t}+u\frac{\partial B_x}{\partial x}+v\frac{\partial B_x}{\partial y}-B_x\frac{\partial U}{\partial x}-B_y\frac{\partial U}{\partial y}=\eta\frac{\partial^2 B_x}{\partial y^2} \tag{3.2.17}$$

The equation (3.2.15) shows that $\frac{p}{\rho}+\frac{B_x^2}{2\mu_e\rho}$ is independent of y in the boundary layer and may be taken to be the same as that outside the boundary layer where it is determined by the inviscid flow. We may therefore write

$$-\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\frac{B_x^2}{2\mu_e\rho}\right)=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}-\frac{B_{x\infty}}{\mu_e\rho}\frac{\partial B_{x\infty}}{\partial x} \tag{3.2.18}$$

where U is the potential flow velocity and $B_{x_\infty}$ is the external applied magnetic field along the direction of x-axis.

The boundary conditions are usually taken as

$$y=0: u=0,\ v=0,\ B_y=0,$$

$$y\to\infty: u=U(x,t),\ H_x=H_{x_\infty}(x,t) \tag{3.2.19}$$

The above set of equations are known as *Prandtl—MHD boundary layer equations for a strong interaction of the magnetic field.* It is to be noted that in addition to R, $R_m$ should also be large.

## 3.3 TWO-DIMENSIONAL MHD BOUNDARY LAYER EQUATIONS FOR FLOW OVER A PLANE SURFACE FOR FLUIDS OF VERY SMALL ELECTRICAL CONDUCTIVITY

In many aeronautical engineering problems, it is found that the magnetic Reyrolds number $R_m$ is usually very small and then the thickness ($\delta_B$) of the magnetic boundary layer is very large. In such a case the approximations applied in section-3.2 are no longer valid and the induced magnetic field due to the flow may be neglected in comparison to the applied magnetic field $\mathbf{B_0}$. Here the transverse component of the magnetic field affects the fluid motion appreciably.

Assuming the fluid properties to be constant, the equations governing the motion of the conducting fluid are

$$\nabla . v = 0, \tag{3.3.1}$$

$$\rho \frac{Dv}{Dt} = -\nabla p + v\rho\nabla^2 v + j \times B_0, \tag{3.3.2}$$

$$\nabla \ B_0 = 0, \tag{3.3.3}$$

$$\nabla \times E = 0, \tag{3.3.4}$$

$$j = \sigma(E + v \times B_0) \tag{3.3.5}$$

We consider two-dimensional flow in at ransverse magnetic field. Thus

$$v = \{u(x,\ y),\ v(x,\ y),\ 0\},$$

$$B_0 = \{0,\ B_{y0}(x),\ 0\},$$

$$E = \{0,\ 0,\ E_z\},$$

$$j = \{0,\ 0,\ j_z\}, \tag{3.3.6}$$

where $E_z$ is constant and $j_z = \sigma(E_z + uB_{y0})$. For fixed magnetic field on a stationary surface, (3.3.6) satisfy the equations (3.3.3) to (3.3.5).

When the magnetic field is in the direction of x-axis moving with velocity $U_0$, then

$$j_z = \sigma\{E_z + (u - U_0)B_{y0}\} \tag{3.3.7}$$

We may now write equations (3.3.1) and (3.3.2) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.3.8}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{B_{y_0} j_z}{\rho} \tag{3.3.9}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial v} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3.3.10}$$

Considering Prandtl boundary layer approximations, noting that the electro-magnetic force is of comparable magnitude with the inertia force and then dropping the negligibly small terms, the boundary layer equations for velocity distribution are obtained from (3.3.8) to (3.3.10) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \tag{3.3.11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2} - B_{y0} j_z \tag{3.3.12}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{3.3.13}$$

where $j_Z$ is given either by $j_Z = \sigma(E_Z + UB_{y0})$ or by (3.3.7) depending on the orientation of the magnetic field. Since the pressure in the boundary layer is the same outside the layer so this is determined by the invisicid flow theory, viz.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \left(\frac{B_{y0} j_z}{\rho}\right)_{u=V} \tag{3.3.14}$$

The boundary conditions for the problem are

$$y = 0 : u = 0,\ v = 0,$$

$$y \to \infty : u = U(x,\ t) \tag{3.3.15}$$

we now consider the following two cases :

(i) When the magnetic field is fixed to the surface which is taken as stationary, then taking $E_Z = 0$, we have from (3.3.14)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{B_{y0}^2 \sigma U}{\rho} \tag{3.3.16}$$

In this case, the velocity U at the outer edge of the boundary layer is affected by the magnetic field. In general, the pressure gradient of the inviscid ordinary flow is prescribed and the value of U is determined. For example, in steady flow with zero pressure gradient, i.e., for flow past a flat plate

$$\frac{\partial U}{\partial x} = -\frac{\sigma B_{y0}^2}{\partial t}$$

If $B_{y0}$ is constant, then,

$$U = U_\infty - \frac{\sigma B_{y0}^2 . x}{\rho}$$

This shows that the magnetic field deaccelerates the inviscid flow. In other words, an adverse pressure gradient is generated in the flow by the magnetic field which retards the fluid motion in the boundary layer.

(ii) If the magnetic field moves with velocity U(i.e. if $U_0 = U$), of the inviscid flow; in other words, the magnetic field is fixed in thefluid, then we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}.$$

In this case the velocity U is not affected by the magnetic field. For zero pressure gradient we have in the case of steady flow $U = U_\infty$.

The above set of equations are *MHD boundary layer equations for a weak interaction of the magnetic field.*

## 3.4 MHD BOUNDARY LAYER FLOW PAST A FLAT PLATE IN AN ALIGNED MAGNETIC FIELD

Let us consider the steady two dimensional motion of a viscous incompressible electrically conducting fluid past a semi-infinite rigid flat plate. We assume that the fluid properties are constant and the applied magnetic field $B_\infty$ is uniform in the direction of the undisturbed stream to the plate and perpendicular to its edge and having a velocity $U_\infty$. In the present case, we have

$$\frac{\partial}{\partial t}(\ ) = 0,\ B_{y\infty} = B_\infty(\text{const.}),\ U = U_\infty(\text{const.})$$

Then the equation (3.2.18) gives

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \frac{B_x^2}{2\mu_e\rho}\right) = 0. \tag{3.4.1}$$

The boundary layer equations (3.2.13) to (3.2.17), therefore, give

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.4.2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\mu_e\rho}\left(B_x\frac{\partial Bx}{\partial x}+B_y\frac{\partial B_x}{\partial y}\right)+v\frac{\partial^2 u}{\partial y^2}, \tag{3.4.3}$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0, \tag{3.4.4}$$

$$u\frac{\partial B_x}{\partial x}+v\frac{\partial B_y}{\partial y}-B_x\frac{\partial u}{\partial x}-B_y\frac{\partial u}{\partial y}=\eta\frac{\partial^2 Bx}{\partial y^2}. \tag{3.4.5}$$

The corresponding boundary conditions are

$$y = 0\ ;\ u = 0,\ v = 0,\ B_y = 0,$$

$$y \to \infty\ ;\ u = U_\infty,\ B_x = B_\infty. \tag{3.4.6}$$

To solve the equation (3.4.1) to (3.4.5), we introduce the stream function $\psi$ and the magnetic potential function $\phi$ by

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x},$$

$$B_x=\frac{\partial\phi}{\partial y},\ B_y=-\frac{\partial\phi}{\partial x} \tag{3.4.7}$$

Then the equations (3.4.2) and (3.4.4) are identically satisfied and the equations (3.4.3) and (3.4.5) reduce respectively to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{1}{\mu_e\rho}\left(\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial y^2}\right)+v\frac{\partial^3\phi}{\partial y^3}, \tag{3.4.8}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\phi}{\partial y^2}-\frac{\partial\phi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial\phi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\eta\frac{\partial^3\phi}{\partial y^3}, \tag{3.4.9}$$

Integrating (3.4.9) with respect to y, we get

$$\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}=\eta\frac{\partial^3\phi}{\partial y^2}, \tag{3.4.10}$$

where the arbitrary function of x, due to integration, vanishes if the electric field E is zero everywhere.

Thus our basic equations are (3.4.8) and (3.4.10) which are to be solved subject to the boundary conditions

$$y = 0 : \psi = 0, \ \frac{\partial \psi}{\partial y} = 0, \ \frac{\partial \phi}{\partial x} = 0,$$

$$y \to \infty : \ \frac{\partial \psi}{\partial y} = U_\infty, \frac{\partial \phi}{\partial y} = B_\infty, \tag{3.4.11}$$

Let us now introduce the following transformations due to Blasins for dependent and independent variables :

$$\psi = \sqrt{(vU_\infty x)}\ f(\xi), \phi = B_\infty \sqrt{\frac{vx}{U_\infty}} g(\xi), \xi = \frac{1}{2} y \sqrt{\frac{U_\infty}{vx}} \tag{3.4.12}$$

Then

$$u = \frac{1}{2} U_\infty f'(\xi), \ \ v = \frac{1}{2}\sqrt{\frac{U_\infty v}{x}}\{\xi f'(\xi) - f(\xi)\},$$

$$B_x = \frac{1}{2} B_\infty g'(\xi), \ \ B_y = \frac{1}{2} B_\infty \sqrt{\frac{v}{U_\infty x}}\{\xi g'(\xi) - g(\xi)\} \tag{3.4.13}$$

and the equations (3.4.8) and (3.4.10) reduce respectively to

$$f''' + f'' - P_m g g'' = 0, \tag{3.4.14}$$

$$g'' + P_{rm}(fg' - f'g) = 0 \tag{3.4.15}$$

where the prime denotes differentiation with respect to $\xi$ and

$$P_m = \frac{B_\infty^2}{\rho \mu_e U_\infty^2} \text{ (magnetic pressure number)},$$

$$P_{rm} = \frac{v}{\eta} = \mu_e v \sigma \text{ (magnetic Prandtl number)} \tag{3.4.16}$$

The magnetic pressure number is interpreted as the square of the ratio of the Alfvén speed to the undisturbed fluid flow speed.

The boundary conditions (3.4.11) then give

$$\xi = 0 : \ f = 0, \ f' = 0, \ g = 0$$

$$\xi \to \infty : \ f' = 2, \ g' = 2 \tag{3.4.17}$$

Greenspan and carrier (1959) have pointed out that for $P_m > 1$, the whole formulation of the problem breaks down, because then the alfvén speed becomes greater than the fluid speed and the disturbances penetrate upstream ahead of the plate. Thus it is not possible to describe the flow in terms of the Blasius variables which are based on leading edge. They obtained the solution for $P_{rm} = 1$ and $(1 - P_m) < 1$ and concluded that the entire flow is plugged, i.e., brought to rest at the critical value $P_m = 1$.

The existence and uniqueness of the solutions of equations (3.4.14) and (3.4.15) have been studied thoroughly by various authors in different directions. However, we consider here a series solution given by Glauert (1961).

**Glauert's Series Solution :**

Let us suppose that $P_m < 1$ and $(1 - P_m)$ is not small. Glauert obtained solutions for large and small magnetic Prandtl number.

**Case I**—*Perfectly conducting fluid*

In this case $\sigma \to \infty$ so that $P_{rm} \to \infty$ and then the equation (3.4.15) reduces to

$$fg' - f'g = 0 \tag{3.4.18}$$

whose only solution satisfying the boundary conditions (3.4.17) is

$$g(\xi) = f(\xi) \tag{3.4.19}$$

which when substituted in (3.4.14) leads to

$$f''' + (1 - P_m)ff' = 0 \tag{3.4.20}$$

This equation is to be integrated under the boundary conditions (3.4.17).

The solution of (3.4.20) is related to the blasius solution $F(\xi)$ [i.e. the solution of (3.4.20) with $P_m = 0$ and the boundary conditions (3.4.17) as in the case of ordinary fluid dynamics] by the relation

$$f(\xi) = (1 - P_m)^{-\frac{1}{2}} F\{(1 - P_m)^{-\frac{1}{2}} \xi\} \tag{3.4.21}$$

where

$$f'(0) = (1 - P_m)^{-\frac{1}{2}} F''(0) = 1.328(1 - P_m)^{\frac{1}{2}}. \tag{3.4.22}$$

However, the result breaks down when $P_m = 1$. Because, in this case the inertial forces are calcelled by the electromagnetic force as a result of which the boundary layer approximations no longer hold. On the other hand, for $0 < P_m < 1$, the solution (3.4.21) shows that the effect of magnetic field is to thicken the viscous boundary layer and reduce the skin friction and the result is valid for all values of $P_{rm}$.

**Case II :** *Large magnetic Prandtl number :*

A series solution of equations (3.4.14) and (3.4.15) were obtained by Glauert by matching the inner and outer expansions of f(ξ) and g(ξ) for large values of $P_{rm}$ and noted that the skin-friction $\tau_\omega$ at the plate and the tangential component $B_t$ of the magnetic field are given by

$$\tau_\omega = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{4}\rho\left(\frac{U_\infty^3 v}{x}\right)^{\frac{1}{2}} f'(0)$$

$$= 0\cdot 33206\rho\left(\frac{U_\infty^3 \nu}{x}\right)^{\frac{1}{2}}(1-P_m)^{\frac{1}{2}}$$

$$\times\left\{1 - 0\cdot 4444P_m \cdot \frac{1}{P_{rm}} \times \log\left(\frac{P_{rm}}{1-P_m}\right) + 0\cdot 577\frac{P_m}{P_{rm}} + 0\left(P_{rm}^{-2}\log^2 P_{rm}\right)\right\}$$

(3.4.23)

and

$$B_t = \frac{1}{2}B_\infty g'(0)$$

$$= 0\cdot 5549B_\infty(1-P_m)^{\frac{1}{3}} P_{rm}^{\frac{1}{3}}\{1-(0\cdot 4148P_m + 0\cdot 1778)\times$$

$$P_{rm}^{-1}\log\left(\frac{P_{rm}}{1-P_m}\right) + (0\cdot 583P_m - 0\cdot 026)P_{rm}^{-1} + 0\left(P_{rm}^{-2}\log^2 P_{rm}\right)\}$$

(3.4.24)

**Case III—***Small magnetic Prandtl number :*

for small electrical conductivity, i.e. for small values of $P_{rm}$, the corresponding values of $\tau_\omega$ and $B_t$ obtained by Glauert, are

$$\tau_\omega = 0\cdot 33206\left(\frac{U_\infty^3 \nu}{x}\right)^{\frac{1}{2}}\left\{1 - \frac{2\cdot 2875P_m \cdot P_{rm}^{\frac{1}{2}}}{(1-P_m)} + 1\cdot 1104P_m \cdot P_{rm} \cdot \log\{(1-P_m)\cdot P_{rm}\} + \right.$$

$$\left. + 11\cdot 814P_m \cdot P_{rm} + 0\cdot 606\cdot\frac{P_m^2 \cdot P_{rm}}{(1-P_m)} + 0\left(P_{rm}^{3/2}\log P_{rm}\right)\right\} \quad (3.4.25)$$

and $B_t = B_\infty \{1 - 1 \cdot 5250(1 - P_m)^{-\frac{1}{2}} P_{rm}^{\frac{1}{2}} + 3 \cdot 028 P_{rm} - 0 \cdot 117$

$$\frac{P_m}{(1 - P_m)} P_{rm} + 0\left(P_{rm}^{3/2} \log P_{rm}\right)\Big\}. \qquad (3.4.26)$$

**Note :** For details of the results considered in this section, the reader should consult with the paper—*Glauert, M.B. : A study of MHD Boundary Layer on a Flat Plate, Journal of Fluid Mechanics, Vol. 10, Part II, pages 276-288 (1961).*

## SUMMARY

This unit is dealt with an outline of steady two-dimensional MHD boundary layer flow. The discussions are limited to the large and small electrical conductivity. Equations are established for different types of flows of which the solutions are given only for one problem.

## MODEL QUESTIONS

**Short questions :**

1. Define : velocity boundary layer (or viscous boundary layer) and magnetic boundary layer, magnetic Prandtl number.

2. What are meant by thickness of velocity boundary layer and that of magnetic boundary layer from the wall?

3. How does the MHD boundary layer theory differ from ordinary boundary layer theory for incompressible viscous fluid?

**Broad questions :**

1. Deduce the Prandtl—MHD boundary layer equations for a strong interaction of the magnetic field.

2. Deduce the MHD boundary layer equations for a weak interaction of the magnetic field.

3. Deduce the basic equations for MHD boundary layer flow past a flat plate in an aligned magnetic field and give a sketch of Glauert's series solutions of these equations for (a) perfectly conducting fluid, (b) large magnetic Prandtl number and (c) small magnetic Prandtl number.

# Unit 4 ❑ Magnetohydrodynamic Shock Waves

**Structure**



## 4.1 GENERAL CONSIDERATIONS

Of the many possible types of magnetohydrodynamic wave motion, as in ordinary gas dynamics, the simplest one is the class of one-dimensional flows which can exist adjacent to a region of constant state, i.e. in a region in which the density ρ, velocity **v,** magnetic field **B** etc. all have constant values. In this case, the flows describe the basic transition flows of magnetohydrodynamics such as compression and expansion processes that must take place if a gas undergoes a change from one constant state to another. These waves represent simple solutions of the magnetohydrodynamic characteristic equations and the behaviour of all the physical quantities is determined by a single ordinary differential equation. It can be shown that it is a direct consequence of the result that the dependent variables ρ, **v**, **B,** involved in the wave motion adjacent to the constant state may all be expressed as functions of one of these dependent variables, say ρ, Magnetohydrodynamic waves having this special property is called *magnetohydrodynamic simple waves*.

For simple waves in a perfect gas, it is seen that under certain conditions the wave profiles of the dependent variables can steepen until at a certain time $t = t_c$ (say), they develop an infinite gradient. It is also noted that the solution is not unique at time $t_c$ and that it is sufficient to take this non-uniqueness in the form of an ordinary jump discontinuity in the dependent variables when crossing the wave front. Such type of wave for which the dependent variables experience finite jumps across the wavefront, is called a shock wave and the shock waves are *strong discontinuities* while the wave fronts considered in Unit-3 are *weak discontinuities.*

This situation is obviously a mathematical idealisation of a physical process involving real gases with dissipative effects in which there are large changes in physical variables within a very thin region of the flow. It is found experimentally that the thickness of a shock wave in a real gas is of the order of a few mean free paths. Thus a mathematical idealisation of a shock wave in which the dependent variables experience a finite jump across a geometrical surface is a good approximation in reality. In our discussions, we shall neglect the dissipative effects of viscosity and electrical resistivity.

Before we establish an important theorem regarding the rate of change for volume integral, in which the bounding surface is moving, let us first recall the general differential form of a conservation law given by

$$\frac{\partial U}{\partial t} + \nabla.\mathbf{F} = \tilde{G} \qquad (4.1.1)$$

By integrating this equation over a volume element dV and using the definition of the divergence operator, we can intepret this equation physically. We find that the sum of the rate of change of a scalar U contained in a volume element dV and flux of the vector **F** into dV is equal to the contribution from the source distribution G throughout dV per unit time.

Now let us consider a general theorem which is important in the study of shock waves. For this, we must take into account the possibility of discontinuities across a moving surface which we now call *shock front*. Suppose an arbitrary surface S(t) is moving with velocity **q** that bounds a volume V(t) in which a differentiable scalar function U is defined.

Let

$$I = \int_{v(t)} U dV \qquad (4.1.2)$$

So that in time increment δt, the integrand of I becomes, to the first order, $U + \frac{\partial U}{\partial t}\delta t.$ But during this time increment δt, the volume bounded by S(t) changes as V(t) changes. In order to find the effect of this change we note that the vector surface element d**S** of S(t) moves a distance **q**δt in the time increment δt, and so the corresponding element of volume change is **q**.d**S** δt. Thus the corresponding increment in the integrand of I due to this is thus $U_{\mathbf{q}}.d\mathbf{S}\delta t$. Combing all these results, we have

$$I + \delta I = \int_{V(t)} \left\{U + \frac{\partial U}{\partial t}\delta t\right\} dV + \int_{S(t)} U\mathbf{q}.\,d\mathbf{S}\delta t \qquad (4.1.3)$$

Subtracting (4.1.2) form (4.1.3), dividing the result by $\delta t$ and then taking the limit as $\delta t \to 0$ we have the following *volume rate of change theorem*

$$\frac{D}{Dt}\int_{v(t)} U dV = \int_{v(t)} \frac{\partial V}{\partial t} dV + \int_{s(t)} U\mathbf{q}.dS \tag{4.1.4}$$

i.e. $$\frac{D}{Dt}\int_{V(t)} U dV = \int_{V(t)} U dV = \int_{V(t)} \left\{\frac{\partial U}{\partial t} + \nabla.(U\mathbf{q})\right\} dV$$

(By Gauss divergence theorem) (4.1.5)

We now use this theorem to derive the jump conditions that are permitted by a conservation law of the form (4.1.1). Suppose a discontinuity surface exists and that an arbitrary part S*(t) of it divides the volume V(t) into volumes $V_0(t)$ and $V_1(t)$ and the surface S(t) into surfaces $S_0(t)$ and $S_1(t)$ respectively. We denote the value of functions on adjacent sides of, and arbitrarily close to S*(t) by the suffixes 0 and 1 according as S*(t) is approached from $V_0(t)$ and $V_1(t)$ respectively.

Now assuming that neither U nor G have any singularities, we have from (4.1.4) by using (4.1.1)

$$\frac{D}{Dt}\int_{V(t)} U dV = \int_{S(t)} (U\mathbf{q} - \mathbf{F}) \cdot dS + \int_{v(t)} \tilde{G} dV \tag{4.1.6}$$

So, by subtracting from this equation the corresponding equation in which V(t) is identified with $V_0(t)$ and $V_1(t)$, we have

$$\int_{S*(t)} (U_{\mathbf{q}} - \mathbf{F})_0 \, dS_0 * + \int_{S*(t)} (U\mathbf{q} - \mathbf{F})_1 \cdot d\mathbf{S}_1 * = 0 \tag{4.1.7}$$

where $dS_1*$ is the outward directed vector surface element of S*(t) with respect to $V_1(t)$. But $dS_0* = - dS_1* = \hat{n} dS*$, where $\hat{n}$ is the outward drawn normal to S*(t) with respect to $V_0(t)$ and S*(t) is an arbitrary part of a discontinuity surface. Thus, we have from (4.1.7)

$$(U\mathbf{q} - \mathbf{F})_0 . \hat{\mathbf{n}} - (U\mathbf{q} - \mathbf{F})_1 . \hat{\mathbf{n}} = 0 \tag{4.1.8}$$

If we denote the jump $X_1 - X_0$ in the quantity X across S*(t) by [X], then the equation (4.1.8) can be written as

$$\left[\bar{\lambda} U - F.\hat{n}\right] = 0 \tag{4.1.9}$$

where

$$\bar{\lambda} = \mathbf{q}.\hat{\mathbf{n}} \tag{4.1.10}$$

is the normal speed of propagation of the discontinuity surface.

As a special case, if the source term $\tilde{G}$ of equation (4.1.1) is a divergence of some vector **G**, i.e, $\tilde{G} = \nabla.\mathbf{G}$, which is also discontinuous across S*(t), then the volume integral in the last term of equation (4.1.6) can also be transformed into a surface integral by Gauss divergence theorem and, by the same argument as above, we obtained the jump condition

$$\left[\bar{\lambda}U - \mathbf{F}.\hat{\mathbf{n}}\right] + \left[\mathbf{G}.\hat{\mathbf{n}}\right] = 0 \tag{4.1.11}$$

*Equation (4.1.11) is the compatibility condition to be satisfied by jumps in the terms U and G of the conservation law (4.1.1) across each element of area of a general curved discontinuity surface which moves with local normal velocity* $\bar{\lambda} = \mathbf{q}.\hat{\mathbf{n}}$.

## 4.2 MAGNETOHYDRODYNAMIC SHOCKS

In order to determine the jump relations in magnetohydrodynamic shocks, we first display the constitutive equations in conservation form. Equation (1.2.4) is already in conservation form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{4.2.1}$$

The momentum equation (1.2.3) can easily be transformed in the following component form :

$$\frac{\partial}{\partial t}(\rho v_x) + \nabla \cdot (\rho v_x v) = -\frac{\partial}{\partial x}\left(p + \frac{B^2}{2\mu_e}\right) + \frac{1}{\mu_e}\nabla \cdot (B_x \mathbf{B}) \tag{4.2.2}$$

and similar two equations. Also for large magnetic Reynobls number, the equation (1.4.4) gives

$$\frac{\partial B_x}{\partial t} + \nabla.(B_x v) = \nabla(v_x \mathbf{B})$$

and similar two equations.

The energy conservation equation (1.9.9) with no dissipative effects reduces to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_e}\right) + \nabla.\left\{v\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_e}\right)\right\}$$

$$= -\nabla.\left(p\mathbf{v} + \frac{B^2}{2\mu_e}\mathbf{v} - \frac{(\mathbf{v}.\mathbf{B})}{\mu_e}\mathbf{B}\right) \tag{4.2.4}$$

Applying the result (4.1.11) to the above equations, the following jump conditions are obtained :

*(a) mass conservation*

$$\left[\bar{\lambda}\rho - \rho \mathbf{v}.\hat{\mathbf{n}}\right] = 0 \tag{4.2.5}$$

*(b) momentum conservation*

$$\left[\bar{\lambda}\rho v - p\mathbf{v}(\mathbf{v}.\hat{\mathbf{n}})\right] = \left[p + \frac{B^2}{2\mu_e}\right]\hat{n} - \frac{1}{\mu_e}\left[(\mathbf{B}.\hat{\mathbf{n}})\mathbf{B}\right] \tag{4.2.6}$$

*(c) magnetic field*

$$\left[\bar{\lambda}\mathbf{B} - (\mathbf{v}.\hat{\mathbf{n}})\mathbf{B}\right] = -\left[(\mathbf{B}.\hat{\mathbf{n}})\mathbf{v}\right] \tag{4.2.7}$$

*(d) energy conservation*

$$= \left[\bar{\lambda}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_e}\right) - \left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_e}\right)(\mathbf{v}.\bar{\mathbf{n}})\right]$$

$$= \left[p(v.\hat{n}) + \frac{B^2}{2\mu_e}(\mathbf{v}.\hat{\mathbf{n}}) = \frac{(\mathbf{v}.\mathbf{B})}{\mu_e}(\mathbf{B}.\hat{\mathbf{n}})\right] \tag{4.2.8}$$

*(e) Solenoidal jump condition*

$$[\mathbf{B}.\hat{\mathbf{n}}] = 0 \tag{4.2.9}$$

Sometimes, it is useful to express these results in terms of the normal fluid velocity component

$$\tilde{v}_n = \mathbf{v}.\hat{\mathbf{n}} - \bar{\lambda} \tag{4.2.10}$$

relative to the velocity of the idscontinuity surface and in such a case, the jump conditions are :

*(a) mass conservation*

$$\left[\rho\tilde{v}_n\right] = 0 \tag{4.2.5$'$}$$

*(b) momentum conservation*

$$\left[\rho\tilde{v}_n v + p * \hat{n}\right] = \frac{1}{\mu_e}[B_n\mathbf{B}] \tag{4.2.6$'$}$$

*(c) magnetic field*

$$\left[v_n\mathbf{B} - B_n v\right] = 0 \tag{4.2.7$'$}$$

*(d) energy conservation*

$$\left[\tilde{v}_n\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_e}\right) + v_n p * - \frac{(\mathbf{v.B})}{\mu_e} B_n\right] = 0 \qquad (4.2.8')$$

*(e) solenoidal jump condition*

$$[B_n] = 0, \qquad (4.2.9')$$

where $p* = p + \frac{B^2}{2\mu_e}$ is the total pressure and the suffix n denotes the normal component.

Equation (4.2.4′) shows that the *mass flow m, say, through the discontinuity surface is constant,* so that

$$m = \rho_0 \tilde{v}_{n_0} = \rho_1 \tilde{v}_{n_1} \qquad (4.2.11)$$

in which 0 and 1 denote opposite sides of the discontinuity surface. If m ≠ 0, then a discontinuity surface will be called a *shock*. Hence *fluid particles must cross a shock front.* It also follows from (4.2.8) that $B_n$ is continuous across a shock front.

In our discussions, we shall assume that shock propogation is steady and that the shock front (discontinuity surface) is plane, so that the jump conditions are uniformly true across the entire shock front for all time. The constant value of $\bar{\lambda}$ determines how the shock moves relative to the given reference frame. In particular, if $\bar{\lambda} = 0$, the shock is stationary; whereas if $\bar{\lambda} = \tilde{v}_{n0}$, then $v_{n0} = 0$ and the shock propagates with speed $\tilde{v}_{n0}$ into the gas of region 0 which is at rest.

## 4.3 THE GENERALISED HUGONIOT CONDITION

Let us rewrite the energy equation (4.2.8′) in the form

$$\frac{1}{2}m\left[v^2\right] + m[e] + \frac{m}{2\mu_e}\left[\tau B^2\right] + \left[v_n p\right] +$$

$$\frac{1}{2\mu_e}\left[v_n B^2\right] - \frac{B_n}{2\mu_e}[v.\mathbf{B}] = 0 \qquad [4.3.1]$$

where $\tau = 1/\rho$ is the specific volume of the fluid. But $R = C_p - C_v$ and the *adiabatic exponent* $\gamma = C_p/C_v$ while for a polytropic gas $e = C_v T$ and, therefore, we can write the equation (1.9.11) as

$$e = \frac{p\tau}{\gamma - 1} \qquad [4.3.2]$$

Then the energy equation (4.3.1) becomes

$$\frac{1}{2}m[v^2] + \frac{m}{\gamma-1}[p\tau] + \frac{m}{2\mu_e}[\tau B^2] + [v_n p] + \frac{1}{2\mu_e}[v_n B^2] - \frac{B_n}{2\mu_e}[v.B] = 0 \qquad [4.3.3]$$

We can replace the first term of this equation by forming the scalar product of equation (4.2.6′) with < v >, where $< Q >= \frac{1}{2}(Q_0 + Q_1)$ denotes the average value of Q. The transformed equation of (4.3.3) is then

$$\frac{m}{\gamma-1}[p\tau] + [v_n p] - [p] < v_n > + \frac{m}{\mu_e}[\tau B^2] + \frac{1}{2\mu_e}[v_n B^2]$$

$$- \frac{B_n}{\mu_e}[\mathbf{v}.\mathbf{B}] - \frac{1}{2\mu_e}[B^2] < v_n > + \frac{B_n}{u_e}[\mathbf{B}].< v >= 0 \qquad [4.3.4]$$

Now the speed $\bar{\lambda}$ of the discontinuity surface must be continuous across the discontinuity surface and, therefore, it follows from equation (4.2.9) that

$$[\tilde{v}_n] = [v_n] \qquad [4.3.5]$$

$$\text{i.e.,} \quad m[\tau] = [v_n] \qquad [4.3.6]$$

So applying the identity [PQ] = < P > [Q] + < Q > [P] to the second and third terms of equation (4.3.4), using equation (4.3.6) and re-writing the remaining terms containing the magnetic field vector we find that

$$\frac{m}{\gamma-1}[p\tau] + m < p > [\tau] + \frac{m}{\mu_e}[\tau B^2] + \frac{1}{2\mu_e}[v_n B^2]$$

$$- \frac{B_n}{\mu_e}[\mathbf{v}.\mathbf{B}] - \frac{1}{2\mu_e}[B^2] < v_n > + \frac{B_n}{\mu_e}[B].< v >= 0, \qquad [4.3.7]$$

Expanding terms of the form [PQ] in equation (4.3.7) and using the identity $< P^2 > -[P]^2 \equiv \frac{1}{4}[P^2]$ we get, together with the result

$$m < \tau > [B] . <B> + m[\tau] <B>^2 = B_n[v].<B> \qquad [4.3.8]$$

which is obtained by forming the scalar product of equation (4.2.6′) with <B>

$$\frac{1}{\gamma-1}[p\tau] + \langle p \rangle [\tau] + \frac{1}{2\mu_e}[\tau][\mathbf{B}]^2 = 0 \qquad [4.3.9]$$

i.e. $$[e + \langle p \rangle \tau] = -\frac{1}{2\mu}[\tau][B]^2 \qquad [4.3.10]$$

provided that m ≠ 0. The relation (4.3.10) with right hand side zero is the *Hugoniot relation* of ordinary gas dynamics and, for this reason, the equation (4.3.10) is known as *generalised Hugoniot relation.*

If the magnetic field **B** is normal to the plane of the shock then the conducting fluid behaves as an ordinary fluid for then, as $\mathbf{B} = B_n\hat{\mathbf{n}}$ and since $H_n$ is continuous across the shock front, we must have $[\mathbf{B}] = \mathbf{0}$.

We can express the pressure ration $p_{1/}p_0$ across the shock front from the equation (4.3.9) in terms of the ratio $r = \tau_0/\tau_1$ $(= \rho_1/\rho_0)$ as

$$\frac{p_1}{p_0} = \frac{(\gamma+1)r-(\gamma-1)}{(\gamma+1)-(\gamma-1)r} + \frac{[B]^2}{2\mu_e p_0}\frac{(\gamma-1)(r-1)}{(\gamma+1)-(\gamma-1)r}. \qquad (4.3.11)$$

## 4.4 THE COMPRESSIVE NATURE OF MAGNETOHYDRO-DYNAMIC SHOCKS

The jump conditions (4.2.5′) and (4.2.9′) relate values on adjacent sides of a discontinuity surface. But these do not determine the senses of the jumps involved (i.e. the increase or decrease). Since in a physical situation, a solution must be unique, it is clear that some extra condition must be imposed on the jump conditions so that we may have real jump conditions. For this we require the thermodynamical requirement that the *entropy cannot decrease across a shock front.* This supplementary condition is imposed from outside the framework of MHD and that it is implied by the second law of thermodynamics.

First, we suppose that the direction of the normal to the shock front is such that

$$\tilde{v}_n = v_n - \overline{\lambda} > 0 \qquad (4.4.1)$$

and denote the quantities by the suffix 1 on the side of the shock front into which $\hat{\mathbf{n}}$ is directed and the other side by 0. Thus the fluid particles leave region 0 and cross the shock front to enter into the region 1. In general, we refer to the side of the shock front through which the fluid enters as the *front* of the shock or the side *ahead* of the shock, while the other side is called the *back* of the shock or the side *behind* the shock.

Now since by the second law of thermodynamics, the entropy cannot decrease across a shock front

$$S_1 \geq S_0 \tag{4.4.2}$$

which also implies that $\rho_1 > \rho_0$ and $p_1 > p_0$. Thus the second law of thermodynamics imposes the requirement that compressive shocks are allowed.

Again from equations (1.9.14) and (1.9.15), it follows that

$$S_1 - S_0 = c_v \log\left(\frac{p_1 \tau_1^\gamma}{p_0 \tau_0^\gamma}\right) \tag{4.4.3}$$

i.e. $$S_1 - S_0 = c_v \log\left(\frac{p_1}{p_0}\right) - \gamma c_v \log r \tag{4.4.4}$$

where $r = \tau_0/\tau_1$. Now putting $k^2 = \dfrac{[\mathbf{B}]^2}{2\mu_e p_0}$, it follows from (4.3.11) that

$$\frac{p_1}{p_0} = \frac{(1+k^2)\{(\gamma+1)r - (\gamma-1) - k^2 r}{(\gamma+1) - (\gamma-1)r} < (1+k^2)\cdot\frac{\{(\gamma+1)r - (\gamma-1)\}}{\{(\gamma+1) - (\gamma-1)r\}}, \tag{4.4.5}$$

since $k^2$ is non-negative. Noting that the pressure ratio is inherently positive, the numerator and denominator of the inequality in (4.4.5) must be of the same sign. Also the fact that $\gamma > 1$ shows that

$$\frac{\gamma-1}{\gamma+1} < r < \frac{\gamma+1}{\gamma-1} \tag{4.4.6}$$

We now proceed to show that if

$$\frac{p_1}{p_0} = \frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1) - (\gamma-1)r}, \tag{4.4.7}$$

then the entropy condition (4.4.2) implies that $r > 1$. Using (4.4.7), we have from (4.4.4)

$$S_1 - S_0 = C_v \log\left\{\frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1) - (\gamma-1)}\right\} - \gamma C_v \log r \tag{4.4.8}$$

Let us assume that the state 0 ahead of the shock to be a fixed state so that $r = r(S_1)$. Then from (4.4.8) we get

$$\frac{dS_1}{dr} = \frac{C_v \gamma(\gamma^2 - 1)(r-1)^2}{r\{\gamma+1)r - (\gamma-1)\}\{(\gamma+1)-(\gamma-1)r\}} \tag{4.4.9}$$

Now the numerator is positive and since both the factors in the denominator have the same sign (by (4.4.6)), so

$$\frac{dS_1}{dr} > 0 \tag{4.4.10}$$

Also the condition (4.4.2) implies that the entropy cannot decrease across the shock and the condition (4.4.10) implies that $S_1$ and r change in the same sense, so $r(S_1)$ increases across a MHD shock. As $r(S_0) = 1$, so $r > 1$ and hence *magnetohydrodynamic shocks are compressive*. Inequality (4.4.7) must then be modified to

$$1 < r < \frac{\gamma+1}{\gamma-1} \tag{4.4.11}$$

The mass conservation jump condition (4.2.5′) shows that

$$\tilde{v}_0 = r\tilde{v}_1 \tag{4.4.12}$$

and, therefore, the gas ahead of the shock moves faster than the gas behind the shock relative to the shock front.

The local *Mach number* M = v/a of an ordinary gas flow is defined to be the ratio of the gas speed v and the local speed of sound a. In general, the mach number is a function of position, while in the steady flow across a shock, it will have different constant values on opposite sides of the shock front. If, for a gas, $M < 1$, then its flow is said to be *subsonic* and if $M > 1$, then the flow is called *supersonic*.

Let us now consider a stationary perpendicular MHD shock in which the magnetic field acts normal to the direction of flow. Suppose the normal lies along the direction of x-axis so that $B_n = 0$ and $v_n = v_x = |v|$. We denote the transverse magnetic field by $B_n$. Then equations (4.2.5′) and (4.2.8′) become

$$\frac{\rho_1}{\rho_0} = \frac{v_{x0}}{v_{x1}} \tag{4.4.13}$$

$$\rho_0 v_{x_0}^2 + p_0 + \frac{B_{t0}^2}{2\mu_e} = \rho_1 v_{x_1}^2 + p_1 + \frac{B_{t1}^2}{2\mu_e} \tag{4.4.14}$$

$$\frac{v_{x_0}}{v_{x_1}} = \frac{B_{t_1}}{B_{t_0}} \tag{4.4.15}$$

$$\frac{1}{2}v_{x_0}^2 + \frac{\gamma p_0 \tau_0}{\gamma - 1} + \frac{B_{t_0}^2 \tau_0}{\mu_e} = \frac{1}{2}v_{x1}^2 + \frac{\gamma p_1 \tau_1}{\gamma - 1} + \frac{B_{t1}^2 \tau_1}{\mu_e} \tag{4.4.16}$$

Equations (4.4.13) and (4.4.15) show that

$$r = \frac{\rho_1}{\rho_0} = \frac{\tau_0}{\tau_1} = \frac{B_{t1}}{B_{t0}} = \frac{v_{x0}}{v_{x1}} \tag{4.4.17}$$

and then we can write equations (4.4.14) and (4.4.16) in the form

$$M_0^2\left(1 - \frac{1}{r}\right) = \frac{1}{\gamma}\left\{\left(\frac{p_1}{p_0} - 1\right) + \frac{B_{t0}^2}{2\mu_e p_0}(r^2 - 1)\right\} \tag{4.4.18}$$

and $$M_0^2\left(1 - \frac{1}{r}\right) = \frac{1}{\gamma}\left\{\left(\frac{p_1}{p_0} - 1\right) + \frac{B_{t0}^2}{2\mu_e p_0}(r^2 - 1)\right\} \tag{4.4.19}$$

where we have used the relation $a_0^2 = \gamma p_0 \tau_0$, Noting that MHD shocks must be compressive so that r > 1, we get, by eliminating the ratio $p_1/p_0$ between these equations, the following quadratic equation in r :

$$(2 - \gamma)\frac{B_{t0}^2}{2\mu_e p_0} r^2 + \gamma\left\{\frac{B_{t0}^2}{2\mu_e p_0} + \frac{1}{2}(\gamma - 1)M_0^2 + 1\right\} r - \frac{1}{2}\gamma(\gamma + 1)M_0^2 = 0 \tag{4.4.20}$$

Consequently, since r > 1, the equation (4.4.20) yields the inequality

$$\frac{1}{2}\gamma(\gamma + 1)M_0^2 > (2 - \gamma)\frac{B_{t0}^2}{2\mu_e p_0} + \gamma\left\{\frac{B_{t0}^2}{2\mu_e p_0} + \frac{1}{2}(\gamma - 1)M_0^2 + 1\right\},$$

provided $\gamma < 2(\gamma = 5/3)$ for plasma. More precisely, we can write the above inequality as

$$\gamma_x^2 > b_{t0}^2 + a_0^2, \tag{4.4.21}$$

where $b_{t0}$, the Alfvén speed ahead of the shock, is given by

$$b_{t0}^2 = \frac{B_{t0}^2}{\mu_e \rho_0} \tag{4.4.22}$$

By analogy with the Mach number, we may define *Alfvén number* A as the ratio of the, fluid speed v to the Alfvén speed b, i.e.

$$A = \frac{v}{b} \tag{4.4.23}$$

and, then, the inequality (4.4.21) gives

Thus the flow ahead of a perpendicular MHD shock is *super-Alfvénic* relative to the conditions ahead of the shock front. Similarly, the flow behind the shock front is *sub-Alfvénic* relative to the conditions behind the shock front. The ratio $v_x \big/ \left(b_t^2 + a^2\right)^{\frac{1}{2}}$ is called the *magnetic Mach number*.

## 4.5 CLASSIFICATION OF MAGNETOHYDRODYNAMIC SHOCK WAVE

We have seen that a continuous wave motion can tend to a shock wave under certain conditions. This fact, when coupled with different types of wave motions, e.g., fast and slow waves, transverse waves and entropy waves leads to classification for shock waves. To establish such a classification for MHD shock waves we first use (4.2.9′) and (4.2.1) to write the jump relation (4.2.6′) in the form

$$m[v] + \frac{[p]}{[\tau]}[\tau]\hat{n} + \frac{1}{\mu_e} < \mathbf{B} > \cdot[\mathbf{B}]\hat{n} - \frac{B_n}{\mu_e}[\mathbf{B}] = \mathbf{0} \tag{4.5.1}$$

Then, using the scalar equation (4.3.6), viz.

$$m[t] - [v_n] = 0$$

which was derived from the continuity of the shock front velocity $\bar{\lambda}$, the jumps relation (4.2.7′) gives

$$B_n[v] - < \mathbf{B} > [v_n] - m<t> [\mathbf{B}] = 0 \tag{4.5.2}$$

The two vector equations (4.5.1, 2) and the scalar equation (4.3.6′) then represent seven homogeneous scalar equations for seven scalar, jump quantities, viz, the six scalar components of [**v**] and [**B**] and scalar quantity [τ]. These equations will be consistent and a non-trival solution exists, provided that the determinant of the coefficients of these jump quantities vanishes, i.e.

$$<\tau>^2 m\left\{<\tau> m^2 - \frac{B_n^2}{\mu_e}\right\}\Big\{<\tau> m^4 + (<\tau>[\tau]^{-1}[p]$$

$$- \frac{<\mathbf{B}>^2}{\mu_e} m^2 - [\tau]^{-1}[p]\frac{B_n^2}{\mu_e}\Big\} = 0 \qquad (4.5.3)$$

We may regard this either as an equation for mass flux m through the shock front, or as an equation for the shock velocity. In fact, by writing equation (4.4.1) in the form $m\tau = v_n - \bar{\lambda}$, and averaging across the shock front we find that

$$\bar{\lambda} = <v_n> - m <\tau> \qquad (4.5.4)$$

The vanishing of different factors of equation (4.5.3) corresponds to different modes of MHD shock wave propagation.

*(a) Fast and slow shocks*

Suppose the last factor of (4.5.3) vanishes so that

$$<\tau> m^4 + (<\tau>[\tau]^{-1}[p] - \frac{<\mathbf{B}>^2}{\mu_e} m^2 - [\tau]^{-1}[p]\frac{B_n^2}{\mu_e} = 0$$

i.e., $$(m^2) + [\tau]^{-1}[p]\left(m^2 - <\tau>^{-1}\frac{B_n^2}{\mu_e}\right) = \frac{m^2(\tau)^{-1}}{\mu_e}(<\mathbf{B}> - B_n^2) \qquad (4.5.5)$$

This is a quadratic equation in $m^2$ having two roots $m_s$ (smaller one) and $m_f$ (larger one). Since the right hand side of this equation is positive, each factor on the left hand side must be of the same sign and so

$$m_s^{\ 2} \le -[\tau]^{-1}[p] \le m_f^{\ 2} \qquad (4.5.6)$$

and $$m_s^{\ 2} \le \frac{B_n^2}{\mu_e <\tau>} \le m_f^2 \qquad (4.5.7)$$

Now we know from thermodynamical considerations of section-4.4 that $[p] > 0$ across a shock wave and $[\tau] < 0$ and, therefore, the middle term in (4.5.6) is positive. The roots $m_f$ and $m_s$ of equation (4.5.5) describe the mass flow or the shock velocity of *fast and slow MHD shock waves.*

The use of the equation (4.5.5) and the jump relations (4.2.5) to (4.2.9) give the jump of quantities across MHD fast and slow shocks as

$$[\mathbf{B}] = \varepsilon_{f,s} m^2 (<\mathbf{B}> - B_n \hat{n}), \qquad (4.5.8)$$

$$[v] = \varepsilon_{f,s} m \left( \frac{B_n}{\mu_e} < \mathbf{B} > - < \tau > m^2 \hat{\mathbf{n}} \right), \tag{4.5.9}$$

$$[\tau] = -\varepsilon_{f,s} \left( < \tau > m^2 - \frac{B_n^2}{\mu_e} \right), \tag{4.5.10}$$

in which $\varepsilon_{f,\,s}$ is a parameter characterising the strength of the jump across a fast (f) or a slow (s) shock.

Noting that $[B^2] = 2 < \mathbf{B}.[\mathbf{B}]$, it follows from (4.5.8) that

$$[B]^2 = 2\varepsilon_{f,s} m^2 \left( < \mathbf{B} >^2 - B_n^2 \right) \tag{4.5.11}$$

and, therefore, taking scalar product of (4.2.6′) with $\hat{\mathbf{n}}$ and using this result we get

$$[p] = \varepsilon_{f,s} m^2 \left( < \tau > m^2 - \frac{< \mathbf{B} >^2}{\mu_e} \right) \tag{4.5.12}$$

Finally, eliminating $\varepsilon_{f,\,s}$ between equations (4.5.10) and (4.5.11), we find that

$$[B^2] = -2m^2 [\tau] \{ < \mathbf{B} >^2 - B_n^2 \} \Big/ \left\{ < \tau > m^2 - \frac{B_n^2}{\mu_e} \right\}. \tag{4.5.13}$$

Since $[\tau] < 0$ across a shock wave, it follows from the equation (4.5.13) and the inequality (4.5.7) that the *magnetic field increases across a fast shock, but decreases across a slow shock*. Also the equation (4.5.8) shows that the *magnetic field experiences a tangential jump discontinuity on crossing the shock front.*

Expanding equation (4.5.8) we get

$$\mathbf{B}_{11} = \left( \frac{1 + \frac{1}{2}\varepsilon_{f,s} m^2}{1 - \frac{1}{2}\varepsilon_{f,s} m^2} \right) \mathbf{B}_{t0}$$

in which the suffix t denotes the transverse component. Consequently,

$$[\mathbf{B}] = [\mathbf{B}_1] = \left( \frac{\varepsilon_{f,s} m^2}{1 - \frac{1}{2}\varepsilon_{f,s} m^2} \right) \mathbf{B}_{t0}$$

This shows that the jump $[\mathbf{B}]$ is parallel to $\mathbf{B}_{t0,}$ while we see from equation

(4.5.8) that the sense of the jump is the same as that of the tangential component of $<B>$.

*(b) Transverse shocks*

Here we suppose that the second factor of (4.5.3) vanishes and this gives the result

$$m = \pm\left(\frac{B_n^2}{\mu_e < \tau}\right) \tag{4.5.14}$$

Such types of disturbances are called *transverse shock waves*. It follows from the jump equations (4.2.5′) to (4.2.9′) and the equation (4.5.14) that in transverse shock waves

$$[\mathbf{B}] = \varepsilon m < B > \times \hat{n}, \tag{4.5.15}$$

$$[v] = \varepsilon \frac{B_n}{\mu_e} < \mathbf{B} > \times \hat{n}, \tag{4.5.16}$$

$$[\tau] = 0, \tag{4.5.17}$$

$$[p] = 0. \tag{4.5.18}$$

It is evident from (4.5.15) and (4.5.16) that the jumps [**B**] and [v] are parallel and lie in the plane of the transverse shock, but the pressure and density remain constant across it. Noting that the density remains constant, we get from (4.5.15) and (4.5.16) the relation

$$[v] = \pm\, [\mathbf{B}] \frac{1}{\sqrt{\mu_e \rho}} \tag{4.5.19}$$

Also the equation (4.5.15) implies that

$$[B^2] = 2 <\mathbf{B}>.\ [\mathbf{B}] = 0 \tag{4.5.20}$$

Thus the *magnetic field remains unchanged across a transverse shock* and this field simply rotates on crossing the plane of the shock.

Equations (4.5.17) and (4.5.18) together with the equation (4.4.3) yields

$$[S] = 0. \tag{4.5.21}$$

*(c) Contact discontinuities*

The vanishing of the only remaining factor in equation (4.5.3), namely

$$m = 0 \tag{4.5.22}$$

leads to the *contact discontinuity* which has no flow across the discontinuity surface.

If $B_n \neq 0$, the jump relations (4.2.5′) to (4.2.9′) are given by

$$[B] = 0 \tag{4.5.23}$$

$$[v] = 0 \tag{4.5.24}$$

$$[p] = 0 \tag{4.5.25}$$

but $[\tau]$ may be arbitrary.

However, if $B_n = 0$, then

$$[p^*] = \left[p + \frac{B^2}{2\mu_e}\right] = 0 \tag{4.5.26}$$

and the tangential components of the jumps [**B**] and [v] may be arbitrary.

*(d) Weak shocks*

Here we note that although **B, v,** $\tau = 1/\rho$ and p are continuous across the discontinuity surface, their first derivatives are discontinuous there. Consequently, we make the following changes in notations :

$$[v] \to \delta v, [\tau] \to -\frac{1}{\rho^2}\delta\rho, \ [\mathbf{B}] \to \delta\mathbf{B}$$

$$\text{and} \ \ m \to \mp\rho c_n, <\tau> \to \frac{1}{\rho}, -\frac{[p]}{[\tau]} \to \rho^2 a^2, <\mathbf{B}> \to \mathbf{B}$$

in which $\delta$ denotes the jump in the normal derivative of the quantity associated with it on crossing the wave-front and $a^2 = \gamma p\tau$. This result implies that *shock waves and weak discontinuities propagate at different speeds*.

Now for a weak shock, we can write the density ratio as $r = 1 + \varepsilon$, $0 < \varepsilon < 1$. Thus from (4.4.9), we have

$$\frac{dS_1}{d\varepsilon} \simeq \frac{1}{4} C_v \gamma(\gamma^2 - 1)\varepsilon^2,$$

integration of which leads to

$$S_1 - S_0 = \frac{1}{12} c_v \gamma(\gamma^2 - 1)\varepsilon^3.$$

Thus the *entropy change $S_1 - S_0$ across a weak shock is of third order with respect to the change in r*. It also follows that *the entropy increase across a shock is not greater than third order with respect to the change in* **B**.

## SUMMARY

In this Unit, an elementary idea of magnetohydrodynamic shock waves has been given. The propagation of the discontinuity surface, jump conditions, the generalised Hugoniot condition have been carried out. The compressive nature of MHD shocks and their classification are also shown.

## MODEL QUESTIONS

**Short questions :**

1. Define : shock wave, strong and weak discontinuities, shock front, front (ahead) and back (behind) of the shock, Mach number, supersonic and subsonic flows, Alfvén number, magnetic Mach number, super and sub-Alfvénic flows, fast and slow shocks, transverse shock waves, contact discontinuity, weak shock.

2. When is a discontinuity surface called a shock? Show that fluid particles cross a shock front.

3. Show that fluid enters on the side of the region of the shock front into which the normal $\hat{\mathbf{n}}$ is directed, but leaves on the opposite side.

4. Show that the jumps [**B**] and [v] are parallel to one another for transverse shock and lie in its plane, but the pressure and density remain constant across it.

5. Show that the magnetic field remains unchanged across a transverse shock.

6. Show that the entropy change across a weak shock is of third order with respect to the change of density ratio.

7. Show that the magnetic field increases across a fast shock and decrease a slow shock

8. Show that for fast and slow shocks, the jump [**B**] is parallel to $B_{t0}$.

9. Find the limits in which the density ratio lies.

10. Show that, relative to the shook front, the gas ahead of the shock moves faster than the gas behind the shock.

11. Deduce the equation describing the mass flow (or the shock velocity) of fast and slow MHD shock waves.

12. Derive the jump conditions across fast and slow MHD shocks, transverse shocks and contact discontinuities.

13. Deduce the equation describing the mass flux (or the shock velocity) through the shock front.

14. If the magnetic field **B** ($\neq$ **0**) is normal to the plane of the shock, then show that the conducting fluid behaves as an ordinary fluid.

**Broad questions :**

1. Deduce the compatibility condition to be satisfied by the jumps in terms of the conservation law across each element of area of a general curved discontinuity surface moving with local normal velocity.

2. Establish the jump relations that are permitted in MHD shocks. Hence give physical interpretations of these relations.

3. Deduce the generalised Hugoniot condition for MHD shock waves. Hence show that if the magnetic field is normal to the plane of the shock, the conducting fluid behaves as an ordinary fluid.

4. Show that MHD shocks are compressive in nature.

5. Show that the flow ahead of a perpendicular MHD shock is super-Alfvénic relative to the conditions ahead of the shock front.

6. Deduce the equation for the mass flow (shock velocity) through the shock front. Hence classify MHD shock wave propagation according to the vanishing of different factors of this equation.

7. Using the generalised Hugoniot relation, the equation of state $p\tau = RT$ and the fact that MHD shocks are compressive, prove that the temperature behind a shock is greater than the temperature ahead of a shock.

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