NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS

POST GRADUATE

PG (MT) : X B(1)

(Pure Mathematics)

Advanced Functional Analysis

PREFACE

In the curricular structure introduced by this University for students of Post- Graduate diploma programme, the opportunity to pursue Post-Graduate Diploma course in any Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate Diploma level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Cooperation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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PG (MT)—XB(1)
(Pure Mathematics)
Advanced Functional Analysis

Unit 1

(**Contents** : Vector spaces, convex sets, their algebra, convex hull of a set, its representation Theorem, Symmetric sets, balanced sets, absorbing sets, Linear operators over a vector space, space of linear operators, Isomorphism between vector spaces, Topological vector spaces, Translation and multiplication operators as homeomorphism, Bounded sets in TVS, Fundamental properties of TVS.)

§ 1.1 Linear/Vector spaces :

Let R (₵) denote the field of reals (complex) scalars.

Definition 1.1.1 A vector-space X is a collection of objects called vectors satisfying following conditions :-

I. X is additively an Abelian Group, the additive identity $\underline{O}$ in X being called the zero vector in X.

II. For every ordered pair (α, x), where α is a scalar and $x \in X$, there is a vector donoted by αx, called scalar multiple of x in X satisfying

(a) $1.x = x$ for every $x \in X$.

(b) $\alpha(x + y) = \alpha x + \alpha y$ for all scalars α and for all vectors x and y in X,

(c) $(\alpha + \beta) x = \alpha x + \beta x$ for all scalars α and β for all vectors $x \in X$,

and (d) $\alpha(\beta x) = (\alpha\beta)x$ for all scalars α, β and for all vectors $x \in X$.

For example, the Euclidean n-space R^n consisting of all ordered n-tuples $(x_1, x_2, ..., x_n)$ of reals x_i forms a vector space with scalars as reals, where vector addition and scalar multiplication are taken as :—

$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ and $\alpha(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$.

Similarly, the collection $C[a, b]$ of all real-valued continuous functions over the closed interval $[a, b]$ $(a < b)$ forms a vector space over reals by taking vector-sum and scalar multiplication as :—

$$(f + g)(t) = f(t) + g(t) \text{ in } a \le t \le b \text{ for } f, g \in C[a, b]$$

and $(\alpha f)(f) = \alpha f(t)$ in $a \le t \le b$ for all scalars α and for all $f \in C[a, b]$. We need only remember that sum of two continuous functions over $[a, b]$ and a scalar multiple of a continuous function over $[a, b]$ are also continuous functions over $[a, b]$.

A vector-space X may also be termed as a real or complex vector-space according as the associated field of scalars is that of reals or complex numbers.

Let A and B be two subsets of a vector space X and let α be a scalar. Then

$A + B = \{a + b : a \in A \text{ and } b \in B\}$;

$\alpha A = \{\alpha a : a \in A\}$.

Clearly $A + B = B + A$.

But $A - B$ may not be the same as $B - A$; because take $A = \{(1, 0)\}$ and $B = \{(0, 0)\}$ as singleton in Euclidean 2-space R^2; where we find $A - B = \{(1, 0)\}$, and $B - A$ $\{(-1, 0)\}$. So $A - B \neq B - A$.

Also $A - B$ may not be the same as $2A|$. Let us take $A \subset R^2$ with $A = \{(1, 0), (0, 1)\}$. Then we have $2A = \{(2, 0), (0, 2)\}$ and $A + A = \{(2, 0), (0, 2), (1, 1)\}$. So here $2A \neq A + A$. For $x \in X$, the subset $x + A = \{x + a : a \varepsilon A\}$ is called a translate of A.

Example 1.1.1 : The Unitary space $\not{C}^n$ = collection of all ordered n-tuples of complex numbers forms a vector-space where vector addition and scalar multiplication are given by

$$\underset{\sim}{z} + \underset{\sim}{w} = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$$

and $\lambda \underset{\sim}{z} = (\lambda z_1, \lambda z_2, \ldots, \lambda z_n)$ where

$\underset{\sim}{z} = (z_1, z_2, \ldots, z_n)$, $\underset{\sim}{w} = (w_1, w_2, \ldots, w_n) \in \not{C}^n$ and λ is a complex scalar.

Definition 1.1.2. : A finite set of vectors $x_1, x_2, \ldots, x_n$ in $X\{$ is said to be linearly dependent it there are scalars $c_1, c_2, \ldots, c_n$ not all zero such that $c_1x_1 + C_2x_2 + \ldots + c_nx_n = \underset{\sim}{O}$.

If the set $(x_1, x_2, \ldots, x_n)$ is not linearly dependent, it is called (vectors $x_1, x_2, \ldots, x_n$ are called) linearly independent.

Explanation : If $(x_1, x_2, ..., x_n)$ is linearly independent, then $\sum_{i=1}^{n} \lambda_i x_i = \underline{O}$ implies that scalars $\lambda_i = 0$ for $i = 1, 2, ..., n$. Also a finite subset of X that contains a linearly dependent set of vectors becomes linearly dependent and any subset contained in a linearly independent set of vectors becomes linearly independent. Plus, any subset of X that includes the zero vector becomes linearly dependent.

Definition 1.13 : An infinite set E of vectors in X is said to be linearly independent *if every finite subset of E becomes Linearly dependent.*

Example 1.1.1 : For any real θ show that $\{\sin(x + \theta), \cos(x + \theta)\}$ is a linearly independent set in Vector-space $C[0, 2\pi]$.

Solution : Let $\lambda \sin(x + \theta) + \mu \cos(x + \theta) = \underline{O}$

Or, $\lambda(\sin x \cos\theta + \cos x \sin\theta) + \mu(\cos x \cos\theta - \sin x \sin\theta) = \underline{O}$

Or, $(\lambda\cos\theta - \mu\sin\theta)\sin x + (\lambda\sin\theta + \mu\cos\theta)\cos x = \underline{O}$

This is true for $O \le x \le 2\pi$, Taking $x = 0$ and $x = \frac{\pi}{2}$ we have

$$\lambda\cos\theta - \mu\sin\theta = o = \lambda\sin\theta + \mu\cos\theta$$

or, $\frac{\lambda}{\mu} = -\frac{\mu}{\lambda}$ or, $\lambda^2 + \mu^2 = 0$ giving $\lambda = \mu = 0$.

That means, Given set of vectors is Linearly independent.

Definition 1.1.4 : A vector space X is said to be finite dimensional if for some +ve integer n, X contains a set consisting of n vectors that are linearly independent, while every subset of $(n + 1)$ number of vectors in X becomes linearly dependent.

In this case we say Dimension of $X = n$.

or, $\text{Dim}(X) = n$.

X is said to be infinite dimensional ($\text{Dim}(X) = \infty$) if X is not finite dimensional.

Example 1.1.2 : The Vector-space $\wp[a, b]$ consisting of all real-valued polynomials in $[a, b]$ $(a < b)$ is not a finite Dimensional vector-space.

Solution : Let $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$, ..., $p_n(t) = t^n$, ...where $a \leq t \leq b$. Then $\{p_0, p_1, ..., p_n, ...\}$ is a subset of $\wp[a, b]$. It suffices to show that any finite part of $\{p_0, p_1, ..., p_n, ...\}$ becomes a linearly independent set in $\wp[a, b]$

Suppose there are scalars $a_0, a_1, ..., a_n$ not all zero such that $a_0p_0 + a_1p_1 + ... + a_np_n = 0$

or $\quad a_0p_0(t) + a_1p_1(t) + ... + a_np_n(t) = 0 \quad$ in $a \leq t \leq b$

or $\quad a_0 + a_1t + a_2t^2 + ... + a_nt^n = 0 \quad$ in $a \leq t \leq b$ ...(1)

Now l.h.s. of above is polynominal of degree $\leq n$; so (1) corresponding polynomial equation does not have more than n roots : a contradiction that (1) in an Identity. Hence the infinite set $\{p_0, p_1, p_2, ..., p_n, ...\}$ is a linearly independent set in $\wp[a, b]$ and Dimension of $\wp[a, b]$ is not finite.

Remarks 1. The vector-space $\wp_{n+1}[a, b]$ consisting of all polynomials over$[a, b]$ of degree $\leq n$ (n being a +ve integer) is however finite dimensional, and

$$\text{Dim}\ (\wp_{n+1}[a, b]) = n + 1.$$

Remarks 2. The Eudidean 2-space R^2 over reals has Dim $(R^2) = 2$; The Euclidean plane taken as argand plane of all complex numbers $x + iy$ (x, y reals) and treated as a vector-space over $\mathbb{C}$ has Dimension = 1.

§ 1.2 Definition 1.2.1. A subset C of a Vector-space X is called a convex set if for any two members x, y ∈ C, and for any real, $x, y \in C$, with $0 \leq \alpha \leq 1$, $(\alpha x + (1 - \alpha)y) \in C$.

lor, equivalently, for any α, β with $0 \leq \alpha, \beta \leq 1$, and $= \alpha + \beta = 1$, $(\alpha x + \beta y) \in C$.

lor, equivalently, $\alpha C + \beta C \in C$.

For examples, any interval in space R of reals; any circular disc in space R^2 are examples of convex sets. A circle in R^2 is not a convex set in R^2.

Theorem 1.2.1. (a) Intersection of any number of convex sets in a vector space X is a convex set in X.

(b) If C is a convex set in X and λ any scalar, than λC is a convex set in X.

I(c) If A and B are two convex sets in X, then $A + B$ is a convex set in X.

Proof : The proofs are easy workout and left out.

Remarks 2. Union of two convex sets may not be a convex set in a vector-space. For instance, a figure like $\underline{\overline{X}}$ as a union of two triangular pieces in R^2 is not a convex set, through any triangular part of R^2 is a convex set.

Theorem 1.2.2. A subset B in a Vector space X is a convex set if and only if $sB + tB = (s + t)\ B$ for all +ve scalars s and t.

Proof : We have (always) $(s + t)\ B \in sB + tB$ .....(1)

Now if B is convex, and s, t are +ve scalars, we have

$$\frac{s}{s+t}B+\frac{t}{s+t}B\subset B$$

or $\quad sB + tB \in (s + t)\ B$ .....(2)

(1) and (2) give $sB + tB = (s + t)\ B$.

Conversely, let $(s + t)\ B = sB + tB$ hold for all +ve and s and t. Taking $0 \leq s \leq 1$ and $t = 1 - s$ we find for any subset B of X.

$sB + (1 - s)\ B = B$; Hence B is convex.

Convex hull of a set : Let S be a non-empty subset of a vector-space X.Then convex hull of S, denoted by conv. huss (S) = intersection of all convex subsets if Xleach of which contains S.

So conv. hull (S) is always a convex set, irrespective of S being cnvex or not; and it is the smallest size convex set to cover S.

Theorem 1.2.3 (Representation Theorem for convex hull).

Conv. hull (S) Consists of all vectors $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$, where $x_1, \ldots, x_n \in S$; $\alpha_1 \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, and then index n is not fixed.

Proof : We apply Induction Principle. Let Tl be the collection of all vectors $x \sum_{i=1}^{n} \alpha_i x_i$; $\alpha_1 \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$ and n is not fixed. For $n = 1$, we find every member of S belongs to T. So $S \subset T$.

We show that T is convex. If $u = \alpha_1 x_1 + \alpha_2 x_2 + ... +$ and $\vartheta = \beta_1 y_1 + \beta_2 y_2 + ... + \beta_n y_n$ are two members of T and if $0 \leq \gamma \leq 1$, then $\gamma\mu + (1 - \gamma)\ \vartheta = \gamma\alpha_1 x_1 + \gamma \alpha_2 x_2 + ... + \gamma\alpha_m + (1 - \gamma)\ \beta_1 + ... + (1 - \gamma)\ \beta_n = \gamma \sum_{k=1}^{n} \alpha_i + \beta_k = \gamma + (1 - \gamma) = 1$; where $\gamma\alpha_j x_1 + \gamma\alpha_2 + ... + \gamma\alpha_m + (1-\gamma)\ \beta_1 + ... + (1 - \gamma)\ \beta_n = \gamma \sum_{i=1}^{m} \alpha_i + (1-\gamma) \sum_{k=1}^{n} \beta_k = \gamma + (1-\gamma) = 1$. Therefore $\gamma u + (1-\gamma)\vartheta \in T$. So T is convex such that $S \in T$.

Therefore conv hull $(S) \subset$ T

Now let W be any convex set containing S.

For $n = 1$, points of T are those of S, and therefore are tose of W. Let us suppose that points of T are those of W if points of T are like $y = \sum_{i=1}^{n} \alpha_i + x_i$; where $\alpha_1 \geq 0$l with $\sum_{i=1}^{n} \alpha_i = 1$, and $\alpha_i \in S$. We now verify that the statement is valid for $(n + 1)$, and induction will be complete.

Take $x = \sum_{i=1}^{n+1} \beta x_i$, where $\beta_i \geq 0$ with $\sum_{i=1}^{n+1} \beta_i = 1$ and $x_i \in S$; Then $x \in T$.

Put $l = \beta_1 + \beta_2 + ... + \beta_n$ and $l_i = \frac{\beta_i}{l}$, $i = 1, 2..., n$.

Then $l_i \geq 0$ such that $\sum_{i=1}^{n} l_i = 1 \in$l and by assumption $y \in T$l where $y = l_1 x_1 + l_2 x_2 + ... + l_n x_n \in W$.

Now $\quad x = l(l_1 x_1 + l_2 x_2 + ... + l_n x_n) + (1 - l)\ x_{n+1}$

$\quad\quad = ly + (1 - l) x_{n+1}$

which is a member of W because Wl is convex containing y and x_{n+1}.

Therefore $x \in W$. The conclusion is $T \subset W$.

Since W is any convex set containing S we have $T \in$ conv. hull (S) .....(2)

Combing (1) and (2) we have T = conv. hull (S)

Example 1.2.1. : If a, b, c are position vectors of three non-collinear points A, B and C in Euclidean 2-space R^2, then conv. hull $\{a, b, c\}$ equals to the triangular region ABC of vector space R^2.

Difiniton 1.2.2. : A Subset S in a vectorspace X is called symmetric if $-S \subset S$.

For example, a circular disc centered at $\underset{\sim}{O}$ of R^2 (Euclidean 2 space) is a symmetric set.

Remark 1. A symmetric set may not contain $\underset{\sim}{O}$ of the space. Consider the annulus $\{z : r_1 <| z| < r_2 ; r_1$ and r_2 are two +ve reals$\}$ is a symmetric set of R^2 and it does not contain $\underset{\sim}{O}$.

Remark 2. Intersection of any number of symmetric sets in X is a symmetric set.

Remark 3. Union of any number of symmetric sets in X is a symmetric set.

Remark 4. If S *is a symmetric set,* λS is a symmetric set for any scalar λ.

Remark 5. If A and B are two symmetric sets, then $A + B$ is a symmetric set.

Definition 1.2.3. A subset B in a Vector-space X is called balanced if $\alpha B \subset B$ for all scalars α with $| \alpha | \leq 1$.

Explanation : A balanced set is always symmetric. It also contains $\underset{\sim}{O}$ of the space. But there is a symmetric set without being balanced. Look at the set $\{z : 2 <| z| < 3\}|$ of Euclidean 2-space R^2. It is symmetric, it is not balanced $\underset{\sim}{O} \notin$ this set.

Remark. Intersection of any number of balanced sets is X is balanced if and only if C is symmetric.

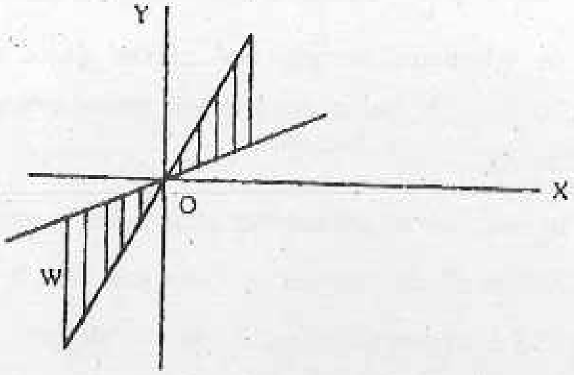
Theorem 1.2.4. A convex set C in a vector-space X is balanced if and only if C is symmetric.

Proof : Let C be a convex set in X and let C be balanced. Then of course Definition 1.2.2. says that C is symmetric.

Conversely, let C he a convex set that is symmetric. Then $-x \in C$ if $x \in C$.| Then $\frac{1}{2}x + \frac{1}{2}(-x) \in C$ or, $\underset{\sim}{O} \in C$. Take $|\alpha| \leq 1$, and say, $0 \leq \alpha \leq 1$; For $x \in C$, we have $\alpha x + (1 - \alpha)\, \underset{\sim}{O} \in C$ because C is convex. Thus $\alpha x \in C$. Similarly, we can show that $\alpha x \in C$ whenever $-1 \leq \alpha \leq 0$. Therefore $\alpha x \in C$ whenever $|\alpha| \leq 1$. Hence $C|$ is balanced.

Example 1.2.2 Give an example of a balanced set that is not convex.

Solution : In Euclidean 2-space R^2 consider a wedge. W vertexed at $\underline{O}$. This subset W is a balanced set, but it is not a convex set.



Definition 1.2.4. A subset A in a vector space X is called *absorbing* if each $x \in X$ corresponds some $\varepsilon > 0$ such that $\alpha x \in A$ if $0 < |\alpha| \le \varepsilon$.

Or, equivalently, if to each $x \in X$, there corresponds some $r > 0$ such that $x \in \alpha A$ for $|\alpha| \ge r$.

Theorem 1.2.5. In a vector space X a balanced set B is absorbing if and only if to each $x \in X$, there corresponds some $\alpha \neq 0$ such $\alpha x \in B$.

Proof : Let a balanced set B be absorbing, and take $x \varepsilon X$. Then we find $r > 0$ such that $x \in \alpha B$ for $|\alpha| \ge r$. Clearly $x \in \lambda B$ if $\lambda > r$ (>0).

or, $\frac{x}{\lambda} \in B$; Take $\alpha = \frac{1}{\lambda}$ and we have $\alpha x \in B$ $(\alpha \neq 0)$.

Conversely, suppose B is balanced where condition holds. Take $x \in X$. By supposed condition let $\beta \neq 0$ such that

$$\beta x \in B.$$

Since B is symmetric, we have

$$|\beta| x \in B \qquad \text{..... (1)}$$

Choose $0 < \varepsilon = |\beta|$. Then $\varepsilon > 0$; such that if $0 < |\alpha| \le \varepsilon$,

We have $\left|\frac{\alpha}{\varepsilon}\right| \le 1$. As B is balanced we have from (1)

$$\frac{\alpha}{\varepsilon}.|\beta| x \in B.$$

or, $\alpha x \in B.$

So, B is absorbing

§ 1.3 Linear operators over a vector (Linear) space :

Let X and Y denote two vector spaces over same field of scalars (reals or complex).

Definition 1.3.1. : A mapping $T : X \to Y$ is called a linear operator if

(i) $T(x + x') = T(x) + T(x')$ for all $x, x' \varepsilon$ X,

and (ii) $T(\lambda x) = \lambda T(x)$ for all $x \varepsilon X$l and for all scalars λ.

(Operator, mapping, map, function, transformation are all synonyms-meaning the same).

Explanation : Condition (i) Says under a Linear Operator image of sum of two vectors is the sum of their images, and Similar conclusion holds for condition. (ii) Further taking $\lambda = 0$ (scalar), one sees that a Linear operator always sends the zero vector in X to the zero vector in Y. Also readily it follows that a Linear operator transforms a Linearly dependent set of vectors in X to a similar set of vectors in Y. However, if does not send a linearly independent set of vectors in X into a similar set of vectors in Y. For instance, the zero operator (which is a linear operator : $X \to Y$ transforming every member of X to the zero vector in Y) sends a linearly independent set of vectors in X into a Linearly dependent set in Y.

Example 1.3.1. In Euclidean 2-space R^2 consider $T : R^2 \to R^2$ given by $T(x, y) = (x, 0)$ for $(x, y) \in R^2$. Then T is a Linear operator.

Theorem 1.3.1. If $T : X \to Y$ is a Linear operator; Then $T(X)$ is a sub-space of Y.

Proof : If $y_1, y_2 \in T(X)$, and λ is a scalar; we find $x_1, x_2 \in X$ such that $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Since T is Linear we have $T(x_1 + x_2) = T(x_1) + T(x_2) = y_1 + y_2$! Therefore, $y_1 + y_2$ is a member of $T(x)$; and similarly we can show that λy_1 is a member of $T(X)$. Thus $T(X)$ is a sub-space of Y.

Remark : Like Theorem 1.3.1 it is easy to check that a linear operator transforms any sub-space of X into a sub-space of X into a sub-space of Y.

Theorem 1.2.3. If $T : X \to Y$ is a Linear operator which is onto, then T^{-1} exists if and only if $x = \underset{\sim}{O}$ is the only solution of $T(x) = \underset{\sim}{O}$ and if T^{-1} exists, then T^{-1} : $Y \to X$ is a Linear operator.

Proof : Let $T : X \to Y$ (X, Y are vector-space over the same scalar field) be a linear operator that is onto and let $T^{-1} : Y \to X$ exists; Then $T(x) = \underset{\sim}{O}$ gives $T(x) = \underset{\sim}{O}$ $= T\underset{\sim}{O}$; Here T is $1 - 1$; so we have $x = \underset{\sim}{O}$.

Conversely, Suppose the condition holds. We check that T is 1–1; If $x_1, x_2 \in X$ such that $T(x_1) = T(x_2) = \underset{\sim}{O}$ i.e., $T(x_1 - x_2) = \underset{\sim}{O}$ i.e., $T(x_1 - x_2) = \underset{\sim}{O}$; By supposed condition $x_1 - x_2$ or $x_1 = x_2$. Now $T : X \to Y$ being onto and $1 - 1$ we see that T^{-1} : $Y \to X$ exists.

Now since $y \in Y$ there is exactly one $x \in X$ with $T(x) = y$. i.e., $x = T^{-1}(y)$. Take $y_1, y_2 \in Y$ and let $x_1, x_2 \in X$ with $T(x_1) = y_1$ and $T(x_2) = y_2$. So $T(x_1 + x_2) = T(x_1) + T(x_2) = y_2 + y_2$ giving $T^{-1}(y_1 + y_2) = x_1 + x_2 = T^{-1}(y_1) + T^{-1}(y_2)$. Similarly for any scalar α we deduce that $T^{-1}(\alpha y_1) = \alpha T^{-1}(y_1)$.

Notation : Let $\mathcal{L}(X, Y)$ denote the collection of all Linear operators : $X \to Y$.

Theorem 1.3.3 $\mathcal{L}(X, Y)$ forms a vector-space with same scalar field as that of $X(Y)$.

Proof : If T_1 and T_2 are two members of $\mathcal{L}(X, Y)$, then $(T_1 + T_2) : X \to Y$ is defined as $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for all $x \in X$. Then it is a routine exercise to see that $T_1 + T_2$ is also a Linear operator and so $(T_1 + T_2) \in \mathcal{L}(X, Y)$. For any scalar λ by a similar argument we see $\lambda T_1 \in \mathcal{L}(X, Y)$ where of course $(\lambda T_1) : X \to Y$ is given by $(\lambda T_1)(x) = \lambda T_1(x)$ for all $x \in X$. So $\mathcal{L}(X, Y)$ is a vector-space with zero operator as the zero vector in $\mathcal{L}(X, Y)$.

Definition 1.2.3. Two vector space X and Y with same scalar field are said to be isomorphic if there is a Linear operator $T : X \to Y$ that is $1 - 1$ and onto (bijective).

Remark : If is not difficut to see that relation of isomorphism is an equivalence relation over the family of all vector-spaces over the same scalar field.

Theorem 1.3.3. Every real vector-space X of dimension n is isomorphic to the Euclidean n-space R^n.

Proof : Here Dim $(X) = n$. Let $(u_1, u_2 ..., u_n)$ form a basis in X so that every member $x \varepsilon X$ has a unique representation $x = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$ where α_1's are reals. Let us define $T : X \to R^n$ by the rule :-

$$T(x) = (\alpha_1, \alpha_2, ..., \alpha_n) \in R^n \text{ where } x = \sum_{i=1}^{n} \alpha_i u_i.$$

It is easy to verify that T is a linear operator such that $x = \sum_{i=1}^{n} \alpha_i u_i$ and $y = \sum_{i=1}^{n} \beta_i u_i$ I are two member of X with $x \neq y$, then we have n-tuples (α_i) and (β_i) I are not the same i.e. $(\alpha_1, \alpha_2, ..., \alpha_n) \neq (\beta_1, \beta_2, ..., \beta_n)$ in R^n.

Or, $T(x) \neq T(y)$. Thus TIis $1-1$; Further, if $(\gamma_1, \gamma_2, ..., \gamma_n) \varepsilon R^n$, we find

$\omega = x = \sum_{i=1}^{n} \gamma_i u_i \in X$ satisfying

$T(\omega) = (\gamma_1, \gamma_2, ..., \gamma_n)$. So T is into.

That is, T is an isomorphism between X and R^n.

Corollary : Any two real vector-spaces with same finite dimension are isomorphic.

§ 1.4 Topological Vector Spaces (TVS).

Suppose X is a vector-space and X is also Topological space such that its Topology satisfies T_1 separation axiom.

Definition 1.4.1. X is said to be a Topological Vector space or simply TVS if the mappings :–

i) $X x X \to X$ defined by $(x, y) \to x + y;\ x, y \in X$ and

ii) $R ¢\ x X \to X$ defined by $(\alpha, x) \to \alpha x;\ x \in R$ (¢) and $x \in X$ are both continuous mappings.

Explanation : Conditions (*i*) and (*ii*) say that principal compositions (operations) in Vector-space X i.e. vector addition and scalar multuplication are rendered continuous with respect to the assigned Topology, so that X becomes a TVS.

Further, Topology in X is assumed to T_1. That debars X from being a TVS in case we give X the indiscrete Topology which is not T_1.

In TVS X to explain further, in respect of (i) corresponding to any neighbourhood W of $x + y$ in X, we have to find a neighbourhood say V_1 of x and a neighbourhood V_2 of y in X such that

$$V_1 + V_2 \subset W.$$

Again in respect of (ii) corresponding to a neighbourhood W of αx in X, we find a neighbourhood of α, say $(\alpha - \delta, \alpha + \delta)$ ($\delta > 0$, in case of α being real scalar) an open interval of R and a neighbourhood V of x in X such that

$$\beta x' \in W \text{ whenever } \beta \in (\alpha - \delta, \alpha + \delta) \text{ and } x' \in V,$$

or, equivalently $\beta V \subset W$ for $|\beta - \alpha| < \delta$. .... (*)

(If scalar field is that of complex numbers, one has to take an open circular disc (in place of open interval above) centred at $\alpha \in \mathbb{C}$).

So this stipulation debars X from being a TVS whenever we assign discrete Topology on X, because (*) fails in this case where one may take singletons in role of neighbourhoods.

Let X be a TVS and a be a fixed member of X, Let us define an operator T_a : $X \to X$ by the rule :—

$$T_a(x) = x + a \text{ for } x \in X.$$

This operator T_a is known as a translation operator. If $a = \underline{O}$ in X, T_a equals to the Identity operator, I.

For a non-zero scalar λ, let us define an operator $M_\lambda : X \to X$ as

$$M_\lambda (x) = \lambda x \text{ as } x \in X.$$

This operator M_λ is known as a multiplication operator on X. If $\lambda = 1$, then M_λ equals to the identity operator, I.

Theorem 1.4.1. T_a and M_λ are homeomorphisms over TVS X.

Proof : $T_a : X \to X$ may be described as a mapping : $Xx\{a\} \to X$ where $(x, a) \to x + a$ as $x \in X$. And this mapping is restriction of vector-addition operation that is continuous. So T_a is continuous such that it is $1 - 1$ and onto. Therefore a T_a^{-1} : $X \to X$ exists. It is equal to T_{-a} which is again a translation operator, and consequently T_a^{-1} is continuous. Therefore T_a is a bijective and bicontinuous mapping. So T_a is a Homeomorphism.

Using the fact that scalar multiplication operation is continuous is TVS X, we prove by a similar argument that M_λ is a Homeomorphism where $M_\lambda^{-1} = M_\lambda^{-1}$.

Corollary 1. If G in an open set in TVS X, its translate $a + G$ is open set in X| where $a \in X$.

Proof : By continuity of T_{-a} we have $T_{-a}^{-1}(G)$ | is an open set in X.

Now $T_{-a}^{-1}(G) = T_{-(-a)}(G) = T_a(G) = a + G$. So $a + G$| is open.

Corollary 2. If A is any subset in TVS X, and G is an open set in X, then $A + G$ is an open set.

Because, $A + G = \bigcup_{a \in G} (a + G) =$ | an arbitrary union of open sets in X = an open set in X.

Corollary 3. For any non-zero scalar α, | and for any open set G, αG is an open set.

By a similar argument one can prove that translate of a closed set in X is a closed set.

Definition 1.4.2. A neighbourhood base of $\underline{O}$ in TVS X is called a local base in X.

If $\boldsymbol{B}$ is a Local base in X and $a \in X$, then the family $\{a + B : B \in B\}$ becomes a neighbourhood base at a.

Definition 1.4.3. A TVS X is called locally convex if X has a Local base consisting of members that are each a convex set.

Theorem 1.4.2. A TVS X has a balanced Local base (i.e., a Local whose members are balanced).

Proof : We prove by taking X as a real TVS. Let U be a neighbourhood if $\underline{O}$ in X. As $0. \underline{O} = \underline{O}$ by continuity if scalar multiplication we have a neighbourhood V of $\underline{O}$ and a+ ve δ such that

$\beta V \subset U$ for $|\beta| < \delta$.

Put $W = \bigcup_{|\beta|<\delta} \beta V$; Then W is a neighbourhood of $\underline{O}$ I such that $W \subset U$. We are ready to check that W is balanced. Take a scalar λ with $|\lambda| \leq 1$. Then $|\lambda\beta| = |\lambda||\beta| < \delta$ because $|\beta| < \delta$; and $\lambda\beta V \subset W$ and this being true for all β with $|\beta|<\delta$, we have

$$\lambda = \bigcup_{|\beta|<\delta} \beta V \subset W$$

i.e., λW W. So W is balanced, and proof is complete.

Theorem 1.4.3. Every neighbourhood of $\underline{O}$ is TVS X contains an absorbing neighbourhood of $\underline{O}$ is X.

Proof. Let W be a neighbourhood of $\underline{O}$ of TVS X, and Theorem 1.4.2 says that there is a balanced neighbourhood V of $\underline{O}$ such that

$$V \subset W$$

Take $x \in X$; we have $0.x = \underline{O}$ and by continuity of scalar multiplication we find

$$\frac{1}{n} x \in V \text{ for large } n.$$

or, $\quad x \in nV$ for some n.

By Theorem 1.2.5 V is absorbing and the proof is complete.

Definition 1.4.4. A subset E of a TVS X is said to be bounded if every neighbourhood V of $\underline{O}$ in X, corresponds a scalar $s > O$ such that $E \subset tV$ for $t > s$.

Theorem 1.4.4. In a TVS X following statements are equivalent.

(a) A subset E of X its bounded

(b) If $\{x_n\}$ is any sequence in E and $\{\alpha_n\}$ is a sequence of scalars such that $\lim_{n\to\infty} \alpha_n = 0$; then $\lim_{n\to\infty} \lambda_n x_n = \underline{O}$ in X.

Proof : Let statement (a) holds, and let E be a bounded set in X. Suppose V is a neighbourhood of $\underline{O}$ in X, and we may assume V to be balanced. Then we have some +ve scalar t such that,

$$E \in tV \qquad \text{..... (2)}$$

Since $\lim_{n\to\infty} x_n = 0$, there is an index N such that $|\alpha_n| < \frac{1}{t}$ for $n \geq N$ or $t\,|\alpha_n|<1|$ for $n \geq N$. from (1) we have $x_n \in tV$ or, $\frac{x_n}{t} \in V$; and since V is balanced, and $|\alpha_n t|<1$ we find $\alpha_n t. \in V$ for $n \geq N$.

i.e. $\alpha_n x_n \in V$ for $n \geq N$.

That means, $\lim_{n\to\infty} \alpha_n x_n = \underline{O}$ is X. Thus (b) holds. Let (b) hold. Suppose E is not bounded. Then there is a neighbourhood V of $\underline{O}$, and sequence $\{r_n : r_n > 0\}$ with $\lim_{n\to\infty} r_n = \infty$ such that $E \not\subset r_n V$, $n = 1, 2, \ldots$..

Take $x_n \in E \setminus r_n V$. Then $x_n \notin r_n V$ or, $\frac{x_n}{r_n} \notin V$; That means $\left\{\frac{1}{r_n} x_n\right\}$ does not converge to $\underline{O}$ in X, although $\{x_n\} \in E\ V$ and $\lim_{n\to\infty} \frac{1}{r_n} = 0$ | : — a contradiction. Hence Theorem is proved.

Example 1.4.1. If A and B are two bounded sets in a TVS X. Show that $A + \hat{B}$ is bounded set in X.

Solution : Let V be a neighbourhood of $\underline{O}$ in X. Since $\underline{O}+\underline{O} = \underline{O}$, using continuity of vector addition we find neighbourhoods V_1 and V_2 of $\underline{O}$ | in X such that

$$V_1 + V_2 \subset V \quad \ldots(1)$$

Since A is bounded, we find a +ve s_1 such that

$$A \subset tV_1 \text{ for all } t > s_1 \quad \ldots(2)$$

Similarly, there is a +ve s_2 such that

$$B \subset tV_2 \text{ for all } t > s_2 \quad \ldots(3)$$

Taking $s = \max(s_1, s_2)$, we see that $A + B \subset tV_1 + tV_2$

or, $A + B \subset t(V_1 + V_2) \subset tV$ when $t > s$ from (1), (2) and (3). Therefore $A + B$ is bounded in X.

Theorem 1.4.5. Let X be a TVS.

(a) If $A \subset X$, then $= \overline{A} = \{\cap\{A + V :$ where V runs through all neighbourhoods of $\underline{O}$ }, bar denoting the closure.

(b) If A and B are two subsets of X, then $\overline{A}+\overline{B}\subset\overline{(A+B)}$.

(c) If Y is a sub-space of X, then its closure $\overline{Y}$ is a sub-space of X.

(d) If B is a bounded set in X, then its closure $\overline{B}$ is bounded in X.

(e) If C is a convex sets in X, then IntC (Interior of C) and $\overline{C}$ (closure of C) are convex sets in X.

(f) If E is a balanced set in X, then its closure $\overline{E}$ is balanced. Further, if $\underline{O}\in int\ E$, then $int\ E$ is balanced.

Proof : (a) By Translation property $x\in\overline{A}$ if and only if $(x+V)\cap A\neq\phi$ for any neighbourhood V of $\underline{O}$ X and this happens if and only if $x\in A-V$. Now $-V$ is a neighbourhood of $\underline{O}$ if and only if V is a neighbourhood of $\underline{O}$ X; Therefore

$x\in\overline{A}$ if and only if $x\in(A+V)$, that is $x\in\cap\{A+V : V$ is any neighbourhood of $\underline{O}$ in $X\}$. Thus,

$\overline{A}=\cap\{A+V$: where V runs through all neighbourhood of $\underline{O}$ in $X\}$

(b) Let $a\in\overline{A}$ and $b\in\overline{B}$ and Let W be a neighbourhood of $(a+b)$. Using continiuty if vector addition we find a neighbourhood W_1 of a and a neighbourhood W_2 of b such that

$$W_1+W_2\subset W.$$

Since $a\in\overline{A}$, we have $A\cap W_1\neq\phi$, | and similarly

$B\cap W_2\neq\phi$. Take $x\in(A\cap W_1)$ and $y\in(B\cap W_2)$.| Then we have

$(x+y)\in(A\cap W_1)+(B\cap W_2)\subset(A+B)\cap W$. | Therefore

$$(A+B)\cap W\neq\phi.$$

So $(a+b)\in\overline{(A+B)}$. Therefore we have shown $\overline{A}+\overline{B}+\overline{(A+B)}$.

(c) Let Y be a sub-space of X. Then Y is closed with respect to vector addition and scalar multiplication operation. These operation are each continuous operation in a TVS. Therefore $\overline{y}$ becomes a sub-space of X.

(d) Let B be a bounded set in X, and V be a neighbourhood of $\underline{O}$ in X. As $O+\underline{O}=\underline{O}$, continuity of vector addition gives a neighbourhood W of $\underline{O}$ such that

$W + W \subset V.$ ..... (1)

Let $x \varepsilon \overline{W}$, then $x + (-W)$ is neighbourhood of x ;

So $(x - W) \cap W \neq \phi$. Take $y \cap (x - W) \cap W$; Then $y = x - w$ for some $w \in W$; So $x = y + w \in W + W$, and by (1) $x \in V$. Therefore we have $\overline{W} \subset V$.

Since B is bounded, there is $\lambda > 0$ such that

$B \subset tW$ for $t \geq \lambda$.

So, $\overline{B} \subset \overline{tW} = t\overline{W} \subset tV$ for all $t \geq \lambda$.

That means $\overline{B}$ is bounded in X.

(e) Let C be a convex set in X and $0 \leq \alpha \leq 1$.

Now $Int C \subset C$, and $\alpha Int C + (1 - \alpha)\, Int\ C$

$\subset \alpha C + (1 - \alpha)\, C \subset C$, because C is convex.

Since α *Int* $C + (1 - \alpha)$ *Int* C is an open subset of C, we have α *Int* $C + (1 - \alpha)$ *Int* $C \subset$ *Int* C,

Therefore *Int* C is a convex set.

Again $\alpha\overline{C} + (1-\alpha)\overline{C} = \overline{\alpha C} + \overline{(1-\alpha)C} \subset \overline{(\alpha C + (1-\alpha)C)} \subset \overline{C}$ because C is convex $(\alpha C + (1 - \alpha)\, C \subset C)$. Thus $\overline{C}$ is convex.

(f) Let E be a balanced set; for a scalar α with $|\alpha| \leq 1$,

We have $\alpha E \subset E$. So $\overline{\alpha E} \subset \overline{E}$ or, $\alpha\overline{E} \subset \overline{E}$.

So $\overline{E}$ is balanced.

Now if $0 < |\alpha| \leq 1$, since $x \to \alpha x$ $(\alpha \neq 0)$ is a homeomorphism α Int E = Int (αE) $\subset \alpha E \subset E$ because E is balanced. Since α *Int* E is open, it follows that α *Int* $E \subset$ *Int* E. It Int E contains $\underline{O}$, then α *Int* $E \subset$ *Int* E even for $\alpha = 0$.

The proof if now complete.

Theorem 1.4.6. Let V be a neighbourhood of $\underline{O}$ in s TVS X.

(a) If $o < r_1 < r_2 < \ldots$ with $\lim_{n\to\infty} r_n = \infty$, then $X = \bigcup_{n=1}^{\infty} r_n V$.

(b) If $\delta_1 > \delta_2 > \ldots$ with lim $\lim_{n\to\infty} \delta_n = 0$, and if V is bounded, then $\{\delta_n V\}$ | is a Local base in X.

Proof : (a) Take a fixed $x \in$ X. Since scalar multiplication is a continuous operation in TVS X, from $0.x = \underline{O}$ we find a +veδ such that $\alpha x \in V$ whenever $|\alpha| < \delta$.| Given $0 < r_n$ with $\lim_{n\to\infty} r_n = \infty$, we have $\frac{1}{r_n} \to 0$ as $n \to \infty$, and therefore for $-\delta < \frac{1}{r_n} < \delta$, we have $\frac{1}{r_n} x \in V$; or, $\alpha \in r_n V$. That means

$$X \subset \bigcup_{n=1}^{\infty} r_n V.$$

or, $$X = \bigcup_{n=1}^{\infty} r_n V.$$

(b) Let V be a bounded neighbourhood of $\underline{O}$ in X. Given any neighbourhood U of $\underline{O}$ | in X, we find $s > 0$ such that

$$V \subset tU \quad \text{for } t \geq s.$$

Since, $\{\delta_n\} \downarrow 0$, we have $\left\{\frac{1}{\delta_n}\right\} \uparrow$ with $\lim_{n\to\infty} \frac{1}{\delta_n} = +\infty$.

We see $\frac{1}{\delta_n} > s$ for large values of n. So we then have

$$V \subset \frac{1}{\delta_n} U$$

or, $$\delta_n V \subset U.$$

Hence, $\{\delta_n V\}$ becomes a neighbourhood base at $\underline{O}$ in X.

or, $\{\delta_n V\}$ is a Local base in X.

Corollary : If K is compact set in TVS X, then K is bounded in X.

Proof : Given any neighbourhood V of $\underline{O}$ we find a balanced neighbourhood W of $\underline{O}$ with $W \subset V$. Then by part (a) of Theorem 1.4.6 we have $K \subset \bigcup_{n=1}^{\infty} nW$. By compactness of K, there are integers $n_1 < n_2 < \ldots < n_s$ such that

$$K \subset n_1 W\, n_2 W \cup n_2 W \cup \ldots\ldots \cup n_s W = n_s W.$$

If $t > n_s$, we have $K \subset tW \subset tV$. Hence K is bounded.

EXERCISE-A

Short answer type questions

1. In a TVS X show that a subset of a bounded set is bounded.
2. Show that translate of a closed set is a closed set in TVS X.
3. If V is bounded neighbourhood of $\underline{O}$ in a TVS X, show that $\left\{\frac{1}{n}V\right\}$ is a Local base in X.
4. Show that $A + B$ is a bounded set when each if A and B is a compact in a TVS X.

EXERCISE-B

1. In a TVS X if B is a Local base, Show that every member of B contains the closure of some member of B.
2. Every TVS X is Hausdroff (T_2).
3. Show that convex hull of an open set in TVS X is an open set.
4. In a TVS X if A is compact and B is closed, show that $A + B$ is closed.

Unit -2

(**Contents :** Separation Theorem in TVS, Linear operators on TVS and their continuity; Linear functionals, Locally compact TVS and its finite Dimensional property, Minkowski functional, semi-norms in TVS, Locally convex TVS, Kolmogorov Theorem on normability if a TVS.)

Let X denote a Topological vector space (TVS) with scalar field as that of reals/complex numbers.

Theorem 2.1.1 (Separation Theorem)

In a TVS X if K and C respectively non-empty compact and closed sets with $K \cap C = \phi$, then there is a neighbourhood V of zero ($\underline{O}$) in X such that

$$(K + V) \cap (C + V) = \phi.$$

To prove this Theorem we first prove a Lemma.

Lemma 2.1.1. If W is a neighbourhood of $\underline{O}$ in X then there is a symmetric neighbourhood V of $\underline{O}$ such that $V + V \subset W$.

Proof : Since $\underline{O} + \underline{O} = \underline{O}$ we use continuity of vector addition to find neighbourhoods V_1 and V_2 of $\underline{O}$ satisfying $V_1 + V_2 \subset W$. Put $V = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$. Then V is a neighbourhood of $\underline{O}$ and V is symmetric such that $V + V \subset V_1 + V_2 \subset W$. (By repeating argument one can increase the number by two each time; So that we find a symmetric neighbourhood V of with $\underline{O}$ $V + V + V + V \subset W$.)

Proof of Theorem 2.1.1. Let $K \cap C = \phi$, where K is compact and C is closed in X. Take $x \in K$; As C is closed with $x \notin C$, we find a neighbourhood H of $\underline{O}$ in X with $(x + H) \cap C = \phi$. By Lemma 1.1 find a symmetric neighbourhood U_x of $\underline{O}$ such that

$$U_x + U_x + U_x + U_x \subset H,$$

Then $x + U_x + U_x + U_x + U_x \subset x + H$, and hence

$(x + U_x + U_x + U_x + U_x) \cap C = \phi$; I This gives

$$(x + U_x + U_x + U_x) \cap (C + U_x) = \phi$$

Taking U_x as open we see the family $\{x + U_x\}_{x \in k}$ as an open cover for K. By compactness of K, we find a finite number of points $x_1, x_2, ..., x_n \in X$ such that

$$K \subset \bigcap_{i=1}^{n} (x_i + U_{x_i})$$

Put $V = U_{x_1} \cap U_{x_2} \cap ... \cap U_{x_n}$. Then

$$K + V \subset \bigcup_{i=1}^{n} (x_i + U_{x_i}) + V \subset \bigcup_{i=1}^{n} (x_i + U_{x_i} + U_{x_i})$$, where none of r.h.s cuts $C + V$.

Therefore $(K + V) \cap (C + V) = \phi$. The proof is now complete.

Corollary 1. Under hypothesis of Theorem 2.1.1 there is a symmetric neighbourhood V of $\underline{O}$ such that

$(\overline{K+V}) \cap (C+V) = \phi$,| because here $C + V$ is open.

Corollary 2. If $\mathcal{B}$ is a Local base in X, then for every members $U \in$ B, there is a member $U \in$ B such that

$\overline{V} \subset U$.

Here Let us take $K = \{\underline{O}\}$ and $U \in \mathcal{B}$ as an open neighbourhood of $\underline{O}$ in X.

Taking $C = X \mid U$; So C is a closed set with $\underline{O} \notin C$.

Now Corollary 1 says that there is a symmetric neighbourhood V of $\underline{O}$| such that $V \in B$ and

$$(\overline{O+V}) \cap ((X \mid U) + V) = \phi$$

or $\overline{V} \cap (X \mid U) = \phi$

or $\overline{V} \subset$U, this is what is wanted.

Corollary 3. TVS X is T_2 (Hausdorff)

Because essentially Corollary 2 says that TVS X is regular, and a Topological space being regular plus T_1 is, of course T_2 (Hausdroff).

§ 2.2 Linear operator | functional over a TVS X.

Let X and Y denote TVS over the same scalar field.

Theorem 2.2.1 : A linear operator $T : X \to Y$ is continuous if and only if T is continuous at $\underset{\sim}{O} \in X$.

Proof : Let a linear operator $T : X \to Y$ | be continuous at $\underset{\sim}{O} \in X$ and $x \neq \underset{\sim}{O}$ | is any other point of X. Let W be a neighbourhood of $T(x)$ in Y. There is a neighbourhood U of $\underset{\sim}{O}$ in Y such that

$T(x) + U \subset W$. Now $T(\dot{\underset{\sim}{O}}) = \underset{\sim}{O}$ in Y and T is continuous at $\underset{\sim}{O} \in X$.| Therefore, we find a neighbourhood V of $\underset{\sim}{O}$ in X such that.

$$T(V) \subset U.$$

As T is Linear we have $T(x + V) = T(x) + T(V) \subset T(x) + U \subset W$. This shows that T is continuous at x, and hence T is shown as continuous over X. The other part being obvious, proof is complete.

Definition 2.2.1 : A scalar-valued Linear operator over X is called a linear functionas denoted by f over X.

Theorem 2.2.2 : Let f be a non-zero linear functional over X; Then following statements are equivalent.

(a) f is continuous.

(b) Null space $N(f) = \{x \in X : f(x) = 0\}$ is closed.

(c) $N(f)$ is not dense is X.

(d) f is bounded in some neighbourhood of $\underset{\sim}{O}$ in X.

Proof : We assume field of scalers as that of reals.

(a) $\Rightarrow$ **(b)** let (a) be true. Write $N(f) = (f)^{-1}(0)$ =| Inverse image of a closed set under a continuous mapping = a closed set in X.

(b) $\Rightarrow$ **(c)** Let (b) hold. Suppose $N(f)$ is dense in X i.e.,

$$\overline{N(f)} = X$$

i.e. $N(f) = X$, because $N(f)$ is closed.

That means f is the zero functional—a contradictin. Hence (c) stands.

(c) $\Rightarrow$ (d) Since $N(f)$ is not dense is X. Then $X \setminus N(f)$ has a non-empty interior. So there is $x \in X$, and a balanced neighbourhood V of $\underset{\sim}{O}$ | satisfying

$(x + V) \subset (X \setminus N(f))$

i.e. $(x + V) \cap N(f) = \phi$ ....(1)

There arise two cases to consider :—

Case I $f(V)$ is bounded; in that case we have finished.

Case II $f(V)$ is unbounded. In this case we show that $f(V)$ = whole scalar field $= R$. Let λ be any real number. Since $f(V)$ is unbounded in R, we find $v \in V$| such that $f(v) > \lambda$. Take scalar α such that $\alpha f(v)=\lambda$. Then $|\alpha| = |\frac{\lambda}{f(v)}|<1$; Since V| is balanced, $\alpha v \in V$ and $f(\alpha v) = \alpha f(v) = \lambda$. Thus λ is attained at some point $v \in V$.

Then $-f(x)$ is attained at some point $y \in V$ i.e., $f(y) = -f(x)$

or, $f(x) + f(y) = 0$, or, $f(x+y) = 0$; Showing thereby $(x+y) \in N(f)$.| But $(x+y) \in (x+V)$. Hence $(x+V) \cap N(f) \neq \phi$–|a contradiction of (1). So *(d)* is established.

(d) $\Rightarrow$ (a) Let f be bounded in some neighbourhood V of $\underline{O}$ is X. Let $|f(v)| \leq M$ as $v \in V$ $(M > 0)$. If $\varepsilon > 0$ is given, Let us put $W = \frac{\varepsilon}{M} V$; Then W is neighbourhood of $\underline{O}$ is X such that if $x \in W$, we have

$$|f(x)| = \left| f\left(\frac{\varepsilon}{M} v\right) \right| \text{ for some } v \in V.$$

$$= \frac{\varepsilon}{M} |f(v)| < \varepsilon.$$

That means, f is continuous at $\underline{O}$ of X, and as seen in proceeding theorem f is continuous over X. The proof is now complete.

Definition 2.2.2 : A TVS X is called Locally compact if $\underline{O}$ | of X had a neighbourhood whose closure is compact.

Theorem 2.2.3 : Let Y be a sub-space of TVS X; If Y is Locally TVS (with relativised Topology), then Y is a closed sub-space of X.

Proof : Since Y is Locally compact, there is a neighbourhood of $\underline{O}$ is Y whose closure (relative to Y), say $= K$ is compact in Y. So we find a neighbourhood U of in X satisfying $(U \cap Y) \subset K$.

Take a symmetric neighbourhood V of $\underline{O}$ in X satisfying.

$\bar{V} + \bar{V} \subset U$.

Then for every $x \in X$ we show that $Y \cap (x + \bar{V})$ is compact. Let $y_0 \in (Y \cap (x + \bar{V}))$ and be kept fixed. Then $y \in Y \cap (x + \bar{V})$, we have $y - y_0 = (y - x) + (x - y_0) \in \bar{V} + \bar{V} \subset U$; Also $(y - y_0) \in Y$ as Y is a sub-space. Therefore, $(y - y_0) \in (U \cap Y) \subset K$. That means, $Y \cap (x + \bar{V}) \subset y_0 + K$, which is a compact set in Y; Now $Y \cap (x + \bar{V})$| is compact in X.

Now fix $x \in \bar{Y}$; Let $\mathfrak{B}$ denote the collection of all open sets W of X such that $O \in W$ and $W \subset V$. Associate with each $W \in \mathfrak{B}$, the set $E_w = Y \cap (x + \bar{W})$. Since $W \subset V$, | each E_w is compact, As $x \in \bar{Y}$, We have

$$(x + \bar{W}) \cap Y \neq \phi.$$

Thus each $E_w \neq \phi$. Since intersection of a finite number of members of $\mathfrak{B}$ belongs to $\mathfrak{B}$, it follows that $\{E_w\}$ is a family of compact sets possessing finite intersection property. Hence $\cap E_w \neq \phi$; Let $z \in \cap E_w$. Clearly $z \in Y$; but $z \in (x + \bar{W})$ for every member $W \in \mathfrak{B}$; Hence $z = x$ i.e., $x \in Y$; Consequently $\bar{Y} = Y$; So Y is a closed sub space of X.

Theorem 2.2.4. Suppose X is a complex TVS Y a subspace of X such that Dim $(Y) = n$. Then

(a) Every Isomorphism of $\mathfrak{C}^n$ onto Y is a homeomorphism; and

(b) Y is closed

Proof : Since Unitary space C^n is Locally compact and by homeomorphism as contemplated in (a) Y is Locally compact. So Theorem 2.2.3 applies here. Thus (b) follows from (a) We now establish (a) by Induction.

Let P_n denoter the statement as in (a) above. First we check that P_1 is true.

Let $h : C \to Y$ be an Isomorphism; Then h is Linear, 1 – 1 and onto mapping. Put $h(1) = u_0 \in Y$. For any complex scalar $z \in C$, we have $h(z) = h(1 \cdot z) = zh(1) = zu_0$. Now continuity of scalar multiplication in Y show that h is continuous. Further h^{-1} exists as a Linear functional over Y, such that Null space $N(h^{-1}) = \{O\}$, and that is a closed set so (by earlier Theorem) h^{-1} is continuous. Thus h is a bicontinuous and objective mapping. So h is a homeomorphism. Therefore P_1 is established.

Let us now assume that P_{n-1} $(n>1)$ hold. Let $h : ¢^n \to Y$ be an Isomorphism. Take

$$(e_1, e_2, \ldots, e_n) \text{ as a basis } C^n, \text{ where } e_k = (\underline{0} \quad \overset{k-th\ place}{1} \quad \underline{0}), \quad k = 1, 2, \ldots, n.$$

Put $h(e_k) = u_k$ $(i = 1, 2, \ldots n)$. For $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in ¢^n$, We find $h(\underline{\alpha})$

$$= h\left(\sum_{k=1}^{n} \alpha_k h(e_k)\right) = \sum_{k=1}^{n} \alpha_k h(e_k) = \sum_{k=1}^{n} \alpha_k u_k.$$

Now continuity of vector sum and scalar multiplication in Y shows that h is a continuous mapping. As h is an Isomorphism, $\{h(e_k) = u_k\}k = 1, 2, \ldots, n\}$ shall form a basis in X. So we find linear functional $\gamma_k(k = 1, 2, \ldots, n)$ over Y such that every $x \in Y$ I has a unique representation

$$x = \gamma_1(x)u_1 + \gamma_2(x)\, u_2 + \ldots + \gamma_n(x)\, u_n.$$

Each γ_k has Null-space $N(\gamma_k)$ as a sub-space of dimensin $(n-1)$ in X. By assumed truth of P_{n-1}, each $N(\gamma_k)$ is closed, and consequenty each γ_k $(k = 1, 2, \ldots, n)$ becomes continuous. Since $h^{-1}(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_n(x))$, it follows that h^{-1} is also continuous i.e. h is bicontinuous nad bijective mapping i.e. h is a homeomorphism. So P_n stands true. Our induction is complete, and Theorem is proved.

Theorem 2.2.5. Every Locally compact TVS is finite dimensional.

Proof : Let X be a Locally compact TVS. So $\underline{0}$ in X has a neighbourhood V whose closure $\overline{V}$ is compact. The $\overline{V}$ is bounded (see corollary of Theorem 1.4.6), and therefore V is bounded. Then $\left\{\frac{V}{2^n}\right\}$ forms a Local base of X.

The collection $\left\{x + \frac{1}{2}V\right\}_{x \in X}$ (without loss of generality V is taken as open) forms an open cover for X, and it is also so for $\overline{V}$. Compactness of $\overline{V}$ I, we find a finite number of points, say, $x_1, x_2, \ldots, x_m$ in X such that

$$\overline{V} \subset \left\{x_1 + \frac{1}{2}V\right\} \cup \left(x_2 + \frac{1}{2}V\right) \cup \ldots\ldots \cup \left(x_m + \frac{1}{2}V\right).$$

Consider the sub-space Y spanned by $\{x_1, x_2, ..., x_m\}$ i.e., Y = Lin. hull $\{x_1, x_2, ..., x_m\}$ Clearly Dim (Y) m. Hence Theorem 2.2.4 applies here. Thus Y is closed. Now we have

$$V \subset \overline{V} \subset \left\{x_1 + \frac{1}{2}V\right\} \cup \left(x_2 + \frac{1}{2}V\right) \cup \text{......} \cup \left(x_m + \frac{1}{2}V\right).$$

$\subset Y + \frac{1}{2}V$, because Y is the sub-space generated by $x_1, x_2, ..., x_m$; and $\left(x_i + \frac{1}{2}V\right)$ $\subset \left(Y + \frac{1}{2}V\right)$.

Further, for any scalar λ, $\lambda Y \subset Y$. So $\frac{1}{2}V \subset \left(Y + \frac{1}{2^2}V\right)$

So one has $V \subset \left(Y + \frac{1}{2}V\right) \subset Y + Y + \frac{1}{2^2}V \subset Y + \frac{1}{2^2}V$. We continue the argument to produce

$$V \subset Y + \frac{1}{2^n}V (n = 1, 2, ...)$$

so $$V \subset \bigcap_{k=1}^{\infty} (Y + \frac{1}{2^n}V)$$

Since $\left\{\frac{V}{2n}\right\}$ is a Local base in X, it follows that

$$V \subset \overline{Y}$$

i.e., $V \subset Y$ because Y is closed.

Since Y is a sub-space, for any +ve integer k we have

$kV \subset Y$, $k = 1, 2, ...$

As V is bounded we have $X = \bigcup_{k=1}^{\infty} kV$, and therefore,

$X \subset Y$, where Dim (Y) $\leq m$.

Hence X is finite dimensional.

§ 2.4 Minkowski functional

Definition 2.4.1. Let K be a convex absorbing set containing $\underline{O}$ of TVS X. and let $x \in X$.

Put $A_x = \{\alpha > 0 : x \in \alpha K\}$

Since K is absorbing, we have $A_x \neq 0$. Let us put

$P_k(x) = Inf\ A_x$.

Then $pk : X \to$ Reals is called Minkowski functional for K.

Note : For any $x \in X$, $0 \leq P_x(x) < \infty$.

Theorem 2.4.1. Suppose K is a convex absorbing set containg $\underline{O}$ of TVS X, and let p_k be the Minkowski functional for K.

Then (a) $p_k(\underline{O}) = 0$

(b) $p_k(x+y) \leq p_k(x) + p_k(y)$ for all $x, y \in X$.

(c) $p_k(\lambda x) = \lambda p_k(x)$ for all scalars $\lambda > 0$, and for all $x \in X$.

(d) If K is balanced then $p_k(\lambda x) = |\lambda|\, p_k(x)$ for all scalars λ and for all $x \in X$.

Proof : (a) Here $\underline{O} \in K$; For every +ve integer n we have $\underline{O} \in \frac{1}{n}K$, giving $p_k(\underline{O}) \leq \frac{1}{n}$; and hence $p_k(\underline{O}) = 0$.

(b) Let $x, y \in X$, and take $\alpha \in A_x$ and $\beta \in A_y$ arbitrarily, then we have $x \in \alpha K$ and $y \in \beta K$; so that

$(x + y) \in \alpha K + \beta K = (\alpha + \beta) K$ (since K is convex), and this gives $(\alpha + \beta) \in A_{x+y}$, and hence

$p_k(x + y) \leq \alpha + \beta$; As α, β are arbitrary members of A_x and A_y respectively, we have

$p_k(x + y) \leq p_k(x) + p_k(y)$.

(c) Take $\lambda > 0$, and $x \in X$; If $\alpha \in A_x$ we have $x \in \alpha K$; So $\lambda x \in \lambda ak$, giving $p_k(\lambda x) \leq \lambda\alpha$ and hence $p_k(\lambda x) \leq p_k(x)$.

Again for $\beta \in A_{\lambda x}$ We have $\lambda x \in \beta K$; giving $x \in \frac{\beta}{\lambda} K$, and hence $p_k(x) \leq \frac{\beta}{\lambda}$,

Since β is an arbitrary members of $A_{\lambda x}$ We have

$$p_k(x) \leq \frac{1}{\lambda} \; p_k(\lambda x)$$

or, $\lambda p_k(x) \leq p_k(\lambda x)$ .... (2)

Combining (1) and (2) we deduce $p_k(\lambda x) = \lambda p_k(x)$.

(d) Suppose K is balanced; Take $\lambda \neq 0$, and $x \in X$. If $x \in A_x$, we have $x \in \alpha K$.|

i.e., $\frac{x}{\alpha} \in K$. As K| is balanced, We have $\frac{\lambda}{|\lambda|} \frac{x}{\alpha} \in K$ This gives $\lambda x \in |\lambda| \alpha k$ fromwhere we have $P_k(\lambda x) \leq |\lambda| \alpha$ and this gives $p_k(\lambda x) \leq |\lambda|. p_k(x)$ .... (3)

Again take $\alpha \in A_{\lambda x}$ giving $\lambda x \in \alpha K$.

So $\frac{\lambda}{\alpha} x \in K$; As K is balanced we get

$$\frac{|\lambda|}{\alpha} x \in K$$

This gives $\alpha \in \frac{\alpha}{|\lambda|} K$, and hence $p_k(x) \leq \frac{\alpha}{|\lambda|}$

or, $|\lambda| P_k(x) \leq \alpha$; As α is any member of $A_{\lambda x}$, we have

$|\lambda| P_k(x) \leq P_k(\lambda x)$ .... (4)

Combining (3) and (4) we get $p\lambda(\lambda x) = |\lambda| p_k(x)$.

Theorem 2.4.2. Let K be a convex absorbing set containing $\underline{O}$ of TVS X and p_k be the Minkowski functional for K. Then (i) $x \in K$ implies $p_k(x) \leq 1$.

(ii) $p_k(x) < 1$ implies $x \in K$.

Proof : (i) Let $x \in k$; So $1 \varepsilon A_x$ and hence $P_k(x) \leq 1$.

(ii) Let $p_k(x) < 1$ for some $x \in K$. So we find $\alpha \in A_x$ satisfying $0 < \alpha < 1$. Now $\underline{O} \in K$, and $x \in \alpha K$

i.e., $\frac{x}{\alpha} \in K$; By convexity of K we find

$$\alpha . \frac{x}{\alpha} + (1-\alpha) . \underline{O} \in K.$$

or, $x \in K$.

Theorem 2.4.3. Let K be a convex absorbing set containing $\underline{O}$ of TVS X, and p_k be the Minkowski functional for K.

Suppose $K_1 = \{x \in X : p_k(x) < 1\}$, and
$K_2 = \{x \in X : p_k(x) < 1\}$.

Then (a) Int $K \subset K_1 \subset K \subset K_2 \subset$ closure $\overline{K}$

(b) $K = K_1$ if K is open

(c) $K = K_2$ if K is closed

(d) If p_k is continuous, then $K_1 =$ Int K, and $K_2 = \overline{K}$.

(e) p_k is continuous if and only if $\underline{O} \in$ Int K.

Proof : (a) Take $x \in$ Int K i.e. $1.x \in$ Int K which is open. By continuity of scalar multiplication in X, we find scalar α near but > 1 such that

$$\alpha x \in \text{Int } K \subset K.$$

So, $x \in \frac{1}{\alpha} K$.

This gives $P_k(x) \leq \frac{1}{\alpha} < 1$; Hence $x \in K_1$.

Therefore Int $K \subset K_1$. Theorem 2.4.2 says $K_1 \subset K \subset K_2$.

To complete the desired chain of inclusion let us take $x \in K_2$.

So $p_k(x) \leq 1$. If $pk(x) < 1$, we have finished. Let $p_k(x) = 1$.

We find $\{\alpha_n\} \subset A_x$ satisfying $\lim_{n \to \infty} \alpha_n = 1 (\alpha_n > 1)$.

So $x \in \alpha_n K$ or Now $\left(\frac{x}{\alpha_n}\right) \in K$.| and therefore $\lim_{n \to \infty} \frac{x}{\alpha_n} = x$ | and therefore $x \in \overline{k}$. Thus inclusion chain arrived at is

$\text{Int } K \subset K_1 \subset K \subset K_2 \subset \overline{K}$.

(b) and (c) are straight consequences; their proofs are left out.

(d) Using contunity of p_k we find $K = \{x : p_k(x) < 1\} = P_k^{-1}(0,1)$ which becomes an open set. This gives $K_1 = Int\ K$, and by a similar argument $K_2 = \overline{K}$.

(e) Suppose p_k is continuous; then Int $K = K_1$.

Now $p_k(\underline{O}) = 0 < 1$; Hence $\underline{O} \in$ Int K.

Conversely let $\underline{O} \in$ Int K. Suppose U be a neighbourhood of $\underline{O}$ with $\underline{O} \in U \subset$ Int K; So we have $p_k(x) < 1$ for $x \in U$. Now if $\varepsilon > 0$ is given advance, εU is a neighbourhood of $\underline{O}$ is Xl, and we have for $x \in \varepsilon U$, and $p_k(x) = p_k(\varepsilon u)|$ (for some $u \in U) = \varepsilon p_k(u) < \varepsilon$ as $p_k(u) < 1$. That shows, p_k is continuous at $\underline{O}$ and hence p k becomes continuous everywhere in X. The proof is not complete.

§ 2.5 Semi-norms :

Let X be a vector space.

Definition 2.5.1 $p : X \to$ Reals is called a semi-norm if

(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$, and

(ii) $p(\alpha x) = |\alpha|\, p(x)$ for all $x \in X$ and for all scalars α.

Explanation : Taking $\alpha = 0$ in (ii) We find $p(\underline{O}) = 0|$ and from (i) taking $x = -y$ we find $p(x) \geq 0$ for all $x \in X$. Thus a semi-norm p becomes a norm over X if $p(x) \neq 0|$ whenever $x \neq \underline{O}$.

Theorem 2.5.1 Let p be a semi-norm over X. Then

(1) $|p(x) - p(y)| \leq p(x-y)$ for all $x, y \in X$

(2) $\{x : p(x) = 0\}$ is a sub-space of X

(3) If $K = \{x \in X : p(x) < 1\}$, then K is convex, balanced and absorbing with its Minkowski functional $p_k = p$.

Proof : (1) Let $x, y \in X$, we have $p(x) = p(x - y + y) \leq p(x - y) + p(y)$

or, $p(x) - p(y) \leq p(x-y)$; and interchanging x and y we have

$p(y) - p(x) \leq p(y-x) = p(-1)(x-y) = |(-1)|\, p(x-y) = p(x-y)$.

Therefore we have $| p(x) - p(y) | \leq p(x-y)$.

(2) The proof is easy and left out.

(3) Let $K = \{x \in X : p(x) < 1\}$. If $x_1, x_2 \in K$ and $0 \leq t \leq 1$, we have, $p(tx_1 + (1 - t) x_2) \leq tp(x_1) + (1 - t) p(x_2) < 1$

This shows that K is convex.

Suppose $|\lambda| \leq 1$, if $u \in K$, then $p(\lambda u) = |\lambda| p(u) < 1$ because $p(u) < 1$. Therefore $\lambda u \in K$. and K is balanced.

Also, let $x \in K$, and take $s > p(x)$. Then $p\left(\frac{x}{s}\right) = \frac{1}{s} p(x) < 1$; So $\frac{x}{s} \in K$ or, $x \in sK$; That means K is absorbing (K is balanced). Finally, let P_k be Minkowski functional of K over X, and let $x \in X$; Take $s > p(x)$. As before we see $x \in sK$. Then $p_k(x) \leq s$. This is true for all $s > p(x)$, So we deduce

$$pk(x) \leq p(x) \qquad \text{...... (1)}$$

Now take t with $0 < t \leq p(x)$. Then $p\left(\frac{x}{t}\right) = \frac{1}{t} p(x) \geq 1$.

Clearly, $\frac{x}{t} \notin$ i.e., $x \notin tK$

Therefore $pk(x) > t$, this being true for $0 < t \leq p(x)$, we deduce

$$pk(x) \geq p(x) \qquad \text{...... (2)}$$

From (1) and (2) we get $p(x) = p_k(x)$ So $p \equiv p_k$

Definition 2.5.2 : A family $\wp$ of semi-norms f over TVS X is said to be separating it for $x \neq O$ in X, there is a member $f \in \wp$ such that $f(x) \neq 0$.

Theorem 2.5.2. Let B be a convex balanced local base of TVS X. For each $V \in \mathcal{B}$ let p_v denote its Minkowski functional ove X. Then the family $\{p_v\}_{V \in \mathcal{B}}$ is a separating family of continuous semi-norms in X.

Proof : For $V \in \mathcal{B}$, V is a convex, balanced and absorbing set with $O \in V$. So its Minkowski functional p_v is a seminorm over X. Take $x \in X$ with $x \neq 0$. So $x \notin$ V for some $V \in \mathcal{B}$, and then $p_V(x) \geq 1$; So $p_V(x) \neq 0$. Thus the family $\{p_v\}_{V \notin \mathcal{B}}$ seminorms is a separating family.

Also if $x \in V$, then $tx \in V$ for some $t > 1$, because V is taken as open. So $x \in \frac{1}{t} V$

That gives, $p_V(x) \leq \frac{1}{t} < 1$

So, $p_V(x) < 1$ for $x \in V$.

Given $\varepsilon > 0$, and $z \in (\varepsilon V)$, we have $p_v(z) = p_V(\varepsilon v)$ for some $v \in V$. So $p_V(z) = \varepsilon Pv(v) < \varepsilon.1 = \varepsilon$.

As $p_v(\underline{O}) = 0$ we find $| Pv(z) - p_V(\underline{O}) | < \varepsilon$ whenever $z \in (\varepsilon V)$.

That shows p_V is continuous at $\underline{O}$, and hence by sub-additivity p_V is continuous everywhere in X.

Theorem 2.5.3 : suppose $\wp$ is a separating family of seminorms in a vector-space X. For each $p \in \wp$ and for each +ve integer n put $V(p, n) = \left\{x \in X : p(x) < \frac{1}{n}\right\}$.

Let $\mathcal{B}$ denote the collection of all finite intersections of members $V(p, n)$.| Then $\mathcal{B}$ forms a convex, balanced Local base for a Topology τ to make X a locally convex TVS such that (i) every member $p \in \wp$ is continuous and

(ii) a subset E of X is bounded if and only if every member $p \in \wp$ | is bounded over E.

Proof : We call a subset A in X to be open if and only if A is a union of translates of members of $\mathcal{B}$. The collection of open sets so designated is a Topology τ on X. It is a routine check up to see that every member of $\mathcal{B}$ is convex, balanced and $\mathcal{B}$ forms a Local base for τ.

We verify that τ is T_1. Take a non-zero x in X. By separating property of $\wp$, we find a member $p \in \wp$ such that $p(x) \neq 0$ i.e., $p(x) > 0$. Take a +ve integer n such that

$$np(x) > 1$$

Then $x \in V(p, n)$.

i.e., $\underline{O} \in x - V(p, n)$; $x - V(p, n)$ being a neighbourhood of x we see that x is not a limut point of $\{\underline{O}\}$. i.e., $x \notin \{\underline{O}\}$; That means singletor $\{\underline{O}\}$ | is closed and hence every singleton is closed in X. So τ is T_1.

Next comes verificatiion of continuity of vector addition and scalar multiplication operations in X. Let U be a neighbourhood of $\underline{O}$ in X. There is $p_1, p_2, \ldots\ldots, p_m \in \wp$ and + ve integers $n_1, n_2, \ldots\ldots, n_m$ satisfying.

$V(P_1, n_1) \cap V(p_1, n_2) \cap \ldots\ldots \cap V(P_m, n_m) \subset U.$

Put $V = \bigcap_{n-1}^{m} V(p_i, 2p_i)$, take x, y from V. Then we have

$$pi(x) < \frac{1}{2n_i} \text{ and } p_i(y) < \frac{1}{2n_i} \quad (i = 1, 2, \ldots\ldots, m).$$

By sub-additivity of p_i we find

$$p_i(x+y) \le p_i(x) + p_i(y) < \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i} \quad (i = 1, 2, \ldots., m).$$

$$\text{Hence } (x+y) \in \bigcap_{n=1}^{m} V(p_i, n_i) \subset U.$$

i.e., $V + V \subset U$.

So vector addition is continuous.

Next take $x \in X$, and α a scalar, and Let U be a neighbourhood of $\underline{O}$. So there are $p_1, p_2, \ldots., p_m$ and +ve integers $n_1, n_2, \ldots., n_m$ such that

$$\bigcap_{i=1}^{m} V(p_i, n_i) \subset U.$$

Put $V = \bigcap_{i=1}^{m} V(p_i, 2n_i)$. We have $x \in sV$ for some $s > 0$.

Now take $t = \dfrac{s}{1+|\alpha|s}$. Then $x + tV$ is a neighbourhood of x. If $y\ x + tV$ and $|\beta - \alpha| < \dfrac{1}{s}$, we find $\beta y - \alpha x = \alpha(y - x) + (\beta - \alpha)x \in |\beta|\, tV + |\beta - \alpha|\, sV$

$\subset V + V \subset U,$

Since V is balanced and $|\beta|\, t \le 1$.

$$\left(|\beta| \le \beta - \alpha| + |\alpha| < \frac{1}{s} + |\alpha| = \frac{1+|\alpha|s}{s} = \frac{1}{t} \right)$$

So $\beta y \in \alpha x + U$ which is a neighbourhood of αx. This holds when $|\beta - \alpha| < \dfrac{1}{s}$ and $y \in x + tV$.

That means scalar multiplication is continuous in X.

Hence X becomes a Locally convex TVS.

From Definition of $V(p, n)$ it follows that p is continuous at $\underline{O}$. and hence p becomes continuous everywhere on X.

Finally, let E be a bounded set in X. For $p \in \wp$, $V(p, 1)$ is neighbourhood of $\underline{O}$ in X. So $E \subset kV(p, 1)$ for some $k < \infty$. For $x \in E$, let $x = k\xi$ where $\xi \in V(p, 1)$.

i.e., $p(\xi) < 1$

or, $p\left(\frac{x}{k}\right) < 1$

or, $p(x) < k$.

This being true for all $x \in E$, we, find p to be bounded over E.

Conversely, E satifies the condition. Let U be a neighbourhood of $\underline{O}$, and let $V(p_i, n_i) \cap V(p_2, n_2) \cap \ldots\ldots \cap V(p_m, n_m) \subset U$,

Where $p_1, p_2, \ldots\ldots, p_m \in \wp$ and $n_1, n_2, \ldots\ldots, n_m$ are +ve integers.

Since each p_i is bounded over E, we find +ve numbers M_i such that

$p_i(x) < M_i$ for $x \in E$ $(i = 1, 2, \ldots\ldots, m)$.

Take $\gamma > \max\limits_{1 \le i \le m} [M_i n_i]$; So $n_i p_i(x) < n_i M_i < \gamma$

i.e., $p_i(x) < \frac{\gamma}{n_i}$

So, $p_i\left(\frac{x}{\gamma}\right) < \frac{1}{n_i}$

This gives $\frac{x}{\gamma} \in V(p_i, n_i)$. This being true for $1 \le i \le m$.

We deduce that $\frac{x}{\gamma} \in \bigcap\limits_{i=1}^{m} V(p_i, n_i) \subset U$.

i.e., $x \in \gamma U$.

and this true for all $x \varepsilon E$, we have $E \subset \gamma U$.

Therefore E is shown to be bounded. The proof is now complete.

Theorem 2.5.4 (Kolmogorov Theorem) A TVS X is normable if and only if there is a convex bounded neighbourhood of $\underline{O}$ in X.

Proof. Let U be a bounded convex neighbourhood of $\underline{O}$ in TVS X. Without loss of generality U is taken to be symmetric.

For $x \in X$, let us define $\| x \| = Inf \{\lambda > 0 : x \in \lambda U\}$.

We now verify that $\| \|$ is indeed a norm function over X. Clearly $\| x \| \geq 0$ always. As $\underline{O} \in \lambda U$ for all +ve λ, we find $\|\underline{O}\| = 0$.

If x is non-zero member of X, we find a +ve integer n_0 such that $x \notin \frac{1}{n_0} U$.

Infact, if $x \in \bigcap_{n=1}^{\infty} \frac{1}{n} U$, then $\{y_n = nx\}$ is a sequence is U where U is bounded such that $\lim_{n \to \infty} \frac{1}{n} y_n = \underline{O}$ gives $x = \underline{O}$ — a contradiction. So $\| x \| \geq \frac{1}{n_0} > 0$.

Thus Norm axiom (N. 1) is O.K.

For (N. 2) we have $x \in \lambda U$ if and if $-x \in \lambda U$ (U is symmrtric) i.e., $\| -x \| = \| x \|$. Take $\alpha > 0$, and $x \in \gamma U$; then $\alpha x \in \alpha\gamma U$, and conversely, $\alpha x \varepsilon \alpha\lambda U$ implies $x \varepsilon \lambda U$. So

$$\begin{aligned} \| \alpha x \| &= \text{Inf } \{\mu > 0 : \alpha x \in \mu U\} \\ &= \text{Inf } \{\alpha\lambda : \alpha x \in \alpha\lambda U\} \\ &= \alpha \text{ Inf } \{\lambda : x \in \lambda U\} \\ &= \alpha \| x \| \end{aligned}$$

In General, $\| \alpha x \| = \| \pm | \alpha | x \| = | \alpha | \| x \|$.

Therefore (N. 2) is established.

For triangle inequality, let $x, y \in X$, and let $\| x \| = \alpha$ and $\| x \| = \beta$. Without loss of Generality, suppose x, y are non-zero. Then $\| \frac{x}{\alpha} \| = \| \frac{y}{\beta} \| = 1$.

Now. $\frac{x}{\alpha} \in (1 + \varepsilon)\, U$

and $\frac{y}{\beta} \in (1 + \varepsilon)\, U$ for $\varepsilon > 0$.

As U is convex, $(1 + \varepsilon)\, U$ is convex, So

$$\left(\frac{\alpha}{\alpha+\beta}\frac{x}{\alpha}+\frac{\beta}{\alpha+\beta}\frac{x}{\beta}\right)\in(1+\varepsilon)U$$

or, $$\frac{x+y}{\alpha+\beta}\in(1+\varepsilon)U$$

or, $(x + y) \in (\alpha + \beta)(1 + \varepsilon) U.$

Therefore $\| x + y \| \leq (\alpha + \beta)(1 + \varepsilon)$

As $\varepsilon > 0$ is arbitrary, we deduce that

$$\| x + y \| \leq \alpha + \beta = \| x \| + \| y \|.$$

Therefore $(X, \| \|)$ becomes a Normed Linear space (*NLS*). To complete the proof we will show that for any neighbourhood V of $\underline{O}$ in TVS X, there is an open ball $B_r(\underline{O})$ of *NLS* X such that $B_r(O) \subset V$ and vice-versa.

Take V as a neighbourhood of $\underline{O}$ in TVS X. Since U is a bounded neighbourhood of $\underline{O}$ in TVS X, there is a +ve r satisfying

$$U \subset \frac{1}{r} V.$$

or, $rU \subset V$

Now open Unit ball B_1 (O) of $(X, \| \|)$ satisfies $B_1 \, \underline{O} \subset U.$

So, $B_r(\underline{O}) \subset r\,U \subset V.$

Conversely, let $B_\varepsilon(\underline{O})$ be an open ball centred at $\underline{O}$ of $(X, \| \|)$. If $0 < \varepsilon' < \varepsilon$, then $\varepsilon' U \subset B_\varepsilon(\underline{O})$ where $\varepsilon' U$ = a neighbourhood of $\underline{O}$ in TVS X.

Thus we have finished proving sufficient part of the proof; the necessary part is rather obvious.

Unit-3

(**Contents** : Representation Theorem for bounded linear functional over spaces R^n, $l_p(1 < p < \infty)$ and $C[0, 1]$; Banach Steinhauss Theorem and its application, weak convergence in NLS. Comparison with Norm convergence, notions being identical in a finite Dimensional NLS. Approximation in NLS, Existence of best approximation, strictly convex norm, uniqueness of the best approximation.

§ 3.1 Let X denote a Normed Linear space (NLS) over reals /complex numbers. Then X^* denotes the Banach space of all bounded Linear functionals over X. It is the first conjugate space X; and Second conjugate space X^{**} is obtained. We now examine how conjugate (Dual) spaces look like. This opens an area of theory of representations of bounded Linear functionals over a given NLS X. We show right now that the Dual space $(R^n)^*$ of the Euclidean n-space R^n as a NLS is isomorphic with R^n.

Theorem 3.1.1. The Dual space of R^n is isomorphic with R^n. (Dual of R^n is $R^n|R^n$ is self-dual).

Proof : Let us take $e_1 = \underbrace{(1,0,0...,0}_{n\ places}$, $e_2 = (0,1,...0),...,$ $e_n = (0,0,0,...,1)$ to form a basis of NLS R^n. So that any member $\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n) \in R^n$ can be expressed uniquely as

$$\underset{\sim}{x} = \sum_{k=1}^{n} \xi_k e_k \, .$$

If $f \in (R^n)^*$, then $f(\underset{\sim}{x}) = f\left(\sum_{k=1}^{n} \xi_k e_k\right) = \sum_{k=1}^{n} \xi_k f(e_k) = \sum_{k=1}^{n} \xi_k \gamma_k$ | where $\gamma_k = f(e_k)(k = 1, 2,......, n)$; and $(\gamma_1, \gamma_2, ..., \gamma_n) \in R^n$.

Conversely, if $(c_1, c_2,...., c_n) \in R^n$, Let us define $\varphi : R^n \to R$ where $\varphi(\underset{\sim}{x}) = \sum_{k=1}^{n} \xi_k c_k$;. and $x = (\xi_1, \xi_2, ... \xi_n) \in R^n$.

Then it is easy to see that φ is a bounded linear functional over R^n, and $\varphi \in (R^n)^*$.

This correspondence $(R^n)^* \to R^n$, we show presently, is an Isomorphism. This correspondence is, of course, linear plus 1 – 1 and onto.

Further, Let $f \in (R^n)^*$, then its corresponding member as we see above is $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^n$ such that if $x \in R^n$, then

$$|f(x)| = |\sum_{k=1}^{n} \xi_k \gamma_k| \leq \sqrt{\sum_{k=1}^{n} \xi_k^2} \cdot \sqrt{\sum_{k=1}^{n} \gamma_k^2} . \; (x = (\xi_1, \xi_2, \ldots, \xi_n))$$

by $C - S$ inequality $\|x\| \|c\|$ where $c = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^n$, and therefore, $\|f\| \leq \|c\|$....... (i)

Taking $x = c$, we find $f(c) = \sum_{k=1}^{n} \gamma_{k^2}$ and $\|f\| \geq \frac{\|f(c)\|}{\|c\|} = \sqrt{\sum_{k=1}^{n} \gamma_k^2} = \|c\|$

$\therefore \|f\| = \|c\|$

Hence $R_n^* \simeq R^n$ ($\simeq$ denoting isomorphism).

Let $\{x_n\}$ be a sequence in NLS X.

If $S_n = x_1 + x_2 + \ldots + x_n$, and if there is a member $s \in X$ such that $\|S_n - s\| \to 0$ an $n \to \infty$ or equivalently $\lim_{n \to \infty} S_n = s \in X$, then the infinite series $x_1 + x_2 + \ldots = \sum_{k=1}^{\infty} x_k$ is said to converge to the sum = S (in norm in X) in X; and we write $x_1 + x_2 + \ldots = S$.

$$\text{or, } \sum_{k=1}^{\infty} x_k = S.$$

If there is a sequence $\{e_n\}$ of vectors in X such that for every member $x \in X$ there is a unique sequence $\{\alpha_n\}$ of scalars satisfying

$$\sum_{k=1}^{\infty} \alpha_k e_k = x$$

Then $\{e_n\}$ is called a Schauder basis in X.

Example 3.1.1. The sequence space $l_p (1 < p < \infty)$ consisting of all real sequence $x = (x_1, x_2, \ldots, x_n, \ldots)$ with $\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} < \infty$ is a Banach space with respect to the norm $\|x\|$ given by $\|x\|_p = \sum_{k=1}^{\infty} |x_k|^p ; x = (x_1, x_2 \ldots) \in l_p$.

This Banach space l_p has a Schauder basis consisting of $\{e_n\}$, where

$$\| x \| = \begin{pmatrix} - & 1 & 0 \\ 0 & n^{th} place & - \end{pmatrix}, n = 1, 2, \ldots$$

Theorem 3.2.1. (Riesz representation Theorem). The conjugate (Dual) space of l_p is isomorphic to the sequence space l_q where $1 < p,\ q < \infty$ and $p^{-1} + q^{-1} = 1$.

Proof. Let $f \in l_p^*$. Since f is Linear and bounded if $x \in l_p$ with $\underset{\sim}{x} = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ we have

$$f(\underset{\sim}{x}) = f\left(\sum_{k=1}^{\infty} \xi_k e_k\right) = \sum_{k=1}^{\infty} \xi_k f(e_k) = \sum_{k=1}^{\infty} \xi_k \gamma_k \quad \text{where } \gamma_k = f(e_k)$$

Let q satisfy $p_1^{-1} + q^{-1} = 1$; consider $\underset{\sim}{x}_n = \{\xi_k^{(n)}\},\ n = 1, 2, \ldots$ where

$$\xi_k^{(n)} = \begin{Bmatrix} |\gamma_k|^q / \gamma_k & if \quad k \le n \quad and \quad \gamma_k \ne 0 \\ 0 & if \quad k > n \quad or \quad \gamma_k = 0 \end{Bmatrix}$$

So, $f(\underset{\sim}{x}_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q$ $\qquad \ldots \ldots \ldots (1)$

or, $|f(\underset{\sim}{x}_n)| \le \| f \| \| \| = \| f \| \left(\sum_{k=1}^{n} |\xi_k^{(n)}|^p \right)^{1/p}$

$$= \| f \| \left(\sum_{k=1}^{n} |\gamma_k|^{(q-1)p} \right)^{1/p} = \| f \| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p}$$

Therefore, from (1) $\sum_{k=1}^{n} |\gamma_k|^q = f(\underset{\sim}{x}_n) \le \| f \| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p}$

Dividing both sides by $\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p}$ we have

$$\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1 - 1/p} \le \| f \|$$

i.e. $\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/q} \le \| f \|$; r.h.s. being independent of n, we proceed as $n \to \infty$ to obtain

$$\left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{1/q} \le \| f \|$$

This shows that $\{\gamma_k\} \in l_q$.

Again $\{\beta_k\} \in l_q$ gives rise to a member $g \in l_q^*$ | in following ways :—

Define $g : l_p \to$ Reals, where

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \text{ where } \underset{\sim}{x} = (\xi_1, \xi_2, \ldots) \in l_p.$$

The infinite series on r.h.s. is convergent by Hölders inequality because

$$\sum_{k=1}^{\infty} |\xi_k \beta_k| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |\beta_k|^q\right)^{1/q}, \text{ and therefore } \{\xi_k\} \in l_p.$$

Therefore, g is linear, and its boundedness follows from Hölders inequality. So, $g \in l_p^*$.

Therefore, there is an 1 – 1 correspondence between the elements of l_p^* and those of l_q and this correspondence between them is also onto.

Finally, we verify that the correspondence preserves the norm. To that end if $f \in l_p^*$ we have

$$|f(\underset{\sim}{x})| = \left|\sum_{k=1}^{\infty} \xi_k \gamma_k\right| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} \text{ (by H örders' inequality)}$$

$$= \|(\underset{\sim}{x})\| \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q}$$

Hence $\|f\| = \|(\underset{\sim}{x})\|^{\text{Sup}} = 1 |f(\underset{\sim}{x})| \leq \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q}$

Also as we have seen above $\|f\| = \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q} = \|\underset{\sim}{c}\|$ (say) ... ... (2)

where $\underset{\sim}{c} = (\gamma_1, \gamma_2, \ldots \gamma_n, \ldots) \in l_q$, where $\gamma_k = f(e_k)\ (k = 1, 2, \ldots, n, \ldots)$

Now the mapping : $l_p^* \to l_q$ defined by correspondence $f \in l_p^* \to c \in l_q$ is Linear and bijective, and (2) says that it is norm-preserving. Therefore it is an isomorphism.

Corollary. The Banach space $l_p (1 < p < \infty)$ has its second Dual (conjugate) space l_p^{**} isomorphic to it self.

Because $l_p^* \simeq l_q$ and by same reasoning (Theorem 3.1.2)

$$l_q^* \simeq l_q (p^{-1} + q^{-1} = 1), \text{ and } \simeq \text{ denoting isomorphism.}$$

Proceeding same way as in Theorem 3.1.2 one can prove following theorem, the proof of which is left over.

Theorem 3.1.3. The Dual space l_1^* of the sequence space l_1 is isomorphic to l_∞(l_∞ is the space of all bounded sequence of with sup norm).

Theorem 3.1.4. Let $C[0, 1]$ be the Banach space of all real valued continuous functions over the closed unit interval $[0,1]$ with sup norm and $f \in C^*[0,1]$, then f can be represented as a stieltjes integral $f(x) = \int_0^1 x(t)dg(t); x \varepsilon C[0,1]$ where g is a function of bounded variation over $[0,1]$ such that

$$\| f \| = \overset{1}{\underset{0}{\vee}} g .$$

Proof : The space $C[0,1]$ is a sub-space of the space $M[0, 1]$ consisting of all real-valued bounded functions over $[0, 1]$.

Take f as a bounded Linear functional over $C[0, 1]$, and then by Hahn-Banach Theorem f can be extended to F as a bounded Linear functional over $M[0,1]$ with $\| F \| = \| f \|$.

Let $0 \le t \le 1$; consider the function

$$u_t(\xi) = \begin{cases} 1 & for \quad 0 \le \xi \le t \\ 0 & for \quad t \le \xi \le 1 \end{cases}$$

Clearly u_t's are bounded functions over $[0, 1]$ i.e., $u_t \in M[0,1]$.

Put $g(t) = F(u_t(\xi)$, in $0 \le t \le 1$

We now verify that g is a function of bounded variation in $[0, 1]$. Let us take a partition

$$t_0 = 0 < t_1 < t_2 < \ldots < t_n - 1 < t_n = 1 \text{ of } [0, 1].$$

Put $\varepsilon_i = Sgn[g(t_i) - g(t_i - 1)]$.

Then the sum $\sum_{i=1}^{n} | g(t_i) - g(t_i - 1) |$

$$= \sum_{i=1}^{n} \varepsilon_i [g(t_i) - g(t_i - 1)]$$

$$= \sum_{i=1}^{n} \varepsilon_i [F(u_{ti}) - F(u_{ti} - 1)]$$

Therefore, $\sum_{i=1}^{n} | g(t_i) - g(t_i - 1) | \le \| F \| \| \sum_{i=1}^{n} \varepsilon_i (u_{ti} - u_{t\,i-1}) \|$

$$\le \| F \|$$

because $\| F \| = \| f \|$ and $\| \sum_{i=1}^{n} \varepsilon_i (u_{ti} - u_{t\,i-1}) \| - 1$.

Thus $\sum_{i=1}^{n} | g(t_i) - g(t_i - 1) | \le \| f \|$ ... ... (*)

r.h.s. being independent of choice of partition $[t_0, t_1, \ldots, t_n]$ of $[0, 1]$
We find g is a function of bounded variation in $[0, 1]$.
Now take any member $x \in C[0, 1]$, and construct Z_n, as follows :—

$$z_n(t) = \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[u_{\frac{k}{n}}(t) - u_{\frac{k-1}{n}}(t)\right]; 0 \le t \le 1.$$

i.e., each Z_n is a step function in $[0, 1]$, We have

$$F(Z_n) = F\left(\sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[u_{\frac{k}{n}}(t) - u_{\frac{k-1}{n}}(t)\right]\right)$$

$$= \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[F\left(u_{\frac{k}{n}}(t)\right) - F\left(u_{\frac{k-1}{n}}(t)\right)\right]$$

$$= \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right]$$

Therefore, $\lim_{n\to\infty} F(Z_n) = \lim_{n\to\infty} \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right]$

Now Summation $\sum_{k=1}^{n}$ represents Stielijes sum arising out of a partition of $[0,1]$ and since x is continuous and g is a function of bounded variation we know that

$$\lim_{n\to\infty} \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right] \text{ is actually } \int_0^1 x \, dg.$$

Therefore $\lim_{n\to\infty} F(Z_n) = \int_0^1 x \, dg \left(= \int_0^1 x(t) dg(t) \right)$.

On the other hand looking on L.H.S. We find as $n \to \infty$, $\{Z_n(t)\}$ converges uniformly to $x(t)$ in $[0, 1]$; So,

$\| Z_n - x \| \to 0$ as $n \to \infty$. By continuity of F, we conclude that $\lim_{n\to\infty} F(Z_n) = F(x)$. Hence we obtain

$$F(x) = \int_0^1 x(t) \, dg(t)$$

Since F is an extension of f form $C[0, 1]$ to $M[0, 1]$, we have for $x \in C[0,1]$ $F \equiv f$ and form above we write

$f(x) = \int_0^1 x(t)\, dg(t)$, and this is the representative formula for f as a Stieltjes Integral.

Finally to determine $\| f \|$, We have from (*) above

$\sum_{k=1}^{n} | g(t_i) - g(t_i - 1) | \le \| f \|;|$ Since r.h.s. does not change by changing partition of [0,1] we conclude that

Sup $\sum | g(t_i) - g(t_{i-1}) |$ Partitions over [0, 1] $\le \| f \|$.

$$\text{or, } \bigvee_0^1 g \le \| f \| \qquad \text{... ... ... (1)}$$

Again from $f(x) = \int_0^1 x(t)\, dg(t)$. We find

$$| f(x) | = | \int_0^1 x(t)\, dg(t) | \le \sup_{0 \le t \le 1} | x(t) | \bigvee_0^1 g$$

$$= \| x \| \bigvee_0^1 g$$

$$\text{This gives } \| f \| \le \bigvee_0^1 g \qquad \text{... .... (2)}$$

(1) and (2) together give $\| f \| = \bigvee_0^1 g,|$ as wanted. The proof is now complete.

§ 3.2 One of the corner stones of functional analysis in Normed Linear spaces is very often marked by Banach-Steinhaus Theorem or Uniform boundedness Principle theorem, others being the Hahn-Banach Theorem, open mapping Theorem, and closed graph Theorem.

Theorem 3.2.1. (Banach-Steinhaus Theorem). Let X be a Banach space and Y a NLS over the same scalarfield and $T_n \in Bd\mathcal{L}(X, Y)$ (T_n be a bounded Linear operator : $X \to Y$), $n = 1, 2,...$ such that

$\{\| T_n(x) \|$ is bounded for each $x \in X$,

Then $\{\| T_n \|\}$ is bounded.

Proof : For any +ve integer k let

$$A_k = \{x \in X : \| T_n(x) \| \le k \quad \text{for all } n\}.$$

We verify that A_k is closed, Because, for any $x \varepsilon \overline{A_k}$. We find a sequence $\{x_j\}$ in A_k such that $\lim_{j \to \infty} \| x_j - x \| = 0$

For a fixed n, we have $\| T_n(x_j) \| \leq k$, by continuity of T_n, We have $\|T_n(x)\| \leq K$. This is true for all n; Hence $x \in A_k$. And $A_k k$ is closed.

Also $X = \bigcup_{k=1}^{\infty} A_k$; Since X is Banach space, by completeness in X, Baire Category Theorem applies here. So as an application we find some $A_k = A_{k_0}$ contains an open ball, say $B_r(x_0) \subset A_{k_0}$.

Take $x \in X$ as arbitrary non-zero member, and put

$$z = x_0 + \gamma x \text{ where } \gamma = \frac{r}{2\|x\|} \qquad \text{... ... ... (2)}$$

Then $\| z - x_0 \| < r$ i.e. $z \in B_r(x_0)$,| and from (1) we have

$$\| T_n(z) \| \leq k_0 \text{ for all } n.$$

Further, $\| T_n(x_0) \| \leq k_0$ since x is centre of $B_r(x_0)$. So from (2) get,

$$x = \frac{1}{\gamma}(z - x_0).$$

This yields for all n, $\| T_n(x_0) \| = \frac{1}{\gamma} \| T_n(z - x_0) \|$

$$\leq \frac{1}{r}(\| T_n(z) \| + \| T_n(x_0) \|) \leq \frac{4}{r} \| x \| k_0.$$

Hence for all n, $\| T_n \| = \sup_{\|x\|=1} \| T_n(x) \| \leq \frac{4}{r} k_0$.

That means $\{T_n\}$ is norm-bounded; that is what was wanted. The proof is complete.

An application : NLS $\wp$ of all real polynomials with $\| p \| = \max_j \| a_j \|$ where $p \equiv a_0 + a_1 t + \ldots + a_n t^n$ is not a Banach space.

The conclusion is derived by applying uniform Boundedness principle of Theorem 3.2.1. So we make appropirate seting of Theorem 3.2.1.

We may put $p \in \wp$ in a form :

$$p = a_0 + a_1 t + ... + a_n t^n (a_n \neq 0)$$

$$= a_0 + a_1 t + ... + a_n t^n + a_{n+1} t^{n+1} + ... \text{ with } a_{n+1} = a_{n+2} = ... = 0$$

$$= \sum_{k=1}^{\infty} a_j t^j$$

(It does not matter if p is the zero polynomial). Now define a sequence of functionals $\{f_n\}$ in the following way :—

f_n(zero poly) = 0

and $f_n(p \neq 0) = a_0 + a_1 + ... + a_{n-1}$

Then it is routine exercise to check that each f_n| is linear; each f_n is bounded because $|a_j| \leq \| p \|$.

So that $| f_n(x) | \leq n \| x \|$

Further more, for fixed $p \in \wp \{f_n(p)\}$

satisfies $| f_n(p) | \leq (N_x + 1) \max_j | a_j |$ where N_x = degree of p.

That means at each $p \in \wp \{f_n\}$ is bounded; However we will show right now that $\{f_n|\}$ is not norm (operator-norm) bounded.

Let $p(t) = 1 + t + t^2 + ... + t^n$; So, $\| p \| = 1$.

And $f_n(p) = \underbrace{1 + 1 + ... + 1}_{n \text{ terms}} = n = \| p \|$.

Now $\| f_n \| \geq \left| \frac{f_n(p)}{| p |} \right| = n$, where n is free to take indefinite large values.

Clearly here $\sup_n \| f_n \| = +\infty$, and hence.

Uniform boundedness Principle (Theorem 3.2.1).| is failing; because $\wp$ is not complete.

So, NLS $\wp$ is not a Banach space.

Example 3.2.1 : Let X be a Banach space and Y and NLS over the same scalars, and Let $T_n \in Bd\mathcal{L}(X,Y)$ such that for each $x \in X$, $\{T_n(x)\}$ is Cauchy in Y. Show that

$$\sup_n \| T_n \| < \infty.$$

Solution : Here for each $x \in X$, image sequence $\{T_n(x)\}$ is Cauchy in Y, and therefore it is bounded there.

or, $\sup_n \| T_n(x) \| < \infty$

Since X is a Banach space, and we apply Uniform boundedness principle Theorem 3.2.1. to conclude that $\{T_n\}$ is operator-norm bounded and we get

$$\sup_n \| T_n \| < \infty$$

§ 3.3 Weak convergence in a NLS *X*.

Let X be a NLS, and X^* denote its dual / Conjugate space of all bounded linear functionals over X.

Definition 3.3.1 A sequence $\{x_n\}$ in X is said to converge weakly to $x \varepsilon X$ if $\lim_{j\to\infty} f(x_n) = f(x)$ for every member $x \varepsilon X^*$.

If $\{x_n\}$ converges weakly to x, we write $\{x_n\} \overset{wk}{\to} x$ or weak $\lim_{n\to\infty} x_n = x$ or simply $\omega - \lim_{n\to\infty} x_n = x$.

Theorem 3.3.1 If $\{x_n\}$ is weakly convergent in X, then $\omega - \lim_{n\to\infty} x_n$ is unique.

Proof : Let $\omega - \lim_{n\to\infty} x_n = u$ and $\omega - \lim_{n\to\infty} x_n = \vartheta$ in X. Then for every bounded Linear functional f over X i.e., $f \in X^*$, We have $\lim_{n\to\infty} f(x_n) = f(u)$ and $\lim_{n\to\infty} f(x_n) = f(\vartheta)$.

That means $f(u) = f(v)$ or $f(u) - f(\vartheta) = 0$ or, $f(u - \vartheta) = 0$

This is true for every member $f \varepsilon X^*$ and hence $(u - \vartheta) = 0$ or, $u = \vartheta$.

Theorem 3.3.2. If $\{x_n\}$ converges to x in norm in X, then $\{x_n\}$ converges weakly to x, but not conversely.

Proof : Let $\{x_n\}$ converge to x in respect of norm in X; We also say $\{x_n\}$ converges strongly to x. That means

$$\lim_{n\to\infty} \| x_n - x \| = 0 .$$

Take $f \varepsilon X^*$ arbitrary; then $| f(x_n) - f(x) | = f(x_n - x) | \le \| f \| \| x_n - x \| \to 0$ as $n \to \infty$. Therefore

$$\omega - \lim_{n\to\infty} x_n = x .$$

Converse is not true. Consider a basis (Schauder basis) $e_k = \begin{pmatrix} - & 1 & 0 \\ 0 & kth\ place & - \end{pmatrix}$, $k = 1,2,...$| in the sequence space l_2| which is a Banach-space. First we verify that

$$\omega - \lim_{n \to \infty} e_n = \underset{\sim}{O} \text{ in } l_2.$$

By representation Theorem we know that every member $f \in l_2^*$ | may be represented as $f(\underset{\sim}{x}) = \sum_{k=1}^{\infty} \xi_k \gamma_k$, where $\underset{\sim}{x} = (\xi_1, \xi_2, ...) \in l_2$; where we also recall $\gamma_k = f(e_k), k = 1,2,...$

Since $\lim_{n \to \infty} \gamma_n = 0$, we find

$$\lim_{n \to \infty} f(e_n) = 0$$

That means $\lim_{n \to \infty} e_n = \underset{\sim}{O}$ in l_2, because f is any member of l_2^*.

Now if $n \neq m$, we at once see that $\| e_n - e_m \|^2 = 2$,| and sequence $\{e_n\}$ does not converge to any element in l_2 in l_2 norm. The proof is now complete.

However situation looks simple in some cases.

Theorem 3.3.3. If X is a finite dimensional NLS, then notion of norm convergence and weak convergence of elements, in X are coincident.

Proof : Actually, we need showing that weak convergence implies norm convergence (strong convergence) if Dim $(X) < \infty$.

Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \to \infty} x_n = x_0 \varepsilon X.$$

Suppose Dim $(X) = k$ and $(e_1, e_2, ..., e_k)$ form a basis in X; Therefore, we write

$$x_n = \xi_1^{(n)} e_1 + \xi_2^{(n)} e_2 + ... + \xi_k^{(n)} e_k, n = 1,2,...$$

and $\quad x_0 = \xi_1^{(0)} e_1 + \xi_2^{(0)} e_2 + ... + \xi_k^{(0)} e_k,$

Define functionals $f_1, f_2, ..., f_k$ over X in the following way :

$f_j(x) = \xi_j$, when $x = \xi_1 e_1 + \xi_2 e_2 + ... + \xi_k e_k \in X \ (j = 1,2,...,k)$.

Also since Dim $(X) < \infty$, each of these linear functionals is continuous. Therefore each $f_j \in X^*$.

Now, $f_j(x_n) = \xi_j^{(n)}$ and $f_j(x_0) = \xi_j^{(n)}$.

Since $\omega - \lim\limits_{n\to\infty} x_n = x_0$, $\omega - \lim\limits_{n\to\infty} f_j(x_n) = f_j(x_0)$

i.e., $\omega - \lim\limits_{n\to\infty} \xi_j^{((n)} = \xi_j^{(0)}, j = 1, 2, \ldots, k$

Put $M = \max\limits_{i \le j \le k} \| e_j \|$; given $\varepsilon > 0$, there is a +ve integer n_0 such that

$| \xi_{j(n)} - \xi_{j(0)} | < \dfrac{\varepsilon}{MK}$ for $n \ge n_0$ and j = 1, 2, ..., k.

Then for $n \ge n_0$, we have

$$\| x_n - x_0 \| = \| \sum_{j=1}^{k} (\xi_{j(n)} - \xi_{j(n)}) \| \le \sum_{j=1}^{k} | \xi_j^{(n)} - \xi_j^{(0)} | .M$$

$$< M . \frac{\varepsilon}{MK} . k = \varepsilon .$$

Consequently, $\{xn\}$ converges in norm to x_0.

i.e., $\lim\limits_{n\to\infty} x_n = x_0$ in norm in X.

The proof is now complete.

Theorem 3.3.4. Let X and Y be two NLS with same scalars. Let $T \in Bd\mathcal{L}(X, Y)$ if $\{x_n\}$ converges weakly to x in X then $\{T(x_n)\}$ converges weakly to $T(x_0)$ in Y.

Proof : Take $\varphi \in Y^*$; Consider the composition $\varphi_0 T : X \to$ I Scalars Then it is an easy exercise to see that $\varphi_0 T$ is a linear functional over X. As $W - \lim\limits_{n\to\infty} x_n = x_0$ in XI; So $\lim\limits_{n\to\infty} (\varphi_0 T)(x_n) = (\varphi_0 T)(x_0)$

or, $\lim\limits_{n\to\infty} \varphi_0(T(x_n)) = \varphi(T(x_0))$ As $\varphi \in Y^*$I is an arbitrary member it follow that $\{T(x_n)\}$ converges weakly to $T(x_0)$ in Y.

Theorem 3.3.5. If $\{x_n\}$ converges weakly in X, then it is norm bounded in X.

Proof : Take $f \in X^*$, So $\{f(x_n)\}$ is a convergent sequence of scalars. Hence it is bounded, Thus I $f(x_n) \le M_f$ where $M_f(> 0)$ a fixed scalar. Via canonical mapping let $x_n^{**} \in X^{**}$I such that

$$x_n^{**}(f \in X^*) = f(x_n) \; n = 1, 2, \ldots$$

So inequality above gives

$$\| x_n^{**}(f) \| \leq M_f$$

Thus $\{x_n^{**}\}$ is a sequence of bounded linear functionals over X^* (which is a Banach space) and this becomes sequence bounded at $f \in X^*$; By Uniform boundadeness Principle Theorem (or Banach–Steinhauss Theorem) We conclude that $\{x_n^{**}\}$ becomes norm bounded i.e.

$$\sup_n \| x_n^{**} \| < \infty .$$

But we know that $\| x_n^{**} \| = \| x_n \|$. I Therefore from above we have

$$\sup_n \| x_n^{**} \| < \infty, \text{ and Proof is complete.}$$

Example 3.3.1. If $w - \lim\limits_{x \to \infty} x_n = x_0$ in x Show that $\lim\limits_{n \to \infty} \| x_n \| \geq \| x_0 \|$ (X is taken a real NLS).

Solution : Given $w - \lim\limits_{n \to \infty} x_n = x_0$ in X. Let $f \in X^*$, I we have $\lim\limits_{n \to \infty} f(x_n) = (x_0)$. By Hahn-Banach Theorem there is a member $f \in X^*$ I such that $\| x_0 \| = f_0(x_0)$ and $\| f_0 \| = 1$. So taking f_0 in place of f in limit above we obtain,

$$f_0(x_0) = \lim_{n \to \infty} f_0(x_n)$$

or, $\quad \| x_0 \| = \lim\limits_{n \to \infty} f_0(x_n)$

Now $f_0(x_n) \leq | f_0(x_n) | \leq \| f_0 \| \, x_n \| = \| x_n \|$

$$\text{So} = \lim_{n \to \infty} f_0(x_n) \leq \frac{\lim}{n \to \infty} \| x_n \| .$$

From (1) we have $\lim\limits_{n \to \infty} \| x_n \| \geq \dfrac{\lim}{n \to \infty} f_0(x_n) = \| x_n \|$.

Example 3.3.2. Let (x_n) converges weakly to x_0 in NLS X and Y = closed sub-space of X spanned by $(x_1, x_2, x_3, \ldots, x_n \ldots)$ Show that $x_0 \in Y$.

Solution : Let $\omega - \lim\limits_{n \to \infty} x_n = x_0$ and Y = closure of Lin. hull $[x_1, x_2, \ldots, x_n, \ldots]$| which is the closed sub-space generated by $(x_1, x_2, \ldots, x_n, \ldots)$ in X. Suppose $x_0 \notin Y$. Then dist (x_0, Y) is +ve, say = $\delta (> 0)$ We apply Hahn-Banach Theorem to obtain a bounded Linear functional f over X i.e., $f \in X^*$ such that

$f(x_0) = \delta$

and $f(y) = 0$ for all $y \in Y$.

Clearly, $f(x_n) = 0$ for all n; Hence $\{f(x_n)\}$I does not converge to $f(x_0) = \delta > 0$, a contradiction that $w - \lim\limits_{n \to \infty} x_n = x_0$. Hence conclusion follows.

§ 3.4. Approximation in NLS

Theory of approximation is a very useful area of study in Mathematics. Numerical Analysis is an extensive area to take care of some approximation Theory.

Let X be a NLS (Normed Linear Space) and let Y be a fixed sub-space of X. Take $x \in X$, and put $\delta = (x, Y) = \operatorname{Inf}_{y \in Y} \| x - y \|$.

Definition 3.4.1. If there is a member $y_0 \in Y$ such that $\| x - y_0 \| = \delta =$ I dist (x, Y).I then y_0 is said to be a best approximation of x out of Y.

Explanation : A best approximation y_0 out Y is thus an element whose distance from x is minimum. Such an element y_0 may or may not be there. Even if it is there problem of uniqueness is of interest too.

Theorem 3.4.1. Let Y be a sub-space of a NLS X with Dim $(Y) < \infty$. Then for each $x \in X$ there is a best approximation to x out of Y.

Proof : Let $x \in X$, and $\tilde{B} = \{y \in Y : \| y \| \leq 2 \| x \|\}$ denote the closed ball in Y. Clearly, $\underline{O} \in \tilde{B}$, and we see that dis $(x, \tilde{B}) = \operatorname{Inf}_{\tilde{y} \in \tilde{B}} \| x - \tilde{y} \| \leq \| x - \underline{O} \| = \| x \|$.

Now if $y \notin \tilde{B}$, then $\|y\| > 2\|x\|$, and $\|x - y\| \geq \|y\| - \|x\| > \|x\| \geq$ dist $(x, \tilde{B})$. ..... (1)

Again if $y \notin \tilde{B}$, then $\| x - y \| \geq$ dist $(x, \tilde{B})$. ..... (2)

Combining (1) and (2) we duduce

$$\text{dist } (x, Y) \geq \text{dist } (x, \tilde{B}) \quad ..... (3)$$

On the other hand $\tilde{B} \subset Y$, therefore

$$\text{dist } (x, Y) \geq \text{dist } (x, \tilde{B})$$

combining (3) and (4) we have

$$\text{dist } (x, \tilde{B}) \text{ dist } (x, Y).$$

Now dist (x, Y) is not equal to $\| x - Y \|$ for any position $y \in (Y \setminus \tilde{B})$l because from (1) we know that in this case $\| x - y \| >$ dist $(x, \tilde{B})$ = dist (x, Y).

So if a best approximation to x exist in $\tilde{B}$, then we have finished. Now $\tilde{B}$| is a bounded closed subset of a finite dimensional NLS, and therefore it is compact, and therefore by continuity of norm function there is a point $y_0 \in \tilde{B}$ such that

$$\| x - y_0 \| = \text{dist}(x, Y).$$

So y_0 is a base approximation to x out of Y.

Example 3.4.1. Let $p_0(t) = 1, = p_t(t) = t, \ldots, p_n(t) = t^n$ in $a \leq t \leq b$ and n be kept fixed. If Y = lin. hull $[p_0, p_1, \ldots, p_n]$ is the sub-space of $C[a, b]$ generated by $[p_0, p_1, \ldots, p_n]$, then Y is a finite Dimensional NLS as a sub-space if $C[a, b]$.

Now given a continuous function x over $[a, b]$ i.e. $x \in C[a, b]$, we apply Theorem 3.4.1. to find a member of Y sat = p_n that $\| p - p_n \|$ = dist (x, Y)

$$\text{or,} = \underset{a \leq t \leq b}{Max} | p(t) - p_n(t) | = \underset{p \in Y}{Min} \| x - p \|$$

$$| \qquad = \underset{p \in Y}{Min} \underset{a \leq t \leq b}{Max} | x(t) - p(t) |$$

We now examine necessity of assumption that sub-space Yl in Theorem 3.4.1 is such that Dim $(Y) < \infty$. Example 3.4.2 serves our purpose.

Example 3.4.2. Cosiden NLS Y consisting of all real polynomial over $\left[0, \frac{1}{2}\right]$ (of any degree). So we have

$$\text{Dim}(Y) = +\infty.$$

Consider a continuous function $x(t) = \dfrac{1}{1-t}$ in $\left[0, \frac{1}{2}\right]$.l Then we have $x(t) = 1 + t + t^2 + \ldots\ldots\ldots$ in $\left[0, \frac{1}{2}\right]$. Let $p_n(t) = 1 + t + \ldots + t^n$; Then we know that

$$\lim_{n \to \infty} p_n = x \text{ in sup norm.}$$

Given $a + ve\ \varepsilon$, we obtain $p(t) = 1 + t + \ldots + t^n$ such that

$$\| x - p_n \| < \varepsilon \text{ for all } n \geq N.$$

i.e., dist $(x, Y) = 0$.

Here x is not a polyminal function and $x \notin Y$, and we see that there in no $y_0 \in Y$ such that $\| x - y_0 \| = 0$.

Theorem 3.4.2. In a NLS X, the set M of best approximations to a given $x \in X$ out of a sub-space Y is a convex set.

Proof : Conclusion is obvious if either $M = \phi$ or $M = a$ singleton. So let M consists of more then one element, and take $u, v \varepsilon M$. Then we have $\| x - u \| x - v\| = \text{dist}(x, Y)$. If, $0 \leq \lambda \leq 1$, and put $\omega = \lambda u + (1 - \lambda)v$. Then $\omega \varepsilon Y$ (because $u, v \varepsilon Y$)

Then $\| x - \omega \|$ dist x, Y. .....(1)

Again $\| x - \omega \| = \| x - (\lambda u + 1 - \lambda v) \| = \|\lambda (x - u) + (1 - \lambda) (x - v) \|$

$\geq \lambda \| x - u \| + (1 - \lambda)\| x - v\| = \|(\lambda + (1 - \lambda) \text{ dist } (x, Y)$

$= \text{dist } (x, Y)$ ..... (2)

From (1) and (2) we have

$\| x - v \| = \text{dist } (x, Y)$ Therefore $\omega \in M$. Hence M is convex.

To examine uniqueness of best approximation we recall Definition of strict convex norm.

Definition 3.4.3. Norm in a NLS X is said to be strict convex if $\| x \| = \| y \| = 1$ imply $\| x + y \| < 2$ $(x \neq y)$.

A NLS with a strict convex norm is called a strictly convex NLS.

Explanation : $\| x \| = \| y \| = 1$ give by triangle inequality, $\| x + y \| \leq \| x \| + \| y \| = 2$, and strictly convexity of the norm demands that $\| x + y \| < 2$ | (equality sign is excluded).

Example 3.4.3. Every Hilbert space H is strictly convex.

Solution : Let $x, y \varepsilon H$; $x \neq y$ and $\| x \| = 1 \| y \|$

Put $\| x - y \| = \alpha$; then $\alpha > 0$. By law of parallelogram, we have

$$\| x + y \|^2 + \| x - y \|^2 = 2 (\| x \|^2 + \| y \|^2)$$

or, $$\| x + y \|^2 = 2(\|x\|^2 + \| y \|^2) - \| x - y \|^2$$

$$= 4 - \alpha^2 < 4; \text{ This gives } \| x + y \| < 2.$$

Theorem 3.4.3. Let X be a NLS with strict convex norm and Y be a sub-space of X. If $x \in$ E then there is atmost one best approximation to x out of Y.

Proof : Suppose for $x \in X$, there is a best approximation to x out Y and let u, v be two distinct best approximations. Then we haven $u, v \in Y$ with $\| x - u \| = \| x - v \| = |$ dist (x, Y).

Putting δ = dist $(x, Y) > 0$; we have $\left\| \frac{x-u}{\delta} \right\| = \left\| \frac{x-v}{\delta} \right\| = 1$ and by strict convex norm of X, we have

$$\left\| \frac{x-u}{\delta} + \frac{x-v}{\delta} \right\| < 2$$

or, $$\left\| x - \frac{u+v}{2} \right\| < \delta = \text{dist } (x, Y) \qquad \text{..... (1)}$$

Now $\frac{u+v}{2} \in Y$ (Y is sub-space), so $\left\| x - \frac{u+v}{2} \right\|$ dist (x, Y) ..... (2)

So (1) and (2) are contradictory, and therefore we have proved Theorem 3.4.3.

Theorem 3.4.4. Let C be a non-empty closed convex subset of a Hilbert space H and let $x \in H$; if

$$\delta = \inf_{y \in C} \| x - y \|,$$

Then there is a unique member y_0 in C satisfying $\| x - y_0 \| = \delta$.

Proof : Let $\{y_n\}$ be a sequence in C such that

$$\lim_{n \to \infty} \| x - y_n \| = \delta .$$

By Law of parallelogram we see

$$\| y_n - y_m \|^2 = 2\| y_n - x \|^2 + 2\| y_m - x \|^2 - \| 2x - y_n - y_m \|^2$$

$$= 2\| y_n - x \|^2 + 2\| y_m - x \|^2 - 4 \left\| x - \frac{y_n + y_m}{2} \right\|^2$$

By convexity of C we have $x - \frac{y_n + y_m}{2} \in C$, and therefore,

$$= \left\| x - \frac{y_n + y_m}{2} \right\|^2 \geq \delta^2.$$

Above gives $\| y_n - y_n \|^2 \leq 2 \| y_n - x \|^2 + 2\| y_m - x \| - 4\delta^2$.

r.h.s tends to O as $n, m \to \infty$, because $\lim_{n\to\infty} \| x - y_n \|^2 = \delta^2$ etc.

Therefore $\{y_n\}$ is Cauchy; Hence there is $y_0 \in H$ such that $\lim_{n\to\infty} y_n = y_0$. Since $y_0 \in C$ and C is closed we have $y_0 \in C$.

Now $\| x - y_0 \| \le \| x - y_n \| + \| y_n - y_0 \|$, Letting $n \to \infty$, we have $\| x - y_0 \| \le \delta$; Also $\| x - y_0 \| \ge \delta$; these together mean $\| x - y_0 \| = \delta$.

For uniqueness of y_0, let $y_1 \in C$ such that $\| x - y_1 \|\ \delta$; Since C is convex, $\frac{1}{2}(y_0 + y_1) \in C$. Therefore.

$$\delta \le \| x - \frac{1}{2}(y_0 + y_1) \| = \| \frac{1}{2}x - \frac{1}{2}y_0 + \frac{1}{2}x - \frac{1}{2}y_1 \|$$

$$\le \frac{1}{2} \| x - y_0 \| + \frac{1}{2} \| x - y_1 \| = \frac{1}{2}.\delta + \frac{1}{2}\delta = \delta.$$

Therefore $\| x - \frac{1}{2}(y_0 + y_1) \| = \delta$.

By Law of parallelogram we have

$$\|y_0 - y_1\|^2 = 2\| x - y_0 \|^2 + 2\| x - y_1 \| - 4\| x - \frac{1}{2}(y_0 + y_1) \|^2$$

$$= 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

So, $y_1 = y_0$. The proof is complete.

EXERCISE-A

Short Answer type questions

1. In a NLS x if $\omega - \lim_{n\to\infty} x_n = x$, and $\omega - \lim_{n\to\infty} y_n = y$ and α any scalar, show that (i) $\omega - \lim_{n\to\infty} (x_n + y_n) = x + y$ and (ii) $\omega - \lim_{n\to\infty} (\alpha x_n) = \alpha x$.
2. Show that the space $\wp_3$ of all real polynomials of degree not exceeding two is isomorphic to Eudidean 3-space R^3.
3. In a NLS X if $f(x) = f(y)$ for every member $f \varepsilon X^*$, Show that $x = y$.
4. Using representation Theorem for a bounded linear functional show that every Hilbert space is self-dual.

EXERCISE-B

1. In a Hilbert space H show that $\omega - \lim_{n\to\infty} x_n = x$ if and only of $\lim_{n\to\infty} \langle x_n, u\rangle = \langle x, u\rangle$ for all $u \in H$.
2. Show that Banach space $C[a, b]$ with sup norm is not strictly convex.
3. In Euclidean 2-space R^2 with norm $\|(x, y)\| = |x| + |y|$ for $(x, y) \in R^2$, Obtain all the best approximation to $(1, -1)$ out of the sub-space $Y \in R^2$ where $Y = \{(x, y) \in R^2 : x = y\}$.
4. If $\{x_n\}$ in a Banach space X is such that $\{f(x_n)\}|$ is bounded for all $f \in X^*$, I show that $\sup_n \| x_n \| < \infty$.
5. In a NLS X which is strictly convex, show that $\| x + y \| = \| x \| + \| y \|$ $(x \neq \underset{\sim}{O}\ y \neq \underset{\sim}{O})$ implies $x = \mu y$ for some +ve scalar μ.
6. If a NLS has a Schauder basis, show that it is separable (converse is false : In 1975 Enflo had constructed a Banach space that is separable without a Schauder basis).

Unit - 4

(**Concepts :** Resolvant set $\rho(T)$ and spectrum $\sigma(T)$| of a bounded Linear operator T, expansion of $(I - T)^{-1}$, properties of $\sigma(T)$: compact linear operators, their algebra, properties of compact operators; spectral properties of bounded self-adjoint operators, formula $\| T \| = \sup |< T(x), x >|$, projection operators, their sum and product; orthogonal projection operators, their properties.)

§ 4.1 Over a finite Dimensional NLS spectral Theory is essentially matrix eigen-value theory. The importance of spectral theory over a NLS rests in its application like solving equations (system of Linear algebraic equations, Differential and integral equations).

Let X be a Normed Linear space (NLS) over real/complex field, and assume X to be nonnull.

Suppose T is a bounded Linear operator : $X \to X$.| For a complex scalar λ| consider another operator.

$T_\lambda : X \to X$ defined by

$T_\lambda (x) = T(x) - \lambda x$ for all $x \in X$

or, $T_\lambda = T - \lambda I$ where I is Identity operator of X.

Definition 4.1.1. (a) λ is called a regular value of T if T_λ has an inverse or equivalently operator $T - \lambda I$ is invertible.

(b) The collection $\rho(T)$ of all regular values of T is called resolvant set of T.

(c) The complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$| is called the spectrum of T, every $\lambda \in \sigma(T)$ is called a spectral value of T.

Explanation : We have $\mathbb{C} = \sigma(T) \cup \rho(T)$ Suppose λ **is an eign value of** T; So there is a **non-zero Vector** $x \in T$ such that $T(x) = \lambda x$

or, $T(x) - \lambda x = \underline{O}$

or, $(T - \lambda I)(x) = \underline{O}$

or, $T_\lambda(x) = \underline{O}$

That means Null-space of $T_\lambda \neq \{\underline{O}\}$ and hence linear operator T_λ has got no inverse i.e. T_λ^{-1} does not exist, Therefore λ **is a spectral value of** T, or $\lambda \in \sigma(T)$.

But converse is not true. Consider Example 4.1.1.

Example 4.1.1. Consider the Hilbert space l_2 where let $T : l_l \to l_2$ be defined by

$$T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots) \text{ as } (\xi_1, \xi_2, \ldots) \in l_2.$$

(This operator is known as a shift operator on l_2)

Then T has a spectral value which is not an eigen value.

Solution : Here T is linear such that

$$\left\| T(\underline{x}) \right\|^2 = \sum_{j=0}^{\infty} \left| \xi_j \right|^2 = \|x\|^2 x \text{ where,}$$

So T is bounded Linear operator over l_2. Here

T is not onto; because $(1, 0, 0, \ldots, 0, \ldots) \in l_2$ has no pre-image in l_2 under T. So $T^{-1} : X \to X$ does not exist. Thus $(T - \lambda I)^{-1}$ with $\lambda = 0$ does not exist. $\lambda = 0$ is a spectral value of T, but $\lambda = 0$ is not an eigen value of T.

Theorem 4.1.1. Let $T \in Bd\mathcal{L}(X, X)$ where X is a Branch space. If $\|T\| < 1$, then $I - T$ is invertible as a bounded linear operator over X and $(I - T)^{-1} = I + T + T^2 +$ ... **(r.h.s. is convergent in operator norm of** $Bd\mathcal{L}(X, X)$**).**

Proof : We have by Induction $\left\| T^j \right\| \leq \|T\|^j$; because, $\|T\| < 1$, geometric series

$\sum_{j=1}^{\infty} \|T\|^j$ is convergent.

As X is complete r.h.s. series as in statement of Theorem is absolutely convergent for $\|T\|<1$ Let.

$$S = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \ldots\ldots$$

We now verify that $S = (I - T)^{-1}$ For a + ve integer n we have

$(I - T)(I + T + T^2 + \ldots + T^n) = (I + T + T^2 + \ldots + T^n)(I - T)$, and

each $= I - T^{n+1}$ As $\|T\|<1$, we have $\lim_{n\to\infty} T^n + 1 = 0$ (zero operator), Therefore from above we obtain.

$(I - T)S = S(I - T) = \lim_{n\to\infty} (I - T^{n+1}) = I,$

That means $S = (I - T)^{-1}$.

Thearem 4.1.2. If $T \varepsilon Bd\mathcal{L}(X, X)$, the resolvant set $\rho(T)$ is an open set.

Proof : If $\rho(T) = \phi$, it is open. Say $\rho(T) \neq \phi$. Take a fixed $\lambda_0 \in \rho(T)$. For any complex scalar λ, we have

$T - \lambda I = T - \lambda_0 I - (\lambda - \lambda_0)I$

$= (T - \lambda_0 I).\ [I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}]$

$= (T - \lambda_0 I)V$ (say) where $V = I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}$

Putting $T_\lambda \equiv T - \lambda I$ we write from above.

$T_\lambda = T_{\lambda 0} V$ where $V = I - (\lambda - \lambda_0)T_{\lambda_0}^{-1}$ .....(1)

Since $\lambda_0 \in \rho(T)$ and T is bounded we have

$T_{\lambda_0}^{-1} \in Bd,(X, X)$

So, $\left\|T_{\lambda_0}^{-1}\right\| < \infty$

From (1) $\|I - V\| = \left\|(\lambda - \lambda_0)T_{\lambda_0}^{-1}\right\| = |\lambda - \lambda_0|\left\|T_{\lambda_0}^{-1}\right\|$

Therefore, $\|I - V\| =< 1$ ..... (2)

whenever, $|\lambda - \lambda_0| < \dfrac{1}{\left\|T_{\lambda_0}^{-1}\right\|}$ ..... (3)

So we find T_λ has inverse for all λ satisfying (3)

i.e. $T_\lambda^{-1} = (T_{\lambda_0}V)^{-1} = V^{-1}\,T_{\lambda_0}^{-1}$ for all λ satisfying (3).

In other words, λ_0 has a neighbourhood for every scalar λ of which T_λ is invertible i.e, this neighbourhood of λ_0 is entirely made up by regular values of T. Hence λ_0 is an interior point of $\rho(T)$; and $\rho(T)$ is open.

Corollary : Spectrum $\sigma(T)$ is a closed set in $\mathbb{C}$.

Theorem 4.1.3. The spectrum $\sigma(T)$ of a bounded Linear operator $T : X \to X$ where X is a Banach space is compact, and $|\lambda| \le \|T\|$ for all $\lambda \in \sigma(T)$.

Proof : For non-zero λ, take $K = \frac{1}{\lambda}$. Then

$$T_\lambda^{-1} = (T - \lambda I) = -\frac{1}{\lambda}(I - KT)^{-1} = -\frac{1}{\lambda}\sum_{j=1}^{\infty}(KT)^j = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}T\right)^j$$

and Theorem, 4.1.1. says, the series on r.h.s. above converges for all λ satisfying $\left\|\frac{1}{\lambda}T\right\| = \frac{1}{|\lambda|}\|T\| < 1$ i.e. for all λ with $= |\lambda > \|T\|$.

Thus such scalar $\lambda \varepsilon \rho(T)$; Thus spectrum $\sigma(T) = \mathbb{C} \setminus P(T)$ and further $\sigma(T)$ must lie in disk $\{\lambda \in \mathbb{C} : |\lambda| \le \|T\|\}$. Moreover $\sigma(T)$ is bounded and a bounded closed set of $\mathbb{C}$ is compact.

Definition 4.1.2. If X is a Banach space and $T \in Bd\mathcal{L}(X, X)$, then spectral redius $r_\sigma(T)$ of T is equal to

$$r_\sigma(T) = \operatorname{Sup}_{\lambda \in \sigma(T)} |\lambda|.$$

We shall later on obtain a formula for $r_\sigma(T)$.

Example 4.1.1. Let X be the Banach space $C[0, 1]$ with sup norm let $T : X \to X$ be given by $T(x) = \vartheta x$, where ϑ is a fixed element $C[0, 1]$ and $C[0, 1]$. Obtain $\sigma(T)$, and verify that it is compact.

Solution : It is a routine check up that T is a Linear operator over $C[0, 1]$. For boundedness, We have $\|Tx\| = \operatorname{Sup}_{0\le t\le 1}|(\vartheta x)(t)| = \operatorname{Sup}_{0\le t\le 1}|\vartheta(t)x(t)| = \operatorname{Sup}_{0\le t\le 1}|\vartheta(t)||x(t)| = \|\vartheta\|\,\|x\|$

Since $\|\vartheta\|$ is a fixed quantity, we see that T is bounded.

For scalar λ consider operator $T - \lambda I$, I being identity operator over $C[0, 1]$. If $x \in C[0, 1]$ we have

$(T - \lambda I)(x) = T(x) - \lambda x$

$= \vartheta x - \lambda x$

Now $(t)x(t) - \lambda x = (\vartheta t) - \lambda)x(t)$ in $0 \le t \le I$.

Thus $T - \lambda I$ **is not** inevitable if $\vartheta(t) - \lambda I$ vanishes in [0, 1].

Now ϑ is continuous in [0, 1] so $\min_{0\le t\le 1} \vartheta(t) = m$ and $M = \min_{0\le t\le 1} \vartheta(t)|$ are finite reals and $m \le M$;

Hence $T - \lambda I$ is not invertible if λ lies in $[m, M]$

Thus, here $\sigma(T)$ equals to the range of ϑ = closed interval $[m, M]$ and this is a compact set of scalars.

Remarks (1). If $\vartheta(t) = t$ then $0 \le t \le 1$, closed interval [0, 1]

(2). If $\vartheta(t)$ = constant = c, then $\sigma(T)$ = singleton $\{c\}$.

Example 4.1.2. Obtain a Linear operator $T : C[0,1] \to C[0, 1]$ such that $\sigma(T) = [a,b]$; $a < b$.

4.2. Let X and Y be two Normed Linear spaces (NLS) over the same scalar field. Theory of compact operators from X to Y owes primarily to Fredholm's famous integral equation related to linear functional equations $T(x) = \lambda x = y|$ with a complex parameter λ.

Definition 4.2.1. A linear operator $T : X \to Y$ is called a compact linear operator (or simply compact) if for every bounded set B is X closuer of $T(B)$ is compact in Y.

Theorem 4.2.1. Every compact operator $T : X \to Y|$ is bounded (hence continuous).

Proof : The unit sphere $S = \{x \in X : \|x\| = 1\}$ is bounded. For a compact operator T we have

$\overline{T(S)}$ is compact, and hence it is bounded, and $T(S)$ is also bounded. That means.

$\|T(x)\| \le M$ for all x with $= \|x\| = 1$

for some +ve constant M.

and therefore $\operatorname{Sup}_{\|x\|=1}\|T(x)\| < \infty$ and therefore T is bounded (hence continuous).

Converse of Theorem 4.2.2. is not true. Because let X be a infinite dimensional NLS. There we know that closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is bounded. Now identity operator $I : X \to X$ is a bounded (hence continuous) continuous operator where $I(B) = B$ is not compact, although B is closed.

Theorem 4.2.2. Let $T : X \to Y$ **be a linear operator.** Then T is compact if and only if T **sends every bounded sequence** $\{x_n\}$ **in** X **to** $\{T(x_n)\}$ **that has a convergent subsequence in** Y.

Proof : Let $T : X \to Y$ be a compact Linear operator, and $\{x_n\}$ be a bounded sequence in X; Then $\overline{\{T(x_n)\}}$ is compact in Y, and hence from sequential compactness it follows that $\overline{\{T(x_n)\}}$ has a convergent subsequence in Y.

Conversely, suppose every bounded sequence $\{x_n\}$ has a subsequence $\{x_{nk}\}$| such that $\{T(x_n)\}$ converges in Y. Now take B any bounded set B is X and $\{y_n\}$ any sequence in $T(B)$; Then let x_n be pre-image of y_n in X under T i.e. $y_n = T(x_n)$. Hence $\{T(x_n)\}$,| by assumption contains a convergent sub-sequence. Therefore $\overline{T(B)}$ is compact. So T is a compact Linear operator.

Corollary (1) Sum of two compact linear operators : $X \to Y$ | is a compact Linear operator.

(2) A scalar multiple of a compact Lineat operator : $X \to Y$ | is a compact Linear operator.

Therefore we have the following Theorem the proof of which is left out.

Theorem 4.2.3. The collection of all compact Linear operators : $X \to Y$ forms a vector space and it is a sub-space of $Bd\mathfrak{L}(X, Y)$.

Theorem : 4.2.4. Let $T : X \to Y$ be a Linear operator.

(a) If T is bounded and Dim $(T(X)) < \infty$ then T is compact.

(b) If X is finite Dimensional NLS, Then T is compact.

Proof : (a) Let A be a non-empty bounded set in X. Because $T : X \to Y$ is a bounded Linear operator, we have $T(A)$ as a bounded set in $T(X)$ which is finite Dimensional, and is therefore closed. Thus $\overline{T(A)} \subset \overline{T(X)} = T(X)$; Hence $\overline{T(A)}$ becomes a bounded and closed subset of a finite Dimensional NLS; Hence $\overline{T(A)}$ is compact; and T is compact.

(b) follows from (a); To see it we first show that Dim $(T(X)) < \infty$ when Dim $(X) < \infty$.

Let Dim $(X) = n$; Take any $n + 1$ elements $y_1, y_2, ..., y_{n+1}$ from $T(X)$. So we find $x_1, x_2, ..., x_n, x_{n+1}$ in X such that $T(x_i) = y_i$ $(i = 1, 2, ..., n + 1)$. Since Dim $(X) = n$, the set of vectors $\{x_1, x_2, ..., x_n, x_{n+1}\}$ becomes linearly dependent and we find scalars $\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n + \alpha_n + {}_1 x_{n+1} = 0$$

By linearity of T, we have

$$T\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) = T(\underline{O}) = (\underline{O})$$

or, $\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n + \alpha_n + {}_1 y_{n+1} = O$ where scalars α_i's are not all zero.

That means $\{y_1, y_2, ..., y_n, y_{n+1}\}$ is a linearly dependent set. Thus $T(X)$ contains no linearly independent subsets of $(n + 1)$ vectors; Hence

$$\text{Dim } (T(X)) \le n.$$

Thus our observation stand OK. and Dim $T(X) < \infty$, and part (a) applies by remembering only that over a finite Dimensional NLS every Linear operator becomes bounded. The proof is now complete.

Example 4.2.1. Let $K(s, t)$ be continuous in the square $0 \le s \le 1,\ 0 \le t \le 1$; and let $T : C[0,1] \to C[0,1]$ be defined by

$$y = T(x) \text{ as } x \in C[0,1] \text{ such that}$$

$$y(s) = \int_0^1 k(s,t)x(t)dt;\ 0 \le s \le 1.$$

Show that T is a compact Linear operator.

Solution : Put $M = \underset{\substack{0\le s\le 1\\0\le t\le 1}}{\text{Max}}|k(s,t)|$

Take a sequence $\{x_n\}$ in $C[0,1]$ with $\|x_n\| \le 1$ for all n. ($C[0, 1]$ taken as Banach space with sup norm).

If $y_n = T(x_n)$; Then $= |y_n(s)| = \left|\int_0^1 k(s,t)x_n(t)dt\right|$

$$\le \underset{\substack{0\le s\le 1\\0\le t\le 1}}{\text{Max}}|k(s,t)|.\underset{0\le t\le 1}{\text{Max}}|x_n(t)|1. = M\|x_n\| = M$$

Therefore, the sequence of functions $\{y_n(s)\}$ is uniformly bounded.

For $\varepsilon > 0$, by uniform continuity of $K(s, t)$ we find a $\delta > 0$ such that

$|k(s_1,t) - k(s_2,t)| < \varepsilon$ whenever $|s_1 - s_2| < \delta$ s_1, s_2 $\in [0,1]$ and for all $t \in [0,1]$ Then for $n = 1, 2,\ldots,$

We get $|y_n(s_1) - y_n(s_2)| \le \int_0^1 |K(s_1,t) - K(s_2,t)|\,|(x_n(t)|dt$

$< \varepsilon$ if $|s_1 - s_2| < \delta$.

Therefore sequence of functions $\{y_n(S)\}$ is equi-continuous. By Arzela-Ascoli Theorem the set consisting of members of the sequence $\{y_n(S) = T(x_n)\}$ is compact, and so $\{T(x_n)\}$ has a convergent sub-sequence. Therefore T is compact Linear operator.

Theorem 4.2.5. Let X be a Banach space and $T : X \to X$ be a compact linear operator and let $S \varepsilon Bd\pounds(X, X)$ then ST and TS are compact.

Proof : Let $\{X_n\}$ be a bounded sequence in X; Then $\{S(X_n)\}$ is bounded, because $\|S(X_n)\| \le \|S\|\,\|x_n\|$, Therefore $\{T(S(X_n))\}$ has a convergent sub sequence, because T is compact. As $(TS)\,(x_n) = T(S(x_n))$, we have verified that TS is compact.

On the other hand, $\{T(x_n)\}$ has a convergent subsequence, say $= \left\{T\left(x_{n_k}\right)\right\}$. Since S is continuous we have $\left\{S\left(T\left(x_{n_k}\right)\right)\right\}$ also convergent. This proves that ST is compact.

Theorem 4.2.6. Let X be a NLS and Y a Banach space over same scalars; and let $\{T_n : X \to Y\}$ be a sequence of compact Linear operators such that $\lim_{n\to\infty}\|T_n - T\| = 0$ where $T \in Bd\pounds(X, Y)$ Show that T is compact.

Proof : The proof rests upon applying 'diagonal method.'

Let $\{x_m\}$ be a bounded sequence in X. We show that $\{T(x_n)\}$ has a convergent sub-sequence, Since T_1 is compact, $\{x_m\}$ has a sub sequence, say, $\{x_{1,\,m}\}$ such that $\{T_1(x_{1,\,m})\}$ is convergent, and hence is Cauchy. Similarly, $\{x_{1,\,m}\}$ has a sub-sequence $\{(x_{2,\,m})\}$ such that $\{T_2(x_{2,\,m})\}$ becomes Cauchy. Continuing this process we arrive at the diagonal sequence $\{y_m = x_{m,\,m}\}$ as a sub-sequence of $\{x_m\}$ such that for every fixed +ve integer n, the sequence $\{T_n(y_m)\}_m = 1, 2, \ldots$ is Cauchy. $\{x_m\}$ is bounded, say, $\|x_m\| \le M$ (> 0) for all m. Hence $\|y_m\| \le M$ for all m. Take $\varepsilon > 0$ Since $\lim_{m\to\infty}\|T_m - T\| = 0$ there is $n = p$ such that

$$\|T - T_p\| < \frac{\varepsilon}{3M}$$

Since $\{T_p(y_m)\}$ is Cauchy, there is an index N satisfying

$$\|T_p(y_j) - T_p(y_k)\| < \frac{\varepsilon}{3} \text{ for } j,\, k \ge N$$

So for $j,\, k \ge N$ we have

$$\|T(y_j) - T(y_k)\| \le \|T(y_j) - T_p(y_j)\| + \|T_p(y_j) - T_p(y_k)\| + \|T_p(yk) - T(y_k)\|$$

$$\le \|T - T_p\|\,\|y_j\| + \frac{\varepsilon}{3} + \|T_p - T\|\,\|y_k\|$$

$$< \frac{\varepsilon}{3M}.M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}.M = \varepsilon$$

Therefore $\{T(y_m)\}$ is Cauchy, and because Y is a Banach space, $\{T(y_n)\}$ is convergent in Y.

Now nothing that $\{y_m\}$ is a subsequence of any bounded sequence $\{x_m\}$ we see that T becomes a compact operator.

Example 4.2.2. Let $T : l_2 \to l_2$ be defined as

$T(x) = y$ for $x = (\xi_1, \xi_2, \ldots) \in l_2$ where $y = (\mu_1, \mu_2, \ldots) \in l_2$ such that $\mu_i = \frac{1}{i}\,\xi_i$ $(i = 1, 2, 3, \ldots)$. Show that T is a compact linear operator.

Solution : Here $T(\xi_1, \xi_2, \xi_3, \ldots) = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots, \frac{\xi_n}{n}, \ldots\right)$ whenever $(\xi_1, \xi_2, \ldots) \in l_2$; and it is easy to see that T is linear. For every +ve integer n let us define an operator $T_n : l_2 \to l_2$ given by $T_n(\xi_1, \xi_2, \xi_3, \ldots) = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots, \frac{\xi_n}{n}, 0,0,0,\ldots\right)$ when $(\xi_1, \xi_2, \ldots) \in l_2$. Then a fixed n we see T_n to be Linear and bounded and because Dim $(T_n(l_2)) < +\infty$, we conclude that T_n is also compact i.e., each T_n is a compact linear operator, Further for $x = (\xi_1, \xi_2, \ldots) \in l_2$ we have

$$\left\|(T - T_n)(\underset{\sim}{x})\right\|^2 = \left\|T(\underset{\sim}{x}) - T_n(\underset{\sim}{x})\right\|^2 = \left\|\underbrace{0,0,\ldots 0}_{n\,\text{places}}, \frac{\xi_{n+1}}{n+1}, \frac{\xi_{n+2}}{n+2}\right\|^2$$

$$= \sum_{j=n+1}^{\infty} |u_j|^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2}|\xi_j|^2 \le \frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} \frac{1}{j^2}|\xi_j|^2 \le \frac{\left\|\underset{\sim}{x}\right\|^2}{(n+1)^2}$$

$$\therefore \quad \sup_{\left\|\underset{\sim}{x}\right\|=1} \left\|(T - T_n)(\underset{\sim}{x})\right\| \le \frac{1}{n+1}$$

or, $$\|T - T_n\| \le \frac{1}{n+1}$$

Therefore $\lim_{n\to\infty} \|T - T_n\| = 0$. Now Theorem 4.2.6 applies for desired conclusion.

Theorem 4.2.7. If X is NLS and $T : X \to X$ is a compact Linear operator Then

(a). Every non-zero spectral value of T ia an eigen value of T.

(b) The set all eigen value of T is atmost countable.

(c) For non-zero eigen value of T the Dimension of any eigen-sub-space of T is finite.

Proof.: Part of the proof is rather deep and lenghthy. For short of space proof is left out.

Remark : The reader may like to consult Kreyszing' book on Functional analysis–chapter on compact linear operators for proof of above theorem and allied matters.

§ 4.3 Spectral properties of bounded self-adjoint operators on a complex Hilbert space *H*.

We know that a bounded Linear operator $T : H \to H$l has its adjoint operator $T^* : H \to H$ satisfying

$< T(x), y > = < x, T^*(y) >$ for all $x, y \in H$; and T is known as self-adjoint if $T = T^*$. Therefore for a self-adjoint operator $T : H \to H$ we always have

$$< T(x), y > = < x\ T(y) \text{ for all } x, y \in H.$$

Theorem 4.3.1. If $T : H \to H$ is a bounded self-adjoint operator. Then

(a) All the eigen values of T (if they exist) are real.

(b) Eigen vectors corresponding to different eigen values of T are orthogonal.

Proof : (a) Suppose λ is an eign value of T and let x be a corresponding eigen vector. Then $x \neq \tilde{O}$ and $T(x) = \lambda x$ By self-adjointness of T we get.

$$\lambda <x, x> = <\lambda x, x> = <T(x), x> = <x\ T(x) > = <x, \lambda x>$$

$$= \bar{\lambda} <x, x>.$$

i.e. $\lambda\|x\|^2 = \bar{\lambda}\|x\|^2$ since $x \neq \underset{\sim}{O}$ it follows that $\lambda = \bar{\lambda}$ So λ is real.

(b) Let λ and μ be two distinct eigen values of T, and let x and y be corresponding eigen vectors. Then we have $T(x) = \lambda x$ and $T(y) = \mu y$. As T is self adjoint and μ is also real, we write.

$$\lambda <x,y> = <\lambda x, y> = <T(x), y > = <x, T(y)>= <x, \mu y> = \mu<x, y>.$$

Since $\lambda \neq \mu$, we have $< x, y > = 0$. Hence x and y are orthogonal.

Remark : For a bounded self-adjoint operator $T : H \to H$ surprising result is that (in connection with Theorem 4.3.1) spectrum $\sigma(T)$ of T consists of real scalars only. Interested readers are advised to look into Advanced Functional text like "Dunford and Schwartz, Linear operators, vols. I & II, Wiley Intersciences, New York, 1963", in this connection.

Theorem 4.3.2. Let $T : H \to H$ be a bounded self-adjoint Linear operator. Then $\lambda \in \rho(T)$ if and only if $\|T_\lambda(x)\| \geq c\|x\|$ for some +ve c and for all $x \in H (T_\lambda = T - \lambda I)$.

Proof : Let $\lambda \in \rho(T)$. Then Inverse $T_\lambda = T - \lambda I$ exists; and T_λ^{-1} is a bounded Linear operator over H; put $\|T_\lambda^{-1}\| = K (> 0)$.

Now for $x \in H$, $\|x\| = \|T_\lambda^{-1} T_\lambda(x)\| \leq \|T_\lambda^{-1}\|\ \|T_\lambda(x)\| \leq K\|T_\lambda(x)\|$

Therefore, $\|T_\lambda(x)\| \geq c\|x\|$ taking $c = \dfrac{1}{k} > 0$.

Converse part is a bit involved. Suppose condition holds. Then we will prove

(i) T_λ is bijective,

(ii) $T_\lambda(H)$ is dense in H,

(iii) $T_\lambda(H)$ is closed

so that $T_\lambda(H) = H$ and T_λ^{-1} is bounded.

(i) For $x_1, x_2 \in H$, we have $\|T_\lambda(x_1) - T_\lambda(x_2)\| = \|T_\lambda(x_1 - x_2)\| \geq c\|x_1 - x_2\|$

So, $T_\lambda(x_1)$ $T_\lambda(x_1)$ implies $x_1 = x_2$.

(ii) We show that $x_0 \perp \overline{T_\lambda(H)}$ implies $x_0 = \underline{O}$; Then $\overline{T_\lambda(H)} = H$. So take $x_0 \perp \overline{T_\lambda(H)}$. Then $x_0 \perp T_\lambda(H)\,(T_\lambda(H) \subset (\overline{T_\lambda(H)})$. So for all $x \in H$ we have $0 = < T_\lambda(x), x_0 > = < T(x), x_0 > - \lambda < x, x_0 >$. As T is self-adjoint we obtain $< x, T(x_0)$ $= < T(x), x_0 > = < x, \bar{\lambda} x_0 >$; So that $T(x_0)$ Now $x_0 \neq \underline{O}$ is impossible, because that would mean $\bar{\lambda}$ is an eigen value of T; so that $\bar{\lambda} = \lambda$ and $T(x_0) - \lambda(x_0) = T_\lambda(x_0) = \underline{O}$, and a contradiction results $0 = \|T_\lambda(x_0)\| \geq c\|x_0\| > 0$.

Therefore $x_0 = O$. Then $\overline{T_\lambda(H)^\perp} = \{\underline{O}\}$, because x_0 was any vector orthogonal to $T_\lambda(H)$. Hence $\overline{T_\lambda(H)} = H$. So $T_\lambda(H)$ is dense in H.

(iii) Finally we show that $y \in \overline{T_\lambda(H)}$ implies $y \in T_\lambda(H)$; so that $T_\lambda(H)$ is closed, and $T_\lambda(H) = H$. Take $y \in \overline{T_\lambda(H)}$. There is a sequence $\{y_n\}$ in $T_\lambda(H)$ such that $\lim_{n \to \infty} y_n = y$. Say, $\lambda_n = T_\lambda(x_n)$ for some $x_n \in H$. Then $\|x_n - x_m\| \leq \|T_\lambda(x_n - x_m)\|$ $= \frac{1}{c}\|y_n - y_m\|$. Therefore $\{x_m\}$ is Cauchy. As H is complete, say $\lim_{n \to \infty} x_n = x \in H$. Now T and hence T_λ is continuous, and $y_n = T_\lambda(x_n)$; So $\lim_{n \to \infty} T_\lambda(x_n) = T_\lambda(x)$. So $T_\lambda(x) \in T_\lambda(H)$. Clearly $T_\lambda(x) = y$ i.e., $y \in T_\lambda(H)$. Hence $T_\lambda(H)$ is closed because $y \in \overline{T_\lambda(H)}$ was arbitrary. Thus $T_\lambda(H) = H$. Hence $T_{\lambda^{-1}}$ is defined on all of H, and is bounded. Therefore $\lambda \in \rho(T)$.

Theorem 4.3.3. The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ lies in the closed interval $[m, M]$ on real axis where $m = \inf_{\|x\|=1} <T(x), x>$ and $M = \sup_{\|x\|=1} <T(x), x>$.

Proof : It is known that spectrum $\sigma(T)$ for T consists of real scalars only. Put $\lambda = M + \varepsilon$ with $\varepsilon > 0$. We show that

$\lambda \in \rho(T)$ (The resolvant set for T).

For every non-zero x, put $\vartheta = \frac{x}{\|x\|}$, So and $x = \|x\|\vartheta$, and

$$< T(x), x > = \|x\|^2 < T(\vartheta), \vartheta > \leq \sup_{\|u\|=1} < T(u), u > = < x, x > M.$$

Hence $- <T(x), x > \geq - <x, x > M$, and by $C - S$ inequality, we have (Putting $T_\lambda \equiv T - \lambda I$)

$= \|T_\lambda(x)\|\|x\| \geq -< T_\lambda x, x >=< T(x), x > +\lambda < x, x >\geq (-M+\lambda) < x, x >$

$= \varepsilon\|x\|^2$

where $\varepsilon = \lambda - M > 0$. Thus from above chain we get $\|T_\lambda(x)\| \geq \varepsilon\|x\|$

Now by Theorem 4.3.2. this implies λ is a member of $\rho(T)$ Similarly, one shows that for any real $\lambda < m$ is a member of $\rho(T)$; Therefore $\sigma(T) \subset [m, M]$.

Theorem 4.3.4. For any bounded Linear operator $T : H \to H$; T is self-adjoint if and only if $< T(x), x >$ is real for each $x \in H$.

Proof : Suppose T is self-adjoint, and $x \in H$; we have $< T(x), x > = <x, T(x)>$ $\overline{< T(x), x >}$ = and so $< T(x), x >$ is real.

Conversely; suppose T is a bounded Linear operator over H such that $<T(x), x>$ is real for each $x \in H$, Then for all $x \in H$.

$< T(x), x > = \overline{< T(x), x >} = <T(x), x>$ ..... (1)

Now if $x, y \in H$, two elementary calculations show that

$4< T(x), y > = < T(x + y) > - <T(x - y), x - y > + i <T(x + iy), x + iy >$
$-i <T(x - iy), x - iy >|$ .....(2)

and $4 < x, T(y) > = < x + y, T(x + y) > - < x, y\ T(x - y) + i < x + iy.$
$T(x + iy) > - i < x - iy, T(x - iy) >$ .....(3)

Equations (1), (2) and (3) show that $<T(x), y> = <x, T(y)>$, therefore T is self-adjoint.

Theorem 4.3.5 If $T : H \to H$ is a bounded Linear self-adjoint operator, then $\|T\| = \operatorname{Sup}_{\|x\|\leq 1} |< T(x), x >|$

Proof : Put $K = \operatorname{Sup}_{\|x\|\leq 1} |< T(x), x >|$. By $C - S$ inequality

We obtain $|< T(x), x >| \leq \|T(x)\|\|x\| \leq \|T\|\|x\|^2$ for all $x \in H$, and therefore $K \leq \|T\|$.

Again $\quad \dfrac{1}{\|x\|^2}|\langle T(x), x\rangle| = \left\langle \dfrac{T(x)}{\|x\|}, \dfrac{x}{\|x\|} \right\rangle = \left|\left\langle T\left(\dfrac{x}{\|x\|}\right), \dfrac{x}{\|x\|} \right\rangle\right| \leq \sup_{\|x\|\leq 1} |< T(u), u >| = K$

i.e. $|< T(x), x >| \leq K.\|x\|^2$ for all $x \in H$ ..... (4)

A simple exercise yields.

$<T(x+y), x+y> - <T(x-y), x-y> = 4RI < T(x), y >$ for all $x, y \in H$. .....(5)

Using (4) and (5) and parallelogram law, we get

$$4\,|Re< T(x), y >| \leq |< T(x+y), x+y >| + |< T(x-y), x-y >|$$

$$= \leq \alpha\left(\|x+y\|^2 + \|x-y\|^2\right)$$

$$= 2\alpha\left(\|x\|^2 + \|y\|^2\right) \text{ for all } x, y \in H. \quad .....(6)$$

Let $x \in H$ with $\|x\| \leq 1$ and $T(x) \neq O$, Put $y = \dfrac{T(x)}{\|T(x)\|}$ in (6) one obtains

$$\|T(x)\| = Re < T(x), \frac{T(x)}{\|T(x)\|} >< \frac{1}{4} 2K\left(\|x\|^2 + 1\right) \leq K.$$

The last inequality is, of course true when $T(x) = O$, so we have

$\|T\| = \sup\limits_{\|x\| \leq 1}\{\|T(x)\|\} \leq K$. Now combining with $K \leq \|T\|$ we get

i.e. $\|T\| = K$

$$= \|x\| \underset{\leq 1}{up}\ |< T(x), x >|.$$

Example 4.3.1. Show that $T : L_2[0, 1] \rightarrow L_2[0, 1]$ given by

$T(x) = y$; $x \in L_2[0, 1]$,

where $y(t) = tx(t)$; $0 \leq t \leq 1$;

is a bounded linear self-adjointed operator wihtout eigen values.

Solution : We know that the space $L_2[0,1]$ consisting of all square integrable functions x over $[0, 1]$ is a Hilbert space with *I.P.* function $<x, y> = \int_0^1 x(t)\ y(t)dt$ (Integration taken in Lebesgue sense) as $x, y \in L_2[0,1]$. So L_2 – norm

$$\|x\| = \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Here T is clearly linear. For boundedness we have for $x \in L_2[0,1], \|T(x)\|^2 = \int_0^1 (tx(t)^2)dt$

$$= \int_0^1 t^2 x^2(t)dt \leq \sup_{o \leq t \leq 1}(t^2)\int_0^1 x^2(t)dt$$

$$= 1.\ \|x\|^2$$

If $x, y \in L_2[0, 1]$ we have $<T(x), y> = \int_0^1 t(x)y(t)dt$

and $< x, T(y) >= \int_0^1 x(t)t\ y(t)dt$

and therefore $<T(x), y> = <x, T(y)>$, and that shows T to be self-adjoint. For λ to be an eigen value of the operator T we have

$T(x) = \lambda x$ gives $tx(t) = \lambda x(t)$ in $0 \leq t \leq 1$

or, $x(t)(\lambda - t) = 0$ in $0 \leq t \leq 1$.

Since x is non-zero member $L_2[0, 1]$ we have $\lambda - t = 0$ in $0 \ 1 \leq t \leq 1$ — a contradiction that λ is a scalar (fixed). Hence no scalar λ exists to satisfy $T(x) = \lambda x(x \neq \underline{O})$ in $L_2[0, 1]$.

i.e., T has got no eigen value.

§ 4.4 Projection operators

If H is a Hilbert space and Y is a closed sub-space of H, then Decomposition theorem says that each member $x \in H$ has a unique representation as

$x = y + z$, where $y \in Y$ and $z\ Y^{\perp}$,

$y^{\prime}$ denoting orthogonal complement of Y where

$$
\begin{aligned}
Y^{\perp} &= \{z \in H : z \perp Y\} \\
&= \{z \in H : < z, y > = 0 \text{ for all } y \in Y\}.
\end{aligned}
$$

And in this case we write

$H = Y \oplus Y^{\perp}$ as a direct sum decomposition of H.

Now consider a mapping $H \to H$ sending $x \in H$ to.

$y \in Y$ ($x = y + z$; $x \in Y \subset H$; $z \in Y^{\perp} \subset H$).

It is well defined because gives x, y is unique.

Definition 4.4.1. The mapping : $H \to H$ | defined as $x \in H$ as $y \in Y$ is called a Projection (or orthogonal projection) on H. It is denoted by P.

Explanation : It is readily seen that a Porjection $P : H \to H$ is a Linear operator where Y is given a closed Linear sub-space of H and $P(x \in H) \in Y$.

If $y \in Y \subset H$; We see $y = y + \underline{O}$ by representation formula and $P(y) = y \in Y$.| Therefore P restricted to Y becomes the Identity operator on Y.

On the other had if $z\, Y^{\perp} \subset H$, We have $z = \underset{\sim}{O} + z$ where $\underset{\sim}{O} \in Y$ and $z \in Y^{\perp}$ we see that $P(z) = \underset{\sim}{O}$ and conversely if $P(u) = \underset{\sim}{O}$ we have $u = \underset{\sim}{O} + u$ showing $u \in Y^{\perp}$.

Therefore Null-space of $P = Y^{\perp}$.

Also we can now write $x = y + z$

$$= P(x) + z$$

$$\text{or, } x = x - P(x) = (I - P)(x).$$

So we treat the operator $I - P$ as the projection of H onto $Y^{\perp}$.

Theorem 4.4.1. A bounded Linear operator $P : H \to H$ (H being a Hilbert space) is a Projection if and only if P is self-adjoint and Idempotent (i.e. $P^2 = P$).

Proof : Let $P : H \to H$ be a Projection operator and denote $P(H) = Y$. If $x \in H$, and $P(x) = y \in Y$, we have $P^2(x) = P(P(x)) = P(y) = y = P(x)$. So $P^2 = P$.

Further, Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ where, $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^{\perp}$. Then $< y_1, z_2 > = <y_2, z_1> = 0$, because $Y \perp Y^{\perp}$. | Finally, $<P(x_1), x_2> = <y_1, y_2 + z_2> = <y_1, z_2> = < y_1 + z_1, y_2> = <x_1, P(x_2)>$ | That means P is self-adjoint.

Conversely, let $P^2 = P$ and denote $P(H) = Y$. Then for every $x \in H$ we have $x = P(x) + (I - P)(x)$.

Now P being self-adjoint we have

$$<P(x), (I - P)(y)> = <x, P(I - P)(y)> = <x, P(y) - P^2(y)>$$

$$= <x, \underset{\sim}{O}> = 0 \text{ because } P = P^2.$$

Now $(I - P)(P(x) = P(x) - P^2(x) = 0$, | shows that

Null space of $I - P$ i.e., $N(I - P)$ satisfies

$$Y \subset N(I - P) \qquad \text{..... (1)}$$

Again $(I - P)(x) = \underset{\sim}{O}$ gives $x = P(x)$ i.e. $x \in Y$.

Thus $N(I - P) \subset Y$ .....(2)

(1) and (2) together give $N(I - P) = Y$. So Y is closed.

Finally, P restricted to Y is Identity operator on Y because $y = p(x)$ gives $p(y) = p(p(x)) = p^2(x) = p(x) = y$.

Theorem 4.4.2. If P is projection on H, then

(a) $< P(x), x >= \|P(x)\|^2$ for all $x \in H$.

(b) $\|P\| \leq 1$ and $\|P\| = 1$ if $P(H) \neq \{\underset{\sim}{O}\}$.

Proof : (a) if $x \in H$, then $<P(x), x> = <P^2(x), x> = <P(x), P(x)>$

$$= \|P(x)\|^2 \geq 0.$$

(b) By $C - S$ inequality,

$$= \|P(x)\|^2 =< P(x), x > \leq \|P(x)\| \, \|x\|$$

or, $\dfrac{\|P(x)\|}{\|x\|} \leq 1$ for every non-zero x in H.

Therefore, $\|P\| = \leq 1$.

Also $\dfrac{\|P(x)\|}{\|x\|} = 1$ if $x \in P(H)$, and $x \neq \underline{O}$.

So (b) is proved.

Definition 4.4.2. Two Projection operators P_1 and P_2 on a Hilbert space H are called orthogonal if their composition P_1P_2 is the zero operator.

If $P_1P_2 = 0$, then $P_2P_1 = 0$ because $0 = (P_1P_2)^* = P_2P_1^* = P_2P_1^*$ (Projection operators are Self-adjoint).

Theorem 4.4.3. Two projection operators P_1 and P_2 over H are orthogonal if and only if their corresponding sub-spaces Y_1 and Y_2 of H are orthogonal ($Y_1 \perp Y_2$).

Proof : If $P_1P_2 = 0$; and $x_1 \in y_1$ and $x_2 \in Y_2$, we have $\langle x_1, x_2 \rangle = \langle P_1(x_1), P_2(x_2) \rangle$ $< x_1, P_1{}^*P_2(x_2)> = < x_1, P_1P_2(x_2) > = < x_1, \underline{O} > = 0$. | Therefore, ($Y_1 \perp Y_2$).

Conversely if $Y_1 \perp Y_2$; for each $x \in H$ we know that $P_2(x) \in Y_2$; So $P_2(x) \perp Y_1$.| Thus writing

$$P_2(x) = \underline{O} + P_2(x) \text{ where } \underset{\sim}{O} \in Y_1, \text{ and } P_2(x) \in Y_1^{\perp}.$$

Therefore, $P_1P_2(x) = \underline{O}$. This is true for all $x \in H$. So, P_1P_2 = the zero operator.

Notation : If a projection operator over H is associated with a closed sub-space Y of H, we write Projection operator as P_Y.

Theorem 4.4.4. If P_{Y_1} and P_{Y_2} are two projection operators over H, their sum is a Projection operator if and only if P_{Y_1} and P_{Y_2} are orthogonal.

Proof : The condition is necessary. Suppose $P_{Y_1} + P_{Y_2}$ is a Projection operator.

Then $(P_{Y_1} + P_{Y_2})^2 = P_{Y_1} + P_{Y_2}$

or, $P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1} = 0$ (zero operator).

or, $P_{Y_1} P_{Y_1} P_{Y_2} + P_{Y_1} P_{Y_2} P_{Y_1} = 0$

or, $P_{Y_1} P_{Y_2} + P_{Y_1} P_{Y_2} P_{Y_1} = 0$ .....(*)

or, $P_{Y_1} P_{Y_2} P_{Y_1} + P_{Y_1} P_{Y_2} P_{Y_1}{}^2 = 0$

or, $P_{Y_1} P_{Y_2} P_{Y_1} + P_{Y_1} P_{Y_2} P_{Y_1} = 0$

or, $P_{Y_1} P_{Y_2} P_{Y_1} = 0.$

So from (*) we get $P_{Y_1} P_{Y_2} = 0$

The condition is sufficient : Suppose $P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1} = 0$

Then $(P_{Y_1} + P_{Y_2})^2 = P_{Y_1}{}^2 + P_{Y_1} P_{Y_2} + P_{Y_2}{}^2 = P_{Y_1} + P_{Y_2}$.

And $(P_{Y_1} + P_{Y_2})^* = P_{Y_1}{}^* + P_{Y_2}{}^* = P_{Y_1} + P_{Y_2}$.

So the sum operator is both self adjoint and Idempotent. And Theorem 4.4.1 applies to say that $P_{Y_1} + P_{Y_2}$ is a Projection operator.

Theorem 4.4.5. The product of two projection operators P_{Y_1} and P_{Y_2} is a Projection operator if and only if

$$P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1}$$

Proof : The condition is necessary : Let $P_{Y_1} P_{Y_2}$ be a Projection operator; So $P_{Y_1} P_{Y_2}$ is self-adjoint. Therefore

$P_{Y_1} P_{Y_2} = (P_{Y_1} P_{Y_2})^* = P_{Y_2}{}^* P_{Y_1}{}^* = P_{Y_2} P_{Y_1}$.

The condition is sufficient : Suppose $P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1}$.

Then $(P_{Y_1} P_{Y_2})^* = P_{Y_2}{}^* \, P_{Y_1}{}^* = P_{Y_2} P_{Y_1} = P_{Y_1} P_{Y_2}$

So $P_{Y_1} P_{Y_2}$ is self-adjoint.

Also $(P_{Y_1} P_{Y_2})^2 = P_{Y_1} P_{Y_2} P_{Y_1} P_{Y_2} = P_{Y_1} P_{Y_1} P_{Y_2} P_{Y_2} = P_{Y_1}{}^2 P_{Y_2}{}^2 = P_{Y_1} P_{Y_2}$

So $P_{Y_1} P_{Y_2}$ is idempotent. Therefore Theorem 4.4.1 applies to conclude that $P_{Y_1} P_{Y_2}$ is a Projection operator.

Example 4.4.1 Let P_{Y_1} and P_{Y_2} be two projection operators over a Hilbert space H with corresponding sub-space Y_1 and Y_2 of H. Show that if $P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1}$ then $P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2}$ is a Projection operator on H.

Solution : Here $(P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2})^*$

$$= (P_{Y_1} + P_{Y_2})^* - (P_{Y_1} P_{Y_2})^*$$

$$= P_{Y_1}{}^* + P_{Y_2}{}^* - P_{Y_2}{}^* P_{Y_1}{}^*$$

$$= P_{Y_1} + P_{Y_2} - P_{Y_2} P_{Y_1}$$

$$= P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2}$$

So, $P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2}$ is self-adjoint.

Again $(P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2})^2 = (P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2})(P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2})$

which on simplification supported by assumption $P_{Y_1} P_{Y_2} = P_{Y_2} P_{Y_1}$ shall give $= P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2}$. That means $(P_{Y_1} + P_{Y_2} - P_{Y_1} P_{Y_2})$ is idempotent. Therefore this operator is a Projection operator on H.

EXERCISE-A

Short answer Type Question

1. If T is self-adjoint operator on a Hilbert space H for any +ve integer show that T^n is self-ajoint.
2. If P is projection operator, on a Hilbert space H with corresponding sub-space Y, show that $I - P$ is a Projection operator with $Y^{\perp}$.
3. Let $T : L_2[0, 1] \rightarrow L_2[0, 1]$ be definied as $T(x) = y$ for $x \in L_2[0, 1]$ where $y(t) = tx(t)$ in $0 \leq t \leq 1$ show that T is a self ajoint operator.
4. Let $T : H \rightarrow H$ be a Self-adjoint operator (H is Hilbert space) and $W \in Bd\mathcal{L}(H, H)$ show that $W^* TW$ is self-adjoint W^* denoting ajoint of W).

EXERCISE-B

1. Show that every linear over a finite dimensional NLS is a compact operator.
2. Show that an operator T is a Projection operator on a Hilbert space H if and only if $T = T^*T$.
3. If $\{P_n\}$ is a Sequence of Projection operators over a Hilbert space H and $\lim_{x \to \infty} P_n = P$, show that P is a Projection operator on H.
4. Find the eigen values of $\psi : R^n \to R^n$ given by

   $\psi(x) = \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_n - 1}{n-1}\right)$ where $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \in R^n$.
5. Let H a Hilbert space and $T : H \to H$l be a compact linear operator. Show that its adjoint T' is compact.
6. If z is any fixed element in an I.P. space X, show that $f(x) = \langle x, z \rangle$ for $x \in X$ is a bounded Linear funtional over X, and find $\|f\|$.

Unit - 5

(**Concepts :** Eigen value, Eigen vector of a Linear operator over NLS X with Dim $(X) < \infty$, Characteristic equation, Characteristic polynomial, Existence Theorem, finite Dimensional Spectual Theorem, Banach algebra X with identity, examples, invertibel and non-invertible elements of X, Topological divisor of zero X; for $x \in X$, I Resolvant set $\rho(x)$, Spectrum $\sigma(x)$ Properties of $\sigma(x)$ Gelfand-Mazur Theorem, Spectral radius formula.)

§ 5.1 Finite Dimensional spectral Theory.

Let X be a Normed Linear space NLS with Dim $(X) < \infty$ and let $T : X \to X$ be a linear operator. Then spectral Theory of such an operator T is simpler than that of its counterpart acting on an infinite dimensional NLS. We know that T has a matrix representation. We see at present that spectral theory of T is essentially eigen-value (eigen vector) theory.

We recall that a scalar λ is called and eigen value of T. if there is a **non-zero vector** $x \in X$, called in eigen vector of T corresponding to eigen valueλ, if $T(x) = \lambda x$.

If M be the collection of all such eigen vectors of T corresponding to eigen value λ, and if we adjoins $\underline{O}$ with M, then M becomes a Linear sub-space of X, called an eigen-sub-space X. We also remember that a Linear operator over X (Dim $(X) < \infty$) becomes a bounded linear operator on X, and hence T is continuous, and consequently eigen M becomes a closed linear sub-space of X. Let Dim $(X) = n$. Let A be the Matrix (a Square matrix of order n) that represents T in respect of an ordered basis of X. Then eigen value—eigen vector relation $T(x) = \lambda x$ is transformed into $A(x) = \lambda x$.

Where (say) $(A) = ((a_{ij}))_{n \times n}$, x = I column matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ x_i, are Scalers and λ is a scalar (eigen-value). Note that X a non-zero matrix. Matrix equation above is $(A - \lambda I)\, X = 0$.

Where r.h.s. zero matrix $0 = \left.\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\}$ n places and I = unit matrix of order n. This

is a homogeneous system of n linear equations in n unknowns $x_1, x_2, ..., x_n$ (co-ordinates of vector X). In order that this system admits of a non-zero solution we have det $(A - \lambda I) = 0$.

$$\text{or,} \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

This equation (1) in λ is called characteristic equation of A| with corresponding polynomial in λ of degree n known as characteristic polynomial of A.

Theorem 5.1.1. Eigen values of $A = ((a_{ij}))_{n\times n}$ are solutions of the characteristic equation (1) and Linear operator T has an eigen value.

Proof : Fundamental Theorem of algebra says that equation (1) has always a solution and second part of Theorem is thus taken care of. Since a polynomial equation with degree of the polynomial n has exactly n roots and no more in complex scalar if follows that A has at most n | numerically different eigen-values.

Example 5.1.1. Find eigen values and eigen vectors of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a and b are real with $b \neq 0$.

Solution : Let λ be a (complex) scalar such that

$$\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0$$

or, $(a - \lambda)^2 + b^2 = 0$ or, $(a - \lambda)^2 = - b^2 = i^2d^2$

$\therefore \quad a - \lambda = \pm ib$

$\therefore \quad \lambda = a \pm ib$

So eigen values are $a - ib$ and $a + ib$.

If $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = (a-ib)\begin{pmatrix} z \\ w \end{pmatrix}$. $\begin{pmatrix} z \\ w \end{pmatrix}$ being a non-zero vector.

We have $az + b\omega = (a - ib)z$

and $-bz + a\omega = (a - ib)\omega$

Now $az + b\omega = az - ibz$ gives $\omega = -iz$ $(b \neq 0)$

and $-bz + a\omega = a\omega - ib\omega$ $z = i\omega$ $(b \neq 0)$

Therefore eigen-vectors corresponding to eigen value $= a - ib$ is given $(z, \omega) = z(1, -i)$. Similarly we show that corresponding to eigen value $= a + ib|$, we have eigen vector $= (z, \omega) = z(1, i)$.

Therefore eigen values and corresponding eigen–vectors are given as $a - ib$, $(1, -i)$, and $a + ib$, $(1, i)$

We have seen that a Linear operator $T : X \to X$ is represented by a square matrix of size n,(X is a NLS with Dim $(X) = n$), and this matrix changes as basis taken in X changes.

Theorem 5.1.2. Let $T : X \to X$ (X a NLS with Dim $(X) = n$) be a Linear operator. Then all matrices representing T corresponding to various basis in X have the same eigen values.

Proof : Let $E = (e_1, e_2, ..., e_n)$ and $F = (f_1, f_2, ..., f_n)$ be two basis for X. Then each f_1 can be written as

$$f_1 = \sum_{j=1}^{n} \alpha_{ij} e_j, i = 1,2,\ldots,n$$

$$\text{So } F = (f_1, f_2, ..., f_n) = \left(\sum_{j=1}^{n} \alpha_{1j} e_j, \sum_{j=1}^{n} \alpha_{2j} e_j, \ldots, \sum_{j=1}^{n} \alpha_{nj} e_j \right)$$

$$= (e_1, e_2, ..., e_n) \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$= EC$, where C is a non-singular matrix.

If $x \in X$ we write $x = \sum_{j=1}^{n} \xi_j e_j$ and $x = \sum_{k=1}^{n} \eta_k f_k$

or, $x = Ex_1$ where $x_1 = (\xi_1, \xi_2, ..., \xi_n)$| and similarly,

$$x = \sum_{k=1}^{n} \eta_k f_k = Fx_2 \text{ where } x_2 = (\eta_1, \eta_2, ..., \eta_n)$$

Thus $x = Ex_1 = Fx_2 = ECx_2$

Thus $x_1 = Cx_2$

Similarly, $T(x) = y = Ey_1 = Fy_2$ we have

$y_1 = Cy_2$.

Now $T(e_j)$ $j = 1, 2, ..., n$ may be expressed as a Linear combination of E. So

$$T(e_j) = \sum_{j=1}^{n} n_{ij}, e_i, j = 1, 2, ... n.$$

Therefore we have from above

$$Ey_1 = T(x) = T\left(\sum_{i=1}^{n} \xi_j, e_i\right) = \sum_{j=1}^{n} \xi_j T(e_i)$$

$$= \xi_1 T(e_1) + \xi_2 T(e_2) + ... + \xi_n T(e_n)$$

$$= \xi_1 \sum_{i=1}^{n} \eta_{i1} e_i + \xi_2 \sum_{i=1}^{n} \eta_{i2} e_i + ... + \xi_n \sum_{i=1}^{n} \eta_{in} e_i$$

$= EB_1 x_1$ where B is the matrix representing T with repsect to E.

Thus $y_1 = B_1 x_1$. Similarity, if B_2 denotes the matrix the represents T with respect to basis F then $y_2 = B_2 x_2$.

Therefore $y_1 = T_1 x_1$ and $y_2 = T_2 x_2$.

Thus, $CB_2 x_2 = Cy_2 = y_1 = B_1 x_1 = B_1 C x_2$.

Multiplying by C^{-1} one obtains

$$B_2 = C^{-1} B_1 C$$

with the help of this, we verify that the characteristic determinant of B_1 and B_2 are equal and from this it follows that eigen values of B_1 and B_2 are equal. As C^{-1} det C = Identity matrix I, we have det $(B_2 - \lambda I)$ = det $(C^{-1} B_1 C - \lambda C^{-1} IC)$.

$= \det (C^{-1}(B_1 - \lambda I)IC)$

$= \det (C^{-1}) \det (B_1 - \lambda I) \det C$

$= \det (B_1 - \lambda I).$

Theorem 5.1.3. (Existence Theorem) : Every Linear operator on a finite Dimensional NLS $\neq \{\underline{O}\}$ has one eigen value. Theorem 5.1.3 is now clear and no proof need be given.

§ 5.2 Spectral Theorem.

Let H be a Hilbert space with complex scalars and let Dim $(H) < \infty$. Without loss of generality let H be non-null and $T : H \to H$ be a (bounded) Linear operator. Suppose $\lambda_1, \lambda_2, ..., \lambda_m$ are all the distint eigen values of T and $M_1, M_2, ..., M_m$ are the corresponding eigen spaces. Let $P_1, P_2, ..., P_m$ be the projections with corresponding sub-spaces $M_1, M_2, ..., M_m$.

Theorem 5.2.1. (Spectrum Theorem). Following statements are equivalent :-

I. The eigen spaces $M_1, M_2, ..., M_m$ are pairwise orthogonal such that linear hull $[M_1, M_2, ..., M_m] = H$

i.e., $H = M_1 \oplus M_2 \oplus ... \oplus M_m$

II. The projection P_i's are pairwise orthogonal and $P_1 + P_2 + ... + P_m = I$ (Identity operator) and $\lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_m P_m = T$.

III. $T^*T = TT^*$

Proof : I $\Rightarrow$ II. Let I hold. Since M_i's are pairwise orthogonal, Projections P_i's are orthogonal. Now we write every member $x \in H$ uniquely as

$$x = x_1 + x_2 + ... + x_m \text{ where } x_i \in M_i, \text{ and } x_i \perp x_j \ (i \neq j). \qquad ...(1)$$

Therefore, $T(x) = T(x_1) + T(x_2) + ... + T(x_m)$

$$= \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_m x_m. \qquad ..... (2)$$

We have $P_i(x) = x_i \ (i = 1, 2, ..., m)$. So

$I(x) = x \quad = x_1 + x_2 + ... + x_m$

$\qquad = P_1(x) + P_2(x) + ... + P_m(x)$

$= (P_1 + P_2 + ... + P_m)(x)$. This is true all $x \in H$. Therefore, $I = P_1 + P_2 + ... + P_m$

$$= \sum_{i=1}^{n} P_i$$

Form (2), $T(x) = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_m x_m$

$$= \lambda_1 P_1(x) + \lambda_2 P_2(x) + ... + \lambda_m P_m(x)$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_m P_m)(x)$$

This being true for all $x \in H$, we produce

$$T = \lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_m P_m = \sum_{i=1}^{m} \lambda_i P_i \qquad ..... (3)$$

This is what was wanted in II

II ⇒ III Suppose II holds; from (3) we have

$$T^* = \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + ... \overline{\lambda_m} P_m, \text{ Therefore}$$

$$TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_m P_m)(\overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + ... \overline{\lambda_m} P_m)$$

$$= \lambda_1 \overline{\lambda_1} P_1^2 + \lambda_2 \overline{\lambda_2} P_2^2 + ... + \lambda_m \overline{\lambda_m} P_m^2. \text{ because } P_i P_j = 0 \ (i \neq j)$$

$$= |\lambda_1|^2 + P_1 + |\lambda_2|^2 P_2 + ... + |\lambda_m|^2 P_m.$$

That shows $TT^* = T^*T$ and III is established.

III ⇒ I This will be proved by help of following Lemimas.

Lemma 5.2.1. If x is an eigen vector of T with eigen-value λ, then x is an eigen vector of T^* with eigen-value $\overline{\lambda}$ and conversely. This result is known.

Lemma 5.2.2. M_j 's are orthogonal.

This is also a known result.

Lemma 5.2.3. Each M_i 's is invariant under T and T^*.

Proof : Let $x \in M_i$, then $T(x) = \lambda_i x$, as M is sub-space $\lambda_i x \in M_i$ and $T(M_i) \subset M_i$. Again $T(x) = \lambda_i x$ gives $T^*(x) = \overline{\lambda_i} x|$ which is also a member of M_i as M_i is a sub-space. Therefore $T^*(M_i) \subset M_i$ Lemma is proved.

Now $M = M_1 + M_2 + ... + M_m$ is a closed sub-space of H and corresponding projection on M is $P = P_1 + P_2 + ... + P_m$ Since each M_i is invariant under T (and under

T^*), we have $TP_i = P_iT$ for each i and $TP = T\left(\sum_{i=1}^{m} P_i\right) = \sum_{i=1}^{m} TP_i = \sum_{i=1}^{m} P_iT = PT$ and therefore M remains invariant under T and under T^*.

So $\quad M^{\perp}$ is invariant under T i.e., $T(M^{\perp}) \subset M^{\perp}$.

we now check that $M^{\perp} = \{O\}$

Suppose no. As $M^{\perp}$ is a closed sub-space $M^{\perp}|$ is a non-null Hilbert space with dim $(M^{\perp}) < \infty$. Now all eigen vectors of T belong to M, so restriction $\frac{T}{M^{\perp}}$ has neither eigen vector nor $\frac{T}{M^{\perp}}$ has an eigen-value :— a contradiction of existence theorem as in Theorem 5.1.3. Therefore proof is now complete.

§ 5.3 Banach Algebra :

In Banach Algebra two apparently diverse trains of thought—topological and algebraic are fused into a single Mathematical system.

Definition 5.3.1. An algebra X over a field of real | complex scalars is a vector-space where multiplication is defined subject to :—

(1) $(xy)z = x(yz)$ for any there elements $x, y, z \in X$.

(2*a*) $x(y + z) = xz + yz$

and (2*b*) $(x + y)z = xz + yz$ for any there elements $x, y, z \in X$,

(3) $a(xy) = (ax)y = x(ay)$ for any scalar α and any element $x, y, z \in X$.

We shall generally deal with complex field C of scalars and term X as Algebral (over $\mathbb{C}$).

An algebra X is said to be commutative if multiplication operation in X is commutative *i.e.*, for all $x, y \in X$, $xy = yx$.

X is said to be an Algebra with identity if there is a member e called the Identity in X such that $xe = ex = x$ holds for all $x \in X$.

It is not difficult to see that identity element e in X is unique.

Definition 5.3.2. and algebra X is said to be a Banach algebra if X is Banach space (over $\mathbb{C}$) which is also an algebra as per Definition 5.3.1. such that for all x, $y \in X$,

$$\|xy\| \leq \|x\| \, \|y\|$$

And if X has the identity e then $\|e\| = 1$,

Example 5.3.1.(a) The space R of reals and space $\mathbb{C}$ of complex numbers are examples of Commutative Banach algebra with identity $e = 1$, the norm being taken as the usual norm of $R|\mathbb{C}$.

(b) The Banach space $C[a, b]$ of all real-valued continuous functions over the closed interval $[a, b]$ with Sup norm is a Cimmutative Banach algebra with identity e = Constant function equal to 1 in $[a, b]$ and with multiplication defined as usual, namely

$$(xy)(t) = x(t)x(t) \text{ in } a \leq t \leq b \text{ as } x, y \in C\,[a, b].$$

Example 5.3.2. Let X be a Banach space $(\neq \{\underline{O}\})$. Then the collection $Bd\mathcal{L}(X, X)$ of all bounded Linear operators : $X \rightarrow X$ forms a Banach algebra with identity.

Solution : We know that $Bd\mathcal{L}(X, X)$ forms a Normal Linear space (NLS) with operator norm $\|T\|$ as $T \in Bd\mathcal{L}(X, X)$ and sicne X is a Banach space $Bd\mathcal{L}(X, X)$ is also a Banach space. Further, if $T_1, T_2 \in Bd\mathcal{L}(X, X)$ defining multiplication $T_1T_2 : X \rightarrow X$ as $(T_1T_2)(x) = T_1(T_2(x))$ as $x \in X$, it is a routine exercise to check that all axioms of algebra as in Definition 5.3.1. above are satisfied here. Further, the Identity operator I becomes the Identity element of this Banach algebra $Bd\mathcal{L}(X, X)$.

Remark : In general, $Bd\mathcal{L}(X, X)$ may not be commutative. Take the case when X = Eucildean n-space R^n which is a Banach space with respect to usual norm. We find by matrix representation theorem that every member of $Bd\mathcal{L}(R^n, R^n)$ is represented by a square matrix of size n over reals, because matrix multiplication is not commutative we conclude that here commutativity fails in Banach algebra $Bd\mathcal{L}(R^n, R^n)$

Theorem 5.3.1. Multiplication operation in a Banach algebra X is a continuous operation.

Proof : Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim\limits_{x\to\infty} x_n = x$ and $\lim\limits_{x\to\infty} y_n = y$ in norm of X.

i.e., $\lim\limits_{x\to\infty}\|x_n - x\| = 0 = \lim\limits_{x\to\infty}\|y_n - y\|$

Now $x_n y_n - xy = (x_n - x)\, y_n + x(y_n - y)$;

So $\|x_n y_n - xy\| \le \|(x_n - x)y_n\| + \|x(y_n - y)\|$

$$\le \|y_n\|\,\|x_n - x\| + \|y_n - y\|.$$

Since $\{y_n\}$ is a convergent sequence in X, we know that it is bounded and Let $\|y_n\| \le M$ for all n, So above reads as

$$\|x_n y_n - xy\| \le M\|x_n - x\| + \|x\|\,\|y_n - y\| \to 0 \text{ as } n \to \infty$$

so $\lim\limits_{x\to\infty}(x_n y_n) = xy$. Hence theorem is proved.

Definition 5.3.3. An element x in a Banach algebra X with identity e is said to be invertible if x^{-1} exists. i.e. if $x^{-1} \in X$ satisfying $x^{-1}\, x = x.x^{-1} = e$.

Otherwis, x is said to be a non-inveritble element in X.

Explanation : (a) If inverse of x exists in X_1, then it is unique. Because, suppose $yx = e = xz$, then we have

$y = ye = y(xz) = (yx)z = ez = z.$

(b) If x and are y both invertible then xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$.

Because, $(xy)(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xex^{-1} = xx^{-1} = e$.

and similarly $(y^{-1} - x^{-1})(xy) = e$.

From this observation one may conclude that *the set G of all invertible elements of X froms a Group.*

§ 5.4 Let X be a Banach algebra with identity e. Then there are invertible elements like e in X. Also $\underline{O} \in X$ is not inveritble element in X.

Theorem 5.4.1. Let $x \in X$ with $\|x\| < 1$. Then $e - x$ is invertible and $(e - x)^{-1} = e + \sum\limits_{j=1}^{\infty} x^j$.

Proof : By induction we have $\|x^j\| \le \|x\|^j$ for all +ve integers j. So the infinite series $\sum_{j=1}^{\infty} \|x^j\|$ is convergent because $\|x\| < 1$. Because X is also a Banach space the infinite series $\sum_{j=1}^{\infty} x^j$ is convergent to some member in X.

Put $S = e + \sum_{j=1}^{\infty} x^j$

We now show that $S = (e - x)^{-1}$. For any natural number n, we have

$$(e - x)(e + x + x^2 + \dots + x^n) = (e + x + x^2 + \dots + x^n)(e - x)$$
$$= e - x^{n+1}. \qquad \text{.....(1)}$$

Because $\|x\| < 1$ we see $\lim_{n\to\infty} x^{n+1} = \underline{O}$ Therefore, we pass on $\lim_{n\to\infty}$ in (1) and because multiplication is continuous we have

$$(e - x)S = S(e - x) = e.$$

This gives $S = (e - x)^{-1}$

Corollary 1. If $x \in X$ with $\|e - x\| < 1$, then x^{-1} exists and $x^{-1} = e + \sum_{j=1}^{\infty} (e - x)^j$.

In Theorem 5.4.1. replace x by $e - x$ then

Theorem 5.4.1 says $(e - (e - x))^{-1} = \sum_{j=1}^{\infty} (e - x)^j$ then

or, $x^{-1} = e + \sum_{j=1}^{\infty} (e - x)^j$.

Corollary 2. Let $x \in X$ and λ be a scalar such that $\|x\| < |\lambda|$. Then $(\lambda e - x)^{-1}$ exists and

$$\lambda(e - x)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} \chi^{n-1} \quad (x^0 = e).$$

Proof : Write $\lambda e - x = \lambda\left(e - \frac{x}{\lambda}\right)$ By Corollay 1 above

$\left\| e - \left(e - \frac{x}{\lambda}\right) \right\| = \left\| \frac{x}{\lambda} \right\| = \frac{1}{|\lambda|}\|x\| < 1$. Therefore

$\left(e-\frac{x}{\lambda}\right)^{-1}$ exists. So $(\lambda e - x)^{-1}$ exists. Then

$$\lambda e - x = \lambda\left(e-\frac{x}{\lambda}\right) \text{ and } (\lambda e - x)^{-1}) = \lambda^{-1}\left(e-\frac{x}{\lambda}\right)^{-1}$$

$$= \lambda^{-1}\left(e+\sum_{n=1}^{\infty}\left[e-\left(e-\frac{x}{\lambda}\right)\right]^{n}\right) = \lambda^{-1}\left[e+\sum_{n=1}^{\infty}(\lambda^{-1}x)^{n}\right]$$

$$= \sum_{n=1}^{\infty}\lambda^{-n}x^{n-1}$$

Theorem 5.4.2. The set G of all invertible elements of X is an open set.

Proof : Let $x_0 \in G$. We find an open ball centered at x_0 so that open ball $\subset G$.

Consider the open ball $B_r(x_0)$ with radius $r = \frac{1}{\|x_0^{-1}\|}$. Then $x \in B_r(x_0)$ if and only if $= \|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$

Let $y = x_0^{-1}x$ and $z = e - y$, Then from (1) we have

$$= \|z\| = \|e-y\| = \|y-e\| = \|x_0^{-1}x - x_0^{-1}x_0\| = \|x_0^{-1}(x-x_0)\|$$

$\leq \|x_0^{-1}\| \; \|x - x_0\| < 1$. Now Theorem 5.3.1. applies and $e - z$l is invertible i.e. y is invertible. So $y \in G$. Now $x_0 \in G$ and $y \in G$, and because G forms a Group, $x_0y \in G$; Now $x_0y = x_0x_0^{-1}x = x \in G$,l and therefore $B^r(x_0) \subset G$ i.e., x_0 is an interior point of Gl and G is shown to be open.

Corollary : The set of all non-invertible elements of X forms a closed set in X.

Theorem 5.4.3. The mapping of taking inverse i.e., $x \to x^{-1} : G \to G$ is continuous.

Proof : Suppose $x_0 \in G$. Consider the set $B_r(x_0) \cap G$ Where $B_r(x_0) =$ l open ball centered at x_0 with radious $r = \frac{1}{2\|x_0^{-1}\|}$.

Now take any $x \in (B_r(x_0) \cap G)$.

Then $\|x_0^{-1}x - e\| = \|x_0^{-1}(x - x_0)\| \leq \|x_0^{-1}\| \; \|x - x_0\| < \frac{1}{2}$ ..... (1)

Therefore $x_0^{-1}x$ is invertible i.e. $x_0^{-1}x \in G$. Further

$$x^{-1}x_0 = (x_0^{-1}x)^{-1} = e + \sum_{n=1}^{\infty}(e - x_0^{-1}x)^n \qquad \text{..... (2)}$$

Now $\|x^{-1} - x_0^{-1}\| = \|(x^{-1}x_0 - e)x_0^{-1}\| \le \|x_0^{-1}\| \, \|x^{-1}x_0 - e\|$

$$= \|x_0^{-1}\| \left\| \sum_{n=1}^{\infty}(e - x_0^{-1} - x)^n \right\| \text{ from (2)}$$

$$= \|x_0^{-1}\| \sum_{n=1}^{\infty}\|e - x_0^{-1}x\|^n = \|x_0^{-1}\| \, \|e - x_0^{-1}x\| \sum_{n=1}^{\infty}\|e - x_0^{-1}x\|^n$$

$$= \frac{\|x_0^{-1}\| \, \|e - x_0^{-1}x\|}{1 - \|e - x_0^{-1}x\|}$$

$\le 2\|x_0^{-1}\|^2 \, \|x_0 - x\|$ (since $\|e - x_0^{-1}x\| = \|x_0^{-1}x - e\| = \|x_0^{-1}(x - x_0^{-1})\|$ $\|x - x_0)\|$ and from (1)

Thus $= \|x^{-1} - x_0^{-1}\| < 2\|x_0^{-1}\|^2 \|x - x_0\|$ gives the continuity of the concerned mapping at x_0. The proof is now complete.

Corollary : The mapping : $G \to G$ given by $x \to x^{-1}$ as $x \in G$ is a homemorphism onto itself.

Definition 5.4.1 (Topological divisor of zero).

An element z is X called a Topology divisor of zero if there is a sequence $\{z_n\}$ of elements z_n in X with $\|z_n\| = 1$ such that either $\lim_{n\to\infty} zz_n = \underline{O}$ or, $\lim_{n\to\infty} z_n z = \underline{O}$.

Explanation : Every divisor of $\underline{O}$| is fo course a Topological divisor of $\underline{O}$. We have denoted the set of all invertible elements in X by G. If Z denotes set of all topological divisor of $\underline{O}$ in X, then presently we see that there is connection between Z and the set $(X \mid G)$ = set of all non-invertible elements in X.

Theorem 5.4.4. Z is subset of $(X \mid G)$ $(Z \subset (X \mid G))$.

Proof : Let $z \in Z$, then we find a sequence $\{z_n\}$ in X with $\|z_n\| = 1$ such that either $\lim_{n\to\infty} zz_n = \underline{O}$ or, $\lim_{n\to\infty} z_n z = \underline{O}$. If possible, let $z \in G$, So $z^{-1} \in G$. Now multiplication is continuous, we have

$$z_n = z^{-1} z.z_n = z^{-1}(zz_n) \to z^{-1}\underline{O} = \underline{O} \text{ as } n \to \infty.$$

This contradicts assumption that $\|z_n\| = 1$ for all n. Therefore we have shown that $z \in (X \setminus G)$.

Theorem 5.4.5. Boundary $(X \setminus G) \subset Z$.

Proof : Since G is open $(X \setminus G)$ is a closed set in X.

So Boundary $(X \setminus G) \subset (X \setminus G)$, Further, if $u \in Bdry(X \setminus G)$. We find a sequence $\{u_n\}$ of elements $u_n \in (X \setminus (X \setminus G)) = G$ such that $\lim_{n\to\infty} u_n = u$.

Now $u_n^{-1} u - e = u_n^{-1} (u - u_n)$ ..... (1)

If $\{\|u_n^{-1}\|\}$ is bounded, then because $\lim_{n\to x} u_n = u$, from (1) it follows that for large values of n.

$$\|u_n^{-1}u - e\| < 1$$

and this would imply $(u_n^{-1} u) \in G$, and hence

$u_n(u_n^{-1} u)$ i.e., $u \in G$—a contradiction that $u \in (X \setminus G)$ hence $\{\|u_n^{-1}\|\}$ is not bounded. We may take $\lim_{n\to\infty} \|u_n^{-1}\| = \infty$ Put $\vartheta_n = u_n^{-1}/\|u_n^{-1}\|$, So $\|\vartheta_n\| = 1$. and

$$u\vartheta_n = \frac{u\,u_n^{-1}}{\|u_n^{-1}\|} = \frac{e+(u-u_n)u_n^{-1}}{\|u_n^{-1}\|} = \frac{e}{\|u_n^{-1}\|} + (u-u_n)v_n$$

Now $\lim_{n\to\infty}\|u_n^{-1}\| = \infty$ and $\lim_{n\to\infty} u_n = u$ with $\|\vartheta_n\| = 1$, we see from above lim $u\vartheta_n = \underline{O}$.

That means $u \in Z$, Hence we have shown Boundary $(X \setminus G) \subset Z$.

§ 5.5 Resolvant set; Spectrum

Here also we take X to be a Banach algebra with identity e, and take $x \in X$.

Definition 5.5.1 (a) The resolvant set $\rho(x)$ of x is equal to the set of all scalars $\lambda \in ⊄$ such that $x - \lambda e$ is invertible. i.e., $\rho(x) = \{\lambda \in ⊄ : (x - \lambda e)$ exists in $X\}$.

(b) The complement $⊄ | P(x) = \{\lambda \in ⊄ : (x - \lambda e)^{-1}$ does not exist in $X\}$ is called the spectrum of x, denoted by $\sigma(x)$.

Any scalar $\lambda \in \sigma(x)$ is called a spectral value of x. So we have $\rho(x) \cup \sigma(x) = \mathbb{C}$ with $\rho(x) \cap \sigma(x) = \phi$.

For a fixed $x \in X$,

Consider the mapping : $\rho(x) \to X$ given by

$\lambda \in \rho(x)$, $\lambda \to (x - \lambda e)^{-1} \in X$. We write $x(\lambda) = (x - \lambda e)^{-1}$

Therefore, the mapping $\lambda \to x(\lambda)$ is well-defined and this function $x(\lambda)$ is called the resolvant function associated with $x \in X$. Thus a resolvant function is a Vector-valued function over $\rho(x)$ with range in a Banach algebra.

Remark : Take $\lambda_1, \lambda_2 \in \rho(x)$. Then $x(\lambda_1) = (x - \lambda_2 e)^{-1}$ and $x(\lambda_2) = (x - \lambda_2 e)^{-1}$ and

$$(x(\lambda_1))^{-1} x(\lambda_2) = (x - \lambda_1 e)\, x(\lambda_2) = ((x - \lambda_2 e) + (\lambda_2 e - \lambda_1 e))\, x(\lambda_2)$$

$$= e + (\lambda_2 - \lambda_1)\, x(\lambda_2)$$

That means, $x(\lambda_2) = x(\lambda_1) + (\lambda_2 - \lambda_1)\, x(\lambda_2) x(\lambda_1)$

$$\text{or, } \frac{x(\lambda_2) - x(\lambda_1)}{\lambda_2 - \lambda_1} = x(\lambda_1)x(\lambda_2)\ .\ (\lambda_2 \neq \lambda_2) \qquad ..... (*)$$

Theorem 5.5.1 The resolvant function $x(\lambda)$ is analytic at every point of $\rho(x)$.

Proof : Take $\lambda, \lambda_0 \in \rho(x)$ with $\lambda \neq \lambda_0$. Then from (*) above we have

$$\frac{x(\lambda) - x(\lambda_0)}{\lambda - \lambda_0} = x\,(\lambda_0)x(\lambda)$$

notice that $\lim\limits_{\lambda \to \lambda_0} x(\lambda) = \lim\limits_{\lambda \to \lambda_0} (x - \lambda e)^{-1} = (x - \lambda_0 e)^{-1} = x(\lambda_0)$.

$$\text{Thus } \lim_{\lambda \to \lambda_0} \frac{x(\lambda) - x(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} (x(\lambda_0)x(\lambda))$$

$$= x(\lambda_0) \lim_{\lambda \to \lambda_0} x(\lambda)$$

$$= x(\lambda_0))^2$$

So derivative $x'(\lambda_0)$ exists, and hence $x(\lambda)$ is analytic as wanted.

Definition 5.5.2. For $x \in X$ the spectral radius of x denoted by $r_\sigma(x) = \sup\limits_{\lambda \varepsilon \sigma(x)} |\lambda|$

Theorem 5.5.2. $r_\sigma(x) \leq \|x\|$

Proof : Take $\lambda \in \mathbb{C}$ with $|\lambda| > \|x\|$, So $\left\|\frac{x}{\lambda}\right\| < 1$ and therefore $e - \lambda^{-1} x$ is invertible.

So $x - \lambda e = -\lambda(e - \lambda^{-1} x)$ is also invertible. That means $\lambda \in \rho(x)$.

Therefore $r_\sigma(x) < \|x\|$.

Theorem 5.5.3. $\sigma(x)$ is a compact set of scalars.

Proof : Theorem 5.5.2. says $\sigma(x)$ is bounded and we need showing that it is closed. This we will proved by showing that is complement $\not{C}\backslash\sigma(x) = \rho(x)$l to be open.

Consider the function $f : \not{C} \to X$ given by

$$f(\lambda) = x - \lambda e \text{ as } \lambda \in \not{C}$$

This is a continuous function of scalar λ. Take $\lambda_0 \in \rho(x)$. So $x - \lambda_0 e$l is invertible. So $(x - \lambda_0 e) \in G$. As G is open, we find an open ball $B_r(x_0 - \lambda e)$l centered at $(x_0 - \lambda_0 e)$l with a +ve radius r such that

$$B_r (x_0 - \lambda e) \subset G.$$

since f is continuous at λ_0 we find a + veδ such that

$$\|f(\lambda) - f(\lambda_0)\| < r \text{ whenever } |\lambda - \lambda_0| < \delta.$$

i.e., $f(\lambda) = (x - \lambda e) \in B\,(x_0 - \lambda e)$ whenever $|\lambda - \lambda_0| < \delta$.

i.e., $\lambda \in \rho(x)$ whenever $|\lambda - \lambda_0| < \delta$.

Hence λ_0 is an interior point of $\rho(x)$, and $\rho(x)$l is shown as an open set.

Theorem 5.5.4. for $x \in X$, spectrum $\sigma(x) \neq \phi$.

Proof : Let $f \in X^*$. For $\lambda \in \rho(x)$, Let $f(\lambda) = f(x - \lambda e)^{-1})$,

$$= f(x(\lambda))$$

since f is continuous it follows that $f(\lambda)$l is a continuous function of λl over $\rho(x)$. We have already had

$\dfrac{x(\lambda) - x(\mu)}{\lambda - \mu} = x(\lambda)\, x(\mu)$ for any scalars λ, $\mu(\lambda \neq \mu)$. Since fl is Linear, we

have

$$\frac{f(x(\lambda) - f(x(\mu))}{\lambda - \mu} = f(x(\lambda)x(\mu))$$

i.e., $\dfrac{f(\lambda) - f(\mu)}{\lambda - \mu} = f(x(\lambda)x(\mu))$

So, $\lim_{\lambda\to\mu} \frac{f(\lambda)-f(\lambda)}{\lambda-\mu} = f(x(\mu)^2)$.

This shows that $f(\lambda)$ is analytic on $\rho(x)$. Further,

$$|f(\lambda)| = |f(x(\lambda))| \le \|f\| \ge \|x(\lambda)\|$$

$$= \|f\| \frac{1}{|\lambda|} \left\| \left(e - \frac{x}{\lambda}\right)^{-1} \right\|$$

For large value of $|\lambda|$, we have $\left\|\frac{x}{\lambda}\right\| = \frac{\|x\|}{|\lambda|} < 1$ and therefore

$\left(e - \frac{x}{\lambda}\right)^{-1} = e + \sum_{j=1}^{\infty} \left(\frac{x}{\lambda}\right)^j$, and therefore

$$= \left\| \left(e - \frac{x}{\lambda}\right)^{-1} - e \right\| \le \sum_{j=1}^{\infty} \left\|\frac{x}{\lambda}\right\|^j = \frac{\left\|\frac{x}{\lambda}\right\|}{1 - \left\|\frac{x}{\lambda}\right\|} = \frac{\left\|\frac{x}{\lambda}\right\|}{1 - \frac{\|x\|}{|\lambda|}} \to 0 \text{ as } |\lambda| \to \infty.$$

So, $\left(e - \frac{x}{\lambda}\right)^{-1} \to e$ as $|\lambda| \to \infty$.

Therefore from Step $|f(\lambda)| \le \|f\| \frac{1}{|\lambda|} \left\| \left(e - \frac{x}{\lambda}\right)^{-1} \right\|$ above

We duduce that $\lim_{|\lambda|\to\infty} |f(\lambda)| = 0$.

If $\sigma(x) = \phi$ we have $\rho(x) = ¢$ and $f(\lambda)$ becomes an entire function. So, by Lioville's theorem $f(\lambda)$ must be a constant function, and from limit above we see this constant = 0

i.e. $f(\lambda) = 0$ for all $\lambda \in ¢ = \rho(x)$

This is true for every f coming from X^*, and therefore it follows that $x(\lambda) = (x - \lambda e)^{-1} = \underline{O}$ in X for any $\lambda \in ¢$. But this is not the case. Because $\|e\| = \|(x - \lambda e)\|. x(\lambda) = \|\underline{O}\| = 0 - a$l contradiction. Therefore conclusion is that $\sigma(x) \neq \phi$.

Theorem 5.5.5. If a Banach algebra X with identity e has every non-zero member inver table, them X is isometrically isomorphic to scalar field $¢$.

(This theorem is due to Gelfand and Mazur who had left memorable marks in Advanced funciton Analysis).

Proof : Take $x \in X$. Then Theorem 5.5.4 says that $\sigma(x) \neq \phi$. So there is a scalar $\lambda \neq ¢$ such that $x - \lambda e$ is not invertible. By assumption that every non-zero element of X is invertible. Therefore $x - \lambda e = O$ or $x = \lambda e$.

Now if λ_1 and λ_2 are two scalars with $x = \lambda_1 e = \lambda_2 e$ then clearly $\lambda_1 = \lambda_2$. Hence x is unique scalar multiple of e.

Now consider the mapping $f : x \to ¢$ given by

$$f(x) = f(\lambda e) = \lambda.$$

Then f is $1-1$ and linear plus f is onto. Therefore f is the desired isomorphism as wanted.

Theorem 5.5.6 If $\underline{O}$ is the only Topological divisor of zero in X, then X is isomietrically isomorphic to scalar field $¢$.

Proof : Take $x \in X$. So $\sigma(x) \neq \phi$, $\sigma(x)$ is also bounded. Let λ be a boundary point of $\sigma(x)$. Then $x - \lambda e$ is a Topology divisor of zero (see Theorem 5.4.5). By assumption $x - \lambda e = \underline{O}$ i.e., $x = \lambda e$. Now one can copy rest of the proof as in proof of Theorem 5.5.5 to conclude that X is isomorphic to $¢$ as desired.

§ 5.6 Spectral radius formula :

Let $x \in X$, $\sigma(x)$ its spectrum. We know that $r_\sigma(x) = \sup\limits_{l \in s(x)} |\lambda|$. Presently we derive formula for $r_\sigma(x)$ like

$$r_\sigma(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$$

Theorem 5.6.1. If $p(t)$ is a polynomial with complex coefficients and $x \in X$, Then $\sigma(p(x)) = p(\sigma(x))$.

Proof : The proof proceeds by stages. First take $p(t)$ to be a constant polynomial, say $p(t) = \alpha_0 = \alpha_0 t^0$ and we have

$$\sigma(p(x)) = \sigma(\alpha_0 e) = \{\lambda : (\alpha_0 e - \lambda e)^{-1} \notin_0 X\}$$
$$= \{\alpha_0\}.$$

Now $p(\sigma(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$

$$= \{\alpha_0 \lambda_0 : \lambda \in \sigma(x)\}$$
$$= \{\lambda_0\}$$

so in this case $\sigma(p(x)) = p(\sigma(x))$.

For any vector z and any scalar α we show that

$$\sigma(\alpha z) = \alpha\sigma(z)$$

This is o.k. for $\alpha = 0$, suppose $\alpha \neq 0$. then

$$\lambda \in \sigma(\alpha z)$$

$$\Leftrightarrow \quad \alpha z - \lambda e \text{ is not inveritible}$$

$$\Leftrightarrow \quad z - \frac{\lambda}{\alpha} e \text{ is not inveritible}$$

$$\Leftrightarrow \quad \frac{\lambda}{\alpha} \in \sigma(z)$$

$$\Leftrightarrow \quad \lambda \in \alpha\sigma\ (x).$$

Let us now consider polynomial with leading coefficient equal to 1, and let

$$p(t) = t^m + \alpha_{n-1}t^m + ... + \alpha_1 t^m + \alpha_0.\ (n \geq 1).$$

and consider $p(t) - \lambda$ for any scalar $\lambda \in ¢$ since field ¢ is algebraically closed, $p(t) - \lambda$ is completely factorisable like

$$p(t) = \lambda(t - \beta_1)\ (t - \beta_2)\ ...\ (t - \beta_n) \qquad \text{l..... (1)}$$

Taking x for t, we get

$$p(x) - \lambda e = (x - \beta_1 e)(x - \beta_2 e)...(x - \beta_n e) \qquad \text{l..... (2)}$$

If $\lambda \in \sigma(p(x))$, then one of factors $x - \beta_j\ e$ must be non-invertible and in that case $\beta_j \in \sigma(x)$.

That implies $p(\beta_j) \in p(\sigma(x)) = (p(\lambda)) = p\{(\lambda) : \lambda \in \sigma(x)\}$. l..... (3)

Taking β_j for t in (1) above we see $p(\beta_j) = \lambda$ and (3) becomes

$$\lambda \in p(\sigma(x)).$$

So we have shown $p(\sigma(x)) \subset p(\sigma(x))$

To obtain opposite inclusion, suppose $\lambda \in p(\sigma(x))$, by definition of $p(\sigma(x))$, there is $\gamma_j \in \sigma(x)$ such that

$\lambda = p(\gamma_j)$ Now from $p(t) - \lambda = (t - \gamma_1)\ .....\ (t - \gamma_j)\ ...\ (t - \gamma_n)$, it being clear that γ_j is a root of $p(t) - \lambda$ Taking x for t we obtain

$$p(x) = \lambda e = (x - \gamma_1 e)\ ...\ (x - \gamma_j e)\ ...\ (x - \gamma_n e) \qquad \text{..... (4)}$$

if $\lambda \notin \sigma(p(x))$ that is, if $p(x) - \lambda e$ were invertible, we could multiply both sides of (4) on left by $(p(x) - \lambda e)^{-1}$ and move $(x - \gamma_j e)$ all the way to the right to get

$$e = (p(x) - \lambda e)^{-1} [(x - \gamma_1 e) \ldots (x - \gamma_n e)] (x - \gamma_j e) \qquad \ldots\ldots (5)$$

to conclude that $(x - \gamma_j e)$ has left inverse. Similarly we show that $(x - \gamma_j e)$ has right inverse—a contradiction that $\gamma_j \in \sigma(x)$. We therefore conclude that $\lambda \in \sigma(p(x))$ and that implies.

$$p(\sigma(x)) \subset \sigma(p(x))$$

And the proof is now complete.

Corollary : $\sigma(x^n) = (\sigma(x))^n$ for any +ve integer n.

Next consider, $r_\sigma(x^n) = \sup\limits_{\lambda \in \sigma(x^n)} |\lambda| = \sup\limits_{\lambda \in (\sigma(x^n))^n} |\lambda|$

$$= \sup_{\lambda \in \sigma(x)} |\mu|^n$$

$$= \left(\sup_{\lambda \in \sigma(x)} |\mu| \right)^n$$

$$= (r_\sigma(x))^n$$

Of course $r_\sigma(x^n) \leq \|x^n\|$

or, $\quad (r_\sigma(x))^n \leq \|x^n\|$

or, $\quad (r_\sigma(x) \leq \|x^n\|^{1/n}$

This gives $r_\sigma(x) \leq \underline{\lim}_{n \to \infty} \|x^n\|^{1/n}$ (*)

Since inferior limit of a Sequence is ≤ superior limit,

if we can show that $r_\sigma(x) = \overline{\lim}_{n \to \infty} \|x^n\|^{1/n}$. ..... (**)

We at once have $= \overline{\lim}_{n \to \infty} \|x^n\|^{1/n} \leq \underline{\lim}_{n \to \infty} \|x^n\|^{1/n}$ from (*)

And this implies $\lim\limits_{n \to \infty} \|x^n\|^{1/n}$ exists and

$$r_\sigma(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Now (**) is obtained by computing the radius of convergence of a power series via Cauchy-Hadamard formula.

Theorem 5.6.2. Prove that following statements are equivalent.

(a) $\|x^2\| = \|x^2\|$ for all $x \in X$,

and (b) $r_\sigma(x) = \|x\|$ for all $x \in X$,

Proof : (a) ⇒ (b). Let (a) hold. Then we have $\|x^4\| = \|(x^2)^2\| = \|x^4\|$ and by induction $\|x^{2k}\| = \|x\|^{2k}$ for +ve integers k.

$$\text{Now } r_\sigma(x) = \lim_{n\to\infty}\|x^n\|^{1/n} \text{ (See above).}$$

$$= \lim_{k\to\infty}\|x^{2k}\|^{1/2k} = \|x\| \text{ and this is (b).}$$

(b) ⇒ (a). Suppose (b) holds. Then $\|x^2\| = r_\sigma(x)^2$

$$= (r_\sigma(x)^2 = \|x\|^2 \text{ which is (a).}$$

Example 5.6.1 In a Banach algebra X with identity e if $x \in X$ satisfies $\|x\| < 1$, Show that

$$\|(e-x)^{-1} - e - x\| \le \frac{\|x\|^2}{1-\|x\|}$$

Solution : Since $\|x\| < 1$. $e - x$ is inverible and $(e-x)^{-1} = e + \sum_{j=1}^{\infty} x^j$.

$$\text{or, } (e-x)^{-1} - e - x = \sum_{j=2}^{\infty} x^j = \lim_{n\to\infty}\sum_{j=2}^{\infty} x^j$$

$$\therefore \|(e-x)^{-1} - e - x\| = \left\|\lim_{n\to\infty}\sum_{j=2}^{n+1} x^j\right\| = \lim_{n\to\infty}\left\|\sum_{j=2}^{n+1} x^j\right\|$$

$$\le \lim_{n\to\infty}\sum_{j=2}^{n+1}\|x\|^j \le \|x\|^2 + \|x\|^3 + \ldots.$$

$$= \|x\|^2(1+\|x\|+\ldots.) = \frac{\|x\|^2}{1-\|x\|},$$ the convergence of infinite series being taken care of $\|x\| < 1$.

Example 5.6.2. Let X be a Banach algebra with identity e. If $x \in X$ and there are $y, z \in X$ such that $yx = e$ and $xz = e$, show that x is invertible and $y = z = x^{-1}$.

Solution : Here $y = ye$

$= yxz$

$= ez$

$= z$

Therefore $yx = e$

$= xz$

$= xy$

Showing x has an inverse equal to y (hence $= z$) i.e., $y = z = z^{-1}$.,

EXERCISE-A

Short answer Type Question

1. If x is an invertible element of a Banach algebra X with identiti e such that x commutes with $y \in X$, show that x^{-1} commutes y.

   Here $xy = yx$; So $x^{-1}xy + x^{-1}yx$; or, $ey = x^{-1}yx$

   or, $y = x^{-1}yx$ or, $yx^{-1} = x^{-1}yxx^{-1}$ or, $yx^{-1} = x^{-1}yx = x^{-1}-y$

   Thus x^{-1} and y commute).

2. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a Banach algebra X, then show that $\{x_n y_n\}$ is a Cauchy sequence X.

3. Let X be a Banach space, for the Identity operator $1 : X \to X$ and $\sigma(1)$.

4. Obtain (i) the eigen values and (ii) eigen vectors of $\begin{pmatrix} 1 & 2 \\ -8 & 11 \end{pmatrix}$

EXERCISE-B

1. Show that eigen vectors $x_1, x_2, \ldots, x_n$ corresponding to different eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a linear operator T on a Normed Linear space X form a linearly independent set.

2. Show that collection of all Lincar operators on a vector-space into itself forms of algebra.

3. In a Banach algebra X with identity e if invertible and yl satisfics $\|yx^{-1}\| < 1$.

   Show that $x - y$ is invitable and $(x-y)^{-1} = \sum_{j=0}^{\infty} x^{-1}(yx^{-1})^j$

4. If G is the set of all invertible clements of a Banach algebra X with identity e and $x \in G$ and $h \in X$ satisfies.

   $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$, then prove that $(x + h) \in G$ and

   $\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \le 2\|x^{-1}\|^3 \|h\|^2$

5. Let $\wp_{n+1}$ denote the vector space of all polynomials with complex coefficients of degree $\le n$(n is a +ve integer), examine if $\wp_{n+1}$ is a Banach algebra with norm. to be specified by you.

Unit-6

(**Contents** : Weak topology, weak* topology, Banach-Alaogulu Theorem)

§ 6.1 In a Normed Linear space (NLS) (X, $\| \|$) we have seen that there are two Topologies, namely Norm Topology $\tau_{\| \|}$ arising out of the norm $\| \|$ in X and weak Topology τ_ω governed by weak convergence in X as described earlier.

Let X^* be the conjugate space of X consisting of all bounded Linear functional f over X. For $x_0 \in X$ and $\varepsilon > 0$, define

$V(x_0; f_1, f_2, ...f_n, \varepsilon) = \{x \in X : | f_i(x) - f_i(x_0) | < \varepsilon\}$, where

$\{f_1, f_2, ..., f_n \in X^*\}$, and n is not fixed.

Then collection of all such members $V(x_0; f_1, f_2, ...f_n, \varepsilon\}$,| We may check, forms a base for a Topology τ_ω Known as weak Topology on X, generated by members of X^*.

Since each member $f_i \in X^*$ is continuous, it follows that member $V(x_0; f_1, f_2, ...f_n, \varepsilon)$ are all open sets of $\tau_{\| \|}$ in X. So we have

$\tau_\omega \subset \tau_{\| \|}$. Hence the name 'Weak Topology' τ_ω for X.

It is now an exercise to verify that notion of weak convergence as described through weak convergence of a sequence in X i.e., $\omega - \lim_{n \to \infty} x_n = x_0$ in X actually coincides with convergence notion arising out of τ_ω in X.

To this end let $\{x_n\}; x_n \in X$ be convergent to x_0 relative to τ_ω |. That means every member of τ_ω that contains x_0 shall include all but a finite number of members of $\{x_0\}$.

Therefore $V(x_0; f, \varepsilon)(\varepsilon > 0, f \in X^*)$ contains all but a finite number of members of x_n.

So $| f(x_n) - f(x_0) | < \varepsilon$ for sufficiently large n. That means, $\omega - \lim_{n \to \infty} x_n = x_0$ as per definition of weak convergence.

Conversely, Let $\omega-\lim_{n\to\infty} x_n = x_0$ in X

Take an open set $0 \in \tau_\omega$ such that $x_0 \in 0$.

As we have seen right above that sets of the form $V(x_0, f_1, f_2, ...f_n, \varepsilon)$ form a basis for τ_ω, we find $V(x_0; f_1, f_2, ..., f_n, \varepsilon) \subset X$,| satisfying

$$V(x_0; f_1, f_2, ..., f_m, \varepsilon) \subset 0.$$

Now $\omega-\lim_{n\to\infty} x_n = x_0$ says that for sufficiently large n.

$|f_i(x_n) - f_i(x_0)| < \varepsilon$ for all $i = 1, 2, ..., m$; showing thereby

$x_n \in V(x_0; f_1, f_2, ...f_m, \varepsilon)$, for sufficiently large n.

Therefore $\lim_{n\to\infty} x_n = x_0$ relative to τ_ω, and the exercise is over.

We shall employ suffices s and w relatively to strong or norm Topology $\tau_{\|\ \|}$ and weak Topology τ_ω on X respectively for convergence or closure / interior etc.

Suppose E is a subset of X, then $\bar{E}$ and $\bar{E}^\omega$ shall be respectively strong and weak closure of E.

Thus if $\bar{E}^\omega - E$ then E, is weakly closed and hence $X \setminus E$ is weakly open. Because $\tau_\omega \subset \tau_{\|\ \|}$ we see that $X \setminus E$ is strongly open. So E is strongly closed

i.e., $\bar{E}^s - E$.

Hence $\bar{E}^\omega = E$ implies $\bar{E}^s - E$. However converse is not true. However we have the following theorem.

Theorem 6.1.1. A sub-space M of a NLS X is weakly closed if and only if it is strongly closed.

Proof : In a NLS X we know that every weakly closed strongly closed. So it remains to check that if M be strongly closed, then M is weakly closed.

i.e., if $\overline{M}^s = M$, we show that $\overline{M}^\omega \subset \overline{M}^s$.

That is to say, we show that $(X \setminus \overline{M}^\omega) \supset (X \setminus \overline{M}^s) = X \setminus M$

i.e., $(X \setminus M) \subset (X \setminus \overline{M}^\omega)$.

Take $x \notin \overline{M}^s (= M)$; Then dist $(x, M) > 0$. By Hahn-Banach Theorem we find a member $f \in X^*$ satisfying

$f(M) = \{0\}$ and $f(x) \neq 0$.

Now the sets $\{y \in X : f(y) > 0\}$ and $\{y \in X : f(y)|$ are each open in $\tau_\omega|$ i.e., the set $\{y \in X : f(y) \neq 0\}$ is τ_ω -open, say $= O_\omega$ and here $O_\omega \cap M = \phi$.

Thus here $O_\omega \in \tau_\omega$ with $x \in O_\omega$; but $O_\omega \cap M = \phi$, because $f(M) = \{0\}$. Therefore $x \notin \overline{M}^s$. Thus we have shown

$(X \setminus M) \subset X \setminus \overline{M}^\omega)$ and the proof ends.

§ 6.2. We have seen that under the Cononical mapping $J : X \to X^{**}$,| where X^{**}| is the second conjugate space of X. We have $J(X) \subset X^{**}$| and in general, this inclusion is proper i.e., $J(X) = X^{**}$. Now look at the first conjugate space X^*| that have (i) the usual norm Topology τ through operator norm $\| f \|$ as $f \in X^*$. (ii) Weak Topology τ_ω as induced by all members $\varepsilon(X^*)^* = X^{**}$. And now X^* has yet another weak Topology, called weak* Topology τ_{ω^*} generated by members in $J(X)$, a part of X^{**} and therefore one has $\tau_{\omega^*} \subset \tau_\omega \subset \tau$.

Let us describe a typical open set in τ_{ω^*}. It is of the form $V(f_0; J_{x_1}, J_{x_2}, \ldots J_{x_n}, \varepsilon)$, where $f_0 \in X^*$, and $\varepsilon > 0$ is given; and where

$$V(f_0; J_{x_1}, J_{x_2}, \ldots, J_{x_n}, \varepsilon)$$

$$= \{f \in X^* : | J_{x_i}(f) - J_{x_i}(f_0) | < \varepsilon, i = 1, 2, \ldots, n\}$$

$$= \{f \in X^* : | (f_{x_i}) - f_0(x_i) | < \varepsilon,\ i = 1, 2, \ldots, n\}$$

Because $J_{x_i}(f) = f(x_i)$.

It is customary to denote $V(f_0; J_{x_1}, J_{x_2}, \ldots, J_{x_n}, \varepsilon)$ by $V(f_0, x_1, x_2, \ldots, x_n, \varepsilon)$ as $x_i \in X$.

Theorem 6.2.1. τ_{ω^*} is a Hausdorff Topology in X^*.

Proof : Take $f_1, f_2 \in X^*$ with $f_1 \neq f_2$.| So there is a member $x \in X$ such that

$f_1(x) \neq f_2(x)$,

or, $J_x(f_1) \neq J_x(f_2)$

Let us take $\varepsilon > 0$ with $0 < 2\varepsilon < | f_1(x) - f_2(x)|$. Then $V(f_1; x, \varepsilon)$ and $V(f_2; x, \varepsilon)$ are τ_{ω^*} -open sets Containing f_1 and f_2 respectively such that

$$V(f_1; x, \varepsilon) \cap V(f_2; x, \varepsilon) = \phi .$$

Therefore τ_{ω^*} is T_2.

The following important Theorem (Banach-Alaogulu Theorem) Pertaining to weak* Topology = τ_{ω^*} in X^* has gained importance for application in theory of maximal ideals of a Banach algebra.

Theorem 6.2.2. (Banach-Alaogulu Theorem).

The closed unit ball in X^* is τ_{ω^*} – compact.

($B^* = \{f \in X^*; \|f\| \leq 1\}$| is compact in weak * Topology τ_{ω^*} of X^*).

Proof. Let $x \in X$, we can associate a compact space C_x = The closed interval in real line or compact circular disc $\{z: \|z\| \leq \|x\|\}$ in $\mathbb{C}$ according as X is a real or complex NLS. So by Tychonoff Theorem the product space $C = \prod_{x \in X} C_x$ | is compact.

Recall that C consists of all functions $g : X \to \bigcup_{x \in X} C_x$ such that $g(x) \in C_x$

Now for $f \in B^*$ we have $|f|(x)| \leq \|f\| \; \|x\| \leq \|x\|$. So

$f(x) \in [-\|x\|, \|x\|]$ i.e. $f(x) \subset C_x$; Hence $B^* \subset C$.

Now Tychonoff (Product) Topology for C is the weakest topology for which all projection are rendered continuous and a typical basic open set is of the form

$V(g_0; x_1, x_2, \ldots, x_n, \varepsilon) = (g \in C : |g(x_i) - g_0(x_i)| < \epsilon, i = 1, 2, \ldots, n\}$ | where $g_0 \in C$, $\varepsilon > 0$ and $Pr_x(g) = g(x)$. and So B^* as a subset of C has an induced Topology, namely that induced by the product Topology of C and a typical basic open set in this topology is

$$V(g_0; x_1, x_2, \ldots, x_n, \varepsilon) \cap B^* = \{g \varepsilon B^* : |g(x_i) - g_0(x_i) \ldots \varepsilon, \; i = 1, 2, \ldots, n\}.$$

However B^* as a part of X^* inherits τ_{ω^*} (weak* topology) of X^*. On examination of basic open sets of τ_{ω^*} we see that these two Topologies coincide in B^*. Since C is compact in Product Topology it follows that B^* is compact in τ_{ω^*} if we show B^* is closed in C. To that end take $g_0 \in \overline{B}^*$. Then every open set of τ_{ω^*} containing g_0 shall cut B^*. So a basic open set of τ_{ω^*}, say = $V(g_0; x, y, x + y, \varepsilon)$, $\varepsilon > 0$ and $x, y, \in X$ shall meet B^*. Suppose

$$f \in V(g_0; x, y, x + y, \varepsilon) \cap B^*.$$

Since $f \in B^* \subset X^*$, by Linearity of f we have $f(x + y) = f(x) + f(y)$, and since $f \in V(g_0; x, y, x + y, \varepsilon)$, we have $|g_0(x) - f(x)| < \varepsilon$, $|g_0(y) - f(y)| < \varepsilon$ and

$|g_0(x + y) - f(x + y)| < \varepsilon$. Therefore,

$$|g_0(x+y) - g_0(x) + g_0(y)| \le |g_0(x+y) - f(x+y)| + |f(x+y) - (g_0(x) + g_0(y))$$

$$| \le |g_0(x+y) - f(x+y)| + |f(x) - g_0(x) + f(y) - g_0(y)| \le |g_0(x+y) - f(x+y)| +$$

$|f(x) - g_0(x)| + |f(y) - g_0(y)| < 3\varepsilon$. As $\varepsilon > 0|$ is arbitrary we find $g_0(x+y) = g_0(x) + g_0(y)$

By a similar argument we show $g_0(\lambda x) = \lambda g_0(x)$ for any scalar λ. Thus g_0 is shown to be Linear.

For any $x \in X$, there is $f_x \in V(g_0; x, \varepsilon) \cap B^*$, so

$$|g_0(x) - f_x(x)| < \varepsilon$$

Thus $|g_0(x)| \le |f_x(x)| + \varepsilon \le \| f_x \| \, \| x \| + \varepsilon = \| x \| + \varepsilon$.

that means $|g_0(x)| \le \| x \|$ ($\varepsilon > 0$ is arbitrary) and hence $\| g_0 \| \le 1|$ showing that $g_0 \in B^*$; Thus B^* is shown to be closed, and the proof is now complete.

Example 6.2.1. In a NLS X show that every weak Cauchy sequence is bounded.

Solution : Let X be a NLS and X^* be its conjugate space. Suppose $\{x_n\}$ is a weak Cauchy sequence in X. That is, $\{f(x_n)\}$ is a cauchy sequence for all $f \in X^*|$. Recall that $J : X \to X^{**}$ is a canonical mapping from X to its second conjugate space X^{**}.

Now $Jx_n(f) = f(x_n)$ if $f \in X^*$

Given $\sup_n |f(x_n)| < \infty$ so

$$\sup_n |Jx_n(f)| < \infty.$$

By Uniform boundedness Principle Theorem we obtain

$$\sup \| Jx_n \| < \infty.$$

Since $\| Jx_n \| = \| x_n \|$ we deduce that $\sup_n |x_n| < \infty$

So $\{x_n\}$ is a bounded sequence.

Example 2. In a NLS X every weakly convergence Sequence is bounded. This is a consequence of Example above.

Definition 6.2.1. A NLS X is said to be weakly complete if every weakly Cauchy sequence in X converges weakly to some member of X.

Theorem : 6.2.3. If a NLS X is reflexive, then if is weakly complete.

Proof : Take $\{x_n\}$ as a weakly Cauchy sequence in X. if $f \in X^*$, it follows that $\{f(x_n)\}$ is a cauchy. If X^{**} is second conjugate space of X and $J : X \to X^{**}|$ is the canonical mapping, we have $Jx_n(f) = f(x_n)$; Hence $\{Jx_n(f)\}$ is a Cauchy sequence of scalars for all $f \in X^*$. As scalars are real or complex take $y(f) = \lim_{n\to\infty} Jx_n(f)$.

Here y becomes a linear functional and we verify that y is bounded. Since $\| Jx_n \| = \| x_n \|$ and $\{x_n\}$ is a weak Cauchy sequence it follows it is bounded. Let M be a positive real satisfying $\|x_n\| \leq M$ for all n.

Therefore we have $|Jx_n(f)| = |f(x_n)|$

$\leq \| f \| x_n \|$ i.e $|Jx_n(f)| \leq M \| f \|$

This gives $\lim_{n\to\infty} Jx_n(f) \leq M \| f \|$

or $|y(f)| \leq M \| f \|$

This is true for all $f \in X^*$. Therefore y is a bounded linear functional over $X^*|$ i.e., $y \in X^{**}$ and by reflexivity of X, we see an element $x \in X$ such that $Jx = y$. Therefore for any $f \in X^*$, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} Jx_n(f) = y(f) = Jx(f) = f(x)$ and this tells us that $\{x_n\}|$ weakly converges to x i.e.,

$\omega - \lim_{n\to\infty} x_n = x$. The proof is now complete.

Example 6.2.2. If A is a subset of a NLS X such that $\operatorname{Sup}_{x \in A} |f(x)| < \infty$ for every member f in X^* (conjugate space of X). Show that A is a bounded set in X.

Solution : If J is the canonical mapping : $X \to X^{**}$ where X^{**} is the second conjugate space, we have $Jx \in X^{**}$ as $x \in X$ such that $Jx(f) = f(x)$ for $f \in X^*$ and $\|J_x\| = \|x\|$.

Let $x \in A$ and $Jx = x^{**} \in X^{**}$ with

$$X^{**}(f) = f(x).$$

given $\sup_{x \in A} | f(x) | < \infty$, and therefore $\sup_{x \in A} | X^{**}(f) | < \infty$

as $f \in X^*$; and ny Uniform boundedness Principal Theorem we have

$$\sup_{x \in A} \| X^{**} \| < \infty \text{ i.e. } \sup_{x \in A} \| J(x) \| < \infty$$

$$\text{i.e. } \sup_{x \in A} \| x \| < +\infty .$$

Hence A is bounded.

EXERCISE-A

Short Answer type questions

1. Let X be a Banach space. Show that infinite series $\sum_{n=1}^{\infty} x_n$ in X converges in X if and only if for each $\varepsilon > 0$ there is an index N such that

   $\| x_{n+1} + x_{n+2} + ... + x_{n+p} \| < \varepsilon$ for all $n > N$ and for $p = 1, 2, 3, ...$

2. Let X be a Normed Linear space. Show that every $x \in X$ induces a Linear functional f_x on X^* such that

   (i) $f_x(A) = A(x)$ as $A \in X^*$, and (ii) $\{f_x\}_{x \in X}$ separates members of X^*.

3. Explain weak Topology τ_ω and weak * Topology τ_{ω^*} on conjugate X^* of a NLS X and show that

   $$\tau_{\omega^*} \subset \tau_\omega .$$

4. Verify that weak * Topology τ_{ω^*} on conjugate space X^* of a NLS X has Hausdorff property.

EXERCISE-B

Short Answer type questions

1. Let $\sum_{n=1}^{\infty} x_n$ be an infinits series in a Banach space X such that $\sum_{n=1}^{\infty} \| x_n \| < \infty$.

   Show that the series $\sum_{n=1}^{\infty} x_n$ is convergent in X and $\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \| x_n \|$.

2. If X is a Banach space. Show that X is reflexive if and only if its conjugate X^* is reflexive.

3. Let X and Y be two NLS with same scalars. Show that $T \in Bd\mathcal{L}(X,Y)|$ is adjoint T^* if and only if its adjoint T^* is an Isometry of Y^* onto X^*.

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1. B. K. Lahiri—Functional analysis
2. Brown and Page—Functional Analysis
3. W. Rudin—Functional Analysis
4. Bachman and Narici—Functional Analysis
5. Kreyzsig—Functional Analysis.

মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— *রবীন্দ্রনাথ ঠাকুর*

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