PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in a subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation. I am happy to note that university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade 'A'.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing, and devising of a proper layout of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts are still experimental— in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Netaji Subhas Open University
Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Mathmatics (HMT)
Course : Differential Equations
Code : CC - MT - 07

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UG-Mathematics (HMT)

Course : Differential Equations

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Differential Equation

Differential Equation

Unit - 1

Structures

1.0 Objective

The objective of this unit is to discuss on basics of ordinary differential equations and their solutions.

1.1 Differential Equation—Genesis, Order and Degree

Differential equations have wide level of applications in various aspects of science and engineering. Many of the principles or laws underlying the behaviour of the natural world are statements of relatios of rates by which things really happen. When expressed in mathematical terms the relations are equations and rates are derivatives. The mathematical statements of facts describing a real world problem is said to be mathematical models. Differential equations play a significant role in framing of mathematical models. During the last part of 17^{th} century, eminent scientists like Issac Newton, Gottfried Leibniz, Jaeques Bernoulli, Jean Bernoulli and Christian Huygens were engaged in solving differential equations. Many of the techniques which they built up are still in use today. During the 18^{th} century the mathematicians like Leonhard Euler, Dainel Bernoulli, Joseph Legrange and others added significantly to tthe enrichment of the subject. The doyens who pioneered tot he development of ordinary differential equations as a branch of modern mathematics are Cauchy, Riemann, Picard, Poincare, Lyapunoy and Birkhoff.

To understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of heat waves or the increase or decrease of population, among many others, it is necessary to know the basics and working theories of differential equations. While applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes or models the problem being investigated.

1.2 Formal Defintion

An equation involving derivatives or differentials of one or more dependent variable (s) with respect to one or more independent variable (s) is called a differential equation.

For example,

$$\frac{dy}{dx} = 5x + 3$$

$$\frac{\partial y}{\partial x} + 4\frac{\partial y}{\partial t} = 3$$

Depending on the nature of differential of dependent variable (s) to the independent variable (s) the differential equation can be classified in two categories.

1. Ordinary Differential Equation (ODE)
2. Partial Differential Equation (PDE)

Definition of ODE and PDE : A differential equation is ordinary differential equation (ODE) if the unknown function or dependent variable depends only on one independent variable. If the unknown function of dependent variable depends on more than one independent variable then the differential equation is said to be a partial differential equation (PDE).

1.3 Order and Degree of ODE

The order of a differential equation is the highest ordered derivative that appears in the equation.

The degree of a differential equation is the greatest exponent of the highest ordered derivative involving in it, when the equation is free from radicals and fractional powers.

To find the degree of a differential equation, the important view is that the differential equation must be a polymomial in derivatives of various orders. Also it can be mentioned nere that the order and degree (if defined) of a differential equation are always positive integers.

Example : Determine the order and the degree of the following ordinary differential equations :

a. $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = c\frac{a^2y}{dx^2}$

b. $\frac{d^2y}{dx^2}+y=\frac{dy}{dx}$

c. $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$

d. $\left(\frac{d^3y}{dx^3}\right)^{3/2}+\left(\frac{d^2y}{dx^2}\right)^{2/3}=0$

Solution : a. Here $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{2/3} = c\frac{d^2y}{dx^2}$

i.e. $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$

So, the order and degree of the equation are two each, since the highest order derivative is two and the exponent of the highest order derivative is also two.

b. Here $\frac{d^2y}{dx^2}+y=\frac{dy}{dx}$

Clearly, the order of the diffrential equation is two and the degree is one.

c. The degree of the differential equation $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$ is not defined as the differential equation is not a polynomial in its derivatives although it has order one.

d. The order is three and degree is nine as the differential equation is a polynomial equation in its derivatives not a polynomial in y.

1.4 Origin of Ordinary Differential Equation

1. Algebraic and Geometric origin.
2. Mechanical origin
3. Physical/Chemical Science origin
4. Population and Demographic origin
5. Economics and other Social Sciences origin
6. Biological origin

In algebraic or geometric field the differerntial equations are formed by eliminating all the arbitrary constants that involved in a relation. The elimination of the arbitrary constants from the resulting equation gives the required differential equation whose order is equal to the number of independent constants actually involved. For example, given a relation

$y = ax^2 + a^2$ (1)

where a is an orbitrary constant. This relation contains only one arbitrary constant, so the order of the ODE is one. Differentiating (1) with respect to x, we have

$$\frac{dy}{dx}=2xa,$$

i.e., $a=\frac{1}{2x}\cdot\frac{dy}{dx}$

Substituting the value of a in (1), we have

$$y = \frac{1}{2x}\cdot\frac{dy}{dx}x^2 + \left(\frac{1}{2x}\cdot\frac{dy}{dx}\right)^2$$

i.e. $$\left(\frac{dy}{dx}\right)^2 + 2x^2\frac{dy}{dx} - 4x^2 y = 0$$

which is the required differential eqution.

There is one very good example drawn from Biology to demonstrate the need of ordinary differential equation. Let us suppose that the rate of increase in the number of bacteria is proportional to the number of bacteria present. Let $N(t)$ = the number of bacteria at time t.

Assuming $N(t)$ to be a differentiable function of t we can describe the above phenomenon as

$$\frac{dN(t)}{dt} = cN(t),$$ where c is a constant.

1.5 Classification of Ordinary Differential Equations

❑ **Linear and non-linear ordinary differential equatins :**

An ordinary differential equation which contains a single dependent variable and its derivatives with respect to a single independent variable as all first degree terms and there is neither any such term involving any form of product between two or more derivatives of different order nor any transcendental form of the depedent variable or any of its derivatives will be called a linear differential equation.

The general form of a linear ordinary differential equation is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_n(x)y = r(x),$$

where a_0, a_1......, a_n and $r(x)$ are the funcitions of x only.

For exmple, $\frac{dy}{dx}+x^2y=e^x$ and $\frac{d^2y}{dx^2}+\sin(x)\frac{dy}{dx}+xy=\sec(x)$ linear ordinary differential equations.

If the condition of linearity as stated in the above definition is violated then the corresponding ordinary differential equation is said to be a non-linear ordinary differential equation.

For example $(x-y)^2\frac{dy}{dx}+5e^y=3x^2$ and $e^y\frac{a^2y}{dx^2}+y^2\frac{dy}{dx}+xy=\sin(y)$ are not in linear form. These are non-linear ordinary differential equations.

1.6 Homogeneous and Non-Homogeneous Ordinary Differential Equation

An ordinary differential equation is said to be homogeneous if there is no isolated term in the equation, i.e, if all the terms are proportional to a derivative of dependent variable or dependent variable itself and there is no term that contains a function of independent variable or constant alone.

An *n*-th order linear differential equation of the form

$$P_0\frac{d^2y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\dots\dots+P_ny=R \qquad (2)$$

where y is the dependent variable, x is the independent variable, P_0, P_1, P_2, ..., P_n and R are either constants or functions of x.

In (2), if R = 0, then (2) is called a homogeneous linear ordinary differential equation. An ordinary differential equation which is not homogeneous is called a non-homogeneous ordinary differential equation.

Remarks : A homogeneous differential equation has several distinct meanings :

1. A first order oridinary different equation of the form $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$ is a particular type of homogeneous equation.

2. A linear differential equation is said to be homogeneous if it has zero as a solution otherwise it is non-homogenous.

3. Generally (2) is written in the form $F(x, y, y', y'', ..., y^{(n)}) = 0$

1.7 Solution of an Ordinary Differential Equation

A function is said to be a **solution** of an ordinary differential equation, over a particular domain of the independent variable, if its substitution into the equation reduces to an identity everywhere within that obtain.

A function ϕ is said to be a soution of ODE $F(x, y, y', y'', ..., y^{(n)}) = 0$ if

$$F\left(x, \varphi(x), \varphi'(x), \varphi''(x), ..., \varphi^{(n)}(x)\right) = 0$$

where $\varphi^{(n)}(x)$ stands for n-th derivative of the function : $x \to \varphi(x)$ with respect to the indenpendent variable x.

The solution of an ordinary differential equation is called general solution if it contains a number of arbitrary constants equal to the order of the differential equation. This solution sometimes called **a complete solution** or a **complete primitive** or a **complete integral.**

If the solution of an ordinary differential equation with y as dependent and x as independent variable can be obtained in the form $y = f(x)$ then that form of solution is said to be an **explicit solution.** An **implicit solution** of an ordinary differential equation is a solution that is not in explicit form rather can be expressed in the form $\varphi(x,y) = 0$.

A solution of a differential equation by giving particular values to the arbitrary constants in its general solution is called a **particular solution** of that equation.

The general solution of any differential equation may not include all possible solutions of the differential equation. There may exist such a solution which cannot be obtained by giving any particular value to these arbitrary constants in the general solution. This is called a **singular solution** of that ordinary differential equation.

Theorem : Any n-th order ordinary differential equation can have only n and not more than n, independent first integrals and so its general solution cannot have more than n arbitrary and independent constants.

1.8 Summary

This unit provides the basic understanding of ordinary differential equation, its order and degree and certain basic classifications.

1.9 Exercises

1. Determine the order and degree of the following differential equation :

a. $\left(\frac{dy}{dx}\right)^2 + 3y^2 = 0$

b. $\left(\frac{d^2y}{dx^2}\right)^2 + xy = \frac{dy}{dx}$;

c. $\sqrt{\frac{dy}{dx}} = 2y$;

d. $\left(\frac{dy}{dx}\right)^{2/3} = 3 + \frac{d^2y}{dx^2}$;

e. $\left(\frac{d^2y}{dx^2} + 1\right)^{2/3} = 3x\frac{dy}{dx}$;

f. $y + \frac{dy}{dx} = e^{\frac{d^2y}{dx^2}}$

Unit - 2

Structures

2.0 Objective

The objective of this unit is to discuss on various types of first order and first degree ordinary differential equations and their salution strategies.

2.1 First Order Ordinary Differential Equations

❑ **First Order and First Degree Ordinary Differential Equations :**

Standard form for a first order ordinary differential equation in the dependent variable is with the independent variable x is $\frac{dy}{dx} = f(x, y)$, where f (x, y) is a continuous real valued function defined on some rectangular region in real *xy*-plane. An ordinary differential equation of first order and first degree $\frac{dy}{dx} = f(x, y)$ can be written as

$$M(x, y)dx + N(x, y)dy = 0$$

2.2 Cauchy-Lipschitz Condition

A function $f(x, y)$ defined on a rectangular region R : $|x - x_0| < a$, $|y - y_0| < b$ is *xy*-plane is said to satisfy Cauchy-Lipschitz condition if there exists a positive constant λ such that.

$|f(x, y) - f(x, y_2)| \leq \lambda \mid y_1 - y_2 \mid$ for all $(x, y_1),(x, y_2) \in R^2$.

The above constant λ is known as Lipschitz constant for the corresponding function.

2.3 Picard's Theorem

The first order and first degree differential equation $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ defined on a rectangular region R : $|x - x_0| < a$, $| y - y_0| < b$ in is xy-plane, will have a unique solution subject to the following conditions :

(i) $f(x, y)$ is continuous in R ;

(ii) $| f(x, y) | \leq M$, where M is a fixed real number, for all (x, y) in R i.e, $f(x, y)$ is bounded in R ;

(iii) $|f(x,y_1) - f(x, y_2)| \leq \lambda |y_1 - y_2|$ for all (x, y_1), $(y, y_2) \in R$, λ being the Lipschitz constant.

2.4 Solution Strategies for First Order and First Degree Differential Equation

We can classify these equations according to the methods by which they are solved.

(i) Equations with Separable Variables

(ii) Homogeneous Equations

(iii) Exact Equations

(iv) Linear Equations

(v) Bermouli Equations

(i) Equations with Separable Variables :

When a first order and first degree differential equation $\frac{dy}{dx} = f(x, y)$ can be arranged in the form $\frac{dy}{dx} = \frac{\phi(x)}{\psi(y)}, \psi(y) \neq 0$ then we have $\psi(y)dy = \phi(x)dx$..

Integrating we have $\int \psi(y)dy = \int \phi(x)dx + c$, where c is an arbitrary constant. This method is known as method of separable variables.

In other words, in standard form Mdx + Ndy =0, Where $M = M(x)$ and $N = N(y)$ then we can apply this method.

Example : Solve $\frac{dy}{dx}=\frac{3x^2}{1+y^2}$,

Solution : Here given one is a first order and first degree differential equation

$\frac{dy}{dx}=f(x,y)$, where $f(x, y) = \frac{3x^2}{1+y^2}$

Now, $f(x, y) = \frac{\phi(x)}{\psi(y)}$, where $\phi(x)$, $= 3x^2$, $\psi(y) = 1 + y^2$

So, we can apply the method of separable variables.

Thus $\int\psi(y)dy = \int\phi(x)dx + c$, where c is an arbitrary constant.

i.e., $\int(1 + y^2)dy = \int 3x^2 dx$

Therefore, $y+\frac{y^3}{3}=x^3+c$, which is the required solution.

Remarks : In the above example, $\frac{dy}{dx}=\frac{3x^2}{1+y^2}$ if we put it in the standard form, we have $3x^2dx + \{-1(1 + y^2)\}dy = 0$.

Comparing this equation with the equation $Mdx + Ndy = 0$, get $M = 3x^2$ and $N = \{-(1 + y^2)\}$.

It is clear $M = M(x)$ and $N = N(y)$. So observing this we can apply the above method.

(ii) Homogeneous Equations :

If a function $f(x, y)$ can be expressed in the form either $x^n\phi\left(\frac{y}{x}\right)$ or $y^n\phi\left(\frac{x}{y}\right)$ then $f(x, y)$ is said to be homogeneous function of degree n in x and y.

When the function M and N are homogeneous functions of x and y of same order, then the differential equation $Mdx + Ndy = 0$ is called a homogeneous differential equation.

There is another way to check the homogeneity of a first order and first degree equation

$\frac{dy}{dx} = f(x, y)$. If $f(tx, ty) = f(x, y)$ for any real t, then $\frac{dy}{dx} = f(x, y)$ is called a homogeneous differential equation.

Remarks : A function $f(x, y)$ is said to be homogeneous of degree n, if $f(tx, ty) = t^n f(x, y)$ in x and y and t be any non-zero real.

For example we take $\frac{dy}{dx} = \frac{3x^2}{x^2 + y^2}$

We put the above in the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y) = \frac{3x^2}{x^2 + y^2}$

Now for any real t(non-zero).

$$f(tx, ty) = \frac{3(tx)^2}{(tx)^2 + (ty)^2} = \frac{3x^2}{x^2 + y^2} = f(x, y)$$

Therefore, the given differential equation is homogeneous.

Again, here we have

$$\frac{dy}{dx} = \frac{3x^2}{x^2 + y^2}$$

So $Mdx + Ndy = 0$, where $M = 3x^2$ and $N = -x^2 - y^2$.

Now, $M = x^2\phi\left(\frac{y}{x}\right)$ and $N = x^2\psi\left(\frac{y}{x}\right)$, where $\phi\left(\frac{y}{x}\right) = 3$ and $\psi\left(\frac{y}{x}\right) = -1 - \frac{y^2}{x^2}$

It is clear that M and V are homogeneous functions in x and y of order 2. i.e., M and V are homogeneous functions of same order.

Hence the given differential eqution $\frac{dy}{dx} = \frac{3x^2}{x^2 + y^2}$ is a homogeneous differential equation.

Problems : Verify whether the following differential equation are homogeneous

(i) $\left(x^2 - 2y^2\right)dx + xydy = 0$,

(ii) $x^2 \frac{dy}{dx} - 3xy - 2y^2 = 0$,

(iii) $x\frac{dy}{dx} = y + 2xe^{\frac{-y}{x}}$,

(iv) $x\sin\frac{y}{x}\cdot\frac{dy}{dx} = y\sin\frac{y}{x} + x$

(v) $x\frac{dy}{dx} = \sqrt{x^2 + y^2}$

(iii) Exact Equations :

The differential eaquation $Mdx + Ndy = 0$ is called exact differential equation if there exists a function $u = u(x, y)$ such that $du = Mdx + Ndy$ and its general solution is $u(x, y) = c$, where c is an arbitrary constant.

Thorem : The necessary and sufficient condition for the ordinary differential equation $Mdx + Ndy = 0$ to be exact on a rectangular region $R : |x - x_0| < a, |y - y_0| < b$ in xy-plane is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R.

Note : $xdx + ydy = d(xy)$

$$\frac{xdy - ydx}{xy} = d\left(\log\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\frac{xdt + ydx}{\sqrt{1 - x^2y^2}} = d\left(\sin^{-1}(xy)\right)$$

Exmaple : Check whether the equation $(x + y)dy + (y - x)dx = 0$ is exact.

Solution : Here we have $(x + y)dy + (y - x)dx = 0$

Comparing the eqution with $Mdx + Ndy = 0$, we have

$M = y - x, N = x + y$

Now, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

By the statement of last theorem the given differential equation is exact.

Example : Check whether the equation $ydx + xdy = xy(dy - dx)$ is exact or not.

Solution : Here we have $ydx + xdy = xy(dy - dx)$

i.e. $(y + xy)dx + (x - xy)dy = 0$

Comparing the equation with $Mdx + Ndy = 0$ we get $M = y + xy$, $N = x - xy$.

Now $\frac{\partial M}{\partial y} = 1 + x$, $\frac{\partial N}{\partial x} = 1 - y$,

So, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence the given equation is not exact.

2.5 Working procedure to solve an exact equation

Step 1. Calculate $\int Mdx$ treating y as constant and omitting arbitrary contant.

Step 2. Calculate $\int Ndy$ treating x as constant and omitting arbitrary contant.

Step 3. Add with the result of step 1, the result of step 2 deleting those terms which are already been taken in step 1.

Step 4. Equating the result in step 3 to an arbitrary constant, we get the general solution of the equation.

Example : Solve $(4x^3 + 3y^2 + \cos x)dx + (6xy + 2)dy = 0$.

Solution : Here we have $(4x^3 + 3y^2 + \cos x)dx + (6xy + 2)dy = 0$.

Comparing this equation with $Mdx + Ndy = 0$, we get

$M = (4x^3 + 3y^2 + \cos x)$, $N = (6xy + 2)$

Now $\frac{\partial M}{\partial y} = 6y$, $\frac{\partial N}{\partial x} = 6y$

So, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and hence the given equation is exact.

Now, $\int Mdx = (4x^3 + 3y^2 + \cos x)dx = x^4 + 3xy^2 + \sin x$, omitting arbitrary constant

$\int Ndy = \int (6xy + 2)dy$

$= 3xy^2 + 2y$, omitting arbitrary constant

Therefore, $x^4 + 3xy^2 + 2y + \sin x = c$, where c is an arbitrary constant, is the required solution.

Example : Solve $\cos x. \sin ydx + \sin x.\cos ydy = 0$.

Solution : Here we have $\cos x. \sin ydx + \sin x. \cos ydy = 0$

i.e. of the form $Mdx + Ndy = 0$, where $M = \cos x \sin y$ and $N = \sin x. \cos y$.

Now $\frac{\partial M}{\partial y} = \cos x. \cos y$ and $\frac{\partial N}{\partial x} = \cos x. \cos y$.

Hence the given differential equation is exact.

Therefore, $\int Mdx = \int \cos x. \sin ydx = \sin x. \sin y$ and $\int Ndy = \int \sin x. \cos y\, dy = \sin x. \sin y$

Hence the required solution is $\sin x. \sin y = c$, where c is an arbitrary constant.

Exercises :

1. Solve : $(x + 2y)dx + (2x + y)dy = 0$.
2. Solve : $(2xy + 3x^2)dx + (x^2 + 2y)dy = 0$
3. Solve : $(6x + y^2)dx + y(2x - 3y)dy = 0$
4. Solve : $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$
5. Solve : $(2xy - y)dx + (x^2 + x)dy = 0$
6. Solve : $(2uv^2 - 3)du + (3u^2v^2 - 3u + 4v)dv = 0$
7. Solve : $(\cos^2 y - 3x^2y^2)dx + (\cos^2 y + 2x \sin^2 y - 2x^2y)dy = 0$
8. Solve : $(1 + xy^2)dx + (x^2y + y)dy = 0$.
9. Solve : $(1 + y^2 + xy^2)dx + (x^2y + y + 2xy)dy = 0$

10. Solve : $(w^2 + wz^2 - z)dw + (z^3 + w^2z - w)dz = 0$

11. Solve : $(2xy - \tan y)dx + (x^2 - x \sec^2 y)dy = 0$

12. Solve : $(\cos x \cos y - \cot x)dx - \sin x \sin y\ dy = 0$

13. Solve : $(r + \sin t - \cos t)dr + r(\sin t + \cos t)\ dt = 0$

14. Solve : $(3xy - 4y^3 + 6)dx + (x^3 - 6x^2y^2 - 1)dy = 0$

15. Solve : $(\sin t - 2r \cos 2t)dr + r \cos (2r \sin r + 1)dt = 0$

16. Solve : $[2x + y \cos (xy)]dx + x \cos (xy)dy = 0$

17. Solve : $2xydy + (y^2 + x^2)dy = 0$

18. Solve : $2xy\ dx + (y^2 - x^2)dy = 0$

19. Solve : $(2x - 3y)dx + (2x - 3x)dy = 0$

20. Solve : $(3x^2y^3 + 2xy)dx + (2x^2y^3 - x^2)dy = 0$

21. Solve : $(x^3 + 3xy^2)dx + (y^2 + 3x^2y)dy = 0$

2.6 Integrating Factor

Let Mdx + Ndy = 0 be a non-exact first order and first degree ordinary differential equation. A non-zero function $\mu = \mu(x, y)$ is called an integrating factor of the equation $Mdx + Ndy = 0$ if $\mu(Mdx + Ndy) = 0$ becomes an exact differential equation

i.e. $\mu(x, y)$ is said to be the integrating factor of the differential equation Mdx + Ndy = 0, if we can find $u = u(x, y)$ such that $\mu(Mdx + Ndy) = du = 0$

Theorem : The number of integrating factors of an equation $Mdx + Ndy = 0$ is infinite.

2.7 Rules for Finding Integrating Factors (I. F.)

Rule 1. If the given equation $Mdx + Ndy = 0$ is a homogeneous such that $Mx + Ny \neq 0$, then $\frac{1}{(Mx+Ny)}$ is an integrating factor $(I.F.)$.

Example : Solve : $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Solution : Here the given equation can be written in the form Mdx + Ndy = 0, where $M = x^2 + y^2$; $N = -xy$.

Now, $\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = -y$

Therefore, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so the given differential equation is not exact.

Now $Mx + Ny = x(x^2 + y^2) + y(-xy) = x^3 + xy^2 - xy^2 = x^3 \neq 0$

So, $\text{I.F} = \frac{1}{Mx + Ny} = \frac{1}{x^3}$

Multiplying I. F to the both sides of the given equation we have

$$\frac{1}{x^3}\left(x^2 + y^2\right)dx - \frac{xy}{x^3}dy = 0$$

or, $\frac{dx}{x} + \frac{y^2}{x^3}dx - \frac{y}{x^2}dy = 0$

or, $d(\log x) + \frac{y}{x}\left(\frac{ydx - xdy}{x^2}\right) = 0$

or, $d(\log x) - \frac{y}{x}d\left(\frac{y}{x}\right) = 0$.

Integrating we get $\log x - \frac{1}{2}\left(\frac{y}{x}\right)^2 = c$, where c is an arbitrary constant.

Example : Solve $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

Solution : Here, $M = \left(x^2y - 2xy^2\right), N = \left(3x^2y - x^3\right)$

Therefore, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ So, the given differential equation is not exact.

Here, $Mx+Ny=x(x^2y-2xy^2)+y(3x^2y-x^3)=x^2y^2\neq 0$

So, $I.F.=\dfrac{1}{x^2y^2}$

Multiplying I. F. to the both sides of the given equation we have

$$\frac{1}{x^2y^2}\left[\left(x^2y-2xy^2\right)dx+\left(3x^2y-x^3\right)dy\right]=0$$

Or, $\left(\dfrac{1}{y}-\dfrac{2}{x}\right)dx+\dfrac{3}{y}dy-\dfrac{x}{y^2}dy=0$

Or, $d\left(\dfrac{x}{y}\right)-2d(\log x)+3d(\log y)=0$

Integrating we get $\dfrac{x}{y}-2\log x+3\log y=c$, where c is an arbitrary constant.

Example : Solve $(y^3-2x^2y)dx+(2xy^2-x^2)dy=0$

Solution : Comparing the given differential equation with $Mdx+Ndy=0$, we get $M=(y^2-2x^2y)$, $N=(2xy^2-x^3)$

Therefore $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, So, the given differential equation is not exact.

Now, $Mx+Ny=x(y^3-2x^2y)+y(2xy^2-x^3)=3xy(y^2-x^2)\neq 0$

So. I. F. = $\dfrac{1}{3xy\left(y^2-x^2\right)}$

Multiplying I. F. to the both sides of the given equation we have

$$\frac{\left(y^3-2x^2y\right)}{3xy\left(y^2-x^2\right)}dx+\frac{\left(2xy^2-x^2\right)}{3xy\left(y^2-x^2\right)}dy=0$$

or, $$\frac{\left(y^3-2x^2y\right)}{3x\left(y^2-x^2\right)}dx+\frac{\left(2xy^2-x^2\right)}{3y\left(y^2-x^2\right)}dy=0$$

or, $$\frac{dx}{x}+\frac{dy}{y}+\frac{ydy-xdx}{y^2-x^2}=0$$

or, $2d(\log x) + 2d(\log y) + d(\log (y^2 - x^2)) = 0$

Integrating we get $\log x^2 + \log y^2 + \log (y^2 - x^2) = \log c$ i.e. $x^2y^2 (y^2 - x^2) = c$, where c is an arbitrary constant.

Rule : 2. If $Mx - Ny \neq 0$ and the equation can be written as $\{f(xy)\}ydx + \{g(xy)\}xdy = 0$, i.e. Mdx + Ndy = 0 then the integrating factor of the given equation is of the form

$$\frac{1}{Mx-Ny}$$

Example : Solve $\left(\text{xy}\sin(\text{xy})+\cos(\text{xy})\right)\text{ ydx}+\left(\text{xy}\sin(\text{xy})-\cos(\text{xy})\right)\text{xdy}=0$

Solution : Here given differential equation is of the form

$f(xy)ydx + g(xy)xdy = 0$

where $f(xy) = (xy \sin (xy) + \cos (xy))$, $g(xy) = (xy \sin (xy) - \cos (xy))$

Here, $M = (xy \sin (xy) + \cos (xy))\, y$ and $N = (xy \sin(xy) - \cos(xy)x$

Now $Mx - Ny = 2xy\cos(xy)$

So, I. F. = $$\frac{1}{2xy\cos(xy)}$$

Multiplying I. F. to the both sides of the given equation we have

$$\frac{\left(xy\sin(xy)+\cos(xy)\right)}{2xy\cos(xy)}ydx+\frac{\left(xy\sin(xy)-\cos(xy)\right)}{2xy\cos(xy)}xdy=0$$

or, $$\frac{1}{2}\left\{\tan(xy)+\frac{1}{xy}\right\}ydx+\frac{1}{2}\left\{\tan(xy)-\frac{1}{xy}\right\}xdy=0$$

or, $\frac{1}{2}\tan(xy)\{ydx+xdy\}+\frac{1}{2}\left\{\frac{dx}{x}-\frac{dy}{y}\right\}=0$

or, tan (xy) d(xy) + d(log x – d(log y) = 0

Integrating we have,

log | see(xy)| + log x – log y = log c

or, x sec (xy) = cy, where c is an arbitrary constant.

Rule : 3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ be a function of x only, say ϕ (*x*), then $e^{\int\phi(x)dx}$ is an integrating factor of the given equation *Mdx* + *Ndy* = 0.

Example : Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Solution : Here M = $(x^2 + y^2 + 2x)$, N = 2y

Therefore, $\frac{\partial M}{\partial y}=2y, \frac{\partial N}{\partial x}=0$

Therefore, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. So, the given differential equation is not exact.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2y}(2y-0)$ = 1 = ϕ (x) (say)

Thus I. F. = $\text{I.F.}=e^{\int\phi(x)dx}=e^{\int 1.dx}=e^{x}$

Multiplying I. F. to the both sides of the given equation we have

$e^x(x^2 + y^2 + 2x)dx + 2ye^x dy = 0$

or, $e^x dx + 2xe^x dx + y^2 e^x dx + 2ye^x dy = 0$

or, $d(e^x x^2) + d(y^2 e^x) = 0$

Integrating we get $e^x x^2 + e^x y^2 = c$, where *c* is an arbitrary constant.

Rule : 4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ be a function of *y* alone. say ϕ (y), then $e^{\int\phi(y)dy}$ is an integrating factor of the given differential equaton *Mdx* + *Ndy* = 0.

Example : Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

Solution : Comparing with the equation $Mdy + Ndy = 0$, we have

$M= (3x^2y^4 + 2xy)$, $N = (2x^3y^3 - x^2)$

Therefore, $\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$, $\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

So, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy}\frac{1}{\left(3xy^2+2\right)}\left(6x^2y^2 - 2x - 12x^2y^3 - 2x\right)$

$= -\frac{2}{y} = \phi(y)$ (say) which is a function of y only.

Thus, I. F. $= e^{-\int -\frac{3}{4}dy} = e^{-2\log y} = \frac{1}{y^2}$

Multiplying I. F. to the both sides of the given equation we have

$$\frac{1}{y^2}\left(3x^2y^4 + 2xy\right)dx + \frac{1}{y^2}\left(2x^3y^2 - x^2\right)dy = 0$$

or, $3x^2y^2dx + 2\frac{x}{y}dx + 2x^3ydy - \frac{x^2}{y^2}dy = 0$

or, $d\left(x^3y^2\right) + \frac{2xydx - x^2dy}{y^2} = 0$

or, $d\left(x^3y^2\right) + d\left(\frac{x^2}{y}\right) = 0$

Integrating we get $x^3y^2 + \frac{x^2}{y} = c$

Rule 5. If $Mdx + Ndy = 0$ can be expressed in the form $x^{\alpha}y^{\beta}(mydx + nxdy) + x^{\lambda}y^{\beta}(m_1ydx + n_1xdy)=0$, where α, β, g, γ, δ, m, n, m_1, n_1, are constant and $mn_1 - nm_1 \neq 0$, then x^hy^k is an integrating factor of the given equation $Mdx + Ndy = 0$, where

$$\frac{\alpha+h+1}{m}=\frac{\beta+k+1}{n} \text{ and } \frac{\gamma+h+1}{m_1}=\frac{\delta+k+1}{n_1}$$

Example : Solve $x^2\ (2ydx + 3xdy) + y^2(\ -2ydx + 2xdy) = 0$

Solution :

We can rewrite the iven equation in the following form :

$x^2y^6(2ydx + 3xdy) + x^6y^2\ (-\ 2ydx + 2xdy) = 0$

i.e., $x^{\alpha}y^{\beta}\ (mydx + nxdy) + x^{\lambda}y^{\delta}(m_1ydx + n_1xdy) = 0$

where, $a = 2$, $b = 0$, $y = 0$, $d = 2$, $m = 2$, $n = 3$, $m_1 = -\ 2$, $n_1 = 2$.

Therefore, I. F. = x^hy^k where

$$\frac{\alpha+h+1}{m}=\frac{\beta+k+1}{n},\quad \frac{\gamma+h+1}{m_1}=\frac{\delta+k+1}{n_1}$$

i.e. $$\frac{2+h+1}{2}=\frac{0+h+1}{3},\quad \frac{0+h+1}{-2}=\frac{2+k+1}{2},$$

Solving the above equations we have $h = -\ 3$ and $k = -\ 1$.

Hence, I. F = $x^{-3}\ y^{-1}$

$$x^{-3}y^{-1}\cdot x^2\left(2ydx+3xdy\right)+x^{-3}y^{-1}\cdot y^2\left(-2ydx+2xdy\right)=0$$

i.e $$2\frac{dx}{x}+3\frac{dy}{y}+2\frac{y}{x}\left(\frac{-ydx+xdy}{x^2}\right)=0$$

or, $d(2\ \log\ x) + d(3\ \log\ y) + d\left(\dfrac{y^2}{x^2}\right) = 0$

Integrating above we get $2\log x + 3\log y + \dfrac{y^2}{x^2} = c$. where c is an arbitrary constant.

(iv) Linear first order ODE :

A particular type of first order and first degree ordinary differential equation of the form $\frac{dy}{dx}+Py=Q$, where each of P and Q is either a function of x only or a constant, is called a Linear Ordinary Differential Equation of first order in y.

For the above form of ODE $e^{\int Pdx}$ is an interating factor (I.F) i.e. the given ODE can be integrated on multiplying this factor to both the sides. This can be evident from the following analysis.

Multiplying both sides of the iven ODE by $e^{\int Pdx}$ we have

$$e^{\int Pdx}.\frac{dy}{dx}+e^{\int Pdx}.Py=e.Q$$

which gives

$$\frac{d}{dx}\left(y.e^{\int Pdx}\right)=e^{\int Pdx}.Q$$

or, $$d\left(y.e^{\int Pdx}\right)=\left(e^{\int Pdx}.Q\right)dx$$

Integrating above we can have the desired solution through the following step :

$$y.e^{\int Pdx}=\int\left(e^{\int Pdx}.Q\right)dx+c$$

i.e $$y.(I.F)=.\int(I.F.).\,Q\,dx+c$$

where ‘c’ is an arbitrary constant.

We can summarize the steps involved in solving such equations.

Step 1. Put the equation in the form $\frac{dy}{dx}+Py=Q$

Step 2. Obtaint I.F. as $e^{\int Pdx}$.

Step 3. Simplify $y.(I.F)=\int(I.F).Qdx+c,$ where c is an integration constant.

Example : Solve $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{\left(x^2+1\right)^3}$

Solution : Here $P+\frac{4x}{x^2+1}, Q=\frac{1}{\left(x^2+1\right)^3}$

Here integrating factor is given by

I.F. $= e^{\int Pdx} = e^{\int\frac{4x}{x^2+1}dx} = e^{\log\left(x^2+1\right)^2} = \left(x^2+1\right)^2$

Hence we have, $y.(I.F.)=\int(I.F).Qdx+c$

ie. $y.\left(x^2+1\right)^2=\int\left(x^2+1\right)^2.\frac{1}{\left(x^2+1\right)^3}dx+c$

or, $y.\left(x^2+1\right)^2=\tan^{-1}x+c$

(v) Bernoulli's Equations :

The first order ordinary differential equation of the form $\frac{dy}{dx}+Py=Qy^n$ where P and Q are continuous function of x and n is a real number, is known as Bernoulli's Equation.

From $\frac{dy}{dx}+Py=Qy^n$

we have $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$

If we put $y^{1-n}=v$ then we can have $(1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$

Thus the quation transforms to $\frac{dv}{dx}+(1-n)Pv=(1-n)Q$ which is a first order liner ODE in v, its integrativing factor being $e^{\int(1-n)\,Pdx}$.

Then its solutionis given by

$v.(I.F.) = (1-n)\int Q.(I.F.)\,dx + c$

ie. $v.e^{\int (1-n)^{Pdx}} = (1-n)\int Q.\,e^{\int (1-n)^{Pdx}} dx + c$

or, $y^{1-n}.\,e^{\int (1-n)\,Pdx} = (1-n)\int Q.e^{\int (1-n)\,Pdx} dx + c$

where c is an arbitrary constant.

Example : Sove $2x^2 \frac{dy}{dx} = 4xy + y^2.$

Solution : Here $\frac{dy}{dx} + \left(\frac{-2}{x}\right) y = \frac{1}{2x^2}.y^2$

Therefore, $\frac{dy}{dx} + Py = Qy^2$. where $P = \frac{-2}{x}, Q = \frac{1}{2x^2}$

We put $v = y^{1-n} = y^{1-2} = y-1$.

So $\frac{dv}{dx} = -\frac{1}{y^2}.\frac{dy}{dx}$ Now we can have $-\frac{1}{y^2}.\frac{dy}{dx} - P.\frac{1}{y} = -Q,$

i.e., $\frac{dv}{dx} - P.v = -Q,$

Which is a first order lincar ODE in *v*.

Therefore integrating factor of the above is $I.F. = e^{\int (-P)dx} = e^{\int \frac{2}{x}dx} = e^{\log x^2 = x^2}$

Hence $v.x^2 = \int \frac{(-1)}{2x^2}.x^2 dx$

i.e, $\frac{1}{y}.x^2 = -\frac{x}{2} + c$

$\frac{x^2}{y}+\frac{x}{2}=c$ where c is an integrating constant.

2.8 Summary

The present unit emphasizes on first order and first degree ordinary differential equations with the conditions of haring unque solution and different working procedures to solve them analytically.

2.9 Exercises

(A) Solve the following exact equations :

1. $(x+2y)dx+(2x+y)dy=0$
2. $\left(2xy+3x^2\right)dx+\left(x^2+2y\right)dy=0$
3. $\left(6x+y^2\right)dx+y(2x-3y)dy=0$
4. $\left(y^2-2xy+6x\right)dx-\left(x^2-2xy+2\right)dy=0$
5. $(2xy-y)dx+\left(x^2+x\right)dy=0$
6. $v\left(2uv^2-3\right)du+\left(3u^2v^2-3u+4v\right)dv=0$
7. $\left(\cos^2 y-3x^2y^2\right)dx+\left(\cos^2 y-2x\sin^2 y-2x^3y\right)dy=0$
8. $\left(1+xy^2\right)dx+\left(x^2y+y\right)dy=0$
9. $\left(1+y^2+xy^2\right)dx+\left(x^2y+y+2xy\right)dy=0$
10. $\left(w^3+wz^2-z\right)dw+\left(z^3+w^2z-w\right)dz=0$
11. $(2xy-\tan y)dx+\left(x^2-x\sec^2 y\right)dy=0$

12. $(\cos x \cos y - \cot x)dx - \sin x \sin y\, dy = 0$

13. $(r + \sin t - \cos t)dr + r(\sin t + \cos t)dt = 0$

14. $x\left(3xy - 4y^3 + 6\right)dx + \left(x^3 - 6x^2y^2 - 1\right)dy = 0$

15. $\left(\sin t - 2r\cos^2 t\right)dr + r\cos t(2r\sin t + 1)dt = 0$

16. $[2x + y\cos(xy)]dx + x\cos(xy)dy = 0$

17. $2xydx + \left(y^2 + x^2\right)dy = 0$

18. $-2xy\, dx + \left(y^2 - x^2\right)dy = 0$

19. $(2x - 3y)dx + (2y - 3x)dy = 0$

20. $\left(3x^2y^3 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0$

21. $\left(x^3 + 3xy^2\right)dx + \left(y^3 + 3x^2y\right)dy = 0$

B. Solve the following Equation :

1. $(1 + y)dx = (1 - x)dy$

2. $x\cos^2 ydx = y\cos^2 xdy$

3. $e^y dy = \left(e^x + x^2\right)dx$

4. $ydx + \left(1 + x^2\right)\tan^{-1} xdy = 0$

5. $x\sqrt{1 - y^2}\, dx = y\sqrt{1 - x^2}\, dx$

6. $Sec^2 x.\tan y.dx + \sec^2 y.\tan x.dy = 0$

7. $x\log xdy + \sqrt{1 - y^2 dx} = 0$

8. $x^{-1}\cos^2 y.dy + y^{-1}\cos^2 x.dx = 0$

9. $3e^x.\tan y\, dx - \left(1-e^x\right)\sec^2 y dy = 0$

10. $x\left(e^y + 4\right)dx + e^{x+y}dy = 0$

11. $dy = \sqrt{y-x}\, dx$

12. $\left(e^x + 1\right)y.dy = \left(y^2+1\right)e^x dx$

13. $dy = y.\ secx.\ dx$

(C) Determine whether the given ODE is exact or not and if exact find the solution :

1. $(3x+2y)dx + (2x+y) = 0$

2. $\left(y^2+3\right)dx + (2xy-4)dy = 0$

3. $(2xy+1)dx + \left(x^2+4y\right)dy = 0$

4. $\left(3x^2y+2\right)dx - \left(x^3+y\right)dy = 0$

5. $\left(6xy+2y^2-5\right)dx + \left(3x^2+4xy-6\right)dy = 0$

6. $(6\sec^2 x + sec\, x\tan x)dx + (\tan x + 2y)dy = 0$

7. $\left(\frac{x}{y^2}+x\right)dx + \left(\frac{x^2}{y^3}+y\right)dy = 0$

(D) Solve the followings :

1. $(2xy-3)dx + \left(x^2+4y\right)dy = 0,\ y(1) = 2.$

2. $\left(3x^2y^2 - y^3 + 2x\right)dx + \left(2x^3y - 3xy^2 + 1\right)dy = 0,\ y(-2) = 1$

3. $\left(2y\sin x\cos x + y^2\sin x\right)dx + \left(\sin^2 x - 2y\cos y\right)dy = 0, y(0) = 3$

4. $\left(ye^x + 2e^x + y^2\sin x\right)dx + \left(\sin^2 x - 2y\cos x\right)dy = 0, y(0) = 3$

E. Solve the following differental equation :

1. $\left(x + 2y^3\right)dy = ydx$

2. $\cot y\, dx - \tan xdy = 0$

3. $(x + y)dy + (y - x)dx = 0$

4. $ydx + xdy = xy(dy - dx)$

5. $xdx + ydy + k(xdy - ydx) = 0$

6. $xdy - ydx - \cos\left(\frac{1}{x}\right).\, dx = 0$

7. $\sin x\frac{dy}{dx} + y^2 = y\cos x$

8. $x\frac{dy}{dx} + y = y^2\log x$

9. $x^2\frac{dy}{dx} + xy + 2\sqrt{1 - x^2y^2} = 0$

10. $(xy\cos(xy) + \sin(xy))dx + x^2\cos(xy)dy = 0$

11. $\left(\sin x.\cos y + e^{2x}\right)dx + (\cos x.\sin y + \tan y)dy = 0$

12. $\left(1 + 4xy + 2y^2\right)dx + \left(1 + 4xy + 2x^2\right)dy = 0$

13. $(1 + xy)\,ydx + (1 - xy)\,xdy = 0$

14. $\left(1 + 3x^2 + 6xy^2\right)dx + \left(1 + 3y^2 + 6x^2y\right)dy = 0$

15. $\left(\log y + \frac{1}{x}\right)dx + \left(\frac{x}{y} + 2y\right)dy = 0$

16. $\left(2xy+e^{x}\right)ydx-e^{x}dy=0$

17. $x^{2}ydx-\left(x^{3}+y^{3}\right)dy=0$

18. $\left(x^{2}y^{2}+xy+1\right)ydx+\left(x^{2}y^{2}-xy+1\right)xdy=0$

19. $3\left(x^{2}+y^{2}\right)dx+x\left(x^{2}+3y^{2}+6y\right)dy=0$

20. $\left(xy^{3}+y\right)dx+2\left(x^{2}y^{2}+x+y^{4}\right)dy=0$

(F) Prove that e^{x^2} is an integrating factor of the equation :

$\left(x^{2}+xy^{4}\right)dx+2y^{3}dy=0.$

(G) If $x^{\alpha}y^{\beta}$ be an integrating factor of the equation (2y dx + 3x dy) + 2xy(3y dx + 4x dy) = 0, find α and β.

(H) If $x^{\alpha}y^{\beta}$ be an integrating factor of the equation

$\left(-3x^{-1}-2y^{4}\right)dx+\left(-3y^{-1}+xy^{3}\right)dy=0$, then find the values of α and β.

I. **Solve :** $x.\cos x.\dfrac{dy}{dx}+y\left(x.\sin x+\cos x\right)=1$

J. **Solve :** $\dfrac{dy}{dx}+2xy=e^{-x^{2}}$

K. **Solve :** $\left(x^{2}+1\right)\dfrac{dy}{dx}+2xy=4x^{2}$

L. **Solve :** $\cos^{2}x.\dfrac{dy}{dy}+y=\tan x.$

M. **Solve :** $\left(x^{2}y^{3}+2xy\right)dy=dx$

N. **Solve :** $\dfrac{dy}{dx}+\dfrac{y}{x}.\log y=\dfrac{y}{x^{2}}.(\log y)^{2}$

Unit - 3

Structures

3.0 Objective

The objective of the present unit is to discuss on the various aspects of first order but not of first degree and second order ordinary differential equations; the strategy of series solution and some basic discussions dynamical systems as an application.

3.1 Equation of first order but not of first degree

An ordinary differential equation of first order and n-th degree can be written as—

$$Q_0 p^n + Q_1 p^{n-1} + \text{.........} + Q_{n-1} p + Q_n = 0 \qquad \text{(A)}$$

where $p = \dfrac{dy}{dx}$ and Q_0, Q_1,........Q_n are functions of x and $Q_0 \neq 0$.

There can be three special cases for the above equation :

(a) Solvable for p.

(b) Solvable for x.

(c) Solvable for y.

(a) Solvable for p :

Let us assume that the left hand side of differential equation (A) can be expressed as a product of n-linear factors in p by the following form :

$$\left(p - f_1(x,y)\right).\left(p - f_2(x,y)\right)\text{...........}\left(p - f_n(x,y)\right) = 0$$

i.e. $p = f_1(x,y), p = f_2(x,y)\text{....}, p = f_n(x,y)$]

all of which are first order and first degree equations. Solving each of the equations we can have the solutions as—

$$F_1\left(x,y,c_1\right) = 0,\ F_2\left(x,y,c_2\right) = 0,\ \text{........},\ F_n\left(x,y,c_n\right) = 0 \qquad \text{(B)}$$

where $c_1, c_2 ....., c_n$ are constants.

As the differential equation (A) is of the first order we must have only one arbitary constant in its general solutiou. without loss of generality $c_1, c_2 ....., c_n$ can be replaced by a single arbitrary constant c. Thus the general solution of the differential equation i.e, one parameter soluton of the equation is given by—

$F_1(x, y, c). F_2(x, y, c)............F_n(x, y, c) = 0$, where c is an arbitrary constant.

Example : Solve : $p^2 + 2xp - 3x^2 = 0$

Solution : Now $p^2 + 2xp - 3x^2 = 0$

i.e : $(p-x)\ (p+3x) = 0$

So, $p = x$ and $p+3x = 0$

i.e., $\frac{dy}{dx} = x, \frac{dy}{dx} = -3x$

Integrating we get $y = \frac{x^2}{2} + c_1, y = -\frac{3}{2}x^2 + c_2$

As the given differential equation of the first order, we must have only one arbitrary constant in its general solution i.e. c_1, c_2, can be replace by a single arbitrary constant c.

Hence the general solution is —

$\left(y - \frac{x^2}{z} - c\right)\left(y + \frac{3}{2}x^2 - c\right) = 0$ where c is an arbitrary constant.

(b) Solvable for x :

If the differential equaton (A) be solvable for x, then it may be put in the form $x = f(y,p)$ (C)

Now $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{p}$

Thus differentiating w.r.t. y we get the following form : $\frac{1}{p} = F\left(y, p\frac{dp}{dy}\right)$ (D)

Eliminating p between (C) and (D) , we get the solution of the differential equation as a relation between x,y and arbitrary constant c. If the elimination of p is difficult x and y may be expressed in terms of p where p acts as a parameter.

Example : Solve : $x = a - \frac{p}{\sqrt{1+p^2}}$ (a)

Solution : The given equation can be written as :

This equation of the form $x = f(y,p)$

Differentiating both sides with respect to y we get.

$$\frac{1}{p} = -\frac{1}{\sqrt{1+p^2}} \cdot \frac{dp}{dy} + \frac{p2}{\left(1+p^2\right)^{\frac{3}{2}}} \cdot \frac{dp}{dy}$$

$$\text{i.e., } \frac{1}{p} + \frac{1}{\left(1+p^2\right)^{\frac{3}{2}}} \cdot \frac{dp}{dy} = 0$$

i.e., $dy = -\frac{p}{\left(1+p^2\right)^{3/2}} .dp$

Integrating, we get $y + c = \frac{1}{\sqrt{1+p^2}}$

i.e., $(y+c)^2 = \frac{1}{1+p^2}$, (b)

where c is an arbitrary constant.

Now from (a), $(x-a)^2 = \frac{p^2}{1+p^2}$ (c)

Eliminating p from (b) and (c) we get

$(x-a)^2 + (y+c)^2 = 1$, which is the general salution of (a).

(c) Solvable for *y* :

If the differential equation (A) be solvable for *y* then it may be put in the from $y = f(x,p)$. (E)

Differentiating both sides of (E) with respect of x we have an equation of the form

$$p = F\left(x, p, \frac{dp}{dx}\right)$$

Now it can be solved to get solution of the form $\phi(p, x, c) = 0$

Eliminting p between (E) and (F) we get the general solution of the differential equation (A).

Example : Solve $y = p.\tan p + \log(cosp); p = \frac{dy}{dx}$.

Solution : The equation is of the form $y = f(x, p)$.

Differentiating both sides with respect to *x*, we get

$$p = \left(\tan p + p.\sec^2 p - \tan p\right)\frac{dp}{dx}$$

i.e $p = p.sec^2 p.\frac{dp}{dx}$

i.e. $dx = sec^2 p.\ dp.$

Integrating botht sdid we tet $x+c = tanp$, where c is an arbitrary constant. Then

$p = \tan^{-1}(x+c)$ and $\cos p = \frac{1}{\sqrt{1+(x+c)^2}}$

Thus the general solution is

$$y = (x+c)\tan^{-1}(x+c) + \log\left(\frac{1}{\sqrt{1+(x+c)^2}}\right)$$

Lagrange Equation :

A first order ODE of the form $y = x.\phi(p) + \psi(p)$ (G) where $p=\frac{dy}{dx}$ and $\phi(p)$ and $\psi(p)$ are known functions of p diferentiable on a certain interval, is called Lagrange Equation.

Now differentiatinm (G) with respect to x we have

$$p=\phi(p)+\{x\phi'(p)+\Psi'(p)\}.\frac{dp}{dx.}$$

$$\text{i.e. } \frac{dx}{dp}+\frac{\phi'(p)}{\phi(p)-p}.x=\frac{\psi'(p)}{p-\phi(p)}$$

which is a linear equation in x. This can be solved easily and eliminating p from this solution and the given equation will give us the complete solution.

Example : Solve : $y = 2xp - p^2$

Solution : Here given equation is of the form

$$y=x.\phi(p)+\psi(p) \qquad \text{(a)}$$

where $\phi(p)=2p, \psi(p)=-p^2$

So, it is a Lagrange equation.

Differentiating (a) with respect to x we have $p=\phi(p)+\{x\phi'(p)+\psi'(p)\}\frac{dp}{dx}$

i.e $p=2p+\{2x+(-2p)\}.\frac{dp}{dx}$

or, $-p.\frac{dx}{dp}=2x-2p$

or, $\frac{dx}{dp}+\frac{2}{p}x=2$

which is linear in x.

Therefore integrating factor of the differential equation (b) is given by—

I.F. $= e^{\int \frac{2}{p}dp} = e^{\log} p^2 = p^2$

So, the solution of (b) is—

$$x.p^2 = \int 2p^2 dp + c$$

i.e , $x.p^2 = \frac{2}{3}p^3 + c$, where c is an arbitrary constant.

or, $x = \frac{2p}{3} + \frac{c}{p^2}$

Now putting this value of x in the given equation, we get $y = \frac{2c}{p} + \frac{p^2}{3}$

Thus the general solution is given by $x = \frac{2p}{3} + \frac{c}{p^2}$ and y $= \frac{2c}{p} + \frac{p^2}{3}$, where p is the parameter.

Clairaut's Equation :

An ODE of the form y = px + ϕ(p) (H)

is known as Clairaut's Equaton.

Now differentiating both sides of (H) with respect to x we have.

$$p = p + \{x + \phi'(p)\}.\frac{dp}{dx}$$

i.e $\{x + \phi'(p)\}.\frac{dp}{dx} = 0$

This gives either $\frac{dp}{dx} = 0$ (I)

or, $x + \phi'(p) = 0$ (J)

From (I) we get p=c, where c is an arbitrary constant. Putting this value of p=c in (H) we get $y = cx + \phi(c)$ which is the general solution of this Clairaut's equatiou.

Again eliminating p from (H) and (J) we get another soluton which does not contain any arbitrary constant. This solution is caled the **singular soluton** of the Clairaut's equation (H).

Example : Find the general and singular solution of

$(y-px)(p-1)=p$ where $p=\dfrac{dy}{dx}$

Solution : The given equation $(y-px)(p-1)=p$

can be written as

$y=px+\dfrac{p}{p-1}$, which is a Clairaut's equation. (a)

Then differentiating both sides with respect to x we get

$$p=p+x.\frac{dp}{dx}-\frac{1}{(p-1)^2}.\frac{dp}{dx}$$

i.e. $\left\{x-\dfrac{1}{(p-1)^2}\right\}.\dfrac{dp}{dx}=0$

i.e either $\dfrac{dp}{dx}=0$ or, $x-\dfrac{1}{(p-1)^2}=0$

Now $\dfrac{dp}{dx}=0$ gives $p=c$..................*(b)*

Eliminating p from (a) & (b) we get the gensal solution as $y=cx+\dfrac{c}{c-1}$

where c is an arbitrary constant

Again $x-\dfrac{1}{(p-1)^2}=0$ gives $(p-1)^2=\dfrac{1}{x}$

or $p=\dfrac{\sqrt{x}+1}{\sqrt{x}}$ ....(c)

Eliminating p from (a) and (c) we have $y=\dfrac{\sqrt{x}+1}{\sqrt{x}}.x+\dfrac{\dfrac{\sqrt{x}+1}{\sqrt{x}}}{\dfrac{1}{\sqrt{x}}}$

i.e $(y-x-1)^2 = 4x$. This is the singular solution of the given equation.

Exercises :

1. Find the general and singular solution of $y = xp + p^2$, where $p = \frac{dy}{dx}$
2. Find the general and singular solution of $y = xp + \sqrt{1+p^2}$, where $p = \frac{dy}{dx}$.
3. Sove the folowing differential equations
   i. $x = 4p + 4p^3$
   ii. $x = py - p$
   iii. $y^{\cdot 2}.\log y = xyp + p^2$
   iv. $y = 2px + y^2 p^3$
   v. $xy\left(p^2-1\right) = \left(x^2 - y^2\right)p$
   vi. $xp^2 - 2yp + ax = 0$
   vii. $6p^2y^2 - y + 3px = 0$
   viii. $xp^2 + (y-x).p - y = 0$
4. Solve : $x^2(y - px) = p^2 y$
5. Reduce the differential equation $x^2p^2 + py(2x+y) = 0$ in Clairaut's form by the substitution $y = u$, $xy = v$ and hence solve the differential equation.
6. Use the transformation $u = x^2, v = y^2$ to solve the euation $(px-y)(py+x) = h^2 p$
7. Use the transformation $u = x^2, v = y - x$ to solve the equation $xp^2 - 2yp + x + 2y = 0$
8. Use the transformation $u = \frac{1}{x}, v = \frac{1}{y}$ to solve the equation $y^2(y - px) = x^4 p^2$

3.2 Singular Solution

A **singular solution** is a solution of the given first order higher degree differential equation which is not obtained from the general solution by assigning particular values to the arbitrary constant involved in it. It is the equation of an envelope of the family of curves represented by the general solution.

Let $\phi(x, y, c) = 0$ represent a family of curves. From the notion of envelope it can be found that the c-discriminant of $\phi(x, y, c) = 0$ is the c-eliminant of $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$ provided $\phi(x, y, c), \frac{\partial \phi}{\partial c}$ are continuons in the domain of the differential equation. As for example let the family of curves be

$y^2 = 4cx$. We consider $\phi(x, y, c) = 4cx - y^2$. Then $\frac{\partial \phi}{\partial c}$ = 4x Eliminating c from $\phi(x, y, c) = 0$ and,

$\frac{\partial \phi}{\partial c}$ = 0, we get x = 0, y = 0 i.e. x = y = 0 gives the required c-discriminant.

Let $f(x,y,p) = 0$ denote a first order differential equation. The p-discriminant of the equation $f(x,y,p) = 0$ is defined as the p-eliminant between the equation $f(x,y,p) = 0$ and $\frac{\partial f}{\partial p} = 0$ provided $f(x,y,p)$, $\frac{\partial f}{\partial p}$ are continuous in the domain of the differential equation. The p-discriminant represents the locus for each of the point of which $f(x,y,p) = 0$ has equal values of p. As for example we consider a differential equation $p^2 y + p(x - y) - x = 0$.

Let. $f(x, y, p) = p^2 y + p(x - y) - x = 0$. Then $\frac{\partial f}{\partial p} = 2py + x - y$

Eliminating p from $f(x, y.p) = 0$ and $\frac{\partial f}{\partial p} = 0$, we get.

$$\left(\frac{y-x}{2y}\right)^2 y+\left(\frac{y-x}{2y}\right)(x-y)=x$$

i.e. $(x+y)^2=0$

or, $x+y=0$, which is the required p-discriminant.

Remark :

It is easy to observe that the equations are of the same degree in c and p, and therefore whenever there is a p-discriminant, there is a c-discriminant.

Note :

The singular solutions of a differential equation can be found by exploring the following situations :

(a) p-equation has multiple roots.

(b) c-equation has multiple roots.

Envelope of a system of curves $\phi(x,y,c)=0,$ if it exists, satisfies the differential equation $f(x,y,p)=0$ and this soluton is evidently a singular solution. Thus if $E(x,y)=0$ represents the envelope then E is a factor of both c-discriminant and p-discriminant and also the soluton of the differential equation.

We have already seen that both the p-discriminant and c-discriminant of $f(x,y,p)=0$ and its solution $\phi(x,y,c)=0$ respectively contain the envelope (if it exists) of the system of curves $\phi(x,y,c)=0$. But it can be seen that the c-discriminant and p-discriminant contain other loci which are different from the envelope and generally they do not satisfy the differential equation. These are called **extraneous loci**.

Not the p-discriminant relation gives the locus of such points for which p has at least two equal values. It may so happen that these two equal values of p belong to two distinct curves which are not consecutive but which touch each other at that point of consideraton. This point will satisfy the p-discriminant but not the c-discriminant. Also the point not being on the envelope will not satisfy the differential equation $f(x,y,p)=0$. The locus of such points which are the points of contact of two non consecutive curves at which the p has equal values is called **tac-locus**. So if $T(x,y)=0$ be the locus, then T(x,y) is a factor of p-discriminant but not of c-discriminant.

The c-discriminant relation is the locus of such points for which c has at least two equal values. It may so happen that each curve of the family $\phi(x,y,c)=0$ has a double point whose nature is that of a node and then the locus of the nodes is called the nodal locus. Thus if N(x,y)=0 be the nodal locus, then N(x,y) is a factor of c- discriminant but not of p-discriminant.

If each member of the family $\phi(x, y, c) = 0$ has a cusp then the locus of those cusps is known as cuspidal locus. Thus if C(x,y)=0 be the cuspidal locus, then C(x,y) is a factor of both c-discriminant and p-discriminant but not the soluton of the differential equation.

Here using symbols E,N,T and C for envelope, nodal locus, tac-locus, cuspidal locus respectively we can summarize the results int the following ways :

Discr$_c$ $\varphi(x,y,c)$: E. N^2. C^3 = 0

Discr$_p$ $f(x,y,p)$: ET^2C = 0

Example : Examine for singular solution and extraneous loci, if any for the differential equation

$4xp^2 - (3x - a)^2 = 0$ ....(a)

Solving for p we get $p = \pm \dfrac{3x - a}{2\sqrt{x}}$

i.e, $\dfrac{dy}{dx} = \pm \dfrac{3x - a}{2\sqrt{x}}$

or. $dy = \pm \dfrac{3x - a}{2\sqrt{x}} dx$

Integrating we get $y + c = \pm \left(x^{\frac{3}{2}} - ax^{\frac{1}{2}} \right) = \pm\sqrt{x}(x - a)$

therefore $(y + c)^2 = x(x - a)^2$ ....(b)

i.e. $c^2 + 2cy + y^2 - x(x - a)^2 = 0$ ....(c)

From, (c), Discr$_c$ $\phi(x, y, c)$: $4y^2 - 4\left\{y^2 - x(x - a)^2\right\} = 0$

or, $x(x - a)^2 = 0$ ....(d)

From (a) $\text{Discr}_p\ f(x, y, p)$: $0 - 4.4x. (3x-a)^2 = 0$ ....(e)

So from (d) and (e), x is the common factor. Hence x = 0 is the singular solution of (a).

Again $3x - a = 0$ is a **tac-locus**, since it appears twice in the p-discriminant relation (e) but does not occur in (d).

Also $x-a = 0$ is a nodal-locus since it appears twice in (d) but does not occur in (e).

Exercises :

a. Solve the following equations and find the singular solution, if any :

(i) $y^2\left(1+p^2\right)=a^2$

(ii) $8ap^3 = 27y$

(iii) $p^4 = 4y(xp-2y)^2$, put $y = u^2$

(iv) $p^2(2-3y)^2 = 4(1-y)$

(v) $xp^2 - 2py + 4x = 0$

b. Examine for singular solutions of the equations :

(i) $3y = 2px - 2\dfrac{p^2}{x}$

(ii) $4xp^2 = (3x-1)^2$

(iii) $x^3p^2 + x^2yp + a^2 = 0$

(iv) $y^2(y-xp) = x^4p^2$

(v) $\left(8p^3 - 27\right)x = 12.p^2y$

(vi) $p^3 = y^4(y+xp)$

c. Reducting the differential equation :

$$xp^2 - 2py + x + 2y = 0$$

to Clairaut's form by the transformations $x^2 = u$ and $y - x = v$, find its singular solution, if any.

(d) Reducing the differential equation :

$$x^2p(2p+1) + 2pxy + (p+2)y^2 + (p+1)^2 = 0$$

to Clairaut's form by the transformations $x+y = u$ and $xy-1 = v$, find its singular solution, if any.

3.3 Second Order Differential Equation

A linear ordinary differential equation of *n*th order is given by

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+..........+P_n y=F(x) \qquad (1)$$

In the domain $D \subseteq R$, where each of $P_1, P_2, ...P_n$ is either a constant or a function of x and F is function of x on D.

In P_1, P_2, ..., P_n are all constants then the differential equation

$\frac{d^n y}{dx^n}+P_1\frac{a^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+.......+P_n y=F(x)$ is known a **linear ordinary differential equation with constant coefficients.**

Now in the linear ordinary differential equation with constant coefficients of the above form if we replace $\frac{d}{dx}$ by D in (1) we have

$$(D^n + P_1D^{n-1} + P_2D^{n-2} + ... + P_n)y = F(x) \qquad (2)$$

i.e. $f(D)y = F(x)$ (3)

where $f(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + ... + P_n$.

If $F(x) = 0$, (3) becomes

$$f(D)y = 0 \qquad (4)$$

(4) is called the **correspoinding homogeneous equation** to (1) and solution of (4) is called the **complementary function or complementary solution or C. F** of (1) The solution due to non homogeneous part $F(x)$ is called the particular solution **(PI)** of (1). The complete or general solution of the differential eqution (1) is thus ***y* = C. F. + P. I.**

3.4 Theorem : Existence Theorem

Let P_1, P_2,...... , P_n be some constants and let a point x_0 be in [*a*, b] within *R*. If $\alpha_1, \alpha_2, ..., \alpha_n$ are any *n* constants there exists a solution ϕ of $f(D)y = 0$ on $[a, b]$ satisfying

$$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2,\ ......,\ \phi^{n-1}(x_0) = \alpha_n$$

3.5 Theorem : Uniqueness Theorem

Let x_0 be in [a, b] within R and let α_1, α_2,.... α_n be any n constants. Then there is at most one solution ϕ of f(D) = 0 satisfying

$$\phi(x_0) = x_1, \phi'(x_0) = \alpha_2, ..., \phi^{n-1}(x_0) = \alpha_n .$$

3.6 Wronskian

The **wronskian** of n differentiable functions y_1, y_2 ..... y_n , denoted by $W(x)$ or $W(y_1, y_2, ... y_n)$ or, W (y_1, y_2, ... y_n : x), is defined by

$$W(y_1, y_2,..... y_n : x) = \begin{vmatrix} y_1 & y_2 & .... & y_u \\ y_1' & y_2' & .... & y_n' \\ .... & .... & .... & .... \\ y_1^{n-1} & y_2^{n-1} & .... & y_n^{n-1} \end{vmatrix}$$

Theorem : The function y_1, y_2, ... y_n will be linearly independent solutions of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{a^{n-1} y}{dx^{n-1}} + P_2 \frac{a^{n-2} y}{dx^{n-2}} + ... + P_n y = F(x)$$

if F and P_1 , P_2, ... Pn are analytic in [a, b]

Definition : Any set y_1, y_2 .....y_n of n linearly indepdent solution of the homogeneous linear nth order differential equation $f(D)y = 0$ in [a, b] is said to be a **fundamental set of solutions** in the interval [a,b].

Theorem : If $y = f(x)$ be the general solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{a^{n-1} y}{dx^{n-1}} + P_2 \frac{a^{n-2} y}{dx^{n-2}} + ... + P_n y = 0 \qquad \text{(a)}$$

and $y = \phi(x)$ be a solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{a^{n-1} y}{dx^{n-1}} + P_2 \frac{a^{n-2} y}{dx^{n-2}} + ... + P_n y = x \qquad \text{(b)}$$

then $y = f(x) + \phi(x)$ is the general solution of the equation (b).

Theorem : If $y = y_1$ is a solution of the reduced equation (4) in D, then $y = c_1y_1$ is a solution of (4) as well, where c_1 is an arbitrary constant.

3.7 Theorem : Principle of Superposition

If y_1 and y_2 be two solutions of the differential equation $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$, then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1, c_2.

3.8 Theorem

If y_1 and y_2 be two solutions of the diffrential equation $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$ and if further there is a point where the Wronskian of y_1 and y_2 is non zero, then the family of solutions $y = c_1y_1 + c_2y_2$ with arbitrary coefficients c_1, c_2 includes every solution of the equation

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0.$$

Last theorem states that, as long as the Wronskian of y_1 and y_2 is not every where zero, the linear combination $y = c_1y_1 + c_2y_2$ spans all the solutions of the equation $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$.

In this case the expression $y = c_1y_1 + c_2y_2$ is said to be the general solution. The solutions y_1 and y_2, with non zero Wronskian, are said to form a **fundamental set of solution** of (5).

Now we pay our attention to the equation of the following form :

$$P\frac{d^2y}{dx^2}+Q\frac{dy}{dx}+Ry=0 \qquad (5)$$

where P ($\neq$ 0), Q and R are all constains.

We take the following simple example :

$$\frac{d^2y}{dx^2} - y = 0 \qquad (6)$$

Comparing (6) with (5) we will get P = 1, Q = 0, R = –1. We can easily verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of (6). We can also conclude that the functions $c_1y_1 = c_1e^x$, $c_2y_2 = c_2e^{-x}$ satisfy the differential equation (6) as well. Further the function $y = c_1e^x + c_2e^{-x}$ is also a solution of (6), for any arbitrary values of c_1, c_2. Again the Wronskian in this case is given by

$$W(y_1, y_2, :x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0 .$$

Hence, $y = c_1e^x + c_2e^{-x}$ is the general solution of (6)

As the coefficients c_1, c_2 in the general solution $y = c_1e^x + c_2e^{-x}$ are arbitrary, this expression represents a doubly infinite family of solutions of (6). Based on this observation. we suppose a trial solution of (5) of the form $y = e^{mx}$, where m is the parameter to the determined. Then one can have

$$y = e^{mx}, \ \frac{dy}{dx} = me^{mx}, \ \frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting the above results in (6) we obtain

$Pm^2e^{mx} + Qme^{mx} + Re^{mx} = 0$

$(Pm^2 + Qm + R)\, e^{mx} = 0$

Since $e^{mx} \neq 0$, we have, $Pm^2 + Qm + R = 0$.

Equation (7) is called the **Auxiliary Equation (A. E.)** for the ordinary differential equation (5).

Now we re-write (5) in the following form :

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \qquad (8)$$

where $p = \frac{Q}{P}$ and $q = \frac{R}{P}$. Then the A. E becomes $m^2 + pm + q = 0$. (9)

Now we have three different types of roots of the A. E. (9)

a. Roots are real and distinct

b. Roots are real and equal

c. Roots are complex conjugate

In the corresponding bomogeneous equation (4) for the differential equation (3) we put $y = e^{mx}$ as a trial solution and this gives the auxiliary equation f (m) = 0

Case-i. If $m_1, m_2 .... m_n$ be the distinct real roots of the auxiliary equation f(m) = 0 then the solution of (4) is given by

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$ where, c_1, c_2, ..., c_n are constants.

Case-ii If $m_1, m_2, ..., m_n$ be the real roots of the auxiliary equation $f(m) = 0$ and if further $m_1 = m_2 = ... m_r = m$, then the solution of (4) is

$$y = \left(c_1 + c_2 x + ... + c_r x^{r-1}\right) e^{mx} + c_{r+1} e^{m_{r+1} x} + c_{r+2} e^{m_{r+2} .x} + ... + c_n .e^{m_n .x}$$

Case-iii If $a \pm i\beta$ be the roots of the auxiliary equation $f(m) = 0$, then the solution of (4) must contain the term $e^{ax}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

Note : If $a \pm i\beta$ be the roots of the auxiliary equation f(m) = 0 repeated r times, the solution of (4) contains the term.

$e^{ax}(c_1 + c_2 x + ... + c_r x^{r-1}) \cos(\beta x) + e^{ax}(b_1 + b_2 x + ... + b_r x^{r-1}) \sin(\beta x)$.

The general form of non homogeneous ordinary differential equation with constant coefficients is given by (2) or (3). To solve a non homongeneous linear ordinary differential equation we first solve the corresponding homogeneous equation by the method as discussed above and this will give this corresponding C. F.

To get the P. I we employ the following scheme :

$\text{P.I} = \frac{1}{f(D)} X$ where $X = F(x)$

Now the general method of finding the expression for $\frac{1}{f(D)}X$ is a laborious one. We shall explain below the short methods for finding $\frac{1}{f(D)}X$ for some standard form of functions.

3.9 Method of finding the particular integral (P. I)

Rule 1. If $X = P(x)$, where $P(x)$ is a polynomial of degree n. Then

$$\text{P.I.} = \frac{1}{f(D)}X = \frac{1}{f(D)}P(x)$$

Note : First express, $f(D)$ in the form $(1 + \phi(D))$. Then expanding $(1 + \varphi(D))^{-1}$ as an infinite series in ascending powers of D and then operate on $P(x)$.

Rule 2. If $X = e^{ax}$, 'a' being a constant, then

$$\text{P. I.} = \frac{1}{f(D)}X = \frac{1}{f(D)}e^{ax}$$

$$= e^{ax}\frac{1}{f(a)}, \text{ if } f(a) \neq 0$$

$$= e^{ax}\frac{x}{f'(a)}, \text{ if } f'(a) \neq 0,\ f(a) = 0$$

In general,

$$\text{P.I} = e^{ax}\frac{x^n}{f^n(a)}, \text{ if } f(a) = 0,\ f'(a) = 0,\ \ldots,\ f^{n-1}(a) = 0,\ f^n(a) \neq 0$$

Rule 3. X = $\sin(ax)$ or, $\sin(ax + b)$ or, $\cos(ax)$ or, $\cos(ax + b)$

Let $f(D) = \phi(D^2)$, $\phi(-a^2) \neq 0$.

$$\text{P.I} = \frac{1}{f(D)}X = \frac{1}{f(D)}\sin(ax) = \frac{1}{\phi(D^2)}\sin(ax) = \frac{1}{\phi(-a^2)}\sin(ax)$$

or, $\frac{1}{f(D)}\sin(ax+b)=\frac{1}{\phi(D^2)}\sin(ax+b)$

$= \frac{1}{\phi(-a^2)}\sin(ax+b)$

or, $=\frac{1}{f(D)}\cos(ax) = \frac{1}{\phi(D^2)}\cos(ax) = \frac{1}{\phi(-\alpha^2)}\cos(ax)$

or, $= \frac{1}{f(D)}\cos(ax+b)=\frac{1}{\phi(D^2)}\cos(ax+b)=\frac{1}{\phi(-a^2)}\cos(ax+b)$

If $\phi(-a^2) = 0$, then

P. I. $=\frac{1}{f(D)}X=\frac{1}{f(D)}\sin(ax)=x\frac{1}{f'(D)}\sin(ax)$

or, $= \frac{1}{f(D)}\sin(ax+b) = x\frac{1}{f'(D)}\sin(ax+b)$

or, $= \frac{1}{f(D)}\cos(ax) = x\frac{1}{f'(D)}\cos(ax)$

or, $= \frac{1}{f(D)}\cos(ax+b)=x\frac{1}{f'(D)}\cos(ax+b)$

Rule 4. If $F(x) = c^{ax}\,\psi(x)$ where $\psi(x)$ is a function of x only.

Then P. I. $= \frac{1}{f(D)}X=\frac{1}{f(D)}e^{ax}\psi(x)=e^{ax}.\frac{1}{f(D+a)}\psi(x)$

Rule 5. If $F(x) = x^n\psi(x)$ where $\psi(x)$ is a function of x only.

Then P. I $=\frac{1}{f(D)}X=\frac{1}{f(D)}x^n\psi(x)=\left\{x-\frac{f'(D)}{f(D)}\right\}^n\frac{1}{f(D)}\psi(x)$

3.10 Properties of D-operator

(a) D, D^2, D^3, ... denote the differentiations with respect to x once, twice, thrice..... respectively.

(b) $\frac{1}{D}, \frac{1}{D^2}, \frac{1}{D^3}$, ... denote the indefinite integration with respect to x once, twice, thrice,..... respectively.

(c) $\frac{1}{D} X = \int X dx$

(d) $\frac{1}{D^n} X = \iiint \ldots \int X (dx)^n$

Example : Solve $(D^2 + 2D + 1)y = x^3 + x^2 + x$.

Solution : Let $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation of the given equation. Then the A. E. is of the form

$m^2 + 2m + 1 = 0$

i.e. $m = -1, -1$

Therefore, the C. F of the given differential equation is of the form

C. F. = $(a + bx)e^{-x}$, where a, b are arbitrary constants.

The particular integral is

P. I. $\frac{1}{(D+1)^2}(x^3 + x^2 + x)$

$= (D + 1)^{-2} (x^3 + x^2 + x)$

$= (1 - 2D + 3D^2 - 4D^3 + \ldots\ldots)(x^3 + x^2 + x)$

$= (x^3 + x^2 + x) - 2. (3x^2 + 2x + 1) + 3(6x + 2) = 24$

$= x^3 - 5x^2 + 15x - 20$

Thus the general solution is given by

y = C. F + P. I= $(a + bx) e^{-x} + (x^3 - 5x^2 + 15x - 20)$

Example : Solve : $(D^2 - 3D + 2)y = e^x$

Solution : Let $y = e^{mx}$ be the trial solution of the given equation. Then the A. E. is of the form

$m^2 - 3m + 2 = 0$

i.e. $m = 1, 2$

Therefore, the C. F of the given differential equation is of the form

C. F. = $ae^{ex} + be^{2x}$, where a, b are arbitrary constains.

Now let $f(D) = D^2 - 3D + 2$

The particular integral is

$$\text{P. I} = \frac{1}{f(D)}e^x = \frac{x}{f'(1)}e^x, \text{ since } f'(1) \neq 0, f(1) = 0$$

$= - xe^x$.

Thus the general solution is given by y = C. F. + P. I = $ae^x + be^{2x} - xe^x$.

Problems : (a) Solve : $(D^2 + 4)y = \sin 3x$.

(b) Solve : $(D^2 + 9)y = \sin 3x + 5 \cos 3x$.

(c) Solve : $(D^2 - 2D + 2)y = \cos x + \sin 2x$.

(d) Solve : $(D^2 - 5D + 6)y = e^x \cos x$.

(e) Solve : $(D^2 - 4D + 4)y = xe^{2x} \cos x$.

(f) Solve : $(D^2 - 5D + 6)y = x^2e^{3x}$.

3.11 Homogeneous Linear Differential Equations with Variable Coefficients

A linear ordinary differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{a^{n-1} y}{dx^{n-1}} + ... + P_n y = X \qquad (1)$$

where P_1, P_2, ..., P_n are constants and X is either a constant or a function of x only is called a **homogeneous linear differential equation.** This is also known as **Euler's Equation.**

Now we want to change the independent variable by using the relation

$x = e^z$, i.e., $z = \log x$ (2)

This gives, $dz = \frac{dx}{x}$, i.e, $\frac{d}{dz} \equiv x\frac{d}{dx} \equiv xD \equiv D'$, where $D \equiv \frac{d}{dx}$, $D' \equiv x\frac{d}{dx}$,

Thus $xDy = D'y$

Now, since $\frac{dy}{dz} = x\frac{dy}{dx}$

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + \frac{dy}{dz}$$

So, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dy^2} - \frac{dy}{dz} = D'(D'-1)y$

Similary

$$x^r.\frac{d^ry}{dx^r} = \left[\prod_{i=0}^{r-1}(D'-i)\right]y \quad (3)$$

Now using the relations given by (2) and (3) the differential equation (1) will be changed into the form of a linear differential equation with constant coefficients. Then we can write it in the form $f(D')y = X'$, where X', is a function of z only.

So, we can solve the problem $f(D')y = X'$ by the method of linear differential equation with constant coefficients.

Now let us suppose that a second order differential equation takes the following form :

$$(ax+b)^2\frac{d^2y}{dx^2} + (ax+b)P\frac{dy}{dx} + Qy = F(x) \quad (4)$$

where P, Q, a, b are constants and F is a function of x on $\left(\frac{-b}{a}, \infty\right)$ which is a homogeneous linear differential equation as well.

Example : Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x.\sin(\log x)$

Solution : First we change the independent variable x to z by the transformation $x = e^z$, i.e, $z = \log\ x$.

So, $\dfrac{dy}{dz} = x\dfrac{dy}{dx}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$

The given equation reduces to

$(D'^2 + 1)y = z.\ \sin z$ (a)

Let $y = e^{mx}$ be the trial solution of the reduced equation of (a). Then the corresponding A. E. is of the form $m^2 + 1 = 0$, So, $m = i, -i$.

Therefore the C. F. = A sin z + B cos z, where A, B are arbitrary constants.

Now, $\text{P.I} = \dfrac{1}{(D'^2+1)}(z.\sin z)$

$$= \left\{z - \frac{1}{(D'^2+1)}(2D')\right\}\frac{1}{(D'^2+1)}\sin z$$

$$= \left\{z - \frac{1}{(D'^2+1)}(2D')\right\}\left(z\cdot\frac{1}{2D'}\sin z\right)$$

$$= \left\{z - \frac{1}{(D'^2+1)}(2D')\right\}\left(z\cdot\frac{1}{2}(-\cos z)\right)$$

$$= -\frac{z^2}{2}\cos z + \frac{1}{(D'^2+1)}\left(D'(z\cos z)\right)$$

$$= -\frac{z^2}{2}\cos z + \frac{1}{(D'^2+1)}(\cos z - z\sin z)$$

$$= -\frac{z^2}{2}\cos z + \frac{1}{\left(D'^2+1\right)}\left(\cos z\right) - \frac{1}{\left(D'^2+1\right)}\left(z\sin z\right)$$

$$= -\frac{z^2}{2}\cos z + z\frac{1}{2D'}\left(\cos z\right) - P.I$$

$$= -\frac{z^2}{2}\cos z + z\frac{1}{2}\cdot\sin z - P.I.$$

Therefore, P. I. = $-\frac{z^2}{4}\cos z + z\frac{1}{4}\cdot\sin z$

Therefore the general solution of the equation (a) is given by

$$y = A\sin z + B\cos z - \frac{z^2}{4}\cos z + z\cdot\frac{1}{4},\ \sin z$$

By putting $z = \log x$ the general solution of the given equation is

$$y = A\sin(\log x) + B\cos(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + (\log x)\frac{1}{4}\cdot\sin(\log x),$$

$0 < x < \infty$.

3.12 Method of Undetermined Coefficients

We consider the following problem of the non homogeneous differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

The method of undetermined coefficients is a procedure for finding the particular solution of the equation (1) where R is an exponential, or a sine or cosine, a polynomial, or some combination of such functions.

Now, we are going to study this method of undermined coefficients throug an example.

Suppose $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=e^{ax}$ (2)

If we differentiate e^{ax}, we have the same function with some numeric constant. Now this is the procedure to find the particular integral. So let $y_p=e^{ax}$ be the P. I. of (2), and we guess that

$y_p = Ae^{ax}$ (3)

might be a particular solution. Here A is the undetermined coefficient and it is to be so chosen that (3) satisfies (2). Then

$A\left(a^2+Pa+Q\right)e^{ax}=e^{ax}$

Hence, $A=\dfrac{1}{a^2+Pa+Q}$, if $a^2 + Pa + Q \neq 0$.

Now if $a^2 + Pa + Q = 0$, then 'a' is a root of A. E. We take

$y_p = Axe^{ax}$ (4)

Then from (2) we get $A=\dfrac{1}{2a+p}$, if $2a + P \neq 0$

Again, if $2a + P = 0$, then we take $y_p = Ax^2e^{ax}$ and we repeat the above procedure if the order of the differential equation is more than two.

Therefore : If y_1 and y_2 are two solutions of the non homogeneous differential equation (1) then their difference $y_1 - y_2$ is a solution of the corresponding homogeneous differential equation.

If, in addition, Y_1 and Y_2 determine a fundamental set of solutions of the corresponding differential equation (2), then $Y_1 - Y_2 = c_1y_1 + c_2y_2$, where c_1 and c_2 are certain constants.

Example : Solve by the method of undetermined coefficients, the equation $(D^2 + 1)y = 10e^{2x}$ for the condition $y = 0$, $Dy = 0$ when $x = 0$.

Solution : Here it is given that $(D^2 + 1)y = 10e^{2x}$ *(1)*

Let $y = e^{mx}$ be the trial solution of the reduced differential equation of (a) Then the A. E is

$m^2 + 1 = 0$, i.e., m = i, – i.

The complementary functon is

C. F. = $C_1 \cos x + c_2 \sin x$.

where c_1 and c_2 are certain constants. We assume the particular integral in the form

P. I = Ae^{2x}, where A is a constant to be determined (since 2 is not a root of the A. E).

So, $(D^2 + 1)Ae^{2x} = 10e^{2x}$

i.e. $5Ae^{2x} = 10e^{2x}$

or, $A = 2$

Thus the general solution is given by

$y = c_1 \cos x + c_2 \sin x + 2e^{2x}$

From the condition $y = 0$ when $x = 0$ we get $c_1 = -2$ and from the condition $Dy = 0$ when $x = 0$ we get $c_2 = -4$. So the final complete solution is $y = -2 \cos x - 4 \sin x + 2e^{2x}$.

Working Rule :

(a) R = e^{ax}

(1) When a is not a root of A.E. i.e. e^{ax} is not in the complementary function, take $y_p = Ae^{ax}$.

(2) When a is a simple root of A. E. i.e. e^{ax} is in the complementary function, take $y_p = Axe^{ax}$.

(3) When a is a double root of A. E. i.e. e^{ax} is in the complementary function, take $y_p = Ax^2e^{ax}$.

(b) R = sin (ax) or cos (ax)

(1) When sin(ax) or cos (ax) is not in C. F., take

$y_p = A \sin(ax) + B \cos(ax)$

(2) When sin (ax) or cos (ax) is in C. F., take

$y_p = x.(A \sin(ax) + B \cos(ax))$

(c) $R = a_0 + a_1x + ... + a_nx^n$

(1) if $P \neq 0$, $Q \neq 0$, we take $y_p = A_0 + A_1x + ... \ A_nx^n$

(2) if $P \neq 0$, $Q = 0$, we take $y_p = x(A_0 + A_1x + ...+ A_nx^n)$

(3) if P = 0, Q = 0, we take $y_p = x^2 (A_0 + A_1x + ...+ A_nx^n)$

(d) $R = e^{ax} \sin (bx)$ or $\sin (bx) (a_0 + a_1x + ... + a_nx_n)$ or, $e^{ax} (a_0 + a_1x + ... + a_nx^n)$

Modify y_p accordingly with the help of (a), (b) and (c).

3.13 Method of Variation of Parameters

The main advantage of the method of variation of parameters is that it is a general method. In principle, it can be applied to any ordinary differential equation, and it requires no detailed assuptions about the form of the solution. In fact later in this section we use this method to derive a formula for a particular solution of an arbitrary second order linear non homogeneous differential equation. On the other hand, the method of variation of parameters eventually requires evaluation of certain integrals involving the non homogeneous term in the differential equation.

We seek a method of finding a particular integral of an ordinary differential equation for which the complementary function is known. This is the main objective of the method of variation of parameters.

Now we consider the following second order linear differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r \qquad (1)$$

where p, q, r are given continuous functions in x. We now assume that $c_1y_1 + c_2y_2$,where c_1, c_2 are both constant, be the general solution of corresponding homogeneous equation.

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

i.e. the C. F. of (1)

Now we replace c_1, c_2 by the function A and B respeectively. This gives

$$y = Ay_1 + By_2 \qquad (2)$$

Then we try to determine A and B so that the expression in (3) is a solution of the non homogeneous equation (1) rather than the homogeneous equation (2). This method is known as the **Method of variation of parameters**.

Calculations yield the expressions of the desired functions A and B as $A=-\int \frac{y_2 r}{w(y_1, y_2 : x)}dx$ and $B=\int \frac{y_1 r}{w(y_1, y_2 : x)}dx$. Substituting these two expression of A and B in (3) we get particular integral of the non homogeneous equation (1).

Theorem : If the functions p, q, r are continuous functions in an open interval I and if the functions y_1, y_2 are linearly independent solutions of the homogeneous equation corresponding to the non homogeneous equation $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=r$, then a particular solution of this equation is $y = Ay_1 + By_2$ and the general solution is $y = c_1y_1 + c_2y_2 + Ay_1 + By_2$.

Note that the two solutions y_1, y_2 of the corresponding homogeneous equation (2) are linearly indendpent.

Let us consider a second order differential equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=r \qquad \text{(a)}$$

in which p, q are constants and $r = r(x)$.The corresponding homogeneous equation of the differential equation (a) is as follows

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0 \qquad \text{(b)}$$

Then the general solution of the differential equation (b) i.e. the complementary function of (a) is

$$y_c = A.u + B.v \qquad \text{(c)}$$

where A, B are constants.

Now as u and v are two linearly independent solutions of (b) we have

$$\frac{d^2u}{dx^2} + p.\frac{du}{dx} + qu = 0 \tag{d}$$

$$\frac{d^2v}{dx^2} + p.\frac{dv}{dx} + qv = 0 \tag{e}$$

Let us assume the general solution in the form $y = A.u + B.v$ (f)

Here A and B are treated as functions of x.

Differentiating (f) with respect to x, we get

$$\frac{dy}{dx} = \left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + \left(u\frac{dA}{dx} + v\frac{dB}{dx}\right) \tag{g}$$

Let us choose A and B in such a way that

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \tag{h}$$

Then from (g) we get

$$\frac{dy}{dx} = \left(A\frac{du}{dx} + B\frac{dv}{dx}\right) \tag{i}$$

Differentiating both sides of (1) with respect to x, we get

$$\frac{d^2y}{dx^2} = A\frac{d^2u}{dx^2} + B\frac{d^2v}{dx^2} + \frac{dA}{dx}\frac{du}{dy} + \frac{du}{dx}\frac{dv}{dx} \tag{j}$$

Now putting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (a), get

$$A\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + qu\right) + B\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx} + qv\right) + \left(\frac{dA}{dx}\frac{du}{dx} + \frac{dB}{dx}\frac{dv}{dx}\right) = r$$

So, $\left(\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}\right)=r$ (k)

Now using (h) in (k) we can get

$$\frac{dA}{dx}.\frac{du}{dx}-\frac{u}{v}\frac{dA}{dx}.\frac{dv}{dx}=r$$

i.e., $-\left(u\frac{dv}{dx}-v\frac{dv}{dx}\right)dA=vrdx$

or, $-W(u, v\ ;\ x)\ dA = vrdx$

The expression W $(u, v\ ;\ x)$ = $\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)$ gives the corresponding wronskian.

Integrating we get $A=-\int\frac{v.r}{W(u.v;x)}dx+c_1$, where c_1 is an arbinary constant.

Similary, we have $B=\int\frac{u.r}{w(u.v.x)}dx+c_2$, where c_2 is an arbitrary constant.

Using the above expression of A and B in (f) the general solution takes the following form

$$y=c_1u+c_2v-u\int\frac{v.r}{\mathrm{W}(u,v:x)}dx+v\int\frac{u.r}{\mathrm{W}(u,v:x)}dx$$

Working Rule :

Step 1 : Find the complementary function of the given differential equation (1). Let the complementary function be C. F. = A. u + B. v.

Step 2 : Check Wronskian $W(u, v) \neq 0$.

Step 3 : Suppose $y = A.\ u + B.v$ where A and B are functions of x.

Step 4 : Calculate $A=-\int\frac{vr}{\mathrm{W}(\mathrm{u},\mathrm{v}:\mathrm{x})}dx+c_1$

and $B=\int \frac{u.r}{W(u,v:x)}dx+c_2$, where c_1 and c_2 are arbitrary constant.

Step 5 : Put the values of *A*, *B* in the expression at Step 3 and this will give the general solution of the given differential equation.

Exercises :

a. Solve the following diffirential equations with constant coefficients :

i. $\frac{d^3y}{dx^3}+y=\left(e^x+1\right)^2$

ii. $\frac{d^3y}{dx^3}-y=x^3-x^2$

iii. $\frac{d^2y}{dx^2}-y=x^2\cos x$

iv. $\frac{d^2y}{dx^2}+y$ = cosec x

v. $\frac{d^2y}{dx^2}+2y=x^2e^{3x}+e^x\cos 2x$

vi. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$

vii. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+4y=e^x\cos x$

viii. $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x\left(x+e^x\right)$

ix. $\left(D^2-D+1\right)y=2\sin(3x)$

x. $\left(D^2 - 1\right)y = x\sin(x) + \left(1 + x^2\right)e^x$

b. Solve the following homogeneous linear differential equations :

i. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 5y = 2\log x$

ii. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$

iii. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (1+x)^2$

iv. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = logx.sinx(logx)$

v. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = \cos(logx) + xsin(logx)$

vi. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x + \sin x$

vii. $(5+2x)^2\dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$

viii. $(2+3x)^2\dfrac{d^2y}{dx^2} + 5(2+3x)\dfrac{dy}{dx} - 3y = 1 + x + x^2$

c. Solve the following differential equations, using the method of undertermined coefficients:

i. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 12 + 15x^2$

ii. $\dfrac{d^2y}{dx^2} + y = e^x + x^2$

iii. $\frac{d^2y}{dx^2}-9y=x+e^{2x}-Sin2x$

iv. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=3sinx$

v. $\frac{d^2y}{dx^2}+4y=sin2x$

vi. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=3xe^x$

vii. $\frac{d^2y}{dx^2}+4y=x^2\sin 2x$

d. Solve the following differential equations, using the method of variation of parameters :

i. $\frac{d^2y}{dx^2}+4y=4\tan 2x$

ii. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+9y=\sec(3logx)$

iii. $\frac{d^2y}{dx^2}+y=\sec^3 x.tanx$

iv. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=9e^x$

v. $\frac{d^2y}{dx^2}-y=\frac{2}{1+e^x}$

vi. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$

vii. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^x}{1+e^x}$

viii. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x tanx$

ix. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2, 0 < x < \infty$

x. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x, 0 < x < \infty$

3.14 Simultaneous Linear Differential Equations with Constant Coefficients:

The system of a linear simultaneous ordinary differential equations with constant coefficients is of the following form :

$\phi_{11}(D)\, x_1 + \phi_{12}(D)\, x_2 + \ldots\ldots + \phi_{1n}(D)\, x_n = f_1\,(t)$

$\phi_{21}(D)\, x_1 + \phi_{22}(D)\, x_2 + \ldots\ldots + \phi_{2n}(D)\, x_n = f_2\,(t)$

....

$\phi_{n1}(D)\, x_1 + \phi_{n2}(D)\, x_2 + \ldots\ldots + \phi_{nn}(D)\, x_n = f_n\,(t)$, where $x_1, x_2, \ldots, x_n$ are the dependent variables and ϕ_{ij} (D), *i. j* = 1, 2, ..., *n* are all rational functions of $D \equiv \frac{d}{dt}$ with constant coefficients and $f_i(t)$, $i = 1, 2, \ldots, n$, are the function of the independent variable *t*.

The method of operator :

Let *x*, *y* be the dependent variables and *t* be the independent variable. The equation with involve derivatives of *x* and *y* with respect to *t*. Let us denote the operator $\frac{d}{dt}$ by the symbol *D*.

Let us consider the simultaneous linear differential equation with constant coefficient for two variables as

$$\phi_1(D)x + \phi_2(D)y = f(t) \qquad (1)$$

and $$\psi_1(D)x + \psi_2(D)y = g(t) \qquad (2)$$

where $\phi_1(D), \phi_2(D), \psi_1(D), \psi_2(D)$ are all rational functions of *D* with constant coefficients and *f* and *g* are functions of *t*.

Now we operate both sides of (1) with $\psi_2(D)$ and both side of (2) with $\phi_2(D)$.

We get,

$\psi_2(D)\phi_1(D)x + \psi_2(D)\phi_2(D)y = \psi_2(D) f(t)$

$\phi_2(D)\psi_1(D)x + \phi_2(D)\psi_2(D)y = \phi_2(D) g(t)$

Subtracting we get,

$[\psi_2(D)\phi_1(D) - \phi_2(D)\psi_1(D)x = \psi_2(D) f(t) - \phi_2(D) g(t)$ which is a linear equation in x and can be used to find x as a function of t. Value of y can be obtained as a function of t by substituting the result of x in (1) or (2).

Example : Solve $\frac{dx}{dt} - 7x + y = 0,\ \frac{dx}{dt} - 2x - 5y = 0$

Solution : The given equations are $(D - 7)x + y = 0$ (a)

$(D - 5)y - 2x = 0$ (b)

Putting the value of $y = -(D - 7)x$ in (b), we have

$= (D - 5)(D - 7)x - 2x = 0$

So, $(D^2 - 12D + 37)x = 0$ (c)

Let $x = e^{mt}$ be the trial solution of the equation (c). Then the A. E is of the form

$m^2 - 12m + 37 = 0$

i.e. $m = 6 \pm i$

Therefore, the general solution of the equation (c) is

$x = e^{6t}(A\cos t + B\sin t)$, where A, B are arbitrary constants.

Putting the value of x in (a), we have

$y = -(D - 7)x = -(D - 7)\{(A\cos t + B\sin t)\} = e^{6t}[(A - B)\cos t + (A + B)\sin t]$.

Hence, the solution of the given simultaneous linear equation is given by

$x = e^{6t}(A\cos t + B\sin t)$

and,

$$y = e^{6t}\,[(A - B)\cos t + (A + B)\sin t]$$

Example : Solve $\dfrac{dx}{dt} + y = e^t$ **,** $\dfrac{dy}{dt} - x = e^{-t}$ **.**

Solution : The equations are

$$Dx + y = e^t \qquad \text{(a)}$$

$$-x + Dy = e^{-t} \qquad \text{(b)}$$

Differentiating both sides of (a) with respect to t we get

$$D^2x + Dy = e^t$$

i.e. $D^2x + (x + e^{-t}) = e^t$ [using (b)]

i.e., $(D^2 + 1)x = e^t - e^{-t}$ (c)

Let $x = e^{mt}$ be the trial solution of the reduced equation of (c). Then the A. E is of the form

$(m^2 + 1) = 0$

i.e. $m = \pm i$.

The complementary function of (c) is

C. F. = A cost + B sin t, where A, B are arbitrary constants,

Now, $\text{P.I.} = \dfrac{1}{(D^2+1)}\left(e^t - e^{-t}\right) = \dfrac{e^t}{2} - \dfrac{e^{-t}}{2}$.

Therefore, the general solution of (c) is

x = (A cos t + B sin t) + $\dfrac{e^t}{2} - \dfrac{e^{-t}}{2}$, where A, B are arbitrary constants,

Putting the above expression of x in (a), we have

$$y = e^x - D\left\{(A\cos t + B\sin t) + \frac{e^t}{2} - \frac{e^{-t}}{2}\right\}$$

Therefore, y = A sin t – B cost + $\frac{e^t}{2} - \frac{e^{-t}}{2}$

Hence, the solution of the given simultaneous linear equation is given by

$$x = (A\cos t + B\sin t) + \frac{e^t}{2} - \frac{e^{-t}}{2}$$

and, $$y = A\sin t - B\cos t + \frac{e^t}{2} - \frac{e^{-t}}{2}$$

Exercises:

Solve the following simultaneous linear differential equations :

i. $\frac{dx}{dt} + 5x - 2y = e^t, \frac{dy}{dt} - x + 6y = e^{2t}$

ii. $\frac{dx}{dt} + 4x + 3y = t, \frac{dy}{dt} + 2x + 5y = e^t$

iii. $\frac{dy}{dt} + 4x + 3y = \sin t, \frac{dy}{dt} + 2x + 5y = e^t$

iv. $\frac{dx}{dt} = 5x + 4y, \frac{dy}{dt} = -x + y$

v. $\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y$

vi. $\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y$

vii. $\frac{dy}{dt} + \frac{dy}{dt} + 2x + y = 0, \frac{dy}{dt} + 5x + 3y = 0$

3.15 Series Solution of the Ordinary Diffrential Equations:

The solutions of many differential equations can be expressed in terms of elementary functions, all of whose mathematical properties are well known. When required, the analytical behaviour of solutions that involve elementary functions can be explored by making use of their familiar properties. With either a pocket calculator of a software package, the method of calculating functional values is usually based on a series expansion of the function concerned.

Most of the ordinary differential equations cannot be solved in terms of elementary functions, yet some form of analytical solution is often needed rather than a purely numerical one. So the fundamental question that then arises is how to obtain a solution in the form of a series, when only the differential equation is knonw.

Definition : A function f defined in the interval I containing x_0 is said to be **analytic at** x_0 if $f(x)$ can be expressed as a power series $f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$, which has a positive radius of convergence.

Defintion : Consider the n-th order linear ordinary differential equation

$y^{(n)} + P_{n-1}(x)y^{(n-1)} + P_{n-2}(x)y^{(n-2)} + \ldots\ldots + P_0(x)y = f(x)$

A point x_0 is called all **ordinary point** of the given differential equation if each of the coefficients P_{n-1}, P_{n-2}, ..., P_0 and $f(x)$ are analytic at x_0.

Definition : Consider the n-th order linear ordinary differential equation

$y^{(n)} + P_{n-1}(x)y^{(n-1)} + P_{n-2}(x)y^{(n-2)} + \ldots\ldots + P_0(x)y = 0$ (a)

A point x_0 is called a **singular point** of the given different equation if it is not an ordinary point, that is, not all of the coefficients P_{n-1}, P_{n-2}, ..., P_0 are analytic at x_0

A point x_0 is called a **regular singular point** of the given differential equation if it is not an ordinary point but all $(x - x_0)^{n-k} P_k(x)$ are analytic for k = 0, 1, 2, ... (n – 1) i.e., all the limits given by $\lim_{x\to x_0} (x-x_0)^{n-k} P_k(x)$ exist and finite.

A point x_0 is called an **irregular singular point** of the given differential equation if it is neither an ordinary point nor a regular singular point.

3.16 Note : Test of Singularity at Infinity

To determine whether the point at infinity is a singular point or not, we transform the equation (a) by substituting $x=\frac{1}{t}$

Then $\frac{dy}{dx}=-t^2\frac{dy}{dt}$ and $\frac{d^2y}{dx^2}=t^4\frac{a^2y}{dt^2}+2t^3\frac{dy}{dt}$

Then the differential equation (a) becomes

$y^{(n)}_{(t)} + p'_{n-1}(t)y^{(n-1)}(t) + p'_{n-2}(t)y^{(n-2)}(t) + ... + p'_{n-2}(t)y(t) = 0$ (b)

If t = 0 is a singular point of (b) then the original equation (a) has a singularity at infinity.

Example : Find the ordinary and singular point (if any) of the differential equation

$$2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$$

Solution : The given differential equation

$2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$, can be written as

$$\frac{d^2y}{dx^2}+\frac{7x(x+1)}{2x^2}\frac{dy}{dx}-\frac{3y}{2x^2}=0$$

Comparing the above differential equation with $\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_0(x)y=0$,

we have,

$$P_1(x)=\frac{7(x+1)}{2x},\ P_0(x)=-\frac{3}{2x^2}$$

Since neither $\lim_{x \to 0} p_1(x)$ not $\lim_{x \to 0} p_0(x)$ does exist hence, $p_1(x)$, $p_0(x)$ are not analytic at $x = 0$.

Therefore, $x = 0$ is a singular point

Now, $\lim_{x \to 0} (x - 0)\, p_1(x) = \lim_{x\to 0} x\frac{7(x+1)}{2x} = \frac{7}{2}$ and

$$\lim_{x \to 0} (x - 0)^2\, p_0(x) = \lim_{x\to 0} x^2\left(-\frac{3}{2x^2}\right) = \frac{-3}{2}$$

So both the limits exist and finite and hence the point x = 0 is a regular singular point. All the points x $(\neq 0)$ are ordinary points.

Example : Show that the equation $\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{n(n+1)}{1-x^2}y = 0$ has a singularity at infinity.

Solution : Substituting $x = \frac{1}{t}$ to the given equation we have

$$\frac{dy}{dx} = -t^2\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = t^4\frac{a^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

Using the above results the given equation reduces to

$$t^4\frac{d^2y}{dt^2} + \frac{2t^2}{t^2-1}\cdot\frac{dy}{dt} + \frac{n(n+1)}{t^2-1}y = 0$$

$$\frac{d^2y}{dt^2} + \frac{2}{t(t^2-1)}\cdot\frac{dy}{dt} + \frac{n(n+1)}{t^2(t^2-1)}y = 0 \qquad \text{(a)}$$

Since $t = 0$ is a singular point of the equation (a) thus the given ODE has a singularity at infinity.

3.17 Series Solution about an Ordinary Point :

Theorem : Let x_0 be any real number and suppose that the coefficients $P_{n-1}, P_{n-2}, ...,$ P_0 in

$f(D)y = y^{(n)}(x) + P_{n-1}(x)\, y^{(n-1)}(x) + P_{n-2}(x) y^{(n-2)}(x) + ... + p_0(x)\, y(x)$ have convergent power series expansions in powers of $(x - x_0)$ in an interval $| x - x_0| < r$, $r > 0$.

If $\alpha_1, \alpha_2, ..., \alpha_n$ are any n constants, there exists a solution ϕ of the problem $f(D)y = 0$, such that $y(x_0) = \alpha_1$, $y'(x_0) = \alpha_2$, ..., $y^{(n-1)}(x_0) = \alpha_n$ with a power series expansion $\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ convergent for $|x - x_0| < R$ where the radius of convergence is $R \geq r$.

Theorem : Suppose that x_0 is an ordinary point of the n-th order linear ordinary differential equation $y^{(n)}(x) + P_{n-1}(x)y^{(n-1)}(x) + P_{n-2}(x)\, y^{(n-2)}(x) + ... + p_0(x)y(x) = f(x)$, where the coefficients $P_{n-1}(x) \cdot P_{n-2}(x)$, ..., $p_0(x)$ and $f(x)$ are analytic at $x = x_0$ then it has two non-trivial linearly independent power series solutions of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$, $|x - x_0| < R$, for some $R > 0$, where $a_n' s$ are constants and these power series converges in some interval $|x - x_0| < R$, $R > 0$ about x_0, R being the radius of convergence of the power series.

Example : Find the series solution of the following ordinary differential equation

$$\left(x^2 + 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Solution : The given differential equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{1}{x^2+1}y = 0 \qquad \text{(a)}$$

Comparing the above eqution with the equation $\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0$, we have $p_1(x) = \frac{x}{x^2+1}$, $p_0(x) = -\frac{1}{(x^2+1)}$.

We have for $i = 0, 1$

$p_i(x) = (-1)^{i+1} \cdot x^i \cdot (1+x^2)^{-1}$

$= (-1)^{i+1} \cdot x^i \cdot (1 - x^2 + x^4 - x^6 + \ldots), -1 < x < 1.$

So, $p_i(x)$ for $i = 0, 1$ can be expressed as power series and $x = 0$ that are convergent for $-1 < x < 1$ i.e. all the coefficients $p_1(x)$ and $p_0(x)$ are analytic at $x = 0$.

Hence, $x = 0$ is a ordinary point of the differential equation (a) and we take therefore.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \ (-1 < x < 1) \quad \text{(b)}$$

Now $\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, $-1 < x < 1$.

Putting these expressions of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (a), we have

$$\left(x^2+1\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\alpha} a_n x^n = 0$$

Therefore,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 1 + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the index of summation in the second series by 2 i.e. we replace n by $(n + 2)$ and use the initial value $n = 0$. Also we shift the index of summation in third series by 1 i.e. we replace n by $(n + 1)$ and use the initial value $n = 0$.

Then we get, $2a_2 - a_0 + (6a_3 + a_1)x$

$$+ \sum_{n=2}^{\infty} \{n(n-1)a_n + (n+2)(n+1)a_{n+2} + na_n - a_n\} x^n = 0$$

Equating the coefficients of various power of x to zero. we get

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$6a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6}$$

and, $n(n-1)a_n + (n+2)(n+1)\ a_{n+2} + na_n - a_n = 0$

i. e., $a_{n+2} = \frac{n-1}{n+2} a_n$ for $n \geq 2$.

Now putting n = 2, 3, 4, ... in the above recurrence relation, we get

$$a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0$$

$$a_5 = -\frac{2}{5}a_3 = \frac{2}{5.6}a_1$$

$$a_6 = -\frac{1}{2}a_4 = \frac{1}{16}a_o$$

$$a_7 = -\frac{4}{7}a_5 = -\frac{2.4}{7.6.5}a_1$$

and so on

Substituting the values of a_0, a_1, a_2, ... in (b) we get the required solution as

$$y(x) = a_0\left\{1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5}{128}x^8 + ...\right\} + a_1\left\{-\frac{1}{6}x^3 + \frac{2}{6.5}x^5 - \frac{2.4}{7.6.5}x^7 + ...\right\} ; -1 < x < 1$$

3.18 Series Solution about Regular Singular Point (Frobenius Method)

Theorem : If the point x_0 is a singular point of the differential equation

$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$, then it has at least one non-trivial solution of the form $y(x)=|x-x_0|^r\sum_{n=0}^{\infty}c_n(x-x_0)^n$, and this solution is valid in some interval $|x-x_0|<R$, where r is a certain constant (real or complex) and R > 0.

If $x = 0$ is regular singular point, we shall use this method to find the series solution about $x = 0$.

Consider the differential equation of the form

$$\frac{d^2y}{dx^2}+\frac{P(x)}{x}\frac{dy}{dx}+\frac{Q(x)}{x^2}y=0 \qquad \text{(a)}$$

where the functions P(x) and Q (x) are analytic for all $|x| < R$, $R > 0$.

We assume a trial solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^{n+r}, a_0\neq 0, 0<x<R \qquad \text{(b)}$$

Now $\frac{dy}{dx}=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$ and $\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$

Since $P(x)$ and $Q(x)$ are analytic at $x = 0$, then

$P(x) = c_0 + c_1x + c_2x^2 + ...$, $Q(x) = d_0 + d_1x + d_2x^2 + ...$

Thus

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \left(c_0+c_1x+c_2x^2+.......\right)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r} +$$

$$\left(d_0 + d_1 x + d_2 x^2 + ...\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0 \qquad \text{(c)}$$

Since (c) is an identity, we can equate to zero the coefficients of various power of x. The smallest power of x is x^r, and the corresponding equation is

$\{r(r - 1) + c_0 r + d_0\}a_0 = 0$

Since, by assumption $a_0 \neq 0$, we get,

$r^2 + (c_0 - 1)r + d_0 = 0$

This equation is known as **indicial equation** of (a). Solving this quadratic equation for r. one obtains r_1 and r_2.

Case-I : Let r_1 and r_2 be the roots of the indicial equations and $r_1 - r_2$ is not equal to an integer. Then the complete solution is given by

$y(x) = A.[y(x)]_{r = r1} + B.[y(x)]_{r = r2}$, $0 < x < R$, where A, B are arbitrary constants.

Case-II : Let r_1 and r_2 be the roots of the indicial equations and $r_1 = r_2$. Then complete solution is given by

$$y(x) = A.\left[y(x)\right]_{r=r_1} + B.\left[\frac{\partial y(x)}{\partial r}\right]_{r=r_2}, 0 < x < R$$

Case-III. Let r_1 and r_2 be the roots of the indicial equations and differs by an integer and if some of the coefficients of $y(x)$ become infinite when $r = r_1$, we modify the form of $y(x)$ by replacing a_0 by $b_0(r - r_0)$. Then we obtain two indepdenent solutions by putting $r = r_1$ in the modified form of $y(x)$ and $\frac{\partial y(x)}{\partial r}$, $0 < x < R$. The result of putting $r = r_2$ in $y(x)$ gives a numerical multiple of that obtained by putting $r = r_1$ and hence we reject the solution obtained by putting $r = r_2$ in $y(x)$.

Example : Find the power series solution of the equation using Frobenius method

$2x^2y''(x) + xy'(x) - (x + 1)y(x) = 0$ in powers of x.

Solution : The given differential equation can be written as

$$y''(x)+\frac{1}{2x}y'(x)-\frac{(x+1)}{2x^2}y(x)=0 \qquad \text{(a)}$$

Comparing the above differential equation with $\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_0(x)y=0$, we have

$p_1(x)=\frac{1}{2x}$ and $p_0(x)=-\frac{(x+1)}{2x^2}$. Here the point $x = 0$ is a singular point.

Now $\lim_{x\to x_0}(x-x_0)p_1(x) = \lim_{x\to 0}(x-0)\frac{1}{2x}=\frac{1}{2}$ and

$\lim_{x\to x_0}(x-x_0)^2 p_0(x) = \lim_{x\to 0}(x-0)^2\left\{-\frac{(x+1)}{2x^2}\right\}=\frac{-1}{2}$. So both the limits exist and finite. Hence the point $x = 0$ is a regular singular point.

Let us assume that the trial solution of the given equation is

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r},\ a_0 \neq 0,\ 0 < x < \infty \qquad \text{(b)}$$

Now, $y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2},\ 0 < x < \infty$$

Putting these values in (a), we have

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - (x+1)\sum_{n=1}^{\infty}a_n x^{n+r}=0$$

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+1} - \sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{2(n+r)(n+r-1)+(n+r)-1\} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{(2n+2r+1)(n+r-1)\} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Equating the coefficent of smallest power of x, namely x^r to zero the indicial equation becomes

$\{(2r + 1)(r - 1)\}a_0 = 0$. As $a_0 \neq 0$ the roots of the equation are $r = 1$ and $r = -\frac{1}{2}$.

Here the roots of the indicial equation are distinct and the difference is $1-\left(-\frac{1}{2}\right)=\frac{3}{2}$ which is not an integer, Now equating the coefficient of x^{n+r}, we obtain the recurrence relation as

$$(2n + 2r + 1)(n + r - 1)a_n - a_{n-1} = 0$$

$$a_n = \frac{a_{n-1}}{(2n+2r+1)(n+r-1)}$$

Putting n = 1, 2, 3.... we get

$$a_1 = \frac{a_0}{(2r+3)r}$$

$$a_2 = \frac{a_1}{(2r+5)(r+1)}$$

and so on

Putting these values in (b) we get

$$y(x) = a_0 x^r \left[1 + \frac{x}{(2r+1)r} + \frac{x^2}{(2r+5)(2r+3)r(r+1)} + ...\right] \qquad \text{(c)}$$

Putting $r = 1$ in (c), we get

$[y(x)]_{r=1} = a_0 x\left[1 + \frac{x}{5} + \frac{x^2}{70} + ...\right]$, $0 < x < \infty$. Next putting $r = -\frac{1}{2}$ in (c), we get

$\left[y(x)\right]_{r=\frac{1}{2}} = a_0 x^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} + ...\right]$, $0 < x < \infty$. Hence the required solution is given by

$y(x) = A.\left[y(x)\right]_{r=1} + B.\left[y(x)\right]_{r=-\frac{1}{2}}$, $0 < x < \infty$, where A and B are two arbitrary constants.

Exercise :

1. Use method of Frobenius to solve the following differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

2. Use method of Frobenius to solve the following differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - 1\right)y = 0$$

3. Use method of Frobenius to solve the following differential equation

$$\left(x^2 - x\right)\frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$$

4. Find the series solution of ODE :

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+x^2y=0$$ about the point x = 0.

5. Find the series solution of ODE

$$\frac{d^2y}{dx^2}+y=0$$ about the point x = 0.

6. Find the series soluton of ODE

$$2x^2\frac{d^2y}{dx^2}+\left(2x^2-x\right)\frac{dy}{dx}+y=0$$ about the point x = 0

7. Find the series solution of ODE

$$\frac{d^2y}{dx^2}-3x^2y=0$$ about the point x = 0 and given $y(0)$ = 1 and $y'(0)$ = 1.

8. Find the series solution of ODE

$$\frac{d^2y}{dx^2}-3\frac{dy}{dz}+2y=0$$ about the point x = 0

9. Find the series solution of ODE

$$\left(1-x^2\right)\frac{d^2x}{dx^2}+2x\frac{dy}{dx}-y=0$$ about the point x = 0.

10. Find the series solution of ODE

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$ about the point x = 0.

3.19 Bessel's Equation

The ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

where n is a non-negative real number, is called **Bessel's equation of order 'n'.**

3.20 Application of Bessel's Equation:

Bessel's equation appears in the problems related to Vibrations, electric fields, heat conduction etc.

Regular Sigularity about $x = 0$

The Bessel's equation can be rewritten as $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$

Since $\frac{1}{x}$ and $\left(1 - \frac{n^2}{x^2}\right)$ are not analytic at $x = 0$ i.e. since $\frac{1}{x}$ and $\left(1 - \frac{n^2}{x^2}\right)$ cannot be expressed in power series about $x = 0$, it follows that $x = 0$ is a singular point of Besseel's equation. Again, $\lim_{x\to 0} x.\frac{1}{x} = 1$ and $\lim_{x\to 0} x^2\left(1 - \frac{n^2}{x^2}\right) = -n^2$. So both these limits exist and are finite. Hence $x = 0$ a regular point of Bessel's equation.

3.21 Solution of Bessel's Equation : Bessel's Function

As $x = 0$ is a regular singular point of Bessel's equation we can express its solution in the form of power series about $x = 0$ using Frobenius method. We can take $y = \sum_{m=0}^{\infty} a_m x^{m+r}$, $a_0 \neq 0$. Solving we get

$$y = C_1 J_n(x) + C_2 J_{-n}(x)$$

Here C_1 and C_2 are two arbitrary constants. $J_n(x)$ is called the **Bessel's function of the first kind of order *n*** and it is given by

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n}$$

$J_{-n}(x)$ is called the **Bessel's function of the first kind of order –*n*** and it is given by

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-n+m+1)}\left(\frac{x}{2}\right)^{2m-n}$$

Here 'n' is not an integer.

If 'n' is an integer then the complete solution is

$$y = a_1 J_n(x) + a_2 J_n(x) \int \frac{dx}{x J_n{}^2(x)} = a_1 J_n(x) + a_2 y_n(x)$$

where $Y_{n\backslash}(x) = J_n(x)\int \frac{dx}{xJ_n{}^2(x)}$ and $Y_n(x)$ is called the **Bussel's function of second kind of order *n*** or the **Neumann's function.**

Derivations :

(1) We have $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)}\left(\frac{x}{z}\right)^{2m+n}$

So, $x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)} \frac{(x)^{2(m+n)}}{z^{2m+n}}$

Therefore,

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m.2(m+n)}{m!\Gamma(n+m+1)} \frac{x^{2(m+n)-1}}{2^{2m+n}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m.\cancel{(m+n)}}{m!\Gamma(n+m).\cancel{(n+m)}}.\frac{x^{2(m+n)-1}}{2^{2m+n-1}} \quad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!.\Gamma(n-1+m+1)}\left(\frac{x}{2}\right)^{2m+n-1}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!.\Gamma[(n-1)+m+1]} \left(\frac{x}{2}\right)^{(n-1)+2m}$$

$$\therefore \boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)} \longrightarrow (1)$$

We have

$$(2)\ J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m-n}$$

$$\text{So, } x^{-n} J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-n+m+1)} \frac{(x)^{2(m-n)}}{(2)^{2m-n}}$$

Therefore,

$$\frac{d}{dx}[x^{-n} J_{-n}(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m .2(m-n)}{m!.\Gamma(-n+m+1)} \frac{x^{2(m-n)-1}}{(2)^{2m-n}}$$

$$= \sum_{m-0}^{\infty} \frac{(-1)^m .(m-n)}{m!\Gamma(m-n).(m-n)} \cdot \frac{x^{2(m-n)-1}}{(2)^{2m-n-1}}$$

$$= x^{-n} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{m!.\Gamma(m-n)} . \left(\frac{x}{2}\right)^{2m-n-1}$$

$$= x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!.\Gamma(-n-1+m+1)} \left(\frac{x}{2}\right)^{(-n-1)+2m}$$

$$\therefore \boxed{\frac{d}{dx}\left[x^{-n} J_{-n}(x)\right] = x^{-n} J_{-n-1}} \longrightarrow (2)$$

(3) We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$\text{So, } x^{-n} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)} . \frac{(x)^{2m}}{(2)^{2m+n}}$$

Therefore,

$$\frac{d}{dx}[x^{-n}J_n(x)] = \sum_{m=1}^{\infty}\frac{(-1)^m.2m}{m!\Gamma(n+m+1)}.\frac{(x)^{2m-1}}{(2)^{2m+n}}$$

$$= \sum_{m=1}^{\infty}\frac{(-1)^m}{(m-1)!\Gamma(n+m+1)}.\frac{(x)^{2m-1}}{(2)^{2m+n-1}}$$

$$= x^{-n}\sum_{m=1}^{\infty}\frac{(-1)^m}{(m-1)!\Gamma(n+m+1)}.\left(\frac{x}{2}\right)^{2m+n-1}$$

$$= x^{-n}\sum_{m'=0}^{\infty}\frac{(-1)^{m'+1}}{m'!\Gamma(n+m'+2)}\left(\frac{x}{2}\right)^{2(m'+1)+n-1} \text{[we put } m - 1 = m'\text{]}$$

$$= -x^{-n}\sum_{m'=0}^{\infty}\frac{(-1)^{m'}}{m'!\Gamma[(n+1)+m'+1]}.\left(\frac{x}{2}\right)^{2m'+(n+1)}$$

$$\therefore \boxed{\frac{d}{dx}\left[x^{-n}J_n\right] = -x^{-n}J_{n+1}(x)} \longrightarrow (3)$$

(4) From (1)

$$\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$$

$$\Rightarrow nx^{n-1}J_n(x) + x^nJ'_n(x) = x^nJ_{n-1}(x)$$

i.e. $\frac{n}{x}J_n(x) + J'_n(x) = j_{n-1}(x) \longrightarrow (4)$

From (3)

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow -nx^{-n-1}J_n(x) + x^{-n}J'_n(x) = -x^{-n}J_{n+1}(x)$$

i.e. $\frac{-n}{x}J_n(x) + J'_n(x) = -J_{n+1}(x) \longrightarrow (5)$

Adding (4) and (5) we get,

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\Rightarrow \boxed{J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]} \longrightarrow (6)$$

Subtracting (5) from (4) we get,

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow \boxed{J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]} \longrightarrow (7)$$

From (7) we can have

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n\ \ 1}(x) \longrightarrow (8)$$

Now, for $n = 1$ in (8)

$$\boxed{J_2(x) = \frac{2}{x} J_1(x) - J_0(x)} \longrightarrow (9)$$

Again, for $n = 2$ in (8)

$$J_3(x) = \frac{2\times 2}{x} J_2(x) - J_1(x)$$

$$\Rightarrow J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}[\tfrac{2}{x} J_1(x) - J_o(x)] - J_1(x) \text{ [using (9)]}$$

$$\therefore \boxed{J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_\circ(x)} \longrightarrow (10)$$

Now for $n = 3$ in (8)

$$J_4(x) = \frac{2\times 3}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right] \text{ [using (10) and (9)]}$$

$$= \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

$$\therefore \boxed{J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)} \longrightarrow (11)$$

Legendre's Equation :

The ordinary differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0$$

is called **Legendre's equation of order *n*,** where n is a real number.

***x* = 0 is an ordinary Point**

Legendre's equation can be rewritten as

$$\frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \frac{n(n+1)}{1-x^2}y = 0$$

Now both $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ can be expressed in power series about $x = 0$ (i.e. both are analytic at $x = 0$) and hence $x = 0$ is an ordinary point of the Legendre's equation.

3.22 Solution of Legendre's Equation : Legendre Polynomial

The solution of Legendre's equation can be written in the form $y = \sum_{n=0}^{\infty} a_n x^n$ about $x = 0$.

Solving we get

$$y = a_0 + a_1 x - \frac{n(n+1)}{2!}a_0 x^2 - \frac{(n-1)(n+2)}{3!}a_1 x^3 + \frac{(n-2)n(n+1)(n+3)}{4!}a_0 x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1 x^5 + \ldots\ldots\ldots\ldots$$

$$= a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \ldots\ldots.\right]$$

$$+\ a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5+.......\right]$$

$= a_0y_1(x) + a_1y_2(x)$

So, $y_1(x)$ contains only even powers of x while $y_2(x)$ contain only odd powers of x. We choose the coefficient a_n of the highest power x^n as

$a_n = \dfrac{(2n!)}{2^n(n!)^2}=\dfrac{1.3.5........(2n-1)}{n!}$ (n is a positive integer) and $a_0 = 1$. Then we have $P_n(x)$

$$= \begin{cases} a_0+a_2x^2+......+a_nx^n, & \text{if } n \text{ is even} \\ a_1x+a_3x^3+......+a_nx^n & \text{if } n \text{ is odd} \end{cases}$$

This polynomial $P_n(x)$ is called the **Legendre Polynomial of degree *n***. We can have

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2 - 1)$;

$P_3(x) = \frac{1}{2}(5x^2 - 3x)$; $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ and so on,

Eventually $P_n(1) = 1$ for $n = 0, 1, 2,........$

Rodrigue's Formula :

$$\boxed{P_n(x)=\frac{1}{n!.2^n}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]}$$

Sample Questions :

1. Write down the Bessel's equation.

2. Check whether $x = 0$ is an ordinary point of the Bessel's equation. If no examine whether it is a regular singular point or irregular singular point.

3. Write down the expression of Bessel's function of the first kind of order n.

4. Write down the expression of Bessel's function of the first kind of order $(-n)$.

5. Write down the expression of Bessel's function of the second kind of order n or the Neumann's functions

6. Prove that $\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$

7. Prove that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}\, J_{n+1}(x)$

8. Prove that $\frac{d}{dx}[x^{-n}J_{-n}(x)] = x^{-n}J_{-n-1}$

9. Prove that $J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

10. Prove that $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

11. Express $J_2(x)$ in terms of $J_0(x)$ and $J_1(x)$
12. Express $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$
13. Express $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$
14. Write down the Legendre's equation.
15. Check whether $x = 0$ is an ordinary point of Legendre's equation or not.
16. Write down the expression of Legendre's polynomial
17. State the Rodrigue's formula regarding Legendre's polynomial.

3.23 Application of Ordinary Differential Equation to Dynamical Systems

Dynamical System :

Definition : A dynamical system is a system which changes with time.

Mathematically if a system can be described by means of interaction of finite number of variables all of which change with time and if further this change in each variable with respect to time can be described by means of certain functions involving these variables where time can be present either explicity or implicity is said to be a **dynamical system.** The variables describing a dynamical system are called **state variables.**

Examples : Motion of a particle under certain number of forces, financial markets etc.

3.24 Dimension of a Dynamical System

The number of state variables involved in a dynamical system is said to be the **dimension** of that dynamical system.

Categorization of dynamical system :

If time is implicity present in the governing equation(s) of a dynamical system then that dynamical system is said to be an **autonomous dynamical system.**

If time is explicity present at least once in the governing equation(s) of a dynamical system then that dynamical system is said to be a **non-autonomous dynamical system.**

If all the state variables involved in a dynamical system are discrete in nature then that dynamical system is said to be a **discrete dynamical system** or a **map** or a **cascade.**

If all the state variables involved in a dynamical system are continuous in nature then that dynamical system is said to be a **continuous dynamical system** or a **flow.**

Examples :

(I) Example of a one dimensional autonomous map :

$x_{t+1} = x_t + x_t^2$

[general form : $x_{t+1} = x_t + f(x_t)$]

(II) Example of a one dimensional non-autonomous map :

$x_{t+1} = x_t + (x_t^3 - 1) + e^t$

[general form : $x_{t+1} = x_t + f(t, x_t)$]

(III) Example of a two dimensional autonomous map :

$x_{t+1} = x_t + x_t^2 - 1,\ y_{t+1} = y_t + x_t y_t - 1$

[general form : $x_{t+1} = x_t + f(x_t, y_t),\ y_{t+1} = y_t + g(x_t, y_t)$]

(IV) Example of a two dimensional non-autonomous map :

$x_{t+1} = x_t + t x_t^3 - 1,\ y_{t+1} = y_t + x_t y_t + 1$

[general form : $x_{t+1} = x_t + f(t, x_t, y_t),\ y_{t+1} = y_t + g(t, x_t, y_t)$]

(V) Example of a one dimensional autonomous flow :

$$\frac{dx}{dt} = x + 1$$

[general form : $\frac{dx}{dt} = f(x)$]

(VI) Example of a one dimensional non-autonomous flow :

$$\frac{dx}{dt} = x - 1 + e^t$$

[general form : $\frac{dx}{dt} = f(x, t)$]

(VII) Example of a two dimensional autonomous flow $\frac{dx}{dt} = x + y + 2,\ \frac{dy}{dt} = xy - 1$

[general form : $\frac{dx}{dt} = f(x, y),\quad \frac{dy}{dt} = g(x, y)$]

(VIII) Example of a two dimensional non-autonomous flow :

$\frac{dx}{dt} = x + y + t,\ \frac{dy}{dt} = xy - 1$

[general form : $\frac{dx}{dt} = f(x, y, t),\ \frac{dy}{dt} = g(x, y, t)$]

We can extend the above ideas for three or higher dimensional maps or flows.

N.B. In discrete dynamical system x_t represents the magnitude of x in time t and as derivative does not exist in discrete domain the rate of change of x at t can be equivalently expressed as $\frac{x_{t+1} - x_t}{(t+1) - t} = x_{t+1} - x_t$

As ordinary differential equation plays its role only in continuous dynamical systems of flows we will confine our analysis within the domain of continuous case. Also we will restrict ourselves in autonomous systems only.

3.25 Equilibrium Point of A Flow

One dimension : A point $x = x^* \in D \subseteq R$ is said to be an equilibrium point of a one dimensional flow given by

$$\frac{dx}{dt} = f(x)\ ;\ x \in D \subseteq R \text{ if } \left.\frac{dx}{dt}\right|_{x=x^*} = f(x^*) = 0.$$

Two dimension : A point $(x^*, y^*) \in D_2 \subseteq R^2$ is said to be an equilibrium point of a two dimensional flow given by

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}\right\} (x, y) \in D_2 \subseteq R^2$$

$$\text{if } \left.\begin{aligned} \left.\frac{dx}{dt}\right|_{(x^*, y^*)} &= f(x^*, y^*) = 0 \\ \left.\frac{dy}{dt}\right|_{(x^*, y^*)} &= g(x^*, y^*) = 0 \end{aligned}\right\}$$

Physically, at an equilibrium point of a flow the flow becomes stationary.

Examples :

I) given one dimensional flow :

$$\frac{dx}{dt} = 2x - 1 \ ; \ x \in R$$

For its equilibrium point we must have

$$\frac{dx}{dt} = 0 \text{ i.e. } 2x - 1 = 0 \text{ or } x = \frac{1}{2}$$

So, $x = \frac{1}{2}$ is its only equilibrium point.

II) Given two dimensional flow :

$$\left.\begin{array}{l} \frac{dx}{dt} = x + y - 2 \\ \frac{dy}{dt} = xy - 1 \end{array}\right\} (x, y) \in R^2$$

For its equilibrium point we must have

$$\left.\begin{array}{l} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{array}\right\} \text{ i.e. } x + y - 2 = 0, \ xy - 1 = 0 \text{ or } x = 1, \ y = 1$$

So, (1, 1) is the only equilibrium point of this flow.

There exist certain dynamical systems for which there is no equilibrium point. For example in the one dimensional flow $\frac{dx}{dt} = e^x$; $x \in R$ $\frac{dx}{dt}$ can never be zero as e^x can never be zero for any $x \in R$. Hence this flow has no equilibrium point.

3.26 Analysis of Stability of an Equilibrium Point of a One Dimensional Flow :

Let, $\frac{dx}{dt} = f(x)$, $x \in D \subseteq R$ be a given one dimensional flow, and let $x = x^* \in D \subseteq R$ be an equilibrium point of this flow. Then we must have,

$$\left.\frac{dx}{dt}\right|_{x=x^*} = f\,(x^*) = 0 \qquad (2)$$

We consider a very small amount perturbation 'Δx' about the equilibrium point $x = x^*$. So near the vicinity of this equilibrium point we have

$$x = x^* + \Delta x \qquad (3)$$

Using (3) in (1) we have,

$$\frac{d}{dt}(x^* + \Delta x) = f\,(x^* + \Delta x)$$

$$\frac{d}{dt}(\Delta x) = f\,(x^* + \Delta x)$$

$$= f\,(x^*) + \Delta x\, f'(x^*) + \frac{(\Delta x)^2}{2!} f''(x^*) + \text{.......[using Taylor series expansion]}$$

i.e. $$\frac{d}{dt}(\Delta x) = \Delta x\; f'(x^*) + \frac{(\Delta x)^2}{2!} f''(x^*) + \text{....... [using (2)]} \qquad (4)$$

If 'Δx' is sufficiently small so that we can neglect $(\Delta x)^2$ and other higher powers of Δx then we can have from (4)

$$\frac{d}{dt}(\Delta x) = \Delta x\, f'(x^*)$$

or $$\frac{d(\Delta x)}{\Delta x} = f'(x^*)\; dt$$

Integrating we get $\Delta x = Ke^{f'(x^*)t}$ (5)

where 'K' is a constant of integration.

Now, at $t = 0$ we assume $\Delta x = \Delta x|_{t=0}$

So, $\Delta x|_{t=0} = K$ (6)

Using (6) in (5) we get

$$\Delta x = \Delta x|_{t=0}\; e^{f'(x^*)t} \qquad (7)$$

Case I : $f'(x^*) > 0$:

As $t \to \infty$, $\Delta x \to \infty$ or $-\infty$ according as $\Delta x|_{t=0} > 0$ or < 0 respectively.

In this case, the small perturbation created about the equilibrium point increases with time and thus eventually goes away from the equilibrium point. This situation represents instability and the corresponding equilibrium point $x = x^*$ is said to be an unstable equilibrium point.

Case II : $f'(x^*) < 0$:

As $t \to \infty, \quad \Delta x \to 0$.

In this case, the small perturbation created about the equilibrium point decreases with time and thus tends to return back to the equilibrium point. This situation represents stability and the corresponding equilibrium point $x = x^*$ is said to be a stable equilibrium point

Case III : $f'(x^*) = 0$:

We have, $\Delta x = \Delta x|_{t=0} \ \forall t$.

So, here we fail to determine whether the equilibrium point is stable or unstable. Further investigation is required in this case.

Examples :

1. Given one dimensional flow :

$$\frac{dx}{dt} = x^2 - 3x + 2 \ ; \ x \in \mathrm{R}.$$

Find its equilibrium point (s) and discuss about the stability.

Ans. Given one dimensional flow :

$$\frac{dx}{dt} = x^2 - 3x + 2 \ ; \ x \in \mathrm{R}$$

For its equilibrium point we must have,

$$\frac{dx}{dt} = 0 \text{ i.e. } x^2 - 3x + 2 = 0 \text{ or } x = 1, 2$$

So, the given flow has two equilibrium points viz. $x = 1$ and $x = 2$.

We consider $f(x) = x^2 - 3x + 2$

Hence $f'(x) = 2x - 3$

Now $f'(\mathrm{I}) = 2 \times 1 - 3 = -1 < 0$

So, $x = 1$ is a stable equilibrium point.

Again, $f'(2) = 2 \times 2 - 3 = 1 > 0$

So, $x = 2$ is an unstable equilibrium point.

2. Given one dimensional flow :

$$\frac{dx}{dt} = 2x^2 \; ; \; x \in \text{IR}$$

Find its equilibrium point (s) and discuss about the stability.

Ans. Given one dimensional flow :

$$\frac{dx}{dt} = 2x^2 \; ; \; x \in \text{IR}.$$

For its equilibrium point we must have,

$$\frac{dx}{dt} = 0 \text{ i.e., } 2x^2 = 0 \text{ or } x = 0.$$

So, $x = 0$ is its only equilibrium point.

Now, we have, $f(x) = 2x^2$

$$\text{So } f'(x) = 4x$$

$$\text{and } f'(0) = 0$$

Hence no conclusion can be drawn about the stability of the equilibrium point $x = 0$ from the above. Now, if we consider 'Δx' as the small perturbation about the equilibrium point $x = 0$ we then have near the vicinity of this equilibrium point $x = 0 + \Delta x$ i.e. $x = \Delta x$

Then we get,

$$\frac{d}{dt}(\Delta x) = f(\Delta x)$$

$$= f(0) + \Delta x\, f'(0) + \frac{(\Delta x)^2}{2!} f''(0) + \frac{(\Delta x)^2}{3!} f'''(0) + \ldots\ldots\ldots$$

Now, $f(x) = 2x^2$

$$f'(x) = 4x$$

$$f''(x) = 4 \text{ and } f^{(n)}(x) = 0 \quad \forall n \geq 3$$

So, $\frac{d}{dt}(\Delta x) = 0 + \Delta x.0 + \frac{(\Delta x)^2}{2!} \times 4 + 0$

$$= 2(\Delta x)^2$$

or, $\dfrac{d(\Delta x)}{(\Delta x)^2} = 2dt$

Integrating we get,

$-\dfrac{1}{\Delta x} = 2t + k'$ where k' is a constant of integration,

or, $\Delta x = -\dfrac{1}{2t+k'}$

Now, as $t \to \infty$, $\Delta x \to 0$

So, from the above analysis we get that $x = 0$ is a stable equilibrium point of the given flow.

Analysis of stability of an equilibrium point of a two dimensional flow :

Let,

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}\right\} (x, y) \in D_2 \subseteq R \qquad (1)$$

be a given two dimensional flow and let $(x^*, y^*) \in D_2 \subseteq R$ be an equilibrium point of this flow. Then we must have.

$$\left.\begin{aligned} \left.\frac{dx}{dt}\right|_{(x^*, y^*)} &= f(x^*, y^*) = 0 \\ \left.\frac{dy}{dt}\right|_{(x^*, y^*)} &= g(x^*, y^*) = 0 \end{aligned}\right\} \qquad (2)$$

We consider a very small amount of perturbation given by $(\Delta x, \Delta y)$ about the equilibrium point (x^*, y^*). So, near the vicinity of the equilibrium point (x^*, y^*) we have

$$\left.\begin{aligned} x &= x^* + \Delta x \\ y &= y^* + \Delta y \end{aligned}\right\} \qquad (3)$$

Using (3) in (1) we get,

$$\frac{d}{dt}(x^* + \Delta x) = f(x^* + \Delta x, y^* + \Delta y)$$

$$\frac{d}{dt}\,(y^* + \Delta y) = g(x^* + \Delta x,\ y^* + \Delta y)$$

or, $$\frac{d}{dt}(\Delta x) = f(x^*, y^*) + \Delta x \cdot \frac{\partial f}{\partial x}(x^*, y^*) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2}(x^*, y^*) + \ldots\ldots.$$

$$+ \Delta y \cdot \frac{\partial f}{\partial y}(x^*, y^*) + \frac{(\Delta y)^2}{2!}\frac{\partial^2 f}{\partial y^2}(x^*, y^*) + \ldots\ldots.$$

$$\frac{d}{dt}(\Delta y) = g(x^*, y^*) + \Delta x.\ \frac{\partial g}{\partial x}(x^*, y^*) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 g}{\partial x^2}(x^*, y^*) + \ldots\ldots.$$

$$+ \Delta y \cdot \frac{\partial g}{\partial y}(x^*, y^*) + \frac{(\Delta y)^2}{2!}\frac{\partial^2 g}{\partial y^2}(x^*, y^*) + \ldots\ldots$$

This gives,

$$\left.\begin{aligned} \frac{d}{dt}(\Delta x) &= \frac{\partial f}{\partial x}(x^*, y^*)\cdot\Delta x + \frac{\partial f}{\partial y}(x^*, y^*)\cdot\Delta y \\ \frac{d}{dt}(\Delta y) &= \frac{\partial g}{\partial x}(x^*, y^*)\Delta x + \frac{\partial g}{\partial y}(x^*, y^*)\Delta y \end{aligned}\right\} \qquad (4)$$

Using (2) and considering Δx and Δy sufficiently small so that their squares and other higher powers can be neglected.

(4) Can be equivalently written as

$$\frac{d}{dt}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x^*, y^*)} \begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} \qquad (5)$$

If we take, $\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = \underset{\sim}{X}$ and $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$

which represents the Jacobian of the system as J, we have from (5)

$$\frac{d\underset{\sim}{X}}{dt} = J_{(x^*,y^*)}\underset{\sim}{X} \qquad (6)$$

We have a trial solution of (6) as

$$\underset{\sim}{X} = \begin{bmatrix} C \\ d \end{bmatrix} e^{\lambda t} \qquad (7)$$

Then we have, $\dfrac{d\underset{\sim}{X}}{dt} = \lambda \begin{bmatrix} C \\ d \end{bmatrix} e^{\lambda t}$ (8)

Using (7) and (8) in (6) we get

$$\lambda \begin{bmatrix} C \\ d \end{bmatrix} e^{\lambda t} = \begin{bmatrix} C \\ d \end{bmatrix} e^{\lambda t} \mathrm{J}_{(x^*,y^*)}$$

$$\lambda \begin{bmatrix} C \\ d \end{bmatrix} e^{\lambda t} = e^{\lambda t} \mathrm{J}_{(x^*,y^*)} \begin{bmatrix} C \\ d \end{bmatrix}$$

or, $\mathrm{J}_{(x^*,y^*)} \begin{bmatrix} C \\ d \end{bmatrix} = \lambda \begin{bmatrix} C \\ d \end{bmatrix}$ (9)

From (9) it is clear that λ is an eigen value of $\mathrm{J}_{(x^*,y^*)}$ and $\begin{bmatrix} C \\ d \end{bmatrix}$ is its corresponding eigen vector. The corresponding characteristic equation is

det (J – λI) = 0

i.e. $\begin{vmatrix} \dfrac{\partial f}{\partial x} - \lambda & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} - \lambda \end{vmatrix}_{(x^*,y^*)} = 0$

or, $\left(\frac{\partial f}{\partial x}-\lambda\right)\left(\frac{\partial g}{\partial y}-\lambda\right)-\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial x}=0$

or $\lambda^2-\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)_{(x^*,y^*)}\lambda+\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial x}\right)_{(x^*,y^*)}=0 \quad (10)$

The above is a quadratic equation of λ. We can arrive at the solution for different cases as given below.

Case I : Roots are real and unequal : (say, λ_1 and λ_2)

[The corresponding equilibrium point is said to be a node]

Here we have

$$\left.\begin{aligned}\Delta x &= C_1 a_1 e^{\lambda_1 t}+C_2 a_2 e^{\lambda_2 t}\\ \Delta y &= C_1 b_1 e^{\lambda_1 t}+C_2 b_2 e^{\lambda_2 t}\end{aligned}\right\}$$

Sub case Ia ; **$\lambda_1 > 0$, $\lambda_2 > 0$**.

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$, (if C_1, $C_2 > 0$)

or $\Delta x \to -\infty$, $\Delta y \to -\infty$ (if C_1, $C_2 < 0$)

Hence, the equilibrium point is an unstable node.

Subcase Ib : $\lambda_1 < 0$, $\lambda_2 < 0$:

As $t \to \infty$, $\Delta x \to 0$, $\Delta y \to 0$

Hence, the equilibrium point is a stable node.

Subcase Ic : $\lambda_1 > 0$, $\lambda_2 < 0$ or, $\lambda_1 < 0$, $\lambda_2 > 0$:

As $t \to \infty$, one component tends to infinity and the other component drags it to zero. In this situation the corresponding equilibrium point is said to be a saddle node.

Subcase Id : $\lambda_1 = 0$, $\lambda_2 > 0$ or $\lambda_1 > 0$, $\lambda_2 = 0$

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if C_1, $C_2 > 0$)

or $\Delta x \to -\infty$, $\Delta y \to -\infty$ (if C_1, C_2, < 0)

Hence the equilibrium point is said to be an unstable node.

Subcase Ie : $\lambda_1 = 0, \lambda_2 < 0$ or $\lambda_1 < 0, \lambda_2 = 0$:

As $t \to \infty$, $\Delta x \to C_1$, $\Delta y \to C_1$ or $\Delta x \to C_2$, $\Delta y \to C_2$.

Here, we call the equilibrium point as a pseudo-stable node.

Case II : Roots are real end equal (say λ^* and λ^*)

[Here also the corresponding equilibrium point is said to be a node]

Here we have

$$\left.\begin{aligned} \Delta x &= (C_1' + C_2' t)e^{\lambda^* t} \\ \Delta y &= (C_1' + C_2' t)e^{\lambda^* t} \end{aligned}\right\}$$

Subcase IIa : $\lambda^* > 0$:

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if C_1', $C_2' > 0$) or

$\Delta x \to -\infty$, $\Delta y \to -\infty$ (if C_1', $C_2' < 0$).

Here the equilibrium point is an unstable node.

Subcase IIb : $\lambda^* < 0$:

As $t \to \infty$, $\Delta x \to 0$, $\Delta y \to 0$

Here the equilibrium point is a stable node.

Subcase IIc : $\lambda^* = 0$:

As $t \to \infty$, $\Delta x \to \infty$, $\Delta y \to \infty$ (if C_1', $C_2' > 0$)

or,

$\Delta x \to -\infty$, $\Delta y \to -\infty$ (if C_1', $C_2' < 0$)

Here, the equilibrium point is an unstable node.

Case III : Roots are complex conjugate numbers (say $\alpha \pm i\beta$)

[The corresponding equilibrium point is said to be a focus if $\alpha \neq 0$ and centre if $\alpha = 0$]

Here we have

$$\left.\begin{aligned}\Delta x &= \left[C_1'' \cos(\beta t) + C_2'' \sin(\beta t)\right] e^{\alpha t} \\ \Delta y &= \left[C_1'' \cos(\beta t) + C_2'' \sin(\beta t)\right] e^{\alpha t}\end{aligned}\right\}$$

Subcase III a : $\alpha > 0$ **:**

As $t \to 4$, $|\Delta x| \to 4$, $|\Delta y| \to 4$, Hence the equilibrium point is an unstable focus.

Subcase III b : $\alpha > 0$:

As t $\to 4$; $|\Delta x| \to 0$, $|\Delta y| \to 0$

Hence the equilibrium point is a stable focus.

Subcase III c : $\alpha = 0$ **:**

Here, as t increases Δx and Δy oscillates between two finite values. Here the equilibrium point is said to be a centre.

Example :

Given two dimensional flow :

$$\left.\begin{aligned}\frac{dx}{dt} &= x(4 - x - y) \\ \frac{dy}{dt} &= y(15 - 5x - 3y\end{aligned}\right\} x, y \in \mathrm{R}$$

Find the equilibrium point (s) and discuss about the stability.

Ans. Given two dimensional flow :

$$\left.\begin{aligned}\frac{dx}{dt} &= x(4 - x - y) \\ \frac{dy}{dt} &= y(15 - 5x - 3y\end{aligned}\right\} x, y \in \mathrm{R}$$

For its equilibrium point we must have

$$\left.\begin{aligned}\frac{dx}{dt}=0\\ \frac{dy}{dt}=0\end{aligned}\right\}\text{ i.e. }\left.\begin{aligned}x(4-x-y)=0\\ y(15-5x-3y)=0\end{aligned}\right\}$$

Option 1 : $x = 0, y = 0$.

Hence (0, 0) is an equilibrium point.

Option 2 : $x = 0,\ 15 - 5x - 3y = 0$

i.e. $x = 0, y = 5$

Hence (0, 5) is an equilibrium point.

Option 3 : $y = 0,\ 4 - x - y = 0$

i.e. $x = 4, y = 0$

Hence (4, 0) is an equilibrium point.

Option 4 : $4 - x - y = 0\ ;\ 15 - 5x - 3y = 0$

Solving we get $x = \frac{3}{2},\ y = \frac{5}{2}$

Hence, $\left(\frac{3}{2},\frac{5}{2}\right)$ is an equilibrium point.

Therefore for the given flow we have four equilibrium points viz. (0, 0), (0, 5), (4, 0) and $\left(\frac{3}{2},\frac{5}{2}\right)$

We take,

$f(x, y) = x(4 - x - y)$

$g(x, y) = y\ (15 - 5x - 3y)$

So, $\frac{\partial f}{\partial x} = 4 - 2x - y,\quad \frac{\partial f}{\partial y} = -x\ ;\ \frac{\partial g}{\partial x} = -5y\ ;\ \frac{\partial g}{\partial y} = 15 - 5x - 6y.$

Therefore general Jacobian of the system

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 4-2x-y & -x \\ -5y & 15-5x-6y \end{bmatrix}$$

3.27 Stability Analysis of The Equilibrium Points

I (0, 0) :

Characteristic equation :

det $(J - \lambda I)_{(0,\ 0)} = 0$

i.e. $\begin{vmatrix} 4-\lambda & 0 \\ 0 & 15-\lambda \end{vmatrix} = 0$

or $(4 - \lambda)(15 - \lambda) = 0$ i.e. $\lambda = 4, 15$.

As here both the eigen values are positive (0, 0) is an unstable node.

II. (0, 5) :

Characteristic equation :

det $(J - \lambda I)_{(0,\ 5)} = 0$

i.e. $\begin{vmatrix} -1-\lambda & 0 \\ -25 & -15-\lambda \end{vmatrix} = 0$

i.e. $(-1-\lambda)(-15 - \lambda)$ or, $\lambda = -1, -15$. As here both the eigen values are negative (0, 5) is a stable node.

III. (4, 0) :

Characteristic equation :

det $(J - \lambda I)_{(4,\ 0)} = 0$

i.e. $\begin{vmatrix} -4-\lambda & -4 \\ 0 & -5-\lambda \end{vmatrix} = 0$

i.e. $(-4 - \lambda)(-5 - \lambda) = 0$ or $\lambda = -4, -5$.

As here both the eigen values are negative (4, 0) is a stable node.

IV. $\left(\frac{3}{2}, \frac{5}{2}\right)$ **:**

Characteristic equations :

det $(J - \lambda I)_{\left(\frac{3}{2}, \frac{5}{2}\right)} = 0$

i.e. $\begin{vmatrix} \frac{-3}{2}-\lambda & \frac{-3}{2} \\ \frac{-25}{2} & \frac{-15}{2}-\lambda \end{vmatrix} = 0$

or, $\left(\frac{-3}{2}-\lambda\right)\left(\frac{-15}{2}-\lambda\right)-\left(\frac{-3}{2}\right)\left(\frac{-25}{2}\right) = 0$

or, $\lambda^2 + 19\lambda - \frac{15}{2} = 0$

or, $\lambda = \dfrac{-9 \pm \sqrt{9^2 - 4\times1\times\left(-\frac{15}{2}\right)}}{2\times1} = \dfrac{-9 \pm \sqrt{101}}{2}$

Here one root is negative and the other is positive. Hence, $\left(\frac{3}{2}, \frac{5}{2}\right)$ is a saddle node.

3.28 Summary

This unit presents a very detailed discussions with certain problems on first order but not of first degree and second order ordinary diffenential equations. Different common methods of series solution are discussed and a brief overview of dynamical system are also discussed with a good number examples.

3.29 Exercise

1. Find the equilibrium point(s) and discuss about the stability for the following one dimensional flows : [In all such cases R denotes the set of all real numbers]

(i) $\frac{dx}{dt} = x^2 - 1$; $x \in$ R

(ii) $\frac{dx}{dt} = x^2 - 3x$; $x \in$ R

(iii) $\frac{dx}{dt} = 1 - \sin x$; $x \in$ R ∵

(iv) $\frac{dx}{dt} = 1 - \cos x$; $x \in$ R

(v) $\frac{dx}{dt} = x^3 - 9x^2 + 26x - 24$; $x \in$ R

(vi) $\frac{dx}{dt} = x^3 - 6x^2 + 11x - 6$; $x \in$ R

(vii) $\frac{dx}{dt} = x(1 - x) + \frac{3x}{1+x}$; $x \in$ R

(viii) $\frac{dx}{dt} = 4x^2 + r^2x - rx$; $r \in$ R, $x \in$ R Here r is a parameters.

(ix) $\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right)$; $x \in$ R$^+$ U {0} ; a, K $\in$ R$^+$

Here 'a' and 'K' are two parameters and R$^+$ denotes the set of all positive real numbers.

2. Find the equilibrium points(s) and discuss about the stability for the following two dimensional flows : [In all such Cases R denotes the set of all real numbers]

(i) $\frac{dx}{dt} = x + 1$

$\frac{dy}{dt} = xy - x \; ; \; x, y \in R$

(ii) $\frac{dx}{dt} = 2x + 3$

$\frac{dy}{dt} = y + x - 1 \; ; \; x, y \in R$

(iii) $\frac{dx}{dt} = x(1 - x - y)$

$\frac{dy}{dt} = y(2 - 3x - y) \; ; \; x, y \in R$

(iv) $\frac{dx}{dt} = x - \sin y$

$\frac{dy}{dt} = x - y \; ; \; x, y \in R$

(v) $\frac{dx}{dt} = xy - 1$

$\frac{dy}{dt} = x^2 - 1 \; ; \; x, y \in R$

(vi) $\frac{dx}{dt} = \mu - x^2$

$\frac{dy}{dt} = -y$; $x, y \in R$; $\mu \in R$ and here μ is a parameter.

(vii) $\frac{dx}{dt} = \mu x - x^3$

$\frac{dy}{dt} = -y$; $x, y \in R$; $\mu \in R$ and here μ is a parameter.

(viii) $\frac{dx}{dt} = -\mu x + x^2$

$\frac{dy}{dt} = -y$; $x, y \in R$; $\mu \in R$ and here μ is a parameter.

Further Reading :

1. Ordinary Differential Equations : Principles and Applications — A.K. Nandakumaran, P.S. Datti and R.K. George, Cambridge University Press.

2. Ordinary and Partial Differential Equations — M.D. Raisinghania, S. Chand & Company Ltd.

3. Differential Equations and Dynamical Systems — L. Perks, Springer.