PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific, generic elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade "A".

UGC (Open and Distance Learning Programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
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Netaji Subhas Open University
Under Graduate Degree Programme
Choice Based Credit System (CBCS)

Subject : UG Mathematics (HMT)
Generic Elective Course
Course : Modeling and Simulation
Course Code: GE-MT-41

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UG-Mathematics
(HMT)

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Unit 1 ❑ Introduction

Structure

1.0 Objectives

1.1 What is Mathematical Modeling? (An Introduction)

1.2 History of Mathematical Modeling

1.3 Merits and Demerits of Mathematical Modeling

1.4 Summary

1.5 Exercises

1.0 Objectives

In this unit, we discuss the followings.

- The basic idea and motivation behind mathematical modeling;
- The history of development of mathematical modeling;
- Merits and demerits of mathematical modeling.

1.1 What is Mathematical Modeling? (An Introduction)

Models of systems have become part of our everyday lives. They range from global decisions having a profound impact on our future, to local decisions about whether to cycle to university based on weather predictions. Together with their provision of a deeper understanding of the processes involved, this predictive nature of models, which aids in decision-making, is one of their key strengths.

In particular, many processes can be described with mathematical equations, that is, by mathematical models. Such models have use in a diverse range of disciplines.

There is an aesthetic use, for example, in constructing perspective in paintings or etchings such as is seen in the paradoxical work of Escher. The proportions of the golden mean and the Fibonacci series of numbers, occurring in many natural phenomena such as the arrangement of seed spirals in sunflowers, have been applied to methods of information storage in computers. This well-known mathematical series is also applied in models describing the growth nodes on the stems of plants, as well as in aesthetically pleasing proportions in painting and sculpture and the design of musical instruments. From a philosophical perspective, mathematical logic and rigour provide a model for the construction of argument. In a more practical and analytical mode there is a plethora of applications. Mathematical optimisation theory has been applied in the clothing industry to minimise the required cloth for the maximum number of garments, and to the arrangement of odd-shaped chocolates in a box to minimise the number required to give the impression that the box is full! The mathematics of fractals has allowed the successful development of fractal image compression techniques, requiring little storage for extremely precise images.

Some other areas of application include the physical sciences (such as astronomy), medicine (such as the absorption of medication), and the social sciences (such as patterns in election voting). Mathematical models are used extensively in biology and ecology to examine population fluctuations, water catchments, erosion and the spread of pollutants, to name just a few. Fluid mechanics is another extensive area of research, with applications ranging from the modelling of evolving tsunamis across the ocean, to the flow of lolly mixture into moulds. (Mathematicians were consulted to establish the best entry points for the mixture to the mould in order to ensure a filled nose for a Mickey Mouse lollypop!)

1.2 History of Mathematical Modeling

The word “modeling” comes from the Latin word *modellus*. It describes a typical human way of coping with the reality. Anthropologists think that the ability to build abstract models is the most important feature which gave homo sapiens a competitive edge over less developed human races like homo neanderthalensis.

Although abstract representations of *real-world objects* have been in use since the stone age, a fact backed up by cavemen paintings, the real break through of modeling came with the cultures of the Ancient Near East and with the Ancient Greek.

The first recognizable models were *numbers*. Counting and “writing” numbers (e.g., as marks on bones) is documented since about 30,000 BC. Astronomy and Architecture were the next areas where models played a role, already about 4,000 BC.

It is well known that by 2,000 BC at least three cultures (Babylon, Egypt, India) had a decent knowledge of mathematics and used mathematical models to improve their everyday life. Most mathematics was used in an algorithmic way, designed for solving specific problems.

The development of philosophy in the Hellenic Age and its connection to mathematics lead to the deductive method, which gave rise to the first pieces of *mathematical theory*. Starting with *Thales of Miletus* at about 600 BC, geometry became a useful tool in analyzing reality, and analyzing geometry itself sparked the development of mathematics independently of its application. It is said that Thales brought his knowledge from Egypt, that he predicted the solar eclipse of 585 BC, and that he devised a method for measuring heights by measuring the lengths of shadows. Five theorems from elementary geometry are credited to him:

1. A circle is bisected by any diameter.
2. The base angles of an isosceles triangle are equal.
3. The angles between two intersecting straight lines are equal.
4. Two triangles are congruent if they have two angles and one side equal.
5. An angle in a semicircle is a right angle.

After Thales set the base, *Pythagoras of Samos* is said to have been the first pure mathematician. He is known for developing, among other things, the theory of numbers, and most importantly to initiate the use of *proofs* to gain new results from already known theorems.

Important philosophers like Aristotle, Eudoxos and many more added lots of pieces in the 300 years following Thales. Geometry and the rest of mathematics were developed further. The summit was reached by *Euclid of Alexandria* at about 300 BC when he wrote *The Elements*, a collection of books containing most of the mathematical knowledge available at that time. The Elements held among other the first concise axiomatic description of geometry and a treatise on number theory.

Euclid's books became the means of teaching mathematics for hundreds of years and around 250 BC Eratosthenes of Cyrene, one of the first "applied mathematicians", used this knowledge to calculate the distances Earth-Sun and Earth-Moon and, best known, the circumference of the Earth by a mathematical/geometric model.

A further important step in the development of modern models was taken by *Diophantus of Alexandria* about 250 AD in his books *Arithmetica*, where he developed the beginnings of algebra based on symbolism and the notion of a variable.

For astronomy, *Ptolemy*, inspired by Pythagoras' idea to describe the celestial mechanics by circles, developed by 150 AD a mathematical model of the solar system with circles and epi circles to predict the movement of sun, moon, and the planets. The model was so accurate that it was in use until the time of *Johannes Kepler* in 1619, when he finally found a superior, simpler model for planetary motions, that with refinements due to Newton and Einstein is still valid today.

Building models for real-world problems, especially mathematical models, is so important for human development that similar methods were developed independently in China, India and Persia.

One of the most famous Arabian mathematicians is Abu Abd-Allah ibn Musa Al-*H̱wārizmī* (late 8th century). His name, still preserved in the modern word *algorithm,* and his famous books *de numero Indorum* (about the Indian numbers) and *Al-kitab al-muẖtasar fi hisāb al-ğabr wa'l-muqābala* (a concise book about the procedures of calculation by adding and balancing) contain many mathematical models and problem solving algorithms (actually the two were treated as the same) for real-life applications in the areas of commerce, legacy, surveying and irrigation. The term *algebra*, by the way, was taken from the title of his second book.

In the West, it took until the 11th century to develop mathematics and mathematical models, in the beginning especially for surveying.

The probably first great western mathematician after the decline of Greek mathematics was *Fibonacci, Leonardoda Pisa* (ca. 1170–ca.1240). As a son of a merchant, Fibonacci undertook many commercial trips to the Orient. During that time, he got familiar with the Oriental knowledge about mathematics. He used the algebraic methods recorded in Al-*H̱wārizmī*'s books to improve his success as a merchant, because he realized the gigantic practical advantage of the Indian numbers over the Roman numbers which were still in use in western and central Europe at that time. His highly influential book *Liber Abaci*, first issued in 1202, began with a presentation of the ten "Indian figures" (0, 1, 2, ..., 9), as he called them. This was really important because it finally brought the number *zero* to Europe, an abstract model of *nothing*. The book itself was written to be an algebra manual for commercial use, and explained in detail the arithmetical rules using numerical examples which were derived, e.g., from measure and currency conversion.

Artists like the painter Giotto (1267–1336) and the Renaissance architect and sculptor Filippo Brunelleschi (1377–1446) started a new development of geometric principles, e.g. perspective. In that time, visual models were used as well as mathematical ones (e.g., for Anatomy).

In the later centuries more and more mathematical principles were detected, and the complexity of the models increased. It is important to note that despite the achievements of Diophant and Al-*Ḫwārizmī*, the systematic use of variables was really invented by Vieta (1540–1603). Inspite of that it took another 300 years until Cantor and Russell that the true role of variables in the formulation of mathematical theory was fully understood. Physics and the description of Nature's principles became the major driving force in modeling and the development of the mathematical theory. Later economics joined in, and now an ever increasing number of applications demand models and their analysis.

1.3 Merits and Demerits of Mathematical Modeling

Merits

- They are quick and easy to produce
- They can simplify a more complex situation
- They can help us improve our understanding of the real world as certain variables can readily be changed
- They enable predictions to be made
- They can help provide control - as in aircraft scheduling

Demerits

- The model is a simplification of the real problem and does not include all aspects of the problem
- The model may only work in certain situations

1.4 Summary

In this chapter, we introduce the notion of mathematical modeling and mentioned the areas of its application. Historical perspectives have also been discussed. Moreover, merits and demerits of mathematical modeling have been identified.

1.5 Exercises

Exercise 1.5.1. *Write down some applications of mathematical modeling.*

Exercise 1.5.2. *Which five theorems from elementary geometry are credited to Thales?*

Exercise 1.5.3. *Point out merits and demerits of mathematical modeling.*

Unit 2 ❑ Discrete Models

Structure

2.0 Objectives

2.1 Introduction to difference equations

2.2 Linear difference equations

2.2.1 First order linear homogeneous difference equation with constant coefficients

2.2.2 First order linear non-homogeneous difference equation with constant coefficients

2.2.3 Second-order linear homogeneous difference equation with constant coefficients

2.3 Introduction to Discrete Models

2.4 Linear Models : Exemplifying through a growth model

2.4.1 A growth model

2.5 Steady state solution : Exemplifying through growth models with stocking and harvesting

2.5.1 Growth with stocking

2.5.2 Growth with harvesting

2.6 Linear stability analysis

2.7 Newton's Law of Cooling

2.8 Bank account problem

2.9 Mortgage problem

2.10 Drug Delivery Problems : A decay model and Absorption

2.10.1 A decay model

2.10.2 Absorption

2.11 Harrod Model of Economic growth

2.12 War Model

2.13 Lake pollution model

2.14 Alcohol in the blood stream model

2.15 Arm Race model

2.16 Density dependent growth model with harvesting

2.17 More worked out examples

2.18 Summary

2.19 Exercises

2.0 Objectives

The object of this chapter is to develop and analyse various discrete models on the basis of difference equations. Here we will discuss the followings.

- Notion of difference equations method of their solution;
- a variety of discrete models;
- steady state solution or equilibrium points;
- condition of local stability.

2.1 Introduction to difference equations

In this chapter, we shall discuss systems represented by equations where each variable has a time index $t = 0, 1, 2, ...$ and variables of different time–periods are connected in a non–trivial way. Such systems are called *systems of difference equations* and are useful to describe *dynamical systems with discrete time*.

Let time be a discrete variable denoted by $t = 0, 1, 2, ...$ A function $X = X(t)$ that depends on this variable may be thought simply as a sequence $X_0, X_1, X_2, \ldots$ of vectors of n dimensions (n is any positive integer). These vectors represent evolution of a system in discrete time steps and we assume at each time step the vector may be expressed as some function of the vectors at finitely many previous time steps.

If each vector is connected with the previous vector by means of some function given by $X_{t+1} = f(X_t)$, $t = 0, 1, \ldots$, then we have a *system of first–order difference equations*. In the following definition, we generalize the concept to systems with longer time lags and that can include t explicitly.

Definition 2.1.1. *A k–th order discrete system of difference equations is an expression of the form* $X_{t+k} = f(X_{t+k-1}, \ldots, X_t, t)$, $t = 0, 1, \ldots$ *The system is*

- autonomous, if *f* does not depend on t;
- *linear, if the mapping f is linear in the variables* $X_{t+k-1}, \ldots, X_t$, *otherwise it is nonlinear;*
- *of first order, if* $k = 1$.

2.2 Linear difference equations

A *linear difference equation* or *linear recurrence relation* is a linear polynomial (equated to zero) in various iterates of a variable. Such equation is necessary to explain the evolution of a variable over time, i.e., in terms of the values of the variable over previously measured different time periods or discrete moments. For example, a linear difference equation can be written as

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} - \cdots - a_n y_{t-n} - b = 0$$

i.e.,

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_n y_{t-n} + b \tag{2.1}$$

Here $a_1, a_2, \ldots, a_n$ and b are parameters. The coefficients a_j's are taken to be constant here. We call such equation *autonomous*. However, they may also be polynomials in t. Such equation is called *non- autonomous*.

The equation (2.1) is homogeneous if $b = 0$ and non-homogeneous otherwise.

This is a n–th order difference equation in the sense that y_n can be expressed for with the help of previous n terms. In other words, the longest time lag in equation (2.1) is n.

Using the second principle of mathematical induction, we can say that the linear difference equation (2.1) of order n is uniquely determined by the sequence $\{y_t\}$ once we know the n initial values (i.e., iterates) of y_j 's, i.e., $y_1, y_2, \ldots, y_n$.

Example 2.2.1. *Clearly* $y_t = 3y_{t-1}$, $y_{t+2} = y_{t-1} + y_{t-2} + 5$ *are homogeneous linear difference equation of order* 1 *and non-homogeneous linear difference equation of order* 4 *respectively.*

On the other hand, $y_n = y_{n-1} y_{n-2}$ *is not linear.*

We will now go through the following important observations.

Remark 2.2.1. *In order to find solution of linear homogeneous difference equations, the following observation is very useful.* $y_t = r^t$ *is a solution of the equation*

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \bullet \ldots + a_n y_{t-n} \qquad (2.2)$$

if and only if

$$r^t = a_1 r^{t-1} + a_2 r^{t-2} + \ldots + a_n r^{t-n}$$

or equivalently

$$r^n - a_1 r^{n-1} - a_2 r^{n-2} - \bullet \ldots - a_n = 0 \qquad (2.3)$$

This is called the characteristic equation of the linear homogeneous equation (2.2). The roots of equation (2.3) are called the characteristic roots of the linear homogeneous difference equation (2.2) of order n.

Remark 2.2.2. *Suppose r is any real number that satisfies the equation (2.3). Multiplying both sides of equation (2.3) by* r^{t-n}*, it is easy to check that each term of the sequence* $r, r^2, r^3, \ldots$ *satisfies equation (2.2).*

Conversely, if each term of the sequence $r, r^2, r^3, \ldots$ *satisfies equation (2.2) for some integer r, then r satisfies the equation (2.3).*

Remark 2.2.3. *If both the sequences* $r, r^2, r^3, \ldots$ *and* $s, s^2, s^3, \ldots$ *satisfy equation (2.2), then it is easy to check that the sequence* $\{p_t\}$*, given by* $p_t = Cr^t + Ds^t$, $\forall t \in \mathbb{N} U \{0\}$ *also satisfies the same equation, C and D being arbitrary constants.*

At this point we can state the following theorem, regarding distinct roots of characteristic equation, without proof (the proof is quite easy in fact).

Theorem 2.2.1. *Let* $r_1, r_2, \ldots, r_n$ *be distinct roots of the characteristic equation* $r^n - a_1 r^{n-1} - a_2 r^{n-2} - \bullet \ldots - a_n = 0$ *of the linear homogeneous difference equation* $y_t = a_1 y_{t-1} + a_2 y_{t-2} + \ldots + a_n y_{t-n}$ *with constant coefficients* $a_1, a_2, \ldots, a_n$*. Then the sequence* $\{p_t\}$ *is a solution of the linear homogeneous difference equation if and only if* $\{p_t\}$ *is given by* $A_1 r_1^t + A_2 r_2^t + \ldots + A_n r_n^t$, $\forall t \in \mathbb{N} U \{0\}$*, where* $A_1, A_2, \ldots, A_n$ *are arbitrary constants.*

The following result is for linear homogeneous difference equation with n–th order and with constant coefficients. Here we consider the existence of distinct roots with different multiplicities of the characteristic equation.

Theorem 2.2.2. *Let $r_1, r_2, \ldots, r_k$ be distinct roots, with multiplicities m_1. $m_2, \ldots, m_k$ of the characteristic equation $r^n - a_1r^{n-1} - a_2r^{n-2} - \cdots - a_n = 0$ of the linear homogeneous difference equation $y_t = a_1y_{t-1} + a_2y^{t-2} + \cdots + a_ny_{t-n}$ of order n with constant coefficients $a_1, a_2, \ldots, a_n$ and $m_1 + m_2 + \ldots + m_k = n$. Then the sequence $\{p_t\}$ is a solution of the linear homogeneous difference equation if and only if $\{p_t\}$ is given by $p_t = \left(\alpha_{1,0} + \alpha_{1,1}t + \ldots + \alpha_{1,m_1-1}t^{m_1-1}\right)r_1^t + \left(\alpha_{2,0} + \alpha_{2,1}t + \ldots + \alpha_{2,m_2-1}t^{m_2-1}\right)r_2^t + \ldots + \left(\alpha_{k,0} + \alpha_{k,1}t + \ldots + \alpha_{k,m_k-1}t^{m_k-1}\right)r_k^t$, where $t \in \mathbb{N} \cup \{0\}$, $\alpha_{i,j}$ are constants for $1 \le i \le k$ and $1 \le j \le m_i - 1$.*

2.2.1 First order linear homogeneous difference equation with constant coefficients

A first order linear homogeneous difference equation with constant coefficients is of the following form

$$x_{n+1} = ax_n \tag{2.4}$$

where a is a constant.

It is very obvious that

$$\begin{aligned} x_n &= ax_{n-1} \\ &= a^2x_{n-2} \\ &\ldots \\ &= a^nx_0 \end{aligned}$$

Hence

$$x_n = a^nx_0 \tag{2.5}$$

is the solution to the equation (2.4).

2.2.2 First order linear non- homogeneous difference equation with constant coefficient

A first order linear non- homogeneous difference equation with constant coefficients is of the following form

$$x_{n+1} = ax_n + b \tag{2.6}$$

where a is a constant.

Note that

$$\begin{aligned}
x_1 &= ax_0 + b \\
x_2 &= ax_1 + b \\
&= a(ax_0 + b) + b \\
&= a^2x_0 + ab + b \\
x_3 &= ax_2 + b \\
&= a(a^2x_0 + ab + b) + b \\
&= a^3x_0 + a^2b + ab + b
\end{aligned}$$

Proceeding similarly, $x_n = a^nx_0 + a^{n-1}b + \ldots + a^3b + a^2b + ab + b$

$$= a^nx_0 + b(a^{n-1} + \ldots + a^3 + a^2 + a + 1)$$

Hence the solution to the equation (2.6) is

$$x_n = a^n x_0 + b\frac{a^n - 1}{a - 1} \tag{2.7}$$

Example 2.2.2. *Find the exact solution of* $x_{n+1} = 0.75x_n - 2$ *when* $x_0 = 50$.

Solution: *Put* $a = 0.75$, $b = -2$. *Using* $x_0 = 50$, *we have*

$$x_n = (0.75)^n 50 + (-2)\frac{(0.75)^n - 1}{0.75 - 1}$$

$$= 58(0.75)^n - 8$$

2.2.3 Second-order linear homogeneous difference equation with constant coefficients

Let us focus now on the following second-order linear homogeneous difference equation with constant coefficients.

$$y_t = Ay_{t-1} + By_{t-2} \tag{2.8}$$

for all integers $t \geq 2$.

In order to obtain the non- trivial solution, we take the solution as $y_t = r^t$. Then using equation (2.8), the characteristic equation is given by

$$r^2 - Ar - B = 0 \tag{2.9}$$

Case I: When the roots of the characteristic equation are real and distinct

Let r and s be two distinct roots of equation (2.9).

Then the solution is given by $y_t = Cr^t + Ds^t$, where C and D are arbitrary constants and determined by initial values y_0 and y_1.

The Fibonacci rabbit model: A growth model

Let us consider the rabbit generation model proposed by Fibonacci. A young pair of rabbits, one of each sex, is kept in an island. Each pair breeds only after they are two months old. When they are 2 months old, each pair gives birth to another pair in every month. Let f_n be the number of pairs at the end of n–th month. Then $f_0 = 1 = f_1$. Here f_0 denotes the number of pairs at the beginning of first month.

Now the number of pairs born on $(n + 2)$–th month is same as the number of pairs at n–th month because no pair born on $(n + 1)$–th pair is capable of breeding on the next month. Also the number of already existing (i.e., not newly born) pairs at the end of $(n + 2)$–th month is same as the number of pairs at the end of $(n + 1)$–th month. Hence we have the recurrence relation

$$f_{n+2} = f_{n+1} + f_n, \ \forall n \in \mathbb{N} \cup \{0\} \tag{2.10}$$

Clearly this is a linear homogeneous difference equation with constant coefficients of order two. The associated characteristic equation is

$$r^2 - r - 1 = 0 \tag{2.11}$$

assuming the sequence $\{r^n\}$ to be a solution of equation (2.10).

Now equation (2.11) has two distinct roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Hence the Fibonacci sequence is given explicitly by the formula

$$f_{n+2} = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n, \ \forall n \in \mathbb{N} \cup \{0\} \tag{2.12}$$

where C and D are determined by the condition $f_0 = 1 = f_1$.

Now

$$C + D = 1$$

$$C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solving, we get $C = \frac{1+\sqrt{5}}{2\sqrt{5}}$ and $D = -\frac{1-\sqrt{5}}{2\sqrt{5}}$

Substituting the values of C and D, equation (2.13) becomes

$$f_{n+2} = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

i.e.,

$$f_{n+2} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \forall n \in \mathbb{N} \cup \{0\} \qquad (2.13)$$

Interestingly, in spite of having irrational number in its expression, the Fibonacci sequence has each of its terms integer.

Case II: When the roots of the characteristic equation are real and identical

We assume now the characteristic equation of the linear difference homogeneous equation (2.8) has equal roots. So let the characteristic equation (2.9) has a root r of multiplicity 2.

Then using the observations made in Remark 2.2.2, we can say that the sequence $\{r^n\}$ satisfies the equation (2.8). It can also be easily checked that the sequence $\{nr^n\}$ satisfies the same equation. Hence using Remark 2.2.3, we can have the following theorem.

Theorem 2.2.3. *Let α be the equal root of the characteristic equation $r^2 - Ar - B = 0$ of the linear homogeneous difference equation $y_t = Ay_{t-1} + By_{t-2}$. Then the* *solution is given by* $y_t = (C + Dt)\alpha^t$. *Here C and D are determined by the initial values* y_0 *and* y_1.

Example 2.2.3. *Consider the linear homogeneous difference equation* $b_k = 4b_{k-1} - 4b_{k-2}$ *for integers* $k \geq 2$ *with initial conditions* $b_0 = 1$ *and* $b_1 = 3$.

Here the characteristic equation is $r^2 - 4r + 4 = 0$ *and it has only one root 2 of* multiplicity 2. Using Theorem 2.2.3, we have

$$b_k = C2^k + Dk2^k, \forall k \geq 0 \tag{2.14}$$

For determining C and D, we have

$$b_0 = 1 = C2^0 + D.0.2^0 = C$$

and

$$b_1 = 3 = C2^1 + D.1.2^1 = 2C + 2D$$

i.e., $C = 1$ *and* $D = \frac{1}{2}$.

Substituting the values of C and D in equation (2.14), *we get*

$$b_k = 2^k + \frac{1}{2}k2^k, \ \forall k \geq 0$$

i.e.

$$b_k = 2^k\left(1 + \frac{k}{2}\right), \ \forall k \geq 0$$

Case III: When the roots of the characteristic equation are complex

We assume that the roots of characteristic equation of the linear difference homogeneous equation (2.8) are complex say, $a \pm ib$. *Let* $a = r\cos\phi$ *and* $b = r\sin\phi$. *Then we have* $r^2 = a^2 + b^2$ *and* $\phi = \tan^{-1}\frac{b}{a}$.

Then

$$a + ib = r(\cos\phi + i\sin\phi)$$

$$a - ib = r(\cos\phi - i\sin\phi)$$

So the general solution is

$y_t = A_1(a + ib)^t + A_2(a - ib)^t$

$= A_1 r^t(\cos t\phi + i \sin t\phi) + A_2 r^t(\cos t\phi - i \sin t\phi)$, *using De Moivre's theorem*

$= r^t(C \cos t\phi + D \sin t\phi)$

where A_1, A_2, C, D are arbitrary constants.

Exercise 2.2.1.

1. *Find the explicit formula for the sequence $b_0, b_1, b_2, \ldots$ which satisfies the linear difference equation $b_k = 2b_{k-1} - b_{k-2}$ given $b_0 = 1$ and $b_1 = 2$.*
   (*Ans.* $b_k = 1+k$)

2. *Find the solution to the linear difference equation $a_n = 6a_{n-1} - 11a_{n-2}$ with initial conditions $a_0 = 2$, $a_1 = 5$.*
   (*Ans.* $a_n = \frac{1}{4}\left(\left(4-i\sqrt{2}\right)\left(3-i\sqrt{2}\right)^2 + \left(4+i\sqrt{2}\right)\left(3+i\sqrt{2}\right)^k\right)$

3. *Find the solution to the linear difference equation $a_n = 4a_{n-1} - 5a_{n-2}$ with initial conditions $a_0 = 2$, $a_1 = 1$.*
   (*Ans.* $a_n = \frac{1}{2}\left(\left(2-3i\right)(2-i)^k + \left(2+3i\right)(2+i)^k\right)$)

2.3 Introduction to Discrete Models

The Fibonacci rabbit model in the previous section has already given us a flavour of discrete modeling. We will go now for more formal approach.

Let the variable X_t denotes the state of an ecological or economical or physical system at time *t*. This state variable may be anything like age, weight or number of living organisms of a population or temperature of an environment etc. Now the system state at time $t+\Delta t$ denoted by $X_{t+\Delta t}$ is a function of X_t, i.e.,

$$X_{t+\Delta t} = F(X_t)$$

This function depends on the system under consideration. If the function *F* is explicitly independent of time *t*, then the above equation is called an autonomous difference equation. The difference equation model predicts state of the system at series of equally spaced times, for example say one second or one minute or one year etc. If we know the state at time $t = 0$, we can calculate its state at times $t = \Delta t, 2\Delta t, 3\Delta t, \ldots$.

For non- autonomous systems, the difference equation is of the form

$$X_{t+\Delta t} = F(X_t, t)$$

2.4 Linear Models: Exemplifying through a growth model

Linear models are one of the simplest models. Here the state variable at any time interval is essentially expressed as a linear function of the state variables at previous time intervals. We recall that if the state variable is a linear function of the state variables at $k \geq 1$ previous time intervals, then the linear model is of k–th order. Here we discuss a 1st order linear model.

2.4.1 A growth model

Here we discuss a simple growth model. In Section 2.5, we will see growth models with stocking and harvesting. Those models are also linear.

Suppose a population of cells divides synchronously, with each member producing a daughter cells. Let M_i be the number of cells in i–th generation, where $i = 0, 1, 2, \ldots, n$.

$$M_{n+1} = aM_n \tag{2.15}$$

is the relation between successive generations. Then using equation 2.15, we have

$$M_{n+1} = a^{n+1}M_0 \tag{2.16}$$

Clearly the population grows or dwindles with time depending the magnitude of a. It is easy to understand from our previous discussions that population increases over successive generations if $|a| > 1$, decreases if $|a| < 1$ and remains constant if $|a| = 1$.

Similarly if the per capita birth and death rates of a population are b and d respectively, then setting $r = 1 + b - d$ we can write the population model as

$$P_{n+1} = P_n + bP_n - dP_n = (1+b - d)P_n$$

i.e.,

$$P_{n+1} = rP_n \tag{2.17}$$

where P_i is the population of the i–th generation.

In subsequent sections, we will discuss some more discrete linear models.

2.5 Steady state solution: Exemplifying through growth models with stocking and harvesting

Here we will see an analytic approach to understand the global behavior of our models, especially, their long-term behavior without having to resort to tedious calculations.

Steady state solutions or Equilibrium Values

One of the fundamental object of study in case of mathematical modeling is finding the equilibrium values of the system. An *Steady state solution* or *equilibrium value* is a number, which we denote by P^* in the context of population, at which the system under consideration does not change with time. In other words, P^* is an equilibrium value if setting $P(t-1) = P^*$ results in $P(t) = P^*$ also. We mainly use simple algebraic technique to compute the equilibrium values. The following growth model with stocking or harvesting will help us to understand this. Let us see first what stocking and harvesting are.

Whether intentionally or unintentionally, humans do often have an impact on wildlife populations. There are two types of influence we will see here. One is *harvesting*, i.e., the systematic removal of members from a population, and the other is *stocking*, i.e., the systematic addition of members to a population.

2.5.1 Growth with stocking

Now suppose population of a particular species of birds, say cranes, was 50 in 1980 and was declining (may be due to natural attrition) at an average rate of approximately 6% per year. Also assume 9 birds are introduced to the population every year. We explain below this phenomenon diagrammatically.

Thus if $P(t)$ be the population of birds of the particular species at year t, then our equation becomes

$$P(t) = P(t-1) - 0.06P(t-1) + 9 \qquad (2.18)$$

2.5.2 Growth with harvesting

In contrary to the idea of stocking, some times harvesting becomes a necessity for a growing population. Suppose a certain species of deer population grows shows 26% increase in every year. To prevent over-grazing let us assume *a* deer are harvested.

Therefore if $P(t)$ be the population of birds of the particular species at year t, then our equation becomes

$$P(t) = P(t-1) + 0.26P(t-1) - a \qquad (2.19)$$

In general, the explicit formula for harvesting or stocking is

$$P(t) = P(t-1) + rP(t-1) + a \qquad (2.20)$$

where r is the growth rate (negative or positive) and a is the number that is being added to or subtracted from the population each year. If a is positive then we are stocking, while if a is negative then we are harvesting.

Finding the equilibrium value

We do this using simple algebra. If P^* be the equilibrium value of equation (2.20), then we have

$$P^* = P^* + rP^* + a,$$

i.e., $$P^* = -\frac{a}{r}.$$

We see that finding an equilibrium value for such a model turns out to be a relatively straight for ward calculation—just divide the harvesting or stocking number by the growth rate.

Caution: Mind the minus signs.

Clearly, the equilibrium value for the system represented by equation (2.18) is

$$P^* = -\left(\frac{9}{-0.06}\right) = 150.$$

Remark 2.5.1. *Simplifying equation (2.20), we get*

$$P(t) = (1+r)P(t-1) + a$$

For $t = 1$,

$$P(1) = (1 + r)P(0) + a.$$

For $t = 2$,

$$\begin{aligned} P(2) &= (1+r)P(1) + a \\ &= (1+r)[(1 + r)P(0) + a] + a \\ &= (1+r)^2 P(0) + (1 + r)a + a \end{aligned}$$

For $t = 3$,

$$\begin{aligned} P(3) &= (1 + r)P(2) + a \\ &= (1+r)[(1 + r)^2 P(0) + (1 + r)a + a] + a \\ &= (1+r)^3 P(0) + (1 + r)^2 a + (1 + r)a + a \end{aligned}$$

Proceeding similarly, we get

$$P(t) = (1 + r)^t P(0) + \{(1 + r)^{t-1} a + ... + (1 + r)^2 a + (1 + r)a + a\}.$$

Thus we get the explicit formula for growth with stocking or harvesting

$$P(t) = (1+r)^t \; P(0) + a\frac{(1+r)^t - 1}{r} \tag{2.22}$$

Clearly this is an exponential growth model which may not sustain for long due to scarcity of food and other essentials. Later we will see more practical approach depending on density of the population.

Example 2.5.1. *In a forest, suppose initially there was* 500 *deers. If the deer population grows at a rate of* 10% *per year and* 50 *deers are removed each year from the forest, what will be the population after* 5 *years?*

Solution. Putting $P(0) = 500$, $a = -50$, $r = 0.1$ and $t = 5$ in the equation (2.22), we have the required population $P(5) = 500$.

2.6 Linear Stability Analysis

In mathematical modeling, the stability is of fundamental importance. When a steady state is unstable, great changes may about to happen. For example, an entire population may crash or balance in number of competing groups or species may shift in favour of a few. Thus it is very important to understand the nature of the stability even if an exact analytical solution is not readily available or easy to obtain.

We now discuss the criteria of stability of a steady state solution or equilibrium point of a non-linear first order difference equation

$$x_{n+1} = f(x_n) \tag{2.23}$$

where the function f is a non-linear function of its argument. Let P^* be the equilibrium point of equation (2.23). We are interested in the local stability analysis in the neighbourhood of P^*.

Suppose ξ_n be an infinitesimally small perturbation of the equilibrium point P^* at n-th time interval. Then we write

$$\begin{aligned} x_{n+1} = P^* + \xi_{n+1} &= f\left(P^* + \xi_n\right) \\ &= f(P^*) + \xi_n\, f'(P^*) + O\left(\xi_n^2\right) \\ &= P^* + \xi_n\, f'(P^*) + O\left(\xi_n^2\right) \end{aligned}$$

By neglecting the higher order term $O\left(\xi_n^2\right)$, the equation (2.23) is linearized as follows.

$$\xi_{n+1} = a\,\xi_n \tag{2.24}$$

where

$$a = f'(P^*) = \left.\frac{df}{dx}\right|_{x=P^*} \tag{2.25}$$

Thus the non-linear equation (2.23) has been reduced to the linear equation (2.24). Note that the solution of equation (2.24) decreases and tend to P^*, whenever $|a| < 1$.

Note that if $|f'(P^*)| = 1$, then the sequence $\{\xi_n\}$ and hence the perturbation becomes constant for all n. Hence it fails to give any conclusion.

Condition for local stability

It is evident from equations (2.24) and (2.25) that the equilibrium point P^* is asymptomatically stable if and only if $|f'(P^*)| < 1$.

The equilibrium point P'' is asymptomatically unstable if and only if $|f'(P^*)| > 1$.

Example 2.6.1. *The growth of a population satisfies the difference equation*

$$x_{n+1} = \frac{kx_n}{b + x_n}$$

where $k > b > 0$. *Find the steady state solution (if any). If so, is it stable?*

Solution. Let x^* be the steady state solution. Then we have

$$x^* = \frac{kx^*}{b + x^*}$$

i.e., $x^* = 0, k - b$

Now let $f(x) = \dfrac{kx}{b + x}$. Then $f'(x) = \dfrac{bk}{(b + x)^2}$.

Case I: $x^* = k - b$

Now $f'(x^*) = f'(k - b) = \dfrac{b}{k}$. Then $|f'(x^*)| < 1$. Hence the equilibrium is stable.

Case II: $x^* = 0$

Now $f'(x^*) = f'(0) = \dfrac{k}{b}$. Then $|f'(x^*)| > 1$. Hence the equilibrium is unstable.

2.7 Newton's Law of Cooling

An interesting example arises in modeling the change in temperature of an object placed in an environment held at some constant temperature, such as a cup of tea cooling to room temperature or a glass of lemonade warming to room temperature. If T_0 represents the initial temperature of the object, S the constant temperature of the surrounding environment, and T_n the temperature of the object after n units of time, then the change in temperature over one unit of time is given by

$$T_{n+1} - T_n = k(T_n - S) \tag{2.26}$$

or equivalently

$$T_{n+1} = (k + 1)T_n - S \tag{2.27}$$

where $n = 0, 1, 2, ...,$ and k is a constant which depends upon the object. This difference equation is known as *Newton's law of cooling*. The equation says that the change in temperature over a fixed unit of time is proportional to the difference between the temperature of the object and the temperature of the surrounding environment. Thus large temperature differences result in a faster rate of cooling (or warming) than do small temperature differences. If S is known and enough information is given to determine k, then this equation may be rewritten in the form of a first order-linear difference equation and, hence, solved explicitly. The next example shows how this may be done.

Example 2.7.1. *Suppose a cup of tea, initially at a temperature of* 180^oF*, is placed in a room which is held at a constant temperature of* 80^oF*. Moreover, suppose that after one minute the tea has cooled to* 175^oF*. What will the temperature be after* 20 *minutes? What will be the equilibrium temperature of the room?*

Solution. If we let T_n be the temperature of the tea after n minutes and we let S be the temperature of the room, then we have $T_0 = 180$, $T_1 = 175$ and $S = 80$. Then Newton's law of cooling states that

$$T_{n+1} - T_n = k(T_n - 80) \tag{2.28}$$

where $n = 0, 1, 2, . . .$ and k is a constant which we will have to determine. To do so, we make use of the information given about the change in the temperature of the tea during the first minute. Namely, applying equation (2.28) with $n = 0$, we have

$$T_1 - T_0 = k(T_0 - 80)$$

i.e., $$175 - 180 = k(180 - 80)$$

i.e., $$-5 = 100k$$

Hence, $$k = -0.05$$

Hence from equation (2.28), we have

$$T_{n+1} - T_n = -0.05(T_n - 80)$$

i.e., $$T_{n+1} = 0.95T_n + 4$$

for $n = 0, 1, 2, ...$

Therefore equation (2.28) gives

$$T_n = (0.95)^n 180 + 4\left(\frac{1-(0.95)^n}{1-0.95}\right)$$

$$= 80 + 100(0.95)^n$$

for $n = 0, 1, 2, ...$

In particular, $T_{20} = 80 + 100(0.95)^{20} = 115.85$

where we have rounded the answer to two decimal places. Hence after 20 minutes the tea has cooled to just under 116°*F*. Also, since $\lim_{n\to\infty}(0.95)^n = 0$, therefore $\lim_{n\to\infty} T_n = 80$. Thus the temperature of the tea will approach an equilibrium temperature of 80°*F*, the room temperature.

2.8 Bank Account Problem

Here we discuss the problem of finding the amount deposited for N years in a bank at the interest rate r per annum and principal amount P to be compounded annually. Suppose P_n be the principal at n–th year and $P_0 = P$. Then

$$P_1 = P_0(1 + r)$$

$$P_2 = P_1(1 + r)$$

$$= P_0(1 + r)^2$$

...

$$P_n = P_0(1 + r)^n$$

Hence we have

$$P_N = P_0(1 + r)^N = P(1 + r)^N \tag{2.29}$$

Example 2.8.1. *What will be the amount deposited for 10 years to be compounded annually at the rate* 10% *per annum, if the initial deposit is Rs. 100,000?*

Solution. *Here* $N = 10$, $P = 100,000$ *and* $r = 0.10$. *Therefore the amount deposited after 10 years will be* $P_{10} = P \times 1.1^{10} \approx 259,374$ *(after rounding off).*

2.9 Mortgage problem

Here we discuss the problem of finding the installment for a loan (borrowed from a financial institute like bank taken against some property) at a fixed annual interest rate, for a fixed tenure and compounded over a fixed period of installment (say, monthly or quarterly).

Let $P = P(0)$ be the the amount the borrower has taken as loan at the beginning at an annual interest rate r to be paid off in N regular installments each of span Δt. Also let $P(t)$ be the amount the borrower owes at time t.

$$\xrightarrow{r\Delta t P(t-1)} \boxed{\text{Amount the borrower owes to the bank}} \xrightarrow{M}$$

If M be the payment on each installment, then our difference equation becomes

$$P(t+\Delta t) = P(t) + r\Delta t P(t) - M$$

i.e., $$P(t+\Delta t) = P(t)(1 + r\Delta t) - M$$

Remark 2.9.1. *This is an example of non- autonomous non- homogeneous first order difference equation.*

Now we proceed to solve the above equation. We have

$$P(\Delta t) = P(0)(1 + r\Delta t) - M$$

$$P(2\Delta t) = P(\Delta t)(1 + r\Delta t) - M$$

$$= (P(0)(1 + r\Delta t) - M)(1 + r\Delta t) - M$$

$$= P(0)(1 + r\Delta t)^2 - M[1 + (1 + r\Delta t)]$$

$$P(3\Delta t) = P(0)(1 + r\Delta t)^3 - M[1 + (1 + r\Delta t) + (1 + r\Delta t)^2]$$

...

$$P(n\Delta t) = P(0)(1 + r\Delta t)^n - M[1 + (1 + r\Delta t) + (1 + r\Delta t)^2 + \cdots + (1 + r\Delta t)^{n-1}]$$

Taking $R = 1 + r\Delta t$, we have

$$P(n\Delta t) = P(0)R^n - \frac{M(R^n - 1)}{R-1}$$

Recall that $P = P(0)$. Also note that we must have $P(N\Delta t) = 0$. Hence the installment amount M is given by

$$M = \frac{PR^N (R-1)}{R^N - 1} \tag{2.30}$$

where $R = (1+r\Delta t)$.

Equivalently we have

$$M = \frac{\Pr \Delta t \left(1+r\Delta t\right)^N}{\left(1+r\Delta t\right)^N - 1} \tag{2.31}$$

Exercise 2.9.1. *Suppose someone has borrowed Rs. 100, 000 to buy a property at 10% annually interest, compounded monthly. What would the monthly payment be if he/ she wants to pay off the loan in 30 years?*

Hint: Here $r = 0.1$, $\Delta t = \frac{1}{12}$, $P = 100,000$ and $N = 360$.

2.10 Drug Delivery Problems: A Decay Model and Absorption

2.10.1 A Decay Model

As soon as a drug is ingested, the body begins to eliminate it. This can happen through metabolism, where enzymes break down the drug into different metabolites, or it can happen through excretion, where the drug is passed out of the body through the breath, sweat, or urine.

Here we will not make a distinction between these two processes, opting instead to make the simplifying assumption that treats both possibilities together as a single process that we call *elimination*. It may become necessary or expedient later to consider metabolism and excretion separately, but for now our goal is to keep our model as simple as possible.

For most drugs at usual dosages, elimination takes place at a rate that is a constant proportion of the amount of drug present in the body. This kind of elimination process is called *first-order elimination*. In contrast a drug that is eliminated by a constant amount for each time step is said to undergo zero-order elimination. Many common drugs, including ibuprofen and caffeine, undergo first-order elimination. Alcohol is an example of a drug that is well modeled by *zero-order elimination* (as in the Widmark model), at least for relatively high amounts of alcohol in the body. In this section, we focus on first-order elimination.

We here recognize first-order elimination as an exponential decay model. If we let $B(t)$ be the amount of drug in the body at time t and let r be the elimination rate, then we have the familiar flow diagram

$$\boxed{\text{Drug in body}} \xrightarrow[rB(t-1)]{\textit{outgoing}}$$

Then our model becomes

$$B(t) = B(t-1) - rB(t-1) \tag{2.32}$$

where $B(0)$ is the initial amount of drug in the body.

Drug Half-Life

Drug manufacturers are required to report what is known as the *half-life* of a drug, which is the time it takes the body to eliminate one half of the drug. Thus if a drug has a reported half-life of 4 h and initially 500 mg of the drug is present in the body, there will be 250 mg in the body 4 h later, 125 mg 4 h after that, and so on. We use the symbol $T_{\frac{1}{2}}$ to denote the half-life. Our job as modelers is to deduce the rate of elimination, r, from the half-life.

The next example shows how we can deduce the elimination rate from the half-life by using the explicit formula.

Example 2.10.1. *The half-life for the pain reliever ibuprofen is approximately 2 h. We will determine r, the approximate percentage of the drug that is eliminated from the body each minute.*

We use the explicit formula for the exponential model where t is time in minutes and B(t) is the amount of ibuprofen in milligrams still present in the body at time t. Our explicit formula is $B(t) = (1 - r)^t B(0)$.

By definition if $T_{\frac{1}{2}}$ *is is the half-life of ibuprofen, then* $B\left(T_{\frac{1}{2}}\right) = \frac{1}{2}B(0)$, *where* $B(0)$ *is the initial amount of ibuprofen in the body. Thus with a half-life of 120 min, we have*

$$\frac{1}{2}B(0) = (1-r)^{120}B(0)$$

i.e., $$\frac{1}{2} = (1-r)^{120}$$

i.e., $$r = 1 - \left(\frac{1}{2}\right)^{\frac{1}{120}}$$

Thus r = 0.00574. As a percent we have an elimination rate for ibuprofen of r = 0.574% per min. Note that the initial amount of drug present did not matter in our calculation of r.

2.10.2 Absorption

The preceding model assumed the body was a single compartment, and we focused on the elimination of the drug from the body. For drugs that are administered via direct injection or intravenously, a single-compartment model makes sense because the drug is instantly present in the blood. However, many drugs, especially over-the-counter drugs, are administered orally.

Drugs taken orally do not instantly enter the bloodstream; they must be digested first. This means we need to take into account how quickly the drug is absorbed into the body from the gastrointestinal, or GI, tract. To model absorption we add the GI tract as a second compartment considered as separate from "the body," or central compartment, and we introduce a new parameter, the *absorption rate*, into the model.

Let $GI(t)$ be the amount of drug in the GI tract at time t, and let α be absorption rate. We assume that absorption from the GI tract into the body is a first-order process so that α represents the fixed percentage of the drug being absorbed into the body at each time step.

$$\boxed{\text{Drug in GI}} \xrightarrow{\alpha GI(t-1)} \boxed{\text{Drug in body}} \xrightarrow{rB(t-1)}$$

The corresponding two-compartment model is therefore

$$GI(t) = GI(t-1) - \alpha GI(t-1)$$

$$B(t) = B(t-1) - rB(t-1) + \alpha GI(t-1) \tag{2.33}$$

Example 2.10.2. *Let us assume* 95% *of a drug will be absorbed from the GI tract into the bloodstream within* 30 *min of ingestion. Find the absorption rate* α.

Solution. The explicit formula for the amount of drug remaining in the GI tract is given by $GI(t) = (1 - \alpha)^t GI(0)$, where $GI(0)$ is the initial dose of the drug. The way the absorption of the drug is reported we should have only 5% of the original dose remaining after 30 min, i.e., $GI(30) = 0.05GI(0)$. Hence we have

$$(1-\alpha)^{30} GI(0) = 0.05GI(0) \Rightarrow 1-\alpha = 0.05^{\frac{1}{30}}. \text{ Therefore } \alpha \approx 0.095.$$

2.11 Harrod Model of Economic Growth

The Harrod–Domar model is used in development economics to explain an economy's growth rate. Before we go deep into this, we need to understand the following things.

Gross Domestic Product or GDP

There are different measures to gauge the output of a country. Here we will take GDP as the output of the economy of a country in any given year. The *Gross Domestic Product* or *GDP* is the value of all finished goods and services produced within a country in a year. There are other ways for measuring the GDP. The approach, we are discussing now, is known as the value added approach.

We need to understand what a finished good or service means. A finished good or service is one which will not be sold again as a part of some other good or service. For example, when a bakery purchases flour, eggs or butter, we will not count these sales in GDP as they will be used as intermediate goods to produce the cake. Now the cake is a finished good. On the contrary, the same flour, eggs and butter are considered as finished goods when they are bought by a household consumer for preparing a delicious dish.

There are also goods, which are used to make other goods, but still are considered as finished goods. These are called capital goods. For example, if a company produces a tractor and sells it to an agricultural farm, then the tractor is considered as a finished good and its value is added to the GDP. Although the tractor is used to produce agricultural goods, it will not be sold again as a part of another good.

GDP only counts production in a given year. So if an old house is sold in a given year, its value will not contribute to the GDP since it was not built in that particular year. However, sale of a new house does contribute to the GDP.

Also consideration of geographical or territorial boundary is a must for calculating the GDP. Suppose a manufacturer in Switzerland exports a watch and someone in our country buys that imported watch. Then the value of the watch does not add to our country's GDP but it does contribute to the GDP of Switzerland.

Remark 2.11.1. *To keep things simple, here we neglect the effect of inflation or taxation while defining GDP.*

Remark 2.11.2. *We can also calculate the GDP by adding up the total consumption, investment, government spending and net export (i.e., total export – total import) of a country in a given year. This is known as expenditure approach for calculating GDP. It can be shown that all the four components must add up to the total value of all finished goods and services produced over a certain period of time in a country.*

Remark 2.11.3. *There is another approach for measuring GDP. It is called the income approach. The income approach to calculate the GDP states that all economic expenditures should be equal to the total income generated by the production of all economic goods and services.*

Exercise 2.11.1. *Explain GDP from the value added approach, the expenditure approach* and the income approach.

Capital

Capital is essentially the total resources supplied to a business by the owner. In other words, any financial resource or asset owned by a business (that is beneficial in boosting growth and general revenue). It may include items such as cash or any other assets like machinery, land, equipment, infrastructure, computers, software etc.

Suppose Gita sells jackets in a tourist market. She has a stall there. She has Rs. 3000/- in bank, Rs. 2000/- cash in hand, jackets worth of Rs. 1500/- and fixtures as well as furniture worth of Rs. 2500/- in her stall. So she has capital stock worth of total Rs. 9000/-.

Remark 2.11.4. *Money and capital are not the same thing. Money is used to acquire and sell goods or services within the business itself or between customers and other businesses. This allows businesses to gain money including profits. This is a short time scale phenomenon.*

On the other hand, capital is used to develop and improve the future of the business. The capital is utilized to ensure a sustainable revenue generation. Obviously such activities are long term phenomena.

Exercise 2.11.2. *Explain the difference between money and capital.*

Capital output ratio

If K be the capital and Y be the output of an economy, then $r = \frac{K}{Y}$ is the *Capital Output ratio* of the economy. By this, we try to measure the efficiency of the capital.

Assumptions and their implications

The Harrod–Domar model assumes the followings.

- **The economy of the country is a closed economy.** This means no trade or import- export takes place. So the net export is always zero.
- **There is no government intervention.** This means the factor of government spending is absent in the calculation of GDP.
- **There is always full employment.**
- **The production function is fixed coefficient.** This means the production function Y describes a process which requires inputs to be combined in fixed proportions. Such production function does not allow one factor to be substituted for another when there is a change in the relative prices of inputs.
- **Savings equal to investment.** When people save money, that money is saved in banks and other financial institutions and eventually invested. If companies save money, they can spend it on factories, warehouses and developing infrastructures. Thus if S_t and I_t are the total savings and total investment of a country in t–th year, then we will have

$$S_t = I_t \tag{2.34}$$

- **Investment equals to changes in capital stock.** So if K_t and I_t are the total capital and total investment of a country in t–th year, then

$$I_t = \Delta K_t = K_{t+1} - K_t \tag{2.35}$$

- **The capital output ratio is constant.** Thus if Y_t is the output in the t–th year, then

$$r = \frac{K_t}{Y_t} = \frac{K_{t+1}}{Y_{t+1}}$$

$$= \frac{K_{t+1} - K_t}{Y_{t+1} - Y_t}$$

Hence

$$r = \frac{\Delta K_t}{\Delta Y_t} \tag{2.36}$$

- **Total savings is proportional to the national income.** Hence using the income approach for the GDP, we have

$$S_t = sY_t \tag{2.37}$$

s being the constant of proportionality.

Now using equation (2.36), we have

$$\Delta Y_t = \frac{\Delta K_t}{r}$$

i.e., $Y_{t+1} - Y_t = \frac{I_t}{r}$, using equation (2.35)

$= \frac{S_t}{r}$, using eqnation (2.36)

$= \frac{sY_t}{r}$, using equation (2.37)

Hence the model is given by

$$Y_{t+1} = \left(1 + \frac{s}{r}\right) Y_t \tag{2.38}$$

The the constant $\frac{s}{r}$ is called the *warranted rate of growth*.

Exercise 2.11.3. *What are the assumptions of the Harrod Domar Model?*

Exercise 2.11.4. *Describe the Harrod Domar Model.*

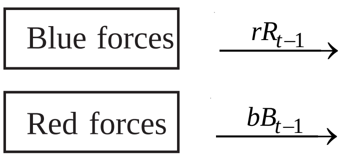
2.12 War Model

Lanchester Combat Model

One of the first mathematical models for analyzing combat was proposed by F. W. Lanchester in 1916 in his book *Aircraft in Warfare: The Dawn of the Fourth Arm* (Engel, 1954). The great strength of the Lanchester combat model and what makes it so compelling is its simplicity. In spite of the fact that the assumptions are too severe to be expected to be satisfied in a real battle, it helps to draw important conclusions regarding tactics and strategy.

Suppose we have two adversaries Blue and Red. Let the number of remaining units of Blue and Red in battle at time t are Bt and Rt. These units can be anything varying from ships, tanks, soldiers, etc. The basic assumption of the Lanchester model is that a side incurs losses at a rate that is proportional to the size of the enemy's force. This means the larger the Red force, the more damage it will do to the Blue force and vice versa. We also assume uniformity of units, that is, that all units for each side are equally capable.

In order to complete the model, we introduce a parameter for *fighting effectiveness*, which we define to be the average number of enemy units put out of action by a single opposing unit during each time step. We can think of fighting effectiveness as a kind of overall measure that is affected by things such as quality of training, weapons technology, and experience with the terrain. We assume b to be the fighting effectiveness of a Blue unit and r to be the fighting effectiveness of a Red unit.



Then at each time interval, both the forces diminishes in proportion to the size of the enemy. Then our model becomes

$$\begin{aligned} B_t &= B_{t-1} - rR_{t-1}, \\ R_t &= R_{t-1} - bB_{t-1}. \end{aligned} \tag{2.39}$$

Exercise 2.12.1. *Suppose Blue begins the battle with 50 units, so* $B_0 = 50$, *and Red begins the battle with 100 units, so* $R_0 = 100$. *Each Blue unit has a fighting effectiveness of* $b = 0.10$, *which means that each Blue unit will inflict 0.10 casualties (units put out of action) on the Red side per time step. Similarly each Red unit has a fighting effectiveness of* $r = 0.20$. *After one time step, how many units of each side remain?*

2.13 Lake pollution model

Consider the case of two lakes connected by a water flow. Suppose also that the measurement of the pollution indicated that p% pollution of the second lake goes to the first lake comes from. On the other hand, q% pollution of the first lake goes to the second lake. This phenomena can be modeled with the help of a system of difference equations. We will also discuss the equilibrium values of the system and try to understand the long term behavior.

To model this situation, consider the following variables. Let n denote the number of years, Let a_n and b_n be the total amounts of pollution in two lakes respectively after n years. In this case

$$\begin{aligned} a_{n+1} &= (1-q)a_n + pb_n \\ b_{n+1} &= qa_n + (1-p)b_n \end{aligned} \tag{2.40}$$

The equilibrium values of this system gives the amount of pollutant that would remain the lakes on the long run.

For this, we assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Thus for sufficiently large n, we have

$$a = (1-q)a + pb$$

$$b = qa + (1-p)b$$

Solving, we get

$$b = \frac{q}{p}a \tag{2.41}$$

This indicates the steady state lies on a straight line. The relation determines the limiting ratio of pollutant in the two lakes.

2.14 Alcohol in the Bloodstream Model

Blood alcohol concentration (BAC) is a measure of how much alcohol, specifically ethanol, is in the body. When alcohol is ingested, it moves rapidly through the stomach to the small intestine. Since alcohol is water soluble, it is absorbed from the small intestine into the body water where it quickly becomes evenly distributed throughout the body. For many drugs, alcohol included, the concentration of the drug in the body is more important than the total amount present because larger bodies need more of the drug in order to achieve the same effect. A 300-pound person, for example, will feel much different after four beers than a 150-pound person would. To calculate BAC, we proceed in stages:

1. we calculate the amount of alcohol ingested,
2. we estimate the amount of water a person's body contains,
3. we calculate the concentration of alcohol in the body water by dividing the amount of alcohol by the amount of water, and
4. we deduce the concentration of alcohol in the blood in light of the fact that blood is 80.6% water.

The question of how much body water a person has is an interesting one that depends on many factors including weight, age, and sex. The amount of body water helps explain observed differences in how males and females respond to the same dose of alcohol. Women in general have a higher percentage of body fat than men, and thus they tend to have less body water than men even when their body weight is the same. Thus a dose of alcohol will typically produce a higher BAC in a woman than in a man of the same weight. As a result, women tend to feel more intoxicated than men when consuming the same amount of alcohol. We proceed with an example of how a basic BAC calculation is done.

Example 2.14.1. *Mark is a 180–pound male who quickly consumes two 12–oz. beers. To estimate Mark's BAC, we assume that all of the alcohol from the two beers is quickly emptied from Mark's stomach and distributed uniformly in his total body water. First we need to know how much alcohol, in grams, Mark consumed. A standard 12-oz. beer contains about 14 g of alcohol (as do a 5-oz. glass of wine or 1.5–oz. shot of 80– proof liquor), so our subject has approximately 28 g of alcohol in his body water. Next we need to calculate how much body water a 180–pound male typically has. In the absence of more specific information, we use* *standard average values for body water percentage. On average males are 58% water, while females are 49% water. The lower percentage of body water for females is due primarily to their typically higher levels of body fat, which contains little water, versus muscle, which contains a lot of water.*

Now we estimate the Mark's BAC.

1. *Begin with body weight in pounds, and change the body weight to kilogram* (1 *kg* = 2.2046 *pounds*)*:* $180\ pounds \bullet \times \dfrac{1\,kg}{2.2046\ pounds} = 81.65\ kg$
2. *Using typical sex percentages, find total body water volume (*1 *l of water weighs* 1 *kg) by multiplying body weight by body water percentage:* $81.65\ kg \bullet \times\ 58\% = 47.36\ kg\ H_2O = 47.36\ l\ H_2O$
3. Calculate the concentration of alcohol in the body water by dividing total amount of *alcohol by total body water:* $\dfrac{28\ g}{47.36\ l\ H_2O} = 0.5912 g\ per\ l\ H_2O.$
4. *Using the fact that blood is* 80.6% *water, calculate BAC from body water concentration:* $BAC = 0.5912\ \dfrac{g}{l\ H_2O} \times 0.806\ \dfrac{l\ H_2O}{l\ blood} = 0.4765\ g\ per\ l\ blood$

The Widmark model

The basic calculations from the previous section provide a way for us to get a rough estimate of a person's BAC. However, these kinds of calculations suffer from being static—they only give us BAC at one moment in time. They also make use of questionable assumptions: that all consumed alcohol is present in the body, and that the alcohol is instantly distributed throughout the blood. In this section we go a step further and discuss a discrete model for predicting BAC over time: the *Widmark model*.

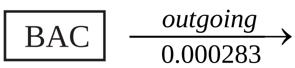
In 1932, Widmark developed a single-compartment model for predicting BAC over time that has become the most widely used and cited BAC model due to its simplicity and its accuracy for a large percentage of the population.

As soon as alcohol is consumed, it begins to be removed from the body primarily by metabolism in the liver. A small percentage of the alcohol is excreted by passing from the body unchanged via the breath, sweat, and urine; another small percentage is metabolized in the stomach. The Widmark model does not differentiate among these different pathways;

instead it treats the body as a single compartment and it treats excretion and metabolism as a single elimination process leading to an overall constant rate of decrease in BAC.

Once consumed, alcohol diffuses rapidly through the body water and hence the blood. Widmark estimated that the rate at which alcohol is then cleared from the body results in a decrease in BAC of about 0.017 each hour, or $\frac{0.017}{60} = 0.000283$ per minute. This rate of elimination varies from individual to individual, and it can range from 0.010 to 0.040 per h with lower values typical for those who do not regularly consume alcohol and higher values for heavy drinkers. In other words, heavy drinkers tend to metabolize alcohol more quickly than others. The average value for a heavy drinker is an approximate 0.020 decrease in BAC per hour.

The Widmark model assumes the rate of change for BAC is a constant that does not depend on the amount of alcohol present.



Thus the model becomes

$$BAC(t) = BAC(t-1) - 0.000283 \qquad (2.42)$$

where time is measured in minutes since the last drink. The initial BAC is calculated as described before.

Note that BAC is decreasing by a constant amount implies that there will be no equilibrium values for this model. A serious drawback of this model is if we project BAC far enough into the future, we will always end up with negative values for BAC which is absurd.

2.15 Arm Race Model

It is unfortunate that even long after the days of cold war are over, war still remains a means for resolving international conflicts. Therefore like it or not, the study of arms races continues to be of practical significance. An arms race may increase the tension between two nations and increase the probability that a minor dispute will end up into war. Even if this kind of escalation does not result in war, increased military expenditure reduces the amount a nation can spend on other pursuits, such as social welfare like education, employment generation, public health etc. Arms races have significant costs independent of whether they lead to war or not.

What causes nations to wage war? History shows that the existence of weapons— large military arsenals— increases the likelihood of violent conflict. Without destructive weapons, perhaps nations sometimes would settle disputes by other means. It was this assumption that led Lewis Fry Richardson to begin his study and analysis of arms races. Richardson was a Quaker and was troubled by both WWI and WWII. His scientific training in physics led him to believe that wars were a phenomena that could be studied and mathematically modeled.

The model

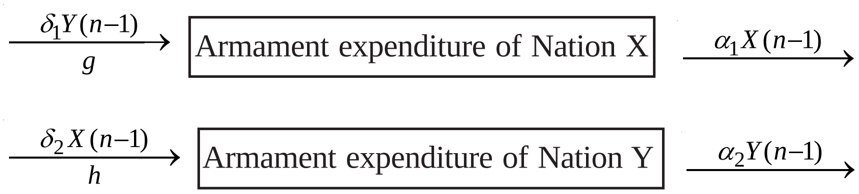
Here we examine the Richardson's Arms Race Model as a system of linear difference equations. We let, $X(n)$ = the expenditure for armament of Nation X at time $t = n$ and $Y(n)$ = the expenditure for armament of Nation Y at time $t = n$.

Now each nation's armament has undeniable effect on the other nation. Let δ_1, δ_2 are constants such that $X(n-1)$ is increased by $\delta_1 Y(n-1)$ at time $t = n$ and similarly $Y(n-1)$ is increased by $\delta_2 X(n-1)$ at time $t = n$, assuming the constants to be positive (however these constants may be negatives as well). These δ_1, δ_2 are termed as *defense coefficients* or how each nation is effected by the strength of the other nation.

We also consider the effect of fatigue due to adverse effect of keeping up an arms race. This fatigue may be due to reduced budget on social welfare schemes like public education, public health programs or steep price hike for essential commodities etc. We assume α_1 and α_2 are *fatigue coefficients* such that $X(n-1)$ is decreased by $\alpha_1 X(n-1)$ at time $t = n$ and similarly $Y(n-1)$ is decreased by $\alpha_2 Y(n-1)$ at time $t = n$.

Finally, grievances or ambitions are added to the model as constants. We let g and h are respective grievances of Nations X and Y .

The following diagram assumes the constants α_i, δ_i (i = 1, 2), g, h to be positive. However the diagram may have to be modified if the signs of the constants are otherwise.



Hence our arm race model becomes

$$X(n) = (1-\alpha_1)X(n-1) + \delta_1 Y(n-1) + g$$

$$Y(n) = (1-\alpha_2)Y(n-1) + \delta_2 X(n-1) + h \qquad (2.43)$$

Example 2.15.1. *Let X(n) and Y (n) are armament expenditures of Nations X and Y respectively in the arm race model. We assume* α_1 = 0.2, α_2 = 0.1, δ_1 = –0.3, δ_2 = 0.2, *g* = 8000, *h* = 2000. *Note that the negative sign of* δ_1 *suggests that Nation X is reducing its armament budget despite the fact that Nation Y is escalating its defense procurement (as* δ_2 *> 0).*

We intend to investigate the system. If (*p*, *q*) *be the equilibrium point, then we have*

$$p = 0.8p - 0.3q + 8000$$

$$q = 0.2p + 0.9q + 2000$$

Solving, we have (*p*, *q*) = (2500, 25, 000). *The coefficient matrix corresponding to given problem has the complex eigenvalues* $0.85 \pm i\sqrt{0.0575}$. *Since* $\left|0.85 \pm i\sqrt{0.0575}\right| \approx 0.883 < 1$, the equilibrium point must be a sink and solutions spiral to it.

2.16 Density Dependent Growth Model with Harvesting

Real populations seldom exhibit exponential growth for long. Certainly there are many examples where populations do grow exponentially for a time, but both experience and common sense tell us that eventually the growth must taper off. As overcrowding develops, resources like food, water, and shelter become more and more scarce, diseases spread more easily, and as a consequence, it becomes more difficult for the population to continue growing. Models that take these growth-limiting effects into account are said to be density dependent.

Discrete logistic model

We begin by assuming that for any population there is a maximum number that a given environment can support. This maximum number is called the carrying capacity, and we follow convention by denoting this number by K. We should note that the carrying capacity depends both on the particular species and on the particular environment in which it is found. A small pond, for example, will have a smaller carrying capacity for goldfish than a large lake. Clearly it is not just the goldfish themselves that determine the carrying capacity. Similarly, a lake will have a larger carrying capacity for minnows than for catfish. Our task in this section is to model a population when its growth is restricted by the carrying capacity of its environment.

First we take note of the following features.

1. The growth rate of the population should decline as the population nears the carrying capacity.
2. The growth rate should be 0 if the population reaches the carrying capacity.

Now suppose the growth rate r is independent of the size of the population, i.e., fixed. Then the model should become

$$P(t) = P(t-1) + rP(t-1)$$

We shall now try to replace the fixed growth rate, r, by an expression that is consistent with properties 1 and 2 above. The simplest idea is to assume the growth rate varies along a straight line that starts with a maximum growth rate of r and decreases to a growth rate of 0 at the carrying capacity K. Here x–axis represents the population and y–axis the growth rate. Thus the straight line passes through the points $(0, r)$ and $(K, 0)$.

So the slope the straight line should be $m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{r}{K}$. Therefore the straight line along which the growth rate should vary is

$$y = -\frac{r}{K}x + r$$

or

$$y = r\left(1 - \frac{x}{K}\right)$$

Hence our desired growth rate becomes $r(t) = r\left(1 - \frac{P(t-1)}{K}\right)$. We refer this growth rate as the *intrinsic growth rate* of the population. Hence the discrete logistic growth model is given by

$$P(t) = P(t-1) + r\left(1 - \frac{P(t-1)}{K}\right)P(t-1) \tag{2.44}$$

Example 2.16.1. *Assume that in 2021 population of Baleen whales is* 75, 000, *the maximum growth rate r is* 5% *per year and the carrying capacity K* = 400, 000 *BWU. What would the discrete logistic growth model predict for the population of baleen whales in 2022 in the Antarctic fishery?*

Solution. Here the population of baleen whales in 2021 is $P(2021) = 75,000$. Also maximum growth rate $r = 0.05$ and the carrying capacity $K = 400,000$ BWU. Hence the population in 2022 will be $P(2022) = P(2021) + r\left(1 - \frac{P(2021)}{K}\right)P(2021) = 78187.5$ BWU, using equation (2.44).

Discrete logistic model with harvesting

Taking a cue from the previous model, we will now discuss the discrete logistic model with harvesting.

We examine logistic growth with harvesting in the context of a fishery model, and we consider two different harvesting strategies. The first is *constant take harvesting*. Here we assume that fishers have a goal (or may be a limit fixed by the authority) for the number of fish they can take each day, regardless of how long it takes them to do so. In this situation we have a constant number of fish that will be harvested each day. The second type of harvesting is *constant effort harvesting*. Here we have fishers who can only fish for, say, 8 h per day, and so the catch will vary depending on how abundant the fish are. In this situation we will have a constant percentage of available fish harvested each day rather than a constant number.

Let us consider the constant take situation first.

A. Constant take harvesting

Let h be the constant number of fish to be harvested in each time period. Then modifying equation (2.44), our model becomes

$$P(t) = P(t-1) + r\left(1 - \frac{P(t-1)}{K}\right)P(t-1) - h \tag{2.45}$$

Finding the equilibrium value

Let the equilibrium value be P^* of equation (2.45). Then from equation (2.45), we have

$$P^* = P^* + r\left(1 - \frac{P^*}{K}\right) - h,$$

i.e., $$r\left(1 - \frac{P^*}{K}\right)P^* - h = 0 \Rightarrow (P^*)^2 - KP^* + \frac{Kh}{r} = 0$$

Hence

$$P^* = \frac{K \pm \sqrt{K^2 - 4\frac{Kh}{r}}}{2} \tag{2.46}$$

Clearly the model's behaviour depends heavily on the discriminant $K^2 - 4\frac{Kh}{r}$. If the discriminant is positive then we get two distinct equilibrium values.

We get two distinct equilibrium values if the discriminant is positive; one unique equilibrium value if the discriminant equals 0 and no equilibrium values if the discriminant is negative (since the value of the equilibrium point would become imaginary then). Thus the value for h that makes the discriminant equal to h gives a harvesting number where the model's behavior changes dramatically. By setting the discriminant equal to 0 and solving for h, we see that this harvesting number is $h = \frac{rK}{4}$.

Let us now consider the growth model of the Baleen whales of Antarctic.

Baleen whales, also known as great whales, are whales that feed by filtering food through baleen plates in their upper jaw. Examples of baleen whales are the blue whale, fin whale, and sei whale. Due to overfishing, baleen whale populations in the Antarctic declined to dangerously low levels in the mid-1900s. In 1946, the International Whaling Commission (IWC) was formed to provide for the proper conservation of whale stocks while ensuring the orderly development of the whaling industry. The commission set limits on the numbers and size of whales which may be taken. Also they prescribed open and closed seasons and areas for whaling.

Prior to 1963, the IWC used the *blue whale unit* (BWU) as its unit in setting whale quotas. In these units we have 1 blue whale = 1 BWU, 1 fin whale = $\frac{1}{2}$ BWU, and 1 sei whale = $\frac{1}{6}$ BWU. Note that the carrying capacity is 400, 000 BWU means that the environment could support as many as 400, 000 blue whales, or 800, 000 fin whales, or 2, 400, 000 sei whales, or any combination of the three species that does not exceed the 400, 000 BWU threshold. Now let us consider the following example.

Example 2.16.2. *If the carrying capacity of an environment is* 300, 000 *BWU, then how* many fin whales the environment can support, assuming no other kind of whales are there?

Solution. Since 1 fin whale = $\frac{1}{2}$ BWU and carrying capacity of the environment is 300,000 BWU, the environment can support 300,000 × 2 = 600,000 fin whales.

Example 2.16.3. *Suppose the carrying capacity of an environment for baleen whale population is* 400, 000 *BWU under discrete logistic model. If the maximum growth rate of the population is* 0.05 *(or* 5%*) and exactly* 3000 *BWU baleen whales are harvested by the whaling industry each year, then find the equilibrium value(s) of the population (if any).*

Solution. Here the carrying capacity K = 400,00 BWU, the maximum growth rate r = 0.05 and the harvesting number h = 3000 BWU per year. Hence using equation (2.46), equilibrium value is $P^* = \dfrac{K \pm \sqrt{K^2 - 4\frac{Kh}{r}}}{2}$. Since the discriminant $K^2 - 4\frac{Kh}{r} = 6.4 \times 10^{10} > 0$, therefore the population has equilibrium values. The equilibrium values are 326, 491 BWU and 73, 509 BWU approximately.

Exercise 2.16.1. *Suppose the carrying capacity of an environment for baleen whale population is* 500, 000 *BWU under discrete logistic model. If the maximum growth rate of the population is* 5% *and exactly* 4000 *BWU baleen whales are harvested by the whaling industry each year, then find the equilibrium value(s) of the population (if any).*

Ans. 400, 000 BWU and 100, 000 BWU

B. Constant effort harvesting

Now we examine an alternate method of harvesting. Instead of setting a quota, we set a limit on the fishing effort expended. As an example of this kind of control, rather than allowing as many boats as necessary to catch a particular number of fish, we could restrict the number or length of time that boats can fish. If we only allow, say, 10 boats to fish for 2 weeks no matter the population, then the catch will not be constant. It will instead be based on how easy it is for those boats to find fish and hence how abundant the fish are.

Consequently, we associate constant effort fishing with a harvest level that corresponds to a proportion of the fish available.

We assume now that we have restricted fishing effort so that a certain percentage of the fish population is harvested in a given time step. We denote this percentage by *e* and we modify our logistic model to reflect this change:

$$P(t) = P(t-1) + r\left(1 - \frac{P(t-1)}{K}\right)P(t-1) - eP(t-1) \tag{2.47}$$

Finding the equilibrium value

To find the equilibrium values, we solve

$$P^* = P^* + r\left(1 - \frac{P^*}{K}\right)P^* - eP^*$$

for P^*.

This implies

$$0 = r\left(1 - \frac{P^*}{K}\right) - h,$$

i.e., $\left(r - r\frac{P^*}{K} - e\right)P^* = 0 \Rightarrow P^* = 0$ (which means extinction) or $\left(r - r\frac{P^*}{K} - e\right) = 0$.

Hence the equilibrium value (other than extinction) for the model described by equation (2.47) is as follows.

$$P^* = K\left(1 - \frac{e}{r}\right) \tag{2.48}$$

Example 2.16.4. *Suppose the carrying capacity of an environment for baleen whale population is* 400, 000 *BWU under discrete logistic model. If the maximum growth rate of the population is* 0.05 *(or* 5%*) and exactly* 1% *population of baleen whales are harvested by the whaling industry each year, then find the equilibrium value of the population (other than the extinction).*

Solution. Here the carrying capacity K = 400, 00 BWU, the maximum growth rate $r = 0.05$ and the harvesting rate $e = 0.01$ per year. Hence using equation (2.48), the non-extinction equilibrium value is $P^* = K\left(1 - \frac{e}{r}\right)$ = 320, 000 BWU.

2.17 More Worked out Examples

Example 2.17.1. *Solve the linear difference equation*

$$a_n = 3a_{n-1},\ a_1 = 2$$

Solution. Here the characteristic equation is $r-3 = 0$ which gives the characteristic root is $r = 3$. So the general solution is $a_n = c_1 3^n$, where c_1 is an arbitrary constant. Using the initial condition $a_1 = 2$, we have $2 = c_1 3$ implying $c_1 = \frac{2}{3}$. Hence $a_n = \frac{2}{3}3^n$ is the solution.

Example 2.17.2. *Solve the linear difference equation*

$$a_n = 5a_{n-1} - 6a_{n-2},\ a_0 = 1,\ a_1 = 0$$

Solution. Here the characteristic equation is $r^2 - 5r + 6 = 0$ which gives the characteristic root is $r = 3, 2$. So the general solution is of the form $a_n = c_1 2^n + c_2 3^n$, where c_1 and c_2 are arbitrary constants. Using the initial conditions $a_0 = 1$, $a_1 = 0$, we have $c_1 + c_2 = 1$ and $2c_1 + 3c_2 = 0$. Solving these two equations, we get $c_1 = 3$ and $c_2 = -2$. Hence the required solution is $a_n = 3.2^n - 2.3^n$.

Example 2.17.3. *Let* $a_1 = 2$ *and* $a_2 = 5$ *and* $a_n = 6a_{n-1} - 9a_{n-2}$ *for* $n \geq 3$. *Solve the difference equation.*

Solution. Here the characteristic equation is $r^2 - 6r + 9 = 0$ which has two identical real roots 3, 3. So the general solution is of the form $a_n = (c_1 + c_2 n)3^n$, where c_1, c_2 are arbitrary coefficients. Using the initial conditions, we have $3c_1 + 3c_2 = 2$ and $9c_1 + 18c_2$

= 5. So $c_1 = \frac{7}{9}$, $c_2 = -\frac{1}{9}$. Hence the solution is $a_n = \frac{7}{9}3^n - \frac{1}{9}n3^n$.

Example 2.17.4. *Solve the difference equation* $a_n = a_{n-1} - a_{n-2}$ *when* $a_1 = 1$ *and* $a_2 = 2$.

Solution. The characteristic equation is $r^2 - r + 1 = 0$ having imaginary roots $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. If z be any root of the above equation, then $|z| = 1$ and $amp(z) = \pm\frac{\pi}{3}$. The general solution is of the form $a_n = 1^n\left(c_1 \cos\frac{n\pi}{3} + c_2 \sin\frac{n\pi}{3}\right) = c_1 \cos\frac{n\pi}{3} + c_2 \sin\frac{n\pi}{3}$, where c_1, c_2 are arbitrary coefficients. Using the initial conditions, we have $\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2} = 1$ and $-\frac{c_1}{2} + \frac{c_2\sqrt{3}}{2} = 2$. Hence $c_1 = -1$ and $c_2 = \sqrt{3}$. Hence the solution is $a_n = -\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}$.

Example 2.17.5. *Suppose two lakes are connected by a canal flowing water through.* 20% *pollutant of the second lake goes to the first lake and* 23% *pollutant of the first lake goes to the second lake. If three tons of pollutant stays in the second lake after a considerably large span of time, find the amounts of pollutant going to stay in the first lake on the long run.*

Solution. Putting $p = 0.2$, $q = 0.23$ and $b = 3$ in equation (2.41), we have the amounts of pollutant going to stay in the first lake on the long run is $a = \frac{p}{q}b$ = 2.61 tons (approx).

2.18 Summary

At the outset, notion of difference equation has been introduced. First and second order linear difference equations have been thoroughly discussed. Later on, discrete modeling has been introduced. Several growth models have been discussed. Stress has been given on stability analysis of the models. Apart from these, various real life problems have been dealt with from a discrete modeling approach.

2.19 Exercises

Exercise 2.19.1. *Solve* $a_n = \frac{a_{n-1}}{2}$ *for* $n \geq 2$ and $a_1 = 4$.

Ans. $a_n = \frac{1}{2^{n-3}}$

Exercise 2.19.2. *If* $a_1 = 3$ *and* $a_2 = 7$, *then solve* $a_n = 2a_{n-1} + 3a_{n-2}$ *for* $n \geq 3$.

Ans. $a_n = \frac{5}{2}3^{n-1} + \frac{1}{2}(-1)^{n-1}$

Exercise 2.19.3. *If* $a_1 = 1$ *and* $a_2 = 2$, *then solve* $a_n = -a_{n-1} - a_{n-2}$ *for* $n \geq 2$.

Ans. $a_n = -\left(3\cos\frac{2n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{2n\pi}{3}\right)$

Exercise 2.19.4. *If* $a_1 = 0$ *and* $a_2 = 2$, *then solve* $a_n = 8a_{n-1} - 16a_{n-2}$ *for* $n \geq 2$.

Ans. $a_n = 2^{2n-3}(n-1)$

Exercise 2.19.5. *Find the non-negative equilibrium of a population given by*

$$x_{n+1} = \frac{2x_n^2}{x_n^2 + 2}$$

and check the stability.

Ans. The required equilibrium points are 0, $1 \pm i$. At 0, the equilibrium is stable and at $1 \pm i$, the equilibria are unstable.

Exercise 2.19.6. *Check the stability of the equilibria of the model given by*

$$x_{n+1} = x_n e^{3-x_n}$$

Ans. The required equilibrium points are 0 and 3. Both the equilibria are unstable.

Exercise 2.19.7. *Let* $X(n)$ *and* $Y(n)$ *are armament expenditures of Nations X and Y respectively in the arm race model. We assume* $\alpha_1 = 0.349$, $\alpha_2 = -0.13$, $\delta_1 = 0.432$, $\delta_2 = 0.195$, $g = 37.1$, $h = -52.9$. *Find the equilibrium point of the model.*

Ans. $\left(\frac{2767580}{12961}, \frac{1122760}{12961}\right)$

Exercise 2.19.8. *Write down the equation of the Harrod Domar Model.*

Exercise 2.19.9. *What is the warranted rate of growth in the Harrod Domar Model?*

Exercise 2.19.10. *Write the mathematical model of constant take harvesting explaining all the parameters.*

Exercise 2.19.11. *Write the mathematical model of constant effort harvesting explaining all the parameters.*

Exercise 2.19.12. *Find the equilibrium value of the model given by equation (2.19) if* 500 *deers are removed every year.*

Ans. Approximately 1923

Exercise 2.19.13. *Suppose the maximum number of a certain species of whales a given environment can support (carrying capacity) is 400,000 BWU. The intrinsic growth rate is 20% (i.e., r = 0.2). If only 15% of the population is permitted for harvesting in every year (i.e., e = 0.15) so that it will not become extinct, then find the population of the species after a sufficiently long time. (Hint. Use the constant effort harvesting model)*

Ans. 100, 000 BWU

Exercise 2.19.14. *Explain the lake pollution model with all its parameters.*

Exercise 2.19.15. *Suppose two lakes are connected by a canal flowing water through.* 20% *pollutant of the second lake goes to the first lake and* 23% *pollutant of the first lake goes to* the second lake. If three tons of pollutant stays in the first lake after a considerably large span of time, find the amounts of pollutant going to stay in the other lake on the long run.

Ans. 3.45 tons.

Exercise 2.19.16. *Roy is a 120 pound male who quickly consumes two* 12–*oz. beers. Assuming a standard twelve oz. beer contains about* 14 *g of alcohol, standard* *average value for body water percentage is* 58% *and the blood is* 80.6% *water, calculate his blood alcohol concentration.*

Ans. 0.71475 g per l blood.

Exercise 2.19.17. *What was the purpose of forming International Whaling Commission?*

Exercise 2.19.18. *What is the Blue Whale Unit or BWU?*

Exercise 2.19.19. *If the carrying capacity of an environment is* 500, 000 *BWU, then how many sei whales the environment can support, assuming no other kind of whales are there?*

Ans. 3, 000, 000 (Hint. 1 sei whale = $\frac{1}{6}$ BWU)

Exercise 2.19.20. *What are the key features of growth rate in a discrete logistic model?*

Exercise 2.19.21. *What is the intrinsic growth rate of a population in a discrete logistic model?*

Exercise 2.19.22. *Assume that in 2021 population of Baleen whales is* 50, 000 *BWU, the maximum growth rate r is* 5% *per year and the carrying capacity K* = 2, 500, 000 *BWU. What would the discrete logistic growth model predict for the population of baleen whales in 2023 in the Antarctic fishery?*

Ans. Approximately 55, 017 BWU. (Hint. First find the population in 2022 using equation (2.44). Then find the population in 2023 by same method. Population in 2022 is 52, 450 BWU.)

Exercise 2.19.23. *Assume that in 2021 population of baleen whales is* 50, 000 BWU, *the maximum growth rate r is* 5% *per year and the carrying capacity K* = 2, 500, 000 *BWU. What would the discrete logistic growth model predict for the population of baleen whales in 2023 in the Antarctic fishery if* 450 *BWU baleen whales are harvested each year by the whaling companies?*

Ans. Approximately 54, 096 BWU. (Hint. First find the population in 2022 using equation (2.44). Then find the population in 2023 by same method. Population in 2022 is 52, 000 BWU.)

Exercise 2.19.24. *Suppose the carrying capacity of an environment for baleen whale population is* 500, 000 *BWU under discrete logistic model. If the maximum growth rate of the population is* 5% *and exactly* 40, 000 *BWU baleen whales are harvested by the whaling industry each year, then find the equilibrium value(s) of the population (if any).*

Ans. There exists no equilibrium value. (Hint. The discriminant is negative)

Unit 3 ❑ Continuous models

Structure

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3.0 Objectives

The object of this chapter is to develop and analyse various continuous models. Here we discuss the followings.

- Notion of continuous models;
- a variety of continuous models;
- steady state solutions or equilibrium points;
- linearization;
- local stability analysis and classification of equilibrium points.

3.1 Introduction to Continuous Models

This chapter introduces the topic of ordinary differential equation models, their formulation, analysis, and interpretation. A main emphasis at this stage is on how appropriate assumptions simplify the problem, how important variables are identified, and how differential equations are tailored for describing the essential features of a continuous process.

Because one of the most challenging parts of modeling is writing the equations, we dwell on this aspect purposely. The equations are written in stages, with appropriate assumptions introduced as they are needed. We begin with a rather simple ordinary differential equation as a model. Gradually, more realistic aspects of the situation are considered.

3.2 Carbon dating

Exponential decay and radioactivity

The process of dating aspects of our environment is essential to the understanding of our history. From the formation of the Earth through the evolution of life and the development of mankind, historians, geologists, archaeologists, palaeontologists and many others use dating procedures to establish theories within their disciplines.

While certain elements are stable, others (or their isotopes) are not, and emit α – particles, β – particles or photons while decaying into isotopes of other elements. Such elements are called *radioactive*.

We make the following assumptions and then, based on these, develop a model to describe the process.

- The amount of an element present is large enough so that we are justified in ignoring random fluctuations.
- The process is continuous in time.
- The rate of decay for an element is fixed.
- There is no increase in mass of the body of material.

Now the rate of change of radioactive material $N = N(t)$ at time t is negative of the rate amount of radioactive material decayed. Hence we have

$$\frac{dN}{dt} = -kN \tag{3.1}$$

where k is a positive constant of proportionality depending on the elements chosen.

Given a sample of a radioactive element at some initial time, say n_0 nuclei at t_0, we may want to predict the mass of nuclei at some later time t. We require the value of k

for the calculations; it is usually found through experimentation. Then, with known k and an initial condition $N(t_0) = n_0$, we have an initial value problem (IVP)

$$\frac{dN}{dt} = -kN \text{, where } N(t_0) = n_0 \tag{3.2}$$

Example 3.2.1. *Solve the initial value problem (IVP) in equation (3.2) with initial condition* $N(t_0) = n_0$.

Solution. Since the differential equation is separable,

$$\int \frac{1}{N}\frac{dN}{dt}dt = \int -kdt$$

$$\Rightarrow \int \frac{1}{N}dN = \int -kdt$$

$$\Rightarrow \ln N = -kt + C$$

since N is a positive quantity. Here C is an arbitrary constant. Taking exponentials of both sides we have

$$N(t) = Ae^{-kt}, \text{ where } A = e^C$$

Note that $N \geq 0$.

Using the initial condition $N(t_0) = n_0$, we get

$$n_0 = Ae^{-kt_0} \text{ and } A = n_0e^{kt_0} .$$

Thus the solution for IVP is

$$N(t) = n_0e^{-k(t-t_0)} \tag{3.3}$$

Example 3.2.2. *Solve the initial value problem (IVP) in equation (3.2) on the interval* $[0, t]$.

Solution. Since the differential equation is separable,

$$\int_0^t \frac{1}{N}\frac{dN}{dt}dt = \int_0^t -kdt$$

$$\Rightarrow \int_{n_0}^{n} \frac{1}{N}\, dN = \int_0^t -k\,dt$$

$$\Rightarrow \ln N - \ln n_0 = -kt + 0$$

$$\Rightarrow \ln \frac{N}{n_0} = -kt$$

since N, n_0 are positive quantities.

Taking exponentials of both sides we have

$$N(t) = n_0 e^{-kt}$$

Remark 3.2.1. *The* half-life τ *of the radioactive nuclei can be used to determine k, where* τ *is the time required for half of the nuclei to decay. The half-life* τ *is more commonly known than the value of the rate constant k for radioactive elements.*

Example 3.2.3. *If the half-life is* τ*, then find k in terms of* τ*.*

Solution. Setting $N(t+\tau) = \frac{N(t)}{2}$, we have $\frac{N(t+\tau)}{N(t)} = \frac{1}{2}$. This gives $e^{-k\tau} = \frac{1}{2}$, using equation (3.3).

Taking logarithms of both sides, $\ln \frac{1}{2} = -k\tau$. Hence

$$\boxed{k = \frac{\ln 2}{\tau}} \tag{3.4}$$

Note that both τ and k are independent of n_0 and t_0.

Radiocarbon dating

We can apply the above theory to the problem of dating paintings by considering the decay process of certain radioactive elements in each.

All living organisms absorb carbon from carbon dioxide (CO_2) in the air, and thus all contain some radioactive carbon nuclei. This follows since CO_2 is composed of a radioactive form of carbon ^{14}C, as well as the common ^{12}C. (^{14}C is produced by the collisions of cosmic rays (neutrons) with nitrogen in the atmosphere, and the ^{14}C nuclei decay back to nitrogen atoms by emitting β particles.) Nobel Prize winner Willard Libby, during the late 1940s, established how the known decay rate and half-life of ^{14}C, together with the carbon remaining in fragments of bones or other dead tissue, could be used to determine the year of death. Because of the particular **half-life of carbon**, internationally agreed upon as 5, 568 ± 30 **years** for ^{14}C, this process is most effective with material between 200 and 70,000 years old.

Carbon dating depends on the fact that for any living organism the ratio of the amount of ^{14}C to the total amount of carbon in the cells is the same as that ratio in the surroundings. Assuming the ratio in air is constant, then so is the ratio in living organisms. However, when an organism dies, CO_2 from the air is no longer absorbed although ^{14}C within the organism continues to undergo radioactive decay.

In the Cave of Lascaux in France there are some ancient wall paintings, believed to be prehistoric. Using a Geiger counter, the current decay rate of ^{14}C in charcoal fragments collected from the cave was measured as approximately **1.69 disintegrations per minute per gram of carbon**. In comparison, for living tissue in 1950 the measurement was **13.5 disintegrations per minute per gram** of carbon.

Example 3.2.4. *How long ago was the radioactive carbon formed and the Lascaux Cave paintings were painted, assuming the half life of ^{14}C to be approximately 5,568 years?*

Solution. Let $N(t)$ be the amount of ^{14}C per gram in the charcoal at time t. We apply the model of exponential decay given by $\frac{dN}{dt} = -kN$. We have $\tau \approx$ 5, 568 years (the half-life of ^{14}C). Using equation (3.4), we have $k = \frac{\ln 2}{\tau} \approx$ 0.0001245. Let $t = t_0 =$ 0 be the current time. Let T be the time that the charcoal was formed, and thus $T < 0$. For $t > T$, ^{14}C decays at the rate

$$\frac{dN}{dt} = -kN \text{ with } N(t_0) = n_0$$

and

$$N(T) = n_0 e^{-kT}$$

or

$$T = -\frac{1}{k}\ln\left(\frac{N(T)}{n_0}\right)$$

But we do not know $N(T)$ or n_0. However,

$$\frac{N'(T)}{N'(0)} = \frac{-kN(T)}{-kN(0)} = \frac{N(T)}{n_0}$$

and we do have $N'(T) = 1.69$ and $N'(0) = 13.5$, as discussed above. Thus $T = -\frac{1}{k}\ln\left(\frac{N(T)}{n_0}\right) \approx$ 16, 690 years.

Exercise 3.2.1. *An artefact was discovered in 1950 from a pre- historic cave. Assume the half life of* ^{14}C *to be approximately 5,568 years. The decay rate of* ^{14}C *in charcoal fragments collected from the cave was measured as approximately 1.85 disintegrations per minute per gram of carbon. In comparison, for living tissue in 1950 the measurement was 13.5 disintegrations per minute per gram of carbon. How long ago was the artefact made?*

Ans. Approximately 15,964 years.

Exercise 3.2.2. *Establish the model of exponential growth of radioactive elements with initial assumptions.*

Exercise 3.2.3. *What is half life of a radioactive element?*

Exercise 3.2.4. *Find the rate of decay per nucleus in unit time in terms of the half life of a radioactive element.*

Hint. See Example 3.2.3.

3.3 Introduction to Compartmental Models

One of the most naturally occurring framework in mathematical modeling is to think of the domain of a process as a compartment where incoming and/ or outgoing of the mass or population take place over time. A compartment may be a polluted lake with provisions of inflow of water carrying mass of pollutants from industries into it and outflow of water carrying some pollutant mass with it **OR** it may be an environment where a population of bacteria may be cultured where the incoming is the birth and outgoing is the death of micro-organisms happened over time. This model is crucial in understanding the decay (outgoing) of some radioactive substance over time (no incoming or input at all!!!) **OR** quantity of drug present in our bloodstream (compartment) where the drug is absorbed from the G. I. tract and excreted through the function of kidneys **OR** any other similar cases.

The very basic idea of the compartmental modeling lies in the following sketch.

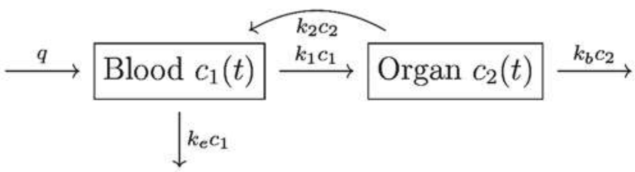
So all we need to do is to work out a **balance law** to compute the rate of change of substance/ population as the difference between the incoming rate and the outgoing rate of the same over time.

net rate of change of substance/ population = incoming rate – outgoing rate

3.4 Drug distribution in the body

This model is a two-compartment model. Assume a drug, which has been taken orally, is present in the intestine during a certain time interval. The drug is absorbed with the constant flow rate q (millimole per litre per second) into the first compartment, the blood plasma. In the blood plasma, the concentration of the drug is $c_1(t)$ (millimole per litre) The second compartment is the organ where the drug is active. Between the first and second compartments, there is an drug exchange with rate $k_1c_1(t)$ (millimole per litre per second) leading to the drug concentration $c_2(t)$ (millimole per litre) in the second compartment. In the organ, the drug is consumed with the rate $k_bc_2(t)$ (millimole per litre per second) and the surplus is sent back to the blood with the rate $k_2c_2(t)$ (millimole per litre per second).

From the blood, finally, there is an elimination of the drug through the kidneys with the rate $k_ec_1(t)$ (millimole per litre per second). Note that k_1, k_2, k_e, k_b are rate constants each with *per second* as its unit.



The above diagram illustrates the blood–organ compartment model of drug distribution.

The model is described by the following system.

$$\frac{dc_1}{dt} = -k_1c_1 + k_2c_2 - k_ec_1 + q$$

$$\frac{dc_2}{dt} = k_1c_1 - k_2c_2 - k_bc_2 \qquad (3.5)$$

where $c_1(0) = 0 = c_2(0)$.

Example 3.4.1. *Suppose an orally taken drug is absorbed with a constant flow* 0.2 *millimole per litre per second into the blood. The drug then moves to the target organ with* 0.15 *per second as rate constant. The target organ consumes the drug with* 0.03 *per second as rate constant and the surplus is sent back to the blood with* 0.10 *per second as rate constant. Finally the drug is eliminated from the blood through the kidneys with* 0.01 *per second as rate constant. Assuming the drug was present neither in the blood nor in that organ initially, write down the mathematical model describing the concentration of the drug in blood as well as in the target organ.*

Solution. Let $x(t)$ and $y(t)$ be the concentrations of the drug in blood and the target organ respectively. Putting $q = 0.2$, $k_1 = 0.15$, $k_2 = 0.10$, $k_b = 0.03$ and $k_e = 0.01$. Then using the system of differential equations (3.5), our desired model becomes

$$\frac{dx}{dt} = -0.16x + 0.10y + 0.2$$

$$\frac{dy}{dt} = 0.15x - 0.13y$$

3.5 Growth and Decay of Current in an L-R Circuit

Here we consider a circuit containing resistance R and inductance L connected in series combination through a battery of constant emf E through a two way switch S. The inductor has been used in this single-loop circuit to stop the current from reaching its maximum value instantaneously. This is described in the following figure 3.1.

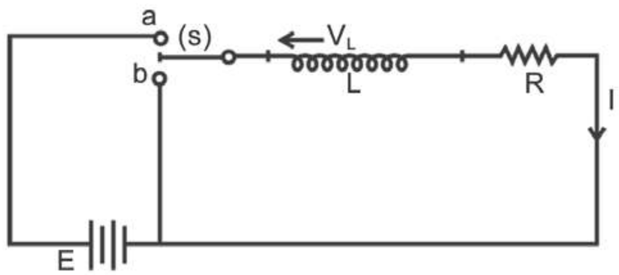


Figure 3.1: LR circuit with two way switch S

Also we make the following assumptions.

- To distinguish the effects of R and L,we consider the inductor in the circuit as resistance less and resistance R as non-inductive
- Current in the circuit increases when the key is pressed and decreases when it is thrown to *b*

A. Growth of current in an L-R Circuit

Suppose in the beginning, we close the switch in the up position as shown in below in the figure 3.2. Since the switch is closed, the battery E, inductance L and resistance R are now connected in series. Because of self induced emf, current will not immediately reach its steady value but grows at a rate depending on inductance and resistance of the circuit.

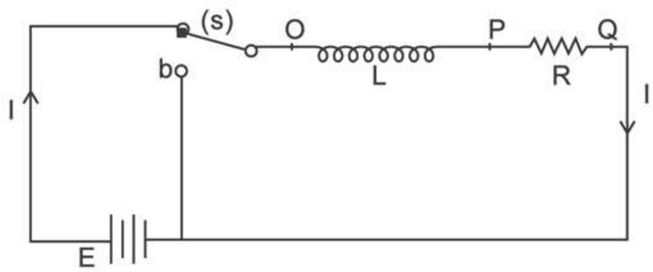


Figure 3.2: Battery included in the LR circuit

Let at any time t, I be the current in the circuit increasing from 0 to a maximum value at a rate $\frac{dI}{dt}$.

Now the potential difference across the inductor is $V_{OP} = L\frac{dI}{dt}$ and across resistor is $V_{PQ} = IR$. Since $V = V_{OP} + V_{PQ}$, therefore

$$V = L\frac{dI}{dt} + IR \tag{3.6}$$

Thus rate of increase of current is

$$\frac{dI}{dt} = \frac{V - IR}{L} \tag{3.7}$$

Clearly in the beginning at t=0 when circuit was closed, current began to grow at a rate $\left(\frac{dI}{dt}\right)_{t=0} = \frac{V}{L}$. Hence greater would be the inductance of the inductor, more slowly the current starts to increase.

When the current reaches its steady state value I, the rate of increase of current becomes zero. Then from equation (3.7) we have, $I = \frac{V}{R}$. Therefore, final steady state current in the circuit does not depend on self inductance. Rather it is same as it would be if only resistance is connected to the source.

From equation (3.6), we have $\frac{dI}{\left(\frac{V}{R}\right) - I} = \frac{R}{L}dt$.

Now we assume $\frac{V}{R} = I_{\max}$, the maximum current in the circuit. So we have

$$\frac{dI}{I_{\max} - I} = \frac{R}{L}dt$$

Integrating on both sides we have,

$$-\ln\left(I_{\max} - I\right) = \frac{R}{L}t + C \tag{3.8}$$

where C is a constant and is evaluated by the value for current at $t = 0$ which is $I = 0$. So, $C = -\ln I_{max}$. Putting this in equation (3.8), we have

$$\ln \frac{I_{\max} - I}{I_{\max}} = -\frac{R}{L}t$$

$$\Rightarrow \frac{I_{\max} - I}{I_{\max}} = e^{-\frac{R}{L}t}$$

Hence we have

$$I = I_{\max}\left(1 - e^{-\frac{R}{L}t}\right) \qquad (3.9)$$

This equation shows the exponential increase of current in the circuit with the passage of time as depicted in figure 3.3.

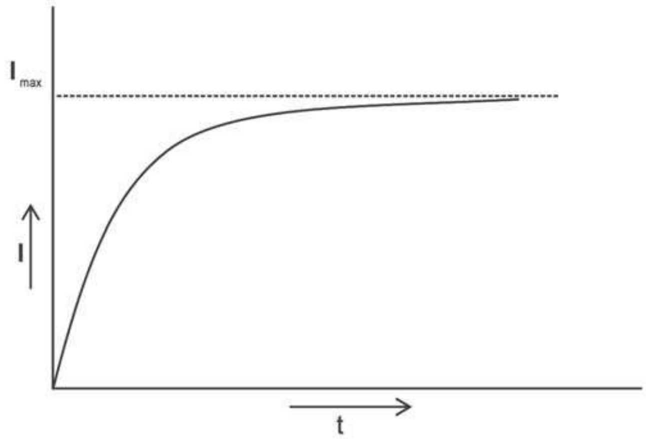


Figure 3.3: Growth of current in LR circuit

B. Decay of current in an L-R Circuit

When the switch S is thrown down to b as shown below in the figure, the L-R circuit is again closed and battery is cut off as depicted in figure 3.5.

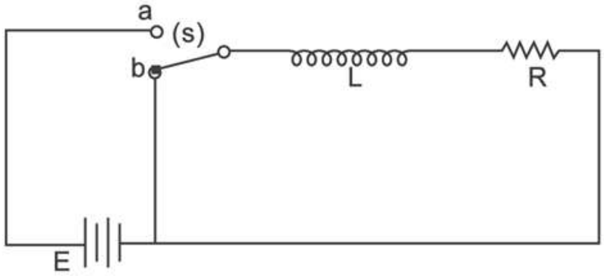


Figure 3.4: Battery is now cut off from the circuit

This time $V = 0$. Therefore from equation (3.6), we can write the equation for decay as

$$L\frac{dI}{dt} + RI = 0$$

$$\Rightarrow \frac{dI}{I} = -\frac{R}{L}dt$$

$$\Rightarrow \int \frac{dI}{I} = -\frac{R}{L}\int dt$$

Hence

$$\ln I = -\frac{R}{L}t + C_1 \tag{3.10}$$

At time $t = 0$, current $I = I_{max}$. So $C_1 = \ln I_{max}$.

Therefore from equation (3.10), we have

$$\ln I = -\frac{R}{L}t + \ln I_{\max} .$$

i.e.,

$$I = I_{\max} e^{-\frac{R}{L}t} \tag{3.11}$$

Hence current decreases exponentially with time in the circuit in accordance with the above equation after the battery is cutoff from the circuit as depicted in the figure

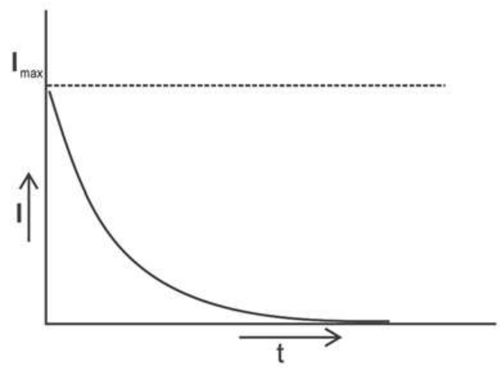


Figure 3.5: Current decreasing exponentially with time

Example 3.5.1. *A* 5 *mH inductor, a* 15 Ω *resistor are connected across a* 12 *V battery with negligible internal resistance in series. What is the maximum current in the circuit?*

Solution. $I_{\max} = \frac{V}{R} = \frac{12}{15} = 0.8A.$

Exercise 3.5.1. *A* 25 *mH inductor, a* 8 Ω *resistor are connected across a* 6 *V battery with negligible internal resistance in series. What is the maximum current in the circuit?*

Ans. 0.75 A

Exercise 3.5.2. *Establish the expression for the current in terms of inductance and resistance in an LR circuit including a resistor with resistance R, an inductance L, and an emf E in series connection.*

Hint. See Section 3.5

Exercise 3.5.3. *Draw the graph of growth of current in LR circuit.*

Hint. See Section 3.5

Exercise 3.5.4. *Suppose we have a circuit including a resistor with resistance R, an inductance L, and an emf E in series connection. Establish the expression for the current in terms of inductance and resistance after the battery is cut off from the circuit. Also draw the graph of decay of current in the LR circuit.*

Hint. See Section 3.5

3.6 Vertical Oscillation

Vertical spring-mass system

We take an ordinary spring that resists compression as well extension and suspend it vertically from a fixed support, as shown in Figure 3.6. At the lower end of the spring we attach a body of mass *m*. We assume *m* to be so large that we can neglect the mass of the spring. If we pull the body down a certain distance and then release it, it starts moving. We assume that it moves strictly vertically.

Now this motion is determined by Newton's second law: Mass × Accelration = my'' = Force, where $y'' = \frac{d^2y}{dt^2}$ and "Force" is the resultant of all the forces acting on the body. We choose the *downward direction as the positive direction*, thus regarding downward Forces as positive and upward forces as negative.

Consider Figure 3.6. The spring is first unstretched. We now attach the body. This stretches the spring by an amount s_0 shown in the figure. It causes an upward force F_0 in the spring. Experiments show that F_0 is proportional to the stretch s_0 say, $F_0 = -ks_0$, by Hooke's law.

$k(> 0)$ is called the *spring constant*. The minus sign indicates that F_0 points upward, in our negative direction. Clearly stiff springs have large k. The extension s_0 is such that F_0 in the spring balances the weight $W = mg$ of the body. Hence $F_0 + W = -ks_0 + mg = 0$. These forces will not affect the motion. Spring and body are again at rest. This is called the *static equilibrium* of the system. We measure the displacement $y(t)$ of the body from this *equilibrium point* as the origin $y = 0$, downward positive and upward negative.

From the position $y = 0$ we pull the body downward. This further stretches the spring by some amount $y > 0$ (the distance we pull it down). By Hooke's law this causes an (additional) upward force F_1 in the spring, i.e.,

$$F_1 = -ky \tag{3.12}$$

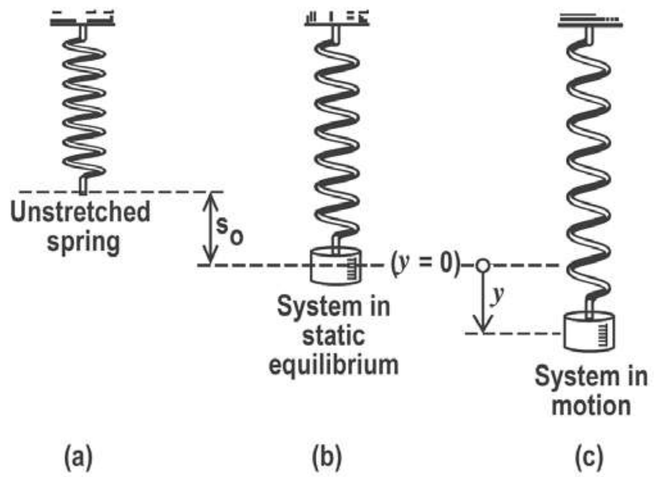


Figure 3.6: A vertical spring-mass system

F_1 is a restoring force. It has the tendency to restore the system, that is, to pull the body back to $y = 0$.

Now neglecting the damping effect, F_1 is the only force causing the motion. Hence from equation (3.12), we have

$$my'' + ky = 0 \tag{3.13}$$

It can be easily checked that the general solution will be

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{3.14}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. The corresponding motion is called a *vertical (harmonic) oscillation*.

The period of the oscillation is given by $\frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$ and the *frequency* is $\frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ cycles per second. Another name for cycles/sec is *hertz (Hz)*.

Example 3.6.1. *If an iron ball of weight W* = 98 *Newtons stretches a spring* 1.09 *m, how many cycles per minute will this mass-spring system execute?*

Solution. We know weight of a 1 kg mass is 9.8 Newtons. Therefore, the mass m of the iron ball is 10 kg. Now initially the stretch s_0 is 1.09 meters. Therefore the spring constant $k = \frac{W}{s_0} \approx$ 89.91 Newton/meter. Then $\omega_0 = \sqrt{\frac{k}{m}} \approx$ 3.00. So the frequency = no. of cycles per second = $\frac{\omega_0}{2\pi} \approx$ 0.48 and hence cycles per minute = frequency × 60 = 28.8.

Exercise 3.6.1. *An iron ball of weight W* = 196 *Newtons stretches a spring 0.25 m. If it is further stretched downwards, find the frequency of the resulting oscillation.*

Ans. 0.996 approx.

3.7 Horizontal Oscillation

Let us consider a cart of mass M attached to a nearby wall by means of a spring as described in figure 3.7. Here $x = x(t)$ is the position of the cart at time t. Using Hooks law as in the previous section, we have $F_s = -kx$, k being the spring constant.

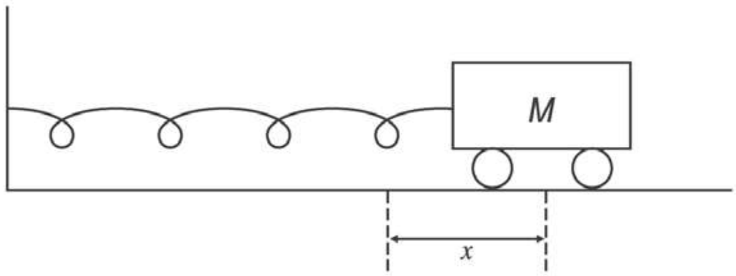


Figure 3.7: Horizontal oscillation

By Newton's second law of motion, which says that the mass of the cart times its acceleration equals the total force acting on it, we have

$$M\frac{d^2x}{dt^2} = F_s \tag{3.15}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0 \tag{3.16}$$

It will be convenient to write this equation of motion in the form

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{3.17}$$

where $\omega_0 = \sqrt{\frac{k}{M}}$. The general solution can be written down as

$$x = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t \tag{3.18}$$

The cart is pulled aside to the position $x = x_0$ and released without any initial velocity at time $t = 0$ so that our initial conditions are $x = x_0$ and $v = \frac{dx}{dt} = 0$ when $t = 0$.

Clearly $c_1 = 0$ and $c_2 = x_0$. So equation (3.18) becomes

$$x = x_0 \cos \omega_0 t \tag{3.19}$$

The amplitude of this simple harmonic vibration is x_0. Since the *period T* is the time required for one complete cycle, we have $\omega_0 T = 2\pi$ and hence

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M}{k}} \tag{3.20}$$

The *frequency f* is the number of cycles per unit time. Therefore *fT* = 1 and hence

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M}} \tag{3.21}$$

Example 3.7.1. *Assume that a cart of mass* 100 *grams is attached to a nearby wall by means of a spring, with spring constant* 9.8 *N/m, and is placed on a smooth horizontal table. You pull the mass* 6 *cm away from its equilibrium position and let it go at t* = 0. *Find an equation for the position of the mass as a function of time t*.

Solution. Lets first find the period of the oscillations, then we can obtain an equation for the motion. The period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.1}{9.8}} \approx 0.635$ sec. At $t = 0$ the mass is at its maximum distance from the equilibrium position. Thus $x(t) = 0.6 \cos \frac{2\pi}{T} t = \approx$ 0.6 cos 9.9*t*.

Exercise 3.7.1. *Assume that a cart of mass* 100 *grams, attached to a nearby wall by means of a spring of spring constant* 9.8 *N/m, is oscillating on a smooth horizontal table. Find the frequency.*

Ans. 1.57 per second

3.8 Damped Oscillation

Now we consider the additional effect of a *damping force* F_d due to the viscosity of the medium through which the cart moves (air, water, oil, etc.) horizontally. We make the specific assumption that this force opposes the motion and has magnitude proportional to the velocity, that is, that $F_d = -c\frac{dx}{dt}$, where c is a positive constant measuring the resistance of the medium. We call c the *damping coeficient*. Equation (3.15) now becomes

$$M\frac{d^2x}{dt^2} = F_s + F_d \tag{3.22}$$

i.e.

$$\frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = 0 \tag{3.23}$$

For the sake of convenience, we write this in the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \tag{3.24}$$

where $b = \frac{c}{2M}$ and $\omega_0 = \sqrt{\frac{k}{M}}$.

The auxiliary equation is

$$m^2 + 2bm + \omega_0^2 = 0 \tag{3.25}$$

and its roots m_1, m_2 are given by

$$m_1, m_2 = -b \pm \sqrt{b^2 - \omega_0^2} \tag{3.26}$$

The nature of the roots of the equation (3.25) determines which would prevail over the other in between the frictional force due to the viscosity and the stiffness of the spring.

Case I: $b^2 - \omega_0^2 > 0$, **i.e.,** $b > \omega_0$ **i.e.,** $c > 2\sqrt{Mk}$ **(Overdamped)**

In loose terms, this amounts to assuming that the frictional force due to the viscosity is large compared to the stiffness of the spring. In other words, The damping force is much stronger than the restoring force due to stiffness of the spring. We call this oscillation *Overdamped*.

It follows that m_1 and m_2 are distinct negative numbers, and the general solution of equation (3.24) is

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3.27}$$

Using the initial conditions $x = x_0$ and $v = \frac{dx}{dt} = 0$ when $t = 0$, equation (3.27) becomes

$$x = \frac{x_0}{m_1 - m_2}\left(m_1 e^{m_2 t} - m_2 e^{m_1 t}\right) \tag{3.28}$$

or

$$x = \frac{x_0}{2\sqrt{b^2 - \omega_0^2}} e^{-bt}\left(m_1 e^{\sqrt{b^2-\omega_0^2}\,.t} - m_2 e^{-\sqrt{b^2-\omega_0^2}\,.t}\right) \tag{3.29}$$

Case II: $b^2 - \omega_0^2 = 0$, **i.e.,** $b = \omega_0$ **i.e.,** $c = 2\sqrt{Mk}$ **(Critically damped)**

In this case, the restoring force and damping force are comparable in effect. Here we have $m_l = m_2 = -b = -\omega_0$ and the general solution of equation (3.24) is

$$x = c_1 e^{-\omega_o t} + c_2 t e^{-\omega_0 t} \tag{3.30}$$

With the initial conditions $x = x_0$ and $v = \frac{dx}{dt} = 0$ when $t = 0$, equation (3.30) becomes

$$x = x_0 e^{-\omega_0 t}(1 + \omega_0 t) \tag{3.31}$$

We call this oscillation *Critically damped*.

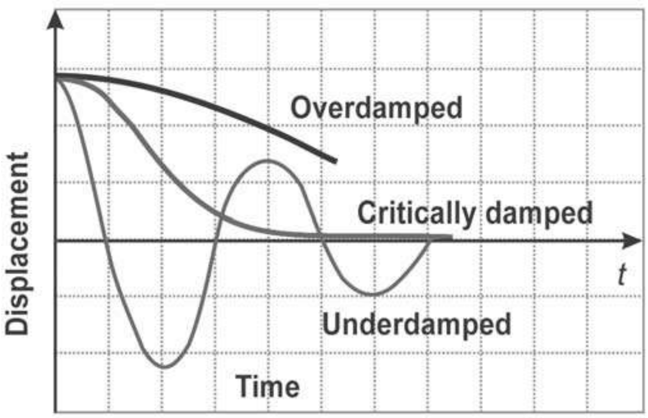


Figure 3.8: Types of displacements in damped oscillation

Case III: $b^2 - \omega_0^2 < 0$, **i.e.,** $b < \omega_0$ **i.e.,** $c < 2\sqrt{Mk}$ **(Underdamped)**

In this case, the restoring force is large compared to the damping force. Here m_1 and m_2 are conjugate complex numbers $-b \pm i\alpha$, where $\alpha = \sqrt{\omega_0^2 - b^2}$. Then the general solution of equation (3.24) is

$$x = e^{-bt}(c_1 \cos \alpha t + c_2 \sin \alpha t) \tag{3.32}$$

With the initial conditions $x = x_0$ and $v = \dfrac{dx}{dt} = 0$ when $t = 0$, equation (3.32) becomes

$$x = \frac{x_0}{\alpha} e^{-bt}(\alpha \cos \alpha t + b \sin \alpha t) \tag{3.33}$$

Putting $\theta = \tan^{-1}\dfrac{b}{\alpha}$, equation (3.33) becomes

$$x = \frac{x_0\sqrt{\alpha^2 + b^2}}{\alpha} e^{-bt} \cos(\alpha t - \theta) \tag{3.34}$$

We call this oscillation *Underdamped*.

Figure 3.8 illustrates the above three phenomena.

Example 3.8.1. *Let a mass of* 1 *kg is attached to a wall by means of a spring with spring constant* 9.8 *N/m. The mass is oscillating horizontally on a rough surface with damping coefficient* 2 *kg/s. Find the nature of the oscillation.*

Solution. Here the mass M = 1 kg, spring constant k = 9.8 N/m and damping coefficient c = 2 kg/s. As $c < 2\sqrt{Mk}$ = 6.26, so the oscillation is underdamped.

Exercise 3.8.1. *Let a mass of* 1 *kg is attached to a wall by means of a spring with spring constant* 9.8 *N/m. The mass is oscillating horizontally on a rough surface. Find the damping coefficient of the surface so that the oscillation is critically damped.*

Ans. 6.26 kg/s.

3.9 Damped forced oscillation

The vibrations discussed above are known as free vibrations because all the forces acting on the system are internal to the system itself. We now extend our analysis to cover the case in which an impressed external force $F_e = f(t)$ acts on the cart. Such a force might arise in many ways: for example, from vibrations of the wall to which the spring is attached, or from the effect on the cart of an external magnetic field (if the cart is made of iron).

Therefore, in place of equation (3.22), we now have

$$M \frac{d^2x}{dt^2} = F_s + F_d + F_e \tag{3.35}$$

Thus we have

$$M \frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = f(t) \tag{3.36}$$

The most important case is that in which the impressed force is periodic and has the form $f(t) = F_0 \cos \omega t$ so that equation (3.36) becomes

$$M \frac{d^2x}{dt^2} + \frac{c}{M}\frac{dx}{dt} + \frac{k}{M}x = f(t) = F_0 \cos \omega t \tag{3.37}$$

We have already solved the corresponding homogeneous equation (3.23), so in seeking the general solution of equation (3.37) all that remains is to find a particular solution.

This is most readily accomplished by the method of undetermined coefficients. Accordingly, we take $x = A \sin \omega t + B \cos \omega t$ as a trial solution. On substituting this into equation (3.37), we obtain the following pair of equations for A and B:

$$\omega c A + \left(k - \omega^2 M\right) B = F_0$$

$$\left(k - \omega^2 M\right) A - \omega c B = 0$$

The solution of this system is

$$A = \frac{\omega c F_0}{\left(k-\omega^2 M\right)^2 + \omega^2 c^2}$$

$$B = \frac{\left(k-\omega^2 M\right) F_0}{\left(k-\omega^2 M\right)^2 + \omega^2 c^2}$$

Our desired particular solution is therefore

$$x = \frac{F_0}{\left(k-\omega^2 M\right)^2 + \omega^2 c^2}\left[\omega c \sin \omega t + \left(k-\omega^2 M\right)\cos \omega t\right] \qquad (3.38)$$

By introducing $\phi = \tan^{-1}\left(\frac{\omega c}{k-\omega^2 M}\right)$, we can write the solution in equation (3.38) in a more useful form

$$x = \frac{F_0}{\sqrt{\left(k-\omega^2 M\right)^2 + \omega^2 c^2}} \cos\left(\omega t - \phi\right) \qquad (3.39)$$

This is our desired particular solution of equation (3.37).

Exercise 3.9.1. *Let a mass M is attached to a wall by a spring with spring constant k.* It is performing a damped forced oscillation (horizontally) through a medium of damping *coefficient c and the external force acting on the mass is* F_0 cos *ωt.*

(i) *Write down and explain the equation of motion of the damped forced oscillation.*

(ii) *Find out the particular solution.*

3.10 Combat Model

Consider now another type of interacting population model which revolves around a destructive competition or battle between two opposing groups or populations. For example, two hostile insect groups or cricket teams or human armies may engage in such interaction. The model we develop here eventually yields a system of two coupled, linear differential equations.

Background

Battles between armies has been a very common natural part of the history of mankind. Ancient battles were fought hand-to-hand and with weapons made of stone, copper, bronze or lately iron. With the invention of gun and artillery, aimed firepower (may be directly with rifles at visible enemy or randomly aimed with artillery at enemy territory) has become an indispensable feature of modern warfare. Although many factors can affect the outcome of a battle, experience has shown that numerical superiority and superior military training are critical. Our model was first developed in the 1920s by F. W. Lanchester who was also well known for his contributions to the theory of flight. Our aim is to develop a simple model that predicts the number of soldiers in each army at any given time, provided we know the initial number of soldiers in each army. (As with epidemics, we consider the number, rather than the density, of individuals.)

Model assumptions

First we make some basic assumptions.

- We assume the number of soldiers to be sufficiently large so that we can neglect random differences between them.
- We also assume that there are no reinforcements and no operational losses (i.e., due to desertion or disease).

In a real battle there will be a mixture of shots: those fired directly at an enemy soldier and those fired into an area known to be occupied by an enemy, but where the enemy cannot be seen. Some battles may be dominated by one or the other firing method. We consider these two idealisations of shots fired as aimed fire and random fire. For the model we assume only aimed fire for both armies.

In the aimed fire idealisation, we assume all targets are visible to those firing at them. If the blue army uses aimed fire on the red army, then each time a blue soldier fires, he/she takes aim at an individual red soldier. The rate of loss of soldiers of the red army depends only on the number of blue soldiers firing at them and not on the number of red soldiers. We see later that this assumption is equivalent to assuming a constant probability of success (on average) for each bullet fired.

For random fire, a soldier firing a gun cannot see his/her target, but fires randomly into an area where enemy soldiers are known to be. The more enemy soldiers in that given area, the greater the rate of wounding. For random fire we thus assume that the rate of enemy soldiers wounded is proportional to both the number firing and the number being fired at.

In summary we make the following further assumptions:

- For aimed fire, the rate of soldiers neutralized (i.e., rendered incapable of fighting by getting wounded or killed) is proportional to the number of enemy soldiers only.
- For random fire, the rate at which soldiers are neutralized is proportional to both numbers of soldiers.

Formulating the differential equations

Let $R(t)$ denote the number of soldiers of the red army and $B(t)$ the number of soldiers of the blue army at any time t. We assume aimed fire for both armies.

We consider two constants a_1 and a_2 measuring the effectiveness of the blue army and red army, respectively, and are called *attrition coefficients* by blue and red armies respectively. So the blue army neutralizes the enemy (i.e., the red army) at per capita rate a_1 and the red army neutralizes the blue army at per capita rate a_2.

We thus assume that attrition rates are dependent only on the firing rates and are a measure of the success of each firing. Thus our model becomes

$$\frac{dR}{dt} = -a_1 B$$
$$\frac{dB}{dt} = -a_2 R \tag{3.40}$$

Example 3.10.1. *Suppose a battle is waging between two countries one having red and another having blue army. Let initially the red and blue armies had R_0 and B_0 armies respectively. Also let a_1 and a_2 be the attrition coefficients by blue and red armies respectively. If $R = R(t)$ and $B = B(t)$ be the number of soldiers in the red and blue armies at time t, find R and B.*

Solution. Clearly the model is given by

$$\frac{dR}{dt} = -a_1 B$$

$$\frac{dB}{dt} = -a_2 R$$

Differentiating w.r.t t, we have

$$\frac{d^2R}{dt^2} = a_1 a_2 R$$

$$\frac{d^2B}{dt^2} = a_1 a_2 B$$

Solving,

$$R = c_1 e^{\sqrt{a_1 a_2}\, t} + c_2 e^{-\sqrt{a_1 a_2}\, t}$$

$$B = d_1 e^{\sqrt{a_1 a_2}\, t} + d_2 e^{-\sqrt{a_1 a_2}\, t}$$

where c_1, c_2, d_1, d_2 are arbitrary constants. Determining their values using the given initial conditions, we have

$$R(t) = \frac{1}{2}\left(R_0 - \frac{a_1 B_0}{\sqrt{a_1 a_2}}\right) e^{\sqrt{a_1 a_2}\, t} + \frac{1}{2}\left(R_0 + \frac{a_1 B_0}{\sqrt{a_1 a_2}}\right) e^{-\sqrt{a_1 a_2}\, t}$$

$$B(t) = \frac{1}{2}\left(B_0 - \frac{a_2 R_0}{\sqrt{a_1 a_2}}\right) e^{\sqrt{a_1 a_2}\, t} + \frac{1}{2}\left(B_0 + \frac{a_2 R_0}{\sqrt{a_1 a_2}}\right) e^{-\sqrt{a_1 a_2}\, t}$$

Exercise 3.10.1. *Who did first develop the combat model?*

Exercise 3.10.2. *What is attrition coefficient?*

Exercise 3.10.3. *What are the basic assumptions of the combat model?*

Exercise 3.10.4. *Establish the combat model.*

Exercise 3.10.5. *During the Battle of Iwo Jima in the Pacific Ocean (1945), daily records were kept of all U.S. combat losses. The values of the attrition coefficients* a_1 *and* a_2 *have been estimated from the data as* $a_1 = 0.0544$ *and* $a_2 = 0.0106$, *and the initial numbers in the red and blue armies, respectively, were* $r_0 = 66{,}454$ *and* $b_0 = 18{,}274$. *Obtain accurate solutions to the differential equations* (3.40).

Ans.

$$R(t) = 12,\ 516.1621e^{0.024t} + 53,\ 937.8379e^{-0.024t}$$

$$B(t) = -5,\ 539.3659e^{0.024t} + 23,\ 813.3659e^{-0.024t}$$

3.11 Mathematical Model of Influenza Infection (within host)

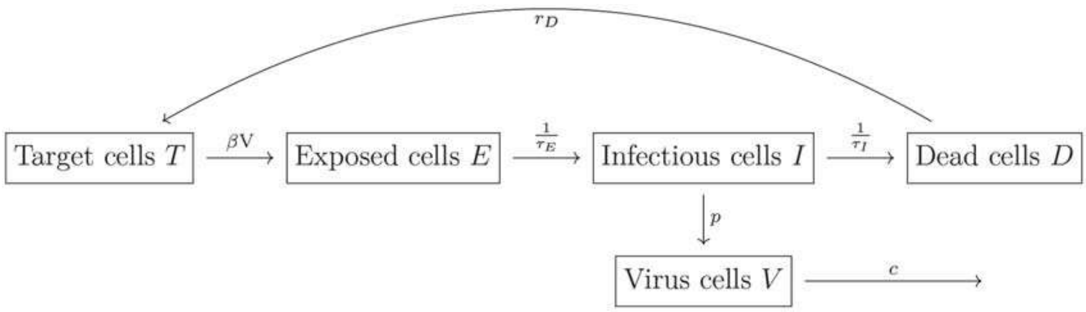
Influenza is a viral infectious respiratory disease that can be seasonal and mild, severe, or chronic. In 2018, there were 3-5 million cases of severe influenza around the world, resulting in approximately 500,000 deaths. Part of what makes Influenza dangerous is that the virus mutates very quickly; in one day it can mutate more than humans have in the past several thousand years. Influenza virus may be contracted via an air-born path by inhaling the cough droplets of an infected individual (in the case of human influenza), or a vector-born virus that is contracted via infected birds (in the case of avian influenza).

Human influenza attacks the upper respiratory tract; however, it is capable of spreading to cells in the lower respiratory tract, cardiovascular system, and nervous system. It is in these secondary locations that it is most dangerous.

Here we model the disease interaction with cells. The cells are grouped into four classes: Target cells T, Exposed cell E, Infectious cells I and Dead cells D. T, represent the cell population susceptible to infection. These cells, after interacting with the virus cells, transition to the exposed class at the per- capita rate β.

Dead cells trigger cellular restoration. This results in increase of target cells at the per-capita rate r_D.

Exposed cells E represent the cells that have been infected but are not yet producing new virons. This class can also be referred to as the latent or eclipse class. This class gains cells from the target population and loses cells to the infectious class at a per-capita rate of $\frac{1}{\tau_E}$. Infectious cells I, represent the class that actively produces new virons. It gains cells from the exposed class and loses cells to infection related death at a per- capita rate of $\frac{1}{\tau_I}$. Finally, Virus V represents the virus. Infectious cells produce new virons at per-capita rate p and cells clear the virus at a per- capita rate c. In the following, we describe this compartmental model.



Here the model becomes

$$\frac{dT}{dt} = -\beta TV + r_D D$$

$$\frac{dE}{dt} = \beta TV - \frac{E}{\tau_E} \qquad (3.41)$$

$$\frac{dI}{dt} = \frac{E}{\tau_E} - \frac{I}{\tau_I}$$

$$\frac{dV}{dt} = pI - cV$$

where $N = D + T + E + I$, N being the total number of cells, or $D = N - T - E - I$.

Exercise 3.11.1. *Describe the model of influenza infection (within host).*

3.12 Epidemic Models (SIR, SIRS, SI, SIS)

3.12.1 SIR Model

Centuries have witnessed devastating epidemics of various diseases. The history of human civilization bears several examples of dreadful diseases like the Black Death, Plague, Small Pox etc. Even after so much advancements in medical sciences, we have to face epidemics like AIDS, Ebola, SARS, MERS. Our present days' grappling to contain the global pandemic Covid 19 has taken the significance of epidemiology to a new height. Evidently, if we can understand the nature of how a disease spreads through a population, then certainly we can equip ourselves with better strategies to contain it through methods like vaccination or quarantine. Sometimes even the biological control of pests may also become handy to curb the spread of disease. For this, it is important to understand the effect of the infesting populace of the pests.

Several diseases, including influenza, measles, chickenpox and present day's Covid 19, spreads by infected persons in the population coming into close contact with susceptible persons. On the other hand, malaria, dengue are transmitted through mosquitoes. Thus these are vector borne diseases. Apart from the variety in mode of transmission, the severity of contagion also varies. Covid 19, influenza and measles are highly contagious, whereas glandular fever is much less so. Interestingly, some diseases, like mumps and measles, confer a lifelong immunity. On the other hand, influenza and typhoid have comparatively much shorter periods of immunity. So the recovered individual may again get infected.

Incubation period: It is the time between infection and the appearance of visible symptoms.

Latent period: It is the period of time between infection and the ability to infect someone else with the disease.

Note that the latent period is shorter than the incubation period. For example, the incubation period of measles is approximately 2 weeks but the latent period is approximately 1 week. As as a result, any infected individual can end up in spreading the disease to others without even knowing it.

Our model of epidemic:

Here we discuss a simple mathematical model for influenza outbreak at a boarding school over a period of about, say, 45 days. During this time interval, we can safely assume that reinfection does not occur.

Basic assumptions:

When studying the outbreak of a disease, the entire population under consideration can be divided into distinct compartments viz. susceptibles of size $S(t)$, infectives of size $I(t)$, and recovered individuals of size $R(t)$ where t denotes time. The susceptibles are those who are vulnerable to the infection, while the infectives are infected individuals capable of spreading the infection to susceptibles. Moreover, $R(t)$ is the number of those who have recovered from the disease and are no longer susceptible (i.e., acquired permanent immunity). Before proceeding further, we now make the following assumptions.

- Population sizes of susceptibles and infectives, i.e., $S(t)$ and $I(t)$ respectively are large enough such that random differences between individuals can be neglected.
- Births and deaths are ignored.
- The infection spreads only by contact.
- The latent period is set to be zero, i.e., an individual can spread the disease immediately after getting infected.
- Every recovered individual is immune to the pathogen i.e., cannot get reinfected (at least within the time period considered).
- At any time t, the population of size $N(t)$ is homogeneous, i.e., the contagious infectives and susceptibles are always randomly distributed over the area in which they reside.

The following is the input-output diagram for the epidemic model of influenza in a school, assuming there is no chance of reinfection, for the time period under consideration.

Susceptibles S(t) $\xrightarrow{\text{infected}}$ Infectives I(t) $\xrightarrow{\text{recovered}}$ Recovered R(t)

Note that, in general, a recovered person does not get life-long immunity against influenza and can get re-infected. But for a period of 15 days, the immunity of a recovered individual may be safely assumed.

Forming the differential equations:

First we consider the number of susceptibles infected by a single infective. The more is the number of susceptibles, the higher is the increase in the number of infectives. Thus the rate of susceptibles infected by a single infective will be an increasing function of the number of susceptibles. We assume now $\lambda(t)$ is the *force of infection*, i.e., it is the per-capita rate at which susceptible individuals become infected. If the number of susceptibles at time t is $S = S(t)$, then the rate in which susceptibles are infected is $\lambda(t)S(t)$. Note that $\lambda(t)$ need not be invariant as the more infectives there are, the higher the risk that a single susceptible will become infected.

Then we have $\frac{dS}{dt} = -\lambda(t)S$ (since there is no ingress to the compartment of susceptibles).

Again, the number of infectives removed from the compartment of infectives to the compartment of recovered in the time interval depends only on the number of infectives.

We assume that the rate at which infectives recover is directly proportional to the number of infectives. If the per- capita rate of recovery is γ, then the rate of infectives recovered is $\gamma I(t)$. γ is known as *recovery rate* or *removal rate*. Note that $\gamma = \frac{1}{D}$, where D is the average duration of the infectious period.

Thus the rate of ingress to the compartment of infectives is $\lambda(t)S$ while the rate of egress from this compartment is $\gamma I(t)$. Hence we have $\frac{dI}{dt} = \lambda(t)S - \gamma I$.

Again, the rate of influx into the compartment of recovered individuals is $\gamma I(t)$. Since we have ignored the possibility of re-infection during the time interval under consideration, there is no outflux from the compartment. Hence $\frac{dR}{dt} = \gamma I$.

Thus we get a system of coupled differential equations

$$\frac{dS}{dt} = -\lambda(t)S$$

$$\frac{dI}{dt} = \lambda(t)S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

where the total population is $N(t) = S(t) + I(t) + R(t)$.

The force of infection, $\lambda(t)$, depends on the current number of infectives $I(t)$ and increases as the proportion of infectives in the population increases. It also depends on the rate that individuals make contacts. Let c be the number of contacts per time and p be the probability that a contact between an infective and a susceptible results in an infection. We can now assume the force of infection to be

$$\boxed{\lambda(t) = cp\frac{I(t)}{N(t)}} \tag{3.42}$$

Let

$$\boxed{\beta = \frac{cp}{N(t)}} \tag{3.43}$$

We call this β the *transmission coefficient*.

Now our model becomes an IVP

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dT}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned} \tag{3.44}$$

with initial conditions $S(0) = s_0$, $I(0) = i_0$ and $R(0) = 0$.

Example 3.12.1. *In a city of twelve lakhs population witnessing the spread of an infectious disease, if the number of contacts per minute is* 3.7 *and* 0.67 *be the probability that a contact between an infective and a susceptible results in an infection, then what will be the transmission coefficient? If at a given point of time, already* 50, 000 *people have been infectives, then what will be the force of infection at that point of time?*

Solution. Here the number of contacts per minute is $c = 3.7$. The probability that a contact between an infective and a susceptible results in an infection is $p = 0.67$. Also, the total population is $N(t) = 1, 200, 000$ and number of infectives $I(t) = 50, 000$. Then using equation (3.43), the transmission coefficient is $\beta = 0.20659 \times 10^{-5}$. Also, using equation (3.42), we have the force of infection $\gamma(t) = 0.10329$.

Exercise 3.12.1. *What is the force of infection?*

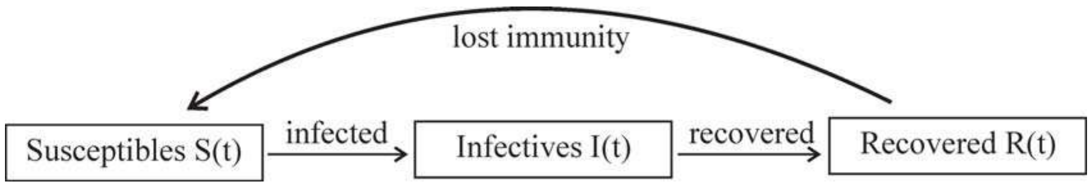
Exercise 3.12.2. *What is the transmission coefficient?*

Exercise 3.12.3. *In a city of six lakhs population witnessing the spread of an infectious disease, if the number of contacts per minute is* 3.7 *and* 0.67 *be the probability that a contact between an infective and a susceptible results in an infection, then what will be the transmission coefficient? If at a given point of time, already* 25, 000 *people have been infectives, then what will be the force of infection at that point of time?*

Ans. 0.41317×10^{-5}, 0.10329

3.12.2 SIRS Model:

Now let us discard the notion of permanent immunity as we assumed in the SIR model. Consider the instance of a influenza outbreak in a boarding school for a period of 45 days. So keeping the other assumptions of SIR model intact, we may now consider the fact that the immunity of the recovered persons wanes with time and they become susceptibles again for the same strain of virus. Let $\xi = \xi(t)$ is the per- capita per rate, per unit time, in which the recovered individuals return to the susceptible state due to loss of immunity.



Reconsidering the influx and outflux of the compartments of susceptibles, infectives and recovered, we have a new system of coupled differential equations

$$\frac{dS}{dt} = -\beta SI + \xi R$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I - \xi R$$

with initial conditions $S(0) = s_0$, $I(0) = i_0$ and $R(0) = 0$.

3.12.3 SI model

The SI model is the simplest form of all disease models. In this model, the population is divided into two compartments viz. susceptibles and infectives. Initially every individual is susceptible, i.e., with no immunity. Individuals are born into the simulation with no immunity. Once infected and with no treatment, the individuals remain infective throughout the rest of the life. Thus the infectious period remains longer than the lifespan of individuals. Also they continue to be in contact with the susceptible ones. Behaviour of diseases like cytomegalovirus (CMV) or herpes are example of this model. As before we assume, at any time t, $S = S(t)$ and $I = I(t)$ are the numbers of susceptible and infective individuals respectively. With β as transmission coefficient, our model becomes

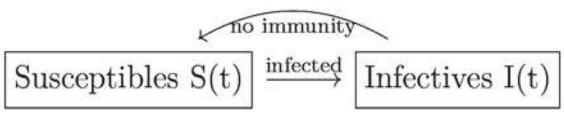
$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

where N= S+I is the total population.

3.12.4 SIS model

In the SIS model, the infected individuals return to the susceptible state immediately after infection. This model is appropriate for diseases that commonly have repeat infections, for example, the common cold (rhinoviruses) or sexually transmitted diseases like gonorrhea or chlamydia.



With β as transmission coefficient and γ as recovery rate, our model becomes

$$\frac{dS}{dt} = -\beta SI + \gamma I$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

Exercise 3.12.5. *Define incubation and latent periods.*

Exercise 3.12.6. *What are the basic assumptions of an epidemic model?*

Exercise 3.12.7. *Establish the SIR, SIRS, SI and SIS models.*

Exercise 3.12.8. *What are the differences between the SIR, SIRS, SI and SIS models?*

3.13 Spreading of Rumour Model

An old saying goes that rumors come true after being repeated a thousand times. In real life, if people are unable to distinguish authenticity, many rumors are deemed to be true after a large number of repetitions. When rumors are widely propagated, people tend to believe the rumor, especially if they lack timely real information. Because of the increased presence of online social networks, rumors are no longer spread by word of mouth over a small area but are spread amongst strangers in different regions and different countries, meaning that rumors are being spread faster and wider than ever before. This sustained and rapid spreading of rumors deepens people's impression about the veracity of the rumor and thus improves the credibility. Rumor spreading, therefore, has the ability to shape public opinion and lead to social panic and instability. For example, the nuclear leakage accidents in Fukushima, after the 2011 Tohoku earthquake, caused a number of rumors in the region. Rumors said that taking materials containing iodine could help ward off nuclear radiation, which led to the fact that many people rushed to purchase iodized salt.

In reality, people hear rumors many times and so have an accumulation of impressions about the rumors, which changes the probability as to when people become rumor spreaders. Therefore, memory effects have a strong time-dependency. Further, the remembering mechanisms can indicate repeatability, which affects the spreading characteristics of the rumor. Even a small amount of memory can affect the rumor spread in small network sizes.

3.13.1 Classification of population

Consider a network with N nodes and E links representing the individuals and their interactions. At each time step, each individual is in one of the following four states:

1. the *unaware*: this individual has not yet heard the rumor;

2. the *lurkers*: this individual knows the rumor but is not willing to spread it because they require an active effort to discern the truth or falseness of the rumor;

3. the *spreaders*: this individual knows this rumor and transmits it to all their contacts;

4. the *stiflers*: this individual neither trusts the rumor nor transmits it.

People generally hear a rumor after many times, and therefore they get an accumulated impression about the rumor, which means that the probability that people become a spreader changes from "will never believe" to "believes." This can be described as the cumulative effect of memory, which affects the probability that an individual becomes a spreader from a lurker in the rumor spreading process. In information spreading theory, a function was established which reflected the probability that a person would approve the information at time t after having received the news m times. This function is $P(m) = (\lambda - T)e^{-b(m-1)} + T$, where $\lambda = P(1)$ is the approving probability of the first receipt of the information and $T \in (0, 1]$ is the upper bound of the probability indicating maximal approval probability.

Now, lurkers do not automatically change their states at time step t. Some may become a stifler or a spreader, while others remain lurkers and may become stiflers or spreaders at a later time. We assume that the new lurkers at each time step have a part of the residuals which last until the end of the rumor spreading. This corresponds with the fact that there are always some people who take a long time to change their state in real life.

Lurkers become spreaders at a variable probability, denoted by $p(t)$ and become stiflers at the rate of p_2.

As the number of times the rumor is received, the probability that a individual agrees to the truth of the rumor grows and infinitely approaches a constant. Thus, as time passes, the number of times the rumor is received for the residual lurkers gradually increases. Because the probability $p(t)$ that an individual becomes a spreader from a lurker is a level that reflects the transformation probability of all lurkers, including the residual old lurkers and the new joined lurkers in each time step, as time passes, the probability increases gradually because of the cumulative effect of memory and infinitely approaches a constant when the accumulated memories achieve a certain degree.

The probability $p(t)$ that lurkers become spreaders affected by memory accumulation at t–th time step is given by

$$p(t) = (p - q)e^{-c(t-1)} + q \tag{3.45}$$

where p,q and c are parameters. These three parameters reflect the characteristics of the variable memory effects rate. p is the initial value of the memory effects function at $t = 1$. The parameter p reflects the importance of an event triggering rumors in the spreading process, and it is the initial probability that an individual becomes a spreader. A larger value for p means that the spreaders more easily remember the rumor because the event is probably more important. $q \leq 1 - p_2,\ q \in (0,1)$, is the maximal transformation probability.

As time passes, $p(t)$ infinitely approaches q. The parameter c can be regarded as the memory speed; namely c captures how quickly $p(t)$ reaches the maximum value q. The memory effects rate $p(t)$ is a probability varying over time t. Here, we do not consider interest decay and assume that the time scale for the rumor spreading is much faster than the memory decay.

3.13.2 Rumor Spreading Model

Denote by $S(t)$, $E(t)$, $I(t)$ and $R(t)$ the density of the unaware, lurkers, spreaders, and stiflers at time t. Thus $S(t) + E(t) + I(t) + R(t) = 1$.

1. Everyone needs time to determine the authenticity of rumor, so an unaware becomes a lurker with a probability 1 when an unaware individual contacts a spreader. The contact probability k is decided by the specific network topology. Therefore, the reduced speed of the unaware $\frac{dS}{dt}$ is proportional to the product of densities of the unawares $S(t)$ and the spreaders $I(t)$. So the differential equation becomes

$$\frac{dS(t)}{dt} = -kS(t)I(t) \tag{3.46}$$

2. A lurker becomes a spreader at the rate of $p(t)$ and becomes a stifler at the rate of p_2, which depends on cognition. For example, some unaware turned lurker individuals may have strong knowledge structures and logical reasoning abilities. So they may have little interest in rumors. Because an unaware individual becomes a lurker with a probability 1 when an unaware contacts a spreader, the increased speed of the lurkers is given by

$$\frac{dE(t)}{dt} = kS(t)I(t) - p(t)\,E(t) - p_2E(t) \tag{3.47}$$

3. When two spreaders contact each other, both may find the two pieces of information inconsistent, so they stop the spread. When a spreader contacts a stifler, the spreader tries to stop the spread, as the stifler shows no interest in the rumor or denies its veracity. We suppose that the above cases occur at the same probability p_3. Therefore, the reduced speed of the spreaders $\frac{dI(t)}{dt}$ is proportional to $I(t)$ and $R(t) + I(t)$. Additionally, a lurker becomes a stifler at the rate of $p(t)$. Therefore

$$\frac{dI(t)}{dt} = p(t)\,E(t) - kp_3 I(t)\left(I(t) + R(t)\right) \tag{3.48}$$

4. The increasing speed of the stiflers $\frac{dR(t)}{dt}$ is proportional to the existing $I(t)$ and $I(t) + R(t)$ from above. Also a lurker becomes a stifler at the rate of p_2. Therefore

$$\frac{dR(t)}{dt} = kp_3 I(t)\left(I(t) + R(t)\right) + p_2 E(t) \tag{3.49}$$

The equations (3.46), (3.47), (3.48) and (3.49) together with the initial assumptions $S(0) = S_0$, $E(0) = 0$, $I(0) = 1 - S_0 > 0$ and $R(0) = 0$ describes a model of rumour spreading.

Exercise 3.13.1. *In the rumour spreading model, who are the unawares, lurkers, spreaders* and stiflers?

Exercise 3.13.2. *Describe the rumour spreading model.*

3.14 Steady State solutions

Definition 3.14.1. *Let* $\frac{dy}{dt} = f(y)$, *where* $f(y)$ *may not be a linear function of* y. *Then the steady state solutions or critical points or equilibrium points are* $y = y_0$ *where* $f(y_0) = 0$.

On a more general set up, consider the following system of ODEs

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{3.50}$$

Here $f(x, y)$ and $g(x, y)$ are non-linear equations. We also assume that the system of equations (3.50) is an autonomous system, i.e., $f(x, y)$ and $g(x, y)$ do not contain t explicitly.

Now, we can have a velocity field $\bar{F} = f(x, y)\,\hat{i} + g(x, y)\,\hat{j}$ corresponding to the system (3.50). From geometric viewpoint, the solutions $x(t)$ and $y(t)$ together give trajectories of the field of $\bar{F}$. This means they give curves everywhere having the right velocity at every point.

Definition 3.14.2. *A* steady state solution *or* critical point *is a point* $P(x_0, y_0)$ *where* $f(x_0, y_0) = 0 = g(x_0, y_0)$.

From the viewpoint of solutions, $x = x_0$, $y = y_0$ give constant solution. On the other hand from viewpoint of a vector field, at such points $\bar{F} = 0$, i.e., there is no velocity at $P(x_0, y_0)$.

Example 3.14.1. *Let us consider a system of ODEs which will be discussed later in detail in section* 3.20.

$$\frac{dX}{dt} = \beta_1 X - c_1 XY$$

$$\frac{dY}{dt} = c_2 XY - \alpha_2 Y \qquad (3.51)$$

where c_1, c_2, α_2, β_1 *are positive constants.*

This system of equations is known as the Lotka–Volterra prey- predator system. We will find the equilibrium solutions or critical points of the system (3.51).

Solution. We set $\frac{dX}{dt} = 0$ and $\frac{dY}{dt} = 0$ in system of ODEs (3.51). So we have

$$X\left(\beta_1 - c_1 Y\right) = 0 \qquad (3.52)$$

and

$$Y\left(-\alpha_2 + c_2 X\right) = 0 \qquad (3.53)$$

From equation (3.52) there are two possible solutions: $X = 0$ or $\beta_1 - c_1 Y = 0$.

Putting $X = 0$ in equation (3.53), we have $Y = 0$. Thus (0, 0) is an equilibrium point of the system (3.51).

Taking the other case, $\beta_1 - c_1 Y = 0$, we have $Y = \frac{\beta_1}{c_1}$. Putting this in equation (3.53), we have $X = \frac{\alpha_2}{c_2}$. Thus $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$ is another equilibrium point of the system (3.51).

Therefore (0, 0) and $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$ are two equilibrium points of the system (3.51).

3.15 Linearization

In the section 3.14, we have been introduced to the notion of steady state solutions or equilibrium points for a system represented by a single ODE as well as a system represented by a coupled system of ODEs. Here we will approximate the non-linear ODE or system of ODEs with a linear ODE or system of ODEs close to the equilibrium point. This process is called *linearization*.

3.15.1 Linearization of an ODE

Consider the differential equation

$$\frac{dx}{dt} = f(x) \tag{3.54}$$

As we have seen earlier, the equilibrium solutions of the equation (3.54) are the solutions $x = x_e$ such that $f(x_e) = 0$.

We let $x(t) = x_e + \xi(t),$, with $0 < |\xi(t)| \ll 1$, where the new variable ξ represents the small perturbation from the equilibrium solution. For the differential equation (3.54), if we expand the RHS about the equilibrium solution by letting $x = x_e + \xi$, then the differential equation for the variable ξ is

$$\frac{dx}{dt} = \frac{d\left(x_e + \xi\right)}{dt} = f\left(x_e + \xi\right) \approx f\left(x_e\right) + \xi f'\left(x_e\right)$$

ignoring the higher order terms of ξ.

Since $f(x_e) = 0$, by the definition of an equilibrium point, then the original differential equation is approximated, close to the equilibrium solution, by

$$\frac{d\xi}{dt} = \xi f'(x_e) \tag{3.55}$$

for small values of ξ.

Equivalently this can be written as

$$\frac{dx}{dt} = \left(x - x_e\right) f'(x_e) \tag{3.56}$$

This is the linearization of equation (3.54) at the equilibrium point x_e.

Example 3.15.1. *Linearize the differential eqaution*

$$\frac{dx}{dt} = x^2 - 3x + 2$$

at their points of equilibrium.

Solution. Let $f(x) = x^2 - 3x + 2$. Clearly $f(x) = 0 \Rightarrow x = 1, 2$. Thus the equilibrium points are 1 and 2.

Now $f'(x) = 2x - 3$.

So linearization at $x = 1$ is $\frac{d\xi}{dt} = \xi f'(1)$, i.e., $\frac{d\xi}{dt} = -\xi$. Also linearization at $x = 2$ is $\frac{d\xi}{dt} = \xi f'(2)$, i.e., $\frac{d\xi}{dt} = \xi$.

3.15.2 Linearization of coupled system of ODEs

Consider a general system of two nonlinear differential equations

$$\begin{aligned} \frac{dX}{dt} &= F(X, Y), \\ \frac{dY}{dt} &= G(X, Y). \end{aligned} \tag{3.57}$$

Let (x_e, y_e) be any equilibrium point for the system (3.57), not necessarily at (0, 0), and then $F(x_e, y_e) = 0 = G(x_e, y_e)$.

Consider solutions close to the steady-state (equilibrium) solutions $X(t) = x_e + \xi(t)$, $Y(t) = y_e + \eta(t)$, where $\xi(t)$ and $\eta(t)$ are small and approach zero when X and Y approach the equilibrium point. These $\xi(t)$ and $\eta(t)$ are perturbations of the steady state.

We now change the variables in the system from X and Y to ξ and η respectively. Then

$$\frac{d\left(x_e + \xi\right)}{dt} = F\left(x_e + \xi, y_e + \eta\right),$$
$$\frac{d\left(y_e + \eta\right)}{dt} = G\left(x_e + \xi, y_e + \eta\right), \tag{3.58}$$

where ξ and η are functions of t. But we have, since x_e and y_e are constants,

$$\frac{dX}{dt} = \frac{d\left(x_e + \xi\right)}{dt} = \frac{d\xi}{dt},$$
$$\frac{dY}{dt} = \frac{d\left(y_e + \eta\right)}{dt} = \frac{d\eta}{dt}, \tag{3.59}$$

Comparing systems (3.58) and (3.59), we have

$$\frac{dX}{dt} = \frac{d\xi}{dt} = F\left(x_e + \xi, y_e + \eta\right),$$
$$\frac{dY}{dt} = \frac{d\eta}{dt} = G\left(x_e + \xi, y_e + \eta\right), \tag{3.60}$$

We now apply the Taylor series expansion in two variables to expand $F\left(x_e + \xi, y_e + \eta\right)$ and $G\left(x_e + \xi, y_e + \eta\right)$. Then we take a linear approximation for each. Applying the Taylor series expansion in two variables, we find

$$\frac{d\xi}{dt} = F\left(x_e, y_e\right) + F_\xi\left(x_e, y_e\right)\xi + F_\eta\left(x_e, y_e\right)\eta + \text{ terms of higher order,}$$

$$\frac{d\eta}{dt} = G\left(x_e, y_e\right) + G_\xi\left(x_e, y_e\right)\xi + G_\eta\left(x_e, y_e\right)\eta + \text{ terms of higher order.} \tag{3.61}$$

where $F_\xi = \dfrac{\partial F}{\partial \xi}$, $F_\eta = \dfrac{\partial F}{\partial \eta}$ and likewise for G.

Recall that since (x_e, y_e) is a equilibrium point for the system (3.57), therefore $F(x_e, y_e) = 0 = G(x_e, y_e)$. Now taking the linear approximation of each Taylor series expansion (i.e., ignoring all terms of higher order), we have

$$\frac{d\xi}{dt} = F_\xi\left(x_e, y_e\right)\xi + F_\eta\left(x_e, y_e\right)\eta,$$

$$\frac{d\eta}{dt} = G_\xi\left(x_e, y_e\right)\xi + G_\eta\left(x_e, y_e\right)\eta. \tag{3.62}$$

Or equilvalently,

$$\begin{pmatrix} \dfrac{d\xi}{dt} \\ \dfrac{d\eta}{dt} \end{pmatrix} = \begin{pmatrix} F_\xi & F_\eta \\ G_\xi & G_\eta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{3.63}$$

Note that ξ and η are not variables of the original equation. However, $X = x_e + \xi$ and $Y = y_e + \eta$ so that $\dfrac{\partial F}{\partial \xi} = \dfrac{\partial F}{\partial X}\dfrac{\partial X}{\partial \xi} = \dfrac{\partial F}{\partial X}\dfrac{d(x_e + \xi)}{d\xi} = \dfrac{\partial F}{\partial X}$. Similarly $\dfrac{\partial F}{\partial \eta} = \dfrac{\partial F}{\partial Y}$, $\dfrac{\partial G}{\partial \xi} = \dfrac{\partial G}{\partial X}$ and $\dfrac{\partial G}{\partial \eta} = \dfrac{\partial G}{\partial Y}$.

This means that we have

$$\begin{pmatrix} F_\xi & F_\eta \\ G_\xi & G_\eta \end{pmatrix} = \begin{pmatrix} F_X & F_Y \\ G_X & G_Y \end{pmatrix}.$$

Thus system (3.63) becomes

$$\begin{pmatrix} \dfrac{dX}{dt} \\ \dfrac{dY}{dt} \end{pmatrix} = \boldsymbol{J}\begin{pmatrix} X - x_e \\ Y - y_e \end{pmatrix} \tag{3.64}$$

where $\boldsymbol{J} = \begin{pmatrix} F_X & F_Y \\ G_X & G_Y \end{pmatrix}$ is the *Jacobian matrix* of the system (3.57).

Example 3.15.2. *We have discussed in example 3.14.1 about the equilibrium points of the system of ODEs representing prey- predator model (will be discussed later in detail in section* 3.20). *Linearize the model.*

Solution. The prey- predator model (described in section 3.20) is given by

$$\begin{aligned} \frac{dX}{dt} &= \beta_1 X - c_1 XY \\ \frac{dY}{dt} &= c_2 XY - \alpha_2 Y \end{aligned} \tag{3.65}$$

where c_1, c_2, α_2, β_1 are positive constants. In example 3.14.1, we have seen the equilibrium points of the prey- predator model (described in section 3.20) are (0, 0) and $\left(\dfrac{\alpha_2}{c_2}, \dfrac{\beta_1}{c_1}\right)$.

We will now linearize the system of differential equations (3.65) at these equilibrium points (using equation (3.64)). We set $F(X, Y) = \beta_1 X - c_1 XY$ and $G(X, Y) = c_2 XY - \alpha_2 Y$. Then the Jacobian matrix is

$$\boldsymbol{J} = \begin{pmatrix} F_X & F_Y \\ G_X & G_Y \end{pmatrix} = \begin{pmatrix} \beta_1 - c_1 Y & -c_1 X \\ c_2 Y & -\alpha_2 + c_2 X \end{pmatrix}$$

Case I:

When the equilibrium point is (0, 0), then $\boldsymbol{J}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}$. Then the linearized system at (0, 0) is

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Case II:

When the equilibrium point is $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$, then the Jacobian becomes

$$\boldsymbol{J}_2 = \begin{pmatrix} 0 & -\frac{c_1\alpha_2}{c_2} \\ \frac{\beta_1 c_2}{c_1} & 0 \end{pmatrix}$$

Then the linearized system at $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$ is

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c_1\alpha_2}{c_2} \\ \frac{\beta_1 c_2}{c_1} & 0 \end{pmatrix} \begin{pmatrix} X - \frac{\alpha_2}{c_2} \\ Y - \frac{\beta_1}{c_1} \end{pmatrix}$$

3.16 Local stability analysis

3.16.1 Local stability analysis of an ODE

In section 3.15 we have seen that the linearization of the differential equation

$$\frac{dx}{dt} = f(x) \tag{3.66}$$

at a equilibrium point x_e is

$$\frac{d\xi}{dt} = \xi f'(x_e)$$

where ξ represents the small perturbation from the equilibrium solution.

By local stability of an equilibrium point, we mean that any solution close to the equilibrium solution will tend towards the equilibrium solution and by unstable equilibrium, the solution will not get closer to the equilibrium point.

We can now interpret what happens without actually solving this differential equation. Suppose $f'(x_e) < 0$. Now for $\xi = \xi(t) > 0$, $\frac{d\xi}{dt} < 0$, so $x(t) = x_e + \xi(t)$ approaches the equilibrium point x_e. Similarly, for $\xi = \xi(t) < 0$, then $x(t) = x_e + \xi(t)$ increases towards the equilibrium solution. Thus the solution is attracted to the equilibrium solution.

By a similar argument, when $f'(x_e) > 0$ the solution moves away from the equilibrium solution after a small perturbation.

Thus we have equilibrium solution is stable if $f'(x_e) < 0$ and unstable otherwise.

Example 3.16.1. *Find all equilibrium points for the differential equation*

$$\frac{dC}{dt} = \frac{F}{V}c_{in} - \frac{F}{V}C$$

where F and V are positive constants. Also determine if the equilibrium solution is stable or unstable.

Solution. Setting $\frac{dC}{dt} = 0$, we obtain

$$\frac{F}{V}\left(c_{in} - C\right) = 0$$

$$\Rightarrow C = c_{in}$$

Thus c_{in} is the equilibrium solution.

Now considering $f(C) = \frac{F}{V}\left(c_{in} - C\right)$ and $C_e = c_{in}$, we have $f'(C_e) = -\frac{F}{V}$.

Since F and V are positive parameters, this means that the equilibrium solution $C_e = c_{in}$ is always stable.

3.16.2 Local stability analysis of linear system of ODEs based on eigen values

Solving the system using linear algebraic technique

Let us consider the general pair of linear first-order equations:

$$X' = a_1 X + b_1 Y$$

$$\tag{3.67}$$

$$Y' = a_2 X + b_2 Y$$

which has an equilibrium point at the origin, i.e., $(x_e, y_e) = (0, 0)$. In vector notation, we can write

$$x' = \mathbf{A}x \tag{3.68}$$

Suppose we have found the eigenvalues λ_1 and λ_2, as well as the associated eigen vectors for **A**, namely $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

We define **U** to be the matrix whose columns are the eigen vectors. Thus **U** = (**u v**) $= \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$.

From the definition of eigenvectors and eigenvalues, we have

$$\mathbf{Au} = \lambda_1 \mathbf{u} \ \text{ and } \ \mathbf{Av} = \lambda_2 \mathbf{v}$$

which implies that

$$\mathbf{A}(\mathbf{u\ v}) = (\lambda_1 \mathbf{u}\ \lambda_2 \mathbf{v}) = (\mathbf{u\ v})\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

i.e.,

$$\mathbf{AU} = \mathbf{UD}$$

where $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is a diagonal matrix.

Assuming that **U** is invertible, we can write

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{D} \tag{3.69}$$

We will use this equation below.

First we express **x** as a linear combination of the eigen vectors and, assuming this is possible, we have

$$\mathbf{x} = z_1\mathbf{u} + z_2\mathbf{v} \tag{3.70}$$

Letting $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$,

$$\mathbf{x} = \mathbf{U}\mathbf{z}$$

Since X and Y are functions of time, and the eigen vectors are not (since **A** is not a function of time), therefore z_1 and z_2 must also be functions of time. We now establish two expressions for $\mathbf{x}'$.

$$\mathbf{x} = \mathbf{U}\mathbf{z} \text{ so } \mathbf{x}' = \mathbf{U}\mathbf{z}'$$

and also

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \text{ so } \mathbf{x}' = \mathbf{A}\mathbf{U}\mathbf{z}$$

Equating these two expressions for $\mathbf{x}'$, we have

$$\mathbf{U}\mathbf{z}' = \mathbf{A}\mathbf{U}\mathbf{z}$$

Then using equation (3.69), we have

$$\begin{aligned} \mathbf{z}' &= \mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{z} \\ \text{i.e., } \mathbf{z}' &= \mathbf{D}\mathbf{z} \end{aligned} \tag{3.71}$$

Expanding equation (3.71), we have

$$z_1' = \lambda_1 z_1$$

$$\& \quad z_2' = \lambda_2 z_2$$

Thus we obtain two equations that are easy to solve. They are the equations for exponential growth and decay with which, by now, we are familiar. We have as solutions

$$z_1 = k_1 e^{\lambda_1 t}$$
$$\& \quad z_2 = k_2 e^{\lambda_2 t} \tag{3.72}$$

where k_1 and k_2 are arbitrary constants.

Using these in equation (3.70), we have

$$\mathbf{x} = k_1 e^{\lambda_1 t}\mathbf{u} + k_2 e^{\lambda_2 t}\mathbf{v}$$
$$= e^{\lambda_1 t}\hat{\mathbf{u}} + e^{\lambda_2 t}\hat{\mathbf{v}}$$

where $\hat{\mathbf{u}} = k_1\mathbf{u}$ and $\hat{\mathbf{v}} = k_2\mathbf{v}$ are two eigen vectors (as any scalar multiple of an eigen vector is again an eigen vector) and so

$$X = e^{\lambda_1 t}\hat{u}_1 + e^{\lambda_2 t}\hat{v}_1$$
$$\& \quad Y = e^{\lambda_1 t}\hat{u}_2 + e^{\lambda_2 t}\hat{v}_2 \tag{3.73}$$

This is the solution of linear system (3.67).

Equilibrium point classifications

For the systems described above, we had the origin (0, 0) as the equilibrium (or critical) point. What we will discuss now is the behaviour of the trajectories of the solution close to this point, using the techniques of eigenvalues and eigen vectors. We will mainly focus on the eigen values as we can see the trajectories are given by $z_1 = k_1 e^{\lambda_1 t}$ and $z_2 = k_2 e^{\lambda_2 t}$ (see solution (3.72)). Equation (3.70) as well as the fact that **u** and **v** are independent of t make it very clear that the trajectory given by ($X(t)$, $Y(t)$) depends heavily upon $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. Now we will see how the natures of λ_1 and λ_2 influence the behaviours of the trajectories.

Case I: When $\lambda_1 < 0$ and $\lambda_2 < 0$ (eigen values real and negative)

As $\lim_{t\to\infty} z_1 = \lim_{t\to\infty} k_1 e^{\lambda_1 t} = 0$ and $\lim_{t\to\infty} z_2 = \lim_{t\to\infty} k_2 e^{\lambda_2 t} = 0$, therefore all trajectories approach the equilibrium point at the origin. Such a point is called a *stable node* and is illustrated in figure 3.9.

Case II: When $\lambda_1 > 0$ and $\lambda_2 > 0$ (eigen values are real and positive)

We have both z_1 and z_2 approaching ∞ (diverging) as t increases and therefore all trajectories diverge from the equilibrium point. Such a point is called an *unstable node* (see figure 3.9).

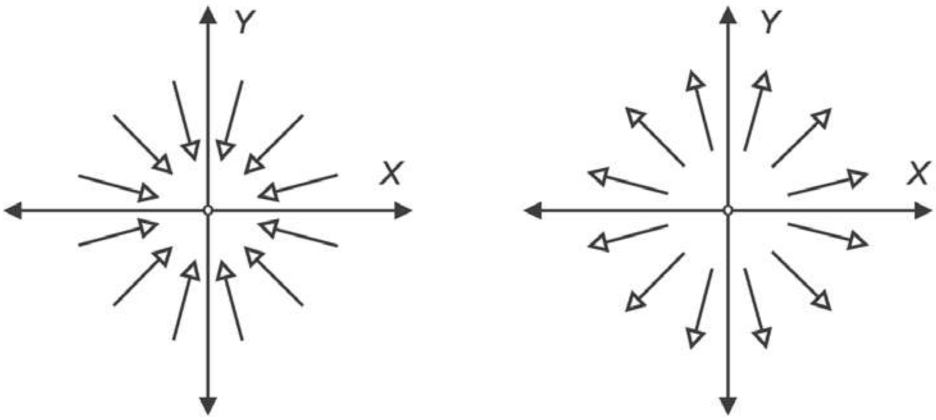


Figure 3.9: Trajectory behaviour close to a stable node (left) and an unstable node (right)

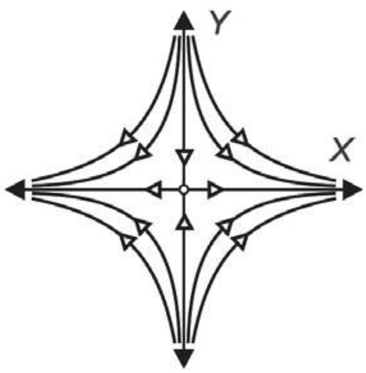


Figure 3.10: Trajectory behaviour close to a (unstable) saddle point

Case III: When $\lambda_1 > 0$ and $\lambda_2 < 0$ (eigen values are real and of different sign)

We have $\lim_{t\to\infty} z_1 = \lim_{t\to\infty} k_1 e^{\lambda_1 t} \to \infty$ and $\lim_{t\to\infty} z_2 = \lim_{t\to\infty} k_2 e^{\lambda_2 t} = 0$. Therefore the trajectories approach zero along one axis and approach ∞ along the other axis. Such a point is called a *saddle* or an *unstable saddle point* and is illustrated in figure 3.10.

Case IV: When $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha \neq 0 \neq \beta$ (eigen values are complex conjugates)

In this case, the solutions can be written in the form $z_1 = e^{\alpha t} \cos \beta t, z_2 = e^{\alpha t} \sin \beta t$. Here the trajectories spiral around the equilibrium point. If α < 0, then they spiral inwards towards the equilibrium point. Such a point is called a *stable focus*. If α > 0, then they spiral outwards and away from the equilibrium point. Such a point is called an *unstable focus*. These have been illustrated in figure 3.11.

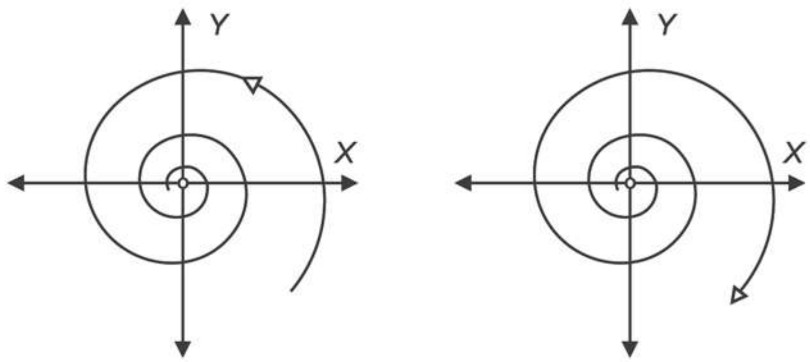


Figure 3.11: Trajectory behaviour close to a stable focus (left) and an unstable focus (right)

Case V: When λ_1 and λ_2 are purely imaginary (eigen values are purely imaginary)

In this case, the solutions can be written in the form $z_1 = \cos \beta t, z_2 = \sin \beta t$. Therefore the trajectories form closed loops enclosing the equilibrium point. Such a point is called a *centre* and the solutions are called periodic. This has been illustrated in figure 3.12.

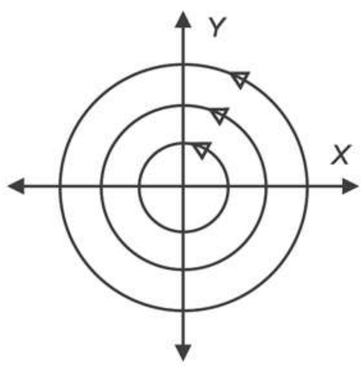


Figure 3.12: Trajectory behaviour close to a centre

3.17 Exponential growth

The growth of a population may take place with discrete jumps in breeding (e.g. fishes, insects etc. as they have fixed breeding season) or continuous breeding process (e.g. humans). Even for discrete cases, if the time gap between successive breeding jumps is negligible (e.g. bacteria) in comparison to the time span under observation, the model can arguably be treated as a continuous growth model. Taking a cue from the notion of compartmental model, we will develop and analyse the continuous model in the following under limited resources. But instead of jumping straight into the core, we will try to keep this model as simple as possible in the beginning and then add further complexities to it gradually.

Suppose we are dealing with a large population of bacteria. While dealing with a large population, we may ignore the random fluctuation in breeding and dying for individual micro-organism and therefore each individual bacterium may be considered as identical. Thus for a large time interval, each of these micro-organism may be supposed to have equal probability of breeding and dying. Here comes the idea of *per capita birth rate* β i.e., birth rate per member of the population per unit time (rate of incoming into the compartment) and *per capita death rate* α i.e., death rate per member of the population per unit time (rate of outgoing from the compartment). We assume these rates to be constant and $\beta > \alpha$. If $X = X(t)$ be the number of bacteria at any given time t, then the birth and death rates per unit time are βX and αX respectively. Assuming birth and death to be continuous with time, we have

$$\frac{dX}{dt} = \beta X - \alpha X \tag{3.74}$$

Note that we have neglected the effects of overcrowding which may take place eventually as well as immigration and emigration.

Let $r = \beta - \alpha$. Then r (> 0) is the *growth rate* of this population, then we can rewrite the equation (3.74) as

$$\frac{dX}{dt} = rX \tag{3.75}$$

Applying the method of seperation of variables, the general solution of the differential equation (3.75) is $X = ce^{rt}$. Applying initial condition $X(0) = x_0$, we have the following solution of the *Initial Value Problem* (IVP)

$$X = x_0 e^{rt} \tag{3.76}$$

The figure 3.13 depicts the behaviour of the solution (3.76).

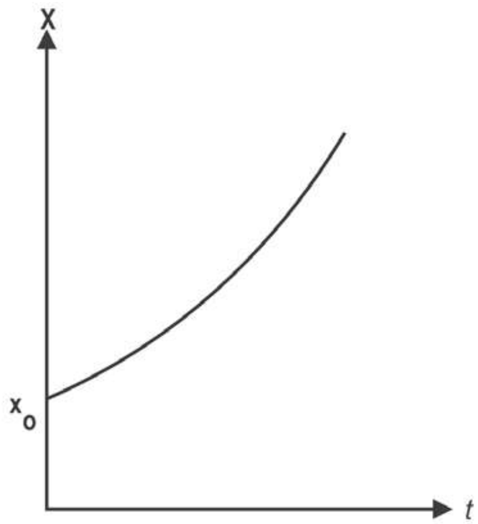


Figure 3.13: Exponential growth curve

3.18 Logistic growth

Here we revisit the previous model in Section 3.17 in the light of an overcrowded population struggling due to the scarcity of resources.

The carrying capacity:

Thus it is quite evident that if we ignore the effect of overcrowding on the growth of population, we will have an exponentially growing population. But when the resources are limited, this picture is far from reality. This is because the competition due to scarcity of resources increases the per capita death rate in an overcrowded population. Thus it can be safely said that only a limited number of micro-organism can sustain in any given environment. We call this number the *carrying capacity* of the population in the given environment and denote this by K. Whenever the population size X exceeds K, the per capita birth and death rates become equal, ignoring the other external factors like possibility of interaction with another population. This carrying capacity plays a crucial role in stabilizing the population.

Understanding the logistic growth:

We suppose the per capita death rate to depend linearly on the size of population. Then we can take the per capita death rate as $\alpha + \gamma X(t),\ \alpha, \gamma > 0$, where α is the per capita death rate due to natural attrition and γ is the per capita dependence of deaths on the population size. As $X \to 0$, the per capita death rate tends to α. Hence the death rate per unit time is $X\left(\alpha + \gamma X\right)$. Here the per capita birth rate is assumed to be same as in Section 3.17, i.e., β. Hence the birth rate over time is βX. Therefore the density dependent growth rate over time of this population is given by

$$\frac{dX}{dt} = \left(\beta - \alpha\right) X - \gamma X^2 \tag{3.77}$$

Letting $r = \beta - \alpha$, the equation (3.77) becomes

$$\frac{dX}{dt} = rX - \gamma X^2 \tag{3.78}$$

Here we assume $\gamma = \frac{r}{K}, K$ being the carrying capacity. Then equation (3.78) becomes

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) \tag{3.79}$$

Hence we can say that when $X = K$, $\frac{dX}{dt} = 0$, i.e., rate of change of the population becomes zero. In other words, whenever the population size $X = K$, the per capita birth and death rates become equal.

Example 3.18.1. *Solve the equation (3.79), i.e.,* $\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$ *with initial condition* $X(0) = x_0$.

We rewrite the equation (3.79) as

$$\frac{dX}{X\left(1 - \frac{X}{K}\right)} = r\,dt$$

Solving this, we have

$$X(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}} \tag{3.80}$$

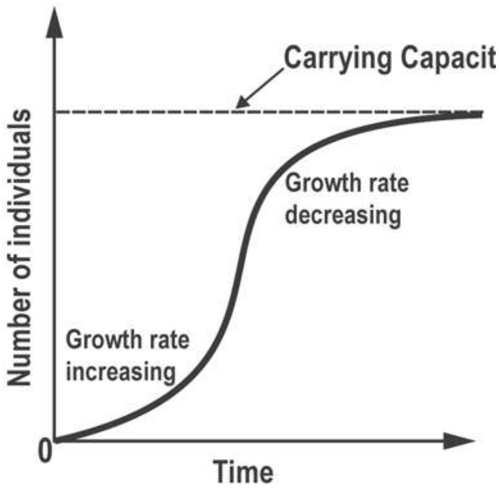


Figure 3.14: Logistic growth curve

Clearly the solution (3.80) implies the population size approaches the carrying capacity K when $t \to \infty$. Also $X \to x_0$, when $t \to 0$. These have been illustrated in figure 3.14.

3.19 Gomperzian Model

We have seen in Section 3.17 that if the population grows exponentially then eventually it will become ridiculously large. Since in reality no population goes to infinity, the exponential model needs to be modified in more realistic manner. Keeping this in mind, the Gompertz model has been devised.

We assumed the growth rate to be constant in the exponential model. In our present model, the growth rate varies with time. Let us recall the equation (3.75) in Section 3.17, i.e.,

$$\frac{dX}{dt} = rX$$

The growth rate r in the above equation changes with time t in the following way.

$$\frac{dr}{dt} = -\alpha r \tag{3.81}$$

where $\alpha > 0$ is a decaying coefficient of r. With the initial condition $r(0) = r_0$, the solution of equation (3.81) becomes

$$r = r_0 e^{-\alpha t} \tag{3.82}$$

Putting this in equation (3.81), we have

$$\frac{dX}{dt} = r_0 e^{-\alpha t} X$$

$$\Rightarrow \frac{dX}{X} = r_0 e^{-\alpha t} dt$$

$$\Rightarrow \log X = \frac{r_0 e^{-\alpha t}}{-\alpha} + \log A, \text{ A being arbitrary constant}$$

$$\Rightarrow X = A \exp\left(-\frac{r_0}{\alpha} e^{-\alpha t}\right)$$

Putting the initial condition $X(0) = x_0$, we have

$$X = x_0 \exp\left(\frac{r_0}{\alpha}\left(1 - e^{-\alpha t}\right)\right) \tag{3.83}$$

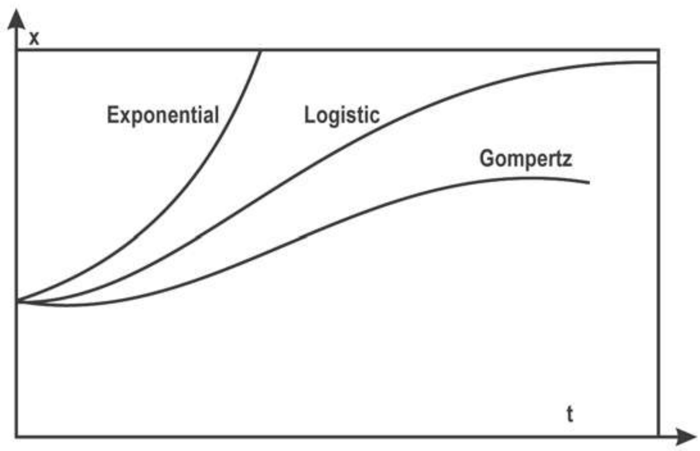


Figure 3.15: A comparison among exponential, logistic and Gomperzian growth curves

This is the Gomperzian growth model.

The figure 3.15 gives a comparison among exponential, logistic and Gomperzian growth models.

3.20 Prey Predator Model

We now develop a simple prey- predator model based on the growth of a population of small insect pests, namely cottony cushion scale insects, that interact with another population of ladybird beetle predators. In the late nineteenth century, these scale insects, which accidentally came from Australia, almost destroyed American citrus industry. To contain the insects (the prey), their natural predators ladybird beetles were also imported from Australia.

Initial assumptions: We make a few preliminary assumptions.

- Initially we assume the populations are sufficiently large so that we can neglect random differences between individuals.
- We ignore the effect of any pesticide like DDT.
- There are only two populations, viz. the predator and the prey in the ecosystem we are considering.
- The prey population grows exponentially in the absence of a predator.

Suppose $X = X(t)$ and $Y = Y(t)$ are the number of prey and predators respectively in the ecosystem, at any time t. The per-capita birth rates give the rate of births from an individual. Suppose the per-capita birth rate for the prey (i.e., the scale insect) is a constant b_1. Therefore rate of birth of the prey in the ecosystem per unit time is $b_1X(t)$. Note that this rate has nothing to do with the activities of the predators.

On the other hand, the death of the prey population has two factors, one is natural cause and another is being killed by the predators. The greater the density of predators, the more likely it is that an individual prey will be eaten. Suppose the natural per-capita death rate of the scale insect is a constant a_1. Again, the per-capita death rate of prey due to being killed by the predators is a function of the population density of the predators. Let's make the simplest assumption that this per-capita rate of insects being killed is c_1Y

Thus the per-capita death rate of the scale insects is $a_1 + c_1Y$. So the death rate per unit time is $(a_1 + c_1Y)X$. Using the compartmental model, we have

$$\frac{dX}{dt} = b_1X - a_1X - c_1XY.$$

Obviously, the per-capita death rate for the predators (the ladybird beetles) is independent of the prey density. So we assume it to be a constant a_2. For the birth rate of predators, it is interesting to observe that it increases with availability of more food, i.e., the population density of prey. Therefore the birth-rate for the predators is the sum of a natural rate and an additional rate that is proportional to the rate of prey killed. Let the per-capita natural birth rate of predators is a constant say b_2. Thus natural birth rate of predators is b_2Y . Again from the above discussion we can see, the rate at which the prey insects are eaten by the beetles is c_1XY . Hence the additional rate of birth, which is proportional to the rate of prey killed, may be assumed to be fc_1XY . Therefore we have

$$\frac{dY}{dt} = b_2Y - fc_1XY - a_2Y.$$

Assuming $\beta_1 = b_1 - a_1, -\alpha_2 = b_2 - a_2$ and $c_2 = fc_1$, we have

$$\frac{dX}{dt} = \beta_1X - c_1XY$$
$$\frac{dY}{dt} = c_2XY - \alpha_2Y \qquad (3.84)$$

This system of equations is known as the *Lotka–Volterra prey- predator system.*

The parameters c_1 and c_2 are known as interaction parameters as they describe the manner in which the populations interact. Since there are positive and negative terms on the RHS of each differential equation, it is natural to anticipate that the populations could either increase or decrease. These differential equations are coupled since each differential equation depends on the solution of the other. The differential equations are also non-linear since they involve the product XY . One interpretation of the product *XY* is that it is proportional to the rate of encounters (contacts) between the two species.

For this two-species model, we would expect that, in the absence of any predators, the prey would grow without bound (since we have not included any growth limiting effects other than the predators). Also, in the absence of prey, we would expect the predators to die out.

3.21 Competition Model

Here we study behaviour of two competing species who are up against each other for limited resources like food or territory in their ecosystem. This phenomenon has two interesting facets: one is exploitation, when the competitor uses the resource itself and the other is interference, where the population tries to prevent its competitor from utilising the same resource.

Initial assumptions:

Our basic assumptions are as follow.

- We assume the populations to be sufficiently large so that random fluctuations can be ignored.
- The ecosystem has only two competing populations.
- Each population grows exponentially in the absence of the other competitor population.

Let $X = X(t)$ and $Y = Y(t)$ be the two population densities (number per unit area) at any time t. Let β_1 and β_2 are their respective per-capita birth rates. Unlike in the predator-prey model as we have seen before, neither population is dependent on the other as far as growth rates are concerned. Hence we can assume β_1 and β_2 to be constant. On the other hand, the two populations are competing for the same resource. Therefore, the density of each population has a restraining effect on the other. Suppose the per-capita death rate for Y is proportional to X, and that for X is proportional to Y. So we can write death rate of species X is $(c_1 Y)X$ and death rate of species Y is $(c_2 X)Y$, where c_1 and c_2 are the constants of proportionality for this restraining effect.

Hence our model becomes

$$\frac{dX}{dt} = \beta_1 X - c_1 XY$$

$$\frac{dY}{dt} = \beta_2 Y - c_2 XY \tag{3.85}$$

These equations are known as *Gause's equations* and are a coupled pair of first-order, non-linear differential equations.

Remark 3.21.1. *This system has striking similarity with the predator-prey model of the section 3.20 although the terms describing the interaction between the species differ.*

3.22 More Worked out Examples

Example 3.22.1. *Let us consider the following system of differential equations*

$$\begin{aligned} \frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= \quad -3y \end{aligned} \tag{3.86}$$

Solution. Clearly the given system can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.87}$$

i.e.,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.88}$$

where $A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$

Clearly the only equilibrium point is (0, 0). In order to find the general solutions of the system (3.88), we find the eigen values and eigen vectors of the matrix *A*. To find the eigen values, we have

$$\det(A - \lambda I) = (\lambda + 1)\,(\lambda + 3) = 0, \text{ i.e., } \lambda = -1, -3.$$

Thus the eigen values are real and both of negative sign. Hence the equilibrium point is a stable node.

Example 3.22.2. *Consider the combat model, discussed in section 3.10, given by*

$$\begin{aligned}\frac{dR}{dt} &= -a_1 B \\ \frac{dB}{dt} &= -a_2 R\end{aligned} \tag{3.89}$$

where a_1 and a_2 are positive constants. Determine the nature of the equilibrium point(s).

Solution. Clearly the equations (3.89) may be rewritten in matrix form as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{3.90}$$

where $\boldsymbol{A} = \begin{pmatrix} 0 & -a_1 \\ -a_2 & 0 \end{pmatrix}$. Clealy the only equilibrium point is (0, 0). Now the eigen values of **A** are $\pm\sqrt{a_1 a_2}$, i.e., the eigen values are real and of opposite signs. Hence the equilibrium point is a saddle point.

Example 3.22.3. *Determine the nature of the equilibrium points of the prey-predator model given by*

$$\begin{aligned}\frac{dX}{dt} &= \beta_1 X - c_1 XY \\ \frac{dY}{dt} &= c_2 XY - \alpha_2 Y\end{aligned} \tag{3.91}$$

Solution. In example 3.14.1, we have seen the equilibrium points of the prey-predator model described in section 3.20 are (0, 0) and $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$. Further we have linearized the system at these equilibrium points in example 3.15.2.

When the equilibrium point is (0, 0), the linearized system is

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \mathbf{J}_1 \begin{pmatrix} X \\ Y \end{pmatrix}$$

where $\mathbf{J}_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}$.

The eigen values of $\mathbf{J}_1$ are $\beta_1 > 0$ and $-\alpha_2 < 0$. Hence the equilibrium point (0, 0) is a saddle point.

When the equilibrium point is $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$, the linearized system is

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \mathbf{J}_2 \begin{pmatrix} X \\ Y \end{pmatrix}$$

where $\mathbf{J}_2 = \begin{pmatrix} 0 & -\frac{c_1\alpha_2}{c_2} \\ \frac{\beta_1 c_2}{c_1} & 0 \end{pmatrix}$.

The eigen values of $\mathbf{J}_2$ are $\pm i\sqrt{\alpha_2\beta_1}$, i.e., purely imaginary. Hence the equilibrium point $\left(\frac{\alpha_2}{c_2}, \frac{\beta_1}{c_1}\right)$ is a centre.

Example 3.22.4. *Find the time T required for the population with exponential growth to double.*

Solution. Clearly $X(t + T) = 2X(t)$. Then $\frac{X(t+T)}{X(t)} = 2 = \frac{x_0 e^{r(t+T)}}{x_0 e^{rt}}$ by equation (3.76). Hence $T = \frac{\ln 2}{r}$.

3.23 Summary

This chapter introduces and deals with continuous modeling. Notion of compartmental modeling has been discussed. Several physical and real world phenomena including carbon dating, oscillation, spreading of infections etc. are discussed from a continuous modeling approach. Equilibrium points or steady state solutions have been discussed. Learners have also learned about of linearization techniques and stability analysis without actually solving the problem.

3.24 Exercises

Exercise 3.24.1. *Linearize the differential equation*

$$\frac{dx}{dt} = x^2 - 13x + 36$$

at their points of equilibrium.

Ans. $\dfrac{d\xi}{dt} = \pm 5\xi$

Exercise 3.24.2. *Linearize the system of differential equations*

$$\frac{dX}{dt} = X - XY$$

$$\frac{dY}{dt} = XY - Y$$

Ans. At (0, 0),

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

At (1, 1),

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X-1 \\ Y-1 \end{pmatrix}$$

Exercise 3.24.3. *Find the nature of the critical points of the following system of ODEs*

$$\frac{dx}{dt} = -x - y$$

$$\frac{dy}{dt} = 2x - 3y$$

Ans. Stable focus (Eigen values of the coefficient matrix are $-2 \pm i$).

Exercise 3.24.4. *What is the growth rate of a population?*

Exercise 3.24.5. *What is the carrying capacity of a population?*

Exercise 3.24.6. *Draw the graph of exponential growth of a population.*

Exercise 3.24.7. *For a population with exponential growth and growth rate r, find the time required for the population to grow three times.*

Ans. $\frac{\ln 3}{r}$

Exercise 3.24.8. *Establish the model of exponential growth of a population.*

Exercise 3.24.9. *What is the per capita death rate of a population with logistic growth?*

Exercise 3.24.10. *Establish the logistic growth model.*

Exercise 3.24.11. *Draw the graph of logistic growth model.*

Exercise 3.24.12. *Find the equilibrium point(s) of the system given by equation (3.79). If* $r > 0$ *and* $k > 1$, *then find the nature of the equilibrium points.*

Ans. 0 and k. Both are unstable.

Exercise 3.24.13. *Linearize the system represented by the differential equation*

$$\frac{dx}{dt} = rX\left(1 - \frac{X}{K}\right)$$

Ans. $\frac{dx}{dt} = rX$ at $X = 0$ and $\frac{dx}{dt} = r(k - 1)(X - k)$ at $X = k$.

Exercise 3.24.14. *Establish the Gomperzian growth model.*

Exercise 3.24.15. *Compare the growth rates in exponential, logistic and Gomperzian growth models.*

Exercise 3.24.16. *Draw the graph of Gomperzian growth model.*

Exercise 3.24.17. *Draw the comparative graphs among exponential, logistic and Gomperzian growth models.*

Exercise 3.24.18. *What is per- capita birth rate?*

Exercise 3.24.19. *What is per- capita death rate?*

Exercise 3.24.20. *Establish Lotka–Volterra prey- predator system with its initial assumptions.*

Exercise 3.24.21. *Find the equilibrium points of the Lotka–Volterra prey-predator system given by*

$$\frac{dX}{dt} = X - XY$$

$$\frac{dY}{dt} = XY - Y$$

Ans. (0, 0), (1, 1)

Exercise 3.24.22. *What are the basic (initial) assumptions of the Competition model?*

Exercise 3.24.23. *Establish Competition model with its initial assumptions.*

Exercise 3.24.24. *Find the points of equilibrium of the competition model represented by system of equations* (3.85).

Ans. (0, 0) and $\left(\frac{\beta_2}{c_2}, \frac{\beta_1}{c_1}\right)$

Exercise 3.24.25. *Linearize the competition model represented by system of equations* (3.85).

Ans. $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$ at (0, 0) and $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c_2\beta_2}{c_2} \\ -\frac{c_2\beta_1}{c_1} & 0 \end{pmatrix}\begin{pmatrix} X - \frac{\beta_2}{c_2} \\ Y - \frac{\beta_1}{c_1} \end{pmatrix}$

at $\left(\frac{\beta_2}{c_2}, \frac{\beta_1}{c_1}\right)$.

Unit 4 ❑ Further Models

Structure

4.0 Objectives

4.1 Introduction

4.2 Heat flow through a small thin rod

4.3 Wave equation: Vibrating string

4.4 Traffic flow

4.5 Theory of Car-following

4.6 Crime Model

4.7 More worked out examples

4.8 Summary

4.9 Exercises

4.0 Objectives

The followings have been discussed here.

- Modeling of heat flow and wave equation using partial differential equations;
- Two different approaches on automobile traffic flow modeling;
- Crime model.

4.1 Introduction

In this unit, first we will see how the heat flows in a thin rod which is entirely insulated except at the two ends and which has no source of heat within. Later on, two different approaches on automobile traffic flow modeling and then a model about evolution of crime in a certain region will be discussed.

4.2 Heat flow through a small thin rod

In this section, we will try to understand the flow of heat in a thin rod which is entirely insulated except at the two ends and which has no source of heat within. Suppose we have a thin rod that is given an initial temperature distribution, then insulated on the sides. The ends of the rod are kept at the same fixed temperature; e.g., suppose at the start of the experiment, both ends are immediately plunged into ice water. We are primarily trying to understand how the temperature along the rod varies with time. Suppose that the rod has a length l (in meters). We set up a coordinate system along the rod as illustrated in figure 4.1.

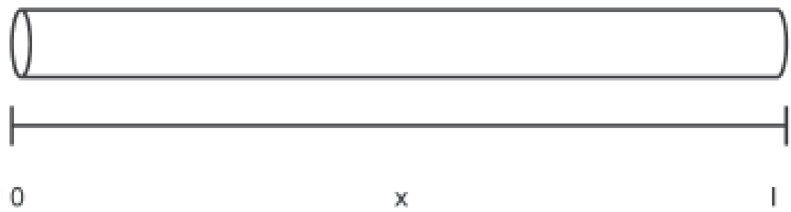


Figure 4.1: The variation of temperature in an insulated rod

Now, the heat energy H of a body of mass m can be measured as the following

$$H = msT \tag{4.1}$$

where s is the specific heat i.e. the energy required to raise a unit mass of the substance 1 unit in temperature. Also $T = T(x, t)$ is the temperature of the body. As $\mathbf{ML^2T^{-2}}$ is the dimension of energy, so the dimension of the specific heat is $\mathbf{L^2T^{-2}U^{-1}}$. Here **M**, **L**, **T**, **U** are the dimensions of mass, length, time and temperature respectively.

Now heat flows from of high temperature area to low temperature area. According to the Fourier's law of heat transfer, the rate of heat transfer per unit area i.e., heat transferred per unit time per unit area is proportional to negative temperature gradient. Therefore we have

$$\frac{Rate\ of\ \ heat\ transfer}{area} = -K_0\frac{\partial T}{\partial x} \tag{4.2}$$

where K_0 is said to be the *thermal conductivity* having dimension $\mathbf{MLT^{-3}U^{-1}}$.

Now we consider our rod to be uniform, i.e., the density ρ, specific heat s, thermal conductivity K_0, cross-sectional area A all are invariant throughout the rod. Consider an arbitrary thin slice of the rod of width Δx between x and $x + \Delta x$. The slice is so thin that the temperature throughout the slice is $u(x, t)$. Therefore, by equation (4.1), we have

Heat energy of the slice = $\rho A\Delta x \times s \times T(x, t)$

Using conservation of energy,

change of heat energy of the slice in time Δt = heat in from the left boundary – heat out from the right boundary

Using equation (4.2),

$$s\rho A\Delta xT\left(x,t+\Delta t\right)-s\rho A\Delta xT(x,T)=\Delta tA\left(-K_0\frac{\partial T}{\partial x}\right)_x-\Delta tA\left(-K_0\frac{\partial T}{\partial x}\right)_{x+\Delta x}$$

This implies

$$\frac{T(x,t+\Delta t)-T(x,t)}{\Delta t}=\frac{K_0}{s\rho}\left(\frac{\left(\frac{\partial T}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial T}{\partial x}\right)_x}{\Delta x}\right)$$

Taking $\Delta t, \Delta x \to 0,$ we have our heat equation

$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial x^2}, 0<x<l \tag{4.3}$$

where

$$\kappa=\frac{K_0}{s\rho} \tag{4.4}$$

is called the *thermal diffusivity* with dimension $\mathbf{L^2/T}$. It depends on the thermal conductivity of the material composing the rod, the density of the rod, and the specific heat of the rod.

Now initially the temperature distribution in the rod is $T(x, 0) = f(x)$, $0 < x < l$, say. This gives us the *initial condition.*

Since we have assumed both the ends of the rod is always kept at same temperature 0^o C, so $T(0, t) = T(l, t) = 0$, $t>0$. This gives us the *boundary conditions*.

4.3 Wave equation: Vibrating string

We will discuss here the derivation of the wave equation in one dimensional space. We will be modeling the vibrations of a wire or a string that is stretched between two points. A violin string is a very good example.

The derivation

We assume the string is stretched from $x = 0$ to $x = L$. We are looking for the function $u(x, t)$ that describes the vertical displacement of the wire at position x and at time t. We assume the string is fixed at both endpoints, so $u(0, t) = u(L, t) = 0$ for all t. We will ignore the force of gravity, so at equilibrium we have $u(x, t) = 0$ for all x and t. This means that the string is in a straight line between the two fixed endpoints.

To derive the differential equation that models a vibrating string, we have to make some simplifying assumptions. In mathematical terms the assumptions amount to assuming that both $u(x, t)$, the displacement of the string, and $\frac{\partial u}{\partial x}$, the slope of the string, are small in comparison to L, the length of the string.

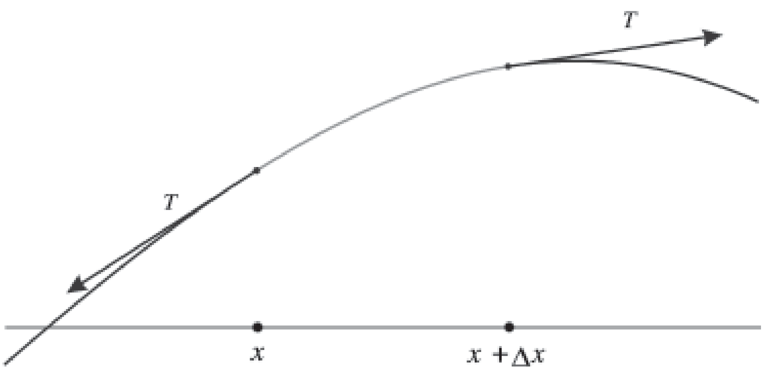


Figure 4.2: The forces acting on a portion of a vibrating string

Consider the portion of the string above the small interval between x and $x + \Delta x$, as illustrated in Figure 4.2. The forces acting on this portion come from the tension T in the string. The tension is a force that the rest of the string exerts on this particular part. For the portion in Figure 4.2, tension acts at the endpoints. We assume that the tension is so large that the string acts as if it were perfectly flexible and can bend without the requirement of a bending force. With that assumption, the tension acts tangentially to the string.

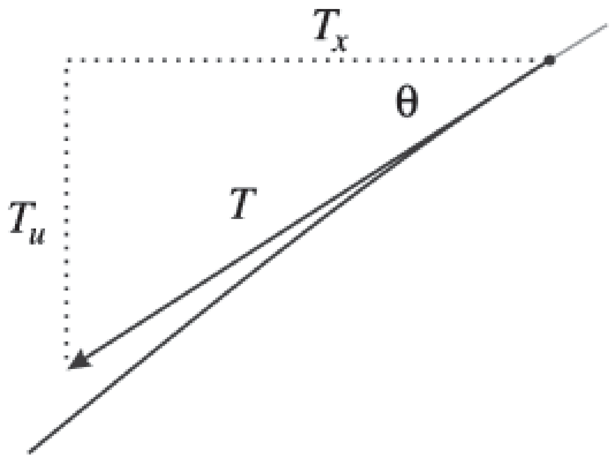


Figure 4.3: The resolution of the tension at the point *x*

The tension at the point x is resolved into its horizontal and vertical components in Figure 4.3. We are assuming that the positive direction is upward. The vertical component is $T_u = -T \sin\theta$, and the horizontal component is $T_x = -T \cos\theta$ ¸. The slope of the graph of *u* at the point *x* is $\frac{\partial u}{\partial x} = \tan\theta$ ¸.

We are assuming that the slope is very small, so θ is small. Therefore $\cos\theta \approx 1$ and $\tan\theta \approx \sin\theta$. As a result, we have

$$T_u \approx -T\frac{\partial u}{\partial x}(x, t)$$

and

$$T_x \approx -T$$

In a similar manner, we find that horizontal component of the force at $x + \Delta x$ is approximately *T*, which cancels the horizontal component at *x*. More interesting is the fact that the vertical component of the force at $x + \Delta x$ is approximately $T\frac{\partial u}{\partial x}(x + \Delta x, t)$. So the total force acting in the vertical direction on the small portion of the string is $F \approx T\left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right)$.

The length of the segment of string is close to Δ*x*. If the string is uniform and has linear mass density ρ, then the mass of the segment is $m = \rho\Delta x$. The acceleration of the segment in the vertical direction is $\dfrac{\partial^2 u}{\partial t^2}$.

By Newton's second law, we have $F = ma$. This translates into

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right)$$

Dividing by Δx and taking the limit as Δx goes to zero, we have

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right) = T \frac{\partial^2 u}{\partial x^2}$$

If we set $c^2 = \dfrac{T}{\rho}$, the equation becomes

$$u_{tt} = c^2 u_{xx} \tag{4.5}$$

This is the wave equation in one space variable. The constant c has dimension length/time, so it is a velocity.

4.4 Traffic Flow

When one thinks of modeling automobile traffic, it is natural to reason from personal experience and to visualize the car and driver as a coupled system, the driver responding to the surrounding vehicles and operating the car to make it become a part of the flow of freeway and city traffic. Thus the traffic is not just a mechanical process but one in which human decisions are involved, decisions which we have all experienced and can understand.

In our study of traffic, we shall however step back from this personal view to take a broader perspective. Let us think of a traffic helicopter pilot looking down on a metropolitan highway grid.

Looking at four miles of highway, the pilot will see a line of cars moving with various speeds. On some stretches, the traffic may be light and fast, on other stretches heavy and slow. To this observer, the individual vehicles are not as important as the sense of overall flow of the cars. The reason why the cars in the lighter traffic move faster is clear to any driver, but to the observer in the helicopter, it seems to be a property of the spacing of the cars. The closer the cars are together, the slower they move. Models of traffic flow try to exploit these observations and use them to formulate a set of assumptions to produce relevant models. The purpose of these models is to understand the peculiar and often frustrating experience of daily driving.

In the scenario, the cars are viewed in the large, almost as a moving gas or liquid. This kind of picture we will call a *continuum model of traffic flow*. In this section, We shall focus on this point of view. There is however another kind of traffic theory based upon the point of view of the individual driver responding to surrounding traffic- just the way we would naturally want to think about driving. This kind of study is called *car following theory* which we will discuss later.

Formulation

Ultimately the traffic engineer is interested in how fast cars move through the traffic grid. Every car has a speedometer, and we all want to know how long it will take to go from location *A* to location *B*.

Certainly, one of the main quantitative measures of traffic is the speed of the cars in the traffic. Consider, for the sake of argument, a one-lane highway with cars in a line moving in the same direction. Since there is no passing, and cars cannot move through each other, the order of the cars is preserved, although they can move at slightly different speeds.

Let the velocity of the *i*–th car be u_i. If the *x*–axis coincides with the road and the position of this car is $x_i(t)$ at time *t*, then we have

$$u_i = \frac{dx_i}{dt} \tag{4.6}$$

Any discussion of traffic on our single- lane road must deal with a collection of vehicles, with positions $x_i(t)$, $i = 1, 2, \ldots, N$ and velocities $u_i \frac{dx_i}{dt}$, $i = 1, 2, \ldots, N$. The continuum approach to traffic takes the view that this collection of discrete objects should be replaced by a "moving continuum", a kind of fluid of vehicles. Such a fluid has a velocity at every value of *x* and at every time *t*, and so we may define a *velocity field* by a function $u(x, t)$. The idea is that the variation of $u(x, t)$ with *x* should be on a scale of length (say, a hundred yards) which is large compared to the size of a typical vehicle.

Thus the value of $u(x, t)$ at a certain time t^* and a certain position x^* on the road should be the velocity of cars on that particular part of the road at that time.

If we know the velocity field for our road, how do we find the movement of an individual car? First we must specify the car. One way to do that is to choose a particular time, say $t = t_0$, and a particular position on the road, say $x = x_0$, , and identify a car as being at that spot at that time. If we then want to know where this car is located at time $t > t_0$, we must use our knowledge of the velocity field, which tells us how fast any car is going when at position x and time t. Thus if $x(t)$ is the position of our car, we know that $x(t_0) = x_0$ but also that

$$\frac{dx}{dt} = u\left(x(t), t\right) \tag{4.7}$$

This last equation is the crucial one, since it relates the overall velocity field to the function $x(t)$ for the particular car which was located at x_0 at time t_0. We will call $x(t)$ the *Lagrangian coordinate* of the car. Note that the problem of locating the position of our car, summarized as

$$\frac{dx}{dt} = u\left(x(t), t\right), \text{ where } x(t_0) = x_0 \tag{4.8}$$

where $u(x(t), t)$ is a given function, amounts to solving an ordinary differential equation of first order with an initial condition at the time t_0.

4.5 Theory of Car-following

We now introduce car-following theory. This model is in contrast to the previously discussed continuum model. We assume a given vehicle responds only to the car immediately in front of it (again restricting ourselves to the case of a single lane with no passing). One useful approach is to assume that the n–th car responds to the car in front of it, i.e., $(n+1)$–th car, according to the difference of their two velocities u_n and u_{n+1} respectively.

Let a fraction λ of the *velocity difference* of the two cars be eliminated by acceleration (or deceleration) of the n–th car. Clearly deceleration will apply if $u_n > u_{n+1}$. If a_n is acceleration, we should have

$$a_n = -\lambda\left(u_n - u_{n+1}\right) \tag{4.9}$$

In terms of car positions,

$$\frac{d^2x_n}{dt^2}(t) = -\lambda\left(\frac{dx_n}{dt}(t) - \frac{dx_{n+1}}{dt}(t)\right) \tag{4.10}$$

A somewhat more accurate model is to take into account a time delay *T* of the response of the driver in the *n*–th car

$$\frac{d^2x_n}{dt^2}(t+T) = -\lambda\left(\frac{dx_n}{dt}(t) - \frac{dx_{n+1}}{dt}(t)\right) \tag{4.11}$$

If all cars move at the same speed *u* and are equally spaced at a distance *d* apart, so that *d* + *L* is the front to front distance between cars (*L* = car length), then integrating (4.10) we have,

$$u = -\lambda\left(x_n - x_{n+1}\right) + C = \lambda(L+d) + C = \frac{\lambda}{\rho} - \frac{\lambda}{\rho_{\max}} \tag{4.12}$$

where $\rho = \dfrac{1}{L+d}$ is the *uniform car density*.

Here we have chosen the constant of integration to make $u = 0$ at $\rho = \rho_{\max}$. This gives us a velocity-density relation from a car-following theory. Since it goes to infinity as $\rho \to 0$, we need to again cut this off and take

$$u(\rho) = \begin{cases} u_{\max} & \text{for } 0 < \rho < \rho_{\min} \\ \lambda\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_{\max}}\right) & \text{for } \rho_{\min} < \rho < \rho_{\max} \end{cases} \tag{4.13}$$

Here $\rho_{\min}$ is defined in terms of $u_{\max}$, $\rho_{\max}$, λ by $u_{\max} = \lambda\left(\dfrac{1}{\rho_{\min}} - \dfrac{1}{\rho_{\max}}\right)$

Let's examine the likely value of λ. It is useful here to deal with the unit feet and seconds, since we are talking about interactions between cars on the scale of seconds. It would seem reasonable to assume that a driver would try to eliminate the velocity difference in about 5 seconds, or about $\dfrac{1}{5}$–th of the difference per unit time, making $\lambda = \dfrac{1}{5}$. To see how this plays out in a driving situation, we consider the following example.

Example 4.5.1. *Suppose that the n–th and* (n+1)*–th cars both are moving at* 100 *ft/ sec and* t = 0 *are separated by* 200 *ft., with n–th car at* x = 0. *At this moment, the* (n+1)-*th* car begins a constant deceleration, so that

$$u_{n+1}(t) = 100 - 20t \tag{4.14}$$

So it will come to a stop in five seconds. We shall neglect the reaction time of the n–th driver (i.e., the delay T). Find the position of the n–th car.

Solution. The equation (4.11), with $\lambda = \frac{1}{5}$, gives

$$\frac{d^2 x_n}{dt^2}(t) + \frac{1}{5}\frac{dx_n}{dt}(t) = \frac{1}{5}(100 - 20t) \tag{4.15}$$

Using the conditions $x_n(0) = 0$ and $\frac{dx_n}{dt}(0) = 100$, we get (verify!)

$$x_n(t) = 200t - 10t^2 + 500\left(e^{-\frac{t}{5}} - 1\right) \tag{4.16}$$

Remark 4.5.1. *Also we see by an integration on equation (4.14), using* $x_{n+1}(0) = 200$,

$$x_{n+1}(t) = 100t - 10t^2 + 200 \tag{4.17}$$

At t = 5 *seconds the* (n + 1)*–th car has come to rest at* x_{n+1} = 450 *feet. We can see that n–th car is still moving and in fact will collide with* (n+1)*–th car shortly after* 5 *seconds unless the n–th driver hits the brakes harder and harder into the stop.*

4.6 Crime Model

In this section, we discuss a model that describes the evolution of crime in a certain area. We will take consideration of two different types of criminals, serious and minor. Let $\rho_1(t)$ and $\rho_2(t)$ be the respective number of serious and minor criminals active in an area at time t. We also assume the behaviour of the criminals is driven by a quantity which we will refer to as the *attractiveness* of the area. One may think of the attractiveness as an indicator of how probable it is for a criminal to act at a specific time. The attractiveness of area depends not only on the behaviour of the active criminals but also on other factors such as time, characteristics of the area examined, or the type of crime committed. With this in mind, we split the attractiveness as follows:

$$\text{Attractiveness} = A(t) + B(t)$$

where $A(t)$ denotes the 'intrinsic' part of the attractiveness that depends on factors other than the behaviour of criminals and $B(t)$ represents the 'dynamic' part of the attractiveness that is caused by criminal activity. To be more concrete, let us suppose that knowledge of crimes being committed in an area tends to encourage more crimes to take place. This effect would then be represented by the dynamic part $B(t)$. Conversely, if the number of police officers patrolling a certain area changes according to the number of crimes taking place, that would be a negative effect represented again by $B(t)$. On the other hand, changes in attractiveness due to factors not affected by criminal activity (e.g. time of day or seasonality) will be accounted for by the intrinsic attractiveness $A(t)$.

We will now discuss the behaviour of the criminals ρ_1 and ρ_2. Let us assume that at a certain time t, a number of individuals commit a crime. Some of those are arrested and therefore removed from the system, whereas others appear in the system, perhaps due to release from prison or through people becoming criminals. We first consider how the number of criminals evolve. We take the rate of lost criminals, through arrest and conviction, to be a constant multiple of the rate at which crimes are committed, namely $k_i\rho_i(A + B)$, $i = 1, 2$. Because of the way attractiveness is defined, we assume that the total number of crimes of type i committed at time t is proportional to the product of the total attractiveness by the number of criminals, resulting in a contribution to the rate of change of the form

$$-k_i c_i \rho_i \left(A(t) + B(t)\right), \; i = 1, 2$$

where each k_i, c_i are constants of proportionality. We also assume the number of new serious and minor criminals in that area at any time t to be $\gamma_1(t)$ and $\gamma_2(t)$ respectively. Hence we can write

$$\begin{aligned} \frac{d\rho_1}{dt} &= \gamma_1 - k_1 c_1 \rho_1 (A + B) \\ \frac{d\rho_2}{dt} &= \gamma_2 - k_2 c_2 \rho_2 (A + B) \end{aligned} \tag{4.18}$$

Let us now examine the behaviour of the dynamic part of the attractiveness i.e., $B(t)$. Every crime, that is committed, increases $B(t)$. Therefore we assume the dynamic attractiveness is boosted by a term proportional to the total number of crimes of both categories committed. We use the term

$$(\lambda_1\rho_1 + \lambda_2\rho_2)(A+B)$$

to model this boost, where λ_1 and λ_2 are constants. Note that we have implicitly assumed that the dynamic attractiveness $B(t)$ is global rather than local, in the sense that criminals may exchange information about crimes committed. We further assume that B decays exponentially in time. Hence, the evolution equation for this part of the attractiveness is

$$\frac{dB}{dt} = (\lambda_1\rho_1 + \lambda_2\rho_2)(A+B) - \omega B \tag{4.19}$$

where ω is the (constant) decay rate. This equation, together with equations (4.18), forms a 3×3 non-linear coupled system of ODEs.

4.7 More worked out examples

Example 4.7.1. *Consider the units of x to be in miles. On the stretch of road $0 < x < 4$ cars are accelerating from a red light, and the velocity field is found to be $u(x, t) = 10x+30t$ miles per hour where $t > 0$ is measured in hours. What is the Lagrangian coordinate of the car which was located at $x = 1.5$ at time $t = 0$?*

Solution. To answer this we must solve

$$\frac{dx}{dt} = 10x + 30t, \text{ where } x(0) = 1.5 \tag{4.20}$$

Using the integrating factor e^{-10t}, we have $xe^{-10t} = -(0.3 + 3t)e^{-10t} + C$. Using the initial condition $x(0) = 1.5$, we have $x(t) = -(0.3 + 3t) + 1.8e^{10t}$.

Example 4.7.2. *Let the cars' trajectories be given by $x = t^2 + 2tx_0 + x_0$. Note that $x(0) = x_0$, identifying the parameter x_0 as the initial position. Find the velocity field for this flow.*

Solution. To do this first compute the velocity, then use the two equations to eliminate x_0. Thus we have $\frac{dx}{dt} = u = 2t + 2x_0$, where the first equation tells us that $x_0 = \frac{x-t^2}{1+2t}$.

Therefore $u(x, t) = 2t + \frac{2x-2t^2}{1+2t} = \frac{2t+2t^2+2x}{1+2t}$.

4.8 Summary

In this unit, we have learned about heat flow through a small thin rod and wave equation for vibrating string using partial differential equations. We have also discussed the modeling of traffic flow from two different approach, viz., traffic flow model and car following model. Another interesting model about evolution of crime have also been discussed.

4.9 Exercises

Exercise 4.9.1. *What is thermal conductivity? What is its dimension?*

Exercise 4.9.2. *What is thermal diffusivity? What is its dimension?*

Exercise 4.9.3. *Establish the model of Heat flow through a small thin rod.*

Exercise 4.9.4. *Establish the wave equation of a vibrating string.*

Exercise 4.9.5. *Let the cars' trajectories be given by* $x = t^2 + x_0$. *Note that* $x(0) = x_0$, *identifying the parameter* x_0 *as the initial position. Find the velocity field for this flow.*

Ans. $u(x, t) = 2t$

Exercise 4.9.6. *Let the cars' trajectories be given by* $x = t^2 + tx_0$. *Note that* $x(0) = x_0$, *identifying the parameter* x_0 *as the initial position. Find the velocity field for this flow.*

Ans. $u(x,t) = \frac{t^2+x}{t}$

Exercise 4.9.7. *Consider the units of* x *to be in miles. On the stretch of road* $0 < x < 4$ *cars are accelerating from a red light, and the velocity field is found to* *be* $u(x, t) = x + 5t$ *miles per hour where* $t > 0$ *is measured in hours. What is the Lagrangian coordinate of the car which was located at* $x = 2$ *at time* $t = 0$*?*

Ans. $x(t) = -5(t + 1) + 7e^t$

Exercise 4.9.8. *Suppose that the n–th and* $(n+1)$*–th cars both are moving at* 200 *ft/ sec and* $t = 0$ *are separated by* 200 *ft., with n-th car at* $x = 0$*. At this moment, the* $(n+1)$*–th car begins a constant deceleration, so that*

$$u_{n+1}(t) = 200 - 25t$$

Find the position of the n–th car.

Ans. As $(n + 1)$–th car will come to a stop in eight seconds, so $\lambda = \frac{1}{8}$. $x_n = (400 - 25t) - 400e^{-\frac{1}{8}}$.

Exercise 4.9.9. *What is the attractiveness of an area w.r.t the crime model?*

Exercise 4.9.10. *Describe the crime model.*

Unit 5 ❑ Numerical Solution of the model and its graphical representation using EXCEL

Structure

- **5.0** **Objectives**
- **5.1** **Introduction**
- **5.2** **Growth Model: Long–term Behaviour**
- **5.3** **Bank Account Problem**
- **5.4** **Affine Discrete Dynamical System and equilibrium point**
- **5.5** **Antibiotic in the Bloodstream**
- **5.6** **Discrete Logistic Model**
- **5.7** **A Linear Predator–Prey Model**
- **5.8** **A non-Linear Predator–Prey Model**
- **5.9** **Continuous Dynamical Models**
- **5.10** **Euler's Method**
- **5.11** **Logistic Equation**
- **5.12** **System of Differential Equations**
- **5.13** **Summary**

5.0 Objectives

- Define and solve discrete dynamical systems
- Analyse the long–term behaviour of discrete dynamical systems and Continuous dynamical systems numerically and graphically
- Model different scenarios with linear and non-linear discrete dynamical systems and differential equation for continuous dynamical models

5.1 Introduction

The main goal of this chapter is to present different ways of building and analysing mathematical models in a format that can be read by students, not just instructors. This is not a text on how to use Excel. Rather, Excel is seen as a tool to further the goal of building and analysing mathematical models. No prior knowledge or experience with Excel is required to use this text. Excel is chosen as the only software used to implement and analyse models for two main reasons:

1. It is easy to use and almost everyone is familiar with it, so it takes very little time to become comfortable with the software.
2. It is everywhere. Students will have access to Excel for every mathematical modelling project they encounter inside and outside of academics. Each section contains step-by-step instructions for building the models in Excel.

Discrete dynamical systems

Definition 5.1: A dynamical system is simply a system that changes over time. The bacterial growth model is one such example. When time is measured in discrete increments, such as in the bacterial growth model, the system is called a ***discrete dynamical system***.

5.2 Growth Model: Long–term Behaviour

Let's graphically examine the long–term behaviour of a linear dynamical system

$$a_{n+1} = b\, a_n$$

for various values of *b*. For different values of *b* the behaviour of a_n are shown in Table 5.2.1

Table 5.2.1

Value of b	**Behaviour of an**
$b < -1$	Oscillates between positive and negative, $\|a_n\|$ grows without bound
$b = -1$	Oscillates between $-a_0$ and $+a_0$
$-1 < b < 0$	Oscillates between positive and negative, $\|a_n\|$ approaches 0
$b = 0$	$a_n = 0$ for $n > 0$
$0 < b < 1$	a_n approaches 0
$b = 1$	$a_n = a_0$ for all n
$b > 1$	a_n grows without bound

Example 5.2.1. Take a_0 = 0.1.

Working process in EXCEL

1. Rename a blank worksheet **"Linear**" and format it to look like Figure 5.2.1. Copy the formulas in A3:B3 down to row 1 as shown in Table 5.2.2. This will give the first 15 values of a_n $(0 \le n \le 15)$ with b = 0.5 as given in Table 5.2.3. Then draw the graph by using EXCEL as shown in Fig 5.1. It is observed that the graph shows decreasing behaviour for b = 0.5.

Table 5.2.2

	A	B	C
1	n	a_n	b
2	0	0.1	0.5
3	=A2+1	=B2* $C $2	

Table -5.2.3

n	**a_n**	**b**
0	0.1	0.5
1	0.05	
2	0.025	
3	0.0125	
5	0.00625	
5	0.003125	
6	0.0015625	
7	0.00078125	
8	0.000390625	
9	0.000195313	
10	9.76563E-05	
11	5.88281E-05	
12	2.55151E-05	
13	1.2207E-05	
15	6.10352E-06	

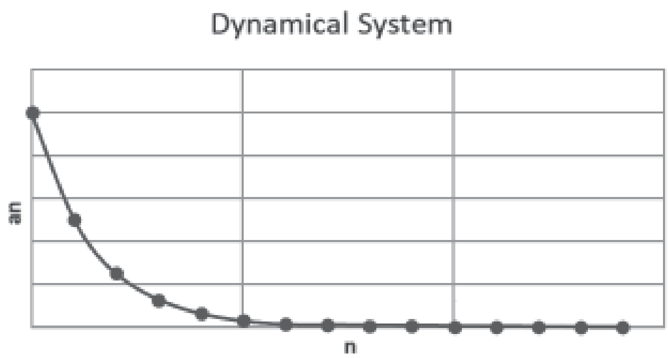


Fig 5.2.1

5.3 Bank Account Problem

Now consider a savings account that pays 5% interest compounded yearly. We know that a model for an account with an interest rate r is

$$a_{n+1} = (1 + r)\, a_n.$$

Example 5.3.1

Take $r = 0.05$, so our model is

$$a_{n+1} = 1.05\, a_n$$

Here the value of b is 1.05

Table -5.3.1

n	a_n	b
0	0.1	1.05
1	0.105	
2	0.11025	
3	0.1157625	
5	0.121550625	
5	0.127628156	
6	0.135009565	
7	0.150710052	
8	0.157755555	
9	0.155132822	
10	0.162889563	
11	0.171033936	
12	0.179585633	
13	0.188565915	
14	0.19799316	
15	0.207892818	

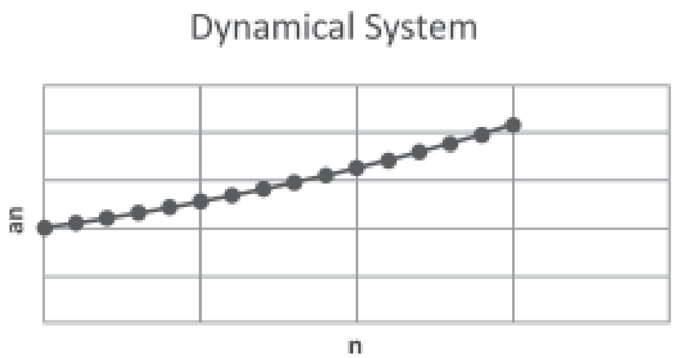


Fig 5.3.1

The increasing behaviour is observed in graph for $b > 1$.

5.4 Affine Discrete Dynamical System and Equilibrium Point

Definition 5.4.1 (Affine Discrete Dynamical System). An *affine discrete dynamical system* is a sequence of numbers { a_n | n = 0, 1,...} described by a relation of the form

$$a_{n+1} = b\, a_n + m$$

where $b \neq 0$. Central to the analysis of the long–term behaviour of any dynamical system are equilibrium values (also called fixed points)

Definition 5.4.2 (Equilibrium Value). A number a is called an *equilibrium value* for the dynamical system $a_{n+1} = f(a_n)$ if $a_n = a$ for all *n* whenever $a_0 = a$. To find equilibrium values, note that if *a* is an equilibrium value, we must have

$$a_{n+1} = a_n = a$$

$$\Rightarrow f(a) = a$$

So finding equilibrium values simply requires us to solve the equation $f(a) = a$. For an affine system, we have

$$a = b.a + m$$

$$\Rightarrow a = m/(1-b)$$

Example 5.4.1

Suppose now that we want to withdraw Rs.2,000 at the end of each year to supplement our income. We want to know how much money we need to deposit now so that we never run out of money.

To answer this question, we will analyse a slightly more general problem:

What happens to the amount in the account in terms of the initial deposit? First we will construct our model. The amount in the account grows at 5% compounded yearly but we are withdrawing Rs.2,000 each year. A dynamic model that describes this scenario is

$$a_{n+1} = 1.05\, a_n - 2000.$$

As before, a_n is the amount in the account at the end of year n. We are also assuming that there is no penalty for withdrawing money each year and that we withdraw the money after the interest from the previous year has been added. This system is an example of an **affine dynamical system**.

Solution:

In this example, b = 1.05 and m = –2000, so the equilibrium value is a = –2000/(1–1.05) = 50,000. Thus, if we start with Rs.50,000 in the account and withdraw Rs.2,000 at the end of each year, we will always have the same amount in the account at the end of each year.

We will take a graphical approach to analyse what happens for initial values other than the equilibrium value of Rs.50,000.

Working process in EXCEL

1. Rename a blank worksheet "**Savings**" and format it as in Table 5.4.2. Copy the range A3:B3 from Table 5.4.1 down to row 27 to model the account over the first 25 years. Now draw the graph fig 5.4.1 using EXCEL.

Table 5.4.1

		B	C	D
1	N	a_n	r	m
2	0	50000	0.05	2000
3	=A2+1	=(1 + $C $2)* B2 – $D $2		

Table 5.4.2

	a_n	**r**	**m**
0	50000	0.05	2000
1	50000		
2	50000		
3	50000		
5	50000		
5	50000		
6	50000		
7	50000		
8	50000		
9	50000		
10	50000		
11	50000		
12	50000		
13	50000		
15	50000		
15	50000		
16	50000		
17	50000		
18	50000		
19	50000		
20	50000		
21	50000		
22	50000		
23	50000		
25	50000		
25	50000		

Fig . 5.4.1

In above example we saw that the long–term behaviour of the system changed quite dramatically with a small change in a_0. In situations like this we say that the system is sensitive to the initial condition.

Also note that if $a_0 \neq 50{,}000$, the system either approaches 0 or increases without bound. The equilibrium value of 50,000 is an example of an unstable or repelling equilibrium. We see this in Table 5.4.3 by taking a_0 = 58000. This is illustrated in the fig 5.4.2. The graph in the figure shows increasing nature.

Table 5.4.3

n	a_n	r	m
0	58000	0.05	2000
1	58500		
2	58820		
3	59261		
5	59725.05		
5	50210.25		
6	50720.77		
7	51256.8		
8	51819.65		
9	52510.63		
10	53031.16		
11	53682.71		
12	55366.85		
13	55085.19		
15	55839.55		
15	56631.53		
16	57563		
17	58336.15		
18	59252.95		
19	60215.6		
20	61226.38		
21	62287.7		
22	63502.09		
23	65572.19		
24	65800.8		
25	67090.85		

Fig. 5.4.2

Next add a scroll bar. Set the linked cell to B2 and the min and max to 0 and 80,000, respectively. This will allow us to vary the value of a between Rs.0 and Rs.80,000 in increments of Rs.1.

Table 5.4.4

n	a_n	r	m
0	30000.00	0.05	2000
1	29500.00		
2	28975.00		
3	28523.75		
5	27855.95		
5	27237.18		
6	26599.05		
7	25929.00		
8	25225.45		
9	25586.72		
10	23711.05		
11	22896.61		
12	22051.55		
13	21153.51		
15	20200.68		
15	19210.72		
16	18171.25		
17	17079.82		
18	15933.81		
19	15730.50		
20	13567.02		
21	12150.37		
22	10757.39		
23	9285.76		
25	7759.00		
25	6136.55		

Fig 5.4.3

Move the slider on the scroll bar left and right and observe how the long–term behaviour of the system changes. Specifically, note that amount eventually decreases to 0 when the deposited amount is less than Rs.50,000, particularly taking Rs.30,000 shown in figure 5.4.3. When the deposited amount in greater than Rs.50,000, the amount grows without bound.

5.5 Antibiotic in the Bloodstream

An infant is given an antibiotic to treat an ear infection. When taking an antibiotic, it is important to keep the amount of the drug in the bloodstream fairly constant. If it gets too low, the bacteria can begin to regrow. If it gets too high, it could cause other complications.

Example 5.5.1 Suppose the half-life of the drug is 1 day (meaning that half the drug remains in the blood after each 1-day period) and a dosage of 0.1 mg is given at the end of each day. We want to examine what happens to the amount of the drug in the bloodstream in the long–run.

Solution: A simple affine model for this system is

$$a_{n+1} = 0.5\ a_n + .1$$

where a_n = the amount of the drug in the blood at the end of day n. Since the problem did not specify the initial dosage, a_0, we need to experiment with different values.

Working process in EXCEL

1. Rename a blank worksheet "**Antibiotic**" and format it as in Table 5.5.2. Copy the range A3:B3 from Table 5.5.1 down to row 15 to model the system from day 0 to day 15.
2. Now draw the graph fig 5.5.1 using EXCEL.
3. Notice that even with an initial dosage of 0 mg, the amount of antibiotic in the blood appears to approach 0.2 mg at the end of each day. Note that this does not mean that the amount eventually equals 0.2 mg at every point in time, only that is equals 0.2 mg at the end of every day.

Table 5.5.1

	A	B
1	n	a_n
2	0	0
3	=A2+1	=0.5*B2+0.1

Table 5.5.2

n	a_n
0	0
1	0.1
2	0.15
3	0.175
5	0.1875
5	0.19375
6	0.196875
7	0.198538
8	0.199219
9	0.199609
10	0.199805
11	0.199902
12	0.199951
13	0.199976
14	0.199988
15	0.199995

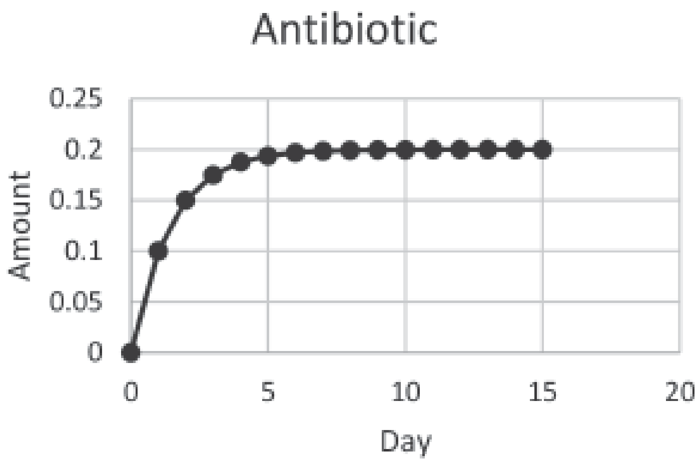


Fig 5.5.1

4. Next, add a scroll bar, set the min to 0, the max to 100, and the linked cell to C1. Add the formula in table 5.5.1 to allow us to vary the initial dosage from 0 to 1 mg in increments of 0.01 mg.

5. Move the slider on the scroll bar left and right and observe the long–term behaviour of the system. Specifically note that when a_0 = 0.2, the system remains at 0.2, meaning that 0.2 is an equilibrium value. Also note that no matter what the value of a_0 is, the system appears to always approach 0.2. This shows that 0.2 is an attracting equilibrium.

5.6 Discrete Logistic Model

Definition 5.6.1 (Discrete Logistic Equation). A discrete logistic equation (also called a logistic map or a constrained growth model) is an equation of the form

$$a_{n+1} = a_n + b(c - a_n)\, a_n$$

where b and c are constants. This type of equation is often used to model population growth where a_n is the population at time n . The constant b is called the **intrinsic growth** rate and c is called the **carrying capacity**

5.6.1 Bacteria Growth model

Example 5.6.1

Table 5.6.1 gives the number of bacteria in a Petri dish, a_n, at the end of each hour n. This data is graphed in Figure 5.6.1. We want to model a_n in terms of n. When modelling a dynamical system, it is often convenient to think about the way the variable(s) change between time periods. Specifically, we consider the change between time periods

$$\Delta a_n = a_{n+1} - a_n$$

The values of Δa_n for the first 7 values of n are given in Table 5.6.2. Notice that as an increases, Δa_n also increases. This suggests that Δa_n is proportional to a_n, which leads to the equation

$$\Delta a_n = a_{n+1} - a_n = r a_n \tag{5.3}$$

Table 5.6.1

n	0	1	2	3	5	5	6	7	8	9
a_n	10.3	17.2	27.	55.3	80.2	125.3	176.2	255.6	330.8	390.5
n	10	11	12	13	15	15	16	17	18	19
a_n	550	520.5	560.5	600.5	610.8	615.5	618.3	619.5	620	621

Table 5.6.2

n	0	1	2	3	4	5	6
a_n	10.3	17.2	27	45.3	80.2	125.3	176.2
Δa_n	6.9	9.8	18.3	34.9	45.1	50.9	79.4

Working process in EXCEL

Rename a blank worksheet "**Bacteria Population**" and format it as in Figure 5.6.1. Enter the data from Table 5.6.3 in columns A and B and draw the figure 5.6.1

Table 5.6.3

n	a_n
1	10.3
2	17.2
3	27
5	55.3
5	80.2
6	125.3
7	176.2
8	255.6
9	330.8
10	390.5
11	550
12	520.5
13	560.5
15	600.5
15	610.8
16	615.5
17	618.3
18	619.5
19	621

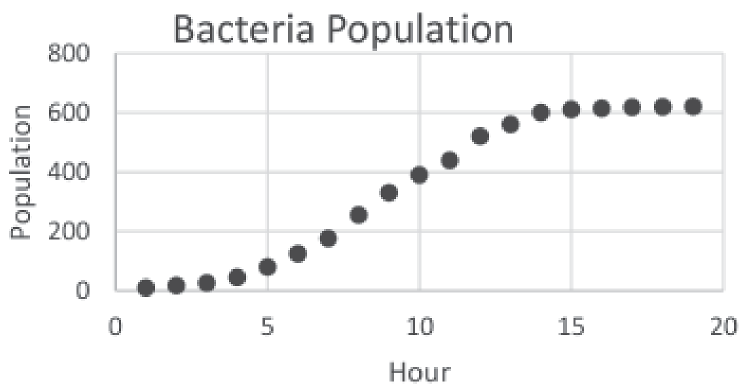


Figure 5.6.1

Example 5.6.2

For **discrete logistic equation**, redefine the above model by introducing carrying capacity. So instead of assuming a constant growth rate r, we assume a growth rate that approaches 0 as the population approaches carrying capacity given by 621. An equation implementing this assumption is given by

$$\Delta a_n = a_{n+1} - a_n = b\left(621 - a_n\right)a_n \qquad (5.6.1)$$

where $b > 0$ is a constant. Solving for a_{n+1} yields the model

$$a_{n+1} = a_n + b(621 - a_n)a_n \qquad (5.6.2)$$

To implement the model (5.5.2) we need to find the value of b. Equation (5.5.1) predicts that $\left(a_{n+1} - a_n\right)$, is proportional to $(621 - a_n)a_n$. If a graph of $\left(a_{n+1} - a_n\right)$ vs $(621 - a_n)a_n$ is approximately a straight line through the origin, then the assumption is reasonable and the slope of the line is the value of *b*.

Working process in EXCEL

1. Rename a blank worksheet "**Bacteria**" and format it as in Table 5.6.5. Enter the data from Table 5.6.1 in columns A and B and copy range D2:E2 down to row 20.

Create a graph of the transformed data in columns D and E of Table 5.6.5 and fit a straight line through the origin as in Figure 5.6.2. We see that the line fits the data well, so our model appears to be reasonable.

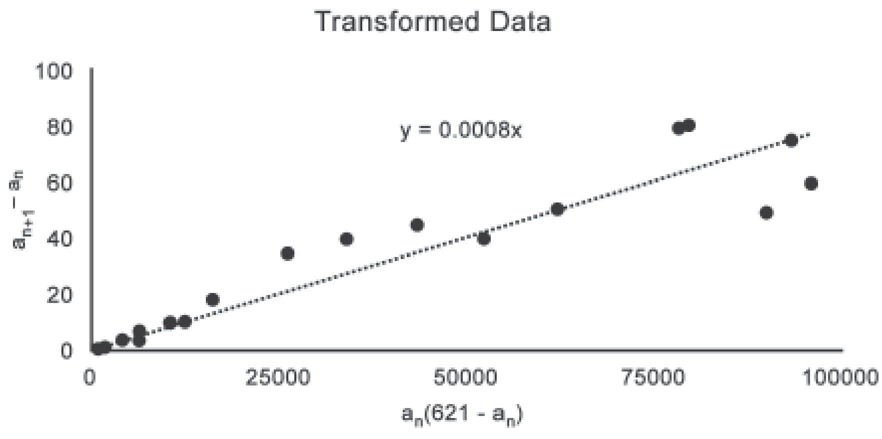


Figure 5.6.2

Using the slope of the line in Figure 5.6.2, our model is

$$a_{n+1} = a_n + 0.0008\left(621 - a_n\right)a_n$$

	C
1	**Predicted**
2	10.3
3	= C2 + 0.0008* (621-C2)*C2

Table 5.6.5

	A	B	C	D	E
1	n	a_n	predicate	an(621 - an)	an+1 - an
2	0	10.3	10.3	=B2*(621-B2)	= B3-B2
3	=A2+1		= C2+0.0008*(621-C2)*C2		

Table 5.6.5

n	an	Predicate	an(621 - an)	Δ an=a(n+1)-a(n)
1	10.3	10.3	6290.21	6.9
2	17.2	15.33217	10385.36	9.8
3	27	22.76113	16038	18.3
5	55.3	33.6555	26079.21	35.9
5	80.2	59.56781	53372.16	55.1
6	125.3	72.08577	62111.21	50.9
7	176.2	103.7509	78373.76	79.5
8	255.6	156.6696	93396.25	75.2
9	330.8	202.3255	95998.16	59.6
10	390.5	270.0925	90026.25	59.6
11	550	355.9153	79650	80.5
12	520.5	522.0392	52352.25	50
13	560.5	589.2156	33960.25	50.1
15	600.5	550.7917	12310.25	10.3
15	610.8	575.5925	6230.16	3.7
16	615.5	596.5539	3995.25	3.8
17	618.3	608.1609	1669.51	1.2
18	619.5	615.5075	929.25	1.5
19	621	617.6579	0	-621

Use the data in columns A, B, and C from Table 5.6.5 to form a graph as in Figure 5.6.3. Notice that the "shape" of the predicted values is relatively close to the shape of the observed values, so the reasonableness of our model is verified.

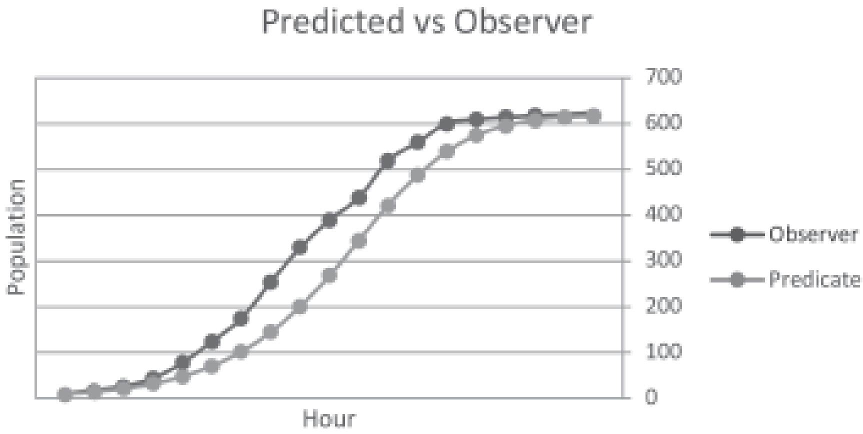


Figure 5.6.3

5.7 A Linear Predator–Prey Model

Consider a forest containing foxes and rabbits where the foxes eat the rabbits for food. We want to examine whether the two species can survive in the long–term. A forest is a very complex ecosystem. So, to simplify the model, we will use the following assumptions:

1. The only source of food for the foxes is rabbits and the only predator of the rabbits is foxes.
2. Without rabbits present, foxes would die out.
3. Without foxes present, the population of rabbits would grow.
4. The presence of rabbits increases the rate at which the population of foxes grows.
5. The presence of foxes decreases the rate at which the population of rabbits grows.

We will model these populations using a discrete dynamical model. Each state of the system consists of the populations of foxes and rabbits at a point in time. Since this state consists of two components, this is a two–dimensional discrete dynamical system. To create our model, we first need to define some variables. Let

F_n = population of foxes at the end of month n

R_n = population of rabbits at the end of month n

As in the bacteria model, the assumptions are stated in terms of rates of change,

$$\Delta F_n = F_{n+1} - F_n \tag{5.7.1}$$

and $$\Delta R_n = R_{n+1} - R_n \tag{5.7.2}$$

There are many ways we could model these rates of change with the assumptions. In this section we will create a linear model. In the next section we will create a nonlinear model. Assumptions 2 and 3 deal with the rates of change of each population in the absence of the other. A reasonable way to model these is to say that the rates are proportional to the populations. This yields the difference equations

$$\Delta F_n = F_{n+1} - F_n = -a\,F_n \tag{5.7.3}$$

$$\Delta R_n = R_{n+1} - R_n = d\,R_n \tag{5.7.4}$$

where both a and d are between 0 and 1. Note that the coefficient of proportionality in (5.7.3) is negative to reflect the fact that the foxes would die out (a negative rate of change) without rabbits. The coefficient in (5.7.5) is positive because the population of rabbits grows (a positive rate of change) without foxes. Now, assumptions 5 and 5 say that these rates in Equations (5.7.3) and (5.7.5) either increase or decrease in the presence of the other species. So, to incorporate these assumptions, we will simply add one term to each of Equations (5.7.3) and (5.7.5) yielding:

$$F_{n+1} - F_n = -a\,F_n + b\,R_n \tag{5.7.5}$$

$$R_{n+1} - R_n = -\,c\,F_n + d\,R_n \tag{5.7.6}$$

where b and c are non-negative. Note that the added term in (5.7.5) is positive to reflect the fact that the presence of rabbits increases the rate at which the population of foxes grows. The added term in (5.7.6) is negative since the presence of foxes decreases the rate at which rabbits grow. Rewriting Equations (5.7.5) and (5.7.6) yields our model in the form of a system of linear equations

$$F_{n+1} = (1-a)F_n + b\,R_n \tag{5.7.7}$$

$$R_{n+1} = -\,c\,F_n + (1+d)\,R_n \tag{5.7.8}$$

Because our model has the form of a system of linear equations, it is called a two-dimensional linear discrete dynamical system. The model could be written in matrix form as

$$\begin{pmatrix} F_{n+1} \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} 1-a & b \\ -c & 1+d \end{pmatrix} \begin{pmatrix} F_n \\ R_n \end{pmatrix} \tag{5.7.9}$$

The parameters (1 – a) and b are called the fox death and birth factors, respectively, while the parameters – c and (1 + d) are called the rabbit death and birth factors, respectively

Working process in EXCEL

Rename a blank worksheet "**Linear Predator–Prey**" and format is as Table 5.7.1. The initial values of the parameters and populations are shown in the Table. Copy row 8 down to row 37 to model 30 months as shown in Table 5.7.2

Table 5.7.1

	A	B	C
1		Factors	
2		Death	Birth
3	Foxes	0.5	0.5
5	Rabbits	-0.17	1.1
6	Month	Foxes	Rabbits
7	0	500	200
8	=A7+1	=$.B $3*B7+C7*$C $3	=B7*$B $5+C7*$C $5

Table 5.7.2

	Birth	Death
Foxes	0.5	0.5
Rabbits	-0.17	1.1

Month	Foxes	Rabbit
0	500	200
1	330	135
2	219	92.5
3	156.56	65.51
5	98.995	55.9528
5	67.87812	33.191
6	57.5267	25.55173
7	33.93505	20.05536
8	25.98577	16.28001
9	19.00539	13.6606
10	15.96655	11.79592
11	12.20159	10.53122
12	10.27328	9.500068
13	8.896667	8.593617
15	7.88578	7.950556
15	7.119109	7.395018
16	6.517161	6.923171
17	6.027859	6.507571
18	5.616953	6.133595
19	5.261915	5.792071
20	5.957785	5.576753
21	5.665595	5.183305
22	5.505619	5.908655
23	5.166271	5.650565
25	3.953361	5.507355
25	3.735622	5.177719
26	3.538399	3.960605
27	3.353551	3.755137
28	3.178775	3.560566
29	3.013615	3.376231
30	2.857299	3.201539

1. Next, plot the graphs . The graphs of rabbits versus month and foxes versus month are called time plots shown in fig (5.7.1) and fig (5.7.2) respectively. The curve in the graph of rabbits versus foxes is called a trajectory of the system shown in Fig 5.7.3. The plane on which a trajectory is drawn is called the phase plane. Notice that the trajectory tends toward the origin (0 foxes and 0 rabbits). This means that both species eventually die out. This is also shown in the time plots. If we change the initial populations, we note that the trajectories always tend toward the origin. This indicates that the populations always die out, regardless of the initial populations. As in a one-dimensional discrete dynamical system, two-dimensional systems can have an equilibrium

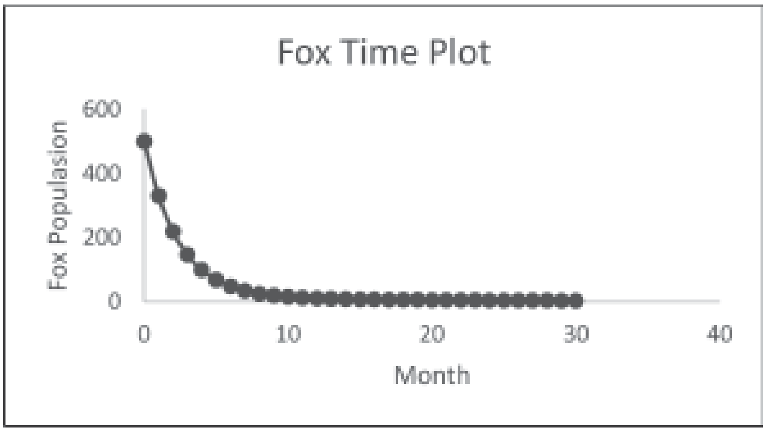


Fig 5.7.1

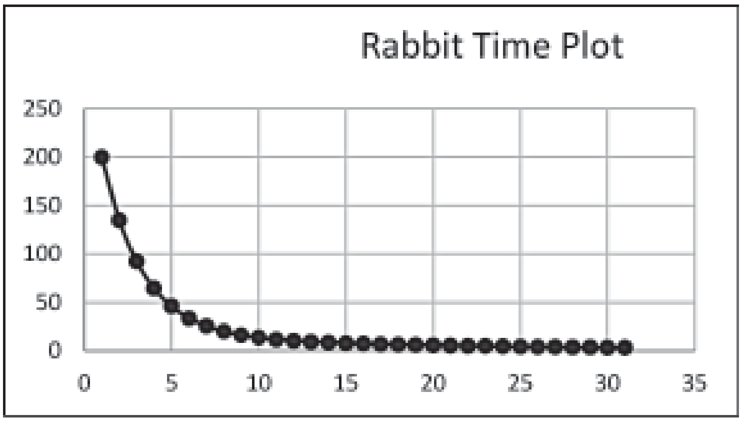


Fig 5.7.2

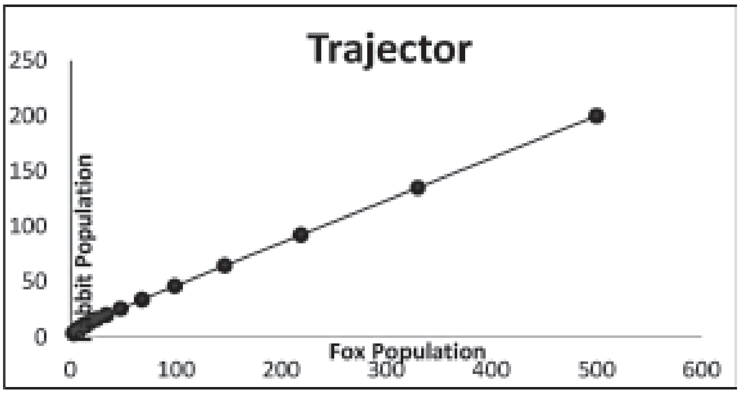


Fig 5.7.3

5.8 A nonLinear Predator–Prey Model

Lotka-Volterra model: Let's consider a similar population of foxes and rabbits along with the same set of assumptions as in previous section, but we will model the assumptions differently. We will start with modelling assumptions 2 and 3 the same way:

$$\Delta F_n = F_{n+1} - F_n = -a\, F_n \tag{5.8.1}$$

$$\Delta R_n = R_{n+1} - R_n = d\, R_n \tag{5.8.2}$$

where $0 < a \leq 1$ and $0 < d \leq 1$. In Section 5.7, the coefficients of F_n and R_n were kept constant. In this section we will model them as increasing or decreasing in the presence of the other population. Assumption 5 says that the presence of rabbits increases the rate of growth of foxes.

so, we write

$$F_{n+1} - F_n = \left(-a + b\, R_n\right) F_n \tag{5.8.3}$$

where $b \geq 0$. Likewise, assumption 5 says that the presence of foxes decreases the rate of growth of rabbits, so, we have

$$R_{n+1} - R_n = \left(d - c\, F_n\right) R_n \tag{5.8.4}$$

where $c \geq 0$. Rewriting (5.7.3) and (5.7.5) we get our model:

$$F_{n+1} = \left(1 - a\right) + b\, R_n\, F_n \tag{5.8.5}$$

$$R_{n+1} = -c\, F_n\, R_n + (1+d)\, R_n \qquad (5.8.6)$$

This type of model is called a **Lotka-Volterra model**, named after the researchers that first devised it in the 1920s and 1930s. Note that both equations have a term involving $R_n F_n$; thus, the model in nonlinear. This term can be interpreted as modelling the number of interactions of the two species. These interactions increase the number of foxes while decreasing the number of rabbits. Also note the similarities between this nonlinear model and the linear model in (5.7.10).

Working process in EXCEL

We will refer to the parameters in this model using the same names as in the linear model. This model can easily be implemented in Excel.

Rename a blank worksheet "**Nonlinear Predator–Prey**" and format it as in Table 5.8.1. Copy row 8 down to row 507 to model 500 months. (Note that the parameters in this model do have similar meanings as in the linear model, but they do have different values. Also we have different initial populations.

Table 5.8.1

A	B	C
1	Factors	
2	Death	Birth
3 Foxes	0.88	0.0001
5 Rabbits	-0.0003	1.039
6 Month	Foxes	Rabbits
7 0	110	900
8 =A7+1	=$B $.3*B7 + B7*C7* $C $3	=B7*C7 $B $5 + C7* $C $5

Create graphs similar to those in Figure 5.7.1. This model predicts that the populations oscillate with the same period of oscillation, but with a phase shift, meaning they don't reach their peaks at the same time. These oscillations cause the spiralling nature of the trajectories in the graph of rabbits versus foxes. Oscillations such as this are actually observed in nature; thus, this model appears to be more reasonable than the linear model.

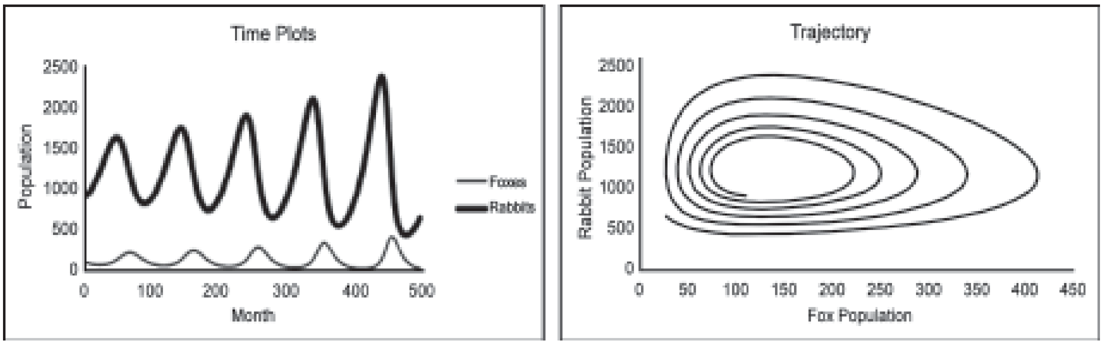


Fig 5.8.1

Now let's calculate the equilibrium point of the system. Suppose (f, r) is an equilibrium point. By definition, this point must satisfy the system of equations

$$f = 0.88f + 0.0001fr$$

$$r = -0.0003fr + 1.039r$$

Assuming that $f \neq 0 \neq r$ yields the solution $f = 130$ and $r = 1{,}200$. Another equilibrium is (0, 0). Note that the point (130, 1200) is at the center of the spiral in the phase plane. If we change the starting populations in the worksheet to 130 foxes and 1200 rabbits we note that the populations do not change, as expected.

To determine if this equilibrium is attracting or repelling, we need to consider starting populations near the equilibrium. Changing the initial populations to 129 foxes and 1201 rabbits yields the trajectory shown in Figure 5.8.2. Notice that the trajectory moves away from the equilibrium. Trying other initial populations yields similar results. The fact that the trajectories move away from the equilibrium is evidence that the equilibrium is repelling.

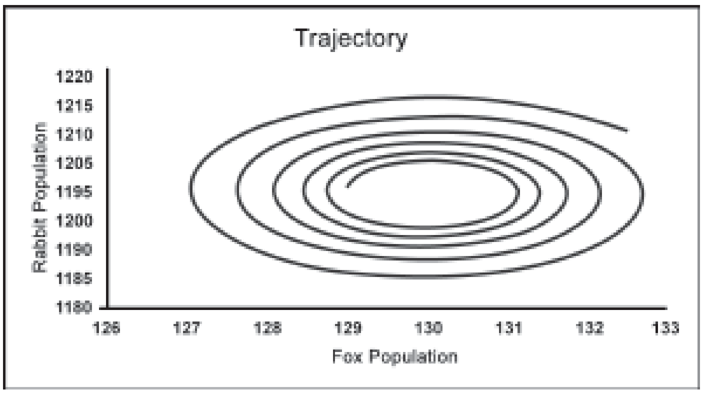


Fig 5.8.2

5.9 Continuous Dynamical Models

In reality, time is continuous so using discrete time units is a simplification. It is a convenient simplification because a difference equation is very easy to solve for a_{n+1} in terms of a_n giving a recursive solution. When measuring time continuously, we describe change with a differential equation. Differential equations are formed in the same basic way as difference equations, but finding their solutions can be much more complicated. To illustrate how differential equations are formed, consider the following observation:

When a hot cup of coffee is set on a desk, it initially cools very quickly. As the coffee gets closer to room temperature, it cools less quickly. This simple observation is an example of Newton's Law of Cooling:

The rate at which a hot object cools (or a cold object warms) is proportional to the difference between the temperature of the object and the temperature of its surrounding medium. This law can be translated into the following differential equation:

$$dy/dt = k(y - T)$$

where $y(t)$ = temperature of the object a time t

T = temperature of the medium (assumed to be constant)

k = constant of proportionality

This differential equation can be solved using basic techniques yielding the general solution:

$$y(t) = T + Ce^{kt}$$

where C is an arbitrary constant.

Example 5.9.1 (Newton's Law of Cooling)

Consider a cup of coffee that is initially 100 ^{o}F, cools to 90 ^{o}F in 10 minutes, and sits in a room whose temperature is a constant T = 60 ^{o}F. The general solution to Newton's Law of Cooling is $y(t) = T + Ce^{kt}$. To find the specific solution in this case we need to find the values of the constants C and k. The initial condition y (0) = 100 gives

$$100 = 60 + Ce^{k.0} \Rightarrow C = 50$$

The condition

$$y\ (10) = 90 \text{ gives } 90 = 60 + 50e^{k.10}$$

$$\Rightarrow k = -0.02877$$

Thus the model is :

$$y(t) = 60 + 50e^{-0.02877t}. \qquad (5.9.1)$$

Working process in EXCEL

A graph of this model is shown in Figure 5.9.1. This curve is called the solution curve.

$$y(t) = 60 + 40e^{-0.02877t}.$$

t	Temp
0	100
10	89.9995
20	82.5992
30	76.8751
50	72.6553
50	69.5913
60	67.1185
70	65.3387
80	65.0039
90	63.0029
100	62.2521
110	61.6891
120	61.2668
130	60.9501
150	60.7125
150	60.5355

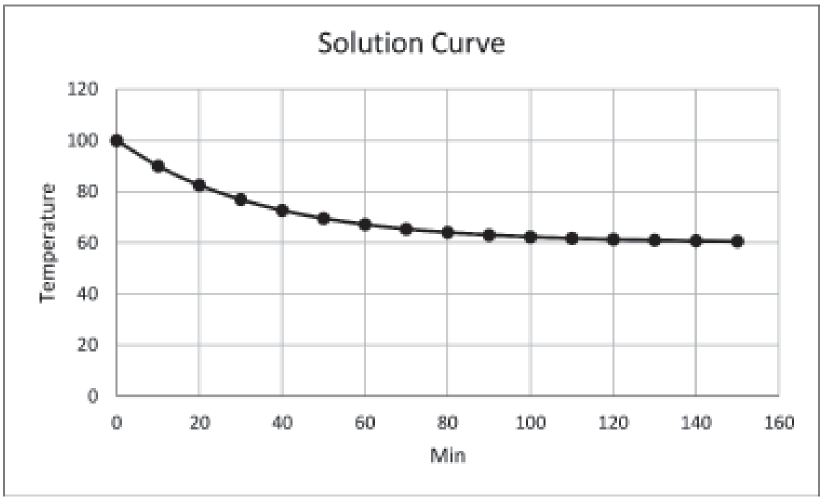


Fig 5.9.1

In this chapter, we do not analytically solve differential equations as done in previous section. Instead, we use a technique called Euler's Method to numerically approximate solution curves and then graphically analyse the results

5.10 Euler's Method

Euler's method is a technique for approximating points on the solution curve of a differential equation. To illustrate the method, consider a differential equation of the form

$$dy/dt = F(y) \qquad (5.10.1)$$

Along with the initial condition $y(t_0) = y_0$ where t_0 and y_0 are some given values. As shown in Figure 5.8.1, the point (t_0, y_0) is a point on the solution curve. Now, let h be some small positive quantity and define time t_1 to be $t_1 = t_0 + h$. Our goal is to approximate the y – coordinate of the point $(t_1, y(t_1))$ on the solution curve

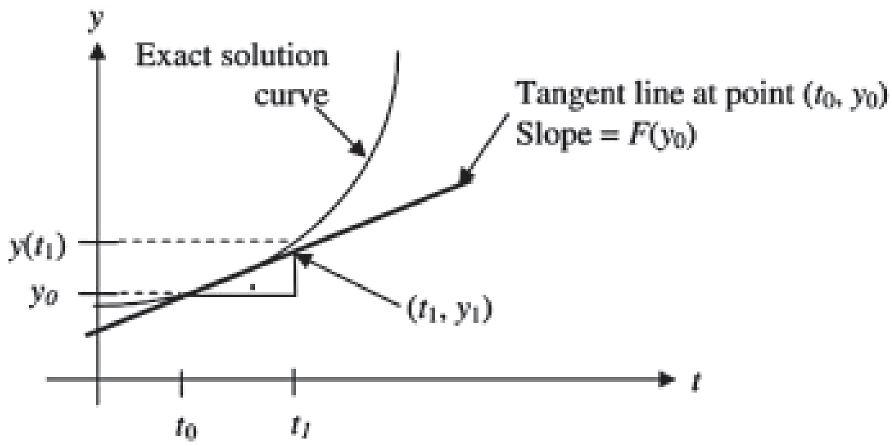


Fig 5.10.1

In the triangle in Figure 5.10.1, the base has length h and the hypotenuse is on a line with slope $F(y_0)$. Therefore, the height is

height = $h\,F\,(y_0)$

The y-coordinate of the base of the triangle is y_0. Thus the y-coordinate of the top of the triangle is

$$y_1 = y_0 + h\,F\,(y_0) \qquad (5.10.2)$$

This y-coordinate is an approximation of y (t_1). To approximate y (t_2) where $t_2 = t_1 + h$, we can repeat this process, replacing y_0 with y_1. We continue to repeat this process as follows:

$$t_1 = t_0 + h \qquad y_1 = y_0 + h\,F(y_0)$$

$$t_2 = t_1 + h \qquad y_2 = y_1 + h\,F(y_1)$$

$$\ldots \qquad \ldots$$

$$t_{n+1} = t_n + h \qquad y_{n+1} = y_n + h\,F(y_n)$$

This algorithm is called *Euler's method*. The results from Euler's method can be interpreted in at least two ways:

a. Numerically: For each $y_n \approx y(t_n)$.

b. Graphically: Each point $(y_n,\ t_n)$ is approximately a point on the solution curve.

Example 5.10.1 (Applying Euler's Method)

Working Process in EXCEL

Euler's method is easy to implement in **Excel**. Here we apply it to the Newton's law of cooling problem in Example 5.9.1 and examine how the value of *h* affects the approximation.

Rename a blank worksheet "**Euler**" and format it as in Table 5.10.1. Copy row 5 down to row 120 to calculate values at 115 different time values.

Table 5.10.1

	A	B
1	h =	0.5
2		
3	Time	Approximate
4	0	100
5	= A4+$B$1	=B4+$B$1*(-0.02877*(B4-60))

Fig 5.10.1

h= 1 Euler's Method

Time	Approximate	
0	100	
1	98.8592	
2	97.73150852	
3	96.65597302	
5	95.59166837	
5	95.56769607	
6	93.57318356	
7	92.60728297	
8	91.66917155	
9	90.75805938	
10	89.8731503	
11	89.01369005	
12	88.17896619	
13	87.36825733	
15	86.58087257	
15	85.81615086	
16	85.07351059	
	…………..	
109	61.66022772	
110	61.61256297	
111	61.56607251	
112	61.52101651	
113	61.57725686	
115	61.53575618	
115	61.39357825	
116	61.35338788	
117	61.31555091	
118	61.27663516	
119	61.23990539	exact
120	61.20523331	61.2668

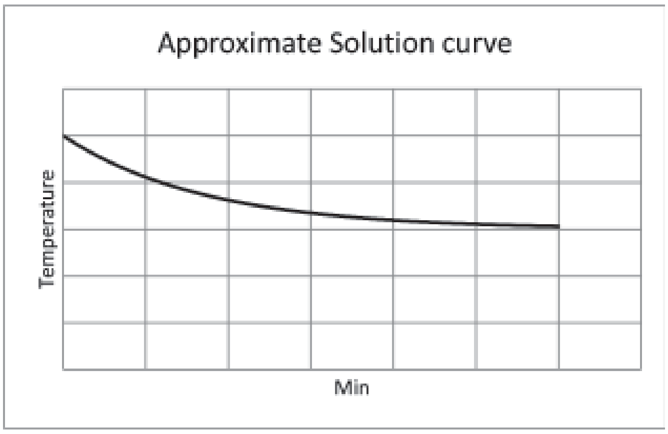


Fig 5.10.1

5.11 Logistic Equation

Here we are trying to explain the Logistic equation with the help of an example.

Example 5.11.1: Suppose that 25 panthers are released into a game preserve. Initially the population grows at a rate of approximately 25% per year, but because of limited food supplies, the preserve is believed to support only 200 panthers. We want to model the population over time. Note that the information given deals with the rate of change. This suggests we create a differential equation to model the rate of change of the population. If $y(t)$ represents the population at year t,

$$dy/dt = 0.25\,y$$

However, this model does not take into account the fact that the preserve can support only 200 panthers. It seems reasonable to assume that the rate of growth will decrease as y approaches 200. One way to model this is

$$\frac{dy}{dt} = 0.25\left(1 - \frac{y}{200}\right)y \qquad (5.11.1)$$

Note that as $y \to 200,\ 1 - \frac{y}{200} \to 0$ meaning that $\frac{dy}{dt} \to 0$. Equation (5.11.1) is called a ***logistic differential equation***. Also note that this logistic differential equation is very similar to the logistic difference equation we derived for the bacteria population in 5.6. The general form of a logistic equation is

$$\frac{dy}{dt} = k\left(1 - \frac{y}{L}\right)y$$

The parameter L is called the carrying capacity and the parameter k is called the unconstrained (or intrinsic) growth rate.

Working Process in EXCEL

To approximate the solution curve of Equation (5.10.1), rename a blank worksheet "**Logistic**" and format it as in Table 5.11.1. Copy row 5 down to row 129 to model 25 years.

Table 5.11.1

	A	B
1	h =	0.2
2		
3	Year	Population
4	0	25
5	= A4+$B$1	=B4+$B$1*(0.25*(1-B4/200)*B4)

Next, create a graph as in Fig 5.10.1. Figure 5.10.1 shows that the rate of growth slows down as the population approaches 200, as expected. The population reaches the carrying capacity by year 25. Also note that this graph looks very similar to the graph of the bacteria population in Example 5.3.1

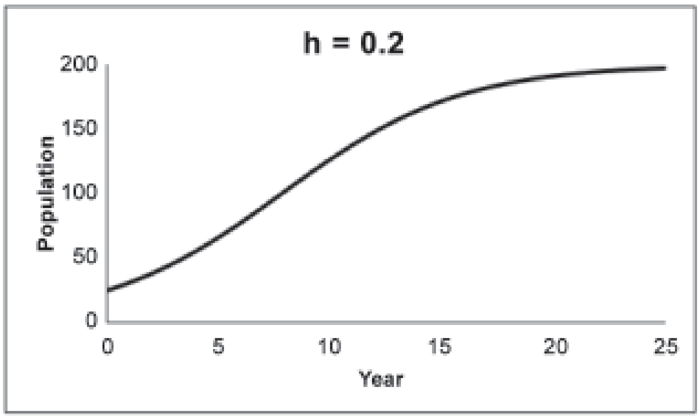


Fig 5.11.1

Non-autonomous differential equations (meaning equations where the right-hand side explicitly depends on t) of the form

$$\frac{dy}{dt} = F(t, y)$$

arise frequently in applications. Euler's method can be easily adapted to these types of differential equations. The basic algorithm is given by

$$t_{n+1} = y_n + h, \;\; y_{n+1} = y_n + h\,F\left(y_n, t_n\right).$$

The next example illustrates an application of a ***non-autonomous differential equation***

Example 5.11.2 (Bacteria Growth) Let $y(t)$ denote the population of bacteria in a Petri dish t days after the bacteria begin growing. Suppose $y(t)$ is described by the differential equation

$$\frac{dy}{dt} = 150\sqrt{t}$$

for t between 0 and 10. If the initial population is 500, approximate the solution curve over the interval $0 \le t \le 10$ and approximate the population at time $t = 7$

Working Process in EXCEL

Rename a blank worksheet **"Bacteria"** and format it as in Table 5.11.2. Copy row 5 down to row 105.

Table 5.11.2

	A	B
1	h =	0.1
2		
3	Day	Population
4	0	500
5	= A4+$B$1	=B4+$B$1*150*SQRT(A4)

Create a graph of the solution curve as in Figure 5.11.2. Note that as time increases, the population grows faster.

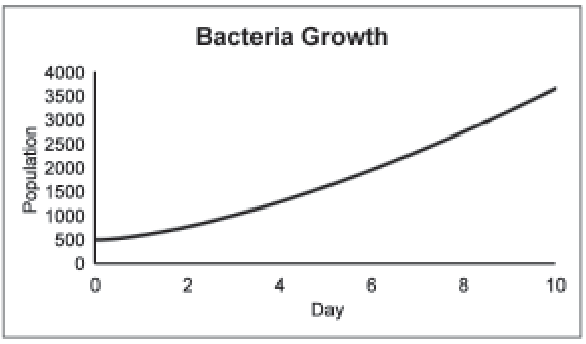


Fig 5.11.2

To determine if this approximate solution curve is accurate, we change the value of h in cell **B1** to 0.05, copy row 5 down to row 205, and graph the resulting approximate solution curve. Observe that this curve looks very similar to that in Figure 5.11.2. This indicates that h = 0.1 yields accurate results. Now note that for h = 0.1, the calculations give $y(7) \approx 2331$. We interpret this result by saying that at the beginning of day 7, there will be approximately 2300 bacteria.

Exercise Let $y(t)$ denote the population of rabbits (in thousands) in a certain forest at time t (in months). Suppose ÁÛUß is described by the differential equation

$$\frac{dy}{dt} = 1 + 3\cos\left(5\sqrt{t} - 9\right).$$

a) Graph an approximate solution curve over the interval $0 \leq t \leq 10$ if the initial population is 3000.

b) Describe, in words, the behavior of the population over this interval of time.

c) What is the approximate population at time t = 5?

5.12 System of Differential Equations

A system of differential equations is a set of two or more related differential equations involving two or more unknown functions. In this section we restrict ourselves to a set of two equations with the general form

$$\frac{dx}{dt} = F(x, y),$$

$$\frac{dy}{dt} = G(x, y)$$

along with the initial conditions $x(t_0) = x_0,\ y(t_0) = y_0$. Euler's method for a system such as this is:

$$t_1 = t_0 + h \qquad x_1 = x_0 + hF\left(x_0, y_0\right) \qquad y_1 = y_0 + hG\left(x_0, y_0\right)$$

$$t_2 = t_1 + h \qquad x_2 = x_1 + hF\left(x_1, y_1\right) \qquad y_2 = y_1 + hG\left(x_1, y_1\right)$$

... ...

$$t_{n+1} = t_n + h \qquad x_{n+1} = x_n + hF\left(x_n, y_n\right) \qquad y_{n+1} = y_n + hG\left(x_n, y_n\right)$$

Example 5.12.1 (Connected Tanks)

Consider the two connected tanks filled with salt water shown in Figure 5.12.1. Let $x(t)$ and $y(t)$ denote the masses of salt (in kg) in the tanks at time t where $x(0) = 5$ and $y(0) = 2$. We assume perfect mixing in both tanks. The goal of this example is to describe the long-term behaviour of x and y

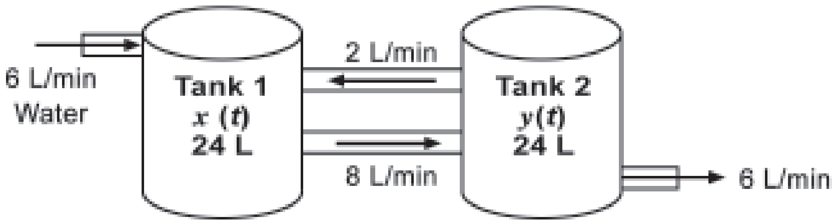


Fig 5.12.1

To set up the system of differential equations, we use the following principle:

Overall rate of change = Rate in – Rate out.

First, observe that each tank is losing solution at the overall rate of 8 L/min and gaining solution at the rate of 8 L/min, so the volume of each tank is not changing. Now consider tank 1. This tank has pure water entering on the left at 6 L/min and solution from tank 2 entering on the right at 2 L/min. Therefore,

$$Rate\ in = \frac{0\ kg}{L} \times \frac{6\ L}{\text{min}} + \frac{y\ kg}{24L} \times \frac{2L}{\text{min}} = \frac{y\ kg}{12\ \text{min}}.$$

Likewise, tank 1 has solution leaving on the right at the rate of $\frac{8L}{\text{min}}$, so

$$Rate\ in = \frac{x\ kg}{24L} \times \frac{8\ L}{\text{min}} = \frac{x\ kg}{3\,\text{min}}$$

Therefore, the differential equation for tank 1 is

$$\frac{dx}{dt} = \frac{y}{12} - \frac{x}{3}$$

By a similar argument, the differential equation for tank 2 is

$$\frac{dy}{dt} = \frac{x}{3} - \frac{y}{3}.$$

Working Process in EXCEL

To numerically solve this system using Euler's method with a step size of h = 0.2, rename a blank worksheet "**Connected Tanks**" and format it as in Table.5.12.1 Copy row 5 down to row 205

Table 5.12.1

	A	B	C
1	h =	0.2	
2			
3	t	x	y
4		4	2
5	= A4+$B$1	=B4+$B$1*(C4/12-B4/3)	=C4+$B$1*(B4/3-C4/3)

To graphically analyse the results, create graphs of *x vs. t* and *y vs. t* as in Figure 5.12.1. These graphs are called time plots. In these graphs, we see that the mass of salt in tank 1 drops to 0 by about time 20 min. The mass of salt in tank 2 initially increases, but then drops to 0 by about time 30 min.

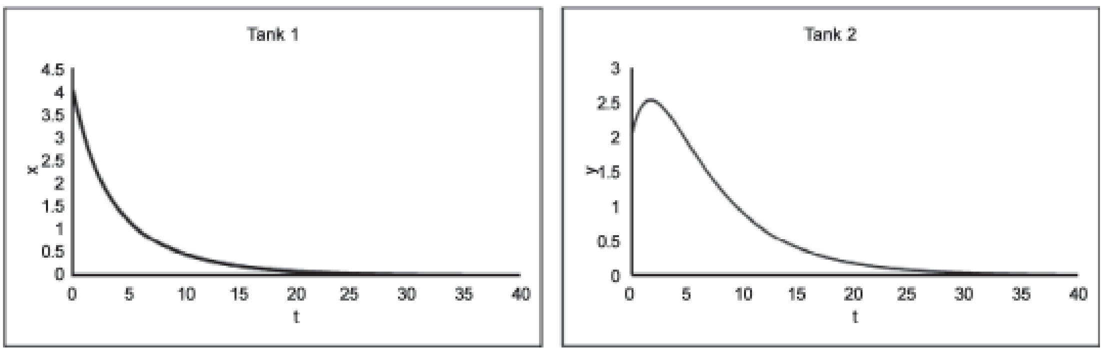


Fig 5.12.1

We can combine the two time plots into a single graph by graphing *y vs. x* as in Figure 5.12.2. The $x - y$ plane in this graph is called the phase plane and the curve is called a trajectory. The trajectory shows that the system starts at the point (5, 2) (the initial condition). Moving along the trajectory to the left, we see that x decreases while y initially increases, but then begins to decrease. Both *x* and *y* eventually approach 0. This is exactly what we saw in the time plots.

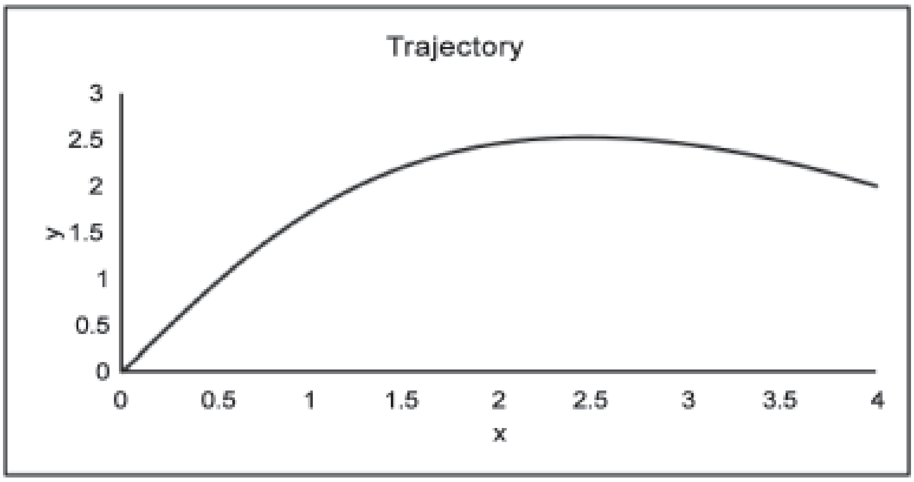


Fig 5.12.2

In simpler terms, an equilibrium point is a point on the phase plane where if we start there, we stay there forever. As with discrete dynamical systems, equilibrium points of systems of differential equations are points on the phase plane which typically attract or repel trajectories. Equilibrium points that attract trajectories are called attracting, stable, or asymptotically stable. Equilibrium points that repel trajectories are called unstable or repelling

	B	C
3	x	y
4	= RANDBETWEEN (–5, 5)	= RANDBETWEEN (–5, 5)

On the graph of the trajectory, change the axes mins and maxes to “5 and 5 as in Figure 5.12.3. Press the F9 key several times. Each time, a new set of initial conditions is generated. Observe that the trajectory always approaches the point (0, 0). This is graphical evidence that (0, 0) is an attracting equilibrium point.

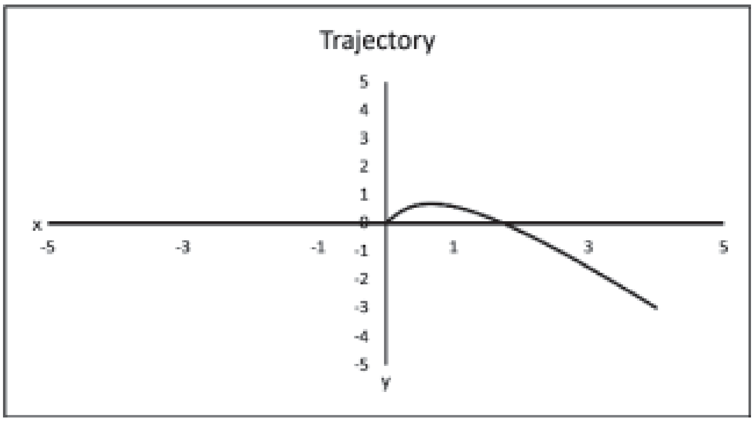


Fig 5.12.3

we need to set both F (x, y) and G (x, y) equal to 0 and solve for x and y. In Example 5.12.1, this yields the system of linear equation

$$\frac{y}{12} - \frac{x}{3} = 0$$

$$\frac{x}{3} - \frac{y}{3} = 0$$

Solving this system using elementary linear algebra techniques yields the only solution x = y = 0. Therefore, (0, 0) is the only equilibrium point of the system.

5.13 Summary

In this Unit we have explained some of the basic terminology and tools used to build the models. These explanations apply directly to Office Excel 2016, although most of them apply to other versions of Excel. We have analysed the long-term behaviour of discrete and continuous dynamical system using working process in EXCEL. Model different scenarios with linear and nonlinear discrete dynamical systems and differential equation for continuous dynamical models also studied numerically and presented graphically.

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