PREFACE

With its grounding in the “guiding pillars of Access, Equity, Equality, Affordability and Accountability,” the New Education Policy (NEP 2020) envisions flexible curricular structures and creative combinations for studies across disciplines. Accordingly, the UGC has revised the CBCS with a new Curriculum and Credit Framework for Undergraduate Programmes (CCFUP) to further empower the flexible choice based credit system with a multidisciplinary approach and multiple/ lateral entry-exit options. It is held that this entire exercise shall leverage the potential of higher education in three-fold ways – learner’s personal enlightenment; her/his constructive public engagement; productive social contribution. Cumulatively therefore, all academic endeavours taken up under the NEP 2020 framework are aimed at synergising individual attainments towards the enhancement of our national goals.

In this epochal moment of a paradigmatic transformation in the higher education scenario, the role of an Open University is crucial, not just in terms of improving the Gross Enrolment Ratio (GER) but also in upholding the qualitative parameters. It is time to acknowledge that the implementation of the National Higher Education Qualifications Framework (NHEQF) and its syncing with the National Skills Qualification Framework (NSQF) are best optimised in the arena of Open and Distance Learning that is truly seamless in its horizons. As one of the largest Open Universities in Eastern India that has been accredited with ‘A’ grade by NAAC in 2021, has ranked second among Open Universities in the NIRF in 2024, and attained the much required UGC 12B status, Netaji Subhas Open University is committed to both quantity and quality in its mission to spread higher education. It was therefore imperative upon us to embrace NEP 2020, bring in dynamic revisions to our Undergraduate syllabi, and formulate these Self Learning Materials anew. Our new offering is synchronised with the CCFUP in integrating domain specific knowledge with multidisciplinary fields, honing of skills that are relevant to each domain, enhancement of abilities, and of course deep-diving into Indian Knowledge Systems.

Self Learning Materials (SLM’s) are the mainstay of Student Support Services (SSS) of an Open University. It is with a futuristic thought that we now offer our learners the choice of print or e-slm’s. From our mandate of offering quality higher education in the mother tongue, and from the logistic viewpoint of balancing scholastic needs, we strive to bring out learning materials in Bengali and English. All our faculty members are constantly engaged in this academic exercise that combines subject specific academic research with educational pedagogy. We are privileged in that the expertise of academics across institutions on a national level also comes together to augment our own faculty strength in developing these learning materials. We look forward to proactive feedback from all stakeholders whose participatory zeal in the teaching-learning process based on these study materials will enable us to only get better. On the whole it has been a very challenging task, and I congratulate everyone in the preparation of these SLM’s.

I wish the venture all success.

Professor Indrajit Lahiri
Vice Chancellor

Netaji Subhas Open University

Four Year Undergraduate Degree Programme

Under National Higher Education Qualifications Framework (NHEQF) &
Curriculum and Credit Framework for Under Graduate Programmes

B. Sc. Mathematics (Hons.)

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Course Type : Discipline Specific Core (DSC)

Course Title : Real Analysis-I

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First Print : March, 2025

Memo No. SC/DTP/25/015 Dated 06.03.2025

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Netaji Subhas Open University

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Unit 1 ❑ Preliminaries

Structure

1.1 Objectives

The aim of this unit is to recall some definitions and useful results for studying and understanding clearly the next units 2, 3 and 4.

1.2 Introduction

Real analysis is a development of the set of real numbers and real valued functions. Therefore the concept of set and function are very much needed to study real analysis. For that purpose, in this unit, some basic terms and results about set and function are discussed.

1.3 Sets

A set is a well defined collection of distinct objects. Here well defined means it must be possible to tell without any ambiguity whether a given object belongs to that collection or not. Sets are usually denoted by capital letters A, B, S, ...etc.

If an object x is a member of a set S, then we write $x \in S$ and read as 'x belongs to S' or 'x is a member of a set S'. If y is not an element of S, we write $y \notin S$ and read as 'y does not belongs to S'.

Example : The collection of the letters of the word 'logic' is a set as it is a well defined collection of distinct objects. If we denote this set by S, then

$S = \{l, o, g, i, c\}$.

We can also write the set as

S = {x : x = a letter of the word logic}.

The first form of S is known as tabular form, where the second one is known as set-builder form of S. The order, in which the objects of a set are taken is immaterial.

Some special sets are denoted as

$\mathbb{N}$ = the set of all natural numbers.

$\mathbb{Z}$ = the set of all integers.

$\mathbb{Q}$ = the set of all rational numbers.

$\mathbb{R}$ = the set of all real numbers.

$\mathbb{C}$ = the set of all complex numbers.

Finite and Infinite set : If the number of elements of a set is finite (respectively infinite) then set is called finite (respectively infinite) set.

For example, the collection of all prime numbers between 10 and 20 is a finite set. If we denote this set by P, then P = {11, 13, 17, 19}, which contains only four (finite) elements. Again the set F = {x : x is a fraction and 0 < x < 1} is an infinite set as it contains infinite number of elements. Above $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are all infinite sets.

Null Set : A set is called null (or void or empty) set if it has no member in it. It is denoted by ϕ and written as $\phi = \{\ \}$.

For example, the set of all prime numbers between 32 and 36 is a null set.

Sub set and super set of a set : If every element of a set A is also an element of a set B, then A is said to be a subset of B. We write this as $A \subseteq B$. Here A is contained in B.

Thus $A \subseteq B$ if $\forall x \in A \Rightarrow x \in B$.

If $A \subseteq B$ then B is said to be a superset of A. We write this as $B \supseteq A$. Here B contains A.

For any set A, we have $\phi \subseteq A$ and $A \subseteq A$. The sets ϕ and A (entire set) are called improper subsets of A. Any other subset of A, if exists, is called a proper subset of A.

It may be clear that a set S is called a proper subset of A, written as $S \subset A$, if for any $x \in S \Rightarrow x \in A$, but $\exists\, y \in A$ such that $y \notin S$.

Moreover, two sets A and B are said to be equal, written as A = B, if $A \subseteq B$ and $B \subseteq A$.

Singleton set : If a set consists of exactly one element then it is called singleton set.

For example, the set $\{1\}$ is a singleton set.

Universal set : If all the sets under study are subsets of a particular set, then that particular set is called the universal set.

Power set of a set : Let A be any set. The set of all subsets of the set A is called the power set of A and it is denoted by P(A).

For example, if $A = \{a, b, c\}$ then

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, A\}$.

Note that if A contains 'n' elements then P(A) contains 2^n elements.

Set operations : Some important operations on sets are :

Union and Intersection of sets : If $\wedge$ is any arbitrary index set then $\{A_i : i \in \wedge\}$ is called an arbitrary collection or family of sets. The union of the above arbitrary family of sets, denoted by $\bigcup_{i\in\wedge} A_i$, is defined by

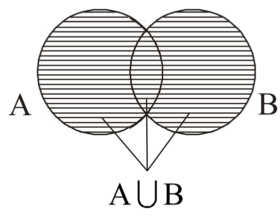
$$\bigcup_{i\in\wedge} A_i = \{x : x \in A_i \text{ for at least one } i \in \wedge\}$$

and the intersection of the above arbitrary family of sets, denoted by $\bigcap_{i\in\wedge} A_i = \{x : x \in A_i \text{ for every } i \in \wedge\}$.

Thus for any two sets A and B, the union of A and B, denoted by $A \cup B$, is defined by

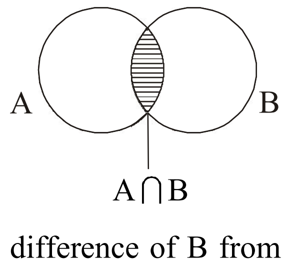
$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{both A and B}\}$.

The Venn-diagram representation of it as



The intersection of A and B, denoted by $A \cap B$, is defined by $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

It's Venn-diagram representation is

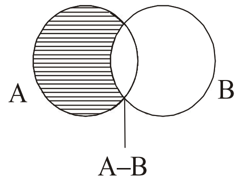


Disjoint sets : Two sets A and B are called disjoint if $A \cap B = \phi$. That means the disjoint sets have no common element.

Difference of sets : Let A and B be any two sets. The difference of B from A, denoted by A – B, is defined by

$A - B = \{x : x \in A \text{ but } x \notin B\}$.

Its Venn-diagram representation is



For example, if A = {0, 1, 2, 3, 4} and B = {2, 4, 6, 8} then

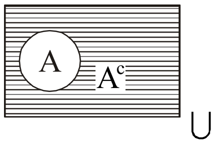
A – B = {0, 1, 3} and B – A = {6, 8}.

Thus A – B ≠ B – A and it is true in general. The symmetric difference of A and B, denoted by A Δ B, is defined by A Δ B = (A – B) ∪(B – A).

Complement of a set : Let ∪ be an Universal set and A ⊂ U. The complement of A, denoted by A' (or A^c), is defined by

$A^c = \{x : x \in \cup \text{ and } x \notin A\}$.

Its Venn-diagram representation as



It is clear that

Also for any two sets A and B, $A \subseteq B \Rightarrow B^c \subseteq A^c$.

Laws of Algebra of sets

(i) Idempotent laws : For any set A, $A \cup A = A, A \cap A = A$.

(ii) Identity laws : For any set A, $A \cup \phi = A, A \cap \cup = A$.

(iii) Commutative laws : For any sets A and B, we have

$$A \cup B = B \cup A, A \cap B = B \cap A.$$

(iv) Associative laws : For any three sets A, B and C, we have

$$(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$$

(v) Distributive laws : For any three sets A, B and C, we have

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

(vi) De-Morgan's laws : For any two sets A and B, we have

$$(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c.$$

Cardinality of a set : For any set A, the number of elements of A is called the cardinality of A and it is denoted by n(A).

It may be noted that n(ϕ) = 0 and n(B) = ∞ for an infinite set B.

For any two finite sets A and B, we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Also for any three finite sets A, B, and C, we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$
$$- n(A \cap C) + n(A \cap B \cap C).$$

Cartesian product of sets : The cartesian product of any two sets A and B, denoted by A × B, is defined by $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Similarly we can define $B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$.

In general, $A \times B \neq B \times A$.

For any set A, we have $A \times \phi = \phi = \phi \times A$.

The cartesian product of any three sets A, B and C can be similarly defined as

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

Similarly the cartesian product of n sets A_1, A_2, ..., A_n is defined as

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i, 1 \leq i \leq n\},$$

where $(a_1, a_2, ..., a_n)$ is known as an ordered n-tuple.

If IR is the set of real numbers, then $IR^2 = IR \times IR = \{(x, y) : x \in IR, y \in IR\}$ represents the set of all ordered pairs of real numbers, i.e., the cartesian plane.

Similarly $IR^3 = IR \times IR \times IR = \{(x, y, z) : x \in IR, y \in IR, z \in IR\}$ represents the three dimensional space, i.e., the Euclideam space.

And $IR^n = \underset{\text{(n times)}}{IR \times IR \times ... \times IR} = \{(x_1, x_2, ..., x_3) : x_i \in IR, 1 \leq i \leq n\}$ represents the n-dimensional Euclidean space.

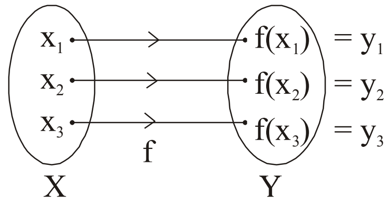
1.4 Functions or Mappings

Let X and Y be any two sets. A function or mapping f of X to Y is a rule which associates to each element x in X, a unique element y in Y and it is written as $f : X \to Y$. Here X and Y are called respectively the domain and codomain of f. Also y is called the f-image of x and written as y = f(x), while x is called pre-image of y. The set of all f-images of (the elements of) X, denoted by f(X), is called the image of X under f or range of f. of course $f(X) \subseteq Y$.

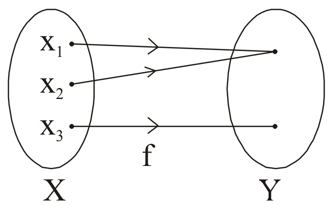
Types of functions : There are many kind of functions such as :

One-one function A function $f : X \to Y$ is said to be one-one (or injective) if distinct elements of X have distinct images. Thus f : X → Y is injective if for all

$x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

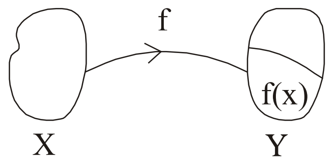


Many one functions : A function f : X → Y is called many one function if $\exists\, x_1, x_2$ in X, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.



Into function : A function f : X →Y is called an into function if $f(X) \subset Y$.

In this case, we say that f maps X into Y.



Onto function : A function f : X → Y is said to be onto (or surjective) function if f(X) = Y.

In this case, we say that f maps X onto Y.

A function f : X → Y is called bijective if f is injective and surjective, i.e., one-one and onto.

Constant function : A function f : X → Y is called constant if

$f(x) = c\ \forall x \in X,$ where c is an element in Y. Here f(X) is a singleton set.

Identity function : A function f : X → Y is said to be identity function if

$f(x) = x \forall x \in X$. Such a function on X is denoted by I_x or simply I.

Equal functions : Two functions $f : X \to Y$ and $h : X \to Y$ are said to be equal if $f(x) = h(x)\ \forall x \in X.$ In this case, we write $f \equiv h.$

The sum, difference and the product of two functions $f : X \to Y$ and $h : X \to Y$ are defined as

$(f + h)(x) = f(x) + h(x)\ \forall x \in X$

$(f - h)(x) = f(x) - h(x)\ \forall x \in X$

and $(fh)(x) = f(x)\, h(x)\ \forall x \in X$.

If $h(x) \neq 0\ \forall x \in X$, the quotient f/h is defined as

$(f / h)(x) = f(x) / h(x)\ \forall x \in X.$

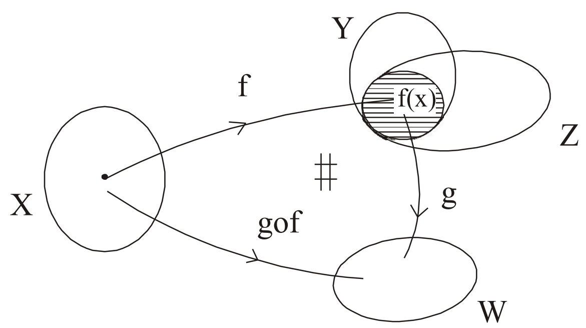
Also $(cf)(x) = cf(x), c \in IR.$

Restriction and Extension of a function : Let $f : X \to Y$ be a function and $A(\neq \phi) \subset X.$ The function $h : A \to Y$ defined by $h(x) = f(x) \forall x \in A,$ is called the restriction of f to A and it is denoted by f/A. Thus h = f/A.

If $h : A \to Y$ is a restriction of $f : X \to Y$ then f is called an extension of h to X.

As the f-images of the elements of X – A can be choosen arbitrarily, the extension f of h to X is not unique.

Composite function : Let $f : X \to Y$ and $g : Z \to W$ be two functions such that $f(x) \subseteq Z.$



Then the composite of f and g is a function $g of : X \to W$ defined by

$(gof)(x) = g(f(x))\ \forall x \in X.$

Thus the composite function $g of : X \to W$ is defined only when f(X) is a subset of the domain of g.

The existence of gof does not ensure the existence of fog.

Property of composite functions : Some important properties regarding composite functions are as follows :

(1) For two functions $f : X \to Y$ and $g : Y \to X$, both $gof : X \to X$ and $fog : Y \to Y$ are defined. However, $gof \neq fog$, in general, i.e., the operation of composite function is not commutative.

(2) For three functions $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h\,o\,(gof) = (hog)\,of$, i.e. the operation of composite function is associative.

(3) If $f : X \to Y$ and $g : Y \to Z$ are both bijective functions then gof is also bijective function. However, the converse of this statement may not be true.

Inverse of a function : Let $f : X \to Y$ be a bijective function. Then f is said to be invertible if $\exists$ a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$. This g is called the inverse of f and written as $g = f^{-1}$.

It may be noted that the inverse of an invertible mapping is unique. Also if $f : X \to X$ is an invertible mapping then $fo\overset{-1}{f} = I = \overset{-1}{f}of$, where I is the identity mapping on X.

Properties of Inverse functions :

(1) For an invertible mapping $f : X \to Y$, $\left(\overset{-1}{f}\right)^{-1} = f$.

(2) Let $f : X \to Y$ and $g : Y \to Z$ be two bijective mappings and $\overset{-1}{f} : Y \to X$ and $\overset{-1}{g} : Z \to Y$ be their respective inverse functions. Then the function $gof : X \to Z$ is also invertible and $(gof)^{-1} = \overset{-1}{f}\,o\,\overset{-1}{g}$.

1.5 Summary

- Sets are well defined collection of distinct objects.
- If a set contains no element then it is called empty set.
- The complement of complement of a set is itself.
- The number of elements of a set is called the cardinality of that set.
- For any two sets A and B, $A \times B \neq B \times A$, in general.

- Functions are, in all, of four kinds :
  (i) One-one into functions
  (ii) One-one into functions
  (iii) Many-one into functions
  (iv) Many-one onto functions.

1.6 Keywords

Sets, union, intersection of sets, complement, cardinality of a set, cartesian product of sets, Function or mapping, injective and bijective mappings, restriction and extension of a mapping, composite functions, inverse of a function.

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Unit 2 ❑ Real Numbers

Structure

2.1 Objectives

One of the important branch of mathematics is real analysis which is consisted with set of real numbers. Thus to study real analysis it is necessary to know the properties of real numbers. That is why the object of this unit are as :

- To study algebraic, order and completeness properties of $\mathbb{R}$.
- To study the concept of rational numbers, irrational numbers and construction of real numbers from system of rational numbers.
- To know the concept of neighbourhood of a point, limit point of a set, open set, closed set in $\mathbb{R}$.
- To study Blozano Weierstrass theorem which states the sufficient condition for the existence of limit points of a set.

2.2 Introduction

It is know that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, where $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are respectively the set of natural numbers, integers and rational numbers. The concept of real numbers $\mathbb{R}$ is systematically developed from $\mathbb{N}$ via the construction of $\mathbb{Z}$ and $\mathbb{Q}$. The set of real numbers and their properties are discussed in this unit. The set of real numbers can be described as a complete ordered field. *The analysis, due to set of real numbers is known as real analysis*, which is one important branch of mathematics. We discuss the limit point of a set, open set, closed set etc. as a basic part of real analysis. It is known that a finite set has no limit point, while an infinite set may or may not have a limit point. Thus the necessary condition for the existence of a limit point is that set must be infinite. We have studied Bolzano Weierstrass theorem, which tells the sufficient condition for the existence of limit point of a set.

2.3 Algebraic and Order properties of $\mathbb{R}$

This section deals with some algebraic and order properties of real numbers, which can be derived by Field axioms and order axioms.

Field Axioms : It is known that the set of real numbers $\mathbb{R}$ is a field with respect to two operations addition and multiplication, denoted by '+' and '.' respectively. That means these two operations '+' and '.' on $\mathbb{R}$ satisfying the following axioms, known as Field axioms.

Addition Axioms :

(A_1) Closure law : $a + b \in \mathbb{R}, \forall\, a, b \in \mathbb{R}$.

(A_2) Associative law : $a + (b + c) = (a + b) + c, \ \forall\, a, b, c \in \mathbb{R}$.

(A_3) Existence of additive identity : The real number 0, called the additive identity such that $a + 0 = a = 0 + a, \ \forall\, a \in \mathbb{R}$.

(A_4) Existence of additive inverse : For each $a \in \mathbb{R}, \exists$ an element $-a \in \mathbb{R}$, called the additive inverse of a such that $a + (-a) = 0 = (-a) + a$.

(A_5) Commutative law : $a + b = b + a, \ \forall\, a, b \in \mathbb{R}$.

Multiplication Axioms :

(M_1) Closure law : $a.b \in IR, \forall a, b \in IR$

(M_2) Associative law : $a.(b. c) = (a. b). c, \forall a, b, c \in IR$

(M_3) Existence of multiplicative identity : The real number 1, called the multiplicative identity satisfies a.1 = a = 1. a, $\forall a \in IR$.

(M_4) Existence of multiplicative inverse : For each $a \in IR$, $\exists$ an element $a^{-1} \in IR$, called the multiplicative inverse of a such that a . a^{-1} = 1 = a^{-1}.a. Here we may also denote a^{-1} by $\frac{1}{a}$.

(M_5) Commulative law : a.b = b.a, $\forall a, b \in IR$.

Distributive laws

(D_1) a. (b + c) = a.b + a.c $\forall a, b, c \in IR$.

(D_2) (b + c) . a = b.a + c.a $\forall a, b, c \in IR$.

Subtraction and Division in IR

The subtraction of a real number 'b' from a real number 'a', denoted by a – b, is defined by a – b = a + (–b).

The division of a real number 'a' by a non-zero real number 'b' denoted by a/b, is defined by a/b = a.b^{-1}.

Algebraic property of IR

The set of real number satisfies Field axioms. Moreover, some algebraic properties of IR are as follows :

For a, b, c, $\in IR$, we have

(i) a + c = b + c $\Rightarrow$ a = b and c + a = c + b $\Rightarrow$ a = b,

(ii) a + b = 0 $\Rightarrow$ b = –a,

(iii) –(–a) = a,

(iv) if c $\neq$ 0 then a . c = b . c $\Rightarrow$ a = b and c.a = c.b $\Rightarrow$ a = b,

(v) a. b = 1 $\Rightarrow$ b = a^{-1},

(vi) if a $\neq$ 0 then $(a^{-1})^{-1} = a$,

(vii) $a.\ 0 = 0$,

(viii) $a \neq 0,\ b \neq 0 \Rightarrow a.\ b \neq 0$,

(ix) $a.\ b \Leftrightarrow 0 = a = 0$ or / and $b = 0$,

(x) $a.\ (-b) = -(a.b) = (-a).\ b$,

(xi) $(-a)\ (-b) = a\ .\ b\ ;\ (-1).a = -a$,

(xii) $(a.b)^{-1} = a^{-1}.b^{-1}$, provided $a \neq 0,\ b \neq 0$.

(xiii) the equation $x + a = b$ has a unique solution $x = b - a$ in $\mathbb{R}$.

(xiv) for $a \neq 0$, the equation $a.x = b$ has a unique solution $x = b/a$ in $\mathbb{R}$.

Order Axioms : The set of real numbers $\mathbb{R}$ is an ordered field, i.e., $\mathbb{R}$ is ordered with respect to order relation '>', called greater than. That means the relation '>' between pairs of real numbers satisfies the following axioms, known as order axioms.

(O_1) Law of trichotomy : For all $a, b \in \mathbb{R}$, one and only one of the following is true

$a > b,\ a = b,\ b > a$.

(O_2) Transitivity law : For all $a, b, c \in \mathbb{R}$, $a > b$ and $b > c \Rightarrow a > c$.

(O_3) Monotone property for addition :

For all $a, b, c \in \mathbb{R}$, $a > b \Rightarrow a + c > b + c$.

(O_4) Monotone property for multiplication :

For all $a, b, c \in \mathbb{R}$ and $c > 0$, $a > b \Rightarrow a.c > b.c$.

Remark : (1) The order relation '<', called less than, is defined as $a < b$ if $b > a$. The order axiom can also be stated with the relation '<' instead of '>'.

(2) The relation $a \leq b$ means either $a < b$ or $a = b$ and $a \geq b$ means either $a > b$ or $a = b$.

(3) A real number 'a' is said to be positive or negative according as $a > 0$ or $a < 0$. The set of positive (respectively negative) real numbers is denoted by $\mathbb{R}^+$ (respectively $\mathbb{R}^-$).

Order property of $\mathbb{R}$: Beside the order axioms, $\mathbb{R}$ satisfies the following order properties :

(i) For each real number a, one and only one of the following holds :

$a > 0,\ a = 0,\ -a > 0$.

(ii) $a < 0 \Leftrightarrow -a > 0$

(iii) $a > b \Leftrightarrow a - b > 0$ for all $a, b \in \mathbb{R}$.

(iv) For all a, b, c $\in$ IR, if c < 0 then a > b $\Rightarrow$ ac < bc.

(v) For a, b, $\in$IR+, $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$.

Extended real number system : We can extend the system of real numbers by adjoining ∞ and $-\infty$. The enlarged set is called the extended real number system.

If a $\in$IR, we have $-\infty < a < \infty$.

$a + \infty = \infty = \infty + a$, $a - \infty = -\infty = -\infty + a$,

$$\frac{a}{\infty} = 0, \frac{\infty}{a} = \infty \times a = a \times \infty = \begin{cases} \infty, \text{if } a > 0 \\ -\infty, \text{if } a < 0 \end{cases}$$

Also $\infty \times \infty = \infty = (-\infty) \times (-\infty) = \infty + \infty$.

$\infty \times (-\infty) = -\infty = (-\infty) \times \infty = -\infty - \infty$.

However, $\infty - \infty$, $-\infty + \infty$, $0 \times \infty$, $\infty \times 0$, $\frac{\infty}{\infty}$ are meaningless.

2.4 Countable sets, Uncountable sets and Uncountability of IR.

This section deals with countable sets and uncountable sets through which infinite set may classify two ways :

Countably infinite set and Uncountably infinite set.

Countable and Uncountable sets : A set S is said to be enumerable or denumerable if $\exists$ a bijection from IN onto the set S.

A set S is called countable if either S is finite or S is enumerable. A set S is called uncountable if it is not countable. Thus an uncountable set must be infinite. The empty set ϕ is countable as it is assumed a finite set.

Examples :

(1) The set E = {2n : n $\in$IN} is denumerable, as there is a bijection f : IN $\rightarrow$ E. Here E is the set of even natural numbers. It is an infinite set, but countable. So, E is countably infinite set.

(2) The set (of odd natural numbers) O = {2n – 1 : n $\in$ IN} is also denumerable and hence O is countable.

(3) The set Z of all integers is countable as Z is denumerable.

(4) The set of real numbers in the interreal (0, 1) is uncountable.

Theorem : Any subset of a countable set is a countable set.

Proof : Let B be a subset of a countable set A. We show that B is countable.

If possible, let us suppose that B is uncountable. Then every injective function $f : \mathbb{N} \to B$ must be into, not onto, i.e. $f(\mathbb{N}) \subset B$. Since $B \subset A$, therefore $f(\mathbb{N}) \subset A$.

Thus for every injective function $f : \mathbb{N} \to A$, $f(\mathbb{N}) \neq A$.

So, A is an uncountable set, which is a contradiction. Hence B must be countable.

Theorem : A countable union of countable sets is countable.

Proof : Let $\{A_i : i \in \mathbb{N}\}$ be a countable collection of countable sets and let $A = \bigcup_{i=1}^{\infty} A_i$.

Each countable set A_i, $i \in \mathbb{N}$ may be represented as

$A_1 = \{a_{11}, a_{12}, a_{13}, ..., a_{1n}, ...\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, ..., a_{2n}, ...\}$

... ...

$A_m = \{a_{m1}, a_{m2}, a_{m3}, ..., a_{mn}, ...\}$

... ...

There are two cases arises :

Case I : If the sets A_1, A_2, ..., A_m, ... are disjoint, the elements of A can be arranged as

$A = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, ...\}$.

We may construct a one-one function f from $\mathbb{N}$ onto A

such that $f\left(\frac{1}{2}(m+n-1)(m+n-2)+m\right) = a_{mn}$.

Then $f(1) = a_{11}$, $f(2) = a_{12}$, $f(3) = a_{21}$, ...

So, A is countable.

Case II : If the sets A_1, A_2, ... are not all disjoint, consider the sets $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash A_1 \cup A_2$, ..., $B_m = A_m \backslash \bigcup_{i=1}^{m-1} A_i$.

Then the sets B_1, B_2, ..., B_m are disjoint and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.

So, by Case I, $\bigcup_{i=1}^{\infty} B_i$ is countable and hence $\bigcup_{i=1}^{\infty} A_i$ is countable.

Corollary : The union of two enumerable sets is enumerable.

Corollary : The union of an enumerable number of enumerable sets is enumerable.

Theorem : The Cartesian product of two countable sets is countable.

Proof : Let A and B be two countable sets. We have to show that A × B is countable.

Since A and B are countable, we can write

$A = \{a_1, a_2, ..., a_i, ...\}$ and $B = \{b_1, b_2, ..., b_j, ...\}$, i, j, ∈ IN.

Then $A \times B = \bigcup_{i=1}^{\infty} P_i$ where $P_i = \{(a_i, b_1), (a_i, b_2), ..., (a_i, b_j), ...\}$.

The jth member of P_i is (a_i, b_j). Clearly P_i is countable for each i. So, by previous theorem, A × B is countable.

Remark : If $A_1, A_2, ..., A_n$ are countable sets them the cartesian product $A_1 \times A_2 \times .... \times A_n$ is also countable.

Proof : It can be proved by method of mathematical induction.

Rational Numbers : If a real number can be expressed in the form of $\frac{p}{q}$, where p, q ∈ Z, q ≠ 0 such that gcd (p, q) = 1 (i.e. p and q are prime to each other), then it is called a rational number. Otherwise, it is called an irrational number. The set of rational numbers is denoted by $\mathbb{Q}$.

Let x, y ∈ $\mathbb{Q}$, then we can write $x = \frac{a}{b}, y = \frac{c}{d}$,

where a, b, c, d ∈ Z, b ≠ 0, d ≠ 0.

We now define the operations addition, subtraction, multiplication and division are as

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad x - y = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd},$$

$$xy = \frac{a}{b} . \frac{c}{d} = \frac{ac}{bd} \text{ and } \frac{x}{y} = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}, \text{ provided } c \neq 0.$$

Also x = y i.e. $\frac{a}{b} = \frac{c}{d}$ iff ad = bc.

Properties of $\mathbb{Q}$: Some important properties of $\mathbb{Q}$ are as follows :

(1) Algebric Property : The set $\mathbb{Q}$ forms a field with respect to addition and multiplication defined as above. That means $\mathbb{Q}$ satisfies the field axioms, mentioned in section 2.3. Here '–a' is the additive inverse of 'a' $\in \mathbb{Q}$ and $\frac{1}{b}$ is the multiplicative inverse of b($\neq$ 0) $\in \mathbb{Q}$. The zero element and unity are respectively 0 and 1.

(2) Order Property : Further, one can check that $\mathbb{Q}$ satisfies the order axioms, discussed in section 2.3. Thus $\mathbb{Q}$ becomes an ordered field.

(3) Density Property : It is known that if x, y $\in \mathbb{Q}$ with x < y then $\frac{x+y}{2} \in \mathbb{Q}$ and $x < \frac{x+y}{2} < y$. That means between any two rational numbers there exists another rational number $\frac{1}{2}(x+y)$. By similar way, it can be check that between x and $\frac{1}{2}(x+y)$ (as $x < \frac{x+y}{2}$), $\exists$ another rational number. Proceeding in this way, we can conclude that between any two rational numbers x and y (where x < y) $\exists$ infinitely many rational numbers. This property of $\mathbb{Q}$ is known as the **density property** of $\mathbb{Q}$. In this case, we can say that $\mathbb{Q}$ is dense.

Problem : Show that there does not exist a rational number x such that $x^2 = 2$.

Solution : If possible, let there exist a rational number x such that $x^2 = 2$.

Since x is rational, so $\exists\, p, q \in \mathbb{Z}, q \neq 0$ such that gcd (p, q) = 1 and x = $\frac{p}{q}$.

$\therefore \frac{p^2}{q^2} = x^2 = 2$

$\Rightarrow p^2 = 2q^2$, which implies that p^2 is even and hence 'p' is even.

Let p = 2m, where 'm' is an integer.

Then $p^2 = 2q^2 \Rightarrow q^2 = 2m^2$, which also implies that q is even.

Thus p and q are both even which contradicts our assumption that gcd (p, q)=1. Therefore, there is no rational number whose square is 2.

Problem : Show that the set of all rational numbers is countable.

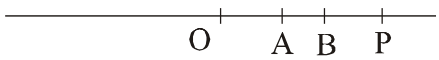
Solution : Let $\mathbb{Q}$ be the set of all rational number. Then we can write $\mathbb{Q}$ as

$$\mathbb{Q} = \left\{\left(\frac{0}{1}, \pm\frac{1}{1}\right), \left(\pm\frac{1}{2}\right), \left(\pm\frac{1}{3}, \pm\frac{2}{3}\right), ..., \left(\pm\frac{1}{n}, \pm\frac{2}{n}, ... \pm\frac{n-1}{n}\right), ...\right\}$$

$$= \{a_1, a_2, a_3, ..., a_n, ...\},$$

where a_n contains all rational numbers where denominator is n. Hence the set $\mathbb{Q}$ is countable.

Geometrical representation of rational number, irrational number and real number : Consider any directed straight line extending indefinitely on both sides. We divide it into two parts and mark middle point by O. The right part of O is called positive and left part of O is called negative.



Take any point 'A' on the positive part. Assume that the point O and A represent rational numbers 0 and 1, so that the distance OA is unity.

Let $p \in \mathbb{Z}$ and $q \in \mathrm{IN}$ and let us divide OA into 'q' equal parts. Then take 'p' numbers such subparts and we represent the rational number $\frac{p}{q}$ by the point P on the directed line.

So, $OP = \frac{p}{q}$.

If p > 0 then P lies to the right part of O and if p < 0 then P lies to the left part of O. When p = 0, the point P lies on 'O'.

Thus the point 'P' corresponds to the rational number $\frac{p}{q}$ and vice-versa. This representation is unique. Here 'P' is known as rational points.

Note that between any two rational points closely enough on the line, there are many points which does not represent rational numbers. Such points are called irrational points and the corresponding numbers are called irrational numbers.

For example, if we consider the point B on the line such that OB is the diagonal of a square of the side unity (i.e. OA) does not correspond to any rational number, as there is no rational number whose square is 2. Thus we may conclude that

Dedekind— Cantor Axiom : Every real number corresponds to a unique point on a directed line and every point on the directed line corresponds to a unique real number.

Remark : The above axiom shows that the set of real numbers is a continuum (means without any gap). That is why the set of real numbers is called the Arithmetical continuum and the set of points on a straight line is called linear or geometric continuum.

Dedekind section of rational number : Let Q be the set of all rational numbers. Å partition of Q into two subsets L and R (called classes) satisfying the following conditions is called a Dedekind section of rational numbers.

(i) $L \neq \phi, R \neq \phi$

(ii) $L \bigcup R = Q$

(iii) $\forall \alpha \in L$ and $\forall \beta \in R \Rightarrow \alpha < \beta$.

There are three types of Dedekind section.

Type-1 : Let us divide the set of all rational number Q into two classes L and R as follows :

$L = \{x : x \in Q \text{ and } x \leq 2\}$

$R = \{x : x \in Q \text{ and } x > 2\}$

Clearly it is a Dedekind section, because $2 \in L$ and $3 \in R$ such that $L \bigcup R = Q$.

Also $\forall \alpha \in L \Rightarrow \alpha \leq 2$ and $\forall \beta \in R \Rightarrow \beta > 2$

$\therefore \alpha < \beta$.

In this section L class has greatest number 2, but R class has no least number.

Type-2 : Let us divide Q into two classes L and R as follows :

$L = \{x : x \in Q \text{ and } x < 3\}$

$R = \{x : x \in Q \text{ and } x \geq 3\}$

It is Dedekind section and in this section, least number of R class is 3 but L class has no greatest number.

Type-3 : Let us divide Q into two classes L and R as follows :

$L = \{x : x \in Q \text{ and either } x \leq 0 \text{ or } x > 0 \text{ but } x^2 < 2\}$

$R = \{x : x \in Q \text{ and } x > 0, x^2 > 2\}$.

Clearly $0 \in L$ and $2 \in R$

$\therefore L \neq \phi$ and $R \neq \phi$.

As there is no rational number whose square is 2, it follows that LUR = Q.

$\forall\alpha \in L \Rightarrow$ either $\alpha \le 0$ or $\alpha > 0$ but $\alpha^2 < 2$ and $\forall\beta \in R \Rightarrow \beta > 0$ and $\beta^2 > 2$

When $\alpha \le 0$ then $\alpha < \beta$ and when $\alpha > 0$ then $\alpha^2 < 2$.

$\alpha^2 < \beta^2$ and hence $\alpha < \beta$.

Thus it is a Dedekind section.

Now we shall show that L class has no greatest number and R class has no least number.

If possible, let 'm' be the greatest number of L class, then $m > 0$ and $m^2 < 2$.

Let us take $n = \dfrac{4+3m}{3+2m}$

Then $n^2 - 2 = \dfrac{m^2-2}{(3+2m)^2} < 0$ and hence $n \in L$.

Now $n - m = \dfrac{4+3m}{3+2m} - m = \dfrac{4-2m^2}{3+2m} > 0$, i.e. $n > m$, which is a contradiction.

Therefore, L class has no greatest number.

If possible let 'r' be the least number of R class.

$\therefore$ $r > 0$ and $r^2 > 2$ i.e. $r^2 - 2 > 0$.

Let us put $s = \dfrac{4+3r}{3+2r}$.

Then $s > 0$ and $s^2 - 2 = \left(\dfrac{4+3r}{3+2r}\right)^2 - 2 = \dfrac{r^2-2}{(3+2r)^2} > 0$ as $r^2 - 2 > 0$

$\Rightarrow s^2 > 2$.

$\therefore$ $s \in R$ class

Now $r - s = r - \dfrac{4+3r}{3+2r} = \dfrac{2(r^2-2)}{3+2r} > 0$.

Therefore, $r > s$, which is a contradiction and hence R class has no least number.

Remark : Type -3 of Dedekind section about rational number shows that the system of rational number has gaps. To fill up these gaps, Dedekind introduced new numbers which are called irrational numbers. Thus irrational numbers are introduced by section of rational numbers as follows :

Modified section of rational numbers : A division of set of all rational numbers into two classes L and R satisfying the following condition is called the modified section of rational numbers.

(i) $L \neq \phi, R \neq \phi$

(ii) LUR = Q,

(iii) $\forall \alpha \in L$ and $\forall \beta \in R \Rightarrow \alpha < \beta$

(iv) L class has no greatest number.

Definition of real number by section of rational number : Every modified Dedekind section defines a real number α. If the section is (L, R), then we write $\alpha \equiv$ (L, R).

The real number 'α' is called the real rational number if 'α' is the least number of R-class and 'α' is called an irrational number if R class has no least number.

Exercise : Define the following real numbers by Dedekind section of rational numbers.

(i) 2 (ii) $\sqrt{3}$ (iii) $7^{1/3}$.

Section : (i) We define the real numbers 2 as $2 \equiv (L, R)$,

where L = $\{x : x \in Q \text{ and } x < 2\}$

and R = $\{x : x \in Q \text{ and } x \geq 2\}$.

(ii) We define the real number $\sqrt{3}$ by $\sqrt{3} \equiv (L, R)$,

where L = $\{x : x \in Q, \text{ either } x \leq 0 \text{ or } x > 0 \text{ and } x^2 < 3\}$

and R = $\{x : x \in Q, x > 0 \text{ and } x^2 > 3\}$.

(iii) We define $7^{1/3} \equiv (L, R)$,

where L = $\{x : x \in Q \text{ and } x^3 < 7\}$ and R = Q – L.

Relative magnitude of real numbers : Let $\alpha \equiv (L_1, R_1)$ and $\beta = (L_2, R_2)$ be two real numbers defined by section of rational numbers. We define $\alpha < \beta$ if and only if L_1 is a proper part of L_2 i.e., $\forall x \in L_1 \Rightarrow x \in L_2$ and there is $y \in L_2$ but $y \notin L_1$.

We also define $\alpha = \beta$ if and only if $L_1 \equiv L_2$

and $\alpha > \beta$ if and only if $\beta < \alpha$

i.e., if and only if L_2 is a proper part of L_1.

Exercise : Prove that the following by Dedekind section :

(i) $1 < \sqrt{2}$ (ii) $\sqrt{2} < \sqrt{3}$ and (iii) $\sqrt{3} < 2$.

Ans. (i) Let $1 \equiv (L_1, R_1)$, where $L_1 = \{x : x \in Q \text{ and } x < 1\}$ and

$\sqrt{2} \equiv (L_2, R_2)$, where $L_2 = \{x : x \in Q, \text{ either } x \le 0 \text{ or } x > 0 \text{ and } x^2 < 2\}$.

Then, $\forall x \in L_1 \Rightarrow x < 1$ and $\forall x \in L_2 \Rightarrow$ either $x \le 0$ or $x > 0$ and $x^2 < 2$.

Thus $\forall x \in L_1 \Rightarrow x \in L_2$.

Let us take a number $y = \frac{5}{4}$. Then $y^2 = \frac{25}{16} < 2$ and hence $y \in L_2$.

But $y = \frac{5}{4} > 1$, so $y \notin L_1$.

$\therefore$ L_1 is a proper part of L_2. Consequently $1 < \sqrt{2}$.

(ii) Let $\sqrt{2} \equiv (L_1, R_1)$, where $L_1 = \{x : x \in Q, \text{ either } x \le 0 \text{ or } x > 0 \text{ and } x^2 < 2\}$

and $\sqrt{3} \equiv (L_2, R_2)$, where $L_2 = \{x : x \in Q, \text{ either } x \le 0 \text{ or } x > 0 \text{ and } x^2 < 3\}$.

Then $\forall x \in L_1 \Rightarrow$ either $x \le 0$ or $x > 0$ and $x^2 < 2$

and $\forall x \in L_2 \Rightarrow$ either $x \le 0$ or $x > 0$ and $x^2 < 3$.

Thus $\forall x \in L_1 \Rightarrow x \in L_2$.

Let us take $y = \frac{3}{2}$. Then $y^2 = \frac{9}{4} < 3$, so $y \in L_2$, but $y^2 = \frac{9}{4} > 2$, i.e. $y \notin L_1$.

Thus L_1 is a proper part of L_2 and hence $\sqrt{2} < \sqrt{3}$.

(iii) Let us consider $\sqrt{3} \equiv (L_2, R_2)$,

where $L_2 = \{x : x \in Q, \text{ either } x \le 0 \text{ or } x > 0 \text{ but } x^2 < 3\}$

and $2 \equiv (L_1, R_1)$, where $L_1 = \{x : x \in Q, x < 2\}$.

It can be proved that L_2 is a proper part of L_1 and hence $\sqrt{3} < 2$.

Addition of two real numbers : Let $\alpha \equiv (L_1, R_1)$ and $\beta \equiv (L_2, R_2)$ be two real numbers, given by Dedekind section of real numbers. We define the number $\alpha + \beta \equiv (L, R)$,

where $L = \{x : x = x_1 + x_2, x_1 \in L_1, x_2 \in L_2\}$ and $R = Q - L$.

Reciprocal of a positive real number : Let $\alpha > 0$ be a real number, where $\alpha \equiv (L_1, R_1)$.

We define $\frac{1}{\alpha} \equiv (L, R)$, where $L \equiv \{x :$ either $x \leq 0$ or $x > 0$ and

$\frac{1}{x} \in R_1$ so that $\frac{1}{x}$ is not the least number of $R_1\}$.

Dedekind's Theorem on real number : If we divide the set of all real numbers IR into two classes L and R satisfying the following conditions :

(i) $L \neq \phi, R \neq \phi$

(ii) $L \cup R = IR$

(iii) $\forall\alpha \in L$ and $\forall\beta \in R \Rightarrow \alpha < \beta$,

then there is a number λ separating the two classes such that all numbers $< \lambda \in L$ class and all numbers $> \lambda \in R$ class.

The number λ may belong to either class. If $\lambda \in L$ then λ is the greatest number of L class and if $\lambda \in R$ then λ is the least number of R class.

2.5 Summary

This chapter discusses the algebraic and order properties of the real numbers R examines countable and uncountable sets, and proves the uncountability of R.

2.6 Keywords

Real numbers, algebraic properties, order properties, countable sets, uncountable sets, uncountability of R.

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2.8 Model Questions

1. What are the algebraic properties of R?
2. Describe the order properties of R.
3. Explain the difference between countable and uncountable sets.
4. Prove that R is uncountable.
5. How do the algebraic and order properties of R relate to each other?
6. Provide an example of a countable set and an uncountable set.
7. Suppose that S and T are sets and TCS. Show that if T is infinite, then S is also infinite.
8. Show that the power set P(N) of N is uncountable.
9. It F[Q] be the set of polynomices having rattional coefficients. Show that F[Q] is countable.

Unit 3 ❑ Bounded property

Structure

3.1 Objectives

This chapter aims to introduce fundamental concepts of real analysis, including intervals, bounded and unbounded sets, and supremum and infimum. Students will learn to classify different types of intervals, distinguish between bounded and unbounded sets, and understand the least upper and greatest lower bounds. These concepts form the foundation for further studies in real analysis, particularly in limits, continuity, and sequences. By mastering these topics, students will develop essential analytical skills for advanced mathematical reasoning.

3.2 Introduction

Real analysis provides a rigorous foundation for calculus and higher mathematics. In this chapter, we explore intervals, which describe subsets of real numbers, and the distinction between bounded and unbounded sets. Further, we examine supremum (least upper bound) and infimum (greatest lower bound), which are crucial for understanding limits and convergence. These fundamental ideas play a significant role in defining continuity, differentiability, and integration. Through this chapter, students will strengthen their logical thinking and problem-solving abilities.

3.3 Intervals

Let a, b $\in$ IR such that a < b. Then the open interval and closed interval are respectively defined as

$(a, b) = \{x \in IR : a < x < b\}$ and $[a, b] = \{x \in IR : a \le x \le b\}$.

The points 'a' and 'b' are known as end points. The closed interval contains end points, while the end points are not included in open interval.

Also the sets

$[a, b) = \{x \in IR : a \le x < b\}$ and $(a, b] = \{x \in IR : a < x \le b\}$ are called semi-open or semi-closed intervals. There are also known as closed-open and open-closed intervals respectively.

Since the length of each above intervals is equal to b – a ; which is a finite positive real number, there above intervals are called finite intervals. **Infinite intervals** are the intervals of infinite length.

For instance, the sets $(a, \alpha) = \{x \in IR : x > a\}$ and $(-\infty, a) = \{x \in IR : x < a\}$ are known as infinite open intervals, while

$[a, \infty) = \{x \in IR : x \ge a\}$ and $(-\infty, a] = \{x \in IR : x \le a\}$ all known as infinite closed intervals. The entire set IR is also considered as an infinite open interval by taking IR = $(-\infty, \infty)$.

Absolute value of a real number : The absolute value (or modulus) of $x \in IR$, denoted by |x| is defined as

$$|x| = \begin{cases} x, \text{ if } x \ge 0 \\ -x, \text{ if } x < 0 \end{cases}$$

For example $|5| = 5$ and $|-5| = -(-5) = 5$.

For any $x, y \in IR$ the distance between x and y is |x – y|.

Observations

(i) $|x| \ge 0$ and $|x|^2 = x^2$

(ii) $|-x| = |x|$

(iii) $|x| = \max\{x, -x\}$

(iv) $-|x| = \min\{x, -x\}$

(v) $|x \cdot y| = |x|\ |y|$

(vi) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$

(vii) $|x \pm y| \leq |x| + |y|$

(viii) $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$

Consequently, $|x - y| \geq |(|x| - |y|)|$

(ix) $|x - a| < \in \Rightarrow x \in (a - \in, a + \in)$ and $|x - a| \leq \in \Rightarrow x \in [a - \in, a + \in]$

(x) $x = y \Rightarrow |x| = |y|$, while the reverse implication does not hold.

3.4 Bounded and Unbounded sets

Let $S \subset IR$. If $\exists M \in IR$ such that $x \leq M \forall x \in S$, then S is called bounded above. This M is called an upper bound of S.

Again, if $\exists m \in IR$ such that $x \geq m \ \forall x \in S$ then S is called bounded below and such m is called a lower bound of S.

If S is bounded above as well as bounded below then S is said to be bounded. Thus S is bounded if $\exists\, m, M \in IR$ such that

$$m \leq x \leq M, \forall x \in S \qquad (2.6.1)$$

If $M \geq 0$, taking m = –M, the relation (2.6.1) reduces to

$|x| \leq M \ \forall x \in S.$

Hence S is said to be bounded if $\exists\, M \geq O$ such that

$|x| \leq M \ \ \forall x \in S.$

Consequently, a subset S is called unbounded or not bounded if it is either not bounded above or not bounded below.

Examples :

(1) The set $\left\{\frac{1}{n} : n \in IN\right\}$ is bounded. Here 0 and 1 are lower bound and upper bound respectively.

(2) Let $a, b \in IR$. Then (a, b), [a, b], (a, b] and [a, b) are bounded.

(3) The set IN is bounded below by 1 but not bounded above.

(4) The set $IR^+ = \{x \in IR : x \geq 0\}$ is bounded below but not bounded above, whereas the set $IR^- = \{x \in IR : x \leq 0\}$ is bounded above but not bounded below.

(5) The set Q, IR are unbounded.

Greatest and Smallest element of a set : Let $\phi \neq S \subset IR.$ If S contains a largest element M, i.e. $x \leq M \ \forall x \in S,$ then M is called the maximum (or largest or greatest) element of S. And if S contains a smallest element m, i.e., $x \geq m \ \forall x \in S,$ then 'm' is called the minimum (or smallest or least) element of S. In this case, we write Max S = M and min S = m.

For example, if S = {0, 2, 4, 6, 8, 10}, then max S = 10 and min S = 0.

Again for S = $\left\{\frac{1}{n} : n \in IN\right\}$, max S = 1, while S does not contain the minimum element.

Remark :

(1) For S = [a, b] $a, b \in IR, a < b,$ max S = b, which is also upper bound of S. And min S = a, which is also lower bound of S.

(2) The set S = (a, b) does not contain the maximum and minimum element, though S is bounded.

(3) Note that a bounded set S of IR may not contain an upper bound and (or) a lower bound. But if S has an upper bound (respectively a lower bound) then it will have infinitely many upper bounds (respectively lower bounds), because if M is an upper (respectively lower) bound of S, then every number greater (respectively less) than M is also an upper (respected, a lower) bound. Thus we get a set of upper bounds (respectively lower bounds) for a bounded above (respectively bounded below) set of IR.

We now define the following :

3.5 Supremum and Infimum

Let $\phi \neq S \subset IR.$ If M is an upper bound of S and any real number less than M is not an upper bound of S, then M is called supremum or least upper bound (lub) of S. Here, we write sup S (or lub S) = M.

Hence a real number M is supremum of S if

(i) M is an upper bound of S

and (ii) M $\leq$ K for every upper bound K of S.

Similarly, if 'm' is a lower bound of S and any real number greater than 'm' is not a lower bound of S then 'm' is called greatest lower bound (glb) or infimum of S.

Here, we write inf S (or glb S) = m.

Hence a real number m is infimum of S if

(i) m is a lower bound of S

and (ii) $m \geq k$ for every lower bound k of S.

Note : The supremum and infimum of a non empty subset of IR are unique, if they exist.

Examples :

(1) Let $a, b \in \mathbb{R}$ with a < b and S = [a, b] and T = (a, b).
Then sup S = b = Sup T and inf S = a = inf T.

(2) For S = $\left\{1+\frac{1}{n} : n \in \mathbb{N}\right\}$, sup S = 2 and inf S = 1.

(3) The supremum of $\mathbb{N}$ does not exist, while inf $\mathbb{N}$ = 1.

(4) The set □ has neither supremum nor infimum.

Theorem : Let $\phi \neq S \subset \mathbb{R}$ and let $M \in \mathbb{R}$. Then M is the supremum of S if and only if

(i) $x \leq M \ \forall x \in S.$

and (ii) for each $\in > 0, \exists$ a real number $x \in S$ such that x > M – $\in$.

Proof : Let $\in > 0$ be arbitrary. Since $M - \in < M,$ by definition of supremum, it follows that M = sup S

$\Leftrightarrow$ M is an upper bound of S and M – $\in$ is not an upper bound of S,

i.e. $\Leftrightarrow x \leq M \ \forall x \in S$ and $x > M - \in$ for some $x \in S.$

Theorem : Let $\phi \neq S \subset \mathbb{R}$ and $m \in \mathbb{R}$. Then m is the infimum of S if and only if (i) $x \geq m \ \forall x \in S$ and

(ii) for each $\in > 0, \exists$ a real number $x \in S$ such that $x < m + \in$.

Proof : Since $m < m + \in$ for arbitrary $\in > 0.$ So, by definition of infimum, the result follows :

Problem : Find the supremum and infimum of the following sets :

(i) $S = \{-2, 2\} \cup \left\{1+\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{-1-\frac{1}{n} : n \in \mathbb{N}\right\}$

$$=\left\{-2,2,1+\frac{1}{n},-1-\frac{1}{n}:n\in IN\right\}=\left\{-2,2,\frac{3}{2},-\frac{3}{2},\frac{4}{3},-\frac{4}{3},\frac{5}{4},-\frac{5}{4,}......\right\}.$$

Let $\in > 0$ be arbitrary.

$\forall x \in S \Rightarrow x \leq 2$ and $2 \in S, 2 > 2-\in$

$\therefore$ sup $S = 2$

Similarly we find that inf S = –2.

(ii) Consider $T=\left\{(-1)^n\left(1+\frac{1}{n}\right):n\in IN\right\}$ $=\left\{-2,\frac{3}{2},-\frac{4}{3},\frac{5}{4},-\frac{6}{5},\frac{7}{6},...\right\}$.

We find that $\forall\ x \in T \Rightarrow x \leq \frac{3}{2}$.

Let δ be an arbitrary small + ve number. Then $\frac{3}{2} > \frac{3}{2} - \delta$ and $\frac{3}{2} \in T$.

So, sup $T = \frac{3}{2}$ and inf $T = -2$.

Properties of Supremum and infimum : From the definition and above results one can prove the following :

(i) For any bounded set S, inf S $\leq$ sup S.

(ii) Sup S = max S, if max S exists and inf S = min S, if min S exists.

(iii) For $\phi \neq A \subset IR$ and $\phi \neq B \subset IR$,

$\inf(A \bigcup B) = \min\{\inf A, \inf B\}$

and sup $(A \bigcup B) = \max\{\sup A, \sup B\}$

Further if $A \subset B$ then inf B ≤ inf A ≤ sup A ≤ sup B

Problem : Let $\phi \neq S \subset IR$ and $T = \{x : -x \in S\}$.

Show that supT = inf S and inf T = – sup S.

Solution : Let sup S = B and inf S = b.

Let $\delta > 0$ be arbitrary small number. Then

$\forall x \in S \Rightarrow x \leq B \Rightarrow -x \geq -B$

and there is a member $x \in S$ such that $x > B - \delta \Rightarrow -x < -B + \delta.$

Thus $\forall -x \in T \Rightarrow -x \geq -B$

and there is some $-x \in T$ such that $-x < -B + \delta$.

$\therefore \inf T = -B = -\sup S$.

Similarly we can prove that sup T = – b = – inf S.

3.6 Summary

This chapter covered key concepts in real analysis, including intervals, pounded and: unbounded sets, and the notions of supremum and infimum. Intervals were. categorized based on their endpoints, while bounded sets were distinguished from unbounded ones. The concepts of least upper bound and greatest lower bound were explored, emphasizing their importance in mathematical analysis. These ideas are essential for deeper studies in real analysis and provide a basis for understanding limits, continuity, and convergence in various mathematical contexts.

3.7 Keywords

Intervals, Sunrernurn. Infimum. Open intervals

3.8 References

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- **Real Analysis: Modern Techniques and Their Applications** - Gerald B. Folland, Wiley
- **Real Analysis** - H.L. Royden, P.M. Fitzpatrick, Pearson
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- **Real Analysis for Graduate Students** - Richard F. Bass, American Mathematical Society
- **Introduction to Real Analysis** - Robert G. Bartle, Donald R. Sherbert, Wiley

3.9 Model Questions

1. Define an interval. Give examples of open, closed, and half-open intervals.
2. What is the difference between bounded and unbounded sets? Provide an example of each.
3. Explain the concepts of supremum and infimum with suitable examples.
4. Why is the supremum considered the least upper bound and not just any upper bound?
5. State and explain the completeness property of real numbers.
6. Determine whether the set $S = \{x \in R \mid O < x < 5\}$, is bounded. Justify your answer.
7. Find the supremum and infimum of the set
$$A = \{\frac{1}{n} \mid n \in N\}.$$
8. Is the set of natural numbers N bounded? Explain your reasoning.
9. Give an example of a set that has a supremum but no maximum element.
10. If a set has an infimum, does it always have a minimum? Justify with an example.

Unit 4 ❑ Bounded property

Structure

4.1 Objectives

This chapter introduces intervals, the completeness property of R\mathbb {R} R, and the Archimedean property. Students will understand these fundamental concepts and their significance in real analysis, particularly in limits and convergence.

4.2 Introduction

Real analysis builds upon rigorous foundations of real numbers. This chapter explores intervals, the completeness property ensuring every bounded set has a supremum, and the Archimedean property, highlighting the density of natural numbers in R\mathbb{R} R.

4.3 Completeness property of IR

In similar to field axioms and order axioms, the set of real numbers satisfies another important axiom, Known as **Completeness axiom,** as follows :

Every non-empty bounded above subset of IR has a supremum in IR.

With the above axiom, we can say that the set of real numbers is a complete ordered field. As a consequence of completeness axiom, we have the following theorem :

Theorem : Every non empty bounded below subset of IR has an infimum.

Proof : Let $\phi \neq S \subset \text{IR}$ such that S is bounded below. Then $\exists K \in \text{IR}$ such that $x \geq K \quad \forall x \in S$

Define a set $T \subset \text{IR}$ by $T = \{x : -x \in S\}$.

Clearly $T \neq \phi$ as $S \neq \phi$. Then by just previous problem, T is bounded above by –K. So, by completeness axiom, T has a supremum in IR, say M and by previous problem, –M is the infimum of S. Hence, the theorem is complete.

We have already seen that the set of rational numbers Q is an ordered field. However, Q does not satisfies the completeness axiom. Thus Q is not a complete ordered field. For this, it is sufficient to construct a non-empty bounded above subset of Q which does not have a supremum in Q.

Define $A = \{x \in Q^+ : x^2 < 2\}$, where Q^+ is the set of all positive rational numbers.

$\forall x \in A \Rightarrow x \in Q^+$ and $x^2 < 2 \Rightarrow x < 2$, which implies that 2 is an upper bound of A. Thus A is a non-empty bounded above subset of Q.

If possible let $\alpha(\in Q)$ be the supremum of A. Then $\alpha \geq 1$ and so $\alpha \in Q^+$.

There are three cases arises :

$$\alpha^2 = 2, \alpha^2 > 2, \alpha^2 < 2.$$

The case $\alpha^2 = 2$ is not possible as there is no rational number whose square is 2. So, $\alpha \neq \sup A$ in this case.

Now choose $\beta = \dfrac{3\alpha + 4}{2\alpha + 3} \in Q^+$.

Then $\alpha - \beta = \alpha - \dfrac{3\alpha + 4}{2\alpha + 3} = \dfrac{2(\alpha^2 - 2)}{2\alpha + 3}$ and $2 - \beta^2 = 2 - \left(\dfrac{3\alpha + 4}{2\alpha + 3}\right)^2 = \dfrac{2 - \alpha^2}{(2\alpha + 3)^2}$

If $\alpha^2 > 2$ then from above we get $\beta < \alpha$ and $\beta^2 > 2$, which implies that $\alpha \neq \sup A$.

Again if $\alpha^2 < 2$ then by similar way as above, it follows that $\alpha < \beta$ and $\beta^2 < 2$, which implies that α is not an upper bound of A. Thus $\alpha \neq \sup A$. Hence the supremum of A does not exist in Q. Consequently, Q is not complete.

Remark : The completeness axiom distinguishes between Q and IR as IR is complete while Q is not.

4.4 Archimedean property of IR

If x and y are any two positive real numbers with $y < x$ then $\exists n \in \mathrm{IN}$ such that $ny > x$.

Proof : If possible let $ny \le x$.

Set $S = \{ny : n \in \mathrm{IN}\}$. Then $S \ne \phi$ as $y \in S$. Also S is bounded above by x. So, by the completeness property of IR, sup S exists, say = M.

Now we have $ny \le M \ \forall n \in \mathrm{IN} \Rightarrow (n+1)y \le M$ as $n+1 \in \mathrm{IN}$

$\Rightarrow ny \le M - y \ \forall n \in \mathrm{IN}$.

This means that $M - y$ is an upper bound of S and $M - y < M$, which is a contradiction to the fact that M = sup S.

So $ny > x$ for some $n \in \mathrm{IN}$.

From the above property, the following results are immediately holds :

Result 1 : If y is a +ve real number and x is any real number then there exists a positive integer 'n' such that $ny > x$.

Result 2 : For any real number x, there exists a +ve integer n such that $n > x$.

Theorem 4.4.1 For any $x \in \mathrm{IR}, \exists\, m, n \in \mathbb{Z}$ such that $m < x < n$.

Proof : From Result 2, we have for any $x \in \mathrm{IR}, \exists\, n \in \mathbb{Z}^+$ (set of +ve integers) such that $x < n$ ....(2.9.1)

Since $x \in \mathrm{IR}, -x \in \mathrm{IR}$, so $\exists$ a +ve integer p such that $p > -x$.

i.e. $-p < x \Rightarrow m < x$ by taking $-p = m$. ...(2.9.2)

From (2.9.1) and (2.9.2), the result follows.

Theorem 4.4.2 For any $x \in \mathrm{IR}$, there exists a unique integer n such that $n \le x < n+1$.

Proof : Set [x] = n, where [x] is the integral part of x.

Then $n \le x$ ...(2.9.3)

We claim that $x < n + 1$. If not, $x \ge n+1$, which is an integer.

So, $[x] \ge n+1 \Rightarrow n \ge n+1$, which is absurd.

Thus $x < n + 1$ ...(2.9.4)

The result follows from (2.9.3) and (2.9.4).

Theorem 4.4.3 For any $x \in \mathrm{IR}$, there exists a unique integer n such that $x - 1 \le n < x$.

Proof : By theorem 2.9.1, for $x \in \mathrm{IR}$, $\exists$ two integers m and p such that $m < x < p$.

Choose $n = \max\{r \in N : r < x\}$ ...(2.9.5)

Then by Theorem 2.9.2, we get $n+1 \geq x$ i.e. $x-1 \leq n$ ...(2.9.6)

(2.9.5) and (2.9.6) gives the theorem.

By density property of Q we have seen that there are infinitely many rational numbers between any two rational numbers, which can be extended as the following :

Theorem 4.4.4 : There is at least one rational number and hence infinitely many rational numbers between any two distinct real numbers.

Proof : Let $x, y \in IR$ such that $x \neq y$ and $x < y$.

So, y – x > 0.

By Archimedean property for y – x and $1 \in IR, \exists$ a +ve iteger n such that n(y–x) >1

i.e. n x + 1 < ny ...(2.9.7)

It is clear that $nx \in IR$. So, by theorem 2.9.2, there exist a +ve integer 'm' such that

$m-1 \leq nx < m$ ...(2.9.8)

$\Rightarrow m \leq nx+1 < ny,$ ...(2.9.9)

using (2.9.7)

From (2.9.8) and (2.9.9), we get nx < m < ny

i.e. x < r < y, where $r = \dfrac{m}{n} \in Q$

Thus we get a rational number lying between x and y. By similar argument, we get rational number r_1 between x and r and another rational number r_2 between r and y such that

$x < r_1 < r < r_2 < y$.

Proceeding in this way, we can find inifintely many rational numbers lying between x and y.

For the case of irrational numbers,

Theorem 4.4.5 : There is at least one irrational number and hence infinitely many irrational numbers between any two distinct real numbers.

Proof : Let $x, y \in IR$ such that $x \neq y$ and $x < y$. Then $x-p < y-p$ for arbitrary irrational number 'p'. Since $x-p, y-p \in IR$ and $x-p \neq y-p, \exists$ a rational number r such that

x – p < r < y – p, by just previous theorem, i.e. $x < K(= r + p) < y$.

Here K must be irrational number as it is the sum of a rational number and an irrational number.

Thus we get an irrational number K between x and y. By similar argument as above, we get irrational number K' between x and K and another irrational number K" between K and y such that

$x < K' < K < K'' < y$.

Proceeding in this way, we can find infinitely many irrational numbers lying between x and y. Hence the proof of the theorem is complete.

By virtue of Theorem 2.9.4 and Theorem 2.9.5, we can state the following :

Theorem 4.4.6 There is at least one real number and hence infinitely many real numbers between any two distinct real numbers.

4.5 Summary

This chapter covered intervals, the completeness property ensuring supremum and infimum exist, and the Archimedean property stating no infinitely large or small real numbers exist, reinforcing foundational real analysis principles.

4.6 Keywords

- Open and closed intervals
- Supremum and infimum
- Completeness property
- Least upper bound property
- Archimedean property
- Bounded sets
- Rational and irrational numbers

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4.8 Model Questions

1. Define an interval. Give examples of different types of intervals.
2. State and explain the completeness property of $\mathbb{R}$.
3. What is the Archimedean property? Provide an example.
4. Prove that there is no smallest positive real number.
5. Explain how the completeness property relates to the existence of supremum and infimum.

Unit 5 ❑ Limit point

Structure

5.1 Objectives

- Understand different types of intervals and their properties.
- Define and explore neighborhoods of a point in a metric space.
- Identify and analyze limit points of a set.

5.2 Introduction

This chapter explores fundamental concepts in real analysis, including intervals, neighborhoods, and limit points. These ideas form the basis for continuity, convergence, and the structure of real numbers.

5.3 Neighbourhood of a point

A set N is called a neighbourhood (abbreuiated by nbd) of a point $p \in \mathrm{IR}$ if there exists an open interval I containing p and contained in N, i.e., $p \in I \subset N$.

The set $N-\{P\}$ is called a deleted neighbourhood of p.

Examples :

(1) The set IR is a nbd of each of its points, because

$\forall x \in \mathrm{IR},\ x \in (x-\in, x+\in) \subset \mathrm{IR}$ for every $\in > 0$. The open interval $(x-\in, x+\in)$ is known as $\in -$nbd of x.

(2) The set Q of rational numbers is not a nbd of any of its points, since if $x \in Q$, then $(x-\in, x+\in)$ contains an infinite number of irrational points and hence $(x-\in, x+\in) \not\subset Q$ for every $\in > 0$.

Properties of Neighbourhood

Theorem 5.3.1 : Every open interval is a neighbourhood of each of its points.

Proof : Let 'p' be an arbitrary point of the given open interval (a, b). Since every set is a subset of itself, we can write $p \in (a, b) \subseteq (a, b)$,

which means that (a, b) is a neighbourhood of p. As p is an arbitrary point of (a, b), so (a, b) is a neighbourhood of each of its points.

Corollary : Any closed interval [a, b] is a neighbourhood of each point in it except the points a and b.

Hints : $p \in (a, b) \in [a, b]$.

Theorem 5.3.2. Any superset of a neighbourhood of a point is also a neighbourhood of that point.

Proof : Let N be a neighbourhood of a point p and let $M \supset N$.

Since N is a neighbourhood p, so an open interval (a, b) containing p such that

$$p \in (a, b) \subset N \subset M,$$

which implies that M is a neighbourhood of p.

Since p and M are choosen arbitrarily, the result follows.

Theorem 5.3.3 : The intersection of two neighbourhoods of a point is also a neighbourhood of that point.

Proof : Let N_1 and N_2 be two neighbourhoods of a point p. So, $\exists \in_1 > 0$ and $\in_2 > 0$ such that

$p \in (p-\in_1, p+\in_2) \subset N_1$ and $p \in (p-\in_2, p+\in_2) \subset N_2$.

Choose $\in = \min\{\in_1, \in_2\}$ so that

$p \in (p-\in, p+\in) \subseteq (p-\in_1, p+\in_1) \subset N_1$

and $p \in (p-\in, p+\in) \subseteq (p-\in_2, p+\in_2) \subset N_2$,

which follows that

$p \in (p-\in, p+\in) \subset N_1 \bigcap N_2.$

Hence $N_1 \bigcap N_2$ is also a neighbourhood of p.

Note : By repeated applications of the above theorem, we can state the following :

The intersection of finitely many neighbourhoods of a point is also a neighbourhood of that point.

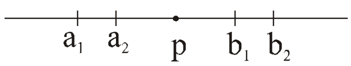
However, the intersection of an infinite number of neighbourhoods of a point may not be a neighbourhood of that point.

For example, for every $n \in IN, \left(-\frac{1}{n}, \frac{1}{n}\right)$ is a neighbourhood of 0.

But $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is not a neighbourhood of 0, as $\{0\}$ is finite set.

Theorem 5.3.4 : The union of two neighbourhoods of a point is also a neighbourhood of that point.

Proof : Let N_1 and N_2 be two neighbourhoods of a point $p \in IR$. So, $\exists$ open intervals (a_1, b_1) and (a_2, b_2) such that $p \in (a_1, b_1) \subset N_1$ and $p \in (a_2, b_2) \subset N_2$.



Choose $a_3 = \min \{a_1, a_2\}$ and $b_3 = \max \{b_1, b_2\}$.

Then $p \in (a_1, b_1) \bigcup (a_2, b_2) = (a_3 b_3)$.

Also $(a_1, b_1) \subset N_1 \bigcup N_2$ and $(a_2, b_2) \subset N_1 \bigcup N_2$

$\Rightarrow (a_3, b_3) = (a_1, b_1) \bigcup (a_2, b_2) \subset N_1 \bigcup N_2.$

Hence $p \in (a_3, b_3) \subset N_1 \bigcup N_2$,

which shows that $N_1 \bigcup N_2$ is a neighbourhood of p.

Note : By repeated applications of the above theorem, we can state the following :

The union of a finite number (or arbitrary number) neighbourhoods of a point is also a neighbourhood of that point.

5.4 Limit points of a set

Let $\phi \neq S \subseteq IR$. A point $p \in IR$ is called a **limit point** (or limiting point) of S if every deleted neighbourhood of p contains atleast one point of S.

Thus a point $p \in IR$ is a limit point of S if

$(N - \{p\}) \cap S \neq \phi,$

where N–{p} is the deleted neighbourhood of p.

A limit point of a set is also sometimes known as an **accumulation point** or a **condensation point** or a **cluster point** of the set.

Isolated point : A point of a set is called an isolated point of the set if it is not a limit point of that set.

Examples : The set $S = \left\{\frac{1}{n} : n \in IN\right\}$ has only a limit point 0, which is not a member of the set. However, each point in the set S is an isolated point of the set.

Remark : A limit point of a set may or may not be a member of the set. Moreover, a set may have no limit point, a unique limit point, or a finite or an infinite number of limit points.

Theorem 5.4.1 : Let $\phi \neq S \subseteq IR$. A point $p \in IR$ is a limit point of S if and only if every neighbourhood of p contains infinitely many points of S.

Proof : At first, let us take that every neighbourhod of p contain infinitely many points of S. So, every neighbourhood of p contains a point of S other than p. Consequently, p is a limit point of S.

Conversely, suppose that p is a limit point of S. We have to prove that every neighbourhood of p contains infinitely many points of S.

If possible, let a neighbourhood N of p contains only finite number of points $p_1, p_2, ..., p_n$ different from p.

Choose $\in = \min\{|p - p_1|, |p - p_2|, ..., |p - p_n|\}$.

The $\in > 0$ and $(p - \in, p + \in)$ is a neighbourhood of p which contains no point of S other than p. So, p is not a limit point of S, which is a contradiction to our assumption. Hence every neighbourhood of p contains infinitely many points of S.

Note : In view of the above theorem, the definition of limit point can be rewritten as :

A point p is a limit point of a non empty set S in IR if every neighbourhood of p contains infinitely many points of S.

Thus the empty set ϕ and a finite set have no limit point. So, a set, having limit point, must be infinite. Though there are so many infinite set which has no limit point. For example, the set of natural numbers has no limit points even though it is an infinite set.

Theorem 5.4.2 : Let $\phi \neq S \subset IR$ and S is bounded above. If S has no maximum member then sup S is a limit point of S.

Proof : Since S is a non empty bounded above subset of IR, the sup S exists (by completeness property) in IR and sup s = p(say). Clearly $p \neq S$ as S has no maximum member.

Let $\in > 0$ be arbitrary number.

Since sup S = p, so $\forall x \in S,\ x \leq p \Rightarrow x < p+\in$

and $\exists$ an $x \in s$ such that $x > p-\in$.

Hence $x \in (p-\in, p+\in)$ and $x \neq p$ as $x \in s$ and $p \notin s$.

This shows that every deleted $\in$-neighbourhood of p contains a point of S and hence p, i.e., sup S is a limit point of S.

Theorem 5.4.3 : Let $\phi \neq S \subset IR$ and S is bounded below. If S has no minimum member then inf S is a limit point of S.

Proof : The proof is similar as above just using the concept of infimum instead of supremum.

Derived set : The set of all the limit points of a set S is called the derived set of S and is denoted by S'.

Examples :

(1) For $S = \left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in IN\right\}, S' = \{-1, 1\}$.

(2) For any finite set A, A', = ϕ and hence $\phi' = \phi$.

(3) (a, b)' = [a, b] and [a, b]' = [a, b].

(4) Q' = IR.

Exercise : Find the derived set of the set $\left\{\frac{1}{m}+\frac{1}{n}+\frac{1}{p} : m, n, p \in IN\right\}$.

Solution : Let $S=\left\{\frac{1}{m}+\frac{1}{n}+\frac{1}{p}: m, n, p \in \mathbb{N}\right\}$

Let δ be an arbitrary small positive number.

Let us keep m, n are fixed and we choose p such that $\frac{1}{p}<\delta$.

Hence $\frac{1}{m}+\frac{1}{n}+\frac{1}{p}<\frac{1}{m}+\frac{1}{n}+\delta$ and also $\frac{1}{m}+\frac{1}{n}+\frac{1}{p}>\frac{1}{m}+\frac{1}{n}-\delta$.

Thus $\frac{1}{m}+\frac{1}{n}-\delta<\frac{1}{m}+\frac{1}{n}+\frac{1}{p}<\frac{1}{m}+\frac{1}{n}+\delta$,

where $\frac{1}{m}+\frac{1}{n}+\frac{1}{p} \in S$,

which implies that $\frac{1}{m}+\frac{1}{n} \in S', \ \forall m, n \in \mathbb{N}$.

Let us keep 'm' fixed and choose integers n and p such that $\frac{1}{n}<\frac{\delta}{2}, \frac{1}{p}<\frac{\delta}{2}$.

Therefore $\frac{1}{n}+\frac{1}{p}<\delta$, which implies that

$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}<\frac{1}{m}+\delta$ and $\frac{1}{m}+\frac{1}{n}+\frac{1}{p}>\frac{1}{m}-\delta$

Thus, $\frac{1}{m}-\delta<\frac{1}{m}+\frac{1}{n}+\frac{1}{p}<\frac{1}{m}+\delta$,

which implies that $\frac{1}{m} \in S', \forall m \in \mathbb{N}$.

Again let us choose m, n, p such that $\frac{1}{m}<\frac{\delta}{3}, \frac{1}{n}<\frac{\delta}{3}$ and $\frac{1}{p}<\frac{\delta}{3}$.

$\therefore \frac{1}{m}+\frac{1}{n}+\frac{1}{p}<\delta$ and hence $0-\delta<\frac{1}{m}+\frac{1}{n}+\frac{1}{p}<0+\delta$.

This shows that $0 \in S'$.

Thus $S' = \left\{0, \frac{1}{m}, \frac{1}{m} + \frac{1}{n} : m, n \in \mathrm{IN}\right\}$

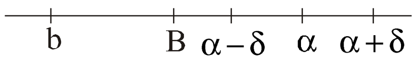
Theorem 5.4.5 : The derived set of a bounded set is bounded.

Proof : Let S be a bounded set, So, sup S and inf S exists and let sup S = B and inf S = b.

Therefore, $\forall x \in S \Rightarrow b \le x \le B$.

We have to show that S' is bounded. If possible, let S' is not bounded above. Then $\exists$ a $\alpha \in S'$ such that $\alpha > B$.

Choose $\delta = \frac{\alpha - B}{2}$.



As $\alpha \in S'$, therefore α is a limit point of S and hence the interval $(\alpha - \delta, \alpha + \delta)$ contains a member $x \in S$, where $x \ne \alpha$.

As x lies in $(\alpha - \delta, \alpha + \delta)$, therefore $x > B$, which is a contradiction to the fact that $x \le B$.

Thus the set S' is bounded above.

Similarly we can prove that the set S' is also bounded below. Hence the derived set of a bounded set is bounded.

Problem : Let A and B be any two subsets of IR such that $A \subset B$.

Show that $A' \subseteq B'$.

Solution : $\forall x \in A' \Rightarrow$ every deleted neighbourhood of x contains at least one point of A.

$\Rightarrow$ every deleted neighbourhood of x contains at least one point of B (since $A \subset B$)

$\Rightarrow$ x is a limit point of B

$\Rightarrow$ $x \in B'$

Hence $A' \subseteq B'$.

Problem : For any two subsets A and B of IR, show that $(A \cup B)' = A' \cup B'$.

Solution : Since $A \subseteq A \cup B$ and $B \subseteq AUB$, by above problem we have that

$A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$.

Thus $A' \cup B' \subseteq (A \cup B)'$ ...(1)

Again $x \in (A \cup B)' \Rightarrow$ every deleted neighbourhood N (say) of x contains at least one point of $A \cup B$

$\Rightarrow$ N contains at least one point of A or B.

$\Rightarrow \; x \in A'$ or $x \in B'$

$\Rightarrow \; x \in A' \bigcup B'$

So, $(A \bigcup B)' \subseteq A' \bigcup B'$ ...(2)

From (1) and (2) it follows thle result.

Problem : Show that $(A \bigcap B)' \subseteq A' \bigcap B'$ for any two subsets A and B of IR.

Solution : Since $(A \bigcap B) \subseteq A$, we have $(A \bigcap B)' \subseteq A'$ and $A \bigcap B \subseteq B$, we have $(A \bigcap B)' \subseteq B'$ Thus $(A \bigcap B)' \subseteq A' \bigcap B'$.

Note : However, $A' \bigcap B' \neq (A \bigcap B)'$ in general. In fact $A' \bigcap B' \not\subset (A \bigcap B)'$, in general For this, let A = (0, 1) and B = (1, 2). Then A' = [0, 1] and B' = [1, 2].

Therefore $A' \bigcap B' = [0,1] \bigcap [1,2] = \{1\}$, while $(A \bigcap B)' = \phi' = \phi$.

5.5 Summary

Intervals classify subsets of real numbers, neighborhoods describe proximity, and limit points determine set boundaries. These concepts are essential for deeper studies in topology, calculus, and mathematical analysis.

5.6 Keywords

Intervals, Open Set, Closed Set, Neighbourhood, Limit Point, Accumulation Point, Metric Space, Real Numbers, Topology, Convergence.

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5.8 Model Questions

1. Find the limit points of the set

   Q = [1, 2, 3, 4, ...].

2. What is the limit point of the set?

$$S = \{\frac{1}{n^2} \mid n \in N\}.$$

3. To the sum of two limit point of a subset of R always a limit point of that subject?

4. For any fund $P \in N$, how may subset, of R can you construct whose limit point is P?

Unit 6 ❑ Open sets and Closed sets

Structure

6.1 Objectives

To explore fundamental concepts in real analysis, including open and closed sets, set closure, and the Bolzano-Weierstrass theorem, essential for understanding the structure and behavior of subsets in metric spaces.

6.1 Introduction

This chapter introduces open and closed sets, their closure properties, and the Bolzano-Weierstrass theorem, providing a foundation for analyzing limits, accumulation points, and compactness in real analysis.

6.3 Open sets and closed sets

Before defining open sets, first of all we define the following :

Interior point : Let $S \subset \mathrm{IR}$ and $p \in S$. Then p is called an interior point of S if $\exists$ a neighbourhood N of p such that $p \in N \subset S$.

The set of all interior points of S is called the interior of S and it is denoted by Int (S) or S^o.

It may be noted that Int(S) $\subseteq S$. Since every neighbourhood of a point contains infinitely many points, so no point of any finite set can be an interior point. Thus Int S = ϕ for any finite set S. Also Int $\phi = \phi$.

Moreover, Int (Int S) = Int S, i.e., $(S^o)^o = S^o$ for any set S.

Examples :

(1) Int (a, b) = (a, b) and Int [a, b] = (a, b) for $a, b \in IR$ with a < b.

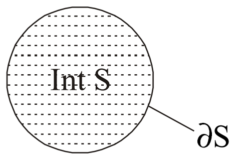
(2) Int IR = IR, since each point of IR is an interior point of IR.

(3) Int IN = ϕ, since every neighbourhood of $P \in IR$. contains points not belonging to IN, i.e. no point 'p' of IN can not be an interior point of IN.

(4) Int Q = ϕ, since every neighbourhood of $p \in Q$ contains rational as well as irrational points, i.e., p can not be an interior point of Q.

Boundary point : Let $S \subset IR$ and $p \in IR$. Then p is called a boundary point of S if every neighbourhood of p can intersect S & S' (same notation for derived and complement of a set). The set of all boundary points of S is called boundary of S and it is denoted by ∂S.

It may be noted that $\partial S = \partial S'$.



Examples :

(1) If S = (a, b) or [a, b], then $\partial S = \{a, b\}$.

(2) If S = $\{(x, y) \in IR^2 : x^2 + y^2 \le 1\}$, then

Int S = $\{(x, y) \in IR^2 : x^2 + y^2 < 1\}$ and $\partial S = \{(x, y) \in IR^2 : x^2 + y^2 = 1\}$.

Remark : A boundary point of a set S may or may not be a point of S.

Open set A non empty set G in IR is called an open set if every point of G is an interior point of G.

Thus a non empty set G in IR is an open set if and only if for each point $p \in G$, $\exists$ a neighbourhood N [i.e., an open interval (a, b)] such that

$p \in N \subset G$ (i.e., $p \in (a, b) \subset G$).

In other words, a non-empty set G in IR is called an open set if G is a neighbourhood of each of its points.

Note that a finite set need not be open.

Examples :

(1) The entire set $IR = (-\infty, \infty)$ is open as for each $x \in IR$, IR is a neighbourhood of x.

(2) Each open interval (a, b) is an open set, because every point of (a, b) is an

interior point, while the closed interval [a, b] is not an open set as a & b are not interior points of [a, b]. Similarly, (a, b] and [a, b) are not open sets.

(3) The null set ϕ is open set, since ϕ contains no points, so ϕ satisfies the defination of open sets.

Theorem 6.3.1 : The intersection of two open sets in IR is open.

Proof : Let G_1 and G_2 be two open sets in IR. We have to show that $G_1 \cap G_2$ is an open set.

If $G_1 \cap G_2 = \phi$ then $G_1 \cap G_2$ is an open set, as ϕ is an open set.

So, let us suppose that $G_1 \cap G_2 \neq \phi$ and $x \in G_1 \cap G_2$.

Then $x \in G_1$ and $x \in G_2$.

Since G_1 and G_2 both are open sets, x is an interior point both of G_1 and G_2 and hence x is an interior point of $G_1 \cap G_2$.

Since x is arbitrary point of $G_1 \cap G_2$, so every point of $G_1 \cap G_2$ is an interior point of $G_1 \cap G_2$. Hence $G_1 \cap G_2$ is an open set.

Theorem 6.3.2 : The intersection of a finite number of open sets in IR is an open set.

Proof : Let G_1, G_2, ..., G_n be n open sets and let $G = \bigcap_{i=1}^{n} G_i$. We have to show that G is open.

If G = ϕ, then G is an open set.

So, let us suppose that $G \neq \phi$ and take $x \in G = \bigcap_{i=1}^{n} G_i$

So, $x \in G_i$ for each i = 1, 2, ..., n.

Since G_i is an open set, so, x is an interior point of G_i for each = 1, 2, ..., n.

Hence x is an interior point of $G = \bigcap_{i=1}^{n} G_i$.

Since x is choosen arbitrarily, every point of $G = \bigcap_{i=1}^{n} G_i$ is an interior point of G. Hence $G = \bigcap_{i=1}^{n} G_i$ is an open set.

Note : The intersection of an arbitrary family of open sets may or may not be an open set.

For example, for each $n \in \mathbb{N}$, let $G_n = (0, n)$. Then each G_n is an open set.

Also $\bigcap_{n=1}^{\infty} G_n = (0,1) \cap (0,2) \cap ..... \cap (0, n) \cap .... = (0, 1),$ which is open.

Again if we consider $G_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$. Then for each $i \in \mathbb{N}, G_i$ is an open set.

However, $\bigcap_{i=1}^{\infty} G_i = (-1,1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap ...... = \{0\}$.

which is a finite set and hence not open set.

Similarly if we take $B_i = \left(0, 1 + \frac{1}{i}\right)$, where i is any positive integer. Then each B_i, being an open interval, is an open set, whereas $\bigcap_{i=1}^{\infty} B_i = (0,1]$ is not open as the point 1 in $\bigcap_{i=1}^{\infty} B_i$ is not an interior point of $\bigcap_{i=1}^{\infty} B_i$.

Theorem 6.3.3 : The union of an arbitrary family of open sets is open set.

Proof : Let $\{G_i : i \in \wedge\}$ be an arbitrary family of open set, where $\wedge$ is an index set.

Put $G = \bigcup_{i \in \wedge} G_i$

We have to show that G is an open set.

Let $x \in G$. Then $x \in G_{i_0}$ for some $i_0 \in \wedge$.

Since G_{i_0} is open, so x is an interior point of G_{i_0} and hence $\exists$ a neighbourhood N of x such that $x \in N \subset G_{i_0}$.

Since $G_{i_0} \subseteq G$, we get $x \in N \subseteq G$, which implies that x is an interior point of G. As x is arbitrarily choosen, so every point of G is an interior point of G. Consequently, G is open.

Corollary : The union of two open sets is an open set.

Theorem 6.3.4 : A subset G of IR is open if and only if it is a union of open intervals.

Proof : Let us suppose that G is open set and $\{G_i : i \in \wedge\}$ be an arbitrary family of open intervals contained in G, where $\wedge$ is an index set.

We have to show that $G = \bigcup_{i \in \wedge} G_i$.

Evidently $\bigcup_{i \in \wedge} G_i \subseteq G$. ... (2.12.1)

Again if $x \in G$, then x is an interior point of G as G is open. So, there exists some open interval G_{i_0} in $\{G_i : i \in \wedge\}$ containing x, i.e. $x \in G_{i_0} \subseteq \bigcup_{i \in \wedge} G_i$

which implies that $G \subseteq \bigcup_{i \in \wedge} G_i$. ...(2.12.2)

From (2.12.1) and (2.12.2), it follows that $G = \bigcup_{i \in \wedge} G_i$.

Conversely, let G be a union of open intervals. Then since each open interval is an open set, G is a union of open sets. Hence G is open.

Theorem 6.3.5 : Let $S \subset$ IR. Then
(i) Int S equals to the union of all open subsets of S.
(ii) Int S is an open set.
(iii) Int S is the largest open subset of S.
(iv) S is open if and only if Int S = S.

Proof : (i) Let $\{G_i\}$ be the collection of all open subsets of S. We have to show that Int $S = \bigcup_i G_i$

Let x $\in$ Int S. Then x must belongs to some open subset, say G_{i_0} of S and hence $x \in \bigcup_i G_i$.

Thus Int $S \subseteq \bigcup_i G_i$. ...(2.12.3)

Now let us suppose $x \in \bigcup_i G_i$ so that $x \in G_{i_1}$, for some i_1. Since G_{i_1} is open, x is an interior point of G_{i_1}.

But $G_{i_1} \subseteq S$ and hence x is an interior point of S, i.e. x $\in$ Int S. Hence

$$\bigcup_i G_i \subseteq \text{Int}\, S. \qquad ...(2.12.4)$$

From (2.12.3) and (2.12.4), we get Int $S = \bigcup_i G_i$.

(ii) From (i) we have Int $S = \bigcup_i G_i$, which is the union of arbitrary family of open sets, so Int S is open.

(iii) Let $\{G_i\}$ be the collection of all open subsets of S. Then $G \in \{G_i\} \Rightarrow G \subseteq \bigcup_i G_i \Rightarrow G \subseteq \text{Int}\, S,$ as Int $S = \bigcup_i G_i$. This shows that Int S is the largest open subset of S.

(iv) If S is open, then $S \subseteq \text{Int}\, S,$ as Int S is the largest open subset of S. Also Int S $\subseteq$ S always. Hence Int S = S.

Conversely, if Int S = S, then S is open as Int S is open by (ii).

Theorem 6.3.6. Let S and T be two sets such that $S \subset T$.

Then $S^o \subset T^o$.

Proof : Let p be an arbitrary point of S^o. Then

$p \in S^o \Rightarrow S$ is a neighbourhood of p.

$\Rightarrow$ T is a neighbourhood of p.

$\Rightarrow p \in T^o$.

Thus $p \in S^o \Rightarrow p \in T^o$ and hence $S^o \subset T^o$.

Theorem 6.3.7 : For any two sets S and T, $(S \cap T)^o = S^o \cap T^o$.

Proof : Since for any two sets S and T,

$S \cap T \subset S$ and $S \cap T \subset T$.

So, we have by Theorem 2.12.6 that

$(S \cap T)^o \subset S^o$ and $(S \cap T)^o \subset T^o$

Hence $(S \cap T)^o \subset S^o \cap T^o$ ...(2.12.5)

Again let p be an arbitrary point of $S^o \cap T^o$.

Then we have

$p \in S^o \cap T^o \Rightarrow p \in S^o$ and $p \in T^o$

$\Rightarrow$ S is a neighbourhood of p and T is a neighbourhood of p.

$\Rightarrow S \cap T$ is a neighbourhood of p.

$\Rightarrow p \in (S \cap T)^o$

Hence $S^o \cap T^o \subset (S \cap T)^o$ ...(2.12.6)

From (2.12.5) and (2.12.6) it follows that $(S \cap T)^o = S^o \cap T^o$

Theorem 6.3.8 : For any two sets S and T, $S^o \cup T^o \subset (S \cup T)^o$.

Proof : Since for any two sets S and T, we have

$S \subset S \cup T$ and $T \subset S \cup T$.

So, by virtue of Theorem 2.12.6, we have that

$S^o \subset (S \cup T)^o$ and $T^o \subset (S \cup T)^o \Rightarrow S^o \cup T^o \subset (S \cup T)^o$.

Remark : In general $S^o \cup T^o \neq (S \cup T)^o$. In fact $(S \cup T)^o \not\subset S^o \cup T^o$, in general.
For this, let us consider S = [0, 1] and T = [1, 3].
Then S^o = (0, 1) and T^o = (1, 3). Also $S \cup T$ = [0, 3] and hence $(S \cup T)^o$ = (0, 3). But $S^o \cup T^o = (0, 1) \cup (1, 3) = (0, 3) - \{1\}$.

Thus $S^o \cup T^o \not\subset (S \cup T)^o$ and hence $S^o \cup T^o = (S \cup T)^o$.

Closed Set : A subset F of IR is called a closed set if all the limit points of F are members of F, i.e. $F' \subseteq F$, where F' is the derived set of F.

Examples :

(1) Any closed interval [a, b] is closed, while (a, b) is not.

(2) The sets [a, b) and (a, b] are neither open nor closed.

(3) Every finite set F is closed, since $F' = \phi \subset F$.

(4) The entire set IR is closed.

(5) The null set ϕ is closed, since $\phi' = \phi \subseteq \phi$.

Remark : The words ‘open’ and ‘closed’ are not antonyms. Any set in IR may be of four types such as

(i) open, for example the open interval (a, b) in IR.

(ii) closed, for example the closed interval [a, b] in IR.

(iii) both open and closed, for example the sets ϕ and IR.

(iv) neither open nor closed, for example the intervals (a, b] and [a, b).

The relationship between open sets and closed set are characterised by the following :

Theorem 6.3.9 : A set F in IR is closed if and only if its complement F^c is open.

Proof : At first, Let us take F is closed. We have to show that F^c is open.

Let p be an arbitrary element of F^c. So, $p \notin F$.

Since F is closed and $p \notin F$, so 'p' is not a limit point of F. So $\exists$ a neighbourhood N containing p such that $F \cap N = \phi$, which means that $p \in N \subseteq F^c$.

Consequently, p is an interior point of F^c. Hence F^c is open.

Conversely, suppose that F^c is open. We show that F is closed. For this, let p be a limit point of F. Then every deleted neighbourhood of p contains at least one point of F. Hence there is no neighbourhood of p, which is contained in F^c. So $p \notin \text{Int}(F^c) = F^c$ as F^c is open, by Theorem 2.12.5(iv). Therefore $p \in F$. Since p is arbitrary, we may conclude that $F' \subseteq F$ and hence F is closed.

Corollary : A set G in IR is open if and only if its complement G^c is closed.

Proof : It follows from above theorem by just taking $F = G^c$ and use $(G^c)^c = G$, i.e. complement of complement of a set is itself.

Theorem 6.3.10 : The derived set of every set is closed.

Proof : Let S be a set and S' be its derived set. We show that S' is closed. For this, let us take α be a limit point of S'. We have to show that $\alpha \in S'$, i.e, α is a limit point of S.

Let $\delta > 0$ be an arbitrary number.

Since α is a limit point of S', the interval $(\alpha - \delta, \alpha + \delta)$ contains an infinite number of members of S' other than α.

Let $\beta \in (\alpha - \delta, \alpha + \delta) \subset S'$ and $\beta \neq \alpha$.

Since $\beta \in S'$, therefore β is a limit point of S. So, the interval $(\alpha - \delta, \alpha + \delta)$ contains an element of S other than α. This shows that α is a limit point of S and hence the theorem is proved.

Theorem 6.3.11 : The intersection of two closed sets is a closed set.

Proof : Let F_1 and F_2 be two closed sets. Then F^c_1 and F^c_2 are open sets (by Theorem 2.12.9) and hence $F_1^c \cup F_2^c$ is an open set as union of two open sets is an open set.

Since $F_1^c \cup F_2^c = (F_1 \cap F_2)^c$, by De Morgan's law. So, $(F_1 \cap F_2)^c$ is an open set and hence $F_1 \cap F_2$ is a closed set.

Theorem 6.3.12 : The intersection of an arbitrary family of closed sets is closed.

Proof : Let $\{F_i : i \in \wedge\}$ be an arbitrary family of closed sets, where $\wedge$ is any index set.

Put $F = \bigcap_{i \in \wedge} F_i$

Using De-Morgan's Law, we have

$$F^c = \left(\bigcap_{i \in \wedge} F_i\right)^c = \bigcup_{i \in \wedge} F_i^c$$

Since each F_i^c is open, so F^c is the union of an arbitrary family of open sets. So, by theorem 2.12.3, F^c is open and hence by Theorem 2.12.9, F is closed.

Theorem 6.3.13 : The union of two closed sets is a closed set.

Proof : Let F_1 and F_2 be two closed sets.

So, F_1^c and F_2^c are open sets, by Theorem 2.12.9.

$\Rightarrow F_1^c \cap F_2^c$ is an open set, by Theorem 2.12.1.

$\Rightarrow (F_1 \cup F_2)^c$ is an open set, by De-Morgan's law.

$\Rightarrow F_1 \cup F_2$ is a closed set, by Theorem 2.12.9.

Theorem 6.3.14. The union of a finite number of closed sets is a closed set.

Proof : Let $F_1, F_2, ..., F_n$ be n closed sets. Then $F_1^c, F_2^c, ..., F_n^c$ are open sets and hence $\bigcap_{i=1}^{n} F_i^c$, the intersection of a finite number of open sets, is an open set.

So, by De-Morgan's law, $\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$ is an open set and hence $\bigcup_{i=1}^{n} F_i$ is a closed set, by Theorem 2.12.9.

Note : The union of an arbitrary family of closed sets may or may not be closed.

For example, for each $n \in \mathrm{IN}$, let $F_n = \left[1, \frac{n+1}{n}\right]$. Then each F_n is a closed set.

Now $\bigcup_{n=1}^{\infty} F_n = [1, 2] \cup \left[1, \frac{3}{2}\right] \cup \left[1, \frac{4}{3}\right] \cup ..... =$ [1, 2], which is a closed set.

Again if we consider $S_n = \left[0, \frac{n}{n+1}\right]$ for each $n \in \mathrm{IN}$. Then each S_n is a closed set.

However $\bigcup_{n=1}^{\infty} F_n = \left[0, \frac{1}{2}\right] \cup \left[0, \frac{2}{3}\right] \cup \left[0, \frac{3}{4}\right] \cup ......$

= [0, 1), which is not a closed set.

Problem : Let G be an open set and F be a closed set in IR. Show that (i) G – F is open and (ii) F – G closed.

Solution : (i) Let $x \in G - F$. Therefore $x \in G$ but $x \notin F$. Since $x \in G$ and G is open, so x is an interior point of G. Thus there is a positive number $\in_1$ such that

$x \in (x - \in_1, x + \in_1) \subseteq G.$

Again since $x \notin F$ and F is closed, so x cannot be a limit point of F. Therefore, there exists a positive number $\in_2$ such that

$(x - \in_2, x + \in_2) \bigcap F = \phi$

Choose $\in = \min\{\in_1, \in_2\}$.

Then $x \in (x - \in, x + \in) \subseteq G$ and $(x - \in, x + \in) \bigcap F = \phi$,

which implies that $x \in (x - \in, x + \in) \subseteq G - F.$

This shows that x is an interior point. Hence G – F is open as x is arbitrary.

(ii) Again let p be a limit point of F – G.

Since F – G $\subseteq$ F, therefore p is a limit point of F.

So, $p \in F$ as F is closed

We now show that $p \notin G$. If possible, let $p \in G$. Then there exists a positive number $\in$ such that

$p \in (p - \in, p + \in) \subseteq G$.

This shows that $(p - \in, p + \in) \bigcap (F - G) = \phi$,

which is a contradiction to our assumption that p is a limit point of F – G.

Thus $p \notin G$ and hence $p \in F - G$, which means that F – G contains all its limit points and hence it is closed.

6.4. Closure of a set

Let S be a subset of IR. The closure of S, denoted by $\overline{S}$, is the intersection of all closed supersets of S, i.e. $\overline{S} = \bigcap\{F : F \text{ is closed and } S \subseteq F\}$.

Note that $S \subseteq \overline{S}$ for any subset S of IR.

Also $\overline{\phi} = \phi$ and $\overline{IR} = IR$.

Theorem 6.4.1 : If S is any subset of IR then

(i) $\overline{S}$ is closed

(ii) $\overline{S}$ is the smallest closed superset of S.

(iii) S is closed $\Leftrightarrow S = \overline{S}$.

Proof : (i) From the definition of $\overline{S}$, it is the intersection of some closed sets containing S. Since intersection of an arbitrary family of closed sets is closed, so $\overline{S}$ is closed.

(ii) By definition of $\overline{S}$ (closure of S) and using above (i), (ii) follows.

(iii) Let us suppose that $S = \overline{S}$. Since $\overline{S}$ is always closed, therefore S is closed.

Converely suppose that S is closed. Then clearly S is the smallest closed superset containing itself. Consequently $S = \overline{S}$.

Note : Since for any set S in IR, $\overline{S}$ is always closed. Thus $\overline{\left(\overline{S}\right)} = \overline{S}$ by above (iii).

Theorem 6.4.2 If S is any subset of IR, then $\overline{S} = S \cup S'$, where S' is the derived set of S.

Proof : We now show that $S \cup S'$ is closed. For this, let x be any limit point of $S \cup S'$. Then x must be a limit point of S and (or) S'.

If x is a limit point of S, then $x \in S'$. Again, if x is a limit point of S' then $x \in S'$ as S' is always closed. Thus, in both the cases, $x \in S'$. Hence $x \in S \cup S'$, and consequently $S \cup S'$ is closed.

Since $S \cup S'$ is a closed superset of S, and $\overline{S}$ is the smallest closed superset of S, we have

$\overline{S} \subseteq SUS'$ ... (2.13.1)

Again, since $\overline{S}$ is closed, we have $\overline{S}' \subseteq \overline{S}$.

Now $S \subseteq \overline{S} \Rightarrow S' \subseteq \overline{S}' \subseteq \overline{S}$ and $S \subseteq \overline{S}$ always, we may conclude that

$S \cup S' \subseteq \overline{S}$. ... (2.13.2)

From (2.13.1) and (2.13.2), it follows that $\overline{S} = S \bigcup S'$

Remark : The above theorem can be used as alternative definition of closure of a set. We can also find the closure of a set using the formula in above theorem. For example,

(1) $\overline{\mathbb{N}} = \mathbb{N} \bigcup \mathbb{N}' = \mathbb{N} \bigcup \phi = \mathbb{N}$ as $\mathbb{N}' = \phi$

(2) $\overline{\square} = \square \bigcup \square' = \square \phi = \bigcup$ as $\square' = \phi \square$

(3) $\overline{IR} = IR \bigcup IR' = IR \bigcup IR = IR$ as $IR' = IR$.

(4) $\overline{Q} = Q \bigcup Q' = Q \bigcup IR = IR$ as $Q' = IR$.

(5) For $S = \left\{\frac{1}{n} : n \in IN\right\}, S' = \{0\}$ and hence $\overline{S} = S \bigcup S' = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$.

Theorem 6.4.3 : For any two sets S and T, $S \subset T \Rightarrow \overline{S} \subset \overline{T}$.

Proof : Let $S \subset T$ and $x \in \overline{S}$.

Then $x \in S \bigcup S' \Rightarrow x \in S$ or $x \in S'$

$\Rightarrow x \in T$ or $x \in T'$, as $S \subset T \Rightarrow S' \subset T'$

$\Rightarrow x \in T \bigcup T' \Rightarrow x \in \overline{T}$

So, $\overline{S} \subset \overline{T}$.

Theorem 6.4.4 : If S and T are two subsets of IR then $\overline{S \bigcup T} = \overline{S} \bigcup \overline{T}$.

Proof : $\overline{S \bigcup T} = (S \bigcup T) \bigcup (S \bigcup T)'$

$= (S \bigcup T) \bigcup (S' \bigcup T')$, as $(S \bigcup T)' = S' \bigcup T'$

$= (S \bigcup S') \bigcup (T \bigcup T') = \overline{S} \bigcup \overline{T}$.

Theorem 6.4.5 : If S and T are two subsets of IR, then $\overline{S \bigcap T} \subset \overline{S} \bigcap \overline{T}$.

Proof : Since $S \bigcap T \subset S$ and $S \bigcap T \subset T$.

Therefore, $\overline{S \bigcap T} \subset \overline{S}$ and $\overline{S \bigcap T} \subset \overline{T}$ by theorem 2.12.17.

$\Rightarrow \overline{S \bigcap T} \subset \overline{S} \bigcap \overline{T}$.

Remark : However $\overline{S \bigcap T} \neq \overline{S} \bigcap \overline{T}$ in general, for any two subsets S and T of IR. For this, let S = (1, 2) and T = (2, 3).

Then $\overline{S} = [1, 2]$ and $\overline{T} = [2, 3]$.

$\therefore \overline{S} \bigcap \overline{T} = \{2\}$ and $S \bigcap T = \phi$, which implies that $\overline{S \bigcap T} = \overline{\phi} = \phi$.

Thus $\overline{S} \cap \overline{T} \neq \overline{S \cap T}$.

Some important sets :

(i) A set S is called dense in IR if $\overline{S}$ = IR

(ii) A set S in IR is called dense-in-itself if S $\subset$ S'.

(iii) A set S in IR is called perfect if S = S'.

For example,

(i) The set Q is dense in IR as $\overline{Q} = \text{IR}$. Also Q is dense-in-itself as $Q \subset Q'$. Similarly IR is dense-in-itself.

(ii) If we consider $S = (a, b) \subset \text{IR}$. Then $S' = [a, b]$. So S $\subset$ S' and hence S is dense-in-itself.

(iii) Let S = [a, b], a, b $\in$ IR. Then S' = [a, b]. So, S is a perfect set.

6.5 Bolzano Weierstrass Theorem for sets

In section 2.11, we have seen that a finite set has no limit point. Also an infinite set may or may not have a limit point. For example, the infinite set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has limit point 0, while the infinite set $\mathbb{Z}$ of integers has no limit point. So, a natural question arises— what is the sufficient condition for the existence of a limit point of an infinite set. The following theorem known as Bolzano— Weierstrass Theorem gives us the said sufficient condition.

Theorem 6.5.1 (Bolzano-Weierstrass Theorem) : Every bounded infinite subset of IR has at least one limit point.

Proof : Let S be a bounded infinite subset of IR. Since S is bounded, So sup S and inf S exists by completeness property of IR.

Let inf S = m and Sup S = M.

Define a subset H of IR by

H = {x $\in$ IR : x exceeds at most finitely many elements of S}.

Then m $\in$ H as m does not exceed any element of S and hence H $\neq \phi$.

However, M exceeds infinitely many elements of S, since S is infinite and Sup S = M. So, there is no number greater than or equal to M in H. Consequently M is an upper bound of H. So, H is a non-empty bounded above subset of IR.

Therefore sup H exists and Sup H = α (say).

We now show that a is a limit point of S.

Choose $\varepsilon > 0$.

Since Sup H = α, so $\exists$ an $y \in H$ such that $\alpha - \in < y$.

So, $\alpha - \in$ exceeds at most finitely many elements of S as $y \in H$.

Also by definition of sup, $\alpha + \in$ can not belongs to H. So $\alpha + \in$ exceeds infinitely many elements of S. Thus for each $\in > 0$, the $\in$-neighbourhood $(\alpha - \in, \alpha + \in)$ of α contains infinitely many elements of S. Hence α is a limit point of S.

Remark : In above theorem, the condition of boundedness is only sufficient condition for the existence of a limit point of an infinite set, while this condition is not necessary for an infinite set may have a limit point. For this, the set of rational numbers Q is an infinite and unbounded set and Q has limit points. In fact Q' = IR.

6.6 Summary

In this unit we have discussed many important properties of IR (set of real numbers) like algebraic property, order property and completeness property. Through this unit, the students can learn the concept of neighbourhood of a point in IR, limit point of a set, open set, closed set in IR etc. The students also can know the sufficient condition for the existence of limit points of a set. Many results regarding the topic are given here. One can study more. For them, a list of references is given in section 2.18. Some important data and results are cited in section 2.16 (summaries) at a glance. For understand the topic clearly, some model questions are given in section 2.19.

- The system of real numbers can be described by means of certain axioms which can be divided into three categories, namely Field axioms, Order axioms and completeness axiom. The system IR of real numbers equipped with above three axioms is called a complete ordered field.
- The set of rational numbers is an ordered field but not a complete ordered field.
- A set is countable if it is either finite or enumerable. A set is uncountable if it is not countable.
- Subset of a countable set is a countable set.
- The cartesian product of two countable sets is countable.
- A real number of the form $\frac{p}{q}$, where p, q $\in \mathbb{Z}$, $q \neq 0$ and gcd (p, q) = 1, is a rational number.
- $\sqrt{m}$, where m is a non-square positive integer, is an irrational number.

- The terms 'real number' and a 'point' on the real line can be used interchangeably.
- (Archimedean property) If x and y are any two positive real numbers with $y < x$ then $\exists\ n \in \mathbb{N}$ such that $ny > x$.
- Between any two distinct real numbers, there exists infinitely many rational numbers, irrational numbers and hence real numbers.
- The set IR is a neighbourhood of each of its points, while each of the set IN, Z, Q and the set of irrational numbers are not a neighbourhood of any of its points.
- A set having limit point must be infinite or in otherwords a finite set has no limit points.
- Every infinite and bounded set in IR has at least one limit point. (Bolzano Weierstrass Theorem).
- The set of all the limit points of a set is known as its derived set.
- A set is open if each point of it is an interior point.
- Any arbitrary union of open sets is an open set.
- The intersection of a finite number of open sets is an open set. However, the intersection of an infinite number of open sets may or may not be an open set.
- Any subset of IR is open if and only if it is a union of open intervals.
- A set is closed if all the limit points of the set are members of that set.
- A set is closed (open) if and only if its complement is open (closed).
- Any arbitrary intersection of closed sets is a closed set.
- The union of a finite number of closed sets is a closed set. However, the union of an infinite number of closed sets may or may not be a closed set.
- The union of a set and its dervied set is the closure of that set.
- For any set S in IR, Int S is the largest open subset of S, while $\overline{S}$ (closure of S) is the smallest closed superset of S.

6.7 Keywords

Real numbers, Field axioms, order axioms, completeness axiom, ordered field, complete ordered field, countable sets, uncountable sets, rational number, irrational number, Archimedean property, Neighbourhood of a point, limit points, open sets, closed sets, Bolzano Weierstrass theorem for sets.

6.8 References

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6.9 Model Questions

[A] Multiple Choice Questions (MCQ) :
(Choose the correct answer each of the following) :

[1] The set of rational numbers is

(a) complete ordered field (b) ordered field but not complete
(c) field but not ordered (d) none of the above.

[2] Let S be a bounded set. Then

(a) inf S < sup S (b) inf S = sup S.
(c) inf S $\leq$ sup S (d) sup S $\geq$ inf S.

[3] The lower bound of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is

(a) 0 (b) 1
(c) n (d) $\frac{1}{n}$

[4] For any two positive real numbers x and y with y < x, there is $n \in \mathbb{N}$ such that

(a) ny $\geq$ x (b) ny $\leq$ x (c) ny > x (d) ny < x.

[5] Between any two distinct real numbers, there exists

(a) only one irrational number (b) finite number of irrational numbers

(c) infinitely many irrational numbers (d) None of the above.

[6] Every non empty bounded above subset of real numbers has

(a) Supremum (b) Infimum

(c) both infimum and supremum (d) neither infimum nor supremum

[7] The derived set of any set is

(a) open (b) closed

(c) both open and closed (d) neither open nor closed.

[8] For any set S, Int S is

(a) open (b) closed

(c) both open and closed (d) neither open nor closed.

[9] For any set S, $\overline{S}$ is

(a) open (b) closed

(c) both open and closed (d) neither open nor closed.

[10] Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then S is

(a) closed (b) dense-in-itself

(c) both closed and dense-in-itself (d) neither closed nor dense-in-itself.

Ans. : [1] (b), [2] (c), [3] (a), [4] (c), [5] (c), [6] (a), [7] (b), [8] (a), [9] (b), [10] (d).

[B] Miscellaneous Questions :

[1] Let a, b ∈ F such that $a \neq 0$ and $a \,.\, b = a$, show that $b = 1$.

Hints : Multiply both sides of $a \,.\, b = a$ by a^{-1} and use the property M_2 and M_4 of section 2.3.

[2] Let F be an ordered field. If a, b, c ∈ F such that $a < b$ and $b < c$ then show that $a < c$.

[3] Given in an ordered field F, $0 \le a \le b$ and $0 \le c \le d$, where a, b, c, d $\in$ F. Show that $0 \le ac \le bd$.

Hints : $0 \le a \le b \Rightarrow 0 \le b - a$ and since $0 \le c$, it follows that $0 \le bc - ac \Rightarrow 0 \le ac \le bc$ ...(1)

Similarly one can show that $bc \le bd$ ...(2)

(1) and (2) gives the result.

[4] Let A and B be two sets such that $A \subseteq B$. If A is an uncountable then show that B is an uncountable, i.e., every superset of an uncountable set is uncountable.

Hints : If possible, let B be a countable set. Then A being a subset of a countable set, must be countable, which is a contradiction. Hence the result.

[5] Let A be the domain of a function f and let A be countable. Show that f(A) is countable.

Hints : Since A is countable, A can be arranged as a_1, a_2, a_3, ... So, f(A) can also be arranged as $f(a_1)$, $f(a_2)$, $f(a_3)$, ..., which means that f(A) has one to one correspondence with IN. Hence f(A) is countable.

[6] Prove that the set IN × IN is countable, where IN is the set of natural numbers.

Hints : Here IN × IN = $\bigcup\{A_n : n \in \mathbb{N}\}$, where

$$A_n = \{(n,1), (n,2), (n,3), ...,(n,n),....\}, n \in \mathbb{N}.$$

Define a mapping $f : A_n \to \mathbb{N}$ by f(n, m) = m, $m \in \mathbb{N}$.

Then f is bijective. Consequently A_n is countable for each $n \in \mathbb{N}$. Hence IN × IN is countable.

[7] Let Z be the set of all integers. Show that Z is countable.

Hints : Define a mapping $f : \mathbb{N} \to Z$ by

$$f(n) = \begin{cases} \frac{1}{2}(n-1), n = 1,3,5, ... \\ -\frac{1}{2}n \quad , n = 2,4,6,.... \end{cases}$$

Show that f is bijective and hence Z is countable.

[8] Prove that union of two countable sets is also countable.

[9] Let 'm' be a non-square positive integer. Show that there is no $r \in Q$ such that $r^2 = m$.

Solution : If possible let $\exists\, r \in Q$ such that $r^2 = m$. So, $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ and gcd (p, q) = 1 such that $r = \frac{p}{q}$.

Since m is a non-square positive integer, $\exists$ two consecutive square integers λ^2 and $(\lambda+1)^2$ such that

$$\lambda^2 < m < (\lambda+1)^2$$
$$\Rightarrow \lambda < \frac{p}{q} < \lambda + 1$$
$$\Rightarrow 0 < p - \lambda q < q \qquad \ldots(1)$$

Now $m(p-\lambda q)^2 = mp^2 - 2\lambda mpq + \lambda^2 mq^2 = (mq - \lambda p)^2$ as $\frac{p^2}{q^2} = m$

Thus $m = \left(\frac{mq-\lambda p}{p-\lambda q}\right)^2$, which implies that m has two representations

$m = \left(\frac{p}{q}\right)^2$ and $m = \left(\frac{mq-\lambda p}{p-\lambda q}\right)^2$.

Since gcd(p, q) = 1, we must have $p - \lambda q > q$, which contradicts to (1). Hence the result.

[10] If p is any prime number, show that $\sqrt{p}$ is not a rational number.

[11] Show that if x is rational and y is irrational then x + y is irrational and if $x \neq 0$ then xy is irrational.

[12] Prove that between any two distinct real numbers, there exists infinitely many real numbers both rational and irrational.
Hints : Density property of IR.

[13] Give examples of sets which are
(i) bounded below but not bounded above
(ii) bounded above but not bounded below
(iii) bounded
(iv) unbounded.

[14] Give an example of an infinite set which is bounded.
Ans. : The open interval (1, 2).

[15] Give an example of a subset of an unbounded set which is not necessarily unbounded.
Ans. : The set IR is unbounded but its subset (0, 1) is bounded.

[16] Find the infimum and supremum, if they exists, of the following sets :

(i) $\left\{\frac{1}{n} : n \in \mathrm{IN}\right\}$

(ii) $\left\{\frac{(-1)^n}{n} : n \in \mathrm{IN}\right\}$

(iii) $\left\{1 + \frac{(-1)^n}{n} : n \in \mathrm{IN}\right\}$

(iv) $\left\{(-1)^n n : n \in \mathrm{IN}\right\}$

(v) $\left\{\frac{n}{n+1} : n \in \mathrm{IN}\right\}$

(vi) $\left\{x \in \square : x^2 \leq 25\right\}$

(vii) $\left\{\Pi + \frac{1}{n} : n \in \mathrm{IN}\right\}$

Solution : (i) Let S = $\left\{\frac{1}{n} : n \in \mathrm{IN}\right\}$. Then max S = 1 and hence sup S =1. And by definition of infimum, inf S = 0.

(ii) The maximum and minimum element of the given set are respectively $\frac{1}{2}$ and –1. So, sup $S = \frac{1}{2}$ and inf S = –1.

(iii) If S = $\left\{1 + \frac{(-1)^n}{n} : n \in \mathrm{IN}\right\}$ then max $S = \frac{3}{2}$ and min S = 0. So, sup $S = \frac{3}{2}$ and inf S = 0.

(iv) Let S = $\{(-1)^n n : n \in \mathbb{N}\}$. Then S = {–1, 2, –3, 4, –5, 6, ...} = {..., –5, –3, –1, 2, 4, 6, ...}.
Clearly the set is neither bounded below nor bounded above. Hence infimum and supremum of S do not exist.

(v) Given $S = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\right\}$

Here sup S = 1 and inf $S = \frac{1}{2}$.

(vi) Let S = $\{x \in \mathbb{R} : x^2 \leq 25\}$. Then sup S = max S = 5 and inf S = min S = –5.

(vii) Let S = $\left\{\Pi + \frac{1}{n} : n \in \mathbb{N}\right\}$. Then sup S = $\Pi + 1$ and inf S = Π by similar to (i).

(17) Show that a non empty finite set can not be a neighbourhood of any of its points.

Hints : Let S ($\neq \phi$) be a finite set and p be an arbitrary point of S. Since for any positive real number $\in$, the open interval $(p - \in, p + \in)$ contains infinitely many points, so $(p - \in, p + \in)$ can not be a subset of the finite set S. Then S is not a neighbourhood of p.

(18) Give an example of
(i) a set which is a neighbourhood of each of its points.
(ii) a set which is not a neighbourhood of any of its points.
(iii) a set which is a neighbourhood of each of its points with the exception of one point.
(iv) a set which is a neighbourhood of each of its points with the exception of two points.
(v) a set which is not an interval but is a neighbourhood of each of its points.

Ans. (i) any open inerval in IR, say (a, b).
(ii) any non empty finite set.
(iii) any semi open interval in IR, say (a, b].
(iv) any closed interval in IR, say [a, b].
(v) $(0, 1) \cup (2, 3)$.

(19) Show that the set of integers is not a neighbourhood of any of its points.

(20) Is the set of natural numbers a neighbourhood of 5 ? Give reasons.

(21) Define limit points and derived set of a set.

(22) Give an example of a set which coincides with its derived set.

(23) Find the limit points of the following sets :
(i) IN (ii)[a, b) (iii) IR — Q (iv) {1, 2, 3, 4}.

(24) Give examples of sets S such that

(i) $S \cap S' = \phi$

(ii) $S' \subset S$

(iii) $S \subset S'$

Ans. (i) $S = \left\{\frac{1}{n} : n \in \text{IN}\right\}$

(ii) $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$

(iii) S = (a, b).

(25) Give example of each of the following :

(i) a bounded set having limit points.

(ii) a bounded set having no limit point.

(iii) an unbounded set having limit points.

(iv) an unbounded set having no limit point.

(v) an infinite set having a finite number of limit points.

Ans. (i) [a, b], (ii) any finite set, (iii) Q, (iv) IN, (v) $\left\{\frac{1}{m} + \frac{1}{n} + \frac{1}{p} : m, n, p \in \text{IN}\right\}$.

(26) Give example of each of the following :

(i) an open set which is not an interval

(ii) a closed set which is not an interval.

(iii) an interval which is an open set.

(iv) an interval which is a closed set.

(v) an interval which is not an open set.

(vi) an interval which is not a closed set.

(vii) a set which is neither an interval nor an open set.

(viii) a set which is neither an interval nor a closed set.

(ix) a set which is open as well as closed

(x) a set which is neither open nor closed.

Ans. (i) $(1, 2) \cup (3, 4)$ (ii) $\{1, 2, 3, 4\}$, (iii) (a, b), (iv) [a, b], (v) [a, b], (vi) (a, b), (vii) IN, (viii) $\left\{\frac{1}{n} : n \in \mathrm{IN}\right\}$, (ix) IR, (x) [a, b).

(27) Verify Bolzano-Weierstrass theorem for the set S in IR, where $S = \left\{\frac{n}{n+1} : n \in \mathrm{IN}\right\}$.

(28) Prove that arbitrary union of open sets in open.

(29) Show that arbitrary intersection of closed sets is closed.

(30) Is the union of an infinite number of closed sets a closed set ? Justify your answer.

(31) Is the intersection of an arbitrary family of open sets an open set ? Give reason.

Unit 7 ❑ Introduction to sequence

Structure

7.1 Objectives

The Object of this unit are as :

- to study sequences, its boundedness and convergence.
- to know about non-convergent sequences.
- to know about the sum, difference, product and quotient of two or more convergent sequences as well as some limit theorems.
- to study a special type of sequence, called monotone sequence and its properties.
- to know monotone convergence theorem through which we get the necessary and sufficient condition of a monotone sequence to be convergent.
- to study subsequence and its properties including Bolzano weierstrass theorem for sequences.
- to study Cauchy sequence and Cauchy's convergence criterian, which states that the necessary and sufficient condition of a sequence to be convergent.

7.2 Introduction

This unit deals with the sequences of real numbers. Its foundation was laid by the French mathematician Augustin Louis Cauchy (1789 – 1857). To the development of sequences of real numbers, the contribution of George Cantor (1845 – 1918) is also significant. A sequence of real numbers is a function from IN to IR. Such functions plays an important in real analysis.

7.3 Sequences

A function $f : \mathrm{IN} \to \mathrm{IR}$ is called a sequence in IR (or a real sequence), where IN and IR are respectively the set of natural numbers and set of real numbers.

The value of the function f at $n \in \mathrm{IN}$ is denoted by f(n). If $f(n) = x_n$ then the sequence is denoted by $\{f(n)\}$ or $\{x_n\}$, i.e., $\{x_1, x_2, ...\}$. Here x_n is called the n^{th} term or general term of the sequence $\{x_n\}$.

Two sequence $\{x_n\}$ and $\{y_n\}$ are said to be equal if $x_n = y_n$ for each $n \in \mathrm{IN}$.

Remark : (1) The domain of every sequence is IN, but its range is $\{f(n) : n \in \mathrm{IN}\} \subseteq \mathrm{IR}$. That means the range of the sequence may be a finite or an infinite set. So, the range of a sequence $\{x_n\}$ is the set consisting of all the distinct elements of the sequence $\{x_n\}$.

(2) We use IN with usual well ordering.

Examples :

(1) Let $f : \mathrm{IN} \to \mathrm{IR}$ be defined by $f(n) = \frac{1}{n}, n \in \mathrm{IN}.$ So, the sequence is $\left\{\frac{1}{n}\right\}$, which can be also written as $\left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$. The range of this sequence is infinite.

(2) Let $f : \mathrm{IN} \to \mathrm{IR}$ be defined by $f(n) = n, n \in \mathrm{IN}.$ So, the sequence is $\{n\}$, i.e. $\{1, 2, 3, 4, ...\}$. Its range is also infinite.

(3) Similarly $\{n^2\}$ is the sequence $\{1^2, 2^2, 3^2, ...\}$

(4) Let $f : \mathrm{IN} \to \mathrm{IR}$ be defined by $f(n) = \frac{n}{n+1}, n \in \mathrm{IN}$. The sequence is $\left\{\frac{n}{n+1}\right\}$,

whose elements are $\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},....\right\}$. Similarly $\left\{\frac{n+1}{n}:n\in\mathbb{N}\right\}$ is the sequence $\left\{2,\frac{3}{2},\frac{4}{3},.....\right\}$. The range of both the sequences are infinite.

(5) Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=(-1)^n$, $n\in\mathbb{N}$. The sequence is $\{(-1)^n\}$, i.e. $\{-1,1,-1,1,-1,....\}$. The range of this sequence is $\{-1,1\}$, i.e., finite.

(6) Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=\sin\frac{n\pi}{2},n\in\mathbb{N}$. So, the sequence is $\left\{\sin\frac{n\pi}{2}\right\}$, i.e. {1, 0, –1, 0, 1, 0, ...}. Its range is {–1, 0, 1}, i.e., finite.

(7) Let $f:\mathbb{N}\to\mathbb{R}$ be defined by $f(n)=3,\ \forall\ n\in\mathbb{N}$. So, the sequence is {3}. i.e., {3, 3, 3, ...}. This sequence is called the constant sequence.

7.4 Bounded Sequence

A sequence $\{x_n\}$ is called **bounded above** if $\exists M\in\mathbb{R}$ such that $x_n\le M,\forall n\in\mathbb{N}$. Here M is known as an upper bound of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ is bounded above as well as **bounded below** if $\exists M\in\mathbb{R}$ such that $x_n\ge m,\forall n\in\mathbb{N}$. Here m is known as a lower bound of the sequence $\{x_n\}$.

If a sequence $\{x_n\}$ is bounded above as well as bounded below then bounded below ther $\{x_n\}$ is called **bounded**. Thus, a sequence $\{x_n\}$ is **bounded** if $\exists m, M\in\mathbb{R}$ such that

$$m\le x_n\le M,\forall n\in\mathbb{N}$$

In other words, a sequence $\{x_n\}$ is **bounded** if there exists a real number M(≥ 0) such that

$$|x_n|\le M,\forall n\in\mathbb{N},$$

that means if the range of the sequence is bounded.

A sequence $\{x_n\}$ is called **unbounded** if it is **not bounded.**

Remark : Every number greater than an upper bound is also an upper bound and every number smaller than a lower bound is also a lower bound.

An upper bound of a sequence is called the supremum (or least upper bound), written as sup or lub, if it is less than or equal to every upper bound of the sequence. Similarly a lower bound of a sequence is called infimum (or greater lower bound), written as inf or glb, if it is greater than or equal to every lower bound of the sequence.

Examples :

(1) The sequence $\{-n\}$ is bounded above by -1, but not bounded below.

(2) The sequence $\{n^2\}$ is bounded below by 1, but not bounded above.

(3) The sequence $\left\{\frac{1}{n}\right\}$ is a bounded sequence, as $0 < \frac{1}{n} \leq 1, \forall n \in \mathbb{N}$. The supremum and infimum of this sequence are 1 and 0 respectively. So, this sequence contains its supremum, but not infimum.

(4) The sequence $\left\{\sin\frac{n\pi}{2}\right\}$ is bounded as $-1 \leq \sin\frac{n\pi}{2} \leq 1, \forall n \in \mathbb{N}$.

(5) The sequence $\{(-1)^n\}$ is a bounded sequence. In this case, the bounds are -1, and 1.

(6) The sequence $\left\{\frac{n}{n+1}\right\}$ is a bounded sequence, as $\frac{1}{2} \leq \frac{n}{n+1} < 1, \forall n \in \mathbb{N}$. The supremum and infimum of this sequence are 1 and $\frac{1}{2}$ respectively. So, this sequence contains its infimum, but not surpemum.

Exercise : Show that the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$ is bounded.

Solution : Here, $x_1 = 1, x_2 = 1 + \frac{1}{2}, x_3 = 1 + \frac{1}{2} + \frac{1}{2^2} + ...,$ that means the sequence is strictly increasing. Consequently the sequence is bounded below by the first term i.e. 1.

Also, $x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} = \dfrac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} < 2, \forall n \in \mathbb{N}.$

Hence the sequence is bounded above also. Thus the given sequence $\{x_n\}$ is bounded.

7.5 Summary

This chapter explores the concept of sequences, focusing on their bounded nature, convergence, and limit points, with applications to real analysis.

7.6 Keywords

Sequence, bounded sequence, convergence, real analysis, limit points, monotonicity.

7.7 References

1. Principles of Mathematical Analysis - Waiter Rudin, McGraw-Hill
2. Real Analysis: Modern Techniques and Their Applications - Gerald B. Folland, Wiley
3. Real Analysis - H.L. Royden, P.M. Fitzpatrick, Pearson

7.8 Model Questions

1. Define a sequence and explain its boundedness.
2. What is the difference between convergent and divergent sequences?
3. State the Bolzano-Weierstrass theorem for bounded sequences.
4. Give an example of a bounded sequence that does not converge.
5. Explain the concept of a limit point for a sequence.
6. What is a Cauchy sequence, and how does it relate to convergence?

Unit 8 ❑ Convergence

Structure

8.1 Objectives

To understand the concept of convergent sequences, their limits, and key theorems governing limits in real analysis.

8.2 Introduction

This chapter explores convergent sequences, their properties, and essential limit theorems. These concepts form the foundation for rigorous analysis and mathematical proofs in calculus and beyond.

8.3 Convergent Sequence

A sequence $\{x_n\}$ is said to be **convergent** if there is a real number ℓ such that for each $\in > 0$, there exists a natural number m (depending on $\in$) satisfying

$$|x_n - \ell| < \in, \forall\ n \geq m. \qquad ...(7.5.1)$$

In this case, we also say that the sequence $\{x_n\}$ converges to ℓ or 'ℓ' is the **limit** of the sequence and we write

$$x_n \to \ell \text{ as } n \to \infty \text{ or } \lim_{n+\infty} x_n = \ell$$

or Simply $\lim x_n = \ell$.

Note : (1) We know that $|x_n - \ell| < \in \Leftrightarrow x_n \in (\ell - \in, \ell + \in)$

So, we may use $x_n \in (\ell - \in, \ell + \in)$ instead of $|x_n - \ell| < \in$ in the above definition. This means that after a finite number of terms from the begining, all the terms of the sequence must lie in the open interval $(\ell - \in, \ell + \in)$.

Thus if $\lim\limits_{n\to\infty} x_n = \ell$ then $\lim\limits_{n\to\infty} x_{n+1} = \ell$.

(2) The choice of m, in the definition, is not unique. As, if the criterian (7.5.1), in above definition, is satisfied then (7.5.1) also holds for any greater natural number of m.

Non-Convergent sequence : A sequence is called non-convergent sequence if it is not convergent. Non-convergent sequences are either 'divergent' or 'oscillatory', as defined below :

Divergent sequence A sequence $\{x_n\}$ is said to diverge to $+\propto$ if for every positive real number K, however large, $\exists$ a natural number m such that $x_n > K, \forall\, n \geq m$.

In this case, we write $\lim\limits_{n\to\infty} x_n = +\infty$ or $\lim x_n = +\infty$ or $x_n \to +\infty$.

Again a sequence $\{x_n\}$ is said to diverge to $-\propto$ if for positive real number K, however large, $\exists$ a natural number m such that $x_n < -K, \forall\, n \geq m$.

In this case, we write $\lim\limits_{n\to\infty} x_n = -\infty$ or $\lim x_n = -\infty$ or $x_n \to -\infty$.

Thus a sequence $\{x_n\}$, which diverges to either $+\infty$, or $-\infty$ is called a divergent sequence.

Oscillatory Sequence : A sequence $\{x_n\}$ is said to be oscillatory if it is neither convergent nor divergent.

In this case, the sequence $\{x_n\}$ oscillates between two numbers as $n \to \infty$.

Also an oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

Examples :

(1) The following sequences are convergent :

(i) $\left\{\frac{1}{n}\right\}$ is convergent and converges to 0, as $\lim \frac{1}{n} = 0$.

(ii) $\left\{\frac{n}{n+1}\right\}$ is convergent and converges to 1, as $\lim \frac{n}{n+1} = 1$.

(iii) $\{x_n\}$, where $x_n = 3$ for all $n \in \mathbb{N}$, is convergent and converges to 3.

(2) Each of the sequences $\{n^2\}, \left\{\frac{n^2+3}{2n+1}\right\}$ diverges to $+\infty$.

(3) Each of the sequences $\{-n\}, \left\{\log\left(\frac{1}{n}\right)\right\}$ diverges to $-\infty$.

(4) The sequence $\left\{(-1)^{n-1}\right\}$ and $\left\{\sin\frac{n\pi}{2}\right\}$ oscillate finitely betdween –1 and 1, whereas the sequence $\left\{(-1)^n n\right\}$ and $\left\{(-1)^n n + \frac{1}{n}\right\}$ oscillate infinitely.

Theorem 8.3.1. The limit of a convergent sequence is unique.

Proof : Suppose $\{x_n\}$ is a convergent sequence. If possible, let $\{x_n\}$ converges to two distinct limits ℓ and ℓ'. Choose $\in = \frac{1}{2}|\ell - \ell'|$. Then $\in > 0$. So, there exists $m_1, m_2 \in IN$ such that—

$|x_n - \ell| < \in, \forall\, n \geq m_1$ and $|x_n - \ell'| < \in, \forall\, n \geq m_2$.

Take m_3 = max $\{m_1, m_2\}$. Then it follows from above that

$|x_n - \ell| < \in$ and $|x_n - \ell'| < \in, \forall n \geq m_3$. ....(3.5.2)

Thus $\forall\, n \geq m_3$, we have

$$\begin{aligned}|\ell - \ell'| &= |(x_n - \ell') - (x_n - \ell)| \\ &\leq |x_n - \ell'| + |x_n - \ell| \\ &< \in + \in, \text{ using (3.5.2)} \\ &= 2\in = |\ell - \ell'|\end{aligned}$$

So, $|\ell - \ell'| < |\ell - \ell'|$, which is absurd and hence our assumption is wrong. Consequently, the limit of a convergent sequence is unique.

Theorem 8.3.2. Every convergent sequence is bounded.

Proof : Let $\{x_n\}$ be a convergent sequence and it converges to ℓ.

Choose $\in = 1$. Then $\exists m \in IN$ such that $|x_n - \ell| < 1, \forall n \geq m$.

Now $|x_n| - |\ell| \le |x_n - \ell| < 1, \forall n \ge m.$ i.e. $|x_n| < 1 + |\ell|, \forall n \ge m.$... (3.5.3)

If M = max $\{1+|\ell|, |x_1|, |x_2|, ..., |x_{m-1}|\}$, then

$|x_n| \le M, \forall n \ge 1, 2, ..., m-1$ ... (3.5.4)

and since $1+|\ell| \le M,$ it follows from (3.5.3) that

$|x_n| \le M, \ \forall\, n \ge m$ ... (3.5.5)

From (3.5.4) and (3.5.5), we see that

$|x_n| \le M, \forall n \in \mathbb{N}.$

Consequently the sequence $\{x_n\}$ is bounded.

Note : The converse of the above theorem is not true. For this, we consider the sequence $\{(-1)^{n-1}\} = \{1, -1, 1, -1, ...\}$, which is bounded but it is not convergent, because lim $(-1)^{n-1}$ oscillates between -1 and 1.

Exercise 3.5.1 : Show that the sequence $\left\{\frac{(-1)^n}{n}\right\}$ is convergent .

Solution : Here $x_{2n} = \frac{1}{2n}$ and $x_{2n+1} = \frac{-1}{2n+1}$.

So, $\lim x_{2n} = 0 = \lim x_{2n+1}$, which implies that the given sequence is convergent and it converges to zero.

Exercise 3.5.2. Show that the sequence $\{x_n\}$, where $x_n = \frac{2n^2+1}{2n^2-1}$, converges to 1.

Solution : Let $\in > 0$ be given, then

$$|x_n - 1| < \in \Leftrightarrow \left|\frac{2n^2+1}{2n^2-1} - 1\right| < \in \ \Leftrightarrow \left|\frac{2}{2n^2-1}\right| < \in$$

$$\Leftrightarrow \frac{2}{2n^2-1} < \in \Leftrightarrow 2n^2 - 1 > \frac{2}{\in} \ \Leftrightarrow n^2 > \frac{2+\in}{2\in} \Leftrightarrow n > \left(\frac{2+\in}{2\in}\right)^{\frac{1}{2}} = \delta, \text{ say}$$

Choose m = $[\delta]$+1, where $[\delta]$ is the greatest integer, but not greater than δ.

Then $n \ge m \Rightarrow n > \delta \Rightarrow |x_n - 1| < \in$,

which means that the sequence $\{x_n\}$ converges to 1.

8.4 Limit Theorems

The sum, difference, product and quotient of two sequences give rise to new sequences. In this section, we show that the sum, difference, product and quotient of two convergent sequences are also convergent and determine their limits.

Theorem 8.4.1 : Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences such that $\lim x_n = x$ and $\lim y_n = y$ respectively.

Then

(i) $\lim(x_n + y_n) = x + y = \lim x_n + \lim y_n$

(ii) $\lim(x_n - y_n) = x - y = \lim x_n - \lim y_n$

(iii) $\lim(cx_n) = cx = c \lim x_n, \forall\ c \in IR$

(iv) $\lim(x_n y_n) = xy = \lim x_n . \lim y_n$.

(v) $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y} = \frac{\lim x_n}{\lim y_n}$, provided $\{y_n\}$ is a non-zero real numbers and $y \neq 0$.

Proof : (i) Let $\in > 0$ be arbitrary small number. Since $\lim x_n = x$ and $\lim y_n = y$, so there exists two natural numbers m_1 and m_2 such that

$$|x_n - x| < \frac{1}{2} \in, \forall n \geq m_1 \qquad ...(3.6.1)$$

and $$|y_n - y| < \frac{1}{2} \in, \forall n \geq m_2. \qquad ...(3.6.2)$$

Choose m = max $\{m_1, m_2\}$. Then (3.6.1) and (3.6.2) hold $\forall n \geq m$.

Thus $\forall n \geq m$, we have

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$$

$$\leq |x_n - x| + |y_n - y| < \frac{1}{2} \in + \frac{1}{2} \in = \in$$

i.e. $|(x_n + y_n) - (x + y)| < \in, \forall n \geq m$,

which implies that the sequence $\{x_n + y_n\}$ is convergent and $\lim(x_n + y_n) = x + y = \lim x_n + \lim y_n$.

(ii) It is similar as above. Only note that $\forall n \geq m$, we have

$$|(x_n - y_n) - (x - y)| = |(x_n - x) - (y_n - y)| \le |x_n - x| + |y_n - y| < \frac{1}{2}\in + \frac{1}{2}\in = \in,$$

which implies that the sequence $\{x_n - y_n\}$ is convergent and

$\lim(x_n - y_n) = x - y = \lim x_n - \lim y_n$.

(iii) If c = 0 then the result is obvious. So, suppose $c \ne 0$.

We know that

$$|cx_n - cx| = |c||x_n - x|. \qquad ...(3.6.3)$$

Let $\in > 0$ be given. Since $\lim x_n = x$, so there exists a nutural number m such that

$$|x_n - x| < \frac{\in}{|c|}, \forall\ n \ge m \cdot \qquad ...(3.6.4)$$

In view of (3.6.4) we have from (3.6.3) that

$|cx_n - cx| < \in \forall\ n \ge m$,

which implies that the sequence $\{c\ x_n\}$ is convergent and

$\lim(cx_n) = cx = c \lim x_n$, for all $c \in IR$.

(iv) We have that

$$\begin{aligned} |x_n y_n - xy| &= |(x_n y_n - x_n y) + (x_n y - xy)| \\ &= |x_n(y_n - y) + y(x_n - x)| \\ &\le |x_n||y_n - y| + |y||x_n - x| \qquad ...(3.6.5) \end{aligned}$$

Since $\{x_n\}$ is convergent, it is bounded. So, there exists $M' \in IR$ such that

$$|x_n| \le M', \forall n \in IN. \qquad ...(3.6.6)$$

Take M = $\max\{M', |y|\}$. Then, in view of (3.6.6), we have from (3.6.5) that

$$|x_n y_n - xy| \le M|y_n - y| + M|x_n - x|. \qquad ...(3.6.7)$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent, so for arbitrary $\in > 0, \exists$ two natural numbers m_1 and m_2 such that

$$|x_n - x| < \frac{\in}{2M}, \forall n \ge m_1 \qquad ...(3.6.8)$$

and $|y_n - y| < \frac{\in}{2M}, \forall n \geq m_2$ ...(3.6.9)

Choose m = max (m_1, m_2). Then the relations (3.6.8) and (3.6.9) hold for all n $\geq$ m.

Thus $\forall\ n \geq m$ we have from (3.6.7) that

$$|x_n y_n - xy| < M.\frac{\in}{2M} + M.\frac{\in}{2M} = \in,$$

which implies that the sequence $\{x_n y_n\}$ is convergent and

$\lim (x_n y_n) = xy = \lim x_n \cdot \lim y_n$.

(v) Since $\lim y_n = y$, so for $\in = \frac{|y|}{2}, \exists$ a natural number m_1 such that

$$|y_n - y| < \frac{|y|}{2}, \forall n \geq m_1 \Rightarrow |y| - |y_n| < \frac{|y|}{2}, \forall n \geq m_1$$

$$\Rightarrow |y_n| > \frac{|y|}{2}, \forall n \geq m_1. \qquad ...(3.6.10)$$

Now $\left|\frac{x_n}{y_n} - \frac{x}{y}\right| = \left|\frac{x_n y - x y_n}{y_n y}\right| = \left|\frac{y(x_n - x) - x(y_n - y)}{y_n y}\right|$

$$\leq \frac{|y||x_n - x| + |x||y_n - y|}{|y_n||y|} < \frac{2}{|y|}|x_n - x| + \frac{2|x|}{|y|^2}|y_n - y|, \qquad ...(3.6.11)$$

$\forall\ n \geq m_1$ by (3.6.10).

Again since $\{x_n\}$ and $\{y_n\}$ are convergent, so for arbitrary $\in > 0, \exists$ two natural numbers m_2 and m_3 such that

$$|x_n - x| < \frac{|y|}{4}\in, \forall n \geq m_2 \qquad ...(3.6.12)$$

and $|y_n - y| < \frac{|y|^2}{4(|x|+1)}\in, \forall n \geq m_3.$ ...(3.6.13)

Choose m = max $\{m_1, m_2, m_3\}$. Then each of the relations (3.6.11) — (3.6.13) hold for all n $\geq$ m.

Thus $\forall\ n \geq m$, in view of (3.6.12) and (3.6.13), we have from (3.6.11) that

$$\left|\frac{x_n}{y_n}-\frac{x}{y}\right|<\frac{2}{|y|}.\frac{|y|}{4}\in+\frac{2|x|}{|y|^2}.\frac{|y|^2}{4(|x|+1)}\in<\frac{\in}{2}+\frac{\in}{2}=\in.$$

Thus we get $\left|\frac{x_n}{y_n}-\frac{x}{y}\right|<\in, \forall n\geq m,$

which implies that the sequence $\left\{\frac{x_n}{y_n}\right\}$ is convergent and

$$\lim\left(\frac{x_n}{y_n}\right)=\frac{x}{y}=\frac{\lim x_n}{\lim y_n}.$$

Note : By virtue of Theorem 8.4.1(V), we can say that

$\lim\left(\frac{1}{y_n}\right)=\frac{1}{y}=\frac{1}{\lim y_n}$, that means if $\{y_n\}$ is a convergent sequence of non-zero real numbers and converges to a non-zero real number y, then the sequence $\left\{\frac{1}{y_n}\right\}$ is also a convergent sequence and converges to $\frac{1}{y}$.

Theorem 8.4.2. : If $\{x_n\}$ is a convergent sequence of real numbers and converges to x, then the sequence $\{|x_n|\}$ is also convergent and converges to $|x|$.

Proof : Let $\in>0$ be an arbitrary small number.

Since $\{x_n\}$ is convergent sequence and converges to x, so $\exists$ a natural number m such that

$$|x_n-x|<\in,\ \forall n\geq m. \qquad ...(3.6.14)$$

Now $\left|(|x_n|-|x|)\right|\leq|x_n-x|<\in,\ \forall n\geq m$, using (3.6.14)

which implies that the sequence $\{|x_n|\}$ is convergent and $\lim|x_n|=|x|$.

Note : The converse of the above theorem is not true. For this, if we consider thle sequence $\{x_n\}=\left\{(-1)^{n-1}\right\}$. Then $|x_n|=1, \forall n\in\mathbb{N}$. So, thle sequence $\left\{|x_n|\right\}$ is a convergent sequence and converges to 1, while the sequence $\{x_n\}$ is not a convergent sequence.

Theorem 8.4.3 : Let $\{x_n\}$ be a convergent sequence of real numbers such that lim $x_n = x$. If $x_n \geq 0\ \forall n \in \mathbb{N}$, then $x \geq 0$.

Proof : We have to show that $x \geq 0$.

If possible, let us suppose that $x < 0$.

Since, lim $x_n = x$, so for a given $\in > 0, \exists$ a positive integer m such that

$|x_n - x| < \in, \forall\ n \geq m$

i.e. $x - \in < x_n < x + \in, \forall\ n \geq m$. ...(3.6.15)

Since $x < 0$, choosing $\in = -\frac{x}{2} > 0$ in (3.6.15) , we get

$$x + \frac{x}{2} < x_n < x - \frac{x}{2}, \forall\ n \geq m$$

i.e. $x_n < \frac{x}{2} < 0, \forall\ n \geq m$,

which is a contradiction to the fact that $x_n \geq 0, \forall n \in \mathbb{N}$. So, our assumption is wrong. Hence we have $x \geq 0$.

Theorem 8.4.4 : Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences and there exists a natural number m such that $x_n \leq y_n, \forall n \geq m.$ Then lim $x_n \leq$ lim y_n.

Proof : Let lim $x_n = x$ and lim $y_n = y$.

Suppose $z_n = y_n - x_n$. Then $\{z_n\}$ is a convergent sequence such that $z_n \geq 0, \forall n \geq m.$ So, by Theorem 8.4.3, it follows that lim $Z_n \geq 0$, and hence lim $x_n \leq$ lim y_n.

Theorem 8.4.5 (Sandwich Theorem) : Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences of real numbers and there is a natural number m such that $x_n < y_n < z_n$, $\forall n \geq m.$ If lin $x_n = \ell =$ lin z_n, then $\{y_n\}$ is convergent and lim $y_n = \ell$.

Proof : Let $\in > 0$. Since lim $x_n = \ell =$ lim z_n, so $\exists$ two natural numbers m_1 and m_2 such that

$|x_n - \ell| < \in, \forall n \geq m_1$ and $|z_n - \ell| < \in, \forall n \geq m_2$.

Choose $m_3 = \max\ \{m_1, m_2\}$. Then it follows from above that

$|x_n - \ell| < \in$ and $|z_n - \ell| < \in, \forall n \geq m_3$.

i.e. $\ell - \in < x_n < \ell + \in$ and $\ell - \in < z_n < \ell + \in, \forall n \geq m_3$ ...(3.6.16)

Also given that $x_n < y_n < z_n$, $\forall n \geq m$. ...(3.6.17)

Again let us choose $K = \max\{m_3, m\}$. Then from (3.6.16) and (3.6.17) we can write
$\ell - \in < x_n < y_n < z_n < \ell + \in, \ \forall n \geq K,$
which implies that $\{y_n\}$ is convergent sequence and $\lim y_n = \ell$.

Examples :

Ex 3.6.1. Prove that $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$.

Solution : Here $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n+1}+\sqrt{n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\lim_{n\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{n}}}=0.\frac{1}{2}=0.$$

Ex 8.3.2 : Prove that $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+.....+\dfrac{1}{\sqrt{n^2+n}}\right)=1$

Solution : Let us take $x_n=\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+....+\dfrac{1}{\sqrt{n^2+n}}$

$$<\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+.....+\frac{1}{\sqrt{n^2+1}}, \text{since } n^2+r>n^2+1 \text{ for } 2 \leq r \leq n.$$

$$=\frac{n}{\sqrt{n^2+1}}, \forall n \geq 2. \qquad ...(3.6.18)$$

Again clearly $\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}>\dfrac{2}{\sqrt{n^2+2}}$.

Similary $\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\dfrac{1}{\sqrt{n^2+3}}>\dfrac{3}{\sqrt{n^2+3}}$.

Proceeding in this way, we get $x_n>\dfrac{n}{\sqrt{n^2+n}}, \forall n\geq 2$. ...(3.6.19)

From (3.6.18) and (3.6.19) we obtain

$$\frac{n}{\sqrt{n^2+n}}<x_n<\frac{n}{\sqrt{n^2+1}}, \ \forall n \geq 2. \qquad ...(3.6.20)$$

Now $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n}}}=1$ and $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\left(\dfrac{1}{n}\right)^2}}=1$.

So, by Sandwich theorem, it follows from (3.6.20) that $\lim\limits_{n\to\infty}x_n=1$.

Ex. 8.3.3. Find the value of $\lim\limits_{n\to\infty}\dfrac{3+2\sqrt{n}}{\sqrt{n}}$

Solution : We have $\lim\limits_{n\to\infty}\dfrac{3+2\sqrt{n}}{\sqrt{n}}=\lim\limits_{n\to\infty}\left(\dfrac{3}{\sqrt{n}}+2\right)=3\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}}+2=3.0+2=2$.

Ex. 8.3.4. Show that $\lim\limits_{n\to\infty}\dfrac{(3n+1)(n-2)}{n(n+3)}=3.$

Solution : We know that $\lim\dfrac{1}{n}=0.$

Now, $\lim\limits_{n\to\infty}\dfrac{(3n+1)(n-2)}{n(n+3)}=\lim\limits_{n\to\infty}\dfrac{\left(3+\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)}{1+\dfrac{3}{n}}=\dfrac{\lim\limits_{n\to\infty}\left(3+\dfrac{1}{n}\right)\lim\limits_{n\to\infty}\left(1-\dfrac{2}{n}\right)}{\lim\limits_{n\to\infty}\left(1+\dfrac{3}{n}\right)}=\dfrac{3.1}{1}=3.$

Theorem 8.3.6. Let $\{u_n\}$ be a sequence such that $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\ell$. If $|\ell|<1$, then $\lim\limits_{n\to\infty}u_n=0$.

Proof : Let ε be an arbitrary small positive number.

Since $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\ell$, so $\exists$ a natural number m such that $\left|\dfrac{u_{n+1}}{u_n}-\ell\right|<\in,\ \forall\ n\geq m.$

As $|\ell|<1$, we choose $\in$ so small such that $|\ell|+\in<1$ and let $|\ell|+\in=r$.
Then $0<r<1$

Now $\left|\dfrac{u_{n+1}}{u_n}\right|=\left|\ell+\left(\dfrac{u_{n+1}}{u_n}-\ell\right)\right|\leq|\ell|+\left|\dfrac{u_{n+1}}{u_n}-\ell\right|<|\ell|+\in,\ \forall n\geq m.$

$\therefore\ \left|\dfrac{u_{n+1}}{u_n}\right|<r,\ \forall n\geq m.$

Hence we have $\left|\frac{u_{m+1}}{u_m}\right| < r, \left|\frac{u_{m+2}}{u_{m+1}}\right| < r, \ldots\ldots\ldots \left|\frac{u_n}{u_{n-1}}\right| < r.$

Multiplying above, we get $\left|\frac{u_n}{u_m}\right| < r^{n-m} = \frac{r^n}{r^m}$

and hence $0 < |u_n| < \frac{|u_m|}{r^m} r^n$, where $0 < r < 1$

Taking limit as $n \to \infty$, we get $|u_n| \to 0$, since $r^n \to 0$ as $n \to \infty$.

This means that $\lim\limits_{n\to\infty} u_n = 0$

Example 8.3.5 Show that for any $x \in \mathrm{IR}$, $\lim\limits_{n\to\infty} \frac{x^n}{|n!|} = 0$.

Solution : Let $u_n = \frac{x^n}{n!}$

So, $\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}$

Hence $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{x}{n+1} = \lim\limits_{n\to\infty} \frac{x.\frac{1}{n}}{1+\frac{1}{n}} = 0 < 1.$

So by above theorem, it follows that $\lim\limits_{n\to\infty} \frac{x}{n!} = 0$.

Example 8.3.6 : Show that $\lim \frac{m(m-1)(m-2)....(m-n+1)}{n!} x^n = 0,\ |x| < 1.$

Solution : Let $u_n = \frac{m(m-1)(m-2).....(m-n+1)}{n!} x^n$

$\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{m-n}{n+1} x = \lim\limits_{n\to\infty} \frac{\frac{m}{n}-1}{1+\frac{1}{n}} x = -x = \ell$, say,

$\therefore |\ell| = |-x| = |x| < 1.$

So, by Theorem 8.4.6. we have $\lim\limits_{n\to\infty} u_n = 0$

i.e. $\lim\limits_{n\to\infty} \dfrac{m(m-1)(m-2).....(m-n+1)}{n!} x^n = 0$

Theorem 8.3.7 If $\{u_n\}$ be a sequence such that $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \ell > 1$, then $\lim\limits_{n\to\infty} u_n = \infty$

Proof : Let $\in > 0$ be arbitrary small number.

Since $\ell > 1$, we choose $\in$ such that $\ell - \in > 1$

Again since $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \ell$, therefore $\exists$ a positive integer m, such that

$$\left|\frac{u_{n+1}}{u_n} - \ell\right| < \in, \forall n \ge m$$

i.e. $\ell - \in < \dfrac{u_{n+1}}{u_n} < \ell + \in, \forall n \ge m.$

So, $\dfrac{u_{n+1}}{u_n} > \ell - \in = K(\text{say})$, where $K > 1$, $\forall\, n \ge m.$

Putting n = m, m + 1, m + 2, ..., n – 1 in above and multiplying them, we get

$\left|\dfrac{u_n}{u_m}\right| > K^{n-m} = \dfrac{K^n}{K^m}$, which means that $|u_n| > \dfrac{|u_m|}{K^m}.K^n$

Since $K > 1$, therefore $K^n \to \infty$ as $n \to \infty$

Hence $\lim\limits_{n\to\infty} u_n = \infty$.

Theorem 8.3.8 : If $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \ell(\text{finite})$ then $\lim\limits_{n\to\infty} (u_n)^{1/n} = \ell$.

Proof : Let $\in > 0$ be an arbitrary small +ve number.

Since $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \ell$, So $\exists$ a natural number m such that

$\left|\dfrac{u_{n+1}}{u_n} - 1\right| < \in$, $\forall n \ge m$ i.e. $\ell - \in < \dfrac{u_{n+1}}{u_n} < \ell + \in$, $\forall n \ge m.$

Thus we get, $\ell - \in < \frac{u_{n+1}}{u_n} < \ell + \in$, $\ell - \in < \frac{u_{m+2}}{u_{m+1}} < \ell + \in$, ... $\ell - \in < \frac{u_n}{u_{n-1}} < \ell + \in$.

Multiplying all these above, we get

$$(\ell - \in)^{n-m} < \frac{u_n}{u_m} < (\ell + \in)^{n-m} \text{ i.e. } \frac{(\ell - \in)^n}{(\ell - \in)^m} < \frac{u_n}{u_m} < \frac{(\ell + \in)^n}{(\ell + \in)^m}$$

i.e. $u_m \frac{(\ell - \in)^n}{(\ell - \in)^m} < u_n < u_m \frac{(\ell + \in)^n}{(\ell + \in)^m}$, as $u_m > 0$.

i.e. $\left[\frac{u_m}{(\ell - \in)^m}\right]^{\frac{1}{n}} (\ell - \in) < u_n^{\frac{1}{n}} < \left[\frac{u_m}{(\ell + \in)^m}\right]^{\frac{1}{n}} (\ell + \in)$

i.e. $A^{\frac{1}{n}} (\ell - \in) < u_n^{\frac{1}{n}} < B^{\frac{1}{n}} (\ell + \in)$, ...(3.6.21)

where $A = \frac{u_m}{(\ell - \in)^m} > 0$ and $B = \frac{u_m}{(\ell + \in)^m} > 0$.

It is known that for p > 0, $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$ and hence $\lim_{n\to\infty} A^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} B^{\frac{1}{n}} = 1$. Consequently it follows from (3.6.21) that

$$(\ell - \in) < u_n^{\frac{1}{n}} < (\ell + \in), \forall n \geq m$$

i.e. $\left| u_n^{\frac{1}{n}} - \ell \right| < \in, \forall n \geq m$ and hence $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \ell$.

Remark (1) In above theorem, if $\ell = \infty$ then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = \infty$.

(2) The converse of the Theorem 8.4.8 is not true. For this, if we consider the sequence $\{u_n\}$, where $u_n = \frac{3+(-1)^n}{2}$. Then $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$ but $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ does not exist.

Example 8.3.7 : Prove that $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$.

Solution : Let $u_n = \frac{n!}{n^n}$. Then $u_n > 0, \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e} > 0.$

So, by virtue of Theorem 8.4.8, it follows that

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{e}, \text{ i.e. } \lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}.$$

Example 8.3.8. Prove that $\lim_{n\to\infty} \frac{\{(n+1)(n+2)....(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}.$

Solution : Let $u_n = \frac{(n+1)(n+2)....(2n)}{n^n}$. Then $u_n > 0, \forall n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2(2n+1)}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{4}{e} > 0.$$

So, by virtue of Theorem 8.4.8, it follows that

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{4}{e}, \text{ i.e. } \lim_{n\to\infty} \frac{\{(n+1)(n+2)....(2n)\}^{\frac{1}{2}}}{n} = \frac{4}{e}.$$

Theorem 8.3.9. (Cauchy's first theorem on limits)

If $\lim_{n\to\infty} a_n = \ell$, then $\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = \ell.$

Proof : Let us take $b_n = a_n - \ell$. ...(3.6.22)

Since $\lim_{n\to\infty} a_n = \ell$, So $\lim_{n\to\infty} b_n = 0$, and hence the sequence $\{b_n\}$ is convergent. Also from (3.6.22), we have that

$$\frac{a_1 + a_2 + ... + a_n}{n} = \ell + \frac{b_1 + b_2 + ... + b_n}{n}.$$

So, to prove the theorem, we have to show that $\lim_{n\to\infty} \frac{b_1 + b_2 + ... + b_n}{n} = 0.$

Since $\{b_n\}$ is convergent, so it is bounded and hence $\exists$ a number K such that

$|b_n| < K, \forall n \in \mathbb{N}$. ...(3.6.23)

Again since $\{b_n\}$ converges to 0, so $\exists$ a natural number m such that

$|b_n| < \frac{1}{2}\in, \forall n \geq m.$ ...(3.6.24)

Now $\left|\frac{b_1 + b_2 + ... + b_n}{n}\right| = \left|\frac{b_1 + b_2 + ... + b_m}{n} + \frac{b_{m+1} + b_{m+2} + ... + b_n}{n}\right|$

$\leq \frac{|b_1| + |b_2| + ... + |b_m|}{n} + \frac{|b_{m+1}| + ... + |b_n|}{n} \leq \frac{mK}{n} + \frac{\in}{2}\frac{(n-m)}{n}, \forall n \geq m,$

$\leq \frac{mK}{n} + \frac{\in}{2}.$ ......(3.6.25), using (3.6.23) and (3.6.24).

Let m_1 be the positive Integer greater than $\frac{2mK}{\in}$ so that $\frac{mK}{n} < \frac{\in}{2}, \ \forall n \geq m_1.$

Thus for all $n \geq \max(m, m_1)$ we have from (3.6.25) that $\left|\frac{b_1 + b_2 + ... + b_n}{n}\right| < \in,$

which means that $\lim\limits_{n\to\infty} \frac{b_1 + b_2 + ... + b_n}{n} = 0.$

Consequently, $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = \ell.$

Note : The converse of the above theorem is not true. For this, let us consider a sequence $\{a_n\}$, where $a_n = (-1)^n$.

Then $\frac{a_1 + a_2 + ... + a_n}{n} = 0,$ if n is even

$= -\frac{1}{n},$ if n is odd.

So, $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = 0,$ but the sequence $\{a_n\}$ is not convergent, i.e. $\lim\limits_{n\to\infty} a_n$ does not exist.

Example 8.3.9 : Show that $\lim\limits_{n\to\infty} \frac{1}{n}\left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + ... + n^{\frac{1}{n}}\right) = 1.$

Solution : Let $a_n = n^{\frac{1}{n}}$. Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

So by Cauchy's first theorem on limits, we have $\lim\limits_{n\to\infty}\dfrac{a_1+a_2+....+a_n}{n}=1$

i.e. $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+....+n^{\frac{1}{n}}\right)=1.$

Example 8.3.10 : Show that $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+...+\dfrac{1}{\sqrt{n^2+n}}\right)=1$.

Solution : Let $a_n=\dfrac{n}{\sqrt{n^2+n}}\cdot$ Then $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1.$$

Thus by Cauchy's first theorem on limits, we have

$$\lim_{n\to\infty}\frac{a_1+a_2+....+a_n}{n}=1$$

i.e., $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+.....+\dfrac{1}{\sqrt{n^2+n}}\right)=1$

i.e., $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+.....+\dfrac{1}{\sqrt{n^2+n}}\right)=1$.

Example 8.3.11 : Show that $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\dfrac{1}{(n+2)^2}+.....+\dfrac{1}{(2n)^2}\right]=0$.

Solution : Now $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\dfrac{1}{(n+2)^2}+.....+\dfrac{1}{(2n)^2}\right]$

$$=\lim_{n\to\infty}\frac{1}{n^2}+\lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+.....+\frac{n}{(n+n)^2}\right]$$

$$=0+\lim_{n\to\infty}\frac{1}{n}\left(a_1+a_2+....+a_n\right), \qquad (3.6.26)$$

where $a_n = \dfrac{n}{(n+r)^2}$ and $\lim\limits_{n\to\infty} a_n = \dfrac{n}{(n+n)^2} = \lim\limits_{n\to\infty} \dfrac{1}{4n} = 0$.

So, by virtue of Cauchy's first theorem on limits, we have

$$\lim_{n\to\infty} \frac{1}{n}(a_1 + a_2 + ... + a_n) = 0$$

and hence it follows from (3.6.26) that

$$\lim_{n\to\infty}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(2n)^2}\right] = 0$$

Theorem 8.4.10 (Cauchy's second theorem on limits)

If $\lim\limits_{n\to\infty} a_n = \ell$, where $a_n > 0, \forall n \in \mathrm{IN}$ and $\ell \neq 0$ then $\lim\limits_{n\to\infty} \sqrt[n]{a_1\, a_2 \ldots a_n} = \ell$.

Proof : Define a sequence $\{u_n\}$, where $u_n = \log a_n$, $\forall n \in \mathrm{IN}$.

Since each $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = \ell > 0$, we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \log a_n = \log\left(\lim_{n\to\infty} a_n\right) = \log \ell.$$

Hence by Cauchy's first theorem on limits, we get

$$\lim_{n\to\infty} \frac{u_1 + u_2 + ... + u_n}{n} = \log \ell.$$

i.e. $\lim\limits_{n\to\infty} \dfrac{1}{n}\left(\log a_1 + \log a_2 + ... + \log a_n\right) = \log \ell \;\Rightarrow \lim\limits_{n\to\infty} \log(a_1\, a_2 \,.....\, a_n)^{\frac{1}{n}} = \log \ell$

$$\Rightarrow \log \lim_{n\to\infty} (a_1\, a_2 \,.....\, a_n)^{\frac{1}{n}} = \log \ell,$$

which yields that $\lim\limits_{n\to\infty} (a_1\, a_2 \,.....\, a_n)^{\frac{1}{n}} = \ell$.

Example 8.3.12.: Show that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Solution : Define a sequence $\{a_n\}$, where

$$a_1 = 1, a_2 = \frac{2}{1}, a_3 = \frac{3}{2}, ..., a_n = \frac{n}{n-1}.$$

Then each $a_n > 0$ and $a_1\, a_2 \,.....\, a_n = n$.

Also $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n}{n-1} = \lim\limits_{n\to\infty} \dfrac{1}{1 - \dfrac{1}{n}} = 1 > 0.$

Therefore by Cauchy's second theorem on limits, we get

$$\lim_{n\to\infty}(a_1\, a_2\, ...\, a_n)^{\frac{1}{n}} = 1, \text{ and hence } \lim_{n\to\infty}\sqrt[n]{n} = 1.$$

Example 8.3.13. : Show that $\lim\limits_{n\to\infty}\left[\frac{2}{1}\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^2....\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e.$

Solution : Let $a_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n.$

Then $a_n > 0, \forall n \in IN$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 0.$

So, by Cauchy's 2nd theorem on limits, we get $\lim\limits_{n\to\infty}\left(a_1\, a_2\, ...a_n\right)^{\frac{1}{n}} = e$

i.e. $\lim\limits_{n\to\infty}\left[\frac{2}{1}\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3....\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = e\cdot$

Example 8.3.14 : Prove that $\lim\limits_{n\to\infty}\left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = e.$

Solution : Let $a_n = \frac{n^n}{n!}.$

Then $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}.\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$

So, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 0.$

Hence by virtue of Theorem 3.6.8, it follows that

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}} = e, \text{ i.e. } \lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = e.$$

8.5 Summary

We define convergent sequences, establish their properties, and prove key limit theorems. These results provide essential tools for studying functions, continuity, and differentiation in real analysis.

8.6 Keywords

Convergent sequence, limit, epsilon-delta, bounded sequence, limit theorems, real analysis.

8.7 References

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8.8 Model Questions

1. Define a convergent sequence with an example.
2. State and prove the uniqueness of limits theorem.
3. Explain the epsilon-N definition of sequence convergence.
4. State and prove the limit theorem for sum of sequences.
5. Prove that a convergent sequence is bounded.
6. Give an example of a sequence that is bounded but not convergent.

Unit 9 ❑ Monotone sequence

Structure

9.1 Objectives

To understand monotone sequences, their properties, and their significance in real analysis, including boundedness and convergence criteria.

9.2 Introduction

Monotone sequences play a crucial role in real analysis. A sequence is monotone if it is either non-decreasing or non-increasing. The Monotone Convergence Theorem ensures their importance in mathematical analysis.

9.3 Monotone Sequences

Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is said to be

(i) **monotonically increasing** if $x_{n+1} \geq x_n, \forall n \in \mathbb{N}$;

(ii) **monotonically decreasing** if $x_{n+1} \leq x_n, \forall n \in \mathbb{N}$.

A sequence $\{x_n\}$ which is either monotonically increasing or monotonically decreasing, is called a monotonic sequence or montone sequence.

Note : If a sequence $\{x_n\}$ such that $x_{n+1} > x_n, \forall n \in \mathbb{N}$ then $\{x_n\}$ is called strictly increasing sequence and if $x_{n+1} < x_n$ then $\{x_n\}$ is called strictly decreasing sequence.

Examples of montonic sequences

(1) The sequence $\{x_n\}$, where $x_n = n$, is a monotonically increasing sequence, as $x_{n+1} > x_n$, $\forall n \in \mathbb{N}$.

(2) The sequence $\{x_n\}$, where $x_n = \frac{1}{n}$ is a monotonically decreasing sequence, as $x_{n+1} < x_n$, $\forall n \in \mathbb{N}$.

(3) The sequence $\{x_n\}$, where $x_n = (-1)^n$ is neither a monotonically increasing sequence nor monotonically decreasing sequence.

Example 9.3.1 : Is the sequence $\{x_n\}$, where

$x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, a monotonic sequence ?

Solution : We have

$$x_{n+1} - x_n = (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n+1}) - (1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) = \frac{1}{n+1} > 0, \quad \forall n \in \mathbb{N} \ \frac{1}{n+1}$$

So, $x_{n+1} > x_n$, $\forall n \in \mathbb{N}$,

which implies that the given sequence $\{x_n\}$ is monotonically increasing and hence monotonic.

Example 9.3.2 : Find the bounds of the sequence $\{x_n\}$, where $x_n = \frac{4n-1}{5n+2}$.

Solution : Here $x_{n+1} - x_n = \frac{4n+3}{5n+7} - \frac{4n-1}{5n+2} = \frac{13}{(5n+7)(5n+2)} > 0$,

which implies that the sequence $\{x_n\}$ is monotonically increasing.

So, a lower bound is the first term of the sequence, i.e., x_1, which is equal to $\frac{3}{7}$.

Moreover, an upper bound is $= \lim_{n\to\infty} x_n = \frac{4}{5}$.

It may be noted that $\frac{3}{7}$ is the greatest lower bound and $\frac{4}{5}$ is the least upper bound.

Theorem 9.3.1 : Every monotonically increasing sequence, which is bounded above, is convergent and converges to its least upper bound.

Proof : Let $\{a_n\}$ be a monotonically increasing sequence which is bounded above.

Let sup $\{a_n\} = B$. Then for given an arbitrary small positive number $\in$, $\exists$ a member a_m of the sequence $\{a_n\}$ such that

$a_m > B - \in$.

Therefore $a_n > B - \in, \forall n \geq m$, ...(3.7.1)

since the sequence is monotonically increasing.

Also $a_n \leq B, \forall n$ i.e. $a_n < B + \in, \forall n$. ...(3.7.2)

From (3.7.1) and (3.7.2), we get

$B - \in < a_n < B + \in, \forall n \geq m$ i.e. $|a_n - B| < \in, \forall n \geq m$.

This shows that the sequence $\{a_n\}$ is convergent and it converges to B, i.e., its supremum.

Theorem 9.3.2. Every monotonically decreasing and bounded below sequence is convergent and converges to its greatest lower bound.

Proof : Let $\{a_n\}$ be a monotonically decreasing sequence, which is bounded below.

Let $\inf\{a_n\} = b$. Then for given an arbitrary small positive number $\in$, there is a number a_m of the sequence $\{a_n\}$ such that

$a_m < b + \in$

Therefore, $a_n < b + \in, \forall n \geq m$, ...(3.7.3)

as the sequence $\{a_n\}$ is monotonically decreasing

Also $a_n \geq b, \forall n$.

Then $a_n > b - \in, \forall n$ ...(3.7.4)

From (3.7.3) and (3.7.4), we get

$b - \in < a_n < b + \in, \forall n \geq m$ i.e. $|a_n - b| < \in, \forall n \geq m$,

which implies that the sequence $\{a_n\}$ is convergent and its limit is b. Thus the sequence converges to its infimum.

By virtue of Theorem 8.3.2, Theorem 9.3.1 and Theorem 9.3.2, we can state the following :

Theorem 9.3.3. (Montone convergence Theorem) : A monotonic sequence is convergent if and only if it is bounded.

Remark : Every monotonically increasing sequence which is not bounded above diverges to ∞ . And every monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Example 9.3.3 Let $a_n = \left(1+\frac{1}{n}\right)^n$. Show that the sequence $\{a_n\}$ is monotonically increasing and bounded above.

If the limit of the sequence is e then show that 2 < e < 3.

Solution : We have

$$a_n = \left(1+\frac{1}{n}\right)^n = 1+\frac{n}{1!}.\frac{1}{n}+\frac{n(n-1)}{2!}.\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}+.....+\text{ to terms }(n+1)$$

$$=1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+......\text{to }(n+1)\text{ terms}.$$

Similarly,

$$a_{n+1} = 1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+......\text{to }(n+2)\text{ terms}$$

Comparing a_n with a_{n+1}, we find that first two terms are equal. From the third term, every term of a_{n+1} is greater than the corresponding term of a_n, and a_{n+1} contains one term more than a_n.

Therefore, $a_{n+1} > a_n, \forall n$;

which implies that $\{a_n\}$ is montonically increasing sequence.

Now, we have $1-\frac{1}{n}<1$

$$\therefore \frac{1}{2!}\left(1-\frac{1}{n}\right)<\frac{1}{2!}$$

Similarly, $\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)<\frac{1}{3!}$ and so on.

Hence, $a_n < 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+....+\frac{1}{n!}$ ....(3.7.5)

Now, $\frac{1}{3!}=\frac{1}{1.2.3}<\frac{1}{2.2}=\frac{1}{2^2}$

Similarly, we can show that $\frac{1}{4!}<\frac{1}{2^3},....\frac{1}{n!}<\frac{1}{2^{n-1}}$

Thus from (3.7.5), we get

$$a_n < 1 + \frac{1}{1\frac{1}{2}} + \frac{2}{2\frac{1}{2}} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots + \frac{1}{2^{n-1}} = 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1 + 2\left[1 - \frac{1}{2^n}\right] = 3 - \frac{1}{2^{n-1}}$$

Thus $a_n < 3 - \frac{1}{2^{n-1}}, \forall n$

and hence $a_n < 3, \forall n,$ which implies that the sequence $\{a_n\}$ is bounded above. Consequently the sequence $\{a_n\}$ is convergent, by Theorem 9.3.1.

If $\lim_{n\to\infty} a_n = e,$ then we have $a_1 < a_n < 3 - \frac{1}{2^{n-1}}$ i.e. $2 < a_n < 3 - \frac{1}{2^{n-1}}.$

Taking limit as $n \to \infty$ in above, we get $2 < e < 3$.

Example 9.3.4 : Show that the sequence f, where

$f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n}$ is convergent.

Solution : Here $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n}$

$\therefore f(n+1) = \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$

Thus $f(n + 1) - f(n) = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2(n+1)(2n+1)} > 0, \forall n \in \mathrm{IN},$

which implies that the sequence $\{f(n)\}$ is monotonically increasing.

Now $\frac{1}{n+1} < \frac{1}{n}, \frac{1}{n+2} < \frac{1}{n}, \ldots., \frac{1}{n+n} < \frac{1}{n}.$

So, $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \ldots. + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \ldots. + \frac{1}{n} = \frac{n}{n} = 1, \forall n$

which means that the sequence $\{f(n)\}$ is bounded above.

Thus by virtue of Theorem 9.3.1, the sequence $\{f(n)\}$ is convergent.

Example 9.3.5 : Prove that the sequence f defined by

$f(1) = \sqrt{7}, f(n+1) = \sqrt{7 + f(n)}$ converges to the positive root of $x^2 - x - 7 = 0$.

Solution : Here $f(1)=\sqrt{7}, f(n+1)=\sqrt{7+f(n)}$

Therefore $\{f(2)\}^2-\{f(1)\}^2=7+\sqrt{7}-7=\sqrt{7}>0$,

which implies that f(2) > f(1). ...(3.7.6)

Now $\{f(n+1)\}^2-\{f(n)\}^2=\left\{\sqrt{7+f(n)}\right\}^2-\left\{\sqrt{7+f(n-1)}\right\}^2$

$= f(n) - f(n - 1)$.

So, f(n+1) > f(n) whenever f(n) > f(n – 1)

i.e. whenever f(n – 1) > f(n – 2)

... ...

i.e. whenever f(2) > f(1), but this is true by (3.7.6).

Thus f(n+1) > f(n), $\forall n$, which means that the sequence {f(n)} is monotonically increasing.

Since f(n) < f(n+1), so $\{f(n)\}^2 < \{f(n+1)\}^2 = 7 + f(n)$

i.e. $\{f(n)\}^2 - f(n) - 7 < 0$. ...(3.7.7)

Consider a quadratic equation $x^2 - x - 7 = 0$, which has two roots, one positive, say α and another is negative, say $-\beta$, such that $\beta > 0$.

So, $x^2 - x - 7 = (x - \alpha)(x + \beta)$

and hence $\{f(n)\}^2-f(n)-7=\{f(n)-\alpha\}\{f(n)+\beta\}$.

So, we have from (3.7.1) that $\{f(n)-\alpha\}\{f(n)+\beta\}<0$

Since $f(n)+\beta>0$,

so, $f(n)-\alpha>0$

i.e. $f(n)<\alpha, \forall n$, which implies that the sequence {f(n)} is bounded above. Consequently, by virtue of Theorem 9.3.1, the sequence {f(n)} is convergent.

Let us take $\lim\limits_{n\to\infty} f(n)=\ell$. Then $\lim\limits_{n\to\infty} f(n+1)=\ell$.

Now, $\{f(n+1)\}^2=7+f(n)$

Taking limit as $n \to \infty$ we get $\ell^2 = 7 + \ell$

i.e. $\ell^2 - \ell - 7 = 0 \Rightarrow (\ell-\alpha)(\ell+\beta)=0$

Since $\ell > 0$, therefore $\ell \neq -\beta < 0$, so $\ell = a$.

Thus the limit of the given convergent sequence is the positive root of the equation $x^2 - x - 7 = 0$.

Example 9.3.6 : Show that the sequence f defined by

$f(1)=\sqrt{2}$ and $f(n+1)=\sqrt{2f(n)}$ converges to 2.

Solution : The members of the sequence {f(n)} are $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2,\sqrt{2\sqrt{2}}}, ...$

We have $2\sqrt{2}>2 \Rightarrow \sqrt{2\sqrt{2}}>\sqrt{2}$, i.e. $f(2)>f(1)$

Supose that f(n + 1) > f(n).

Then $\sqrt{2f(n+1)}>\sqrt{2f(n)} \Rightarrow f(n+2)>f(n+1)$.

Thus f(n + 1) > f(n) => f(n + 2) > f(n + 1), and f(2) > f(1).

So, by mathematical induction, we may conclude that the sequence {f(n)} is monotonically increasing.

Clearly we have f(1) < 2.

Suppose that f(n) < 2.Then f(n + 1) = $\sqrt{2f(n)}<\sqrt{2.2}=2$

Thus f(n) < 2 => f(n + 1) < 2, and f(1) < 2.

So by mathematical induction, we have f(n) < 2, $\forall$n.

This show that the sequence {f(n)} is bounded above.

Consequently the sequence {f(n)} is convergent by virtue of Theorem 9.3.1.

Let $\lim_{n\to\infty} f(n)=\ell$.

Since f(n + 1) = $\sqrt{2f(n)}$, we have $\{f(n+1)\}^2=2f(n)$.

Taking limit of above as $n\to\infty$, we get

$\ell^2=2\ell \Rightarrow \ell(\ell-2)=0$.

But this limit 'ℓ' can not be equal to zero. So, we must have ℓ = 2, i.e., $\lim_{n\to\infty} f(n)=2$.

9.4 Summary

Monotone sequences are classified as increasing or decreasing. A bounded monotone sequence is always convergent. This concept is fundamental for understanding limits, continuity, and convergence of series.

9.5 Keywords

Monotone sequence, increasing sequence, decreasing sequence, bounded sequence, convergence, real analysis.

9.6 References

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9.7 Model Questions

1. Define a monotone sequence with an example.
2. Prove that every bounded monotone sequence is convergent.
3. Give an example of a monotone but unbounded sequence.
4. State and explain the Monotone Convergence Theorem.
5. Show that the sequence $a = \frac{1}{n}$ is monotone and convergent.
6. Discuss the role of monotone sequences in real analysis.
7. $x_n = \frac{n^2+1}{2n^2+3}$. Verify $x_n \to \frac{1}{2}$ as $n \to \infty$.
8. Verify that $\lim\limits_{n \to \infty} \frac{n^2-1}{n^2+n+1} = 1$.

Unit 10 ❑ Subsequence

Structure

10.1 Objectives

To explore the concepts of subsequences and Cauchy sequences, highlighting their significance in convergence analysis and the completeness of real numbers.

10.2 Introduction

Subsequences help analyze convergence behavior, while Cauchy sequences characterize completeness. Understanding these concepts is fundamental to real analysis, ensuring rigorous treatment of limits and continuity.

10.3 Subsequences

Let $\{x_n\}$ be a sequence of real numbers and $\{i_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers, i.e., $i_1 < i_2 < i_3 < \dots$. Then the sequence $\{x_{i_n}\} = \{x_{i_1}, x_{i_2}, x_{i_3}, \dots\}$ is called a subsequence of $\{x_n\}$.

Note : (1) If $\{y_n\}$ is a subsequence of $\{x_n\}$ then each $y_n = x_{i_n}$ for some $i_n \geq n$.

(2) Every sequence can be regarded as a subsequence of itself.

Examples of subsequences.

(1) Each of the sequences

(i) $\left\{\frac{1}{n+2}\right\}$, (ii) $\left\{\frac{1}{2n-1}\right\}$, (iii) $\left\{\frac{1}{n^2}\right\}$ and (iv) $\left\{\frac{1}{(2n)!}\right\}$ are subsequences of the sequences $\left\{\frac{1}{n}\right\}$.

(2) Each of the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ are subsequences of the sequence $\{x_n\}$.

(3) The sequence of prime numbers {2, 3, 5, 7, 11, ...} is a subsequence of natural numbers {1, 2, 3, 4, ...}.

Theorem 10.3.1 : Let $\{y_n\}$ be a subsequence of a sequence $\{x_n\}$. Then

(i) $\{y_n\}$ is bounded if $\{x_n\}$ is bounded.

(ii) $\{y_n\}$ is montonic if $\{x_n\}$ is monotonic.

(iii) $\{y_n\}$ is convergent if $\{x_n\}$ is convergent. Further, if $\{x_n\}$ converges to ℓ then $\{y_n\}$ converges to ℓ.

Proof : Since $\{y_n\}$ is a subsequence of $\{x_n\}$, we have $y_n = x_{i_n}$, where $\{i_n\}$ is a sequence of natural numbers such that $i_n < i_{n+1}$ and $i_n \geq n$, $\forall n \in \mathbb{N}$.

(i) If $\{x_n\}$ is bounded then there exists real numbers m and M such that

$m \leq x_n \leq M, \forall n \in \mathbb{N}$.

So, in particular we have $m \leq x_{i_n} \leq M, \forall n \in \mathbb{N}$.

Consequently the subsequences $\{y_n\}$ is bounded.

(ii) If $\{x_n\}$ is monotonic increasing then

$$i_n < i_{n+1} \Rightarrow x_{i_n} \leq x_{i_{n+1}}$$

i.e. $y_n \leq y_{n+1}$, $\forall n \in \mathbb{N}$,

which implies that $\{y_n\}$ is also monotonic increasing.

Similarly if $\{x_n\}$ is monotonic decreasing then we can prove that $\{y_n\}$ is also monotonic decreasing. Hence if $\{x_n\}$ is monotonic sequence then $\{y_n\}$ is a monotonic sequence.

(iii) Let $\{x_n\}$ be a convergent sequence and converges to ℓ. Then for given an arbitrary small positive number ε, then there exists a positive integer K such that

$|x_n - \ell| < \in \forall n \geq K.$

Since $i_n \geq n$, we have $n > K \Rightarrow i_n \geq K$.

$\Rightarrow |x_{i_n} - \ell| < \in$, i.e. $|y_n - \ell| < \in$.

Thus $\forall n \geq K, |y_n - \ell| < \in$, which implies that the subsequence $\{y_n\}$ is convergent and converges to ℓ.

Note : The converge of (iii) is not true. If there exist two different subsequences $\{x_{i_n}\}$ and $\{x_{j_n}\}$ of $\{x_n\}$such that they converse to two different limits, then the sequence $\{x_n\}$ is not convergent. That means if a sequence $\{x_n\}$ has a divergent subsequence then $\{x_n\}$ is divergent. For example, it is known that $\{y_n\} = \{1, 1, 1, ...\}$ and $\{z_n\} = \{-1, -1, -1, ...\}$ are two subsequences of $\{x_n\}$, where $x_n = (-1)^n$. Then both the subsequences $\{y_n\}$ and $\{z_n\}$ are convergent and they converge to 1 and –1 respectively. However, the sequence $\{x_n\}$ is not convergent.

Example 10.3.1 : Show that the sequence $\left\{\sin\frac{n\pi}{2}\right\}$ is not convergent.

Solution : Let $x_n = \sin\frac{n\pi}{2}$. Then

$$\{x_n\} = \left\{\sin\frac{\pi}{2}, \sin\pi, \sin\frac{3\pi}{2}, \sin 2\pi, \sin\frac{5\pi}{2}, ...\right\}$$

$$= \{1, 0, -1, 0, 1, 0, -1, 0, ...\},$$

which has the subsequences $\{x_{4n-3}\} = \{1, 1, 1, ...\}$,

$\{x_{2n}\} = \{0, 0, 0, ...\}$ and $\{x_{4n-1}\} = \{-1, -1, -1, ...\}$.

Since the subsequences $\{x_{4n-3}\}, \{x_{2n}\}$ and $\{x_{4n-1}\}$

converge to different limits 1, 0 and –1 respectively, the sequence $\{x_n\}$ does not converge.

Corollary 10.3.1 : A sequence $\{x_n\}$ converges to a real number ℓ if and only if its subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converges to the same limit ℓ.

Proof : Suppose the sequence $\{x_n\}$ converges to ℓ. Then by Theorem 10.3.1 (iii), its subsequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ also converges to ℓ.

i.e. $\lim\limits_{n\to\infty} x_{2n} = \ell = \lim\limits_{n\to\infty} x_{2n-1}$. ....(3.8.1)

Conversely, suppose (3.8.1) is true. Then for given an arbitrary small positive number $\in$, there exists two natural numbers m_1 and m_2 such that

$|x_{2n} - \ell| < \in, \forall n \geq m_1$. and $|x_{2n-1} - \ell| < \in, \forall n \geq m_2$.

Choose m = max $\{m_1, m_2\}$. Then from above we get

$\ell - \in < x_{2n} < \ell + \in$ and $\ell - \in < x_{2n-1} < \ell + \in, \forall\ n \geq m$.

Hence $\ell - \in < x_n < \ell + \in, \forall\ n \geq 2m-1$, which is also a natural number. Consequently, $\lim\limits_{n\to\infty} x_n = \ell$.

Note : Any two subsequences of a sequence $\{x_n\}$ converge to the same limit do not imply that the sequence $\{x_n\}$ is convergent.

For this let us consider the sequence $\{x_n\}$, where $x_n = \sin\frac{n\pi}{4}$.

Then $\{x_{8n-7}\} = \left\{\sin\frac{\pi}{4}, \sin\frac{9\pi}{4}, \sin\frac{17\pi}{4}, \ ......\right\}$

and $\{x_{8n-5}\} = \left\{\sin\frac{3\pi}{4}, \sin\frac{11\pi}{4}, \sin\frac{19\pi}{4}, \ ......\right\}$

are subsequences of $\{x_n\}$. Each of $\{x_{8n-7}\}$ and $\{x_{8n-5}\}$ converges to $\frac{1}{\sqrt{2}}$, but the sequence $\{x_n\}$ is not convergent.

Now we have seen that every convergent sequence is bounded, (Theorem 8.3.2), but the converse is not true, i.e., bounded sequence may not be convergent. However, we have the following :

Theorem 10.3.2 (Bolzano-Weierstrass Theorem for Sequences) :
Every bounded sequence has a convergent subsequence.

Proof : Let S be the set of all distinct points of a bounded sequences $\{x_n\}$. Then S is bounded. There are two cases : S may be finite or infinite.

If S is finite, then there must be at least one element. say α, in S, which is infinitely repeated in $\{x_n\}$. Let $\{i_n\}$ be strictly increasing sequence of natural numbers such that $x_{i_n} = \alpha, \forall n \in$ IN. Clearly $\{x_{in}\}$ is a subsequence of $\{x_n\}$ and hence $\{x_{n_1}\}$ converges to

α, as $\{x_{n_1}\}$ is a constant sequence $\{\alpha, \alpha, \alpha, ...\}$. So the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_1}\}$.

Now, if S is infinite, then by Bolzano Weierstrass Theorem for sets, it has a limit point (see, Theorem 2.14.1), say ℓ in IR. We have to construct a subsequence of $\{x_n\}$ which converges to ℓ.

Since ℓ is a limit point of S, the $\frac{1}{m}$ – neighbourhood $I_m = \left(1-\frac{\ell}{m}, 1+\frac{\ell}{m}\right)$ of ℓ contains infinitely many element of S. Hence for each m, there are infinitely many values of n such that $x_n \in I_m$.

Choose $x_{i_1} \in I_1, x_{i_2} \in I_2$ such that $i_2 > i_1$. Then choose $x_{i_3} \in I_3$ such that $i_3 > i_2$ and so on. So, we obtain a subsequence $\{x_{i_n}\}$ of $\{x_n\}$ such that $x_{i_n} \in I_n$ i.e. $|x_{i_n} - \ell| < \frac{1}{n}$, $\forall n \in$ IN.

Consequently $\lim\limits_{n\to\infty} x_{i_n} = \ell$. That means we get a convergent subsequent $\{x_{i_n}\}$ of $\{x_n\}$. Hence the theorem.

Note (1) : In Example 10.3.1, we have seen that the sequence $\{x_n\} = \left\{\sin\frac{n\pi}{2}\right\}$ is bounded (but not convergent), which has three convergent subsequences $\{x_{4n-3}\}, \{x_{2n}\}$ and $\{x_{4n-1}\}$. So, Bolzano Weierstrass Theorem for sequences is verified.

Note (2) : However a bounded sequence may have a divergent subsequence. For this, in the sequence $\{x_n\}$ of Example 10.3.1, the subsequence

$\{x_{2n-1}\} = \{1, -1, 1, -1, 1, -1, ...\}$ is a divergent subsequence of the bounded sequence $\{x_n\}$.

Also an unbounded sequence may have a convergent subsequence. For this we consider a sequence $\{x_n\} = \left\{n^{(-1)^n}\right\} = \left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, ...\right\}$, which is unbounded. The sequence $\{x_{2n}\}$ is a divergent subsequence of $\{x_n\}$, while the sequence $\{x_{2n-1}\}$ is a convergent subsequence of $\{x_n\}$.

10.4 Cauchy Sequences

A sequence $\{x_n\}$ is called a Cauchy sequence if for given an arbitrary small positive number $\in$, there exists a natural number K such that

$|x_n - x_m| < \in, \forall\, n, m \geq K.$

Taking n = m + p, where p = 1, 2, 3, ..., the above condition can also be written as

$|x_{m+p} - x_m| < \in, \forall\, m \geq K.$ and p = 1, 2, 3, ...

Thus a sequence $\{x_n\}$ is cauchy if x_n and x_m are close together when m and n are large w. r. to K.

Example 10.4.1 : Show that the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

Solution : Let $x_n = \frac{1}{n}$. Let $\in$ be an arbitrary small positive number. It is known that $\left\{\frac{1}{n}\right\}$ converges to 0.

So, $\left|\frac{1}{n} - 0\right| < \frac{\in}{2}\ \ \forall n \geq K$ (a natural number)

i.e. $\frac{1}{n} < \frac{\in}{2},\ \forall n \geq K.$

Now $|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \frac{\in}{2} + \frac{\in}{2} = \in, \forall\, m, n \geq K.$

i.e. $|x_m - x_n| < \in, \forall\, m, n \geq K,$ which shows that $\{x_n\}$ is a Cauchy sequence.

Theorem 10.4.1 : Every convergent sequence is a Cauchy sequence.

Proof : Let $\{x_n\}$ be a convergent sequence and $\lim\limits_{n\to\infty} x_n = \ell$.

Then for given an arbitrary small positive number $\in$, $\exists$ a natural number K such that

$$|x_n - \ell| < \frac{\in}{2}, \forall n \geq K, \qquad ...(3.9.1)$$

and hence $|x_m - \ell| < \frac{\in}{2}, \forall m \geq K$. ...(3.9.2)

Thus $\forall n, m \geq K,$ we have

$$\left|x_m - x_n\right| = \left|\left(x_m - \ell\right) - \left(x_n - \ell\right)\right| \leq \left|x_m - \ell\right| + \left|x_n - \ell\right| < \frac{\in}{2} + \frac{\in}{2} = \in,$$

which shows that the sequence $\{x_n\}$ is a Cauchy sequence.

Theorem 10.4.2. : Every Cauchy sequence is bounded.

Proof : Let $\{x_n\}$ be a Cauchy sequence.

Choose $\in = 1$. Then there exists a natural number K such that

$\left|x_n - x_m\right| < 1, \forall n, m \geq K.$

So, in particular taking m = K+1, we have

$\left|x_n\right| - \left|x_{K+1}\right| \leq \left|x_n - x_{K+1}\right| < 1, \forall n \geq K.$

or, $\left|x_n\right| < 1 + \left|x_{K+1}\right| = \lambda(\text{say}), \forall n \geq K$ ...(2.9.3)

Let M = max $\left\{\left|x_1\right|, \left|x_2\right|, ..., \left|x_{K-1}\right|, \lambda\right\}$.

Then it is evident that

$\left|x_n\right| \leq M, \forall n = 1, 2, ..., K-1$ ...(3.9.4)

and also from (3.9.3) we have $\left|x_n\right| < M, \forall n \geq K.$ ...(3.9.5)

From (3.9.4) and (3.9.5) it follows that $\left|x_n\right| \leq M, \forall n \in \mathrm{IN},$

which means that the sequence $\{x_n\}$ is bounded.

Note : The converse of the above theorem is not true, i.e., bounded sequence may not be a Cauchy sequence.

For this, let us consider the sequence $\{x_n\}$, where $x_n = (-1)^n$. Clearly this sequence is bounded as $\left|x_n\right| \leq 1, \forall n \in \mathrm{IN}$.

Now $\left|x_{2m+1} - x_{2m}\right| = \left|(-1)^{2m+1} - (-1)^{2m}\right| = \left|-1-1\right| = 2, \forall m \in \mathrm{IN}$ ....(3.9.6)

Choose $\in = \frac{1}{2}$ and take p = 2m+1, q = 2m then p, q > m.

Then (3.9.6) shows that it is not possible to find any $m \in \mathrm{IN}$ such that $\left|x_p - x_q\right| < \in, \ \forall p, q > m.$

That means the sequence $\{x_n\}$ is not a Couchy sequence.

Theorem 10.4.3 : Every Cauchy sequence in IR is convergent.

Proof : Let $\{x_n\}$ be a Cauchy sequence. So, $\{x_n\}$ is bounded by Theorem 3.9.2. Hence it has a convergent subsequence by Theorem 10.3.2. Let $\{y_n\}$ be a convergent subsequence of $\{x_n\}$ such that $y_n \to \ell$.

We shall show that $\{x_n\}$ also converges to ℓ.

Let $\in$ be an arbitrary shall positive number.

Since $y_n \to \ell, \exists$ a natural number K_1, such that

$$|y_n - \ell| < \frac{\in}{2}, \forall\ n \geq K_1. \qquad ....(3.9.7)$$

Again since $\{x_n\}$ is Cauchy, there exists a natural number K_2 such that

$$|x_n - x_m| < \frac{\in}{2}, \forall\ n, m \geq K_2. \qquad ...(3.9.8)$$

Let $K_3 = \max\ \{K_1, K_2\}$. Then $\forall\ n, m \geq K_3$ we have

$$|x_n - x_m| < \frac{\in}{2} \text{ and } |y_n - \ell| < \frac{\in}{2}. \qquad ...(3.9.9)$$

Since $\{y_n\}$ is a subsequence of $\{x_n\}$, we have

$$y_{k_3} = x_m \text{ for some } m > K_3. \qquad ...(3.9.10)$$

Now, $|x_n - \ell| = |(x_n - x_m) + (x_m - \ell)| = |(x_n - x_m) + (y_{k_3} - \ell)|$

$$\leq |x_n - x_m| + |y_{k_3} - \ell| < \frac{\in}{2} + \frac{\in}{2} = \in, \text{ using } (3.9.9).$$

Thus $\forall n \geq K_3$ we have $|x_n - \ell| < \in$, which implies that the sequence $\{x_n\}$ converges to ℓ. Hence the theorem.

Combining Theorem 3.9.1. and Theorem 3.9.3, we can state the following :

Theorem 10.4.4 : (Cauchy's Convergence Criterion) A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Using the definition of Cauchy sequence, the Cauchy's convergence criterion can be stated equivalently in the form as

A necessary and sufficient condition for the sequence $\{x_n\}$ is Cauchy that for given every arbitrary shall positive number $\in$, there exists a natural number m such that

$$|x_{n+p} - x_n| < \in, \ \forall n \geq m \text{ and } p \in \text{IN}.$$

The above criteria is also known as **Cauchy's general principle of convergence.**

Example 10.4.2. Show that, with the help of Cauchy's general principle of convergence, the sequence $\{x_n\}$,

where $x_n = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}$, is not convergent.

Solution : Here $x_n = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}$,

$\therefore\ x_{n+p} = 1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{n+p}$.

Choose $\in = \frac{1}{2}$

Now, $|x_{n+p} - x_n| = \left|\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{n+p}\right| = \frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{n+p}$

$> \frac{1}{2m}+\frac{1}{2n}+....+\frac{1}{2m}$, taking $p = n = m$

$= \frac{m}{2m} = \frac{1}{2} = \in$.

Thus by Cauchy's criterian for convergence, it follows that the given sequence $\{x_n\}$ is not convergent.

Example 10.4.3 Use Cauchy's general principle of covnergence to prove that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent.

Solution : Let $x_n = \frac{n}{n+1}$. Then for all $p \in$ IN,

$x_{n+p} = \frac{n+p}{n+p+1}$.

Let $\in$ be an arbitrary small positive number. Choose $m \in$ IN such that

$m = \left[\frac{1}{\in}\right]+1$.

Now, $|x_{n+p} - x_n| = \left|\frac{n+p}{n+p+1} - \frac{n}{n+1}\right| = \frac{p}{(n+p+1)(n+1)}$

$< \frac{1}{n+1}$, since $\frac{p}{n+p+1} < 1, \ \forall p \in \mathbb{N}$

$< \frac{1}{n} < \in$, for $n > \frac{1}{\in}$.

Thus $|x_{n+p} - x_n| < \in, \ \forall n \geq m$ and $p \in \mathbb{N}$, which proves that the sequence $\{x_n\}$ is convergent.

10.5 Summary

In this unit we have defined the concept of sequence of real numbers, bounded sequence, montone sequence, Cauchy sequence and their convergence to a limit with examples. We also discussed the subsequence of a sequence of real numbers and their properties with examples. Many important results related to the topic have been presented here. Some problems have also been worked out with help of them. For more study, a list of references is given in section 3.13. The important data and results are also mentioned in section 3.11 as a summary of this unit. Some problems/questions are given at the end of this unit.

- A sequence is a function from IN to IR.
- A sequence is called bounded if it is bounded above as well as bounded below.
- If a sequence is convergent then its limit is unique.
- Every convergent sequence is bounded, but the converse is not true.
- Non-convergent sequences are the sequences which are not convergent.
- Non-convergent sequences are either divergent or oscillatory.
- The sum, difference and product of two convergent sequences are also convergent.
- The quotient of two convergent sequences is also convergent, provided the limit of the sequence & each terms of the sequence in denominator is not equal to zero.
- If a sequcne $\{x_n\}$ is convergent then $\{|x_n|\}$ is also convergent, but the converse is not true.
- A sequence is called monotonic if it is either a monotonically increasing or monotonically decreasing.
- Every monotonic sequence is either bounded above or bounded below.
- Every incresing sequence is bounded below.
- Every discreasing sequence is bounded above.

- A sequence having althernatively positive and negative terms can not be monotonic.
- A monotonic sequence is convergent if and only if it is bounded (Monotone convergence Theorem).
- Every subsequence of a bounded sequence is bounded.
- Each subsequence of a monotonic sequence is monotonic.
- Every subsequence of a convergent sequence is convergent and converges to the same limit of a sequence. However, the converse is not true.
- Every bounded sequence has a convergent subsequence (BolzanoWeierstrass Theorem for sequences). However, a bounded sequence may have a divergent subsequence. Also an unbounded sequence may have a convergent subsequence.
- Every convergent sequence is a Cauchy sequence, but the converse is not true. However, every Cauchy sequence in IR is convergent.
- Every Cauchy sequence is bounded.
- A sequence of real numbers is convergent if and only if it is a Cauchy sequence (Cauchy's General Principle of Convergence)

10.6 Keywords

Sequence, bounded sequence, convergent sequence, divergent sequence, oscillatory sequence, limit of a sequence, monotone sequence, montone convergence theorem, subsequence, Cauchy sequence.

10.7 References

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10.8 Model Questions

[A] Multiple Choice Questions (MCQ) :

(Choose the correct answer each of the following) :

[1] The sequence $\{n\}$ is

(a) bounded above (b) bounded below

(c) bounded (d) unbounded.

[2] The sequence $\{2^n\}$ is

(a) bounded below (b) bounded above

(c) bounded (d) unbounded.

[3] The sequence $\{(-1)^n n\}$ is

(a) bounded below (b) bounded above

(c) neither bounded above nor bounded below

(d) None of these.

[4] The sequence $\left\{1+\frac{(-1)^n}{n}\right\}$ is

(a) convergent (b) divergent

(c) oscillatory (d) none of these.

[5] The value of $\lim\limits_{n\to\infty}\frac{3+2\sqrt{n}}{\sqrt{n}}$ is

(a) 0 (b) 1

(c) 2 (d) 3

[6] The value of $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^n$ is

(a) e (b) e^2

(c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

[7] An example of oscillatory sequence is

(a) $\left\{\frac{(-1)^n}{n}\right\}$ (b) $\left\{(-1)^n n\right\}$

(c) $\left\{(-1)^{n^2}\right\}$ (d) $\left\{(-1)^n n^2\right\}$

[8] A sequence can converges to

(a) one limit (b) finite number of limits

(c) infinitely many limits (d) All of the above.

[9] Every bounded monotonically decreasing sequence is

(a) oscillatory (b) diverges to $+\alpha$

(c) diverges to $-\alpha$ (d) covnergent

[10] Which of the following statement is true ?

(a) a convergent sequence is not bounded

(b) a bounded sequence has no divergent subsequence.

(c) an unbounded sequence may have a convergent subsequence.

(d) None of these above.

Ans. : [1] (b), [2] (a), [3] (c), [4] (a), [5] (c), [6] (b), [7] (b), [8] (a), [9] (d), [10] (c).

[B] Miscellaneous Questions :

[1] Explain the boundedness of the following sequences :

(i) $\{-n^2\}$ (ii) $\left\{\cos\frac{1}{3}n\pi\right\}$ (iii) $\left\{\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}\right\}$, (iv) $\left\{\frac{n+\sqrt{n}}{2n}\right\}$.

[2] Give examples of a sequence which is

(i) bounded above but not bounded below

(ii) bounded below but not bounded above

(iii) bounded

(iv) Neither bounded below nor bounded above.

[3] Show that the sequence $\{(-1)^n\}$ does not converge.

Hints : If $x_n = (-1)^n$, then $x_{2n} = 1$ and $x_{2n+1} = -1$.

[4] Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

[5] Prove that $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n!}}=0$

[6] Show that $\lim\limits_{n\to\infty} p^{\frac{1}{n}}=1,$ where p > 0.

Hints : **Case I ; p = 1.** It is obvious as $p^{\frac{1}{n}}$ is constant sequence.

Case II : p >1. Then $p^{\frac{1}{n}}=1+q_n$ for some $q_n > 0$.

So, $p=(1+q_n)^n \geq 1+nq_n$

i.e. $q_n \leq \dfrac{p-1}{n}, \forall n \in \mathbb{N}$ and hence

$p^{\frac{1}{n}}-1=q_n \leq \dfrac{p-1}{n} \to 0$ as $n \to \infty$, which means $\lim\limits_{n\to\infty} p^{\frac{1}{n}}=1$.

Case III : 0 < p < 1 : Then $p^{\frac{1}{n}}=\dfrac{1}{1+r_n}$ for some $r_n > 0$.

$\therefore p=\dfrac{1}{(1+r_n)^n} \leq \dfrac{1}{1+nr_n} < \dfrac{1}{nr_n} \Rightarrow 0 < r_n < \dfrac{1}{np}, \forall n \in \mathbb{N}$ and hence

$0 < 1-p^{\frac{1}{n}}=\dfrac{r_n}{1+r_n} < r_n < \dfrac{1}{np} \to 0$ as $n \to \infty$, which implies that $\lim\limits_{n\to\infty} p^{\frac{1}{n}}=1$.

[7] Examine, whether the sequence $\left\{\dfrac{n^2+3n+5}{2n^2+5n+7}\right\}$ is convergent or not. Find limit, if it converges.

[8] Show that the sequence $\{x_n\}$, where $x_n=\sqrt{n+1}-\sqrt{n}, \forall n \in \mathbb{N}$, is convergent.

[9] Show that the sequence $\{b_n\}$, where $b_n=\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\ldots..+\dfrac{1}{\sqrt{n^2+n}}$, converges to 1.

[10] Show that $\lim\limits_{n\to\infty}\left\{\dfrac{1}{\sqrt{n+1}}+\dfrac{1}{\sqrt{n+2}}+\ldots..+\dfrac{1}{\sqrt{2n}}\right\}=\infty$.

[11] Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Hints : Use Theorem 3.6.8 for $u_n = n$

[12] Show that $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{2n^2+1}}+\frac{1}{\sqrt{2n^2+2}}+.....+\frac{1}{\sqrt{2n^2+n}}\right]=\frac{1}{\sqrt{2}}$.

Hints : Use Cauchy's first theorem on limits.

[13] Show that $\lim\limits_{n\to\infty}\left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+.....+\frac{1}{\sqrt{2n}}\right]=\infty$.

[14] Prove that $\lim\limits_{n\to\infty}\frac{n}{(n!)^{\frac{1}{n}}}=e$.

Hints : See example 3.6.14 as $\left(\frac{n^n}{n!}\right)^{\frac{1}{n}}=\frac{n}{(n!)^{\frac{1}{n}}}$.

[15] Show that $\lim\limits_{n\to\infty} 2^{-n}n^2 = 0$

Hints : Use Theorem 3.6.6. for $u_n = \frac{n^2}{2^n}$.

[16] Give an example of a sequence in each of the following :

(i) monotonically increasing but not bounded above.

(ii) monotonically decreasing but not bounded below.

(iii) bounded above as well as bounded below but not monotonic

(iv) not monotonic.

[17] Is every bounded sequence a monotonic ?

Hints : No. For this, consider $\{(-1)^{n-1}\}$.

[18] Is the sequence $\left\{\frac{2^n}{n!}\right\}$ monotonically increasing or decreasing ? Find bounds of this sequence, if any.

[19] Show that the sequence f, where $f(n)=\frac{1}{1!}+\frac{1}{2!}+.....+\frac{1}{n!}$ is convergent.

Hints : Use Theorem 9.3.1 by showing that the given sequence is monotonically increasing and bounded below.

[20] Show that the sequence $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, ...$ converges to 3.

[21] Let a_1, b_1 be two distinct positive real numbers and

$a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$ and $b_n = \sqrt{a_{n-1}b_{n-1}}, \forall n \geq 2.$ Show that the sequences $\{a_n\}$ and $\{b_n\}$ are monotonic and convergent.

Also show that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

[22] Define a subsequence. Give an example of a subsequence of a sequence.

[23] Show that the sequence $\left\{\frac{3n}{3n+1}\right\}$ is a subsequence of the sequence $\left\{\frac{n}{n+1}\right\}$.

[24] Prove that the sequence $\{x_n\}$ satisfying the condition

$|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|, \forall n \in \mathrm{IN},$ where $0 < c < 1$, is a Cauchy sequence.

[25] State and prove Cauchy's general principle of convergence.

[26] State and prove Bolzano Weierstrass Theorem for sequences.

[27] Give an example of a bounded sequence that is not a Cauchy sequence.

Unit 11 ❑ Series of Real Number

Structure

11.1 Objectives

The Object of this unit are as :

- to study infinite series, and its convergence.
- to study a special type of series, geometric series & its behaviour.
- to know about Telescoping series.
- to know about convergence Tests like comparison test, D'Abmbert's Ratio test, Cauchy's Root test, Integral test.
- to study about Alternating series & Leibuitz test for alternating series.
- to study Absolute convergence and conditionally convergence.
- to know about power series and radius of convergence of a power series.

11.2 Introduction

In this chapter we shall discuss the techniques of testing the behaviour of infinite series as regards convergence. The most important technique for series, all of whose terms are of the same sign (all positive or all negative), is to compare the given series with another suitably chosen series with known behaviour. So, first of all, comparison tests are discussed, and then some special tests for convergence are considered. Leibnitzs testis for alternating series. At last, power series will be discussed in detail towards the end.

The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$S_{100} = 1+2+3+4+5+...+98+99+100$

It gets much easier if you add two of these sums, and pair the numbers in a special way :

$$2\,S_{100} = 1+2+3+4+...+97+98+99+100$$
$$100+99+98+97+....+4+3+2+1\cdot$$

A straight forward observation that each column on the right side to 101 and that there are 100 such columns yields that

$$2S_{100} = 101.100, \text{ that is } S_{100} = \frac{101.100}{2} = 5050.$$

This can be generalized to any natural number n to get the formula

$S_n = 1+2+3+4+5+....+(n-1)+n = \dfrac{(n+1)n}{2}.$

This procedure indicates that it would be impossible to find the sum

$1+2+3+4+5+.....+n+...$

where the last set of .... indicates that we continue to add natural numbers. The situation is quite different if we consider the sequence

$\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, ..., \dfrac{1}{2^{n!}}$

and start adding more and more consecutive terms of this sequence.

$$\frac{1}{2} = 1-\frac{1}{2} = \frac{1}{2}$$
$$\frac{1}{2}+\frac{1}{4} = 1-\frac{1}{4} = \frac{3}{4}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1-\frac{1}{8} = \frac{7}{8}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 1-\frac{1}{16} = \frac{15}{16}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = 1-\frac{1}{32} = \frac{31}{32}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} = 1-\frac{1}{64} = \frac{63}{64}$$

These sums are nicely illustrated by the following pictures

In this example it seems natural to say that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ equals 1 :

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+.....+\frac{1}{2^n}+.....=1$$

Why does this make sense ? This makes sense since we have seen above that as we add more and more terms of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ..., \frac{1}{2n'}.....$$

we are getting closer and closer to1, Indeed,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$$

and $\quad \lim_{n\to\infty}\left(1-\frac{1}{2^n}\right)=1$.

This reasoning leads to the definition of convergence of an infinite series.

11.3 Infinite Series

Definition : 11.3.1 : Given a sequence (a_n) of real numbers, a formal sum of the form $\sum_{n=1}^{\infty} a_n$ (or $\sum a_n$ for short) is called an infinite series.

For any $n \in \mathbb{N}$, the finite sum $s_n := a_1 + ... + a_n$ is called the (n-th) partial sum of the series $\sum a_n$.

A more formal definition of an infinite series is as follows. By the symbol $\sum_n a_n$ we mean the sequence (s_n) where $s_n := a_1 + ... + a_n$.

We say that the infinite series $\sum a_n$ is convergent if the sequence (s_n) of partial sums is convergent. In such a case, the limit $s := \lim s_n$ is called the sum of the series and we denote this fact by the symbol $\sum a_n = s$.

We may that the series $\sum a_n$ is divergent if the sequence of its partial sums is divergent.

The series $\sum_n a_n$ is said to be absolutely convergent if the infinite series $\sum_n |a_n|$ is convergent. Note that a series $\sum a_n$ of non-negative terms, (that is, $a_n > 0$ for all n) is convergent iff it is absolutely convergent.

If a series is convergent but not absolutely convergent, then it is said to be conditionally convergent.

Let us look at some examples of series and their convergence.

Example 11.3.1 : Let (a_n) be a constant sequence $a_n = c$ for all n. Then the infinite series $\sum a_n$ is convergent iff $c = 0$. For, the partial sums is $s_n = nc$. Thus (s_n) is convergent iff $c = 0$.

Example 11.3.2 : Let an be non-negative real numbers and assume that $\sum a_n$ is convergent. Since $s_{n+1} = s_n + a_{n+1}$, it follows that the sequence (s_n) is increasing. We have seen (Theorem 2.3.2) that (s_n) is convergent iff it is bounded above. Hence a series of non-negative terms is convergent iff the sequence of partial sums is bounded. Note that if $\sum a_n$ is convergent, then $\sum a_n = \text{lub } \{s_n : n \in N\}$.

Example 11.3.3 : (Geometric Series), Let a and r be real numbers. The most important infinite series is

$$a + ar + ar^2 + ar^3 + ... + ar^n + ... = \sum_{n=0}^{+\infty} ar^n$$

This series is called a geometric series. To determine whether this series converges or not we need to study its partial sums :

$S_0 = a,$ $S_1 = a + ar,$
$S_2 = a + ar + ar^2$ $S_3 = a + ar + ar^2 + ar^3,$
$S_4 = a + ar + ar^2 + ar^3 + ar^4,$ $S_5 = a + ar + ar^2 + ar^3 + ar^4 + ar^5,$
$S_n = a + ar + ar^2 + ... + ar^{n-1} + ar^n$

Notice that we have already studied the special case when a = 1 and $r = \frac{1}{2}$. In this special case we found a simple formula for S_n and then we evaluated $\lim\limits_{n\to+\infty} S_n$. It turns out that we can find a simple formula for S_n in the general case as well.

First note that the case a = 0 is not interesting, since then all the terms of the geometric series are equal to 0 and the series clearly converges and its sum is 0. Assume that $a \neq 0$. If r = 1 then $S_n = n\,a$. Since we assume that $a \neq 0$, $\lim\limits_{n\to+\infty} n\,a$ does not exsit. Thus for r = 1 the series diverges.

Assume that $r \neq 1$. To find a simple formula for S_n, multiply the long formula for S_n above by r to get :

$$S_n = a + ar + ar^2 + ... + ar^{n-1} + ar^n$$

$$rS_n = ar + ar^2 + ... + ar^n + ar^{n+1};$$

now subtract, $S_n - r\,S_n = a - ar^{n+1},$

and above for S_n : $S_n = a\dfrac{1-r^{n+1}}{1-r}$

We already proved that if $|r|<1$, then $\lim\limits_{n\to+\infty} r^{n+1} = 0$. If $|r| > 1$, then $\lim\limits_{n\to+\infty} r^{n+1}$ does not exist. Therefore we conclude that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} a\frac{1-r^{n+1}}{1-r} = a\frac{1}{1-r} \text{ for } |r|<1,$$

$$\lim_{n\to+\infty} S_n \text{ does not exist} \quad \text{for } |r| \geq 1,$$

In conclusion

If $|r| < 1$, then the geometric series $\sum\limits_{n=0}^{+\infty} a\,r^n$ converges and its sum is $a\dfrac{1}{1-r}$.

If $|r| \geq 1$, then the geometric series $\sum\limits_{n=0}^{+\infty} a\,r^n$ diverges.

Example 11.3.4 : Prove that the series $\sum\limits_{n=1}\frac{1}{n(n+1)}$ converges and find its sum.

Solution : We need to examine the series of partial sums of this series :

$S_n = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+....+\frac{1}{n(n+1)},\ n = 1,2,3,.....$

It turns out that it is easy to find the S_n if we use the partial fraction decomposition for each of the terms of the series :

$\frac{1}{k(k+1)} = \frac{1}{k}-\frac{1}{k+1}$ for all $k = 1, 2, 3, ...$

Now we calculate :

$$S_n = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+.....+\frac{1}{n(n+1)}$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+.....+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}.$$

Thus $S_n = 1-\frac{1}{n+1}$ for all n = 1, 2, 3, ... Using the algebra of limits we conclude that

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1-\frac{1}{n+1}\right)=1.$$

Therefore the series $\sum\limits_{n=1}^{+\infty}\frac{1}{n(n+1)}$ converges and its sum is 1 :

$$\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}=1.$$

Example 11.3.5 : (Telescoping Series). Let (a_n) and (b_n) be two sequences such that $a_n = b_{n+1}-b_n, n \geq 1$. We note that $s_1 = a_1 = b_2 - b_1$, $s_2 = a_1 + a_2 = (b_2 - b_1) + (b_3 - b_2) = b_3 - b_1$ and

$s_n = a_1 + ... + a_n = (b_2 - b_1) + (b_3 - b_2) + ... + (b_{n+1} - b_n) = b_{n+1} - b_1.$

Thus we see that $\sum a_n$ converges iff $\lim b_n$ exists, in which case we have

$$\sum a_n = -b_1 + \lim b_n.$$

Example 11.2.6 : Consider $\sum_{n=1}^{\infty}\frac{n}{n^4+n^2+1}$. This is one of the series for which we can find the sum! We observe

$$a_n=\frac{n}{n^4+n^2+1}=\frac{n}{(n^2+1)^2-n^2}=\frac{n}{(n^2+1+n)(n^2+1-n)}$$
$$=\frac{1}{2}\left[\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right].$$

Note that the sum in the brackets is a telescoping series with $b_n=\frac{1}{2}\left(\frac{1}{n^2-n+1}\right)$.

Hence we get $s_n=\frac{1}{2}-\frac{1}{2}\left(\frac{1}{n^2+n+1}\right)\to\frac{1}{2}$.

Example 11.3.7 : Let us look at the series $\Sigma_n\frac{1}{n^2}$ of positive terms. Observe that $\frac{1}{n^2}<\frac{1}{n(n-1)}$ for $n \geq 2$. If s_n denotes the partial sum of the series $\Sigma_n\frac{1}{n^2}$ and t_n that of $\Sigma\frac{1}{n(n-1)}$, it follows that $s_n < t_n$. Since (t_n) is bounded above (Example 5.1.6) the sequence (s_n) is bounded above. Hence in view of Example 5.1.3 we see that the series $\sum n^{-2}$ is convergent.

This is a special case of the comparison test to be seen below.

Example 11.3.8 : (Harmonic Series), The harmonic series is

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....$$

The first few terms in the sequence of partial sums are :

$$S_1=1, S_2=\frac{3}{2}, S_3=\frac{11}{6}, S_4=\frac{25}{12}, S_5=\frac{137}{60}, S_6=\frac{49}{20},$$

$$S_7=\frac{363}{140}, S_8=\frac{761}{280}, S_9=\frac{7129}{2520}, S_{10}=\frac{7381}{2520}$$

This series diverges to $+\infty$. To prove this we need to estimate the nth term in the sequence of partial sums. The nth partial sum for this series is

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......+\frac{1}{n}.$$

11.4 Summary

In this chapter, we explored the fundamental concepts of series in real analysis. We began by defining an infinite series as the sum of an infinite sequence. Some examples are also demonstrated.

11.5 Keywords

- Infinite Series
- Partial Sums

11.6 References

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11.7 Model Questions

1. Show that the series

   $\sum \frac{1}{n^p}$, converges and diverges for P > 1 and P ≤ 1 respectively.

2. Show if the sequence $\{S_n\},\ S_n = \sum_{n=1}^{n} a_r$

   oscillating infinitly, then $\sum a_n$ is also oscillates infinitely.

Unit 12 ❑ Convergence Test

Structure

12.1 Objectives

- Understand the concept of an infinite series and its relation to sequences.
- Learn various tests for determining the convergence or divergence of a series.
- Differentiate between absolute and conditional convergence.
- Apply series concepts to mathematical problems and real-world applications.
- Develop rigorous reasoning skills in real analysis.

12.2 Introduction

A series is the sum of the terms of an infinite sequence. It plays a crucial role in mathematical analysis, particularly in understanding limits and approximations. Convergence tests help determine whether a series has a finite sum. This chapter introduces fundamental series concepts and essential tools for their analysis.

12.3 Convergence Tests

Theorem 12.3.1 : (Cauchy Criterian). The series $\sum a_n$ converges iff for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \Rightarrow |s_n - s_m| < \varepsilon$$

Thus, the series $\sum a_n$ converges iff for each $\varepsilon > 0$ there exists $N \in$ such that

$$n > m \geq N \Rightarrow |a_{m+1} + a_{m+2} + ... + a_n| < \varepsilon$$

This cauchy criterian is quite useful when we want to show that a series is convergent without bothering to know its sum. See Theorem 5.1.17 for a typical use.

Proof. Let $\sum a_n$ be convergent. Then the sequence (s_n) of its partial sums is convergent. We know that a real sequence is convergent iff it is Cauchy. Hence (s_n) is convergent iff it is Cauchy. The result follows from the very definition of Cauchy sequences.

Corollary. If $\sum_n a_n$ converges, then $a_n \to 0$.

Proof, We need to estimate $|a_n|$. The key observation is $a_n = s_n - s_{n-1}$ and the fact that (s_n) is convergent and hence is Cauchy. (Here (s_n) is an usual the sequence of the partial sums of the series $\sum a_n$).

Let $\varepsilon > 0$ be given. Since the sum $\sum a_n$ is convergent, the sequence (s_n) of partial sums is convergent and in particular, it is Cauchy. Hence for the given ε there exists such that for $n \geq m \geq N$ we have $|s_n - s_m| < \varepsilon$. Now if we take any $n \geq N+1$, then $a_n = s_n - s_{n-1}$. Note that $n - 1 > N$. Hence we obtain $|a_n| = |s_n - s_{n-1}| < \varepsilon$ for $n \geq N+1$. This proves that $a_n \to 0$.

The converse of the above proposition is not ture.

Remark : Most often we need the following observation on a convergent series $\sum a_n$ .If $\sum_n a_n = s$, then $\sum_{n=N+1}^{\infty} a_n = s - \sum_{k=1}^{N} a_{k.}$.

Now what is the meaning of the symbol $\sum_{n=N+1}^{\infty} a_n$? We define a new sequence (b_k) by setting $b_k := a_{N+k}$.The infinite series associated with the sequence (b_k) is denoted by $\sum_{n=N+1}^{\infty} a_n$ or simply by $\sum_{n \geq N+1}^{\infty} a_n$.

Let s_n denote the partial sums of $\sum a_k$. Let $\sigma_n := \sum_{N+1}^{N+n} a_k = \sum_{k=1}^{n} b_k$. Let $s_N := a_1 + ... + a_N$. Then we have $\sigma_n = s_{N+n} - s_N$. Clearly $\sigma_n \to s - s_N$. The claim follows from this.

An important corollary, which is used most often, is the following.

Corollary. Given $\varepsilon > 0$, there exists $N \in N$ such that the "tail" of the series $\sum_{n=N+1}^{\infty} a_n < \varepsilon$.

Proof : This is easy. Since $s_n \to s$, for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N, s_n \in (s-\varepsilon, s+\varepsilon)$. In particular, $s - \varepsilon < s_N$, that is, $s - s_N < \varepsilon$. By the last remark $\sum_{n\geq N+1} a_n = s - s_N$. Hence the corollary follows.

Exercise 12.3.1 : Given a sequence (a_n), let us assume the associated infinite series $\sum a_n$ is convergent. Let $N \in \mathbb{N}$ be fixed. Let $b_k \in \mathbb{R}, 1 \leq k \leq N$ be given. We form a new sequence (c_n) where $c_k = b_k$ for $1 \leq k \leq N$ and $b_k = a_k$ for $k > N$. Let $s = \sum a_n$ and $b := b_1 + ...+b_N$. Show that $\sum c_n$ is convergent and that $\sum c_n = s + b - s_N$.

Given two series (whether or not convergent) $\sum a_n$ and $\sum b_n$, we may define their sum as the infinite series associated with the sum $(a_n + b_n)$ of the sequences (a_n) and (b_n). Thus, $\sum a_n + \sum b_n := \sum(a_n + b_n)$. Similarly, given a scalar $\lambda \in \mathbb{R}$ we define the scaler multiple $\lambda \sum a_n$ to be the series $\sum(\lambda a_n)$.

Theorem 12.3.2 : (Algebra of Convergent Series), Let $\sum a_n$ and $\sum b_n$ be two convergent series with their respective sums A and B, respectively.

(i) Their sum $\sum(a_n + b_n)$ is convergent and we have $\sum(a_n + b_n) = A + B$.

(ii) The series $\lambda \sum a_n$ is convergent and we have $\lambda \sum a_n = \lambda.A$.

The set of all (real) convergent series is a vector space over R.

Proof, The proofs are straight forward and the reader should go on his own.

Let (s_n), (t_n), and (a_n) be the partial sums of the series $\sum a_n$, $\sum b_n$ and $\sum(a_n + b_n)$. Observe that using standard algebric facts about the commutativity and associativity of addition, we obtain.

$$\sigma_n = (a_1 + b_1) + ... + (a_n + b_n) = (a_1 + ... + a_n) + (b_1 + ... + b_n)$$
$$= s_n + t_n.$$

It follows from the algebra of convergent sequences that $\sigma_n \to A + B$.

(ii) is left to the reader.

Remark The ONLY way to deal with an infinite series is through its partial sums and by using the definition of the sum of an infinite series.

We need to be careful when dealing with infinite series. Mindless algebraic/formal manipulations may lead to absurdities.

Let $s = 1 - 1 + 1 - 1 + ... + (-1)^{n+1} + ...$

(Note that s has no meaning, if we apply our knowledge of infinite series!) Then
$-s = -1 + 1 - 1 + 1 + ... = 1 + (-1 + 1 + ...) - 1 = s - 1$.
Hence s = 1/2. On the other hand
$s = (1 - 1) + (1 - 1) + ... = 0$.
Hence we arrive at the absurdity 0 = 1/2.

Theorem 12.3.3 : The series $\sum_{n=1}^{\infty} u_n$, where $u_n \geq 0, n \geq N \in \mathbb{N}$ converges iff its sequence of partial sums $\{U_n\}$ is bounded, in which case, $U = \sup\{U_n : n \geq N\} = \sum_{n=1}^{\infty} u_n$.

Proof : If $\sum_{n=1}^{\infty} u_n$ converges, then $\{U_n\}$ converges. Since in view of Theorem 7.2 every convergent sequence is bounded, $\sum_{n=1}^{\infty} u_n$ has bounded partial sums. On the other hand, suppose $u_n | \leq M, n \in N$. Since $u_n \geq 0$ for $n \geq N$, U_n is an increasing sequence for $n \geq N$. Now in view of Theorem 8.1(1) every increasing bounded sequence converges to its supremum, it follows that $\sum_{n=1}^{\infty} u_n$ converges to U.

Theorem (Comparison Test), 12.3.4 : Suppose $0 < u_n < v_n$ for large $n \in \mathbb{N}$.

(1). If $\sum_{n=1}^{\infty} v_n < \infty$, then $\sum_{n=1}^{\infty} u_n < \infty$

(2). If $\sum_{n=1}^{\infty} u_n = \infty$, then $\sum_{n=1}^{\infty} v_n = \infty$

Proof : Let $N \in \mathbb{N}$ be so large that $0 \leq u_n \leq v_n, n > N$. Then for the partial sums $U_n = \sum_{k=1}^{n} u_k$ and $V_n = \sum_{k=1}^{n} v_k$, we have $0 \leq U_n - U_N \leq V_n - V_n, n \geq N$. Since N is fixed, U_n is bounded if V_n is bounded, and V_n is unbounded if U_n is unbounded. The result now follows from above Theorem.

Example 4.4.1 : Since $n! \geq 2^{n-1}, n \in \mathbb{N}$, the converges of the series $\sum_{n=1}^{\infty} 1/n!$ immediately follows from Theorem 9.6 and Example 9.1. Similarly the divergence of the series $\sum_{n=1}^{\infty} 1/n^{\varepsilon}, 0 \leq \varepsilon < 1$ follows by comparing it with the harmonic series.

Theorem (Limit Comparison Test) 12.3.5 : Suppose $u_n, v_n > 0$ for large $n \in N$. If $0 < \lim_{n\to\infty} u_n / v_n < \infty$, then $\sum_{n=1}^{\infty} u_n$ converges iff $\sum_{n=1}^{\infty} v_n$ converges.

Proof . Let $\ell = \lim_{n\to\infty} u_n / v_n$. Then there is a large $N \in \mathbb{N}$ such that $(\ell/2)v_n < u_n < (3\ell/2)v_n$ for $n \geq N$. The result now follows from comparison Theorem.

Example 12.3.2 : As an application to above theorem we shall show that $\sum_{n=1}^{\infty}\left[\left(n^3+1\right)^{1/3}-n\right]$ converges. For this, it suffices to consider the convergent series $\sum_{n=1}^{\infty}1/n^2$, and note that $u_n=[(n^3+1)^{1/3}-n]$ and $v_n=1/n^2$ both are positive for all $n\in\mathbb{N}$ and

$$\tfrac{u_n}{v_n}=n^2[(n^3+1)^{1/3}-n]=\frac{n^2[(n^3+1)-n^3]}{(n^3+1)^{2/3}+n(n^3+1)^{1/3}+n^3}$$
$$=\frac{1}{(1+1/n^3)^{2/3}+(1+1/n^3)^{1/3}+1}\to\frac{1}{3}.$$

Example 12.3.3 : Determine whether the series $\sum_{n=1}^{\infty}\frac{n+1}{\sqrt{1+n^6}}$ converges of diverges.

Solution : The dominant term in the numerator is n and the dominant term in the denominator is $\sqrt{n^6}=n^3$. This suggests that this series behaves as the convergent series $\sum_{n=1}^{+\infty}\frac{1}{n^2}$. Since we are trying to prove convergence we will take

$$a_n=\frac{n+1}{\sqrt{1+n^6}}\text{ and }b_n=\frac{1}{n^2}$$

In the Limit Comparison Test. Now calculate :

$$\lim_{n\to+\infty}\frac{\frac{n+1}{\sqrt{1+n^6}}}{\frac{1}{n^2}}=\lim_{n\to+\infty}\frac{n^2(n+1)}{\sqrt{1+n^6}}=\lim_{n\to+\infty}\frac{\frac{n^2(n+1)}{n^3}}{\frac{\sqrt{1+n^6}}{n^3}}=\lim_{n\to+\infty}\frac{1+\frac{1}{n}}{\sqrt{\frac{1}{n^6}+1}}=1.$$

In the last step we used the algebra of limits and the fact that

$$\lim_{n\to+\infty}\sqrt{\frac{1}{n^6}+1}=1$$

which needs a proof by definition.

Since we proved that $\lim_{n\to+\infty}\frac{\frac{n}{\sqrt{1+n^6}}}{\frac{1}{n^2}}=1$ and since we know that $\sum_{n=1}^{+\infty}\frac{1}{n^2}$ is convergent, the Limit Comparison Test implies that the series $\sum_{n=1}^{+\infty}\frac{n+1}{\sqrt{1+n^6}}$ converges.

Theorem 12.3.6 : (d' Alembert's Ratio Test), Let $\sum_n c_n$ be a series of positive reals. Assume that

$$\lim_n c_{n+1}/c_n = r.$$

Then the series $\sum_n c_n$ is (i) convergent if $0 \le r < 1$, (ii) divergent if $r > 1$.

The test is inconclusive if r = 1.

Proof : If r < 1, choose an s such that r < s < 1. Then there exists $N \in \mathbb{N}$ such that $c_{n+1} \le sc_n$ for all $n \ge N$. Hence $c_{N+k} \le s^k c_N$, for $k \in \mathbb{N}$. The convergence of $\sum c_n$ follows.

If r > 1, then $c_n \ge c_N$ for all $n \ge N$ and hence $\sum c_n$ is divergent as the n-th term does not go to 0.

Can you think of why the test is inconclusive when r = 1 ? The failure of the test when r = 1 follows from looking at the examples $\sum_n 1/n$ and $\sum_n 1/n^2$.

Theorem 12.3.7 : (Cauchy's Root Test). Let $\sum_n a_n$ be a series of positive reals. Assume that $\lim_n a_n^{1/n} = a$. Then the series $\sum_n a_n$ is convergent if $0 \le a < 1$, divergent if a > 1 then and the test is inconclusive a =1.

Proof : If a < 1, then choose a such that a < α < 1. Then $a_n < \alpha^n$ for n ≥ N. Hence by comparing with the geometric series $\sum_{n \ge N} \alpha^n$, the convergence of $\sum_n a_n$ follows. If then $a_n \ge 1$ for all large n and hence, the n-th term does not approach zero.

Can you think of why the test is inconclusive when r = 1 ?

The examples $\sum_n 1/n$ and $\sum_n 1/n^2$ illustrate the failure of the test when r = 1.

Exercise set :

(1) Show that $\sum_n \frac{2^n n!}{n^n}$ is convergent.

(2) Is $\sum_n \frac{7^{n+1}}{9^n}$ convergent ?

(3) Use your knowledge of infinite series to include that $\frac{n}{2^n} \to 0$.

(4) Show that the sequence $\left(\frac{n!}{n^n}\right)$ is convergent. Find its limit.

(5) Assume that $\sum a_n$ converges and $\sum a_n = s$. Show that $\sum_n (a_{2k} + a_{2k-1})$ converges and its sum is s.

(6) Let (a_n) be given such that $a_n \to 0$. Show that there exists a subsequence $\left(a_{n_k}\right)$ such that the associated series $\sum_k a_{n_k}$ is convergent.

(7) Show that the series $\sum_n \dfrac{1}{2^n - n}$ is convergent.

(8) Let (a_n) be given. Assume that $a_n > 0$ for all n. Let s_n denote the n-th partial sum of the series $\sum_n a_n$. Show that the series $\sum_n \dfrac{S_n}{n}$ is divergent. Can you say anything more specific ?

Exercise 12.3 Determine whether the series is convergent or divergent. If it is convergent find its sum.

(a) $\sum_{n=1}^{+\infty} 6\left(\frac{2}{3}\right)^{n-1}$ (b) $\sum_{n=1}^{+\infty} \frac{(-2)^{n+3}}{5^{n-1}}$ (c) $\sum_{n=0}^{+\infty} \frac{\left(\sqrt{2}\right)^n}{2^{n+1}}$ (d) $\sum_{n=0}^{+\infty} \frac{e^{n+3}}{\pi^{n-1}}$

(e) $\sum_{n=1}^{+\infty} \frac{2^{2n-1}}{\pi^n}$ (f) $\sum_{n=1}^{+\infty} \frac{5}{2n}$ (g) $\sum_{n=1}^{+\infty} (\sin 1)^n$ (h) $\sum_{n=0}^{+\infty} n^2 + 4n + 3$

(i) $\sum_{n=0}^{+\infty} (\cos 1)^n$ (j) $\sum_{n=2}^{+\infty} \frac{2}{n^2 - 1}$ (k) $\sum_{n=0}^{+\infty} (\tan 1)^n$ (l) $\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$

(m) $\sum_{n=1}^{+\infty} \frac{n}{n+1}$ (n) $\sum_{n=1}^{+\infty} \arctan n$ (o) $\sum_{n=0}^{+\infty} \frac{3^n + 2^n}{5^{n+1}}$ (p) $\sum_{n=2}^{+\infty} \left(\frac{3}{n^2+1} + \frac{\pi}{e^n}\right)$

(q) $\sum_{n=0}^{+\infty} \frac{e^n + \pi^n}{2^{2n-1}}$ (r) $\sum_{n=1}^{+\infty} n \sin\left(\frac{1}{n}\right)$ (s) $\sum_{n=0}^{+\infty} \frac{(n+1)^2}{n^2+1}$ (t) $\sum_{n=0}^{+\infty} (0.9)^n + (0.1)^n)$

12.3 Let $\sum_{n=1}^{\infty} u_n$ be a divergent series of positive numbers. Show that there exists a sequence $\{\varepsilon_n\}$ of positive numbers which converges to zero, but $\sum_{n=1}^{\infty} \varepsilon_n u_n$ diverges.

12.3 Let $\{u_n\}$ be a nonincreasing sequence of positive numbers and converges. Show that $\lim_{n\to\infty} nu_n = 0$. Further, give an example to show that if the sequence $\{u_n\}$ is not nonincreasing then the result is false.

12.4 Suppose $u_n.v_n > 0, n \in N,$ and $\{u_n / v_n\}$, $\{v_n/u_n\}$ are both bounded sequence. Show that the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ either both converge or both diverge.

4.5 Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive real numbers, and there exists an $N \in N$ such that $u_{n+1}/u_n \le v_{n+1}/v_n$ for all $n \ge N$ show that

(i) If $\sum_{n=1}^{\infty} v_n$ converges then $\sum_{n=1}^{\infty} u_n$ converges.

(ii) If $\sum_{n=1}^{\infty} u_n$ diverges, then $\sum_{n=1}^{\infty} v_n$ diverges.

4.6 Suppose that (u_n) is a sequence of positive real numbers, and the series u_n $\sum_{n=1}^{\infty} u_n$ diverges show that the series.

(i) $\sum_{n=1}^{\infty} u_n /(1+n^2u_n)$ converges

(ii) $\sum_{n=1}^{\infty} u_n /(1+nu_n)$ diverges

(iii) $\sum_{n=1}^{\infty} u_n /(1+u_n^2)$ diverges.

(10) Let $\sum a_n$ be absolutely convergent. Assume that $a_n + 1 = 0$ for any n. Show that the series $\sum \frac{a_n}{1+a_n}$ is absolutely convergent.

We shall now state and prove the integral test. We shall use some of the results from the theory of integration, which will be stablished in Chapter 6. (See Page 202).

If $f : [a,b] \to \mathbb{R}$ is continuous with $\alpha \le f(x) \le \beta$ for $x \in [a,b]$ then

$$\alpha(b-a) \le \int_a^b f(x)dx \le \beta(b-a).$$

We can motivate this inequality geometrically by considering a non-negative function f and using the geometric interpretation of the definite integral.

Theorem 4.4.8 : (Integral Test) Assume that $f : [1,\infty] \to [0,\infty)$ is continuous and decreasing. Let $a_n := f(n)$ and $b_n := \int_1^n f(t)dt$ Then

(i) $\sum a_n$ converges if (b_n) converges

(ii) $\sum a_n$ diverges if (b_n) diverges.

Proof . Observe that n ≥ 2, we have $a_n \le \int_{n-1}^{n} f(t)dt \le a_{n-1}$ so that

$$\sum_{k=2}^{n} a_k \le \int_1^n f(t)dt \le \sum_{k=1}^{n-1} a_k.$$

If the sequence (b_n) converges, then (b_n) is a bounded increasing sequence. $\sum_{k=2}^{n} a_k \le b_n$ Hence (s_n) is convergent.

If the integral diverges, then $b_n \to \infty$. Since $b_n \le \sum_{k=1}^{n-1} a_k$, the divergence of the series follows.

In the following examples, you will again have to use results such as the fundamental theorem of calculus to compute the integral.

Exercise Set (Typical application of the integral test).

(1) The p-series $\sum_n n^{-p}$ converges if p > 1 and diverges if p < 1.

(2) The series $\Sigma \dfrac{1}{(n+2)\log(n+2)}$ diverges.

(3) Show that the series $\Sigma \dfrac{\log n}{n^p}$ is convergent if p > 0.

12.4 Summary

In this chapter, we explored the fundamental concepts of series in real analysis. We began by defining an infinite series as the sum of an infinite sequence and introduced the necessary conditions for its convergence. Key topics included partial sums, geometric and arithmetic series, and common tests for convergence, such as the comparison test, ratio test, root test, and alternating series test. By the end of the chapter, readers should be able to determine the convergence or divergence of various series and understand their applications in mathematical analysis.

12.5 Keywords

- Infinite Series
- Convergence and Divergence

- Partial Sums
- Comparison Test
- Ratio Test
- Root Test

12.6 References

- **Principles of Mathematical Analysis** — Waiter Rudin, McGraw-Hill
- **Real Analysis: Modern Techniques and Their Applications** — Gerald B. Folland, Wiley
- **Real Analysis** — H.L. Royden, P.M. Fitzpatrick, Pearson

12.7 Model Questions

1. Invertigate the convergence of the following series.

a) $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^2+\cdots+\left(\frac{n}{2n+1}\right)^2+\cdots$

b) A series whose nth term is $\left(n^3+1\right)^{1/3}-n.$

c) $\sum\frac{n}{2^n}$

d) $1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\frac{1.3.5.7\cdots(2n-1)}{2.4.6.8\cdots(2n)}+\cdots$

e) $\sum_{n=2}^{\infty}\frac{1}{(\log n)^p}$

Unit 13 ❑ Subsequence

Structure

13.1 Objectives

To explore the concepts of subsequences and Cauchy sequences, highlighting their significance in convergence analysis and the completeness of real numbers.

13.2 Introduction

Subsequences help analyze convergence behavior, while Cauchy sequences characterize completeness. Understanding these concepts is fundamental to real analysis, ensuring rigorous treatment of limits and continuity.

13.3 Alternating Series

Let $a_1, a_2, a_3, ...$ be a sequence of positive numbers. A series of the form

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + ...$$

is said to be alternating beacause of the alternating sign pattern. (The series $-a_1 + a_2 - a_3 + ...$ is also alternating, but it is more reassuring to start summation with a positive term.)

The partial susm S_n of an alternating series are evidently not monotone.

$$S_1 > S_2,\ S_2 > S_3,\ S_3 > S_4,\ .....$$

However, the subsequences of odd-numbered and of even-numbered partial sums

$$S_1, S_3, S_5, ...,\quad S_2, S_4, S_6, ...$$

may exhibit monotonic behaviour. In fact, S_{2n+1} and S_{2n} are monotone if and only if the original sequence $a_1, a_2, a_3, ...$is monotone.

If convergent, an alternating series may not be absolutely convergent. For this case one has a special test to detect convergence.

13.3.1. Alternating Series Test (Leibniz). If $a_1, a_2, a_3, ...$ is a sequence of positive numbers monotonically decreasing to 0, then the series

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + ...$$

converges.

It is not difficult to prove Leibniz's test. Indeed, since

$$a_1 \geq a_2 \geq a_3 \geq .....$$

we have

$$a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq ...$$

$$a_1 - a_2 \leq a_1 - a_3 - a_4 \leq a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \leq ...$$

which means that S_{2n+1} is monotone decreasing and S_{2n} is monotone increasing.

Also $S_{2n+1} = S_{2n} + a_{2n+1} > S_{2n}$ for every n, implying that both sequences are bounded and hence convergent. To see that and S_{2n+1} and S_{2n} converge to the same limit, observe that $\lim_{n\to\infty}(s_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$. Proof finished.

13.3.1 Example : The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges by Leibniz's test. Indeed, the sign pattern is + – + – + ... and, as $n \to \infty$ the term $\frac{1}{n}$ monotonically decreases to 0.

To illustrate the error estimate, observe for instance that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \approx .746$$

is larger than the true sum but by no more than 0.1.

13.4 Absolute convergence

Definition : A series $\sum_{n-1}^{\infty} a_n$ is said to converge absolutely, if $\sum_{n-1}^{\infty} |a_n|$ converges.

Theorem 13.4.1 : Every absolutely convergent series converges.

Proof. Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Let s_n and σ_n be the n-th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ respetively. Then, for n > m, we have

$$|s_n - s_m| = \left|\sum_{j=m+1}^{n} a_n\right| \le \sum_{j=m+1}^{n} |a_n| = |\sigma_n - \sigma_m|.$$

Since $\{\sigma_n\}$ converges, it is a Cauchy sequence. Hence, form the above relation it follows that $\{s_n\}$ is also a Cauchy sequence. Therefore, by the Cauchy criterion, it converges.

Definition : A series $\sum_{n=1}^{\infty} a_n$ is said to converge coditionally $\sum_{n=1}^{\infty} a_n$ if converges, but not absolutely.

Example 13.4.1 : We observe the following :

(i) The series $\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}$ is conditionally convergent.

(ii) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

(iii) The series $\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n!}$ is absolutely convergent.

Example 13.4.2. : For any $\alpha \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n^2}$ is absolutely convergent : Note that

$$\left|\frac{\sin(n\alpha)}{n^2}\right| \le \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, $\sum_{n=1}^{\infty} \left|\frac{\sin(n\alpha)}{n^2}\right|$ also converges.

Theorem 13.4.2 : Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series and (b_n) is a sequence obtained by rearranging the terms of (a_n). Then $\sum_{n=1}^{\infty} b_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

13.5 Summary

We analyzed alternating series, the Alternating Series Test, and absolute convergence. Absolute convergence implies convergence, but not vice versa.

13.6 Keywords

Alternating series, Absolute convergence, Alternating Series Test, Conditional convergence, Convergence criteria.

13.7 References

- **Principles of Mathematical Analysis** — Waiter Rudin, McGraw-Hill
- **Real Analysis: Modern Techniques and Their Applications** — Gerald B. Folland, Wiley
- **Real Analysis** — H.L. Royden, P.M. Fitzpatrick, Pearson

13.8 Model Questions

1. To the following alternating sets convergent?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\sqrt{1+\frac{2}{n}}.$$

2. Show the series $\sum \frac{(-1)^n}{1+2n}$ converges to $\frac{\pi}{4}$.

3. Show that $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$ converges.

Unit 14 ❑ Subsequence

Structure

14.1 Objectives

To introduce the concept of power series, explore its properties, and determine its radius of convergence, enabling students to analyze the behavior of infinite series in real analysis.

14.2 Introduction

Power series are infinite series of the form $\sum a_n(x-c)^n$ Understanding their convergence is crucial in mathematical analysis, particularly for function approximations. The radius of convergence determines where a power series converges absolutely, playing a vital role in applications across calculus and complex analysis.

14.3 Power Series

A power series (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

where the a_n are some coefficients. If all but finitely many of the a_n are zero, then the power series is a polynomial function, but if infinitely many of the a_n are nonzero, then we need to consider the convergence of the power series.

The basic facts are these : Every power series has a radius of convergence $0 \le R \le \infty$ which depends on the coefficient a_n. The power series converges absolutely in $|x| < R$ and diverges in $|x| > R$ and the convergence is uniform on every interval $|x| < p$ where $0 \le p < R$. If $R > 0$, the sum of the power series is infinitely differentiable in $|x| < R$, and its derivatives are given by differentiating the original power series term-by-term.

Definition : Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. The power series centered at c with coefficient a_n is the series,

$$\sum_{n=0}^{\infty} a_n (x-c)^n .$$

Here are some power series centered at 0 :

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + ...$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + ...$$

$$\sum_{n=0}^{\infty} (n!) x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + ...$$

$$\sum_{n=0}^{\infty} (1)^n x^{2^n} = x \quad x^2 + x^4 + x^8 + ...$$

and here is a power series centered at 1 :

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + ...$$

The power series in Definition 6.1 is a formal expression, since we have not said anything about its convergence. By changing variables x — (x – c), we can assume without loss of generality that a power series is centered at 0, and we will do so when it's convenient.

14.4 Radius of convergence

First, we prove that every power series has a radius of convergence

Theorem 14.4.1 : Let

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

be a power series. There is an $0 \le R \le \infty$ such that the series converges absolutely for $0 \le |x - c| < R$ and diverges for $x - c > R$. Furthermore, if $0 \le p < R$, then the power series converges uniformly on the interval $|x - c| \leq p$. and the sum of the series is continuous in $|x - c| < R$.

Proof : Assume without loss of generality that c = 0 (otherwise, replace x by x – c). Suppose the power series

$$\sum_{n=0}^{\infty} a_n x_0^n$$

converges for some $x_0 \in$ with $x_0 \neq 0$. Then its terms converges to zero, so they are bounded and there exists M > 0 such that

$$|a_n x_0^n| \leq M \qquad \text{for } n = 0, 1, 2, \ldots$$

If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq Mr^n, \qquad r = \left|\frac{x}{x_0}\right| < 1.$$

Comparing the power series with the convergent geometric series $\sum Mr^n$, we see that $\sum a_n x^n$ is absolutely convergent. Thus, if the power series converges for some $x_0 \in$, then it converges absolutely for every $x \in$ with $|x| < |x_0|$.

Let

$$R = \sup\{|x| \geq 0 : \sum a_n x^n \text{ converges}\}$$

If R = 0 then the series converges only for x = 0. If R > 0, then the series converges absolutely for every $x \in$ with $|x|<R$, because it converges for some $x \in$ with $|x|<|x_0| < R$. Moreover, the definition of R implies that the series diverges for every with $|x| > R$. If $R = \infty$, then the series converges for all $x \in$.

Finally, let $0 \leq p < R$ and suppose $|x| < p$. Choose $\sigma > 0$ such that $p < \sigma < R$. Then $\sum a_n \sigma^n$ converges, so $|a_n \sigma^n| \leq M$, and therefore

$$a_n x^n = a_n \sigma^n \frac{x}{\sigma}^n \leq \left|a_n \sigma^n \frac{p^n}{\sigma}\right| \leq Mr^n,$$

where r = p/q <1. Since $\sum Mr^n < \infty$ the M-test (Theorem 5.22) implies that the series converges unformly on $|x| < p$, and then it follows from Theorem 5.16 that the sum is continuous on $|x| < p$. Since this holds for every $0 \leq p < R$, the sum is continuous in x < R.

Theorem 14.4.2 : Supose that $a_n \neq 0$ for all sufficiently large n and the limit

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

exists or diverges to infinity. Then the power series

$$\sum_{n=0}^{\infty} a_n (x \quad c)^n$$

has radius of convergence R.

Proof. Let

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = |x-c|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

By the ratio test, the power series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result.

The root test gives an expression for the radius of convergence of a general power series.

Theorem 14.4.3 : Hadamard The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is given by $$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

where R = 0 if the lim sup diverges to ∞, and R = ∞, if the lim sup is 0.

Proof. Let $r = \lim_{n\to\infty}\left|a_n (x-c)^n\right|^{\frac{1}{n}} = |x-c|\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}$

By the root test, the series converges if $0 \le r < 1$, or $|x-c| < R$, and diverges if $1 < r \le \infty$, or $|x-c| > R$, which proves the result.

This theorem provides an alternate proof of Theorem 6.2 from the root test ; in fact, our proof of Theorem 6.2 is more-or-less a proof of the root test.

Examples of Power Series

We consider a number of examples of power series and their radii of convergence.

Examples 14.4.1 : The geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

has radius of convergence

$$R = \lim_{n\to\infty}\frac{1}{1} = 1.$$

so it converges for $x < 1$, to $1/(1- x)$, and diverges for $x > 1$. At $x = 1$, the series becomes

$$1+1+1+1+.....$$

and at $x = 1$ it becomes

$$1 \quad 1 + 1 \quad 1 + 1 \; .....$$

so the series diverges at both endpoint $x = + 1$. Thus, the interval of convergence of the power series is $(-1, 1)$. The series converges uniformly on $[-p, p]$ for every $0 < p < 1$ but does not converges uniformlyon $(-1, 1)$ (see Example 5.20. Note that although the function $1/(1 - x)$ is well-defined for all $x \neq 1$, the power series only converges to it when $|x|<1$.

Example 14.4.2 : The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + ...$$

has radius of convergence

$$R = \lim_{n \to \infty} \frac{1/n}{1/(n+1)} = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right) = 1.$$

At $x = 1$, the series becomes the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

which diverges, and at $x = -1$ it is minus the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 + \frac{1}{2} \quad \frac{1}{3} + \frac{1}{4} \quad .....$$

which converges but not absolutely. Thus the interval of convergence of the power series is $[-1, 1)$. The series converges uniformly on $[-p, p]$ for every $0 \leq p < 1$ but does not converge uniformly on $(-1, 1)$.

Example 14.4.3 : The power series

$$\sum_{n=1}^{\infty} \frac{1}{n!} x^n = 1 + \frac{1}{2!} x + \frac{1}{3!} x^3 + \; .....$$

has radius of convergence

$$R = \lim_{n \to \infty} \frac{1/n!}{1/(n+1)!} = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim_{n \to \infty} (n+1) = \infty$$

so it converge for all $x \in \mathbb{R}$ Its sum provides a definition of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ (see Function 6.5.)

Example 14.4.4 : The power series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+....$$

has radius of convergence $R=\infty$, and it converges for all $x\in\mathbb{R}$. Its sum provides a definition of th cosine function cos : $\mathbb{R}\to\mathbb{R}$

Example 14.4.5 : The series

$$\sum_{n=0\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+....$$

has radius of convergence $R=\infty$, and it converges for all $x\in\mathbb{R}$. Its sum provides a defintion of the sine function sin : $\mathbb{R}\to\mathbb{R}$

Example 14.4.6 : The power series

$$\sum_{n=0}^{\infty}(n!)x^n=1+x+(2!)x+(3!)x^3+(4!)x^4+....$$

has radius of convergence

$$R=\lim_{n\to\infty}\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{1}{n+1}=0.$$

so it converges only for x = 0, if $x\neq 0$, its terms grow larger once n > 1/x and $(n!)x^n|\to\infty$ as $n\to\infty$.

Example 14.4.7 : The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3....$$

has radius of convergence

$$R=\lim_{n\to\infty}\frac{(-1)^{n+1}/n}{(-1)^{n+2}/(n+1)}=\lim_{n\to\infty}\frac{n}{n+1}=\lim_{n\to\infty}\frac{1}{1+1/n}=1,$$

so it converges if (x–1) < 1 and diverges if (x–1) > 1. At the endpoint x = 2, the power series becomes the alternating harmonic series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.....$$

which coverages. At the endpoint x = 0, the power series becomes the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}\ \frac{1}{4}+.....$$

which diverges. Thus the interval of convergence is (0, 2)

Example 14.4.8. : The power series

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = x - x^2 + x^4 - x^8 + x^{16} - x^{32} + ...$$

with

$$a_n = \begin{cases} 1 \text{ if } n = 2^k \\ 0 \text{ if } n \neq 2^k \end{cases}$$

has radius of convergence R = 1. To prove this, note that the series converges for |x| < 1 by comparison with the convergent geometric series $\sum |x|^n$, since

$$\left|a_n x^n\right| = \begin{cases} |x|^n & \text{if } n = 2^k \\ 0 \leq |x|^n & \text{if } n \neq 2^k \end{cases}$$

If |x| > 1, the terms do not approach 0 as , so the series diverges. Alternatively, we have

$$\left|a_n\right|^{1/n} = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k \end{cases}$$

so,

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1$$

and the root test gives R = 1. The series does not converge at either endpoint $x = \pm 1$, so its interval of convergence is (–1, 1).

14.5 Summary

In this unit, we have introduced the concept of infinite series, the convergence of series, alternating series, absolutely convergent series, power series, and its radius of convergence. Many essential results, along with their application, have been discussed in this unit. Some problems have been given at the end of this unit.

- A formal sum of a sequence is called a series
- If the sequence of partial sum of the sequence is convergent, then the series is convergent; otherwise, the series is divergent.
- The series is said to be absolutely convergent if the series is convergent.
- If converges then.

- Sum of two convergent serieses is convergent.
- Every absolutely convergent series converges.
- A series of the form is called a power series with center at and coefficient.
- The radius of convergence of a power series is the radius of the largest disk in which the series converges.
- The radius of convergence of a power series is either a non-negative real number or infinite.

14.6 Keywords

Series, convergent series, divergent series, geometric series, d' Alembert's ratio test, Cauchy's root test, alternating series, absolutely convergent series, power series, the radius of convergence.

14.7 References

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14.8 Model Questions

A. (1) Let b_n be a convergent series of non-negative terms. Let (a_n) be sequence such that $|a_n| \le Mb_n$ for $n \ge N$, for a fixed N and M > 0. Show that $\sum a_n$ is convergent.

(2) If (a_n) and (b_n) are sequences of positive terms such that $a_n / b_n \to \ell > 0$. Prove that $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(3) As an application of the last item, discuss the convergence of

(a) $\sum 1/2n$, (b) $\sum 1/(2n-1)$ and (c) $\sum 2/(n^2+3)$.

(4) Assume that $\sum a_n$ is absolutely convergent and (b_n) is bounded. Show that $\sum a_n b_n$ is convergent.

(5) Show that the sum of two absolutely convergent series and a scalar multiple of an absolutely convergent series are again absolutely convergent. Hence conclude that the set ℓ^1 of all absolutely convergent series is a real vector space.

(6) Let $\sum a_n$ be a convergent series of positive terms. Show that $\sum a_n^2$ is convergent. More generally, show that $\sum a_n^p$ is convergent for p > 1.

(7) Let p > 0. Show that the series $\sum_n \frac{n^p}{e^n}$ is convergent. Can we take p = 0?

(8) Find the values of $x \in [0, 2\pi]$ such that the series $\sum \sin^n(x)$ is convergent.

(9) Let $\sum a_n$ and $\sum b_n$ be convergent series of positive terms. Show that $\sum \sqrt{a_n b_n}$ is convergent.

(10) Give an example of a convergent series $\sum a_n$ such that the series $\sum a_n^2$ is divergent.

(11) Give an example of a divergent series $\sum a_n$ such that the series $\sum a_n^2$ is convergent.

(12) Let (a_n) be a real sequence. Show that $\sum (a_n - a_{n+1})$ is convergent iff (a_n) is convergent. If the series converges, what is its sum?

(13) When does a series of the form a + (a+b) + (a+2b) + ... convergent?

(14) Assume that $\left|\frac{a_n+1}{a_n}\right| \le \frac{n^2}{(n+1)^2}$ for $n \in \mathbb{N}$. Show that the series $\sum a_n$ is absolutely convergent.

(15) Prove that if $\sum |a_n|$ is convergent, then $\left|\sum a_n\right| \le \sum |a_n|$.

(16) Prove that is | x | < 1,

$$1 + x^2 + x + x^4 + x^6 + x^3 + x^8 + x^{10} + x^5 + ... = \frac{1}{1-x}.$$

(17) Prove that if a convergent series in which only a finite number of terms are negative is absolutely convergent.

(18) If $(n^2 a_n)$ is convergent, then $\sum a_n$ is absolutely convergent.

(19) Assume that (a_n) is a sequence such that $\sum_n a_n^2$ is convergent. Show that $\sum a_n^3$ is absolutely convergent.

B. Solved Questions :

1. In each of the following cases determine whether or not the series converges.

(a) $$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}.$$

Ans. We could show convergence here by using the ratio or root test or more simply by using the comparison test by noting that

$$0 \le \frac{1}{2^n + 1} \le \frac{1}{2^n}.$$

The upper bound is a term from a convergent geometric series.

(b) $$\sum^{\infty} \frac{4n^2 - n + 3}{n^3 + 2n}.$$

Ans. This is divergent.

$$a_n = \frac{4n^2 - n + 3}{n^3 + 2n} = \frac{1}{n} c_n,\ c_n = \frac{4 - 1/n + 3/n^2}{1 + 2/n^2} \to 4 \text{ as } n \to \infty.$$

$c_n \to 4$ implies that there exists N such that $c_n > 3$ for $n \ge N$. Hence for $n \ge N$ we have $a_n \ge 3/n$ and since $\sum 1/n$ diverges we have by comparison that $\sum a_n$ diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{2n^3-1}$.

Ans. This converges.

$$a_n = \frac{n+\sqrt{n}}{2n^3-1} = \frac{1}{n^2}c_n,\ c_n = \frac{1+1/\sqrt{n}}{2-1/n^3} \to \frac{1}{2} \text{ as } n \to \infty.$$

$c_n \to 1/2$ implies that there exists N such that $c_n < 1$ for $n \geq N$. Hence for $n \geq N$ we have $a_n \leq 1/n^2$ and since $\sum 1/n^2$ converges we have by comparison that $\sum a_n$ diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} n^4 e^{-n^2}$.

Ans. By the root test

$$a_n = n^4 e^{-n^2},\ a_n^{1/n} = \left(n^{1/n}\right)^4 \left(e^{-n^2}\right)^{1/n} = \left(n^{1/n}\right)^4 e^{-n} \to 0 \text{ as } n \to \infty.$$

Here the results is as a consequence of $n^{1/n} \to 1$ and $e^{-n} \to 0$. By the root test the series converges.

2. For each of the following series determine the values of $x \in \mathbb{R}$ such that the given series converges.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Ans. Let $a_k = x^k / k!$ and use the ratio test. We have

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}/(k+1)!}{x^k/k!} = \frac{x}{k+1} \to 0 \text{ as } k \to \infty.$$

By the ratio test the series converges (absolutely) for all $x \in \mathbb{R}$.

(b) In the following $\alpha \in \mathbb{R}$ is not an integer.

$$\sum_{k=0}^{\infty} \left(\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \right) x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \ldots$$

Ans. Let $a_k = \alpha(\alpha-1)...(\alpha-k+1)x^k/k!$. Using the ratio test

$$\frac{a_{k+1}}{a_k} = \frac{\alpha-k}{k+1}x = \frac{\alpha/k-1}{1+1/k}x \to x \text{ as } k \to \infty.$$

Thus the series $\sum a_k$ converges absolutely if $|x| < 1$ which in turn implies that the series converges for $|x| < 1$.

If $|x| > 1$ then the terms of the series are unvounded and thus the series diverges. What happens when $x = -1$ or $x = 1$ needs more refined tests to determine if the series converges or diverges and the outcome depends on α. This will not be considered further here.

(c) $$\sum_{k=0}^{\infty}\frac{k^3x^k}{3^k}.$$

Ans. The root test is the easiest test to use here. With $a_k = k^3x^k/3^k$ we have

$$|a_k|^{1/k} = \frac{\left(k^{1/k}\right)^3|x|}{3} \to \frac{|x|}{3} \text{ as } k \to \infty.$$

By the root test the series converges (absolutely) if $|x| < 3$, it diverges if $|x| > 3$. If $|x| = 3$ then $|a_k| = k^3$ and since these terms become unbounded it follows that the series diverges when $|x| = 3$.

(d) $$\sum_{k=0}^{\infty}k^kx^k.$$

Ans. The root test is the easiest test to use here. With $a_k = k^kx^k$ we have

$$\left|a_k^{1/k}\right| = |kx|.$$

This only converges if $x = 0$ and is unbounded for $x \neq 0$. Hence the series only converges when $x = 0$.

(e) $$\sum_{k=0}^{\infty}a_kx^k = 1+2x+x^2+2x^3+x^4+...,$$

i.e. with $a_{2k}= 1$ and $a_{2k+1}= 2$ for $k = 0, 1, 2, ...$.

Ans. Let $b_k = a_k x^k$. The ratio test does not give any information here as a_{k+1}/a_k does not have a limit as $k \to \infty$. However we can still use the root test. Since

$$1 \le a_k \le 2, \quad 1 \le a_k^{1/k} \le 2^{1/k} \to 1 \text{ as } k \to \infty.$$

Thus $$|b_k|^{1/k} = a_k^{1/k}|x| \to |x| \text{ as } k \to \infty.$$

The series converges (absolutely) if $|x| < 1$ and diverges if $|x| > 1$. By inspection the series diverges if $x = 1$ as the terms of the series do not tend to 0 as $k \to \infty$. It can be shown that the series also diverges when $x = -1$.

Unit- 15 □ Limits of Functions

Structure

15.1 Objectives

This unit gives

- Various types of functions and their classification
- Sequence of real number and its convergence
- Concept of limit of a real function
- Various properties of limit of a function such as algebric operation on limits, sandwich property, etc.

15.2 Introduction

The limit of a function is a fundamental concept in analysis concerning the behaviour of that function near a particular point. Although implicit in the development of calculus of the 17th & 18th centuries, the modern idea of the limit of a function goes back to Bolzano who, in 1817, introduced the basic of the epsilon-delta technique to define limit of functions. The motion of a limit has many applications in modern Calculus. In particular, the many definitions of continuity employ the limit. It also appears in the definition of the derivative.

15.3 Pre requisites

(or Recapitulation of prior elementary ideas that are needed to introduce the concept of limit):

A. Functions

(i) Let A and B be two non-void subsets $\mathbb{R}$ & $f : A \to B$ is a rule of correspondence that assigns to each $x \in A$, a uniquely determined $y \in B$ or $y = f(x)$.

The set of values of x for which f can be defined is known as **Domain** of f, denoted by D_f and the corresponding collection of y's (as mentioned above) is known as Range set of f generally denoted by R_f.

A few examples of f, D_f and R_f :

(i) $$f(x) = \sqrt{\left[\log_c \frac{5x - x^2}{4}\right]}$$

f can be defined for those x for which $\frac{5x - x^2}{4} \geq 1$ and this gives $1 \leq x \leq 4$

so $D_f \equiv [1, 4]$

(ii) $$f(x) = \sqrt{\left(x - \frac{x}{1-x}\right)}$$

f can be defined only when $x - \frac{x}{1-x} \geq 0 \;\Rightarrow\; 1 < x < \infty$ & $D_f = (1, \infty)$

(iii) $f(x) = \cos^{-1} \frac{3}{4 + 2\sin x}$. Here we must have $-1 \leq \frac{3}{4 + 2\sin x} \leq 1$

& for this $D_f \equiv \left[-\frac{\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi\right]$ where $k = 0, \pm 1, \pm 2$,......

Note that D_f may be a closed and bounded interval, may be an open interval (bounded or unbounded), union of intervals and so on.

(Readers are requested to verify the validity of D_f as mentioned in above examples and as well as to look for other functions and their domain).

(i) Consider the function $f:[-1,\ 1]\to\mathbb{R}$ defined by $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Here D_f is an interval $[-1,\ 1]$ but $R_f=\{-1,0,1\}$ which is not an interval.

(ii) Consider the function $f:(-1,\ 1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x^2+1}\ \forall\, x\in(-1,1)$

$$R_f=\left(\frac{1}{2},\ 1\right] \text{ or } \frac{1}{2}<x\leq 1.$$

Note that in D_f, -1 and $+1$ are not included but 1 is included as right hand end point in R_f.

We are interested to learn the reason for such differences of nature of D_f & R_f.

Equal functions : $f,\ g : D\to\mathbb{R}$ are same (or equal) when $f(x)=g(x)$ for each $x\in D$.

Note that x and $\dfrac{x^2}{x}$ are not same.

Operations on Functions : Let f and g be two functions having domain $D_f(\subset\mathbb{R})$ and $D_g(\subset\mathbb{R})$ respectively. If $D_f\cap D_g\neq\phi$, then $f\pm g, fg$ can be defined on $D_f\cap D_g$ by

(i) $(f\pm g)x=f(x)\pm g(x)\ \ \forall x\in D_f\cap D_g$ and

$(fg)(x) = f(x)\ g(x)\ \ \forall x \in D_f \cap D_g$

Again deleting those points of D_g (if any) for which $g(x) = 0$, we can define

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $x \in D_f \cap D_g \setminus \{x : g(x) = 0\}$.

Composition of functions : Let f and g be two functions such that

$x \in D_f \Rightarrow f(x) \in D_g$. In other words $R_f \subset D_g$. Then we can define

$(g \circ f)(x) = g[f(x)] \forall x \in D_f$.

$g \circ f$ is called the composite of two functions *f* and *g*.

Similarly, we can define $(f \circ g)(x)$ with appropriate restrictions.

In general $(f \circ g)(x) \neq (g \circ f)(x)$. For example, $f(x) = x^2, g(x) = \sin x$

Then $(g \circ f)(x) = g(f(x)) = \sin x^2$ & $(f \circ g)(x) = f(g(x)) = f(\sin x) = \sin^2 x$.

Injective (one-one), Surjective (onto) and Bijective functions :

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

If for $x, y \in D$, $f(x) = f(y) \Rightarrow x = y$, *f* is called injective or one-one function

$f(x) = 3x + 4,\ x \in \mathbb{R}$ is Injective but $g(x) = |x|,\ x \in \mathbb{R}$ is not Injective.

Let $f : D \to E$ where $D, E \subset \mathbb{R}$, obviously $f(D) \subseteq E$. But if $f(D) = E$, we say that *f* is surjective or onto function. $f : [1, 2] \to [2, 3]$ defined by $f(x) = x + 1$ is onto function.

But $f : [1, 2] \to [2,\ 4]$, $f(x) = x + 1$ is not so,

$$\frac{7}{2} \in [2,\ 4] \text{ and } \frac{7}{2} = x + 1 \Rightarrow x = \frac{5}{2} \notin [1,\ 2].$$

f is bijective if it is both injective and surjective.

Invertible functions : Let $f : X \to Y$ where $X, Y \subset \mathbb{R}$ be such that for each $y \in Y$, there exists a single value of x such that $f(x) = y$. Then this correspondence defines a function $x = g(y)$. We say that f is invertible and $x = g(y)$ is the inverse function. Note that if f be bijective, then f is invertible.

For example, if $y = \log_a\left(x + \sqrt{x^2+1}\right)$, $a > 0, a \neq 1$, then

$$x = \frac{1}{2}\left(a^y - a^{-y}\right) \text{ or } \sinh(y \ln a)$$

Increasing function & Decreasing function :

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$. If for each pair $x, y \in D$,

$x > y \Rightarrow f(x) \geq f(y)$ or $f(x) > f(y)$, we say that f is increasing function.

But if $x > y \Rightarrow f(x) \leq f(y)$ or $f(x) < f(y)$, we say that f is decreasing function.

$f(x) = \sin x$ is increasing in $\left[0, \frac{\pi}{2}\right]$ but is decreasing in $\left[\frac{\pi}{2}, \pi\right]$

Periodic function :

A function $f : D \to \mathbb{R}(D \subset \mathbb{R})$ is periodic if there exists a number p such that $f(x+p) = f(x) \forall x \in D$.

The smallest positive p for which $f(x+p) = f(x) \forall x$ holds, is called the period of f.

Bounded and unbounded functions :

$f : D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be bounded above if there exists $\lambda \in \mathbb{R}$ such that $f(x) \leq \lambda \forall x \in D$, we say that f is bounded above (by λ). If there exists $\mu \in \mathbb{R}$ such that $f(x) \geq \mu \forall x \in D$, we say that f is bounded below (by μ). If f be both bounded above & bounded below, then *f* is bounded on $D_f(\equiv D)$. In other words, If there exists $K \in \mathbb{R}$ such that $|f(x)| \le K$ for all $x \in D$, we say that *f* is bounded on *D*. For future course of discussion the following concepts are useful.

Let $f : D \to \mathbb{R}(D \subset \mathbb{R})$ be bounded above.

Then $\lambda(\in \mathbb{R})$ is said to be the least upper bound or supremum of *f* in *D* if $\exists\ \lambda \in \mathbb{R}$ such that (i) $f(x) \le \lambda \forall x \in D$ and (ii) for any $\varepsilon > 0, \exists\ y \in D$ such that $f(y) > \lambda - \varepsilon$ (or in other words, no real $< \lambda$ is an upper bound of *f*) this $\lambda =$ sup f. If *f* be bounded above, then sup $f(\in \mathbb{R})$ exists.

If *f* is unbounded above we say that sup $f = \infty$

Let $f : D \to \mathbb{R}$ be bounded below. Then $\mu(\in \mathbb{R})$ is greatest lower bound or infimum of *f* in *D* if

(i) $f(x) \ge \mu$ for all $x \in D$ & (ii) if for any $\varepsilon > 0, \exists\ y \in D$ such that $f(y) < \mu + \varepsilon$, then $\mu =$ inf *f* (in other words, no real $> \mu$ is lower bound of *f*). Then $\mu =$ inf *f*. If *f* be bounded below, then inf $f(\in \mathbb{R})$ exists.

If *f* be unbounded below, we write inf $f = -\infty$

Sup *f* – inf *f* is known as oscillation of function *f* on *D*.

15.4 Sequences in $\mathbb{R}$

(i) *A* function $f : \mathbb{N} \to \mathbb{R}$ is known as a sequence (note that $\mathbb{N}$ is the set of natural numbers).

Examples : $\left\{(-1)^n\right\}_n, \left\{\frac{1}{n}\right\}_n, \left\{\frac{4n+3}{3n+4}\right\}, \left\{n^2\right\}_n$ etc.

Symbolically, $\{a_n\}_n\ (n \to a_n)$. Note that the range set of $\left\{(-1)^n\right\}_n$ is the set $\{-1, 1\}$ where as the range sets of the next three are infinite sets.

A sequence $\{a_n\}_n$ is bounded if its range set is bounded.

Range sets of $\{(-1)^n\}_n, \left\{\frac{1}{n}\right\}_n, \left\{\frac{4n+3}{3n+4}\right\}_n$ are bounded but range set of $\{n^2\}_n$ is not bounded.

(ii) Note that $\mathbb{N}$ is unbounded above, as there is no real $\lambda \in \mathbb{R}$ for which $n \le \lambda \quad \forall n \in \mathbb{N}$.

So an interesting question is that when n becomes arbitrary large without any bound, then what will be the fate of $\{a_n\}_n$?

Consider the above examples : As n becomes larger and larger, $\frac{1}{n}$ becomes smaller & smaller we say that, the difference between $\frac{1}{n}$ and 0 decreases steadily. Neither $\frac{1}{n}$ coincides with zero nor it goes to the left side of 0. We say $\frac{1}{n} \to 0$ (tends to zero) as $n \to \infty$. But note that as n becomes arbitrarily large. n^2 increases more rapidly & we say that $n^2 \to \infty$ as $n \to \infty$. In case of $\{(-1)^n\}_n$, it is either +1 or −1.

Limit of a sequence in $\mathbb{R}$: A sequence $\{a_n\}_n$ is said to converge to a limit $l(\in \mathbb{R})$ if for arbitrary $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $|a_n - l| < \varepsilon$ for all $n \ge m$.

$\lim\limits_{n\to\infty} a_n = \infty$ if for all $G > 0$ there exists $m \in \mathbb{N}$ such that $a_n > G \quad \forall n \ge m$. We say that $\{a_n\}_n$ diverges to ∞.

To explain this definition, we take $a_n = \frac{1}{n}$ as mentioned earlier. We have seen that $\frac{1}{n} \to 0$ as $n \to \infty$.

let $\varepsilon = \frac{7}{1000}$. Then $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}$, if $n > \frac{1000}{7}\left(= 142\frac{6}{7}\right)$

so $m = 143$ & for this $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}, n \geq 143$

Let us change $\varepsilon = \frac{8}{3439}$. Then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n > \frac{3439}{8}\left(= 429\frac{7}{8}\right)$

So $m = 430$ & then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n \geq 430$

These two simple examples exhibit the dependence of m on the arbitrary positive value of ε.

We state the following results without proof at this stage :

(a) A Convergent sequence in $\mathbb{R}$ is necessarily bounded but a bounded sequence may not be convergent $\left(Ex. \left\{(-1)^n\right\}_n\right)$.

(b) Limit of a sequence, if exists, is unique.

(c) Cauchy's general principle of convergence : A necessary & sufficient condition for the convergence of $\{a_n\}_n$ is that given $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $\left|a_{n+p} - a_n\right| < \varepsilon \ \forall\, n \geq m, \quad p \in \mathbb{N}$.

(d) Sandwich rule : Let $a_n < b_n < c_n$ for all $n \geq m$ (or for all n) and $\{a_n\}_n, \{c_n\}_n$ both converge to same limit $l(\in \mathbb{R})$. Then $\lim\limits_{n\to\infty} b_n$ exists & = l.

(iii) Monotonic sequences in $\mathbb{R}$

A sequence $\{a_n\}_n$ in $\mathbb{R}$ is said to be monotonic increasing if $a_{n+1} \geq a_n$ for all n, but if $a_{n+1} \leq a_n$ for all n, $\{a_n\}_n$ is said to be monotonic decreasing sequence in $\mathbb{R}$.

We state the following results without proof :

(a) A monotonic increasing sequence $\{a_n\}_n$ in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded above and $a_n \to \sup a_n$. If $\{a_n\}_n$ be unbounded above, then $\lim\limits_{n\to\infty} a_n = \infty$ (diverges to ∞)

(b) A monotonic decreasing sequence in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded below and and $a_n \to \inf\limits_n a_n$. If $\{a_n\}_n$ be unbounded below, then $\lim\limits_{n\to\infty} a_n = -\infty$ (diverges to $-\infty$)

(iv) The following results are easily deducible following definition and basic results :

If $\lim\limits_{n\to\infty} a_n = l(\in \mathbb{R})$, $\lim\limits_{n\to\infty} b_n = m(\in \mathbb{R})$, then

$\lim\limits_{n\to\infty}(a_n \pm b_n), = l \pm m$, $\lim\limits_{n\to\infty}(a_n b_n) = lm$,

$\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{l}{m}$ provided $b_n \neq 0\ \forall n$ and $m \neq 0$.

(c) Accumulation point (or limit point) of a set

Let $S(\subset \mathbb{R})$ be a set and $\xi \in \mathbb{R}$. ξ is said to be an accumulation point (or limit point) of S if there exists a sequence of distinct elements $\{x_n\}_n$ of S such that $x_n \to \xi$ as $n \to \infty$. '0' is limit point of $S = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$. 1 is limit point of $T\left\{1 + \dfrac{1}{n} : n \in \mathbb{N}\right\}$ etc. Note that $0 \notin S, 1 \notin T$.

Note that a finite set has no accumulation point. The set $U = \{n^2; n \in \mathbb{N}\}$ has no accumulation point in $\mathbb{R}$.

(D) Neighbourhood of a point & Interior point of a set :

(i) Let $x \in \mathbb{R}$. By a δ-neighbourhood of x, we mean the interval $(x-\delta, x+\delta)$ where $\delta > 0$. This is denoted by $N(x,\delta)$ or $N_\delta(x)$.

The set $N(x,\delta)-\{x\}$ is called the deleted δ-neighbourhood $(\text{or } \delta-nbd)$ of x, denoted by $N'(x,\delta)$ or $N_\delta'(x)$. $\mathrm{U}(\subset \mathbb{R})$ is nbd of $x \in \mathbb{R}$ if $\exists$ an open interval I such that $x \in I \subset U$ for example, $\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ is a neighbourhood of 1. The set $\mathbb{R}$ (of all real numbers) is a neighbourhood of each of its points. The situation is different in case of Q, the set of rational numbers for if $\xi \in Q$, then every $(\xi-\delta,\ \xi+\delta)$ contains rational as well as irrational points also. So Q is not a neighbourhood of its points.

(ii) Let $D \subset \mathbb{R}$. We say that $x \in D$ is interior point of D if there exists a neighbourhood of x, say $(x-\delta,\ x+\delta)$, which is contained in D.

For example consider $[a,b]=\{x : a \le x \le b\}$

Let $a<c<b$. we take $0<\delta<\min\{c-a, b-c\}$ & so $(c-\delta, c+\delta) \subset (a,b)$, so c is interior point of the set but a, b are not interior points of it.

Accumulation point can also be defined as follows :

Let $S \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. If every deleted neighbourhood of ξ, $N'(\xi,\delta) \cap S \neq \phi$, then ξ is accumulation point of S.

This can be shown that $N'(\xi,\delta) \cap S$ is an infinite set. On the basis of this approach, it obviously follows that a finite set $(\subset \mathbb{R})$ has no accumulation point.

On the basis of these pre-requisites, we are now in a position to introduce the concept of limit of a function.

15.5 Limit of function

Let $f : D(\subset \mathbb{R}) \to \mathbb{R}$ and p be an accumulation point of D.

(A) Sequential approach : $\lim\limits_{x \to p} f(x) = l(\in \mathbb{R})$ if for every sequence $\{x_n\}_n$, $x_n \in D$ for all n, $x_i \neq x_j$ if $i \neq j$, $x_n \neq p$, converging to p, the sequences $\{f(x_n)\}_n$ converge to same limit $l(\in \mathbb{R})$.

If on the otherhand, $\{f(x_n)\}_n$ converge to different limits for different $\{x_n\}_n$'s we say that the limit does not exist.

To explain the matter, let us consider the following examples :

Example :

(i) $\lim\limits_{x \to 0} \sin\frac{1}{x}$: Note that the sequences $\left\{\frac{2}{2n\pi}\right\}_n$ and $\left\{\frac{2}{(2n+1)\pi}\right\}_n$ both converge to zero. But $\{\sin n\pi\}_n$ converges to zero whereas $\left\{\sin\left(n\pi + \frac{\pi}{2}\right)\right\}_n$ is not convergent, (n even and n odd give different limits). So by above definition, $\lim\limits_{x \to 0} \sin\frac{1}{x}$ does not exist.

(2) $\lim\limits_{x \to 0} \frac{1}{x} \sin\frac{1}{x}$

For $x_n = {}^{1}\!/_{n\pi} (\to 0)$, $\frac{1}{x_n} \sin\frac{1}{x_n} \to 0$ but for $y_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi} \to 0$

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} \left(2n + \frac{1}{2}\right)\pi \ \sin\left(2n + \frac{1}{2}\right)\pi = \infty$$

So $\lim\limits_{x \to 0} \frac{1}{x} \sin \frac{1}{x}$ does not exist.

(B) ($\varepsilon - \delta$ approach) let $\varepsilon > 0$ be any number. If corresponding to such ε, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $x \in \mathbb{N}'(p, \delta) \cap D$. we say that $\lim\limits_{x \to p} f(x)$ exists and $= l (\in \mathbb{R})$.

Here $x \in N'(p, \delta) \cap D$ can be written as $0 < |x - p| < \delta$ or $p - \delta < x < p$, $p < x < p + \delta$, $x \in D$.

(C) The two definitions stated in (A) and (B) are equivalent :

Proof : Let $\lim\limits_{x \to p} f(x) = l (\in \mathbb{R})$ in the sense of $\varepsilon - \delta$ definition.

Then for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ wherever } 0 < |x - p| < \delta \quad \text{(i)}$$

As p is accumulation point of D, so there exists a sequence

$\{x_n\}_n \left(x_n \in D \ \forall n,\ x_i \neq x_j \text{ if } i \neq j,\ x_n \neq p \text{ for all } n\right)$ which converges to p. Hence corresponding to above $\delta > 0$, there exists natural number m such that

$0 < |x_n - p| < \delta$ for all $n \geq m$ (2)

Combining (1) & (2), $|f(x_n) - l| < \varepsilon$ for all $n \geq m$

Note that m depends on ε (as m depends on δ & δ depends on ε).

So $\lim\limits_{n \to \infty} f(x_n) = l (\in \mathbb{R})$ and $\{f(x_n)\}_n$ converges to $l (\in \mathbb{R})$.

Next let $\lim\limits_{x \to p} f(x) = l (\in \mathbb{R})$ following sequential criterion.

If possible let $\lim_{x \to p} f(x) = l$ does not hold in the sense of $\varepsilon - \delta$ definition.

Then for some number $\varepsilon > 0$, the corresponding δ does not exist. That indicates, however small $\delta > 0$ may be, there exists always at least $x'(\neq p)$ for which $0 < |x' - p| < \delta$ nonetheless $|f(x') - l| \geq \varepsilon$.

Let us consider a decreasing positive termed sequence $\{\delta_n\}_n$ converging to zero (in particular, $\delta_n = \frac{1}{n}$ for all $n \in \mathbb{N}$). Then for every δ_n, x'_n can be found such that $0 < |x'_n - p| < \delta_n$ nonetheless $|f(x'_n) - l| \geq \varepsilon$. $\delta_n \to 0 \Rightarrow x'_n \to p$ by Sandwich rule. By assumption, $\{f(x'_n)\}_n$ converges to l. But $|f(x'_n) - l| \geq \varepsilon$.

Thus we arrive at a contradiction. So $\varepsilon - \delta$ definition follows from that of sequential approach. Thus the two definitions are equivalent.

(D) One sided limits

(i) Let p be an accumulation point of D from the left (i.e. $x_n \to p, x_n < p \; \forall n, x_n \in D$ etc) or f has been defined in some left-deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$, we say that $\lim_{x \to p-} f(x) \left(\text{or } \lim_{x \to p-0} f(x) \right)$ exists and $= l_1 (\in \mathbb{R})$. This is commonly known as left hand limit of $f(x)$ as $x \to p$.

(ii) Let p be an accumulation point of D from the right (i.e $x_n \to p, \; x_n > p \; \forall n$, $x_n \in D$ etc.) or f has been defined in some right deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l_2| < \varepsilon$ whenever $p < x < p + \delta$, we say that $\lim_{x \to p+} f(x) \left(\text{or } \lim_{x \to p+0} f(x) \right)$ exists and $= l_2 (\in \mathbb{R})$. This is commonly known as right hand limit of $f(x)$ as $x \to p$.

(E) In this connection, the following result is useful in determining the existence of limit. Let $f: D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and let p be (both sided) accumulation point of D (or f has been defined in both sided deleted neighbourhood of p).

Then $\lim\limits_{x \to p} f(x) = l (\in \mathbb{R})$ if and only if $\lim\limits_{x \to p-0} f(x) = \lim\limits_{x \to p+0} f(x) = l$

Proof : let $\lim\limits_{x \to p} f(x) = l(\in \mathbb{R})$

Following $\varepsilon - \delta$ definition, corresponding to arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - p| < \delta, x \in D$

$\Rightarrow |f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$ as well as $p < x < p + \delta$.

$\Rightarrow \lim\limits_{x \to p-} f(x) = l = \lim\limits_{x \to p+} f(x)$

Converse let $\lim\limits_{x \to p-} f(x) = l = \lim\limits_{x \to p+} f(x)$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta_1 < x < p$ & $|f(x) - l| < \varepsilon$ whenever $p < x < p + \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - p| < \delta$, $| f(x) - l | < \varepsilon \Rightarrow \lim\limits_{x \to p} f(x) = l$

Examples (i) $f(x) = \begin{cases} 3x + 7, & x < 1 \\ 2x + 11, & x > 1 \end{cases}$

Here $\lim\limits_{x \to 1-} f(x) = 10$, $\lim\limits_{x \to 1+} f(x) = 13$ & so $\lim\limits_{x \to 1} f(x)$ does not exist.

(2) $f(x) = \begin{cases} 7x + 3, & x < 2 \\ 8x + 1, & x > 2 \end{cases}$

Here $\lim\limits_{x\to 2-} f(x)=17,\ \lim\limits_{x\to 2+} f(x)=17$

Let $\varepsilon>0$ be any number. Corresponding to ε, there exists $\delta_1>0, \delta_2>0$ such that $|7x+3-17|<\varepsilon$ i.e. $|x-2|<\frac{\varepsilon}{7}$ whenever $2-\delta_1<x<2$ & so $\delta_1=\frac{\varepsilon}{7}$ is admissible & $|8x+1-17|<\varepsilon$ i.e. $|x-2|<\frac{\varepsilon}{8}$ whenever $2<x<2+\delta_2$ & so $\delta_2=\frac{\varepsilon}{8}$ is admissible.

Taking $\delta=\min\{\delta_1,\delta_2\}$, we get $\lim\limits_{x\to 2} f(x)=17$

(F) Cauchy Criterion for the existence of limit

Let $f: D\to\mathbb{R}$ where $D\subset\mathbb{R}$ and p be an accumulation point of D.

A necessary and sufficient condition for the existence of $\lim\limits_{x\to p} f(x)$ is that given $\varepsilon>0$, there exists a deleted neighbourhood of $p, \mathbb{N}'(p,\delta)$ such that

$|f(x)-f(y)|<\varepsilon$ whenever $x,y\in\mathbb{N}'(p,\delta)\cap D$

Proof : Let $\lim\limits_{x\to p} f(x)=l\,(\in\mathbb{R})$

Let $\varepsilon>0$ be any number. Corresponding to ε, there exists a deleted neighbouhood $\mathbb{N}'(p,\delta)$ such that $|f(x)-l|<\varepsilon/2$ whenever $x\in\mathbb{N}'(p,\delta)\cap D$

If moreover $y\in\mathbb{N}'(p,\delta)\cap D, |f(y)-l|<\varepsilon/2$ As a result,

$|f(x)-f(y)|\le|f(x)-l|+|f(y)-l|<\varepsilon$ holds.

Converse : Let for given $\varepsilon>0$, there exists a deleted neighbourhood $\mathbb{N}'(p,\delta)$ such that $|f(x)-f(y)|<\varepsilon$ whenever $x,y\in\mathbb{N}'(p,\delta)\cap D$

Let p be accumulation point of D. So there exists $\{x_n\}_n \left(x_n \in D \, \forall_n, x_i \neq x_j\right.$ if $\left. i \neq j,\ x_n \neq p\right)$ which converges to p. Hence corresponding to above $\delta(>0)$, there exists $m \in \mathbb{N}$ such that $x_n \in \mathbb{N}'(p, \delta) \cap D$ for all $n \geq m$.

Therefore, $|f(x_n) - f(x_k)| < \varepsilon$ for all $n,\ k \geq m$.

So by Cauchy's general principle of convergence of a sequence, $\{f(x_n)\}_n$ is convergent and so $\lim\limits_{x \to p} f(x)$ exists.

Illustration : Let $f:(0,1) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, \text{if } x \text{ is rational} \\ -1, \text{if } x \text{ is irrational} \end{cases}$

Let $a \in (0,1)$. Note that for any $\delta > 0$, $N'(a, \delta) \cap (0,1)$ contains both rational as well as irrational points. If such rational be x & such irrational be y, then $|f(x) - f(y)| = |1 - (-1)| = 2 \nless$ arbitrary $\varepsilon > 0$.

So by Cauchy Criterion, $\lim\limits_{x \to a} f(x)$ does not exist.

(G) Infinite limits and Limit at infinity

(i) Infinite limits :

Let $f: D \to \mathbb{R}$ and p be an accumulation point of $D(\subset \mathbb{R})$. Then $f(x)$ is said to be tend to ∞ as $x \to p$, if given any $G > 0$ (as large as we please), there exists $\delta > 0$ such that

$f(x) > G$ whenever $x \in N'(p, \delta) \cap D$.

If we opt for sequential approach, if for $\{x_n\}_n \left(x_n \in D \, \forall n,\ x_i \neq x_j \text{ if } i \neq j,\ x_n \neq p\right)$ converges to p, $\{f(x_n)\}_n$ diverges to ∞, we say that $\lim\limits_{x \to p} f(x) = \infty$

Illustration : $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$

For any $G > 0, \frac{1}{x} > G$ if $x < \frac{1}{G}(\to 0 \text{ as } G \to \infty)$.

If for given $G > 0$ (as large as we please), there exists $\delta > 0$ such that $f(x) < -G$ whenever $x \in N'(p, \delta) \cap D$, we say that $\lim_{x \to p} f(x) = -\infty$

(ii) Limit at infinity

Let $f : D \to \mathbb{R}$ where D is unbounded above.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (G, \infty) \cap D$,

We say that $\lim_{x \to \infty} f(x) = l(\in \mathbb{R})$ $ex \cdot \lim_{x \to \infty} \frac{1}{x} = 0$.

Next let $f : D \to \mathbb{R}$ where D is unbounded below.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (-\infty, G)$, we say that $\lim_{x \to -\infty} f(x) = l(\in \mathbb{R})$

Illustration (1) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, x \in \mathbb{R}$

To solve this, we will assume the very standard limit of sequence $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e(\in \mathbb{R})$.

We can take $x > 1$. There exists natural number n such that

$n \le x < n + 1$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)\right\} = e \text{ and}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = e$$

So, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

(2) $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = e$

We take $x=-y$ and So $y\to\infty$ and x $\to-\infty$

$$\left(1+\frac{1}{x}\right)^{x} = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y} = \left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)\to e \text{ as } y\to\infty.$$

In this connection, we state the following result :

Let $f:(a,\infty)\to\mathbb{R}$, Then $\lim_{x\to\infty} f(x)$ exists if and only if for every $\varepsilon>0$, there exists $X(>a)$ such that $|f(x)-f(y)|<\varepsilon\,\forall\, x, y>X$.

(iii) Infinite limits at infinity

Let $f:D\to\mathbb{R}$ where $D(\subset\mathbb{R})$ is unbounded above.

Let $G>0$ be any number, as large as we please.

Corresponding to G, there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x>K$, we say that $\lim_{x\to\infty} f(x)=\infty$.

Let D be unbounded below, if corresponding to $G>0$ (as large as we please), there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x<K$, we say that $\lim_{x\to-\infty} f(x)=\infty$.

But if $f(x)<-G$ for all $x<K$, we say $\lim_{x\to-\infty} f(x)=-\infty$.

Example : $\lim_{x\to\infty} \log_a x = \infty,\ a > 1$

Let $G > 0$ be any arbitrary number. If we take $a^G = M$, then

$x > M \Rightarrow \log_a x > G$. Hence $\lim_{x\to\infty} \log_a x = \infty$.

(H) Some standard limits :

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$ where $a > 0, a \neq 1$

(iii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a, a > 0$

(iv) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

(I) Algebra of limits :

Let $g, f : D \to \mathbb{R}$ when $D \subset \mathbb{R}$ and p be an accumulation point of D.

Let $\lim_{x\to p} f(x) = l(\in \mathbb{R}),\ \lim_{x\to p} g(x) = m(\in \mathbb{R})$.

Then (i) $\lim_{x\to p}\{f(x) \pm g(x)\} = l \pm m$

(ii) $\lim_{x\to p}\{f(x)g(x)\} = lm$

(iii) $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{l}{m}$ where $g(x) \neq 0$ and $m \neq 0$.

Proof : (i) Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0,\ \delta_2 > 0$ such that $|f(x) - l| < \frac{\varepsilon}{2}$ whenever $0 < |x - p| < \delta_1,\ x \in D$ and $|g(x) - m| < \frac{\varepsilon}{2}$ whenever

$0 < |x-p| < \delta_2$, $x \in D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So for $0 < |x-p| < \delta$, $x \in D$ both hold.

Hence $\left|\{f(x) \pm g(x)\} - \{l \pm m\}\right| \le |f(x) - l| + |g(x) - m| < \varepsilon$

whenever $0 < |x-p| < \delta$, $x \in D$

$\Rightarrow \lim\limits_{x \to p} \{f(x) \pm g(x)\} = l \pm m = \lim\limits_{x \to p} f(x) \pm \lim\limits_{x \to p} g(x)$

Note : (1) This result can be generalised for finite number of functions.

(2) The converse of the result is not true, in general

Let $f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$ and $g(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational} \end{cases}$

Let $p \in \mathbb{R}$. Every deleted *nbd* of *p* contains both rational (say *a*) and irrational *b* (say) points. Then in case of both *f* and *g*, $|f(a) - f(b)|$ or $|g(a) - g(b)| = 1 \nless$ arb ε.

So neither $\lim\limits_{x \to p} f(x)$ nor $\lim\limits_{x \to p} g(x)$ exists. But $f(x) + g(x) = 1$ and

$\lim\limits_{x \to p} \{f(x) + g(x)\} = 1$.

(ii) To establish it we will first show that as $\lim\limits_{x \to p} g(x)$ exists, so there exists a deleted neighbourhood of *p*, in which *g* is bounded.

There exists $\delta_1 > 0$ such that $|g(x) - m| < 1$ where $0 < |x-p| < \delta_1$, $x \in D$ $\left(\text{or } x \in N'(p, \delta) \cap D\right)$

$\Rightarrow |g(x)| < 1 + |m|$ in $N'(p, \delta_1) \cap D$

$\Rightarrow g$ is bounded in $N'(p, \delta_1) \cap D$

$$|f(x)g(x)-lm| = |g(x)\{f(x)-l\}+l(g(x)-m)| \le |g(x)||f(x)-l|+|l|\,|g(x)-m|$$

.....(1)

As $\lim_{x\to p} g(x)$ exists, so there exists $\delta_1 > 0$ such that $|g(x)| < \lambda$ for some $\lambda \in \mathbb{R}^+$ in $N'(p, \delta_1) \cap D$ .....(1)

Let $\varepsilon > 0$ be any number, corresponding to ε, there exists $\delta_2 > 0, \delta_3 > 0$ such that

$|f(x)-l| < \dfrac{\varepsilon}{2\lambda}$ whenever $x \in N'(p, \delta_2) \cap D$ ....(2)

and $|g(x)-m| < \dfrac{\varepsilon}{2(|l|+1)}$ whenever $x \in N'(p, \delta_3) \cap D$ ....(3)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then in $N'(p, \delta) \cap D$, by (1), (2), (3)

$$|f(x)g(x)-lm| < \lambda \cdot \frac{\varepsilon}{2\lambda} + |l| \cdot \frac{\varepsilon}{2(|l|+1)}$$

$$\Rightarrow |f(x)g(x)-lm| < \varepsilon \text{ in } N'(p, \delta) \cap D$$

$$\Rightarrow \lim_{x\to p} f(x)g(x) = lm = \left(\lim_{x\to p} f(x)\right)\left(\lim_{x\to p} f(g)\right)$$

Note : (1) This result can be generalised for finite number of functions.

(2) $\lim_{x\to 0} \sin\dfrac{1}{x}$ does not exist but $\lim_{x\to 0} x\sin\dfrac{1}{x} = 0$

Let $\varepsilon > 0$ be any number $\left|x\sin\dfrac{1}{x} - 0\right| \le |x| < \varepsilon$ whenever $x \in N'(0, \delta) \cap D_f$ where $\delta \equiv \delta(\varepsilon)$.

(3) If $g(x)$ be bounded on D and $\lim_{x\to p} f(x) = 0$, then $\lim_{x\to p} f(x)g(x)$ exists $= 0$.

(iii) $\left|\dfrac{f(x)}{g(x)}-\dfrac{l}{m}\right|=\left|\dfrac{m\{f(x)-l\}-l(g(x)-m)}{m\,g(x)}\right|$

$$\leq \frac{|m|\,|f(x)-l|+|l|\,|g(x)-m|}{|m|\,|g(x)|} \quad .... (1)$$

As $\lim\limits_{x\to p} g(x)=m(\neq 0)$, there exists $\delta_1>0$ such that

$|g(x)-m|<\dfrac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$ .... (2)

$\Rightarrow |g(x)|>\dfrac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$

Let $\varepsilon>0$ be any number.

As $\lim\limits_{x\to p} f(x)=l$, corresponding to ε, there exits $\delta_2>0$ such that

$|f(x)-l|<\dfrac{\varepsilon|m|}{4}$ whenever $x\in N'(p,\ \delta_2)\cap D$ ... (3)

As $\lim\limits_{x\to p} g(x)=m$, corresponding to ε, there exists $\delta_3>0$ such that

$|g(x)-m|<\dfrac{\varepsilon|m|^2}{4(|l|+1)}$ whenever $x\in N'(p,\ \delta_3)\cap D$ ... (4)

Let $\delta=\min\{\delta_1,\ \delta_2,\ \delta_3\}$. So whenever $x\in N'(p,\ \delta)\cap D$, (2), (3) (4) hold.

Recalling 1. $\left|\dfrac{f(x)}{g(x)}-\dfrac{l}{m}\right|<\dfrac{2}{|m|^2}\left|\left\{\dfrac{\varepsilon.|m|^2}{4}+\dfrac{|l|.\varepsilon|m|^2}{4(|l|+1)}\right\}\right|$

$$\Rightarrow \left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|<\varepsilon \text{ whenever } x\in N'(p,\,\delta)\cap D$$

$$\Rightarrow \lim_{x\to p}\frac{f(x)}{g(x)}=\frac{l}{m}=\frac{\lim\limits_{x\to p} f(x)}{\lim\limits_{x\to p} g(x)}$$

Note : Neither $\lim\limits_{x\to 0}\frac{1}{x}$ nor $\lim\limits_{x\to 0}\sin\frac{1}{x}$ exists, but $\lim\limits_{x\to 0}x\sin\frac{1}{x}$ exists & = 0.

So the Converse of (iii) is not, in general, true.

Illustration : Evaluate (1) $\lim\limits_{x\to 0}\frac{\left(e^x-1\right)\tan^2 x}{x^3}$ (2) $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$

(3) $\lim\limits_{x\to \frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(\sqrt{3}-2\cos x\right)}$

(1) $= \lim\limits_{x\to 0}\left\{\frac{e^x-1}{x}.\left(\frac{\sin x}{x}\right)^2.\left(\frac{1}{\cos x}\right)^2\right\}$

$$=\lim_{x\to 0}\frac{e^x-1}{x}.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2.\lim_{x\to 0}\left(\frac{1}{\cos^2 x}\right)=1 \text{ (As all exist)}$$

So limit is 1.

(2) $\lim\limits_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=\lim\limits_{x\to 0}\left\{\frac{\sin x}{x}.\frac{1}{\cos x}.\frac{2\sin^2\frac{x}{2}}{x^2}\right\}$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}.\frac{1}{\cos x}.\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\right\}\frac{1}{2} = 1.1.1.\frac{1}{2} = \frac{1}{2}$$

(3) (Method of substitution) Put $x-\frac{\pi}{6}=t$ and so $x\to\frac{\pi}{6}\Leftrightarrow t\to 0$.

$$\text{Given limit } = \lim_{t\to 0}\frac{\sin t}{\sqrt{3}-2\cos\left(t+\frac{\pi}{6}\right)} = \lim_{t\to 0}\frac{\sin t}{\sqrt{3}+\sqrt{3}\cos t+\sin t}$$

$$= \lim_{t\to 0}\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{\sqrt{3}\left(2\sin^2\frac{t}{2}\right)+2\sin\frac{t}{2}\cos\frac{t}{2}} = \lim_{t\to 0}\frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}} = 1$$

(J) Neighbourhood properties :

(a) Let $f:D\to\mathbb{R}, D\subset\mathbb{R}$ and p be an accumulation point of D. Let $\lim_{x\to p} f(x) = l(\in\mathbb{R})$. Then

(i) f is bounded in some deleted *nbd* of p

(ii) If l be greater than some real number K, then there exists a deleted *nbd* of p in which $f(x)>K$.

(iii) If l be less than some real number μ, then there exists a deleted *nbd* of p in which $f(x)<\mu$.

Proof : (i) Proved earlier in I(ii)

(ii) Let $0<\varepsilon<l-K$. Corresponding to this ε, then exists $\delta>0$ such that

$|f(x)-l|<\varepsilon$ for all $x\in N'(p,\delta)\cap D$

$\Rightarrow l-\varepsilon < f(x) < l+\varepsilon \forall x \in N'(p,\delta)\bigcap D$

Considering the above choice of $\varepsilon, f(x) > K$ in $N'(p,\delta)\bigcap D$

(iii) As in (ii), taking $0 < \varepsilon < \mu - l$.

(b) Let $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ & p be an accumulation point of D.

Let $\lim\limits_{x\to p} f(x) = A(\in \mathbb{R}), \lim\limits_{x\to p} g(x) = B(\in \mathbb{R})$.

If $A < B$, then there exists a deleted neighbourhood of p in which $f(x) < g(x)$.

Proof : Let $A < C < B$.

As $\lim\limits_{x\to p} f(x) = A$, there exists $\delta_1 > 0$ such that $|f(x)-A| < C-A$ for all $x \in N'(p,\delta_1)\bigcap D$.

As $\lim\limits_{x\to p} g(x) = B$, there exists $\delta_2 > 0$ such that $|g(x)-B| < B-C$ for all $x \in N'(p,\delta_2)\bigcap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta)\bigcap D$, both hold.

In $N'(p,\delta)\bigcap D$, $f(x) < C - A + A = C = B-(B-C) < g(x)$ holds.

(c) Sandwich property :

Let $f, g, h : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$. Let $f(x) < g(x) < h(x)$ for all $x \in D$ & let p be an accumulation point of D. Given that $\lim\limits_{x\to p} f(x) = l$, $\lim\limits_{x\to p} h(x) = l (l \in \mathbb{R})$.

Then $\lim\limits_{x\to p} g(x) = l$.

Proof : Let $\varepsilon > 0$ be any number. Corresponding to this ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x)-l| < \varepsilon$ in $N'(p,\delta_1)\bigcap D$ & $|h(x)-l| < \varepsilon$ in $N'(p,\delta_2)\bigcap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta) \cap D$,

$l - \varepsilon < f(x) < g(x) < h(x) < l + \varepsilon \Rightarrow |g(x) - l| < \varepsilon$ in $N'(p,\delta) \cap D$

So $\lim_{x \to p} g(x) = l$.

(d) $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$, p be accumulation point of D and

let $\lim_{x \to p} f(x) = l(\in \mathbb{R})$, $\lim_{x \to p} g(x) = m(\in \mathbb{R})$. If $f(x) < g(x)$ in D, then $l \le m$.

Proof : If possible let $l > m$ & let $o < \varepsilon < \dfrac{l-m}{10}$, corresponding to such ε, there exists $\delta_1, \delta_2 >$ such that $|f(x) - 1| < \varepsilon$ in $N'(p,\delta_1) \cap D$ & $|g(x) - m| < \varepsilon$ in $N'(p,\delta_2) \cap D$.

If $\delta = \min\{\delta_1, \delta_2\}$, then in $N'(p, \delta) \cap D$, both hold.

In $N'(p, \delta) \cap D$, $l - \varepsilon < f(x) < g(x) < m + \varepsilon \Rightarrow l - m < 2\varepsilon \Rightarrow 10\varepsilon < 2\varepsilon$ — absurd as $\varepsilon > 0$.

So $l \le m$.

[You can take $f(x) = 1 - x$, $g(x) = 1 + x$ where $x > 0$. $f(x) < g(x)$ for all x and $\lim_{x \to 0+} f(x) = 1 = \lim_{x \to 0+} g(x)$.]

K. Infinitesimal :

(a) $f : D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be infinitesimal as $x \to a$ if $\lim_{x \to a} f(x) = 0$.

(b) If $f, g : D \to \mathbb{R}$ are infinitesimals, then $f \pm g$, fg are also so.

(c) If $f : D \to \mathbb{R}$ be infinitesimal as $x \to a$ and $g : D \to \mathbb{R}$ be bounded, then fg is infinitesimal.

(d) We say $f = o(g)$ (or f is of little *–oh* of g over D) if

$f(x)=\alpha(x)g(x)$ where $\alpha(x)$ is infinitesimal.

(e) We say $f=O(g)$ (or f is of big *–oh* of g over D) if $f(x)=\beta(x)g(x)$ where $\beta(x)$ is bounded on D.

(f) The functions f and g are of same order over $D(\subset\mathbb{R})$, if $f=O(g)$ and $g=O(f)$ simultaneously.

15.6 Summary

In this unit, we have defined the term functions and classified various type of functions. We have defined real valued sequences and study limit of a real sequence. We have explaind the concept of limit of functions and study some criterian for the existence of limit. We also introduced the concept of infinite limits, limit at infinity, neighbourhood properties. We have explained the Sandwich property and the concepts of infinitesimal.

15.7 Exercise

1. Find the limits (if exist)

(a) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

(b) $\lim\limits_{x\to 0}\dfrac{\left(2x^2+|x|\right)}{x}$

(c) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+3\right)$, $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+1\right)$ and $\lim\limits_{x\to 0}\left\{\left(\dfrac{1}{x^2}+3\right)-\left(\dfrac{1}{x^2}+1\right)\right\}$

(d) $\lim\limits_{x\to 3}\dfrac{\sqrt{(3x)}-3}{\sqrt{2x-4}-\sqrt{2}}$

(e) Apply Cauchy's principle for the existence of limit to evaluate $\lim\limits_{x\to 0}\dfrac{1+x}{1-x}$.

2. Choose the correct one : $\lim_{x \to 0} \frac{\sin[x]}{[x]}$

(a) the limit exists and is 1
(b) the limit does not exist.
(c) if at $x = 0,\ f(0) = 0$, the limit will exist
(d) if at $x = 0,\ f(0) = 1$, the limit will exist.

Unit- 16 □ Continuity of Functions

Structure

16.1 Objectives

This unit gives

- The concept of continuity of a real fuction
- Classification of discontinuity
- Neighbourhood properties of a continous function
- The behaviour of continuous function in a closed and bounded interval
- The concept of uniform continuity

16.2 Introduction

A general function from $\mathbb{R}$ to $\mathbb{R}$ can be very convoluted indeed, which means that we will not be able to make many meaningful statements about general functions. To develop a useful theory, we must instead restrict the class of functions we consider. Intuitively we require that the functions be sufficiently 'nice', and see what properties we can deduce from such restrictions. The study of continuous functions is a case in point by requiring a function to be continuous, we obtain enough information to deduce powerful theorems, such as the Intermediate value theorem. However, the definition of continuity is flexible enough that there are a wide, and interesting, variety of continuous functions. Indeed, many functions that come up in real-world problems are continuous, which makes the definition pleasing from both an aesthetic and practical point of view.

16.3 Definition

I. (a) A function $f : D \to \mathbb{R} (D \subset \mathbb{R})$ is said to be continuous at $p \in D$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - f(p)| < \varepsilon$ or $f(x) \in N(f(p), \varepsilon)$ whenever $x \in N(p, \delta) \cap D$.

If f is not continuous at p, then f is discontinuous at p.

(b) Let $f : D \to \mathbb{R} (D \subset \mathbb{R})$ and $p \in D$.

(i) If p is an isolated point of D (i.e. not a limit point of D), then f is continuous at p (ii) if p be limit point of D i.e. $p \in D \cap D'$ (D' is the collection of limit points of D) and if $\lim\limits_{x \to p} f(x) = f(p)$, then f is continuous at p.

(c) Continuity in an interval $[a, b]$ or in $\{x : a \le x \le b\}$.

f is continuous in $[a, b]$ if (i) $\lim\limits_{x \to a+0} f(x) = f(a)$ (ii) $\lim\limits_{x \to b-0} f(x) = f(b)$ and (iii) if $a < c < b$, then $\lim\limits_{x \to c-0} f(x) = \lim\limits_{x \to c+0} f(x) = f(c)$.

Examples. 1. Let $f(x) = \begin{cases} x, \text{ for } x \in \left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\} \\ 1, \text{ for } x = 1 \end{cases}$

be defined on $S = \left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{1\}$.

The only accumulation point of S is 1 and all other points of S are its isolated points. Here $\lim_{x\to 1} f(x) = f(1) = 1 \Rightarrow f$ is continuous at 1, f is also continuous at the isolated points $1-\frac{1}{n} : n \in \mathbb{N}$. Hence f is continuous on S.

2. Let $f(x) = \dfrac{1}{(1-x)}$, $x \neq 1$. Find the points of discontinuity of $y = f\left[f\left(f(x)\right)\right]$.

$x = 1$ is a point of discontinuity of $f(x)$.

If $x \neq 1$, $f\left[f(x)\right] = \dfrac{1}{1-\dfrac{1}{(1-x)}} = \dfrac{x-1}{x}$, $x \neq 0 \Rightarrow x = 0$ is a point of discontinuity of $f\left[f(x)\right]$.

If $x \neq 0$, $x \neq 1$, $y = \dfrac{1}{1-\dfrac{x-1}{x}} = x$ is continuous everywhere.

So points of discontinuity of the given composite function are $x = 0,\ x = 1$.

(3) Let $E = \left\{1-\frac{1}{n} | n \in \mathbb{N}\right\} \cup [1,\ 2]$ and $f : E \to \mathbb{R}$ be defined by $f(x) = x^2$.

Each $1-\frac{1}{n}$ is isolated point of E and so by definition, f is continuous at all such points.

Let $p \in [1,\ 2]$. Then $p \in E \cap E'$ (E' derived set of E) and then $x^2 \to p^2$ or $f(x) \to f(p)$. So f is continuous at p.

Thus f is continuous on E.

(Continuation of definiton (d)) $f : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and $p \in D \cap D'$.

f is continuous at p if for every sequence

$\{x_n\}_n \left(x_n \in D \ \forall n,\ x_i \neq x_j \text{ if } i \neq j,\ x_n \neq p\right)$ converging to p, $\{f(x_n)\}_n$ converges to $f(p)$.

Examples (1) Let $A = \{x \in \mathbb{R} | x > 0\}$ and let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is irrational} \\ \dfrac{1}{n}, \text{ if } x = \dfrac{m}{n} \text{ where } m, n \in \mathbb{N} \text{ and } (m, n) = 1 \end{cases}$$

To examine the continuity of f in A.

We require the following lemma :

Let i be any irrational number between 0 and 1.

Let p, q, n be any positive integers such that $p < q \leq n$ and n is fixed. Then there exists a neighbourhood of i which has the property that no rational number of the form $\dfrac{p}{q}$ belongs to it.

Proof of lemma : Let d be the least of the differences $\left| i - \dfrac{p}{q} \right|$ for all p, q such that $p < q \leq n$. Let δ be chosen so that $0 < \delta < d$. Then $(i - \delta,\ i + \delta)$, a nbd of i, which has the property stated above.

Let us now examine the continuity of f.

Let b be any irrational number and let $\varepsilon > 0$.

Now there exists $n_0 \in \mathbb{N}$ such that $n_0 \varepsilon > 1$ (known as Archimedean property of real numbers). By above lemma, $\delta > 0$ can be chosen so small that the nbd $(b - \delta,\ b + \delta)$ contains no rational number with denominator $< n_0$.

If then follows that for $|x-b|<\delta$, $x \in A$, we have

$$|f(x)-f(b)| = |f(x)| \leq \frac{1}{n_0} < \varepsilon \Rightarrow f \text{ is continuous at irrational point } b.$$

Let $a \in A$ be any rational point. Let $\{x_n\}_n$ be any sequence of irrational numbers in A that converges to a. Then $\lim_{n\to\infty} f(x_n) = 0$ where as $f(a) > 0$. Hence f is discontinuous at all rational points.

(2) (Dirichlet's function) $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, \text{ if } x \text{ be rational} \\ 0, \text{ if } x \text{ be irrational} \end{cases}$$

Applying sequential approach, it can be shown that f is discontinuous everywhere.

(3) Let $f(x) = \begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$

To investigate the continuity of f on $\mathbb{R}$.

Let $\varepsilon > 0$ be any number.

$$\left|f(x)-f\left(\frac{1}{2}\right)\right| = \begin{cases} \left|x-\frac{1}{2}\right|, \text{ if } x \text{ is rational} \\ \left|1-x-\frac{1}{2}\right| = \left|x-\frac{1}{2}\right|, \text{ if } x \text{ is irrational} \end{cases}$$

So $\left|f(x)-f\left(\frac{1}{2}\right)\right| = \left|x-\frac{1}{2}\right| < \varepsilon$ whenever $\left|x-\frac{1}{2}\right| < \delta(=\varepsilon)$.

f is continuous at $x = \frac{1}{2}$.

Next let $x \neq \frac{1}{2}$ and x is rational. Let $\{x_n\}_n$ be a sequence of irrationals such that $\lim_{n\to\infty} x_n = x$. So $f(x_n) = 1 - x_n \to 1 - x$ as $n \to \infty$.

As $x \neq \frac{1}{2}$, so $x \neq 1 - x$ and f is discontinuous on $Q - \left\{\frac{1}{2}\right\}$.

Next let x be irrational number and let $\{y_n\}_n$ be a sequence of rational numbers such that $\lim_{n\to\infty} y_n = x$. Here $f(y_n) = y_n \to x$ as $n \to \infty$. But $f(x) = 1 - x$.

So $\lim_{n\to\infty} f(y_n) \neq f\left(\lim_{n\to\infty} y_n\right) \Rightarrow f$ is discontinuous at all irrational points.

Consequently f is continuous only at $x = \frac{1}{2}$.

Classification of discontinuities :

Let f be not continuous at $p(\in D_f)$. This discontinuity of f at p may be due to different reasons which may be classified into two types / kinds.

Definition : (a) Let f be defined in both-sided neighbourhood of point $p(\in D_f)$.

Let $\lim_{x\to p+} f(x)$ and $\lim_{x\to p-} f(x)$ both exist finitely but are unequal, then $x = p$ is known as jump discontinuity of f.

$f(p+o) - f(p-o)$ is known as height of the jump. If f has jump discontiuity on the right at a, the height of jump is $f(a+o) - f(a)$ and similarly at b, it is $f(b) - f(b-o)$, if it is left discontinuous at b.

Example : let $:[0,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}0, & \text{if } x=0\\ \frac{1}{2}-x, & \text{if } 0<x<\frac{1}{2}\\ 1/2, & \text{if } x=\frac{1}{2}\\ \frac{3}{2}-x, & \text{if } \frac{1}{2}<x<1\\ 1, & \text{if } x=1\end{cases}$$

$f(0+)=\frac{1}{2}\neq f(0),\ f\left(\frac{1}{2}-0\right)=0,\ f\left(\frac{1}{2}+0\right)=1$ so $f\left(\frac{1}{2}-0\right)\neq f\left(\frac{1}{2}+0\right)$

$f(1-0)=\frac{1}{2}\neq 1$ so, 0, $\frac{1}{2}$, 1 are points of jump discontinuity of f.

If $f(p-0), f(p+0)$ both exist and are equal but $\neq f(p)$,

then p is removable discontinuity of $f\left(\text{i.e} \lim\limits_{x\to p} f(x)\neq f(p)\right)$

Example : $f(x)=\begin{cases}5x+7, & x<2\\ 13, & x=2\\ 4x+9, & x>2\end{cases}$

$f(2-0)=17=f(2+0)$ but $f(2)=13$

$x=2$ is removable discontinuity. These two types of discontinuity are known as discontinuity of first kind or ordinary discontinuity.

(b) (i) If f is defined in both sided nbd of p including p and at least one of $f(p-0)$ & $f(p+0)$ fails to exist finitely though f is bounded in some deleted neighbourhood of p, then p is discontinutiy of second kind with finite oscillation.

Example : $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Neither $\lim_{x\to 0+} f(x)$ nor $\lim_{x\to 0-} f(x)$ exists & but f is bounded in *nbd* of 0.

(ii) f is unbounded in every nbd p and $\lim_{x\to P+0} f(x)$ or $\lim_{x\to p-0} f(x)$ is $+\infty$ or $-\infty$. Such a discontinuity is known as infinte discontinuity.

Example : $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2, & x = 0 \end{cases}$

16.4 Neighbourhood properties

Let $f : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and p be an accumulation point of D as well as an element of D. Let f be continuous at p.

Then the following results hold :

(i) There exists a neighbourhood of p in which f is bounded.

(ii) If $f(p) \neq 0$, there exists a neighbourhood of p in which $f(x)$ & $f(p)$ have the same sign.

(iii) If in every neighbourhood of $p, f(x)$ assumes both positive & negative values, then $f(p) = 0$

The first two properties follow from the neighbourhood properties for the existence of limit.

For (iii) if $f(p) > 0$, by (ii) there exists *nbd* of p in which $f(x) > 0$ for all $x \in \mathbb{N}(p, \delta) \cap D$. But $f(x)$ have both signs in every *nbd* of p & so $f(p) \not> 0$. By similar logic, $f(p) \not< 0$. Hence $f(p) = 0$.

The converse of (iii) is not true. For example, $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity of some special types of functions

(i) Let $f: D \to \mathbb{R}$ be monotone function (increasing or decreasing). Then at every point c of D, both $f(c+0)$ & $f(c-0)$ exist. So if c be any point of discontinuity, then that discontinuity is of first kind. In other words a monotone function can not have any discontinuity of second kind (for proof, see Apendix).

(ii) Polynomial function $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n \left(a_i \in \mathbb{R}\ \forall\, i, a_0 \neq 0\right)$ is continuous on $\mathbb{R}$. Rational functions $\frac{p(x)}{q(x)}$ are continuous for all $x \in \mathbb{R}$ for which the functions can be defined.

$\sin x$ and $\cos x$ are continuous an $\mathbb{R}$. $\tan x$ & $\sec x$ are continuous for all $x \neq (2n+1)\frac{\pi}{2}$ and $\cot x$, $\operatorname{cosec} x$ are continuous for all $x \neq n\pi$ (n is integer in both cases)

(iii) $a^x, a > 0,$ is continuous for all $x \in \mathbb{R}$. $\log x, x > 0$ is continuous for all $x > 0$.

(iv) For even positive integer n, the function $g: x \to \sqrt[n]{x}$ is continuous for all $x \in [0, \infty)$ and for an odd positive integer n, g is continuous for all $x \in (-\infty, \infty)$.

(v) Limit of composite function :

Let $f:(a, b) \to \mathbb{R}$ be continuous at $c \in (a,b)$. Suppose that $g: I \to (a,b)$ where I is an open interval and $x_\circ \in I$. If $\lim\limits_{x \to x_\circ} g(x)$ exists and is equal to c. then $\lim\limits_{x \to x_\circ} f(g(x)) = f(c)$.

Proof : Continuity of f at c implies that for each pre-assigned $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(c)| < \varepsilon$ whenever $|y - c| < \delta,\ \left(y \in (a, b)\right)$......(1)

As $\lim\limits_{x \to x_\circ} g(x) = c$, so corresponding to above δ, we can find $\eta > 0$ such that $|g(x) - c| < \delta$ for $0 < |x - x_\circ| < \eta$ ....(2)

By (1) and (2) for $0<|x-x_\circ|<\eta$, we have $|f(g(x))-f(c)|<\varepsilon,\ 0<|x-x_\circ|<\eta$

Hence $\lim_{x\to x_\circ} f(g(x))=f(c)$ follows.

Corollary : Let I, J be open intervals, $g:I\to J$ be continuous at $x_0\in I$. If $f:J\to\mathbb{R}$ is continuous at $g(x_0)\in J$ then $f\circ g:I\to\mathbb{R}$ is continuous at x_0. In other words, the composition of two continuous functions is continuous.

Note : Continuity of f at c in (v) is needed.

Let $f,g:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(y)=\begin{cases}3, & y=1\\ 4, & y\neq 1\end{cases}\quad g(x)=1 \text{ for all } x.$$

Note that as $y\to 1, f(y)\to 4$ & $g(x)\to 1$ as $x\to 0$

For all x, $f(g(x))=f(1)=3$ & so it is not true that $f(g(x))\to 4$ as $x\to 0$

Illustration :

To evaluate $\lim_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$

Let $f(x)=\dfrac{1+x}{2+x},\ g(x)=\dfrac{1-\sqrt{x}}{1-x}$

$\lim_{x\to 1} f(x)=\dfrac{2}{3}$ (f is continuous at x =1) & $\lim_{x\to 1} g(x)=\lim_{x\to 1}\dfrac{1}{1+\sqrt{x}}=\dfrac{1}{2}$

Hence $\lim_{x\to 1}\left[f(x)\right]^{g(x)}=\left(\dfrac{2}{3}\right)^{1/2}$

(Note that $\lim_{x\to 1}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to 1} g(x)\ln f(x)}=e^{B\ln A}=A^B$ if

$\lim_{x\to 1} f(x)=A>0$ and $\lim_{x\to 1} g(x)=B$)

(vi) Piecewise Continuous function :

Let $f:[a, b] \rightarrow \mathbb{R}$ be such that it is continuous in $[a, b]$ except for a finite number of points, at each of which f has jump discontinuity. Then f is said to be piecewise continuous function in $[a, b]$

Illustration : Let $f:[0, 3] \rightarrow \mathbb{R}$ be defined by $f(x)=[x]$

$$\text{Then } f(x)=\begin{cases} 0, & 0 \leq x<1 \\ 1, & 1 \leq x<2 \\ 2, & 2 \leq x<3 \\ 3, & x=3 \end{cases}$$

Note that f has jump discontinuity at 1, 2, 3 only & is continuous in (0, 1), (1, 2) and (2, 3)

Example : Let $f(x)=[x],\ x \in \mathbb{R}^{+}$

Then f is not continuous at any point of $\mathbb{Z}$ but is continuous on $\mathbb{R}^{+} \backslash \mathbb{Z}$.

(i) Let $C \in \mathbb{Z}$.

Note that $C-\frac{1}{n} \rightarrow C$ as $n \rightarrow \infty$. $f\left(C-\frac{1}{n}\right)=C-1$ for all $n \in \mathbb{Z}$. But $f(C)=C$. So $\lim\limits_{n \rightarrow \infty} f\left(C-\frac{1}{n}\right) \neq f\left[\lim\limits_{n \rightarrow \infty}\left(C-\frac{1}{n}\right)\right] \Rightarrow f$ is not continuous at any point of $\mathbb{Z}$.

(ii) Let $C \in \mathbb{R}^{+} \backslash \mathbb{Z}$

We take $0<\varepsilon<\min \{C-[C],[C]+1-C\}$

Let $\lim\limits_{n \rightarrow \infty} x_{n}=C$. So corresponding to above ε, $\exists\, n_{\circ} \in \mathbb{Z}$ such that

$\left|x_{n}-c\right|<\varepsilon$ whenever $n \geq n_{0}$

Above choice of ε implies $[C] < x_n < [C]+1$ for all $n \geq n_\circ$

Then $f(x) = [x_n] = [C]$ for all $n \geq n_\circ$

Therefore $f(x_n) \to f(C)$ as $n \to \infty$. Hence the result follows.

Examples of piecewise continuous functions

(i) $f(x) = x - [x], x \in [0, 4]$

(ii) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}+5}{x^{2n}+1}, \ x \in [-2, 2]$

(iii) $f(x) = \begin{cases} 2x+1, & 0 \leq x < 1 \\ 5, & x = 1 \\ 3x+2, & 1 < x \leq 2 \\ 7, & x = 2 \end{cases}$

16.5 Properties of functions continuous in a closed and bounded interval [a, b]

Theorem (1) : Let $f : [a, b] \to \mathbb{R}$ be continuous in the closed and bounded interval [a, b] & $f(a) f(b) < 0$. Then there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.

[To prove this, we require the following result, known as Nested interval property :

If $\{[a_n, b_n]\}_n$ be a sequence of closed and bounded intervals such that each is contained in the preceeding. Then $\bigcap\limits_n [a_n, b_n] \neq \phi$

If more over $\lim\limits_{n\to\infty} (b_n - a_n) = 0$ then if $p \in \bigcap\limits_n [a_n, b_n]$, p is unique.]

Also $\lim\limits_{n\to\infty} a_n = p = \lim\limits_{n\to\infty} b_n$

Proof : We assume that $f(a) < 0, f(b) > 0$

For the sake of convenience, let $[a, b] = [a_1, b_1] \equiv I_1$

Let us bisect I_1 at $c_1 = \dfrac{a_1 + b_1}{2}$, If $f(c_1) = 0$ the result is proved

If $f(c_1) \neq 0$, either $f(c_1) > 0$ or $f(c_1) < 0$

If $f(c_1) > 0$ we take $[a_1, c_1]$ as I_2 so that $f(a_1) f(c_1) < 0$

& if $f(c_1) < 0$, we take $[c_1, b_1]$ as I_2. $I_2 \equiv [a_2, b_2]$

Let us bisect $[a_2, b_2]$ at $c_2 = \dfrac{a_2 + b_2}{2}$ If $f(c_2) = 0$ the result is proved. Otherwise, we assume that sub-interval as $[a_3, b_3] = I_3$ for which $f(a_3) f(b_3) < 0$ This process is continued indefinitely & we get a sequence $\{I_n\}_n$ of closed & bounded intervals $[a_n, b_n]$ for which

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$

(ii) $\lim\limits_{n\to\infty} |I_n| = \lim\limits_{n\to\infty} (b_n - a_n) = \lim\limits_{n\to\infty} \dfrac{b-a}{2^{n-1}} = 0$

Also $f(a_n) f(b_n) < 0$ for all $n \in \mathbb{N}$

By Nested interval property, $\bigcap\limits_n I_n = \{c\}$ Also $\lim\limits_{n\to\infty} a_n = c = \lim\limits_{n\to\infty} b_n$

By construction, $f(a_n) < 0$ and $f(b_n) > 0$ for all n

By continuity of f, $\lim\limits_{n\to\infty} f(a_n) \leq 0$ & $\lim\limits_{n\to\infty} f(b_n) \geq 0$

$\Rightarrow f\left(\lim\limits_{n\to\infty} a_n\right) \leq 0$ & $f\left(\lim\limits_{n\to\infty} b_n\right) \geq 0$

$\Rightarrow f(c) \leq 0$ & $f(c) \geq 0 \Rightarrow f(c) = 0$

Note : This theorem is due to B. P. J. N Bolzano (1781-1848)

Theorem (2) : Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$ and $f(a)\neq f(b)$. If k be any real number such that $f(a)<k<f(b)$ then there exists $c\in(a,b)$ such that $f(c)=k$.

Proof : Let $\phi:[a,b]\to\mathbb{R}$ be defined by $\phi(x)=f(x)-k$

Continuity of f in $[a,b]\Rightarrow$ continuity of ϕ in $[a,b]$

$\phi(a)\,\phi(b)=\{f(a)-k\}\{f(b)-k\}<0$

Then by Bolzano's theorem, there exists $c\in(a,b)$ such that $\phi(c)=0$ i.e $f(c)=k$

Note : (i) This property is known as Intermediate value (I.V.) property of f in $[a, b]$

(ii) I. V. property does not hold in case of functions defined on a set.

Let $S=[0,1]\cup[2,3]$ & $f:S\to\mathbb{R}$ be defined by $f(x)=x$

f is continuous on S but f does not attain the value $\frac{3}{2}$ on S.

(iii) Continunity of f in $[a,b]\Rightarrow$ validity of I V property by f on $[a, b]$ but the converse is not true

Example : $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0, & x=0\\ \frac{1}{2}-x, & 0<x<\frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ \frac{3}{2}-x, & \frac{1}{2}<x<1\\ 1, & x=1\end{cases}$

f assumes every value between $f(0)$ & $f(1)$, f is not continuous in $[0,1]$ & so the validity of I. V property by a function in a closed & bounded interval does not characterise the continuity of the function. In this connection, we state the following two important results :

(1) Let $f : [a, b] \to \mathbb{R}$ obey the Intermediate value property in $[a, b]$ & let f be monotonic in $[a, b]$. Then f is continuous on $[a, b]$.

(2) Let f be stictly monotonic function in the interval $[a, b]$. If $f([a, b])$, the range set is an interval, then f is continous on $[a, b]$.

Theorem (3) : Let $f : [a, b] \to \mathbb{R}$ be continuous and assume each value between $f(a)$ and $f(b)$ just once. Then f is strictly monotonic in $[a, b]$.

Proof : Let $f(a) < f(b)$. We propose to show that f is strictly increasing function.

Let $a < x_1 < b$. As $f(x)$ assumes each value between $f(a)$ and $f(b)$ just once, so $f(x_1) = f(a)$ or, $f(x_1) = f(b)$ is not possible. ....(1)

If $f(x_1) < f(a) (< f(b))$, then by I. V. property $f(x)$ must assume the value $f(a)$ for some $x \in (x_1, b)$. As a result $f(x) = f(a)$, once at $x = a$ & for some $x \in (x_1, b)$. This contradicts the hypothesis that $f(x)$ assumes each value between $f(a)$ & $f(b)$ just once. So $f(x_1) < f(a)$ is not possible. ....(2)

By similar logic, $f(x_1) > f(b)$ is not possible. ....(3)

In that case, $f(x)$ assumes the value $f(b)$ at least twice — once at b & another in (a, x_1) by I. V. property.

By (1), (2) & (3), $f(a) < f(x_1) < f(b)$

This leads to the conclusion that if $a < x_1 < x_2 < b$ then

$f(a) < f(x_1) < f(x_2) < f(b)$

$\Rightarrow f$ is strictly monotonic increasing in $[a, b]$

If at the outset, we assume that $f(a) > f(b)$, then arguing in a similar way f is strictly monotonic decreasing in $[a, b]$.

Examples : $f:[0, 2] \to \mathbb{R}$ be defined by $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n+2} - \cos x}{1 + x^{2n}}$

Show that $f(0)f(2) < 0$ but $f(x)$ is never zero in $(0, 2)$. Explain why.

When 0 <x <1, $x^{2n} \to 0$ & when $1 < x < 2,\ x^{2n} \to \infty$

Here $f(0) = -1$. When $0 < x < 1,\ f(x) = -\cos x$

$f(1) = \frac{1}{2}[1 - \cos 1]$. When $1 < x \le 2,\ f(x) = \lim\limits_{n\to\infty} \dfrac{x^2 - \dfrac{\cos x}{x^{2n}}}{1 + \dfrac{1}{x^{2n}}} = x^2$

$$\text{So } f(x) = \begin{cases} -1, & x = 0 \\ -\cos x, & 0 < x < 1 \\ \dfrac{1}{2}(1 - \cos 1), & x = 1 \\ x^2, & 1 < x \le 2 \end{cases}$$

So $f(0)f(2) = -4 < 0$, but $f(x)$ is never zero in $(0, 2)$. The reason is that f is not continuous in $[0, 2]$ & I. V. Property is not applicable.

(2) Let $f:\left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \log(2 + x), & 0 \le x < 1 \\ \dfrac{1}{2}(\log 3 - \sin 1), & x = 1 \\ -\sin x, & 1 < x \le \pi/2 \end{cases}$$

Here $f(0)\, f\left(\frac{\pi}{2}\right) = (\log 2)(-1) < 0$ but $f(x)$ is never zero in $\left(0, \pi/2\right)$. The reason is $f(x)$ is not continuous in $\left(0, \frac{\pi}{2}\right)$ & so I. V. property is not applicable here.

(3) Let $f : [0, 1] \to \mathbb{R}$ be continuous function and assume only rational values in the entire interval. If $f(x) = 5$ at $x = \frac{2}{3}$, show that $f(x) = 5$ everywhere.

If possible, let there exist $c \in [0, 1]$, $c \neq \frac{2}{3}$ and $f(c) = K \in \mathbb{R}$.

If $K \neq 5$, then by I. V. property of continuous function, $f(x)$ must assume every value between K & 5. Between K & 5, there are rational as well as irrational points also. But $f(x)$ assumes rational values only. So $f(x) = 5$ throughout $[0, 1]$.

(4) Let $f : [0, 1] \to \mathbb{R}$ be continuous function and $f(0) = f(1)$. Show that there exists $y \in [0,1]$, such that $|x - y| = \frac{1}{2}$ and $f(x) = f(y)$.

Let us consider the function $g : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ defined by $g(x) = f\left(x + \frac{1}{2}\right) - f(x)$

Continuity of f in $[0, 1] \Rightarrow$ continuity of g in $\left[0, \frac{1}{2}\right]$.

$$g(0)\, g\left(\frac{1}{2}\right) = \left(f\left(\frac{1}{2}\right) - f(0)\right)\left(f(1) - f\left(\frac{1}{2}\right)\right) < 0$$

By Bolzano's theorem on continuous function, there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$g(c)=0 \Rightarrow f\left(c+\frac{1}{2}\right)=f(c)$ we get $x,\ y \in [0,1],\quad |x-y|=\frac{1}{2}$ for which

$f(x)=f(y)$.

(5) (Fixed point property) Let $f:[a,b]\to[a,b]$ be continuous function. Show that for some $\xi \in [a,\ b],\ f(\xi)=\xi$ holds.

If $f(a)=a$ or $f(b)=b$, the result is established.

We take $f(a)>a,\ f(b)<b.$ (as $f:[a,b]\to[a,b]$)

Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x)=f(x)-x$

Continuity of f in $[a,b] \Rightarrow$ continuity of g in $[a,b]$.

$g(a)\,g(b)=\{f(a)-a\}\{f(b)-b\}<0$. So by Bolzano's theorem, there exists $\xi \in (a,\ b)$ such that $g(\xi)=0$ or $f(\xi)=\xi$.

Notes : (i) The condition of continuity of f can not be dropped

$f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} 1-x, & 0\le x\le\frac{1}{2} \\ \frac{1}{2}-\frac{x}{2}, & \frac{1}{2}<x\le 1 \end{cases}$

(ii) The result may fail if the interval be not closed and bounded :

(a) $f:[0,1)\to\mathbb{R}$ be defined by $f(x)=\frac{1+x}{2}$

(b) $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x)=x+\frac{1}{x}$

(iii) f must be defined on some interval $(\subset\mathbb{R})$

$f:S\to\mathbb{R}$ be defined by $f(x)=-x$ where $x\in S\ (\equiv[-2,-1]\cup[1,2])$

Also $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+1$

Exercise :

1. Show that $x.2^x = 1$ has a solution in $[0, 1]$.

2. Let $f : [a, b] \to \mathbb{R}$ be continuous function & the equation $f(x) = 0$ have finite number of roots in $[a, b]$ & arranging them in the ascending order, these are

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots x_{n-1} < b$

Prove that in each of (x_{r-1}, x_r) $f(x)$ must have the same sign.

3. If $f : [a, b] \to \mathbb{R}$ be a continuous function & $f(x)$ be always a rational number, then $f(x)$ is a constant function.

4. Examine for the continuity of $f : f(x) = \begin{Bmatrix} x^2 - 2x, & \text{when } x \text{ is rational} \\ 3x - 6, & \text{when } x \text{ is irrational} \end{Bmatrix}$

5. Does the equation $\sin \text{x} - x + 1 = 0$ have a root ?

6. Does the equation $f(x) = \dfrac{x^3}{4} - \sin \pi x + 3$ take on the value $2\dfrac{1}{3}$ within the interval $[-2, 2]$?

7. Show that there exists $x \in \left(0, \dfrac{\pi}{2}\right)$ such that $x = \cos x$

Theroem (4) : Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Then f is bounded in $[a, b]$ & attains its bounds in $[a, b]$.

Proof : If possible let f be not bounded in $[a, b]$. So corresponding to $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$.

All such x_n's are in $[a, b]$. So we get a sequence $\{x_n\}_n$ in $[a, b]$. Hence $\{x_n\}_n$ is bounded in $[a, b]$.

By Bolzano-Weierstrass theorem on subsequence, there exists a convergent sub sequence $\{x_{r_n}\}_n$ (say) of $\{x_n\}_n$, which converges to $l(\in \mathbb{R})$. This $l \in [a, b]$ as $[a, b]$ is closed. Due to continuity of f, $\{f(x_{r_n})\}_n$ should converge to $f(x)$. Every convergent sequence is necessarily bounded. So $\{f(x_{r_n})\}_n$ is bounded. But by construction, $|f(x_{r_n})| \geq r_n$ & as $\{r_n\}_n$ is strictly increasing sequence of natural numbers, so $r_n \geq n$. Conequenctly, $|f(x_{r_n})| \geq n$

This contradicts $\{f(x_{r_n})\}_n$ is bounded.

This f is bounded on $[a, b]$

Let $M = \sup\limits_{[a, b]} f$, $m = \inf\limits_{[a, b]} f$

If possible, let there be no point x in $[a, b]$ at which $f(x) = M$. So $f(x) < M$ in $[a, b]$.

We construct $\phi : [a, b] \to \mathbb{R}$ defined by $\phi(x) = \dfrac{1}{M - f(x)}$ for all $x \in [a, b]$.

Continuity of f in $[a, b] \Rightarrow$ Continuity of ϕ in $[a, b]$. So ϕ is bounded in $[a, b]$. Let $G > 0$ be any number, as large as we please.

As $M = \sup\limits_{[a, b]} f$, there exists at least one point $\xi \in [a, b]$ such that

$$f(\xi) > M - \frac{1}{G}$$

$\Rightarrow \dfrac{1}{M - f(\xi)} > G \Rightarrow \phi(\xi) > G$. This contradicts the fact that ϕ is bounded in $[a, b]$

So there exists a point in $[a, b]$ at which $f(x) = M$.

Similarly, it can be shown that there exists a point in $[a, b]$ at which $f(x) = m$ holds.

Corollaries : (i) If $f:[a, b] \to \mathbb{R}$ be a non-constant continuous function, then $f(x)$ assumes every value between its infimum & supremum.

By above theorem, there are points $\xi, \eta \in [a, b]$ such that $f(\xi) = M$, $f(\eta) = m$. By I. V. property of continuous function, applied to f in $[\xi, \eta]$ (or $[\eta, \xi]$) the result follows.

(ii) Let $I\ (\subset \mathbb{R})$ be a closed and bounded interval & let $f: I \to \mathbb{R}$ be non constant continuous function in I.

Then the set $f(I) = \{f(x) : x \in I\}$ is a closed & bounded interval.

If $M = \sup\limits_{[a, b]} f$, $m = \inf\limits_{[a, b]} f$, then $m \le f(x) \le M$ for all $x \in I$

$\Rightarrow f(I) \subseteq [m, M]$ ...1

Let k be any element of [m, M]. Then by Corollany 1, there exists $c \in I$ such that $f(c) = k \in f(I)$

So $[m, M] \subseteq f(I)$ ... (2)

By (1) and (2), $f(I) = [m, M]$

Note : The result fails if the condition of continuity be dropped.

$f: I \equiv [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$f(I)$ is not an interval.

2. The continuous image of an open interval may not be open.

Let $f: (-1, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2 + 1}$

Here $f(I)=(\frac{1}{2},1]$ which is not open interval

3. The continuous image of an unbounded closed interval may not be closed.

Let $f: I \equiv [0,\infty) \to \mathbb{R}$ be defined by $f(x)=\dfrac{1}{x^2+1}$

Here $f(I)=(0,1]$ which is not closed.

Example (1) : Let $f:[a,b] \to \mathbb{R}$ be continuous in $[a,b]$

Let $x_1\, x_2, ...x_n \in [a,b]$. Show that there exists a point ξ in $[a,b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

As f is continuous in $[a,b]$, there are points $\alpha, \beta \in [a,b]$ such that

$f(\alpha) \le f(x) \le f(\beta)$ for all $x \in [a,b]$

$$\Rightarrow nf(\alpha) \le \sum_{i=1}^{n} f(x_i) \le nf(\beta)$$

$$\Rightarrow f(\alpha) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i) \le f(\beta)$$

By I.V. property of continuous functions, there exists $\xi \in [a,\, b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

(2) Let $f, g: \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$. Show that

$A=\{x \in \mathbb{R} \,|\, f(x) > g(x)\}$, $B=\{x \in \mathbb{R} \,|\, f(x) < g(x)\}$, $C=\{x \in \mathbb{R} \,|\, f(x) \ne g(x)\}$

are open sets in $\mathbb{R}$ whereas $D=\{x \in \mathbb{R} \,|\, f(x) \equiv g(x)\}$ is a closed set in $\mathbb{R}$.

Let $\phi(x)=f(x)-g(x)$, $x \in \mathbb{R}$. As f, g are continuous, so $\phi(x)$ is continuous in $\mathbb{R}$.

(i) $A=\{x \in \mathbb{R} \,|\, \phi(x) > 0\}$

Case I : If $\phi(x) \le 0$ in $\mathbb{R}$. Then $A = \phi$ & So A is open set in $\mathbb{R}$.

Case II : If $\phi(x) > 0$ in $\mathbb{R}$. So $A \equiv \mathbb{R}$ & $\mathbb{R}$ being open set, A is open set in $\mathbb{R}$.

Case III : Let $A \subset \mathbb{R}$.

Let $p \in A$, So $\phi(p) > 0$ & by neighbourhood property of continuous function, there exists $\delta > 0$ such that $x \in \mathbb{N}(p, \delta) \Rightarrow \phi(x) > 0$

Thus $\mathbb{N}(p, \delta) \subset A$ & so p is interior point of A. This is true for all $p \in A$. Consequently A is open set in $\mathbb{R}$.

Arguing in a similar way, B is open set in $\mathbb{R}$.

Set $C = A \bigcup B$ so C is union of two open sets in $\mathbb{R}$ & so C is open set in $\mathbb{R}$. D is the complement of open set C & hence D is closed.

(3) Let $I\,(\subset \mathbb{R})$ be a given open interval. Let $f : I \to \mathbb{R}$ be continuous on I. Let α be an arbitrary real constant.

Then $I[f < \alpha] \equiv \{x \in I : f(x) < \alpha\}$ and $J[f > \alpha] = \{x \in I : f(x) > \alpha\}$ are open sets.

If $f(x) = \alpha$ for all x, I and J are void sets & so are open sets in $\mathbb{R}$.

Next let $I\left[f(x) < \alpha\right] \ne \phi$

So there exists $p \in I$ i.e. $f(p) < \alpha$. Let $0 < \varepsilon < \frac{1}{2}\left[\alpha - f(p)\right]$.

Continuity of f at $p \Rightarrow$ corresponding to above chosen ε, there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $x \in N(p, \delta) \bigcap I$ ...(1)

By hypothesis, I is open set & p is interior point of I. By definition of interior point, there exists $r, 0 < r < \delta$, such that $N(p, r) \subset I$ ...(2)

By (1) & (2), $f(x) < f(p) + \varepsilon < f(p) + \frac{1}{2}(\alpha - f(p))$

$\Rightarrow f(x) < \alpha$ where $x \in N(p, r) \Rightarrow N(p, r) \subset I[f < \alpha]$

$\Rightarrow I[f < \alpha]$ is an open set in $\mathbb{R}$.

Following similar argument, $J[f > \alpha]$ is also open set in $\mathbb{R}$.

(iv) Let $f, g : [0, 1] \to [0, \infty)$ be continuous functions satisfying

$$\sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$$

Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$

Continuity of f, in $[0, 1] \Rightarrow$ boundedness & their attainment of bounds in [0,1].

Let $M = \sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$

If both f & g attain M at the same point, the result is established.

Otherwise : Let $f(\xi) = M$ and $g(\eta) = M$ for some $\xi, \eta \in [0, 1]$,

So $g(\xi) < M$, $f(\eta) < M$.

We construct $h : [0, 1] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. Then h is continuous in [0, 1] & by above $h(\xi) = f(\xi) - g(\xi) = M - g(\xi) > 0$ and

$h(\eta) = f(\eta) - g(\eta) = f(\eta) - M < 0$. So $h(\xi)\,h(\eta) < 0$.

$\Rightarrow$ By Bolzano's theorem, there exists $c \in (\xi, \eta) \subset (0, 1)$ such that $h(c) = 0$

or in other words, $f(c) = g(c)$.

Continuity of Inverse function :

Theorem : Let $f : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous on the closed and bounded interval $[a, b]$. Then there exists an inverse function $g : f[a, b] \to \mathbb{R}$ such that (i) g is strictly monotonic in $f[a, b]$ and (ii) g is continuous in $f[a, b]$

Proof : Let f be strictly increasing in [a, b] ..... (1)

Continuity of f in $[a, b] \Rightarrow$ boundedness of f in $[a, b]$ & attainment of bounds in [a, b]. So $\sup_{[a,b]} f = f(b)$, $\inf_{[a,b]} f = f(a)$.

Therefore, here $f([a, b]) = [f(a), f(b)]$... (1)

As f is strictly increasing, so for any distinct pair of points $x_1, x_2 \in [a, b]$, $f(x_1) \neq f(x_2) < \Rightarrow x_1 \neq x_2$. So f is injective. ... (2)

Consequently by (1) & (2) f is bijective. So $f^{-1} = g$ exists where $g : f([a, b]) \to [a, b]$. where $f(x) = y \Rightarrow x = g(y), x \in [a, b], y \in f[a, b]$

Let $y_1, y_2 \in f[(a,b)]$. So there are $x_1, x_2 \in [a, b]$ such that

$y_1 = f(x_1), y_2 = f(x_2)$

f being strictly increasing in $[a, b], y_1 < y_2 \Rightarrow x_1 < x_2$

As a result, $y_1 < y_2 \Rightarrow g(y_1) < g(y_2) \Rightarrow g$ is strictly increasing in $f([a, b])$.

Let y_0 be any point between $f(a)$ and $f(b)$ & x_0 be the corresponding value of x.

Let $\varepsilon > 0$ be arbitrary number such that $x_0 - \varepsilon$, $x_0 + \varepsilon$ are in $[a, b]$. Let $g(y_0 - \eta_1) = x_0 - \varepsilon$ and $g(y_0 + \eta_2) = x_0 + \varepsilon$ such that $\eta_1, \eta_2 > 0$ exist by above.

Let η be such that $0 < \eta < \min\{\eta_1, \eta_2\}$. Then

$|x - x_0| < \varepsilon$ whenever $|y - y_0| < \eta, \eta$ depends on ε.

So $g(y)$ is continuous at y_0. and this is true for all $y_0 \in [f(a), f(b)]$

Hence the result follows :

Note (i) Continuity of Inverse function is preserved only when the domain is closed and bounded.

Let $A = [0, 1) \cup [2, 3]$ and $f : A \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases} x, & x\in[0,1) \\ x-1, & x\in[2,3] \end{cases}$$

$$f^{-1}(x)=\begin{cases} x, & x\in[0,1) \\ x+1, & x\in[1,2] \end{cases} \Rightarrow f^{-1} \text{ is discontinuous at } x = 1.$$

Theorem : If $f:[a,b]\to\mathbb{R}$ be continuous, injective function, then *f* is strictly monotone function.

If possible, let f be not strictly monotone function in $[a,b]$ though f is continuous & injective in $[a,b]$. So we say that there are three points $p,q,r\in[a,b]$ where $p<q<r$ nonetheless $f(q)$ does not lie between $f(p)$ and $f(r)$. Consequently, either $f(r)$ lies between $f(p)$ and $f(q)$ or $f(p)$ lies between $f(q)$ and $f(r)$. For definiteness, let $f(p)$ be between $f(q)$ and $f(r)$.

By hypothesis, *f* is continuous in $[q,r]\subset[a,b]$. By I. V. property, there exists $s\in(q,r)$ such that $f(s)=f(p)$.

So $p<s$ but $f(p)=f(s)$. This contradicts the injectivity of *f*.

Similarly if we assume that $f(r)$ lies between $f(p)$ and $f(q)$, we would arrive at same type of contradiction. So *f* is strictly monotone.

Corollary : A continuous function $f:[a,b]\to\mathbb{R}$ is injective if and only if *f* is strictly monotone in $[a,b]$.

Example : Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(f(x))=f^2(x)=-x$ for all $x\in\mathbb{R}$.

Then *f* can not be continuous.

First we propose to show that *f* is injective.

$$f(x_1)=f(x_2)\Rightarrow f^2(x_1)=f^2(x_2)\Rightarrow -x_1=-x_2\Rightarrow x_1=x_2$$

If f be continuous then it would be either strictly increasing or strictly decreasing. In both cases, f^2 would be increasing.

For if $p<q$, then $f(p)<f(q)$ (in case f is increasing) & $f(p)>f(q)$ (in case f is decreasing). In the first case, $f(f(p))<f(f(q))$ & in the second case $f(f(p))<f(f(q))$. So in any case, $f^2(p)<f^2(q)$

$\Rightarrow -p<-q$ absurd as $p<q$

So f can not be continuous.

Exercise :

1. Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}2x-1, \text{ if } x\in(0,1)\\ 0, \quad \text{if } x=0 \text{ or } 1\end{cases}$

Choose the correct answer :

(a) f is unbounded function (b) f is bounded function and attains its bounds there in (c) f is bounded function but does not attain its bounds.

2. Let $f(x)=\begin{cases}2^x+1, \text{ for } -1\le x<0\\ 2^x, \text{ for } x=0\\ 2^x-1, \text{ for } 0<x\le 1\end{cases}$

Choose the correct answer :

(a) f is bounded in $[-1,1]$

(b) f is unbounded in $[-1,1]$

(c) f is continuous in $[-1,1]$

(d) f has jump discontinuity in $[-1,1]$

16.6 Uniform Continuity

Recall our $\varepsilon-\delta$ definition of continuity of function. The following example will illustrate that the δ mentioned in the definition depends not only on ε, but on the point also.

$f(x)=x^2$ is continuous on $\mathbb{R}$. Let us consider the continuity of x^2 at $x=0$.

Let $\varepsilon=\frac{1}{9}$. Then $|f(x)-f(0)|<\frac{1}{9}\Rightarrow|x|<\frac{1}{3}$. So we get $\delta=\frac{1}{3}$ such that

$|f(x)-f(0)|<\frac{1}{9}$ wherever $|x-0|<\delta$. Our point is that $\delta=\frac{1}{3}$ is permissible here.

Let us examine whether this δ serves for all points of $\mathbb{R}$.

$f(x)$ is continuous at $x=1$. If the above δ serves for $x=1$ also, we would have $|f(x)-f(1)|<\frac{1}{9}$ whenever $|x-1|<\delta\left(=\frac{1}{3}\right)$.

Note that $x=1.3$ satisfy $|x-1|<\frac{1}{3}$. But then

$$|f(x)-f(1)|=.69 \nless \delta\left(=\frac{1}{3}\right)$$

So the δ, obtained in case of $x=0$, does not serve the purpose for $x=1$

Let us consider another example $f(x)=\frac{1}{x}$ in $(0,1)$

If possible, let there exist $\delta>0$ such that $|f(x)-f(y)|<1$ wherever $|x-y|<\delta,\ x,y\in(0,1)$

Let $x=\frac{\delta}{1+\delta}, y=\frac{\delta}{2(1+\delta)}$ $(\text{both}\in(0,1))$.

Note that for these, $|x-y|=\frac{\delta}{2(1+\delta)}<\delta$.

But $|f(x)-f(y)|=\frac{1+\delta}{\delta}>1$

So the above δ is not applicable here.

Our observation is that the δ, appeared in the $\varepsilon-\delta$ definition for continuity, depends both on ε and the point itself. At this stage, our purpose is to investigate whether there exists $\delta>0$ which depends only on ε so that the δ can serve for all points of D_f.

Let $A=\{\delta(p,\varepsilon): p\in D_f\}$ where each $\delta>0$. This set A is non-void bounded below subset of $\mathbb{R}$ and has inf δ_0(say). Then $\delta_0 \geq 0$.

If $\delta_0>0$, then for any $p\in D_f$, $|x-p|<\delta_0 \Rightarrow |f(x)-f(p)|<\varepsilon$. So this δ_0 serves for all points of D_f. As the δ_0 serves for all points of D_f, then the continuity is known as **uniform continuity** & *f* is said to be uniformly continuous on D_f.

(Note : Uniform Continuity is of global character)

Definition : A function $f: D\rightarrow\mathbb{R}(D\subset\mathbb{R})$ is said to be uniformly continuous on *D*, if given $\varepsilon>0$, there exists $\delta>0$. depending on ε only, such that for any pair of points *x, y* of D satisfying $|x-y|<\delta$, we have $|f(x)-f(y)|<\varepsilon$

Example I : if $f: D\rightarrow\mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on *D*.

As $f: D\rightarrow\mathbb{R}$ is a Lipschitz function, there exists a constant $\lambda>0$ such that

$|f(x)-f(u)|\leq\lambda|x-u|$ for all $x,u\in D$

Let $\varepsilon>0$ be any number. Taking $\delta=\dfrac{\varepsilon}{\lambda}$, we get

$|f(x)-f(u)|<\varepsilon$ for all $x,y\in D$ satisfying $|x-u|<\delta$

$\Rightarrow f$ is uniformly continuous on *D*.

Uniform Continuity in closed and bounded interval $[a,b]$.

Theorem : Let $f:[a,\ b]\rightarrow\mathbb{R}$ be continuous in $[a,\ b]$. Let $\varepsilon>0$ be any number. Then the interval $[a,b]$ can be divided into finite number of sub-intervals in such a way that

$|f(x_2)-f(x_1)|<\varepsilon$ whenever x_1 & x_2 are any two points in the same sub-interval.

Proof : If possible, let the theorem be false in $[a, b] \equiv [a_1, b_1]$.

We bisect $[a_1, b_1]$ of $c_1 = \frac{a_1 + b_1}{2}$. Then the theorem is false in at least one of $[a_1, c_1]$ and $[c_1, b_1]$. We designate that sub-interval as $[a_2, b_2]$ in which the theorem is false. Again we bisect $[a_2, b_2]$ at $c_2 = \frac{a_2 + b_2}{2}$ & let $[a_3, b_3]$ be the sub-interval in which the theorem is false.

Proceeding in this way, we obtain a sequence of nested intervals $\{[a_n, b_n]\}_n$ such that (i) each is contained in the preceeding (ii) $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{b-a}{2^{n-1}} = 0$

Also the theorem is false in each $[a_n, b_n]$.

By Nested interval theorem, $\exists \xi \in [a_n, b_n]$ for all n, ξ is unique and $\lim_{n\to\infty} a_n = \xi = \lim_{n\to\infty} b_n$

I. $a < \xi < b$

By hypothesis, f is continuous at ξ. So given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\xi)| < \frac{\varepsilon}{2}$ wherever $|x - \xi| \le \delta$.

As $\lim_{n\to\infty}(b_n - a_n) = 0$ and $\xi \in [a_n, b_n]$ for all n, so for sufficiently large n, say for $n \ge n_0 (\in \mathbb{N}), [a_n, b_n]$ lies wholly in $[\xi - \delta, \xi + \delta]$

Let x_1, x_2 be any two distinct points in $[a_n, b_n]$ for $n = n_0$

so $|f(x_1) - f(\xi)| < \frac{\varepsilon}{2}$, $|f(x_2) - f(\xi)| < \frac{\varepsilon}{2} \Rightarrow |f(x_2) - f(x_1)| < \varepsilon$

So the theorem is true in $[a_{n_0}, b_{n_0}]$. Thus we arrive at a contradiction.

II. Let $\xi = a$

Arguing as before and noting that for sufficiently large values of n. $[a_n, b_n] = [a, b_n] \subseteq [a, a+\delta]$, we will arrive at a similar type of contradiction.

III. Let $\xi = b$

Here for sufficiently large values of n,

$[a_n, b_n] = [a_n, b] \subseteq [b-\delta, b]$ & arguing as before, we will arrive at a similar type of contradiction.

Hence the theorem follows.

Corollaries (I) Let δ be the least of the lengths of the sub-intervals mentioned above.

Let us consider two points x_1, x_2 of $[a, b]$ such that $|x_1 - x_2| < \delta$. Then two cases may arise :

(i) x_1 and x_2 belong to the same sub interval

(ii) x_1 and x_2 belong to two consecutive sub-intervals.

(i) In this case, by the theorem, $|f(x_2) - f(x_1)| < \varepsilon$ holds.

(ii) Let c be the point which separates the two sub intervals.

Then x_1, c are in one sub interval & c, x_2 are in another same subinterval.

So by the theorem, $|f(x_1) - f(c)| < \varepsilon/2$ and $|f(c) - f(x_2)| < \varepsilon/2$

As a result, $|f(x_2) - f(x_1)| \le |f(x_2) - f(c)| + |f(x_1) - f(c)| < \varepsilon$ holds.

So given $\varepsilon > 0$, there exists $\delta > 0$ such that if

$|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$ holds.

(II) Let η_r, denote any sub-interval of $[a, b]$ such that the length of η_r is less then δ, where $\delta > 0$ is as above. If x_1, x_2 be any two points of η_r then $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

let $M_r = \sup_{\eta_r} f$, $m_r = \inf_{\eta_r} f$. As *f* is continuous in η_r, these bounds are attained in η_r. Let x' and x'' be the two points in η_r where $M_r = f(x'), m_r = f(x'')$

Then by above, $M_r - m_r < \varepsilon$.

So if *f* is continuous in $[a, b]$ & ε be any positive number, there exists $\delta > 0$ such that the oscillation of *f* in every sub-interval of length less than δ, is less than ε.

Theorem : If *f* is continuous in the closed and bounded interval $[a, b]$, then *f* is uniformly continuous in $[a, b]$.

Proof : If possible, let *f* be not uniformly continuous in $[a, b]$. Hence there exists $\varepsilon_0 > 0$ for which there is no $\delta > 0$ with the property that

$|f(x_2) - f(x_1)| < \varepsilon_0$ for all pair of points x_1, x_2 of $[a, b]$ satisfying $|x_1 - x_2| < \delta$. In other words, for all each positive integer n, there is a pair x_n', x_n'' of $[a, b]$ such that $|x_n' - x_n''| < \frac{1}{n}$ nonetheless $|f(x_n') - f(x_n'')| \geq \varepsilon_0$ .... (1)

As $x_n' \in [a, b]$ for all n, $\{x_n' - x_n''\}_n$ is a bounded sequence in $\mathbb{R}$. By Bolzano-Weierstrass theorem on subsequence, there is a subsequence $\left\{x_{k_n}'\right\}_n$ of $\{x_n'\}_n$ which converge to x_0 and $x_0 \in [a, b]$ as $[a, b]$ is closed.

Since $\left|x_{k_n}' - x_{k_n}''\right| < \frac{1}{n}$, we see that $x_{k_n}'' \to x_0$ as $n \to \infty$

$\left(\left\{x_{k_n}''\right\}\text{is subsequence of } \{x_n''\}_n\right)$

Due to continuity of *f*, $f\left(x_{k_n}'\right) \to f(x_0), f\left(x_{k_n}''\right) \to f(x_0)$.

So coresponding to above ε_0, there are natural numbers m_1, m_2 such that

$\left|f\left(x'_{k_n}\right)-f\left(x_0\right)\right|<\frac{\varepsilon}{2}$ for all $n \geq m_1$ & $\left|f\left(x''_{k_n}\right)-f\left(x_0\right)\right|<\varepsilon/2$ for $n \geq m_2$.

Hence for all $n \geq m = \max\{m_1, m_2\}$, both hold & we get $\left|f\left(x'_{k_n}\right)-f\left(x''_{k_n}\right)\right|<\varepsilon_0$ for all $n \geq m$.

This last inequality is in contradiction to (1). Hence f is uniformly continuous in $[a, b]$

Uniform Continuity in open interval (*a*, *b*)

Theorem : Let f be continuous in (a, b), then f is uniformly continuous in (a, b) if and only if $\lim\limits_{x\to a+0} f(x)$ and $\lim\limits_{x\to b-0} f(x)$ both exist finitely.

Proof : Let f be continuous in the bounded open interval (a, b) and $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ both exist finitely.

We construct $g:[a, b] \to \mathbb{R}$ as follows :

$$g(x)=f(x) \text{ for all } x\in(a, b)$$

$$g(a)=\lim_{x\to a+} f(x) \text{ and } g(b)=\lim_{x\to b-} f(x)$$

Then $\lim\limits_{x\to a+} g(x)=\lim\limits_{x\to a+} f(x)=g(a)$ and $\lim\limits_{x\to b-0} g(x)=\lim\limits_{x\to b-} f(x)=g(b)$

Along with this, considering the continuity of f in (a, b), we say that g is continuous in $[a,b]$ & so g is uniformly continuous in $[a,b]$ & (a,b). But g and f are identical in (a, b). So f is uniformly continuous in (a, b).

Converse : Let f be uniformly continuous in open interval (a, b).

We propose to show that both $\lim\limits_{x\to a+} f(x)$ & $\lim\limits_{x\to b-} f(x)$ exist finitely.

If possible, suppose that $\lim\limits_{x\to a+} f(x)$ does not exist. Then there is a sequence

$\{x_n\}_n$ in (a,b) with $x_n \to a$ such that the sequence $\{f(x_n)\}_n$ does not converge & hence is not a cauchy sequence in $\mathbb{R}$. Then there exists some $\varepsilon_0(>0)$ with the property that there is no natural number n_0 for which

$$i,j \geq n_0 \Rightarrow \left|f(x_i)-f(x_j)\right| < \varepsilon_0 .$$

Consequently we can find arbitrary large $i, j \in \mathbb{N}$ for which $\left|f(x_i)-f(x_j)\right| \geq \varepsilon_0$. Now since the sequence $\{x_n\}_n$ is a cauchy sequence in $\mathbb{R}$, we have $\lim\limits_{i,\, j\to\infty} \left|x_i - x_j\right| = 0$. Clearly for this ε_0, we can find a pair of points $x_i, x_j \in (a, b)$ which are arbitrarily close and for which $\left|f(x_i)-f(x_j)\right| \geq \varepsilon_0$. This implies that f is not uniformly continuous in (a,b).

A similar argument can be in the case when $\lim\limits_{x\to a+} f(x)$ exists but $\lim\limits_{x\to b-} f(x)$ fails to exist.

Illustration : Let $f(x)=\dfrac{1}{x}$ in $(0, 1)$. $\lim\limits_{x\to 0+} f(x)$ does not exist finitely & so f is not uniformly continuous in $(0, 1)$.

An imporant non-uniform continuous criteria

A function $f: D \to \mathbb{R}(D \subset \mathbb{R})$ is not uniformly continuous on D if and only if there exist sequences $\{x_n\}_n$ and $\{t_n\}_n$ in D such that

(i) $|x_n - t_n| \to 0$ (ii) $|f(x_n) - f(t_n)| \not\to 0$

Examples : (a) $\dfrac{1}{x}$ is not uniformly continuous in $(0, 1)$

choose the sequences $x_n = \dfrac{1}{n}$ and $t_n = \dfrac{1}{2n}, n \in \mathbb{N}$

(b) x^2 is not uniformly continuous on $\mathbb{R}$

choose the sequences $x_n = n,\ t_n = n + \frac{1}{n},\ n \in \mathbb{N}$

(c) $\sin\frac{1}{x}$ is not uniformly continuous in $(0, \infty)$

choose the sequences $x_n = \frac{1}{2n\pi}, t_n = \frac{2}{(4n+1)\pi}, n \in \mathbb{N}$

(d) $\sin x^2$ is not uniformly continuous on $\mathbb{R}$.

choose the sequences $x_n = \sqrt{\frac{\pi}{2}(n+1)}$ & $t_n = \sqrt{\frac{\pi}{2}n}$, $n \in \mathbb{N}$

(e) $e^{1/x}$ is not uniformly continuous on $(0, 1)$.

Choose $x_n = \frac{1}{\ln n},\ t_n = \frac{1}{\ln(n+1)},\ n \in \mathbb{N} - \{1\}$

(f) $x \sin x$ is not uniformly continuous on $(0, \infty)$

choose $x_n = 2n\pi,\ t_n = 2n\pi + \frac{1}{n},\ n \in \mathbb{N}$

Examples :

(1) If $f, g : D \to \mathbb{R}(D \subset \mathbb{R})$ be both uniformly continuous on D & D be bounded, then fg is uniformly continuous on D.

To solve this we will use the following result (which is being stated here without proof).

If $f : D \to \mathbb{R}(D \subset \mathbb{R})$ be uniformly continuous on a bounded set D, then f is bounded on D.

f, g are bounded on D. So there exists $\lambda \in \mathbb{R}^+$ such that $|f(x)| < \lambda, |g(x)| < \lambda$ for all $x \in D$.

Let $x, y \in D$ then

$$|f(x)g(x)-f(y)g(y)| \le |f(x)||g(x)-g(y)|+|g(y)||f(x)-f(y)| \text{ ..... (1)}$$

Let $\varepsilon > 0$ any number. As *f, g* are uniformly continuous on *D*, corresponding to above ε, there exists $\delta_1 > 0, \delta_2 > 0$, both depend on ε only, such that for any pair of points *x, y* of *D* satisfying $|x-y| < \delta_1$, we have $|f(x)-f(y)| < \frac{\varepsilon}{2\lambda}$ and

$$|x-y| < \delta_2 \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2\lambda} \text{ ....(2)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Recalling (1) & (2)

$|f(x)g(x)-f(y)g(y)| < \lambda.\frac{\varepsilon}{2\lambda} + \lambda.\frac{\varepsilon}{2\lambda}$ for any pair of points *x, y* of *D* satisfying $|x-y| < \delta$

$\Rightarrow fg$ is uniformly continuous on D.

Note the result fails if *D* be not bounded. This is evident from the example x^2 on $\mathbb{R}$.

(2) Every uniformly continuous function maps a cauchy sequence onto a cauchy sequence.

Let $\{x_n\}_n$ be a cauchy sequence in $\mathbb{R}$.

Let $\varepsilon > 0$ be any number. Since f is uniformly continuous on D, corresponding to above ε, there exists $\delta > 0$ (δ depends only on ε) such that for any pair of points *x, y* of *D* that satisfy $|x-y| < \delta$, we have $|f(x)-f(y)| < \varepsilon$ ......(1)

Since $\{x_n\}_n$ is a cauchy sequence in $\mathbb{R}$, corresponding to above δ, there exists $m \in \mathbb{N}$ such that $|x_{n+p} - x_n| < \delta$ for all $n \ge m, p \in \mathbb{N}$ ....(2)

By (1) and (2), $|f(x_{n+p}) - f(x_n)| < \varepsilon$ for all $n \ge m, p \in \mathbb{N}$

$\Rightarrow \{f(x_n)_n\}$ is a cauchy sequence.

Note : The result fails if *f* be only continuous on *D*.

Consider $f(x)=\frac{1}{x}$ in $(0,1)$ and $x_n=\frac{1}{n}(n\in\mathbb{N})$.

Here $f(x_n)=n$ and $\{f(x_n)\}_n$ is not cauchy sequence.

16.7 Summary

In this unit, we have defined the terms continuity and discontinuity and given various examples. We have studied various types of discontinuities and their properties. We have explained the most important properties of functions continuous in a closed and bounded interval [a, b], such as, Intermediate value property, Fixed point property. We have also shown the relation between continuity and monotonicity. We have further study the maximum-minimum property. We have introduced the notion of uniform continuity and shown that in a closed and bounded interval [a, b] this concept is same with the concept of continuity. We also studied the uniform continuity on an open interval (a, b), and give an important non-uniform continuity criteria. We have also shown that every uniformly continuous function maps cauchy sequence into a cauchy sequence.

16.8 Exercise

1. Prove or disprove : If $f:S\to\mathbb{R},\ g:T\to\mathbb{R}\,(S,T\subset\mathbb{R})$ are uniformly continuous and $f(S)\subset T$, then the composite function $g\circ f:S\to\mathbb{R}$ is uniformly continuous on S.

2. Show that $e^x\cos\frac{1}{x}$ is not uniformly continuous on $(0,1)$.

(Hints : You can consider the sequences $\left\{\frac{1}{2n\pi}\right\}_n$ & $\left\{\frac{1}{(2n+1)\pi}\right\}_n$)

3. Let $f(x)=\sqrt{x},\ x\in[0,2]$

Choose the correct answer(s) :

(i) f is Lipschitz function in $[0,2]$

(ii) f is not Lipschitz function in $[0,2]$

(iii) f is uniformly continuous in $[0, 2]$

(iv) f is not uniformly continuous in $[0, 2]$

4. Correct or justify : $x \sin^2 x$ is uniformly continuous on $\mathbb{R}$.

5. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \cos \dfrac{\pi}{2x}, & x \neq 0 \\ 0, x = 0 \end{cases}$

Examine whether f is uniformly continuous on $[0, 1]$.

6. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let the equation $f(x) = 0$ have finite number of roots in $[a, b]$. Arrange them in the ascending order.

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots. < x_n < b$

Prove that in each of the intervals $(a_1, x_1), (x_1, x_2), (x_{r-1}, x_r) (x_n, b)$ the function *f(x)* retains the same sign.