PREFACE

With its grounding in the "guiding pillars of Access, Equity, Equality, Affordability and Accountability," the New Education Policy (NEP 2020) envisions flexible curricular structures and creative combinations for studies across disciplines. Accordingly, the UGC has revised the CBCS with a new Curriculum and Credit Framework for Undergraduate Programmes (CCFUP) to further empower the flexible choice based credit system with a multidisciplinary approach and multiple/ lateral entry-exit options. It is held that this entire exercise shall leverage the potential of higher education in three-fold ways – learner's personal enlightenment; her/his constructive public engagement; productive social contribution. Cumulatively therefore, all academic endeavours taken up under the NEP 2020 framework are aimed at synergising individual attainments towards the enhancement of our national goals.

In this epochal moment of a paradigmatic transformation in the higher education scenario, the role of an Open University is crucial, not just in terms of improving the Gross Enrolment Ratio (GER) but also in upholding the qualitative parameters. It is time to acknowledge that the implementation of the National Higher Education Qualifications Framework (NHEQF), National Credit Framework (NCrF) and its syncing with the National Skills Qualification Framework (NSQF) are best optimised in the arena of Open and Distance Learning that is truly seamless in its horizons. As one of the largest Open Universities in Eastern India that has been accredited with 'A' grade by NAAC in 2021, has ranked second among Open Universities in the NIRF in 2024, and attained the much required UGC 12B status, Netaji Subhas Open University is committed to both quantity and quality in its mission to spread higher education. It was therefore imperative upon us to embrace NEP 2020, bring in dynamic revisions to our Undergraduate syllabi, and formulate these Self Learning Materials anew. Our new offering is synchronised with the CCFUP in integrating domain specific knowledge with multidisciplinary fields, honing of skills that are relevant to each domain, enhancement of abilities, and of course deep-diving into Indian Knowledge Systems.

Self Learning Materials (SLM's) are the mainstay of Student Support Services (SSS) of an Open University. It is with a futuristic thought that we now offer our learners the choice of print or e-slm's. From our mandate of offering quality higher education in the mother tongue, and from the logistic viewpoint of balancing scholastic needs, we strive to bring out learning materials in Bengali and English. All our faculty members are constantly engaged in this academic exercise that combines subject specific academic research with educational pedagogy. We are privileged in that the expertise of academics across institutions on a national level also comes together to augment our own faculty strength in developing these learning materials. We look forward to proactive feedback from all stakeholders whose participatory zeal in the teaching-learning process based on these study materials will enable us to only get better. On the whole it has been a very challenging task, and I congratulate everyone in the preparation of these SLM's.

I wish the venture all success.

Professor Indrajit Lahiri
Vice Chancellor

Netaji Subhas Open University

Four Year Undergraduate Degree Programme

Under National Higher Education Qualifications Framework (NHEQF) &

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: Course Writer :

Dr. Biswajit Saha
Assistant Professor
Department of Mathematics
Govt. General Degree College, Muragachha

: Course Editor :

Prof. Sanjib Kumar Datta
Professor
Department of Mathematics
University of Kalyani

: Format Editor :

Dr. Nemai Chand Dawn
Associate Professor of Mathematics,
NSOU

Notification

Netaji Subhas Open University

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Unit-1

THE RIEMANN INTEGRAL

Unit - 1 : The Riemann Integral

Structure

1.0 Objectives

In this chapter we shall learn about some Definitions and Properties of Riemann Integration. Besides we shall discuss necessary and sufficient condition of a function to be Riemann integrable, some Important Inequalities and Fundamantal theormes of Calculus.

1.1 Introduction

A german mathematician Bernhard Riemann (1826-1866) introduced the concept of definite integral from the notion of limit of a sum of which term tends to zero when the numner of terms tending to ∞. Literal meaning of integration is "summation". It can be also considered as the inverse process of differentiation. That is, if $f:[a,b]\to\mathbb{R}$ be a function and of there exists a function $\varnothing$ is called the integral of *f*.

Riemann had defined the integrability of a real valued bounded function on a closed interval $[a, b] \subseteq \mathbb{R}$, using the limit of a sum. Further works have been done in this field. Among these, the theory of integration by Lebesgue (1902) is most noteworthy.

1.1.1. Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ *be a bounded function on* $[a, b]$ *and let M, m be the bounds of f on* $[a, b]$.

Let $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots < x_{n-1} < x_n = b\}$. *be a partition of* $[a, b]$, *where* $x_r, r = 0, \ldots, n$ *are called the points of division of* $[a, b]$, $[x_{r-1}, x_r]$ *is called the rth subinterval of* $[a, b]$. $\delta_r = x_r - x_{r-1}$ *is the length of the r-th subinterval so that*

$$\sum_{r=1}^{n} \delta_r = b - a$$

Let M_r m_r *be the bounds of f in the rth subinterval of* $[x_{r-1}, x_r]$.

Then $O_r = M_r - m_r$ *is called the oscillation of f in* $[x_{r-1}, x_r]$.

Then norm of the partition P, denoted by $||P||$, *is defined by the length of the greatest of all subintervals* $[x_{r-1}, x_r]$. *That is,* $||P|| = \max \{x_1 - x_0, x_2 - x_1, \ldots, x_r - x_{r-1}, x_n - x_{n-1}\}$

The upper and lower sums, denoted by U (P, f) and L (P, f) respectively, are defined by

$$U(P, f) = \sum_{r=1}^{n} M_r \delta r$$

$$L(P, f) = \sum_{r=1}^{n} m_r \delta r$$

Definition 1.1.2 (Refinement of a Partition of $[a, b]$).

A partition P is said to be refinement of a partition*

$$P\{a = x_0 < x_1 < x_2 < ... < x_{r-1} < x_r < ... x_{n-1} < .. x_n = b\} \text{ of } [a, b] \text{ if } P^* \supset P.$$

$x_0 < x_1 < x_2 < .... < x_{r-1} < x_r < ... < x_{n-1} < x_n$

That is, P forms a new partition of* $[a, b]$ containing the points of P as well as some more points of division of $[a, b]$.

Theorem 1.1.1.

If P is a refinement of P* $\{a = x_0 < x_1 < x_2 < ... < x_{r-1} < x_r < ... < x_{n-1} < x_n = b\}$ *then for a bounded function f on* $[a, b]$,

(i) $U(P^*, f) \le U(P, f)$ *and (ii)* $L(P^*, f) \ge L(P, f)$.

*Proof. Let P** contains just one point ξ more than *P* and let $x_{r-1} < \xi < x_r$. Then $P^* = \{a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_{r-1} < \xi < x_r < ... < x_{n-1} < x_n = b\}$. Let M_r, M_r' and M_r'' be the upper bounds of *f* in $[x_{r-1}, x_r]$. $[x_{r-1}, \xi]$ and $[\xi, x_r]$ respectively.

Then $U(P^*, f) - U(P, f)$

$$= \sum_{k=1}^{r-1} M_k (x_k - x_{k-1}) + M_r' (\xi - x_{r-1}) + M_r'' (x_r - \xi) + \sum_{k=r}^{n-1} M_k (x_k + 1 - x_k)$$

$$-\left\{-\sum_{k=1}^{r-1} M_k (x_k - x_{k-1}) + M_r (x_r - x_{r-1}) + \sum_{k=r}^{n-1} M_k (x_{k+1} - x_k)\right\}$$

$$= (M_r' - M_r)(\xi - x_{r-1}) + (M_r'' - M_r)(x_r - \xi)$$

$$\le 0 [\because M_r', M_r'' \le M_r] \Rightarrow U(P^*, f) \le U(P, f)$$

In the same way, we can show that $L(P^*, f) \geq L(P, f)$ □

Note 1.1.1. *By the refinement of a partition P* of $[a,b]$, *the upper sums decreases and the lower sums increases.*

Theorem : 1.1.2. *Let f be bounded function on $[a,b]$ and let M and , m be supremum and infimum of f on $[a,b]$. Then*

$m\ (b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$.

Proof. Let M_r and m_r be the supremum and infimum of *f* in $[x_{r-1}, x_r]$.

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r$$

and

$$L(P, f) = \sum_{r=1}^{n} M_r \delta_r$$

We have

$$m \leq m_r \leq M_r \leq M \quad \forall r = 1, 2, \ldots, n$$

$$\Rightarrow m\delta_r \leq m_r \delta_r \leq M_r \delta_r \leq M \delta_r$$

$$\Rightarrow m \sum_{r=1}^{n} \delta_r \leq \sum_{r=1}^{n} m_r \delta_r \leq \sum_{r=1}^{n} M_r \delta_r \leq M \sum_{r=1}^{n} \delta_r$$

$$\Rightarrow m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

which shows that each of lower sum and upper sum is bounded and also for each partition *P, L(P,f)* is less than or equal to *U(P,f).* □

Definition 1.1.3 (Riemann Integrability)

Let $f : [a, b] \to \mathbb{R}$, $[a, b] \subseteq \mathbb{R}$ *be a bounded function on* $[a,b]$. *Then we have shown that* $m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$. *Further we have observed that for a refinement P* of P of* $[a,b]$, $U(P^*, f) \leq U(P, f)$ *and*

$L(P^*, f) \geq L(P, f)$

Thus for all possible partitions P of $[a,b]$ *we have*

$$\left.\begin{array}{l} m(b-a) \leq \ldots U(P^2, f) \leq U(P^1, f) \leq U(P, f) \\ \text{and } L(P, f) \leq L(P^1, f) \leq L(P^2, f) \ldots \leq M(b-a) \end{array}\right\} \quad (1.1.1)$$

where $P^1, P^2, \ldots$ *are the refinement of P. Hence it follows that the set*

$U = \{U(P, f), P \text{ is any partition of } [a, b]\}$ *of upper sums bounded below by m (b – a) has an infimum and the set* $L = \{L(P, f), P \text{ is any partition of } [a, b]\}$ *of lower sums bounded above by M (b – a) has a supremum followed by 1.1.1.*

The infimum of $U = \{U(P, f), P \text{ is any partition of } [a, b]\}$ *is known as the lower Riemann integral of f on* $[a,b]$ *and is denoted by* $J = \overline{\int_a^b} f(x)dx$.

The supremum of $L = \{L(P, f), P \text{ is any partition of } [a, b]\}$ *is known as the upper Riemann integral of f on* $[a,b]$ *and is denoted by* $I = \underline{\int_a^b} f(x)dx$.

Now f is said to be Riemann integrable on $[a,b]$ *if* $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)\, dx$ *and is denoted by* $R\int_a^b f(x)dx$ *or* $\int \in \mathbb{R}[a, b]$.

Example 1.1.1.

Given $f(x) = \begin{cases} 0, \text{ when } x \text{ is irrational} \\ 1, \text{ when } x \text{ is rational,} \end{cases}$

proven from definition that $\notin \text{IR}[G, b]$ *for any* $a<b$.

Solution :

The function $f(x)$ is bounded on , the least upper bound being 1 and the greatest lower bound 0. In each sub-interval δ_r for all partitions P, the upper and lower bounds will respectively be 1 and 0. Hence

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n} 1.\delta_r=b-a$$

and

$$L(P,f)=\sum_{r=1}^{n} m_r\delta_r=\sum_{r=1}^{n} 0.\delta_r=0.$$

Thus, $J=\inf U(P,f)=b-a$ and $I=\sup L(U,f)=0$

whereby $I\neq J$ unless $b=a$.

Hence $f(x)$ is not integrable on $[a,b]$, where $a<b$.

Note : 1.1.2.

f is bounded but not R-integrable. So every bounded function is not R-integrable.

Example : 1.1.2.

Let $f:[0,a]\to R, a>0$ *be defined by* $f(x)=x^3\ \forall x\in[0,a]$. *Prove that f is R-interable and* $R\int_0^1 f(x)dx=\dfrac{a^4}{4}$.

Solution :

f is bounded $\forall\ x\in[0,a]$.

Let $P=\left\{0,\dfrac{a}{n},\dfrac{2a}{n},\ldots,\dfrac{(n-1)}{n},\dfrac{na}{n}=a\right\}$ be a partition of (o,a).

Then

$$U(P,f)=\sum_{r=1}^{n} M_r\delta_r$$

$$=M_1\left(\frac{a}{n}-0\right)+M_2\left(\frac{2a}{n}-\frac{a}{n}\right)+\ldots+M_n\left(a-\frac{(n-1)a}{n}\right)$$

$$=\left(\frac{a}{n}\right)^3.\frac{a}{n}+\left(\frac{2a}{n}\right)^3.\frac{a}{n}+\ldots+\left(\frac{na}{n}\right)^3.\frac{a}{n}$$

$$= \frac{a^4}{n^4}\left[1^3 + 2^3 + \ldots + n^3\right] = \frac{a^4}{n^4}\left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \frac{a^4}{4}\left\{1 + \frac{2}{n} + \frac{1}{n^2}\right\}$$

$$\therefore \overline{\int_0^a} f(x)dx = \inf U = \lim_{n\to\infty}\left\{\frac{a^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\right\} = \frac{a^4}{4}$$

Now

$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= m_1\left(\frac{a}{n} - 0\right) + m_2\left(\frac{2a}{n} - \frac{a}{n}\right) + \ldots + m_n\left(a - \frac{(n-1)a}{n}\right)$$

$$= 0..\frac{a}{n} + \left(\frac{a}{n}\right)^3.\frac{a}{n} + \left(\frac{2a}{n}\right)^3.\frac{a}{n} + \ldots + \left(\frac{(n-1)a}{n}\right)^3.\frac{a}{n}$$

$$= \frac{a^4}{n^4}\left[1^3 + 2^3 + \ldots + (n-1)^3\right] = \frac{a^4}{n^4}\left\{\frac{n(n-1)}{2}\right\}^2$$

$$= \frac{a^4}{4}\left\{1 - \frac{2}{n} + \frac{1}{n^2}\right\}$$

$$\therefore \int_{\underline{0}}^a f(x)dx = \inf L = \lim_{n\to\infty}\left\{\frac{a^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)\right\} = \frac{a^4}{4}$$

$$\text{So } \overline{\int_0^a} f(x)dx = \int_{\underline{0}}^a f(x)dx = \frac{a^4}{4}$$

Hence f is R-integrable and

$$R\int_0^a f(x)dx = \frac{a^4}{4}.$$

Example : 1.1.3.

Let $f : [a,b] \to R$, *be defined by* $f(x) = e^x$. *Prove that f is R-interable on* $[a,b]$.

Solution :

f is bounded $[a,b]$.

Let $P=(a, a+h, a+2h, \ldots, a+nh)$ where $nh=b-a$. Then P is a Partition of $[a,b]$ diving $[a,b]$ into n Subintervals of equal length. $\|P\|=\frac{b-a}{n}$.

Let

$$M_r=\sup_{x\in[a+(r-1)h,\,a+rh]} f(x),$$

$$m_r=\inf_{x\in[a+(r-1)h,\,a+rh]} f(x),$$

for $r=1,2,\ldots n$

Then $M_r=e^{a+rh}, m_r=e^{a+(r-1)h}$, for $r=1,2,\ldots,n$. Then

$$U(P,f)=h\left[e^{a+h}+e^{a+2h}+\ldots+e^{a+nh}\right]$$

$$=he^{a+h}\left[\frac{e^{nh}-1}{e^h-1}\right]=he^{a+h}\left[\frac{e^{b-a}-1}{e^h-1}\right]$$

$$=\frac{he^h}{e^h-1}\left[e^b-e^a\right];$$

$$\therefore \overline{\int_a^b} f(x)dx=\inf U=\lim_{n\to\infty}U(P,f)=\lim_{h\to 0}\frac{he^h}{e^h-1}\left[e^b-e^a\right]=e^b-e^a$$

$$L(P,f)=h\left[e^a+e^{a+h}+\ldots+e^{a+(n-1)h}\right]$$

$$=he^a\left[\frac{e^{nh}-1}{e^h-1}\right]=he^a\left[\frac{e^{b-a}-1}{e^h-1}\right]$$

$$=\frac{h}{e^h-1}\left[e^b-e^a\right].$$

$$\therefore \underline{\int_a^b} f(x)dx=\sup L=\lim_{n\to\infty}L(P,f)=\lim_{h\to 0}\frac{h}{e^h-1}\left[e^b-e^a\right]=e^b-e^a$$

As $\overline{\int_a^b} f(x)dx=\underline{\int_a^b} f(x)dx$, f is integrable on $[a,b]$ and $\int_a^b f(x)dx=e^b-e^a$.

Theorem : 1.1.3. [*Darboux's theorem*]

For every , there exists a positive number δ *such that* $U(P, f) < \overline{\int_a^b} f(x)dx + \varepsilon$ *and* $L(P, f) > \underline{\int_a^b} f(x)dx - \varepsilon \ \forall P$ *with* $\|P\| \le \delta$.

Proof. Let $U = \{U(P_1, f), P_1 \text{ is a partition of } [a,b]\}$.

$\inf U = \overline{\int_a^b} f(x)\, dx$. So by property of inf U,

$$U(P_1, f) < \overline{\int_a^b} f(x)\, dx + \frac{\varepsilon}{2} \qquad (1.1.2)$$

Here f is bounded. So $|f(x)| \le k \ \forall x \in [a,b]$.

Let $P\{a = x_0 < x_1 < x_2 < ... < x_p = b\}$ be a partition of $[a,b]$ containing $(p-1)$ points in (a,b) such that $\|P\| \le \delta_1, \delta_1$ is a positive number.

Let P_2 is a refinement of P_1 so that $P_2 \supset P_1 \Rightarrow U(P_2, f) \le U(P_1, f)$ by Theorem 1.1.1.

We first suppose that P_2 contains one more point than that of P_1, so one of the subintervals, say, $[x_{r-1}, x_r]$ of P_1 is divided into two subintervals $[x_{r-1}, \xi]$ and $[\xi, x_r]$ respectively such that $x_{r-1} < \xi < x_r, r \le p$.

Let M_r, M_r', M_r'' be the sup of f in $[x_{r-1}, x_r]$, $[x_{r-1}, \xi]$ and $[\xi, x_r]$ respectively such that $\delta_r = \delta_r' + \delta_r''$ where $\delta_r' = \xi - x_{r-1}$ and $\delta_r'' = x_r - \xi$.

Then

$$U(P_1, f) - U(P_2, f) = M_r(x_r - x_{r-1}) - M_r'(\xi - x_{r-1}) - M_r''(x_r - \xi)$$

$$= (M_r - M_r')\delta_r' + (M_r - M_r'')\delta_r'' \qquad (1.1.3)$$

Also

$$|f(x)| \le k \Rightarrow -k \le f(x) \le k$$

$$\Rightarrow -k \le M_r' \le M_r \le k \Rightarrow 0 \le M_r - M_r' \le 2k$$

$$\text{and} -k \le M_r'' \le M_r \le k \Rightarrow 0 \le M_r - M_r' - M_r \le 2k$$

Therefore, from (1.1.3) follows that

$$U(P_1,f)-U(P_2,f)\le 2k\delta_r\left[\because \delta'_r,\delta''_r\le\delta_r\right]$$

$$\le 2k\delta_1\left[\because \|P_1\|\le\delta_1\right] \qquad (1.1.4)$$

Now if P_2 contains at most $(P-1)$ more points that of P_1 then from (1.1.4) it follows that

$$U(P_1f)-U(P_2,f)\le 2k(p-1)\delta_1$$

We choose $\delta_1>0$ such that $\delta_1<\frac{\varepsilon}{4k(p-1)}$ and let P be a division of $[a,b]$ such that $\|P\|\le\delta$ and let $P_2=P_1\cup P$. The $P_2\supset P$. Thus P_2 is a refinement of P, containing at most $(p-1)$ more points than P. Therefore, from (1.1.4)

$$U(P,f)-2k(p-1)\delta\le U(P_2,f)\le U(P_1,f)$$

$$\Rightarrow U(P,f)\le 2k(p-1)\delta+U(P_1,f)$$

$$<2k(p-1)\frac{\varepsilon}{4k(p-1)}+\int_a^{\bar{b}}f(x)dx+\frac{\varepsilon}{2}$$

$$=\int_a^{\bar{b}}f(x)dx+\varepsilon$$

$$\Rightarrow U(P,f)<\int_a^{\bar{b}}f(x)dx+\varepsilon$$

Thus the result (1.1.2) is true $\forall P$ with $\|P\|\le\delta$. The proof of $L(P,f)>\int_{\underline{a}}^{b}f(x)dx-\varepsilon$ is similar to that of the first part. □

Theorem : 1.1.4.

If a function $f:(a,b)\to\mathbb{R}$ be bounded on $[a,b]$, then for any partition $P\{a=x_0<x_1<x_2...<x_n=b\}$ of $[a,b]$ with norm $\|P\|\le\delta,\delta>0$, then

(i) $\lim_{\|P\|\to 0}U(P,f)=\lim_{\|P\|\to 0}\sum_{r=1}^{n}M_r\delta_r=\int_a^{\bar{b}}f(x)dx$.

and

(ii) $\lim\limits_{\|P\|\to 0} U(P,f) = \lim\limits_{\|P\|\to 0} \sum\limits_{r=1}^{n} m_r\delta_r = \int_{\underline{a}}^{b} f(x)\,dx$.

Proof. Let $S_1 = \{U(P,f), \text{P is a partition of } [a,b]\}$.

Then $\overline{\int_a^b} f(x)\,dx$ *being an infimum of* S_1, $U(P,f) \geq \overline{\int_a^b} f(x)\,dx$.

Now from Darboux Theorem 1.1.3

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon \Rightarrow U(P,f) - \overline{\int_a^b} f(x)\,dx < \varepsilon$$

$$\Rightarrow \left| U(P,f) - \overline{\int_a^b} f(x)\,dx \right| < \varepsilon \left[\because U(P,f) - \overline{\int_a^b} f(x)\,dx \geq 0 \right]$$

$$\Rightarrow \lim_{\|P\|\to 0} U(P,f) = \overline{\int_a^b} f(x)\,dx$$

Again, let $S_2 = \{L(P,f), \text{P is a partition of } [a,b]\}$

Then $\int_{\underline{a}}^{b} f(x)\,dx$ being the suprimum of $S_2, L(P,f) \leq \int_{\underline{a}}^{b} f(x)\,dx$.

Now, by Darboux Theorem 1.1.3

$$L(P,f) > \int_{\underline{a}}^{b} f(x)\,dx - \varepsilon \Rightarrow \int_{\underline{a}}^{b} f(x)\,dx - L(P,f) < \varepsilon$$

$$\Rightarrow \left| \int_{\underline{a}}^{b} f(x)\,dx - L(P,f) \right| < \varepsilon \left[\because \int_{\underline{a}}^{b} f(x)\,dx - L(P,f) \geq 0 \right]$$

$$\Rightarrow \lim_{x\to\infty} L(P,f) = \int_{\underline{a}}^{b} f(x)\,dx$$

Another definition based on the notion of the limit of a sum

Definition 1.1.4.

A function Let f Let $: [a,b] \to \mathbb{R}$ *is said to be R-integrable, if f for every partition* $P\{a = x_0 < x_1 \ldots < x_{r-1} < x_r < \ldots < x_{n-1} < x_n = b\}$ *of* $[a,b]$ *and every choice of* ξ_r *in* $[x_{r-1}, x_r]$ *such that*

$$\lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta r$$

exists, then the limit is called the integral of f on $[a,b]$ *and is denoted by* $R\int_a^b f(x)\,dx$. *That is*

$$R\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r$$

1.2 Equivalence of the two Definitions of a Definite Integral

Theorem 1.2.1.

Let a function $f:[a,b]\to\mathbb{R}$ *be bounded on* $[a,b]$. *Then f is R-integrable on* $[a,b]$ *if and only if, for every* $\varepsilon(>0)$, *there exists a positive number* δ *with* $\|P\|\le\delta$, *where* $P\{a=x_0<x_1<x_2<\ldots<x_n=b\}$ *is a partition of* $[a,b]$, *and every choice of* ξ_r *in* $[x_{r-1},x_r]$, *such that* $\left|\sum_{r=1}^{n} f(\xi_r)\delta_r - \int_a^b f(x)\,dx\right| < \varepsilon$.

Proof. Let f be R-integrable on $[a,b]$. Then

$$\int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx \qquad (1.2.1)$$

From Darboux **Theorem 1.1.3**

$$\left.\begin{aligned} U(P,f) &< \int_a^{\bar{b}} f(x)\,dx + \varepsilon = \int_a^b f(x)\,dx + \varepsilon \text{ with } \|P\| \le \delta \\ L(P,f) &> \int_{\underline{a}}^{b} f(x)\,dx - \varepsilon = \int_a^b f(x)\,dx - \varepsilon \text{ with } \|P\| \le \delta \end{aligned}\right\} \qquad (1.2.2)$$

We have $m_r \le f(\xi_r) \le M_r \ \forall \xi_r \in [x_{r-1}, x_r]$

$$\Rightarrow \sum_{r=1}^{n} m_r \delta r \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le \sum_{r=1}^{n} M_r \delta_r$$

$$\Rightarrow L(P,f) \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le U(P,f)$$

Therefore by (1.2.2)

$$\Rightarrow \int_a^b f(x)dx - \varepsilon < L(P,f) \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le U(P,f) < \int_a^b f(x)dx + \varepsilon$$

$$\Rightarrow \int_a^b f(x)dx - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\delta_r < \int_a^b f(x)dx + \varepsilon$$

$$\Rightarrow \left| \sum_{r=1}^{n} f(\xi_r)\delta_r - \int_a^b f(x)dx \right| < \varepsilon \left[\text{with} \|P\| \le \delta \right]$$

$$\Rightarrow \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \int_a^b f(x)dx$$

Conversely, let

$$\left| \sum_{r=1}^{n} f(\xi_r)\delta_r - \int_a^b f(x)dx \right| < \varepsilon \left[\text{ with} \|P\| \le \delta \right]$$

Then

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \int_a^b f(x)dx$$

holds for all values of $f(\xi_r)$

If, in particular, f attains M_r at $\xi_r \in [x_{r-1}, x_r]$ i.e., if $f(\xi_r) = M_r$,

then

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} M_r\delta_r = \int_a^{\bar{b}} f(x)dx$$

By Theorem 1.1.4

$$\Rightarrow \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx$$

$$\Rightarrow \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx \qquad (1.2.3)$$

If, f attains m_r at $\xi_r \in [x_{r-1}, x_r]$, i.e. if $f(\xi_r) = m_r$ then

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} m_r \delta_r = \int_{-a}^{b} f(x)\, dx$$

By Theorem 1.1.4

$$\Rightarrow \int_a^b f(x) dx = \int_a^b f(x) dx \qquad (1.2.4)$$

Thus from (1.2.3) and (1.2.4),

$$\int_a^{\bar{b}} f(x) dx = \int_{\underline{a}}^{b} f(x) dx = \int_a^b f(x) dx$$

Hence f is R-integrable on $[a, b]$. □

1.3 Necessary and sufficient conditions for Integrability

Theorem 1.3.1.

The necessary and sufficient condition is that a bounded function $f : [a, b] \to \mathbb{R}$ *to be R-integrable on* $[a, b]$ *is for every* $\varepsilon (> 0)$, *there exists a positive number* δ *with* $\|P\| \le \delta$ *for every partition* P *of* $[a, b]$,

$$U(P, f) - L(P, f) < \varepsilon$$

Proof. The condition is necessary. Let f be R-integrable on $[a, b]$.

Then

$$\int_a^{\bar{b}} f(x) dx = \int_{\underline{a}}^{b} f(x) dx = \int_a^b f(x) \qquad (1.3.1)$$

Now from Darboux Theorem 1.1.3 and using (1.3.1) we obtain

$$U(P, f) < \int_a^{\bar{b}} f(x) dx + \frac{\varepsilon}{2} = \int_a^b f(x) dx + \frac{\varepsilon}{2}$$

and

$$L(P,f) > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} = \int_a^b f(x)dx - \frac{\varepsilon}{2}$$

$$\Rightarrow -L(P,f) < -\int_a^b f(x)dx + \frac{\varepsilon}{2}$$

Therefore,

$$U(P,f) - L(P,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The condition is sufficient.

For if $U(P,f) - L(P,f) < \varepsilon$ and since

$$L(P,f) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le U(P,f),$$

we have

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(P,f) - L(P,f) < \varepsilon$$

and hence the upper and lower integrals are equal i.e.

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx .$$

Hence the Theorem.

1.4 Integrability Functions

Theorem 1.4.1.

Every continuous function is integrable, i.e., if f(x) be continuous on $[a,b]$*, then f is R-interable on* $[a,b]$.

Proof. Since *f* is continuous on $[a,b]$, it is bounded there and moreover it is uniformly continuous on $[a,b]$.

Let $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta < \frac{\varepsilon}{b-a}$.

Now *f* is uniformly continuous on $[a,b]$, then for a given $\eta > 0$, $\exists$ a $\delta > 0$ such that

$\left|f(x') - f(x'')\right| < \eta$ for any two points $x', x'' \in [a,b]$

$$\text{with } \left|x' - x''\right| < \delta \qquad (1.4.1)$$

where δ depends on η alone.

Let $P\{a = x_0 < x_1 ... < x_{r-1} < x_r < ... < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$, where $\delta_r \equiv x_r - x_{r-1}$, or $\equiv [x_r - x_{r-1}]$ for $r = 1, 2, ..., n$ with $\|P\| < \delta$, then taking

$$M_r = \sup f(x) \text{ in } \delta r, \quad m_r = \inf f(x) \; in \; \delta_r;$$

since *f* is continuous on δ_r, $\exists$ points $\xi_r, \eta \in \delta_r$, such that

$$f(\xi_r) = M_r \; and \; f(\eta_r) = m_r$$

and we have from (1.4.1), the oscillatory sum for the partition P,

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$< \sum_{r=1}^{n} \eta_r \,.\, \delta_r < -\sum_{r=1}^{n} \frac{\varepsilon}{b-a} \,.\, \delta_r$$

$$= \frac{\varepsilon}{b-a}\sum_{r=1}^{n} \delta_r = \frac{\varepsilon}{b-a} \times (b-a) = \varepsilon$$

whereby $f(x)$ is integrable in the closed interval $[a,b]$. □

Theorem 1.4.2.

If $f(x)$ be monotone on the closed interval $[a,b]$, it is integrable there, i.e if f be monotone on [a,b], then *f is R-integrable on* $[a,b]$.

Proof. The function $f(x)$ being monotone on closed interval $[a,b]$ is bounded there. Suppose now, for definiteness, that $f(x)$ is nondecreasing on $[a,b]$. Then for any partition $P(a = x_0, x_1, x_2 ..., x_{r-1}, x_r, ..., x_{n-1}, x_n = b)$ whose norm is δ, we have with usual notations $U(P,f)-L(P,f)=\sum_{r=1}^{n}(M_r-m_r)\delta_r \leq \sum_{r=1}^{n}(M_r-m_r)\delta=\delta\sum_{r=1}^{n}(M_r-m_r)$

Since $f(x)$ is non decreasing, we have in each sub-interval $\delta_r \equiv [x_{r-1}, x_r]$, least upper bound $M_r = f(x_r)$ and greatest lower bound $m_r = f(x_{r-1})$. Thus

$$\sum_{r=1}^{n}(M_r - m_r) = \sum_{r=1}^{n}\{f(x_r) - f(x_{r-1})\} = f(b) - f(a)$$

and hence

$$U(P,f) - L(P,f) \leq \delta\{f(b) - f(a)\}.$$

If $f(a) = f(b)$, then $f(x)$ is constant and the integrability is clear. If however, $f(a) \neq f(b)$, then given $\varepsilon > 0$ we can choose $\delta < \varepsilon / \{f(b) - f(a)\}$ so that the oscillatory sum,

$$U(P,f) - L(P,f) < \varepsilon$$

for every $\varepsilon > 0$ and the integrability follows. Similarly in the case when $f(x)$ is decreasing on $[a,b]$. Hence the theorem. □

Theorem 1.4.3.

Any bounded function which is continuous except for a finite number of discontinuties is integrable.

Proof. Let $f(x)$ be a real valued function bounded on the closed interval $[a,b]$ and with a finite number of discontinuties whose number is p. The function is continuous on all the remaining parts of the closed intervals $[a,b]$. Let M, m be the bounds of $f(x)$ on $[a,b]$ and ε be any positive number, however small.

For the p number of discontinuities alone.

Let us enclose all the points of discontinuity of $f(x)$ on p nonoverlapping number number of subintervals the sum of which taken to be $< \varepsilon / 2(M-m)$. The part of the oscillatory sum coming from this subintervals is.

$$< \frac{\varepsilon}{2(M-m)} \times (M-m) = \frac{\varepsilon}{2}$$

Since the oscillation of f (x) in each of the intervals is ≤ (M-m) For the continuous parts.

Now f (x) is continuous on the remaining portion of [a,b], i.e. on at most (p+1) subintervals of (a,b) excluding those p non-overlapping number of intervals.

Then by Theorem 1.1.7, each of this (p+1) sub-intervals can be further subdivided so that the part of the oscillatory sum arising from these subintervals of each of them seperately is <ε/2 (p+1) Thus the part of the oscillatory sum coming from all these (p+1) continuous part is

$<\varepsilon/2(p+1)\ (p+1) = \frac{\varepsilon}{2}$

For the whole $[a,b]$.

Thus the combined mode of division P, say, for whole of the closed interval $[a,b]$ is such that for it the oscillatory sum

$$U(P,f) - L(P,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whereby $f(x)$ is integrable on $[a,b]$. There remain the possibility that a discountinuous point might coincide with either a or b. The slight modification required in the theorem is obvious. Hence the theore. □

Example 1.4.1.

A function $f:[0,3] \to \mathbb{R}$ *is defined by* $f(x) = x[x] \forall x \in [0,3]$. *Is f R– integrable on* $[0,3]$ *an if so, evaluate* $\int_0^3 f(x)dx$.

Solution : Here $f(x)=x[x] \Rightarrow f(x)=\begin{cases} 0, & 0 \le x < 1 \\ x, & 1 \le x < 2 \\ 2x, & 2 \le x < 3 \\ 9, & x = 3 \end{cases}$ Then

f has finite number of points discontinuities 1, 2 and 3.

Hence *f* is *R*– integrable by Theorem 1.1.1. Now

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= 0 + \int_1^2 x\,dx + \int_2^3 2x\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^2 + \left[x^2\right]_2^3$$

$$= \left(2 - \frac{1}{2}\right) + [9 - 4] = \frac{13}{2}.$$

Example : 1.4.2.

A function $f:[0,1] \to \mathbb{R}$ *is difined by* $f(x) = 2rx$,

when $\frac{1}{r+1} < x \le \frac{1}{r}, \quad r = 1, 2, 3, \ldots$

Show that f is R–integrable on $[0,1]$ *and* $\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

Solution :

Here $f(x) = \begin{cases} 2x, & \frac{1}{2} < x \le 1, \text{for } r = 1 \\ 4x, & \frac{1}{3} < x \le \frac{1}{2} \text{for } r = 2 \\ 6x, & \frac{1}{4} < x \le \frac{1}{3} \text{for } r = 3 \\ 9, & x = 3 \end{cases}$

and so on. Also $f(0)=0$

So f has infinite number of points of discontinuities given by $S=\left\{0,\frac{1}{2},\frac{1}{3},\ldots\right\}$ and also S has a only one limit point 0.

Therefore, f is $R-$ integrable.

Now

$$\int_{\frac{1}{n}}^{1} f(x)\,dx=\int_{\frac{1}{2}}^{1} f(x)\,dx+\int_{\frac{1}{3}}^{\frac{1}{2}} f(x)\,dx+\int_{\frac{1}{4}}^{\frac{1}{3}} f(x)\,dx+\ldots+\int_{\frac{1}{n}}^{\frac{1}{n-1}} f(x)\,dx$$

$$=\int_{\frac{1}{2}}^{1} 2x\,dx+\int_{\frac{1}{3}}^{\frac{1}{2}} 4x\,dx+\int_{\frac{1}{4}}^{\frac{1}{3}} 6x\,dx+\ldots+\int_{\frac{1}{n}}^{\frac{1}{n-1}} 2(n-1)x\,dx$$

$$=\left[x^2\right]_{\frac{1}{2}}^{1}+2\left[x^2\right]_{\frac{1}{3}}^{\frac{1}{2}}+3\left[x^2\right]_{\frac{1}{4}}^{\frac{1}{3}}+\ldots+(n-1)\left[x^2\right]_{\frac{1}{n}}^{\frac{1}{n-1}}$$

$$=1-\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)^2-2\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)^2-3\left(\frac{1}{4}\right)^2$$

$$+\ldots+(n-1)\left(\frac{1}{n-1}\right)^2-(n-1)\left(\frac{1}{n}\right)^2$$

$$=1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2+\ldots+\left(\frac{1}{n}\right)^2$$

$$\therefore \lim_{x\to\infty}\int_{\frac{1}{n}}^{1} f(x)\,dx=\lim_{x\to\infty}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right]$$

$$\Rightarrow \int_0^1 f(x)\,dx=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\infty$$

$$=\frac{\pi^2}{6}$$

Example : 1.4.3.

A function $f:[0,1]\to\mathbb{R}$ *is defined by* $f(x)=\frac{1}{n}$, *for*

$$\frac{1}{n+1}<x\le\frac{1}{n},\quad n=1,2,3,\ldots$$

Show that f R–integrable on $[0,1]$ *and* $\int_0^1 f(x)\,dx=\frac{\pi^2}{6}-1.$

Solution : *Here* $f(x)=\begin{cases}1, & \frac{1}{2}<x\le 1, \text{for } n=1\\ \frac{1}{2}, & \frac{1}{3}<x\le\frac{1}{2}, \text{for } n=2\\ \frac{1}{3}, & \frac{1}{4}<x\le\frac{1}{3}, \text{ for } n=3\end{cases}$

and so on.

The points of discontinuities of *f* are $\frac{1}{2},\frac{1}{3},\frac{1}{4}\cdots$

So *f* has infinite number of points of discontinuities given by $S=\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4}\cdots\right\}$ and also S has a only one limit point 0.

Therefore, *f* is *R*-integrable on $[0,1]$.

Now

$$\int_{\frac{1}{n}}^{1} f(x)\,dx=\int_{\frac{1}{2}}^{1} f(x)\,dx+\int_{\frac{1}{3}}^{\frac{1}{2}} f(x)\,dx+\ldots+\int_{\frac{1}{n}}^{\frac{1}{n-1}} f(x)\,dx$$

$$=\int_{\frac{1}{2}}^{1} 1dx+\int_{\frac{1}{3}}^{\frac{1}{2}}\frac{1}{2}dx+\int_{\frac{1}{4}}^{\frac{1}{3}}\frac{1}{3}dx+\ldots+\int_{\frac{1}{n}}^{\frac{1}{n-1}}\frac{1}{n-1}dx$$

$$=\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\frac{1}{n-1}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$=\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}\right\}-\left\{\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{4}+\ldots+\frac{1}{(n-1)}\cdot\frac{1}{n}\right\}$$

$$=\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}\right\}-\left\{1-\frac{1}{n}\right\}$$

$$\therefore \lim_{x\to\infty}\int_{\frac{1}{n}}^{1} f(x)\,dx=\lim_{x\to\infty}\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots \text{ to } \infty\right\}-1$$

$$\Rightarrow \int_0^1 f(x)dx=\frac{\pi^2}{6}-1$$

1.5 Properties of Integrable Functions

Theorem 1.5.1.

If $f(x)$ is integrable on $a \le x \le b$, then it is integrable on $a \le x \le b$ then it is integrable on $c \le x \le d$ where $a \le c < d \le b$. That is, $f(x)$ is integrable in every subinterval.

Proof. Let us choose a partition P of $[a,b]$ with c and d as ends of certain partial intervals in such a manner that to an arbitrary positive number ε, there corresponds a positive number δ_1 for which $U(P,f)-L(P,f)<\varepsilon$, the norm of the division being $<\delta_1$. Let P_1 be the corresponding partition of $f(x)$ in $[c,d]$. Then, since $U(P,f)-L(P,f)$ contains all terms of $U(P_1,f)-L(P_1,f)$ plus other non-negative terms, we have $0 \le U(P_1,f)-L(P_1,f) \le U(P,f)-L(P,f)<\varepsilon$ whereby $f(x)$ is integrable on $[c,d]$ □

Theorem 1.5.2.

If $f(x)$ be integrable on $a \le x \le c$ and $c \le x \le b$, then it is integrable on

$a \le x \le b$, i.e., $f(x)$ is integrable on the sum of two consecutive intervals.

Proof. Let ε be a positive number. Then there exist partitions of $[a,c]$ and $[c,b]$ for which the corresponding oscillatory sums are each $< \frac{\varepsilon}{2}$. Now the two modes of partitions of $[a,c]$ and $[c,b]$ give rise to a partition of $[a,b]$, for which the oscillatory sum will then be $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence *$f(x)$ is integrable on $[a,b]$* □

Theorem 1.5.3.

If $f(x)$ be integrable on $a \le x \le b$, so also is $\lambda f(x)$ where λ is any real number.

Proof. If $\lambda = 0$ the result is obvious. Taking oscillatory sum for $f(x)$ to be $< \varepsilon / \lambda$, the result follows for positive λ. Similar for negative λ □

Lemma 1.5.1.

Let P be any partition of $[a,b]$ for $f(x)$, $g(x)$, $f(x)+g(x), f(x).\, g(x)$ and let the corresponding supremum and infimum be $(M_r m_r)$, $(M_r^{'}, m_r^{'})$, $(M_r^{''}, m_r^{''})$, $(M_r^{'''}, m_r^{'''})$, in the sub-interval δ_r, then in δ_r

$$M_r^{''} - m_r^{''} \le (Mr - mr) + (M_r^{'} - m_r^{'})$$

$$M_r^{'''} \le M_r M_r^{'}; \; m_r^{'''} \ge m_r m_r^{'}.$$

Theorem 1.5.4.

If $f(x)$ and $g(x)$ be both integrable on $a \le x \le b$, then $f(x) \pm g(x)$ are also integrable on the same interval

Proof. Let $P\{a = x_0, x_1, x_2, ..., x_{r-1}, x_r, ..., x_{n-1}, x_n = b\}$ be any partition of $[a,b]$ applied to all the functions $f(x)+g(x)$, $f(x)$ and $g(x)$.

We take

$$\delta_r \equiv [x_{r-1}, x_r] \ or, \ x_r - x_{r-1} \ (1,2,...,n)$$

and let

$$M_r = \sup\{f(x)+g(x)\}, \qquad M_r' = \sup f(x), \quad M_r'' = \sup g(x)$$

$$m_r = \inf\{f(x)+g(x)\}, \qquad m_r' = \inf f(x), \quad m_r'' = \inf g(x)$$

Then from lemma 1.5.1

$$M_r - m_r \le (M_r' - m_r') + (M_r'' - m_r'')$$

We have

$$\sum_{r=1}^{n}(M_r - m_r)\delta_r \le \sum_{r=1}^{n}(M_r' - m_r')\delta_r + \sum_{r=1}^{n}(M_r'' - m_r'')\,\delta_r$$

i.e.,

$$U(P, f+g) - L(P, f+g)$$
$$\le \{U(P,f) - L(P,f)\} + \{U(P,g) + L(P,g)\} \qquad (1.5.1)$$

Let $\varepsilon > 0$ be given, then since $f(x)$ and $g(x)$ are integrable on $[a,b]$, there exist two partitions P_1 and P_2 such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2} \text{ and } U(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2}$$

Let P be the common refinement of P_1 and P_2. Then

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2} \text{ and } U(P, g) - L(P, g) < \frac{\varepsilon}{2}$$

Hence from (1.5.1),

$$U(P, f+g) - L(P, f+g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whereby $f(x)+g(x)$ is integrable on $[a,b]$

For $f(x)-g(x)$, put $-g(x) = \phi(x)$ and proceed as before. □

Theorem 1.5.5.

The product $f(x)\, g(x)$ *of two integrable functions* $f(x)$ *and* $g(x)$ *on* $a \le x \le b$ *is also integrable on the closed interval* $[a,b]$.

Proof. suppose that $f(x)$ and $g(x)$ are both positive on the whole interval $[a,b]$.

Let $P\{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots x_{n-1}, x_n = b\}$ be any partition of $[a,b]$ for $f(x)\, g(x)$, $f(x)$ and $g(x)$ and let the corresponding least upper and greatest lower bounds be (M_r, m_r), (M'_r, m'_r) and (M''_r, m''_r) in the sub-interval $\delta_r = x_r - x_{r-1}$ for $r = 1, 2, \ldots n$. Then in δ_r, (see Lemma 1.5.1)

$$M_r \le M'_r M''_r \text{ and } m_r \ge m'_r m''_r$$

Thus

$$M_r - m_r \le M'_r M''_r - m'_r m''_r = M'_r\left(M''_r - m''_r\right) + m''_r\left(M'_r - m'_r\right)$$

$$\le M'\left(M''_r - m''_r\right) + M''\left(M'_r - m'_r\right)$$

If M', M''_r be the upper bounds of $f(x)$, $g(x)$ in $[a,b]$. Thus

$$\sum_{r=1}^{n}(M_r - m_r)\delta_r \le M'\sum_{r=1}^{n}\left(M''_r - m''_r\right)\delta_r + M''\sum_{r=1}^{n}\left(M'_r - m'_r\right)\delta_r$$

i.e.,

$$U(P, fg) - L(P, fg)$$

$$\le M'\{U(P,g) - L(P,g)\} + M''\{U(P,f) + L(P,f)\}$$

$$\le k\{U(P,g) - L(P,g)\} + k\{U(P,f) + L(P,f)\} \qquad (1.5.2)$$

where M' and M'' are each less than *k*.

Let $\varepsilon > 0$ be given, then since $f(x)$ and $g(x)$ are both integrable on $[a,b]$, there exists two partitions P_1 and P_2 such that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2k} \text{ and } U(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2k}$$

Let P be the common refinement of P_1 and P_2. Then

$$U(P, f) - L(P, f) < \frac{\varepsilon}{2k} \text{ and } U(P, g) - L(P, g) < \frac{\varepsilon}{2k}$$

Hence from (1.5.2),

$$U(P, fg) - L(P, fg) < k\frac{\varepsilon}{2k} + k\frac{\varepsilon}{2k} = \varepsilon$$

whereby $f(x)g(x)$ is integrable on $[a,b]$ □

Theorem 1.5.6.

If $f(x)$ be integrable on $[a,b]$, so is $|f(x)|$ on $[a,b]$.

Proof. Sience $f(x)$ is integrable on $a \le x \le b$ for a given $\varepsilon > 0$, there exist a positive number δ such that for a partition $P\{a = x_0, x_1, x_2, ..., x_{r-1}, x_r, ... x_{n-1}, x_n = b\}$ of $[a,b]$ with $\|P\| < \delta$, we have

$$U(P, f) - L(P, f) < \varepsilon,$$

where $\delta_r \equiv [x_{r-1}, x_r]$; M_r, m_r being the least upper and greatest lower bounds of $f(x)$ in δ_r.

With the same partition P let $M_r^{'}$, $m_r^{'}$ be the upper and lower bounds of $f(x)$ be δ_r. Then since

$$\big||a| - |b|\big| \le |a - b|,$$

We have

$$M_r^{'} - m_r^{'} \le M_r - m_r$$

whereby

$$U(P,|f|) - L(P,|f|) \le U(P,f) - L(P,f)$$

which gives

$$U(P,|f|) - L(P,|f|) < \varepsilon$$

and $|f(x)|$ becomes integrable on $[a,b]$. □

Note 1.5.1.

But the converse of above theorem is not true, which can be seen from the illustration that follows.

Example : 1.5.1.

Let $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational.} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$.

Then in the closed interval $[a,b]$ *for* $b > a$, *we have*

$$I = -(b-a) \text{ and } j = (b-a)$$

so that $f(x)$ *is not integrable on* $[a,b]$.

But since\f (x)| = 1 for all values of x, I and J for \f(x)| are each equal to (b-a) whence \fx)| becomes integrable on [a,b]

1.6. Properties of the Definite Integral

Theorem 1.6.1.

If $f(x)$ *be integrable on* $[a,b]$ *and c be an intermediate point then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. Since $f(x)$ is integrable both on $[a,b]$, it is integrable both on $[a,c]$ and on $[c,b]$. Let P' be a partition on $[a,b]$ into sub-intervals.

Let $P = P' \cup \{c\}$. Then P *is also partition of* $[a,b]$.

Let P_1 and P_2 be two partitions of $[a,c]$ and $[c,b]$ respectively such that $P_1 \cup P_2 = P$. Then

$$U(P',f) \geq U(P,f) = U(P_1,f) + U(P_2,f)$$

$$\geq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx \qquad (1.6.1)$$

Now for any $\varepsilon > 0$, there are partitions $\overline{P}_1$ and $\overline{P}_2$ of $[a,c]$ and $[c,b]$ *for which*

$$U(\overline{P}_1,f) \leq \overline{\int_a^c} f(x) + \frac{\varepsilon}{2} \, and \, U(\overline{P}_2 f) \leq \overline{\int_c^b} f(x)\,dx + \frac{\varepsilon}{2}$$

since the integrals are the greatest lower bounds of such sums. Now these two partitions together form a partition $\overline{P}$ of $[a,b]$, so that for this divisions, we have

$$U(\overline{P},f) = U(\overline{P}_1,f) + U(\overline{P}_2,f) \leq \overline{\int_a^c} f(x)\,dx + \overline{\int_a^b} f(x)\,dx + \varepsilon$$

But the sum on the left is an upper bound for the upper integrals on $[a,b]$, so that

$$\overline{\int_a^b} f(x)\,dx \leq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx + \varepsilon$$

Since this hold for every $\varepsilon > 0$, we have

$$\overline{\int_a^b} f(x)\,dx \leq \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx \qquad (1.6.2)$$

Thus from (1.6.1) and (1.6.2)

$$\overline{\int_a^b} f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx$$

Since $f(x)$ is integrable on $[a,b]$, i.e. on $[a,c]$ and $[c,b]$, the result follows immediately. □

Theorem 1.6.2

If $f(x)$ be integrable on $[a,b]$, and λ be any real number, then

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx\ .$$

Proof. Since $f(x)$ is integrable on $[a,b]$, $\lambda f(x)$ is also integrable there.

If $\lambda = 0$, the result is obvious. Suppose $\lambda > 0$. Then for any partition P of $[a,b]$,

$$U(P,\lambda f) = \lambda U(P,f).$$

Take infimum of both sides. Then for all P,

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

and the property is established for $\lambda > 0$. If, however, $\lambda < 0$, take $\mu = -\lambda > 0$ and so □

Theorem 1.6.3.

If $f(x)$ and $g(x)$ be integrable on $[a,b]$ then

$$\int_a^b \{f(x)+g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. See that the integrability of $f(x)$ and $g(x)$ on $[a,b]$ implies the integrability of $f(x)+g(x)$ by Theorem 1.5.4

For any partition P of $[a,b]$, we have

$$L(P,f)+L(P,g) \le L(P,f+g) \le U(P,f+g) \le U(P,f)+U(P,g)$$

and

$$\int_a^b \{f(x)+g(x)\}\,dx \le U(P,f+g) \le U(P,f)+U(P,g) \qquad (1.6.3)$$

Let $\varepsilon > 0$ be given. Then there exists two partitions P_1 and P_2 such that

$$U(P_1,f) < L(P_1 f) + \frac{\varepsilon}{2} \le \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

$$U(P_2, f) < L(P_2 f) + \frac{\varepsilon}{2} \le \int_a^b g(x)dx + \frac{\varepsilon}{2}$$

Let P be the common refinement of P_1 and P_2. Then

$$U(P, f) < \int_a^b f(x)dx + \frac{\varepsilon}{2}; \quad U(P, g) < \int_a^b g(x)dx + \frac{\varepsilon}{2}$$

Thus from (1.6.3)

$$\int_a^b \{f(x) + g(x)\}\, dx < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon\,.$$

Since ε is arbitary

$$\int_a^b \{f(x) + g(x)\}\, dx \le \int_a^b f(x)dx + \int_a^b g(x)dx \qquad (1.6.4)$$

Replacing $f(x)$ and $g(x)$ in (1.6.4) by $-f(x)$ and $-g(x)$,

$$\int_a^b \{-f(x) - g(x)\}\, dx \le \int_a^b -f(x)dx + \int_a^b -g(x)dx$$

$$\Rightarrow \int_a^b \{f(x) + g(x)\}\, dx \ge \int_a^b f(x)dx + \int_a^b g(x)dx \qquad (1.6.5)$$

Hence the Theorem is established form (1.6.4) and (1.6.5) □

1.7 Some Important Inequalities

Theorem 1.7.1.

If M, m be the least upper and greatest lower bounds of an integrable function $f(x)$ on $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Proof. Let $P\{a = x_0, x_1, x_2 ..., x_{r-1}, x_r, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ into sub-intervals $\delta_r = [x_{r-1}, x_r]$, $(r = 1, 2, ..., n)$ with norm of division $\| P \| = \delta$. Denating M_r, m_r to be the supremum and infimum of $f(x)$ in δ_r, we have

$$m \le m_r \le M_r \le M$$

i.e.,

$$\sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$

or,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

Let now $\delta \to 0$

$$m(b-a) \le I \le J \le M(b-a)$$

or, $m(b-a) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le M(b-a)$

Since $f(x)$ is integrable, $f(x)$ is integrable, $I = J$ and hence

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)\overline{\int_a^b} dx$$

So, $m(b-a) = \int_a^b f(x)dx \le M(b-a)$

Hence the Theorem. □

Corollary 1.7.1.

If $f(x)$ *be integrable on* $a \le x \le b$, *then there exists a number* μ *lying between the bounds of* $f(x)$ *on* $[a,b]$, *such that*

$$\int_a^b f(x)dx = \mu(b-a).$$

Proof. This is obvious since $m \le \mu \le M$. □

Corollary 1.7.2.

If $f(x)$ *be continuous on* $[a,b]$, *there exists a number* ξ *between a and b such that*

$$\int_a^b f(x)dx = (b-a)f(\xi), \quad a \le \xi \le b.$$

Proof. Since $f(x)$ is continuous on $a \le x \le b$, it takes the value μ where $m \le \mu \le M$ at some point ξ on $[a,b]$, i.e., $f(\xi) = \mu$ □

Theorem 1.7.2.

If $f(x)$ *be integrable on* $[a,b]$ *and* $f(x) \geq 0, \forall x \in [a,b]$, *then*

$$\int_a^b f(x)dx \geq 0, \text{ when } b \geq a$$

$$\leq 0, \text{ when } b \leq a.$$

Proof. Let $b > a$. Now $f(x) \geq 0 \forall x \in [a,b] \Rightarrow m \geq 0.$

Therefore by Theorem 1.7.1

$$\int_a^b f(x)dx \geq m(b-a) \geq 0 [\because b-a > 0]$$

Again, $b < a \Rightarrow a > b$

$$\Rightarrow \int_b^a f(x)dx \geq m(b-a) \geq 0 \quad [\because a-b > 0]$$

$$\Rightarrow -\int_b^a f(x)dx \geq 0 \Rightarrow \int_b^a f(x)dx \leq 0$$

The results are trivial when $a = b$.

Theorem 1.7.3.

If $f(x)$ *and* $g(x)$ are both bounded and integrable on $a \leq x \leq b$, *and* $f(x) \geq g(x)$, *then*

$$\int_a^b f(x) \geq dx \geq \int_a^b g(x)dx.$$

Proof . Since $f(x)$ and $g(x)$ are both bounded and integrable on $a \leq x \leq b, f(x) - g(x)$ is also such and hence by Theorem 1.7.2 $f(x) - g(x)$ being ≥ 0;

$$\int_a^b \{f(x) - g(x)\}dx \geq 0$$

whereby the result follows. □

Theorem 1.7.4.

If $f(x)$ *be interagble on* $[a,b]$, *then*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Proof. We have seen in Theorem 1.5.6 that if $f(x)$ be integrable on $[a,b]$, so is $|f(x)|$ in $|f(x)|$.

Next we have $\forall x \in [a,b]$,

$$-|f(x)| \le f(x) \le |f(x)|$$

and by Theorem 1.7.3, since $a \le x \le b$,

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

$$\left|-\int_a^b |f(x)|dx\right| \le \int_a^b |f(x)|dx.$$

Hence the Theorem. □

1.8 Illustrative Examples

Example 1.8.1.

Given $f(x)$ *defined by* $f(x) = \begin{cases} x^2, & \text{when } 0 \le x \le 1 \\ \sqrt{x}, & \text{for } 1 \le x \le 2, \end{cases}$

evaluate $\int_0^2 f(x)\,dx$.

Solution

x^2 and $\sqrt{x}$ are integrable on their respective given renges, since they are both contiinuous on [0,2] ,

hence $\int_0^2 f(x)\,dx = \int_0^1 f(x)dx + \int_1^2 f(x)\,dx$

$$= \int_0^1 x^2\,dx + \int_1^2 \sqrt{x}dx = \frac{4\sqrt{2}}{3} - \frac{1}{3}.$$

Example 1.8.2.

Show that $f(x) = \frac{1}{2^n}$ *for* $\frac{1}{2^n+1} < x \le \frac{1}{2^n}, n = 0,1,2,\ldots$ *and* $f(0) = 0$ *is integrable* over $[0,\ 1]$ *and* $\int_0^1 f(x)\,dx = \frac{2}{3}$

Solution :

$$\text{Here } f(x) = \begin{cases} 1, & \frac{1}{2} < x \le 1, \quad \text{for } n = 0 \\ \frac{1}{2}, & \frac{1}{2^2} < x \le \frac{1}{2}, \text{ for } n = 1 \\ \frac{1}{2^2}, & \frac{1}{2^3} < x \le \frac{1}{2^2}, \text{ for } n = 2 \end{cases}$$

and so on.

The points of discontinuities of f are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots.$

So f has infinite number of points of discontinuities given by

$S = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$ and also S has a only one limit point 0.

Therefore, f is R – integrable on $[0,1]$.

Now

$$\int_{\frac{1}{2^n}}^1 f(x)dx = \int_{\frac{1}{2}}^1 f(x)\,dx + \int_{\frac{1}{2^2}}^1 f(x)\,dx + \ldots + \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(x)dx$$

$$= \int_{\frac{1}{2}}^1 f(x)\,1dx + \int_{\frac{1}{2^2}}^1 \frac{1}{2}dx + \int_{\frac{1}{2^3}}^{\frac{1}{2^2}} \frac{1}{2^2}dx + \ldots + \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} \frac{1}{2^{n-1}}dx$$

$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left\{1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \ldots + \left(\frac{1}{2^2}\right)^{n-1}\right\}$$

$$= \frac{1}{2}\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}} = \frac{2}{3}\left(1-\frac{1}{4^n}.\right)$$

Let now $n \to \infty$, so that $\int_0^1 f(x)\, dx = \frac{2}{3}$.

Example 1.8.3.

Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx$ *from* $m(b-a) \le \int_a^b f(x) dx \le M(b-a)$.

Solution :

Let $f(x) = \frac{\sin x}{x}$. It decreases in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ since it is continuous there and $f'(x) = \frac{x\cos x - \sin x}{x^2} < 0$.

Hence the minimum value of the function is

$$m = f\left(\frac{\pi}{3}\right) = \left(\frac{\sin\frac{\pi}{3}}{\frac{\pi}{3}}\right) = \frac{3\sqrt{3}}{\pi \cdot 2}$$

and the maximum value is

$$M = f\left(\frac{\pi}{4}\right) = \left(\frac{\sin\frac{\pi}{3}}{\frac{\pi}{4}}\right) = \frac{4}{\pi} \cdot \frac{1}{\sqrt{2}}$$

whereby

$$m(b-a) \le \int_a^b f(x) dx \le M(b-a)$$

or, $$\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx \le \frac{4}{\sqrt{2}\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

i.e., $$\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx \le \frac{4}{\sqrt{2}\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

i.e., $\dfrac{\sqrt{3}}{8} \le \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{\sin x}{x}\, dx \le \dfrac{\sqrt{2}}{6}.$

Example 1.8.4.

Show that $\dfrac{1}{2} < \int_0^1 \dfrac{dx}{\sqrt{4-x^2+x^3}} < \dfrac{\pi}{6}$

Solution :

We have in $0 < x < 1$.

$$4-x^2+x^3 = 4-\left(x^2-x^3\right) < 4$$

also,

$$4-x^2+x^3 = \left(4-x^2\right)+x^3 > 4-x^2.$$

Therefore on $0 < x < 1,\ 4 > 4-x^2+x^3 > 4-x^2$ or,

$$\frac{1}{\sqrt{4}} < \frac{1}{\sqrt{4-x^2+x^3}} < \frac{1}{\sqrt{4-x^2}} \qquad (1.8.1)$$

And at $x = 0, 4 = 4-x^2+x^3 = 4-x^2.$ Also at $x = 1, 4 = 4-x^2+x^3.$

Again $\dfrac{1}{2}, \dfrac{1}{\sqrt{4-x^2+x^3}}$ and $\dfrac{1}{\sqrt{4-x^2}}$ are all continuous on $0 \le x \le 1$ and hence intergable there and satisfy (1.8.1) on $0 < x < 1$. Thus

$$\int_0^1 \frac{1}{2}dx < \int_0^1 \frac{1}{\sqrt{4-x^2+x^3}}dx < \int_0^1 \frac{1}{\sqrt{4-x^2}}dx$$

or, $\dfrac{1}{2} < \int_0^1 \dfrac{1}{\sqrt{4-x^2+x^3}}dx < \left[\sin^{-1}\dfrac{x}{2}\right]_0^1 = \dfrac{\pi}{6}$

1.9. Fundamental Theorem.

Definition 1.9.1.

Let a function $f:[a,b] \to \mathbb{R}$ *be integrable on* $[a,b]$. *Then for each* $x \in [a,b]$, f

is integrable on $[a, x]$. $\int_a^x f(t)\,dt$ *exists and it depends on x. Therefore we can define a function F on* $[a, b]$ *by* $F(x) = \int_a^x f(t)\,dt$

Theorem 1.9.1.

If $f : [a, b] \to \mathbb{R}$ *be integrable on* $[a, b]$ *then the function F defined by* $F(x) = \int_a^x f(t)\,dt, x \in [a, b]$ *is continuous on* $[a, b]$.

Proof. Let x_1, x_2 *be any two points in* $[a, b]$.

$F(x_2) - F(x_1) = \int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt = \int_{x_1}^{x_2} f(t)\,dt$.

Therefore $|F(x_2) - F(x_1)| = \left|\int_a^{x_2} f(t)dt\right|$. *Since f is integrable on* $[a, b]$, *f is bounded on* $[a, b]$. *Therefore there exists a real number* $k > 0$ *such that* $|f(x)| < k$ *for all* $x \in [a, b]$.

If $x_2 > x_1$,

$$\left|\int_{x_1}^{x_2} f(t)\,dt\right| \le \int_{x_1}^{x_2} |f(t)|\,dt \le (x_2 - x_1)k$$

If $x_1 > x_2$,

$$\left|\int_{x_1}^{x_2} f(t)\,dt\right| = \left|\int_{x_2}^{x_1} f(t)\,dt\right| \le \int_{x_2}^{x_1} |f(t)|\,dt \le (x_1 - x_2)k$$

Consequently $|F(x_2) - F(x_1)| \le |x_2 - x_1|k$.

Let us take $\varepsilon > 0$. Then $|F(x_2) - F(x_1)| < \varepsilon$ for all x_1, x_2 in $[a, b]$ satisfying $|x_2 - x_1| < \frac{\varepsilon}{k}$.

Let $\delta = \frac{\varepsilon}{k}$. Then $|F(x_2) - F(x_1)| < \varepsilon$ for all x_1, x_2 in $[a, b]$ satisfying $|x_2 - x_1| < \delta$. This proves that F is uniformly cotinuous on $[a, b]$ and therefore F is continuous on $[a, b]$. This completes the proof. □

Example 1.9.1.

Let $f(x)=\begin{cases}0, & for -1\le x\le 0\\ 1, & for\ 0<x\le 1\end{cases}$.

Prove that f is integrable on [–1, 1]. *Show that the function F defined by* $F(x)=\int_{-1}^{x} f(t)\,dt$ *is continuous on* $[-1,1]$.

Solution :

Here *f* is bounded on $[-1,1]$ and is continuous on $[-1,1]$ except at only one point, 0. Therefore *f* is integrable on $[-1,1]$.

For $-1\le x\le 0,\ F(x)=\int_{-1}^{x} f(t)\,dt=0$

For $0<x\le 1$,

$$F(x)=\int_{-1}^{x} f(t)\,dt=\int_{-1}^{0} f(t)\,dt+\int_{0}^{x} f(t)\,dt$$

$$=0+\int_{0}^{x} f(t)\,dt=x$$

We have $f(x)=\begin{cases}0, & \text{for} -1\le x\le 0\\ x, & \text{for } 0<x\le 1\end{cases}$.

Clearly, F is continuous on $[-1,1]$.

Note 1.9.1. *Here f is not continuous on* $[-1,1]$, *but F is continuous on* $[-1,1]$.

We observe that the function F is continuous on $[a,b]$ when *f* is integrable on $[a,b]$.

If however, *f* be continuous on $[a,b]$ then F will be differentiable on $[a,b]$ as we shall see in the next theorem.

Theorem 1.9.2.

If a function $f:[a,b]\to\mathbb{R}$ *be integrable on* $[a,b]$ *then the function F defined* *by* $F(x)=\int_a^x f(t)\,dt,\ \ x\in[a,b]$ *is diferentiable at any point* $c\in[a,b]$ *at which f is continuous and* $F'(c)=f(c)$.

Proof. Let $c\in[a,b]$ and $\varepsilon>0$. Since *f* is continuous at *c* there exists a positive δ such that $|f(x)-f(c)|<\varepsilon$ for all $x\in[c,c+\delta)$.

Let us choose *h* satisfying $0<h<\delta$. Then $f(c)-\varepsilon<f(x)<f(c)+\varepsilon$ for all $x\in[c,c+h]$.

Therefore

$$\int_x^{c+h}|f(c)-\varepsilon|dx\le\int_x^{c+h}f(x)dx\le\int_x^{c+h}|f(c)+\varepsilon|dx$$

or $|f(c)-c|\cdot h\le F(c+h)-F(c)\le|f(c)+\varepsilon|\cdot h$

or $\left|\dfrac{F(c+h)-F(c)}{h}-f(c)\right|\le\varepsilon.$

This holds for all *h* satisfying $0<h<\delta$.

This implies

$$\lim_{h\to0+}\frac{F(c+h)-F(c)}{h}=f(c)$$

That is, $RF'(c)=f(c)$ (1.9.1)

Let $c\in[a,b]$ and $\varepsilon>0$. Since *f* is continuous at *c* there exists a positive η such that $|f(x)-f(c)|<\varepsilon$ for all $x\in(c-\eta,c]$.

Let us choose *h* satisfying $0<h<\eta$.

Then $f(c)-\varepsilon<f(x)<f(c)+\varepsilon$ for all $x\in[c-h,c]$. Therefore

$$\int_{c-h}^{c}|f(c)-\varepsilon|dx\le\int_{c-h}^{c}f(x)dx\le\int_{c-h}^{c}|f(c)+|dx$$

or $|f(c)-c|\cdot h\le F(c)-F(c-h)\le|f(c)+\varepsilon|\cdot h$

or $\left|\dfrac{F(c-h)-F(c)}{-h}-f(c)\right|\le\varepsilon.$

This holds for all h satisfying 0<h<n.

This implies

$$\lim_{h\to 0-}\frac{F(c+h)-F(c)}{h}=f(c).$$

That is,

$$LF'(c)=f(c)\cdot \qquad (1.9.2)$$

From (1.9.1) and (1.9.2) it follows that *f* is differentialble at any point $c\in[a,b]$ at which *f* is continuous and $F'(c)=f(c)$. □

Theorem 1.9.3.

If a function $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *then the function F defined by* $F(x)=\int_a^x f(t)\,dt,\ x\in[a,b]$ *is differentiable on* $[a,b]$ and $F'(x)=f(x)$ *for all* $x\in[a,b]$.

Proof. **Case 1.** Let $c\in(a,b)$

Let us choose *h* such that $c+h\in[a,b]$. Then

$$F(c+h)-F(c)=\int_c^{c+h} f(t)\,dt.$$

Let $h>0$. Since *f* is continuous on $[c,c+h]$, *f* is bounded on $[c,c+h]$. Let $M=\sup_{t\in[c,c+h]} f(t)$, $m=inf_{t\in[c,c+h]} f(t)$.

Then $m\le f(t)\le M$ for all $t\in[c,c+h]$.

Therefore $mh\le\int_c^{c+h} f(t)\,dt\le Mh$

or, $\int_c^{c+h} f(t)\,dt=\mu h$, where $m\le\mu\le M$.

Since *f* is continuous at $[c,c+h]$, $\mu=f(c+\theta h)$ for sume θ satisfying $0\le\theta\le 1$. Then

$$\frac{F(c+h)-F(c)}{h}=f(c+\theta h).$$

Since f is continuous at c, $\lim_{h\to 0+} f(c+\theta h) = f(c)$.

Therefore we have

$$\lim_{h\to 0+} \frac{F(c+h)-F(c)}{h} = f(c) \qquad (1.9.3)$$

Let $h<0$. Considering the interval $[c+h, c]$,

we have

$$-mh \le \int_{c+h}^{c} f(t)\,dt \le -Mh,$$

where $M = \sup_{t\in[c+h,c]} f(t)$, $m = inf_{t\in[c+h,c]} f(t)$.

or, $\dfrac{f(c+h)-F(c)}{h} = \mu$ where $m \le \mu \le M$.

Since f is continuous on $[c+h, c], \mu = f(c+\theta h)$ for some θ satisfying $0 \le \theta \le 1$.

Taking limit as $h \to 0-$ and noting that $\dfrac{F(c+h)-F(c)}{h} = f(c+\theta h)$.

Since f is continuous at $c, \lim_{h\to 0-} f(c+\theta h) = f(c)$.

Therefore we have

$$\lim_{h\to 0+} \frac{F(c+h)-F(c)}{h} = f(c) \qquad (1.9.4)$$

From (1.9.3) and (1.9.4) we have $F'(c) = f(c)$.

Case 2. Let $c = a$.

Let us choose h such that $a+h<b$. Then

$$F(a+h)-F(a) = \int_{a}^{a+h} f(t)\,dt$$

Considering the interval $[a, a+h]$, we have

$$mh \le \int_{a}^{a+h} f(t)\,dt \le Mh,$$

where

$$M = \sup_{t\in[a,a+h]} f(t), \qquad m = \inf_{t\in[a,a+h]} f(t)$$

or, $\dfrac{F(a+h)-F(a)}{h}=\mu$, where $m\le\mu\le M$.

Since *f* is continuous on $[a,a+h]\mu=f(a+\theta h)$ for some θ satisfying $0\le\theta\le1$.

Taking limit as $h\to0+$ and nothing that $\lim_{h\to0+}f(a+\theta h)=f(a)$, we have

$$\lim_{h\to0+}\frac{F(a+h)-F(a)}{h}=f(a)\ or,\ F'(a)=f(a).$$

Case 3. Let $c=b$

Proof is similar to case 2.

This completes the proof. □

Definition 1.9.2.

A function ϕ is called an antiderivative or a primitive of a function f on an interval I, if $\phi'(x)=f(x)$ for all $x\in I$.

If ϕ be an antiderivative of *f* on *I*, then $\phi+c$, where $c\in\mathbb{R}$, is obviously an antiderivative of *f*. This shows that if *f* admits of an antiderivative on *I*, then there exist many antiderivatives of *f* on *I*.

It follows from the previous theorem that if *f* be continuous on a closed interval $[a,b]$, then *f* possesses an antiderivatrive on $[a,b]$ given by *F*. Therefore continuity of *f* ensures the existence of an antiderivative of *f*.

Note 1.9.2. *It is worthwhile to note that continuity of f is not a necessary condition for the existence of an antiderivative of f.*

for example, let f : $[-1,1]\to\mathbb{R}$ *be defined by*

$$f(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\ne0\\ 0, & x=0\ .\end{cases}$$

Here f is not continuous on $[-1,1]$, 0 *being the point of discontinuity.*

Let $\phi:[-1,1]\to\mathbb{R}$ *be defined by* $\phi(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\ne0\\ 0, & x=0\end{cases}$

Then $\phi'(x)=f(x)$ *for all* $x\in[-1,1]$.

Then ϕ is an antiderivative of f on $[-1,1]$ although f is not continuous on $[-1,1]$.

Theorem 1.9.4.

If $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and $\phi:[a,b]\to\mathbb{R}$ be antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=\phi(b)-\phi(a).$$

Proof. Since *f is continuous on* $[a,b]$, *f* is integrable on $[a,b]$.

Let $F(x)=\int_a^x f(t)dt, x\in[a,b]$.

Since *f* is continuous on $[a,b]$, F is differentiable on $[a,b]$ and $F'(x)=f(x)$ for all $x\in[a,b]$, So F is antiderivative of f on [a,b] on [a,b]

Since $\phi$is an antiderivative of f on [a,b], for all $x\in$ [a,b], for all $x\in$ [a,b], $\phi(x)=$ F'(x) where c is a constant.

So $\phi(a)=F(a)+c=c$ since $F(a)=0$.

Therefore $\phi(x)=F(x)+\phi(a)$, for all $x\in[a,b]$.

Consequently, $\int_a^b f(x)dx=F(b)=\phi(b)-\phi(a)$.

Note 1.9.3. *The theorem states that if $f(x)$ be continuous on $[a,b]$ then the integral $\int_a^b f(x)dx$ can be evaluated in terms of an antiderivative of $f(x)$ on $[a,b]$.*

□

Theorem 1.9.5.

If $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$, and (ii) f possesses an antiderivative ϕ on $[a,b]$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof . Let $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. *Let*

$$M_r = \sup_{x \in [x_r - 1, x_r]} f(x), \qquad \mathrm{M_r} = \inf_{x \in [x_{r-1} - 1, x_r]} f(x), \qquad \text{for } r = 1, 2, \ldots, n.$$

Since $\phi'(x) = f(x)$ for all $x \in [a,b]$, ϕ satisfies all conditions of Langrange's Mean value theorem on $[x_{r-1}, x_r]$, for $r - 1, 2, \ldots, n.$

Therefore for $r - 1, 2, \ldots, n,$

$$\phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)(x_r - x_{r-1}) \text{ for some } \xi_r \text{ in } (x_{r-1}, x_r)$$

$$f(\xi_r)(x_r, x_{r-1})$$

The summation gives

$$\sum_{r=1}^{n} = f(\xi_r)(x_r - x_{r-1}) = \phi(b) - \phi(a)$$

But $m_r \le f(\xi_r) \le M_r$ for $r = 1, 2, \ldots, n$.

Therefore

$$\sum_{r=1}^{n} m_r(x_r - x_{r-1}) \le \phi(b) - \phi(a) \le M_r(x_r - x_{r-1}).$$

Therefore $L(P,f) \le \phi(b) - \phi(a) \le U(P,f)$.

This holds for all partitions P of $[a,b]$. So $\phi(b) - \phi(a)$ is an upper bound of the set $\{L(P,f) : P \text{ is any partition of } [a,b]\}$.

As the supremum of the set is $\int_{\underline{a}}^{b} f(x)dx$, it follows that

$$\int_{\underline{a}}^{b} f(x)dx \le \phi(b) - \phi(a) \tag{1.9.5}$$

Also $\phi(b) - \phi(a)$ is a lower bound of the set $\{U(P,f) : P \text{ is any partition of } [a,b]\}$.

As the infimum of the set is $\int_a^{\bar{b}} f(x)dx \le \phi(b) - \phi(a)$ (1.9.6)

From (1.9.5) and (1.9.6) We have

$$\underline{\int_a^b} f(x)dx \le \phi(b) - \phi(a) \le \overline{\int_a^b} f(x)dx.$$

Since *f* is intrergable on $[a,b]$,

$$\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx.$$

Consequently $\int_a^b f(x)dx = \phi(b) - \phi(a)$. □

1.10. Mean Value Theorem for Integrals.

Theorem 1.10.1. ***[First Mean Value Theorem]***

Let f (x) and $\phi(x)$ *be two bounded functions integrable on* $a \le x \le b$ *and let* $\phi(x)$ *keep the same sign on* $[a,b]$*, then*

$$\int_a^b f(x)\phi(x)dx = \mu\int_a^b \phi(x)dx$$

where $m \le \mu \le M$*, m and M being the greatest lower and least upper bounds of* $f(x)$ *on* $[a,b]$.

Proof. Frist we suppose that $\phi(x)$ *is non-negative, i.e.,* $\phi(x) \ge 0$ *in* $[a,b]$.

Now in $a \le x \le b$,

$m \le f(x) \le M$ *or,* $m\phi(x) \le f(x)\phi(x) \le M\phi(x)$.

Since $m\phi(x), f(x)\phi(x)$ and $M\phi(x)$ are each integrable on $[a,b]$, we have

$$\int_a^b m\phi(x)dx \le \int_a^b f(x)\phi(x)dx \le \int_a^b M\phi(x)dx$$

i.e., $m\int_a^b \phi(x)dx \le \int_a^b f(x)\phi(x)dx \le M\int_a^b \phi(x)dx$

Therefore, $\int_a^b f(x)\phi(x)dx = \mu\int_a^b \phi(x)dx$

where $m \le \mu \le M$.

The case when $\phi(x)$ is negative is similar. Hence the theorem. □

Corollary 1.10.1.

If , in particular, $\phi(x)=1$ *for all* $x\in[a,b]$, *then*

$$\int_a^b f(x)\,dx=\mu\int_a^b dx=\mu(b-a),$$

where $m\le\mu\le M$.

If, moreover, f is continuous on $[a,b]$, *there exists a point* ξ *in* $[a,b]$ *such that*

$$\int_a^b f(x)dx=f(\xi)(b-a).$$

Since $\xi\in[a,b],\xi=a+\theta(b-a)$ *for some* θ *satisfying* $0\le\theta\le1$.

Therefore

$$\int_a^b f(x)dx=(b-a)f(a+\theta(b-a)),$$

where $0\le\theta\le1$.

Example 1.10.1. *Use first mean value theorem to prove that*

$$\frac{\pi}{6}\le\int_0^{\frac{1}{2}}\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx\le\frac{\pi}{6}\cdot\frac{1}{\sqrt{1-k^2/4}},k^2<1.$$

Solution : *Let* $f(x)=\frac{1}{\sqrt{1-k^2x^2}},x\in\left[0,\frac{1}{2}\right]$ and $f(x)=\frac{1}{\sqrt{1-k^2x^2}},x\in\left[0,\frac{1}{2}\right]$. Then

f and ϕ are integrable on $\left[0,\frac{1}{2}\right]$ and $\phi(x)>0$ for all $x\in\left[0,\frac{1}{2}\right]$.

Since *f* is continuous on $\left[0,\frac{1}{2}\right]$, by the first Mean value theorem there exists a point ξ in $\left[0,\frac{1}{2}\right]$ such that

$$\int_0^{\frac{1}{2}}f(x)\phi(x)dx=f(\xi)\int_0^{\frac{1}{2}}\phi(x)dx.$$

or, $$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{(1-k^2x^2)(1-x^2)}}dx=\frac{1}{\sqrt{1-k^2\xi^2}}\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-k^2\xi^2}} \cdot$$

Since $0 \le \xi \le \frac{1}{2}, 1 \le \frac{1}{\sqrt{1-k^2\xi^2}} \le \frac{1}{\sqrt{1-k^2/4}}$

Therefore

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} dx \le \frac{\pi}{6} \frac{1}{\sqrt{1-k^2/4}} .$$

Lemma 1.10.1. ***Abel's Intequality***

If (i) $a_1, a_2, \ldots, a_n$ *is a non-decreasing sequence of n positive numbers.*

(ii) $v_1, v_2, \ldots, v_n$ *is a set of any n numbers and*

(iii) h, H are two numbers such that

$$h < v_1 + v_2 + \ldots v_p < H \text{ for } 1 \le p \le n$$

$$\text{then } a_1 h < a_1 v_1 + a_2 v_2 + \ldots + a_n v_n < a_1 H .$$

Proof. Writing $S_p = v_1 + v_2 + \ldots + v_p$, *we have*

$$\sum_{r=1}^{n} a_r v_r = a_2 S_1 + a_2 (S_2 - S_1) + \ldots + a_r (S_r - S_{r-1})$$

$$+ \ldots + a_n (S_n - S_{n-1})$$

$$= (a_1 - a_2) S_1 + (a_2 - a_3) S_2 + \ldots + (a_{n-1} - a_n) S_{n-1} + a_n S_n .$$

Now by *(i)* $a_1 - a_2, a_2 - a_3 \ldots, a_{n-1} - a_n$ are all non-negative. Also by (*iii*) $h < S_p < H$ for all $p \le n$.

Therefore

$$\sum_{r=1}^{n} a_r v_r < (a_1 - a_2) H + (a_2 - a_3) H + \ldots + (a_{n-1} - a_n) H + a_n H = a_1 H$$

and

$$\sum_{r=1}^{n} a_r v_r > (a_1 - a_2) h + (a_2 - a_3) h + \ldots + (a_{n-1} - a_n) h + a_n h = a_1 h$$

Hence the theorem.

Theorem 1.10.2. ***Second Mean Value Theorem (Bonnet's form)***

Let $f(x)$ *be bounded nonotonic non-increasing and never negative on* $[a,b]$*;* and let $\phi(x)$ *be bounded and integrable on* $[a,b]$. *Then there exists a value* ξ *of* x on $[a,b]$, *such that*

$$\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx,\ a \le \xi \le b.$$

Proof Let $P = \{a = x_0, x_1, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}, x_n = b\}$ be any partition of $[a,b]$ and let M_r, m_r be the supremum and infimum of $\phi(x)$ on $\delta_r \equiv [x_{r-1}, x_r]$. Suppose also that $\xi_1 = a$ and $\xi_r\,(r \ne 1)$ be any arbitrary point of δ_r. Then

$$m_r\delta_r \le \int_{x_{r-1}}^{x_r} \phi(x)dx \le M_r\delta_r$$

and $m_r\delta_r \le \phi(\xi_r)\delta_r \le M_r\delta_r$.

Putting $r = 1,2,3,\ldots,p$ where $p \le n$ and adding we have

$$\sum_{r=1}^{p} m_r\delta_r \le \int_a^{x_p}\phi(x)dx \le \sum_{r=1}^{p} M_r\delta_r$$

and $$\sum_{r=1}^{p} m_r\delta_r \le \sum_{r=1}^{p}\phi(\xi_r)\delta_r \le \sum_{r=1}^{p} M_r\delta_r.$$

Thus $$\left|\sum_{r=1}^{p}\phi(\xi_r)\delta_r - \int_a^{x_p}\phi(x)dx\right| \le \sum_{r=1}^{p}(M_r - m_r)\delta_r \le \sum_{r=1}^{n}(M_r - m_r)\delta_r$$

or, $$\int_a^{x_p}\phi(x)dx - \sum_{r=1}^{n}(M_r - m_r)\delta_r \le \sum_{r=1}^{p}\phi(\xi_r)\delta_r$$

$$\le \int_a^{x_p}\phi(x)dx + \sum_{r=1}^{n}(M_r - m_r)\delta_r$$

Now $\phi(x)$ being integrable, $\int_a^{x_p} \phi(x)dx$ is a continuous function of x and as such must have its supremum and infimum on $[a,b]$ and must attain them.

Let M, m be the respective supremum and infimum of $\int_a^x \phi(x)\,dx$ on [a,b].

Hence

$$m-\sum_{r=1}^{n}(M_r-m_r)\delta_r \le \sum_{r=1}^{p}\phi(\xi_r)\delta_r \le M+\sum_{r=1}^{n}(M_r-m_r)\delta_r$$

Next we apply Abel's Inequality by writing

$$a_r = f(\xi_r), \qquad v_r=\phi(\xi_r)\delta_r,$$

$$h=m-\sum_{r=1}^{n}(M_r-m_r)\delta_r, \qquad H=M+\sum_{r=1}^{n}(M_r-m_r)\delta_r$$

and obtain

$$f(a)\left\{m-\sum_{r=1}^{n}(M_r-m_r)\delta_r\right\}\le\sum_{r=1}^{n}f(\xi_r)\phi(\xi_r)\delta_r$$

$$\le f(a)\left\{M+\sum_{r=1}^{n}(M_r-m_r)\delta_r\right\}.$$

Let now norm of $P\to 0$ so that $\sum_{r=1}^{n}(M_r-m_r)\delta_r\to 0$

whereby $$mf(a)\le\int_a^b f(x)\phi(x)dx\le M f(a)$$

i.e., $$\int_a^b f(x)\phi(x)dx=\mu f(a), \quad m\le\mu\le M$$

But *m*, *M* are the infimum and supremum of the continuous function $\int_a^x \phi(x)dx$, hence it must assume every value intermediate to *m* and *M*. Therefore, there must exist at least one value ξ on $a\le x\le b$ for which $\int_a^b f(x)\phi(x)dx=f(a)\int_a^\xi\phi(x)dx$

This proves Bonnet's form of second mean-value theorem □

Theorem 1.10.3. ***Second Mean Value Theorem (Weierstrass'form)***

Let $f(x)$ be bounded and nonotonic on $[a,b]$. and let ϕ (x) be bounded and integrable on [a,b]. Then there exists at least one value of x, say ξ on $[a,b]$, such that

$$\int_a^b f(x)\phi(x)dx=f(a)\int_a^\xi\phi(x)dx+f(b)\int_\xi^b\phi(x)dx; \qquad a\le\xi\le b.$$

Proof. **Case 1.** Let f be monotonic decreasing on $[a,b]$ and let $\psi(x) = f(x) - f(b), x \in [a,b]$. Then ψ is monotonic decreasing on $[a,b]$ and $\psi(x) \geq 0$ on $[a,b]$.

By Bonnet's theorem there exists a point ξ in $[a,b]$ such that

$$\int_a^b \psi(x)\phi(x)dx = \psi(a)\int_a^\xi \phi(x)dx$$

or, $$\int_a^b [f(x) - f(b)]\phi(x)dx = [f(a) - f(b)]\int_a^\xi \phi(x)dx$$

or, $$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx$$

$$+f(b)\left[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx\right]$$

$$= f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx.$$

Case 2. Let f be monotonic increasing on $[a,b]$ and let $\psi(x) = f(b) - f(x), x \in [a,b]$. Then ψ is monotonic decreasing on $[a,b]$ and $\psi(x) \geq 0$ on $[a,b]$.

By Bonnet's theorem there exists a point ξ in $[a,b]$ such that

$$\int_a^b \psi(x)\phi(x)dx = \psi(a)\int_a^\xi \phi(x)dx$$

or, $$\int_a^b [f(b) - f(x)]\phi(x)dx = [f(b) - f(x)]\int_a^\xi \phi(x)dx$$

or, $$\int_a^b f(x)\phi(x)dx = f(a)\int_a^\xi \phi(x)dx$$

$$+f(b)\left[\int_a^b \phi(x)dx - \int_a^\xi \phi(x)dx\right]$$

$$= f(a)\int_a^\xi \phi(x)dx + f(b)\int_\xi^b \phi(x)dx$$

This completes the proof □

Example 1.10.2.

Show that the second Mean value theorem (Bonnet's form) is applicable to $\int_a^b f(x)\frac{\sin x}{x}dx$ *where* $0<a<b<\infty$. *Also prove that* $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{2}{a}$.

Solution :

Let $f(x)=\frac{1}{x}, x\in[a,b]$. and ϕ (x) Sin x, x∈ [a,b] Then $f(x)$ and $\phi(x)$ are both integrable on $[a,b]$ and $f(x)$ is monotonic decreasing on $[a,b]$ and $f(x)>0$ for all $x\in[a,b]$.

By the Mean value theorem (Bonnet's form) there exists a point ξ in $[a,b]$ such that

$$\int_a^b \frac{\sin x}{x}dx = \left(\frac{1}{a}\right)\int_a^{\xi}\sin x\,dx = \left(\frac{1}{a}\right)\{-\cos\xi+\cos a\}.$$

Therefore $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{2}{a}$.

Example 1.10.3.

Show that the second Mean value theorem (Weierstrass' form) is applicable to $\int_a^b \frac{\sin x}{x}dx$ *where* $0<a<b<\infty$. *Also prove that* $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{4}{a}$.

Solution :

Let $f(x)=\frac{1}{x}, x\in[a,b]$ and $\phi(x)=\sin x, x\in[a,b]$. Then $f(x)$ and $\phi(x)$ are both integrable on $[a,b]$ and $f(x)$ is monotonic decreasing on $[a,b]$.

By the Mean value theorem (Weierstress" form) there exists a point ξ in $[a,b]$ such that $\int_a^b f(x)\phi(x)dx = f(a)\int_a^{\xi}\phi(x)dx + f(b)\int_{\xi}^b \phi(x)dx$.

or, $\int_a^b \frac{\sin x}{x}dx = \left(\frac{1}{a}\right)\int_a^{\xi}\sin xdx + \left(\frac{1}{b}\right)\int_{\xi}^b \sin xdx$

$$= \left(\frac{1}{a}\right)[-\cos\xi+\cos a] + \left(\frac{1}{b}\right)[-\cos b+\cos\xi]$$

Therefore

$$\left|\int_a^b \frac{\sin x}{x}dx\right| \le \left(\frac{1}{a}\right)[-\cos\xi + \cos a] + \left(\frac{1}{b}\right)[-\cos b + \cos\xi]$$

$$\le \left(\frac{1}{a}\right)\{|-\cos\xi| + |\cos a|\} + \left(\frac{1}{b}\right)\{|-\cos b| + |\cos\xi|\}$$

$$\le \frac{1}{a}(1+1) + \frac{1}{b}(1+1)$$

$$\le \frac{4}{a}, a < b\ .$$

1.11. Change of Variable in an Integrals

Theorem 1.11.1.

If (i) $\int_a^b f(x)dx$ *exists*

(ii) $x = \phi(t)$ *is a derivable function on* $[\alpha,\beta]$ *and* $\phi'(t) \ne 0$ *for any value of t and* $\phi(\alpha) = a, \phi(\beta) = b$ *; and*

(iii) $f\{\phi(t)\}$ *and* $\phi'(t)$ *are bounded and integrable on* $[\alpha,\beta]$ *then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f\{\phi(t)\}\phi'(t)dt\ .$$

Proof. Let $P(\alpha = t_0, t_1, t_2, \ldots, t_{r-1}, t_r \ldots, t_{n-1}, t_n = \beta)$ be any partition of $[\alpha,\beta]$ and let $P(a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots x_{n-1}, x_n = b)$ be the corresponding partition of $[a,b]$, where $x_r = \phi(t_r)$.

By Mean value theorem of differential calculus

$$x_r - x_{r-1} = \phi(t_r) - \phi(t_{r-1}) = (t_r - t_{r-1})\phi'(\xi_r),\ \ t_{r-1} < \xi_r < t_r\ .$$

Let $\phi(\xi_r) = \eta_r$. Then

$\phi(\xi_r) = \eta_r$

$$\sum_{r=1}^{n} f(\eta_r)(x_r - x_{r-1}) = \sum_{r=1}^{n} f\{\phi(\xi_r)\}\phi'(\xi_r)(t_r - t_{r-1}).$$

Let norm of $P \to 0$, then the norm of P' also $\to 0$ and by conditions (*i*) and (*iii*)

$$\sum_a^b f(x)dx = \int_\alpha^\beta f\{\phi(t)\}\phi'(t)dt$$

Hence the theorem. □

Example 1.11.1.

Evaluate $\int_{-1}^{1} \frac{1}{1+x^2}dx$ *by the substitution* $x = \tan t$.

Solution:

Let $\phi(t) = \tan t, t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Then ϕ is differentiable and strictly increasing on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, $\phi'(t)$ is integrable on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \phi\left(-\frac{\pi}{4}\right) = -1, \phi\left(\frac{\pi}{4}\right) = 1$.

Let $f(x) = \frac{1}{1+x^2}, x \in [-1,1]$.

Then $\int_{-1}^{1} f(x)dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(\phi(t))\phi'(t)dt$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\tan^2 x}.\sec^2 t dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dt = \frac{\pi}{2}.$$

1.12. Integration by Parts

Theorem 1.12.1.

If $f(x)$ *and* $g(x)$ *be derivable on* $a \le x \le b$ *and if* $f'(x)$ *and* $g'(x)$ *are integrable there, then*

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x)f'(x)dx.$$

Proof. Clearly $\left[f(x)g(x)\right]'$ is integrable since

$$\left[f(x)g(x)\right]' = f'(x)g(x) + f(x)g'(x)$$

is a sum of products of integrable functions.
Then by Fundamental Theorem of integral calculus

$$\int_a^b \left[f(x)g(x)\right]' dx = \left[f(x)g(x)\right]_a^b$$

$$i.e., \int_a^b \{f'(x)g(x) + f(x)g'(x)\} dx = \left[f(x)g(x)\right]_a^b$$

$$i.e., \int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x)f'(x)dx$$

This completes the proof. □

1.13 Summary

1. In Section 1.1 Introduction, Definition of Partition, Refinement and darboux's Theorem has been discussed.

2. In Section 1.2 and 1.3 we study about the equivalence of two definition of Definite Integral, Necessary and sufficient condition for a function to be Riemann Integrable and Integrability of piecewise continuous and monotone functions.

3. In Section 1.4, 1.5, 1.6 and 1.7 we have focussed on Properties of Integrable function, Properties of definite Integration and some important inequalities.

4. In Section 1.9 Fundamental Theorem and in section 1.10 Abel's Inequality, First Mean Value Theorem, Second Mean Value Theorem (Bonnet's Form and Weierstrass's Form) has been taken for discussion.

5. In Section 1.11, and 1.12 we have studied about Change of Variable in an Integral and Integration by Parts.

1.14 Exercise

1. Let $f:[a,b]\to\mathbb{R}$ be bounded and monotone increasing on $[a,b]$. If P_n be the partition of $[a,b]$ dividing into n subintervals of equal length prove that

$$\int_a^b f(x)dx \le U(P_n, f) \le \int_a^b f(x)dx + \frac{b-a}{n}\left[f(b)-f(a)\right].$$

Consider the sequence of partitions $\{P_n\}$ and deduce that $\lim_{n\to\infty} U(P_n, f) = \int_a^b f(x)dx$

Utilise this result to evaluate

(*i*) $\int_0^1 x\,dx$, (*ii*) $\int_0^1 x^2\,dx$, (*iii*) $\int_0^1 e^x\,dx$

2. A function $f(x)$ is defined on $[0,1]$ by $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ x^3, & x \text{ is rational} \end{cases}$.

(*i*) Evaluate $\underline{\int_0^1} f(x)dx, \overline{\int_0^1} f(x)dx$;

(*ii*) Show that $f(x)$ is not integrable on $[0,1]$.

3. Prove that $\lim_{x\to\infty} e^{-x^2}\int_0^x e^{t^2}dt = 0$

4. Show that

(*i*) $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{(1-x^{2n})}} < \frac{\pi}{6}, \forall n > 1$.

(*ii*) $\frac{2\pi^2}{0} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2x}{\sin x} \le \frac{4\pi^2}{9}$.

(*iii*) $0.573 < \int_1^2 \frac{dx}{\sqrt{4-3x^2+x^3}} < 0.595$.

(*iv*) $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x + \cos x}dx < \frac{\pi^3}{24}$.

(*v*) $\frac{\pi^3}{96} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{5+3\sin x}dx < \frac{\pi^3}{24}$.

5. Let *f* be continuous and let $F(x) = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$. Show that

$$F'(x) = x\int_0^x f(t)dt - \int_0^x tf(t)dt.$$

6. State and prove the Fundalental theorem of integral calculus. Deduce that

$$\int_a^b f(x)dx = (b-a)f\{a+\theta(b-a)\}, 0 \le \theta \le 1$$

under suitable conditions on $f(x)$ to be stated by you.

7. If α and ϕ are positive acute angles, then show that

$$\phi < \int_0^{\phi} \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\phi}{\sqrt{1-\sin^2\alpha\sin^2\phi}}.$$

If $\alpha = \phi = \frac{\pi}{6}$, then prove that the integral lies between 0.523 and 0.541.

8. Prove that

$$\int_0^1 \frac{x^3\cos 5x}{2+x^2}dx \text{ lies between } -\frac{1}{2} \text{ and } \frac{1}{2}.$$

9. Show with the help of the function *f* defined by

$$f(x) = \frac{1}{2^n} \text{for } \frac{1}{2^n+1} < x \le \frac{1}{2^n},\ n = 0,1,2,...$$

and $f(0)=0$ that *f* is integrable over the interval $[0,1]$, although it has an infinite number of points of discontinuity. Also show that $\int_0^1 f(x)dx = \frac{2}{3}$.

10. Prove that the following function *f* defined on $[-1,1]$ by

$$f(x) = 2x\sin\left(\frac{1}{x^2}\right) - \left(\frac{2}{x}\right)\cos\left(\frac{1}{x^2}\right),\ x \ne 0$$

cannot be integrated on $[-1,1]$, but has a primitive there.

11. If $f(x)=2x$ for $0 \le x \le 2$ and $f(x)=x^2$ for $2 \le x \le 3$, show that

$$\int_0^3 f(x)dx = \frac{31}{3}.$$

12. if $f(x)$ be continuous on $[a,b]$ and $f(x) \ge 0$ for all x in $[a,b]$ and if $\int_a^b f(x)dx = 0$, prove that $f(x)=0$ for all $x \in [a,b]$.

13. Prove that

$$0 < \int_0^{\frac{\pi}{2}} \sin^{n+1} x dx < \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

and

$$0 < \int_0^{\frac{\pi}{4}} \tan^{n+1} x dx < \int_0^{\frac{\pi}{4}} \tan^n x\, dx$$

14. Let $f(x)=\begin{cases}1, & \text{when } 0\le x\le 1.\\ x, & \text{when } 1<x\le 2.\end{cases}$

Show that F defined by $F(x)=\int_0^x f(t)\,dt, 0\le x\le 2$ is given by

$$F(x)=\begin{cases}x, & 0\le x\le 1\\ \frac{1}{2}\left(1+x^2\right), & 1<x\le 2\end{cases}$$

and hence verify that $F'(x)=f(x)$ on $[0,2]$

15. Evaluate the limits

(*i*) $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+3n}\right]$

(*ii*) $\lim_{n\to\infty}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{n\pi}{n}\right]$

(*iii*) $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+4n^2}\right]$

(*iv*) $\lim_{n\to\infty}\left[\left(1+\frac{n}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$

(*v*) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)^n\right]^{\frac{1}{n}}$

16. Use Bonnet's form of second Mean value theorem to prove that $\left|\int_a^b \sin x^2\,dx\right|\le\frac{1}{a}$ if $0<a<b<\infty$.

17. Discuss tha applicability of the second Mean value theorem to the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos x\,dx$.

18. Verify second Mean value theorem (Weierstrass form) for the function f on the indicated intervals.

(*i*) $f(x)=x\sin x, x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$,

(ii) $f(x) = xe^x, x \in [-1, 1]$,

(iii) $f(x) = x \sin x, x \in [\pi, 2\pi]$.

Answer :

1. (i) $\frac{1}{2}$, (ii) $\frac{1}{3}$, (iii) e,

15. (i) $2\log 2$, (ii) $\frac{2}{\pi}$, (iii) $\tan^{-1} 2$, (iv) $\frac{4}{e}$, (v) $\frac{4}{e}$.

1.15 References

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Unit - 2

IMPROPER INTEGRAL

Unit - 2 : Improper Integrals

Structure

2.0 Objectives

In this chapter we have discussed about the types of Improper Integrals and their Convergence, different methods for test of Convergence, Convergence of the integral of Product and Convergence of Beta and Gamma Functions.

2.1 Introduction

The two important limitations in the definition of definite integral $\int_a^b f(x)\,dx$ are (i) the limits a and b are finite and (ii) the integrand is bounded and integrable in $a \le x \le b$. If either (or both) of these limitations is not fulfilled, that is, when a limit is infinite or the integrand becomes infinite in $a \le x \le b$ then the symbol $\int_a^b f(x)\,dx$ is called an improper integral.

If either one or both the limits of integration become infinite, and the integrand bounded, then $\int_a^b f(x)dx$ is called an improper integral of first kind. If $f(x)$ becomes unbounded on $[a,b]$ and the limits of integration are finite, then we say that $\int_a^b f(x)dx$ is an improper integral of second kind.

A. Improper integrals on an unbounded interval (First Kind).

There are three kinds of unbounded ranges over which integrals may be taken are symbnolised and defened as follows :

(1) Let $f(x)$ be bounded and integrable in $a \le x \le B$ for every $B > a$. Then the symbol $\int_a^\infty f(x)dx$ called the improper integral, is said to coverage or to exists if $\lim_{B\to\infty}\int_a^B f(x)dx$ finitly and we write

$$\int_a^\infty f(x)dx = \lim_{B\to\infty}\int_a^B f(x)dx.$$

The improper integral diverges if the limit tends to infinity with a fixed sign. Finaly when none of these alternatives occurs, we say it is oscilatory.

(2) If $f(x)$ be bounded and integrable in $A \le x \le b$ and $\lim_{A\to-\infty}\int_A^b f(x)dx$ exists finitely then we say that the improper integral $\int_{-\infty}^b f(x)dx$ exists or is convergent and we write $\int_{-\infty}^b f(x)dx = \lim_{A\to-\infty}\int_A^b f(x)dx$.

If the limit tends to plus infinity or to minus infinity, then the improper integral is said to diverge. And if there is no limit, the integral is said to be oscillatory.

(3) If $f(x)$ be bounded and integrable in $A \le x \le a$ for every $A < a$ and in $a \le x \le B$ for every $B > a$ and $\lim_{A\to-\infty}\int_A^a f(x)dx$ and $\lim_{B\to\infty}\int_a^B f(x)dx$ for $A < a < B$ exist finitely then we say that the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ is convergent and we write

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^\infty f(x)dx$$

$$= \lim_{A\to-\infty}\int_A^a f(x)dx + \lim_{B\to\infty}\int_a^B f(x)dx.$$

Example 2.1.1.

Does the improper integral $\int_0^\infty \frac{1}{1+x^2}dx$ *exist?*

Solution : To determine whether this integral is convegent or not, we see that $\frac{1}{1+x^2}$ is bounded and integrable in $0 \le x \le R$ for every $B > 0$ and

$$\lim_{B\to\infty}\int_0^B \frac{1}{1+x^2}dx = \lim_{B\to\infty}\left[\tan^{-1}x\right]_0^B = \lim_{B\to\infty}\left(\tan^{-1}B - \tan^{-1}0\right)$$

$$= \lim_{B\to\infty}\tan^{-1}B = \frac{\pi}{2}$$

Hence the integral $\int_0^\infty \frac{1}{1+x^2}dx$ does exist and its value is equal to $\frac{\pi}{2}$.

Example 2.1.2.

Evaluate $\int_0^\infty \sin x\,dx$, *if it exists.*

Solution : Here

$$\lim_{B\to\infty}\int_a^B \sin x\,dx = \lim_{B\to\infty}\left[-\cos x\right]_a^B = \lim_{B\to\infty}\left(\cos a - \cos B\right),$$

oscillates finitely. Therefore, $\int_a^\infty \sin x\,dx$ is oscillatory.

Example 2.1.3.

Evaluate $\int_0^\infty e^x\,dx$, *if it converges.*

Solution : Here

$$\lim_{B\to\infty}\int_a^B e^x\,dx = \lim_{B\to\infty}\left(e^B - 1\right).$$

Since $\left(e^B - 1\right)$ increases beyond all bounds as $B \to \infty$, this integral does not converge.

Example 2.1.4.

Evaluate $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$, *if it converges.*

Solution : For convenience, we break this infinite range into two parts as

$$I=\int_{-\infty}^{0} xe^{-x^2}\,dx+\int_{0}^{\infty} xe^{-x^2}dx .$$

Now

$$\lim_{A\to-\infty}\int_{A}^{0} xe^{x^2}\,dx+\lim_{B\to\infty}\int_{0}^{B} xe^{x^2}dx$$

$$=\lim_{A\to-\infty}\left[-\frac{1}{2}e^{x^2}\right]_{A}^{0}+\lim_{B\to\infty}\left[-\frac{1}{2}e^{x^2}\right]_{0}^{B}$$

$$=\lim_{A\to-\infty}\left(\frac{1}{2}e^{A^2}-\frac{1}{2}\right)+\lim_{B\to\infty}\left(\frac{1}{2}-\frac{1}{2}e^{B^2}\right).$$

Thus $\int_{-\infty}^{\infty} xe^{-x^2}dx=0$.

B. Improper integrals on a finite interval where the integrand is unbounded (Second Kind).

Here also we have three kinds of integrals may be taken are symbolised and defined as follows :

(1) Let $f(x)$ has an infinite discontinuity only at the left hand endpoint a, then by

$\int_{a}^{b} f(x)dx$ we shall mean $\lim_{\varepsilon\to0+}\int_{a+\varepsilon}^{b} f(x)dx$, $\quad 0<\varepsilon<b-a$.

(2) Let $f(x)$ has an infinite discontinuity only at the right hand end-point b, they by

$\int_{a}^{b} f(x)dx$ we shall mean $\lim_{\varepsilon\to0+}\int_{a}^{b-\varepsilon} f(x)dx$, $\quad 0<\varepsilon<b-a$.

(3) Let $f(x)$ has an infinite discontinuity at the point $x = c$ where $a < c < b$, then by $\int_a^b f(x)dx$ we shall mean $\lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0+} \int_{c+\delta}^b f(x)dx$.

If either of these limits fail to exist, we say that the integrand does not exist.

If however we make $\varepsilon = \delta$ and say that

$$\int_a^b f(x)dx \ \textit{means} \lim_{\varepsilon \to 0+} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right]$$

We heve what is called the Cauchy Principal value of $\int_a^b f(x)dx$ and write it as

$$Pc\int_a^b f(x)dx = \lim_{\varepsilon \to 0+} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right].$$

It may sometimes happen that the Cauchy Principla value of the integral exists when according to the general definition the integral does not exist.

Whenever the appropriate limits exist finitely, an improper integral is said to be convergent. When the appropriate limits fail to exist or tend to infinity with a fixed sign, an improper integral is said to be non-convergent. In the third case both limits must exist and be finite in order that the integral is to converge.

Illustrative Examples

Example 2.1.5.

Evaluate $\int_0^1 \frac{1}{x} dx$, *if it converges.*

Solution : Here $\frac{1}{x}$ has an infinite discontinuity at $x = 0$. So, we evaluate

$$\int_\varepsilon^1 \frac{1}{x} dx = \log 1 - \log \varepsilon = -\log \varepsilon$$

As $\varepsilon \to 0+$, $\log \varepsilon \to -\infty$

Hence $\lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{1}{x} dx$ does not exist and the integral does not converge.

Example 2.1.6.

Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1-\varepsilon)$, *if it converges.*

Solution : Since the integrand becomes infinite as $x \to 1$, we evaluate

$$\int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1-\varepsilon)$$

As $\varepsilon \to 0+$, $\sin^{-1}(1-\varepsilon) \to \sin^{-1}(1) = \frac{\pi}{2}$.

Hence $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

Example 2.1.7.

Prove that $\int_{-1}^1 \frac{1}{x^3}$ *exists in Cauchy Principal value sense but not in general sense*

Solution : The integrand is unbounded as $x \to 0$. Therefore, we evaluate

$$\lim_{\varepsilon \to 0+} \int_{-1}^{-\varepsilon} \frac{1}{x^3} dx + \lim_{\delta \to 0+} \int_{\delta}^{1} \frac{1}{x^3} dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{2x^2} \right]_{-1}^{-\varepsilon} + \lim_{\delta \to 0+} \left[-\frac{1}{2x^2} \right]_{\delta}^{1}$$

$$= \lim_{\varepsilon \to 0+} \left\{ \frac{1}{2} - \frac{1}{2\varepsilon^2} \right\} + \lim_{\delta \to 0+} \left\{ -\frac{1}{2} + \frac{1}{2\delta^2} \right\}.$$

Since $\lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon^2}$ and $\lim_{\delta \to 0+} \frac{1}{2\delta^2}$ do not exist, the original integral does not exist. If however, we consider Cauchy Principal value, we are to find.

$$\lim_{\varepsilon \to 0+} \left[\int_{-1}^{-\varepsilon} \frac{1}{x^3} dx + \int_{\delta}^{1} \frac{1}{x^3} dx \right] = \lim_{\varepsilon \to 0+} \left\{ \left(\frac{1}{2} - \frac{1}{2\varepsilon^2} \right) + \left(-\frac{1}{2} + \frac{1}{2\varepsilon^2} \right) \right\} = 0$$

Since the term involving ε cancel before the limit is taken.

A useful Theorem in Evaluating Improper Integrals

Theorem 2.1.1.

If (i) $f(x)$ *be bounded and integrable in* $0 < x / \leq a$ *leqa and tends to* ∞ *only when* $x \to 0+$ *or* $f(x)$ *is bounded and integrable in* $0 \leq x < a$ *and tends to* ∞ *only when* $x \to a-$ *and (ii)* $\int_0^a f(x)dx$ *converges, then*

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Proof. Let $f(x) \to \infty \text{ as } x \to 0+$. And since $\int_0^a f(x)dx$ we have from definition

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^a f(x)dx$$

exists and finite. Putting $x = a - z$,

$$\int_\varepsilon^a f(x)dx = \int_0^{a-\varepsilon} f(a-z)dz = \int_\varepsilon^{a-\varepsilon} f(a-x)dx$$

and the result follows. Similar in the case where $f(x) \to \infty$ as $x \to a-$. Hence the theorem. □

Remark 2.1.1.

$\int_0^a f(x)dx = \int_0^a f(x)dx + \int_\varepsilon^a f(x)dx$ *when*

$f(x) \to \infty$ *as* $x \to 0+$ *or* $f(x) \to \infty$ *as* $x \to a-$, $f(x)$ *being bounded and integrable in* $0 < x \leq a$ *un the first case and in* $0 \leq x < a$ *in the second case provided* $\int_0^a f(x)dx$ *converges.*

Illustrative Examples

Example 2.1.8.

Assuming the integrals to be convergent show that

$$\int_0^{\frac{\pi}{2}} \log \sin \mathrm{x}\mathrm{dx} = \int_0^{\frac{\pi}{2}} \log \cos \mathrm{x}\mathrm{dx} = \frac{\pi}{2}\log\frac{1}{2}.$$

Solution : The only singularity is at $x = 0$. The integrand has been assumed to be convergent, hence by

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

we have

$$I = \int_0^{\frac{\pi}{2}} \log \sin x\,dx = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \log \cos x dx$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x)\,dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{2}\sin 2x\right)dx$$

[addition is valid, since both integrals are convergent]

$$= \int_0^{\frac{\pi}{2}} \log\frac{1}{2}dx + \int_0^{\frac{\pi}{2}} \log \sin 2x dx = \frac{\pi}{2}\log\frac{1}{2} + \int_0^{\frac{\pi}{2}} \log \sin 2x dx.$$

Next to find the convergent integral ($\log\frac{1}{2}$ being convergent)

$$\int_0^{\frac{\pi}{2}} \log \sin 2x dx.$$

We are to calculate $\int_0^{\frac{\pi}{2}-\delta} \log \sin 2x dx$ when $\varepsilon, \delta \to 0+$

Thus,

$$\int_0^{\frac{\pi}{2}} \log \sin 2x dx = \frac{1}{2}\int_0^{\pi} \log \sin x dx$$

$$\therefore\ 21 = \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}\cdot 2\int_0^{\pi} \log \sin x dx$$

$$= \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \log \sin x dx$$

$$= \frac{\pi}{2}\log\frac{1}{2} + I$$

Thus, $I = \frac{\pi}{2}\log\frac{1}{2}$.

Example 2.1.9.

Assuming the integrals to be convergent show that

$$\int_0^{\pi} \frac{x\tan x dx}{\sec x + \cos x} = \frac{\pi^2}{4}.$$

Solution : The only singularity is at $x = \frac{\pi}{2}$. The integrand $x = \frac{\pi}{2}$ has been assumed to be convergent, hence we have

$$I = \lim_{\varepsilon\to 0+}\int_0^{\frac{\pi}{2}-\varepsilon} f(x)\,dx + \lim_{\delta\to 0+}\int_{\frac{\pi}{2}+\delta}^{\pi} f(x)\,dx$$

$$= \lim_{\varepsilon\to 0+}\int_0^{\frac{\pi}{2}-\varepsilon} \frac{x\tan x}{\sec x + \cos x}dx + \lim_{\delta\to 0+}\int_{\frac{\pi}{2}+\delta}^{\pi} \frac{x\tan x}{\sec x + \cos x}dx$$

$$= \lim_{\varepsilon\to 0+}\int_0^{\frac{\pi}{2}-\varepsilon} \frac{x\sin x}{1+\cos^2 x}dx + \lim_{\delta\to 0+}\int_{\frac{\pi}{2}+\delta}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

Putting $x = \pi - z$ in the first integral and $x = \pi - t$ *in the second integral we get*

$$I = \lim_{\varepsilon\to 0+}\int_{\frac{\pi}{2}+\varepsilon}^{\pi} \phi(\pi - z)\,dz + \lim_{\delta\to 0+}\int_0^{\frac{\pi}{2}-\delta} \phi(\pi - t)\,dt$$

$$= \lim_{\delta\to 0+}\int_0^{\frac{\pi}{2}-\delta} \phi(\pi - x)\,dx + \lim_{\varepsilon\to 0+}\int_{\frac{\pi}{2}+\varepsilon}^{\pi} \phi(\pi - x)\,dx$$

$$= \int_0^{\pi} \phi(\pi - x)\,dx \qquad \text{Where } \phi(x) = \frac{x\sin x}{1+\cos^2 x}$$

i.e.

$$\int_0^{\pi} \frac{x\tan x dx}{\sec x + \cos x} = \int_0^{\pi} \frac{x\sin x dx}{1+\cos^2 x} = \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$$

Hence

$$21 = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx.$$

And

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = a \ \ proper \ \ integral$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left[\frac{\sin x}{1+\cos^2 x} + \frac{\sin(\pi - x)}{1+\cos^2(\pi - x)} \right] dx$$

$$= \frac{\pi}{2} \times 2 \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \pi \times \int_0^1 \frac{dz}{1+z^2} \ \ where \quad z = \cos x$$

$$= \pi \times \left[\tan^{-1} z \right]_0^1 = \pi \times \frac{\pi}{4} = \frac{\pi^2}{4}.$$

2.2. Exercise 1.

1. Show that

(i) $\int_0^{\infty} e^{-x}\, dx = 1$. (ii) $\int_{-\infty}^{0} e^{2x}\, dx = \frac{1}{2}$

(iii) $\int_0^3 \frac{dx}{\sqrt{9 - x^2}} = \frac{\pi}{2}$ (iv) $\int_{-\infty}^{\infty} x e^{-x^2} dx = o$

(v) $\int_{-\infty}^{\infty} \frac{x dx}{x^4 + 1} = o$

2. Prove that $\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2(a+b)}; \ a, b > 0$.

3. Show that $\int_0^{\infty} e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}; \ a > 0$.

4. Show that $\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}; \quad a > b$.

5. Show that $\int_0^{\infty} \frac{dx}{x^2 + 2x\cos\alpha + 1} = \frac{\alpha}{\sin\alpha}; \quad 0 < \alpha < \pi$.

6. Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}; \quad a,b>0$.

7. Prove that $\int_0^{\frac{\pi}{2}} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2} = \frac{\pi}{4ab^2(a+b)}; \quad a,b>0$.

8. Assuming the following integrals are convergent prove that

(i) $\int_0^{\pi} \log(1+\cos x)\,dx = -\pi\log 2$

(ii) $\int_0^{\pi} \log(1+\cos\theta)\,d\theta = -\pi\log\frac{1}{2}$

(iii) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\log\frac{1}{2}$

(iv) $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{\sec x + \operatorname{cosec} x} = \frac{\pi}{2}\left(1+\log\sqrt{2}+1\right)$

(v) $\int_0^{\frac{\pi}{2}} \log(\tan x + \cot x)\,dx = \pi\log 2$

(vi) $\int_0^{\frac{\pi}{2}} \log\tan x\,dx = 0$

9. (i) If $I_n = \int \frac{\sin nx}{\sin x}\,dx$, show that $(n-1)(I_n - I_{n-2}) = 2\sin(n-1)x$. Hence or otherwise prove that $\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx = \pi$ or 0 according as n is odd or even.

(ii) If $I_n = \int_0^{\pi} \frac{\sin^2 nx}{\sin^2 x}\,dx$ when n is a positive integer, show that

$$I_{n+1} = 2I_n - I_{n-1}$$

Hence deduce that $I_n = n\pi$.

(iii) If $I_n = \int_0^{\pi} \sin 2nx \cot x\,dx,\ n \ge 1$, show that $I_{n+1} = I_n$ and that $I_n = \frac{\pi}{2}$.

10. Verify each of the following :

(i) $\lim_{n\to\infty}\left\{\frac{1}{n} + \frac{1}{\sqrt{n^2-1^2}} + \ldots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right\} = \frac{\pi}{2}$.

(ii) $\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}=\frac{\pi}{2}+1$.

(iii) $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}=\frac{1}{e}$.

2.3. Necessary and Sufficient Condition for the Convergence of the First Type Improper Integral $\int_0^\infty f(x)dx$

Let $f(x)$ be bounded and integrable in $a \le x \le B$ *for every* $B > a$ *so that the proper integral*

$$\int_a^B f(x)\,dx$$

exists and is a function of B, say, $F(B)$. Suppose now that $F(B)$ tends to a finite limit I as $B \to \infty$. Then according to the definition of limit we get the following.

Definition 2.3.1.

Let f (x) be bounded and integrable in $a \le x \le B$ *for every B>a. The integral* $\int_a^\infty f(x)dx$ *is convergent and has the value I if for any preassigend positive numbers, however small, there corresponds a positive number X such that.*

$$\left|I-\int_a^B f(x)dx\right|<\varepsilon \quad B>x$$

Also the Cauchy Criterion for existence of the limit of the function f (B), we may come to the following theorem.

Theorem 2.3.1. ***(Cauchy Criterion)***

A necessary and sufficient condition for the convergence of the integral $\int_a^B f(x)dx$ *is that for any preassigned positive number* ε, *however small, there corresponds a positive number X such that*

$$\left| I - \int_a^b f(x)\,dx \right| < \varepsilon$$

for all values x_1, x_2 *for which* $x_2 > x_1 > X$.

Clearly, a similar criterion holds for the existence of an improper integral when the singularity is at the lower limit.

Absolute and Conditional Convergence

Definition 2.3.2.

The integral $\int_a^\infty f(x)\,dx$ *is said to converge absolutely when* $\int_a^\infty |f(x)|\,dx$ *converges and when* $f(x)$ *is bounded and integrable in the arbitrary interval* $a \le x \le B$ *for every* $B > a$. *But if the first integral converges and the second diverges, then we say that* $\int_a^\infty f(x)\,dx$ *is conditionally convergent.*

Theorem 2.3.2.

An absolutely convergent improper integral carries it ordinary convergence. That is, if $\int_a^\infty f(x)\,dx$ *is an absolutely convergent improper integral, it is convergent.*

Proof. Since

$$\left| \int_{x_1}^{x_2} f(x)\,dx \right| \le \int_{x_1}^{x_2} |f(x)|\,dx, \quad \text{for} \quad x_2 > x_1 > X$$

it follows with a two-way use of the Cauchy Criterion that if $\int_a^\infty |f(x)|\,dx$ *converges, so also does* $\int_a^\infty f(x)\,dx$. *Hence the theorem.* □

Observation :

The converse of the theorem is not necessarily thue. That is, an improper integral of this type may converge and yet not necessarily converge absolutely.

For example let us consider $\int_a^\infty \frac{\sin x}{x}\,dx$.

In the first step we will show that $\int_a^\infty \frac{\sin x}{x}\,dx$ is convergent. We note that

$\lim_{x\to 0}\dfrac{\sin x}{x}=1$. Hence the integrand is not singular at $x=0$. The singularity is only at upper limit. Next since $\dfrac{1}{x}$ and $\sin x$ are bounded and integrable in $[x_1, x_2]$, by the second mean value theorem of integral calculus, we have

$$\int_{x_1}^{x_2}\frac{\sin x}{x}dx=\frac{1}{x_1}\int_{x_1}^{\varepsilon}\sin x dx+\frac{1}{x_2}\int_{\varepsilon}^{x_2}\sin x dx,\quad x_1\le\varepsilon\le x_2 .$$

Now $\left|\int_{x_1}^{\varepsilon}\sin x dx\right|=|\cos x_1-\cos\xi|\le|\cos x_1|+|\cos\xi|\le 2$ and similarly $\left|\int_{\varepsilon}^{x_2}\sin x dx\right|\le 2$

Consequently

$$\left|\int_{x_1}^{x_2}\sin x dx\right|\le 2\left\{\frac{1}{x_1}+\frac{1}{x_2}\right\}<\frac{1}{x_1}<\varepsilon,\ x_2>x_1>X$$

Provided $X=\dfrac{4}{\varepsilon}$. Thus the integral is convergent.

Next we show that $\int_0^\infty\left|\dfrac{\sin x}{x}\right|dx=\int_0^\infty\dfrac{|\sin x|}{x}dx$ diverges. First of all let us consider the integral $\int_0^{n\pi}\dfrac{|\sin x|}{x}dx$

where n is any positive integer.

We have

$$\int_0^{n\pi}\frac{|\sin x|}{x}dx=\sum_{r=1}^{n}\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx .$$

Putting $x=(r-1)\pi+t$, we obtain

$$\int_{(r-1)\pi}^{r\pi}\frac{|\sin x|}{x}dx=\int_0^{\pi}\frac{|\sin\{(r-1)\pi+t\}|}{(r-1)\pi+t}dt=\int_0^{\pi}\frac{|\sin t|}{(r-1)\pi+t}dt$$

since $|\sin\{(r-1)\pi+t\}|=|(-1)^{r-1}\sin t|=|\sin t|=\sin t$ since t varies from 0 to π.

Again since $r\pi$ is the maximum value of $(r-1)\pi+t$ in $0\le t\le\pi$, we have

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{r\pi}\int_0^{\pi} \sin t dt = \frac{2}{r\pi}$$

$$\therefore \int_0^{r\pi} \frac{|\sin x|}{x} dx \geq \frac{2}{\pi}\sum_{r=1}^{n}\frac{1}{r}.$$

But the series on the right diverges to ∞ as $n \to \infty$, whereby

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx \to \infty \quad n \to \infty.$$

Let now X be any real number. Then without loss of generality we may assume $n\pi \leq X \leq (n+1)\pi$ for n to be a positive integer.

Thus

$$\int_0^{X} \frac{|\sin x|}{x} dx \geq \int_0^{n\pi} \frac{|\sin x|}{x} dx$$

Let now $X \to \infty$ so that n also $\to \infty$. Thus we see that

$$\int_0^{X} \frac{|\sin x|}{x} dx \to \infty \quad X \to \infty.$$

Hence $\int_0^{\infty} \frac{|\sin x|}{x} dx$ diverges.

2.4. Tests for Convergence (First Type)

(A) Comparison Test

Theorem 2.4.1.

If $f(x)$ be a non-negative integrable function when $x \geq a$ and $\int_a^B f(x)dx$ is bounded above for every $B > a$, then $\int_a^{\infty} f(x)dx$ will converge, otherwise it will diverge to ∞.

Theorem 2.4.2.

If $f(x)$ and $g(x)$ is integrable functions when $x \geq a$ such that $0 \leq f(x) \leq g(x)$, then *(i)* $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges*

(ii) $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

(B) Limit Tests

Theorem 2.4.3.

Let $f(x)$ *and* $g(x)$ *be integrable function when* $x \geq a$ *and* $g(x)$ *be positive. Then if*

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lambda \neq 0,$$

the integrals $F = \int_a^\infty f(x)dx$ *and* $G = \int_a^\infty g(x)dx$ *both converges absolutely or both diverges.*

If $f/g \to 0$ *and G converges, then F converge absolytely. If* $f/g \to \pm\infty$ *and G diverges, then F diverges.*

Comparison Integral 1.

Show that the inproper integral $\int_0^\infty e^{px}dx$, *where p is a constant, converges for p>0 and diverges when* $p \leq 0$.

Proof. We have

$$\int_0^B e^{px}dx = -\frac{1}{p}\left[e^{-px}\right]_0^B = \frac{1}{p}\left\{1-\frac{1}{e^{pB}}\right\}, \quad p \neq 0$$

and $\int_0^B dx = B$ when $p = 0$.

Let $B \to \infty$, then $\int_0^\infty e^{px}dx = \frac{1}{p}$, when $p > 0$ and diverges when $p \leq 0$. □

Comparison Integral 2.

Show that the inproper integral $\int_0^\infty \frac{dx}{x^\mu}dx\,(\mu > 0)$ exists if $\mu > 1$ and does not exist, if $\mu \leq 1$.

Proof. We have

$$\int_0^B \frac{dx}{x^{\mu}} dx = \frac{1}{1-\mu}\left\{B^{1-\mu} - a^{1-\mu}\right\} \text{ when } \mu \neq \perp$$

and

$$\int_0^B \frac{dx}{x} = \log B - \log a \quad \text{ when } \mu = 1$$

Let $B \to \infty$, then $\int_a^{\infty} \frac{dx}{x^{\mu}} = \begin{cases} \dfrac{a^{1-\mu}}{\mu - 1}, & \text{when } \mu > 1 \\ \infty, & \text{when } \mu < 1 \end{cases},$

whereby the result follows. □

Theorem 2.4.4. (***The*** μ ***test for convergence)***

Let $f(x)$ *be an integrable function when* $x \geq a$. *Then* $F = \int_a^{\infty} f(x)dx$ *converges abslutely if*

$$\lim_{x \to \infty} x^{\mu} f(x) = \lambda, \quad \mu > 1$$

and F diverges if

$$\lim_{x \to \infty} x^{\mu} f(x) = \lambda(\neq 0) \quad or \pm\infty, \quad \mu \leq 1.$$

Illustrative Examples :

From Comparison Integrals 1 and 2 it has been clear that $\int_0^{\infty} e^{-\frac{1}{2}x} dx$, $\int_0^{\infty} e^{-x} dx, \int_0^{\infty} e^{-4x} dx, \ldots$ and $\int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}}, \int_1^{\infty} \frac{dx}{x^3}, \int_1^{\infty} \frac{dx}{x^4}, \cdots$ converge; whereas $\int_0^{\infty} e^{\frac{1}{2}x} dx$, $\int_0^{\infty} e^{x} dx, \ldots$ and $\int_1^{\infty} \frac{dx}{x}, \int_1^{\infty} \frac{dx}{\sqrt{x}}, \cdots$ diverge.

Examples 2.4.1.

$\int_0^{\infty} \frac{dx}{e^x + 1}$ *converges by comparison test, since* $0 \leq \frac{1}{e^x + 1} \leq \frac{1}{e^x} = e^{-x}$ *and* $\int_0^{\infty} e^{-x} dx$.

Examples 2.4.2.

$\int_2^\infty \frac{dx}{\log x}$ *diverges by comparison test, since for* $x \geq 2,\ \log x < x, \frac{1}{\log x} > \frac{1}{x}$ *and* $\int_2^\infty \frac{dx}{x}$ *diverges.*

Examples 2.4.3.

$\int_2^\infty \frac{\sin^2 x}{s^2} dx\,(a > 0)$ *converges by comparison test,*

since $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ *when* $x \geq a > 0$ *and* $\int_0^\infty \frac{1}{x^2} dx$ *converges.*

Examples 2.4.4.

$\int_1^\infty e^{-x} x^n dx$ *converges by* $\mu -$ *test for all values of n, since as* $x \to \infty$

$x^2 f(x) = \frac{x^{n+2}}{e^x} \to 0$, *for* $\mu = 2 > 1$.

2.5. Necessary and Sufficient Condition for the Convergence of the Second Type Improper Integral $\int_a^b f(x)\,dx$

Let $f(x)$ be bounded and integrable in $a \leq x \leq b$ and a be the only point of infinite discontinuity of $f(x)$ in a finite interval $[a, b]$. Then the proper integral

$$\int_{a+\varepsilon}^b f(x)dx, \quad 0 < \varepsilon < b - a$$

exists and is a function of ε, say, $\phi(\varepsilon)$. Suppose now that $\phi(\varepsilon)$ tends to a finite limit I as $\varepsilon \to 0+$. Then according to the difinition of limit we get the following.

Definition 2.5.1.

Let $f(x)$ *be bounded and integrable in a* $a \leq x \leq b$ *and a be the only point of infinite discontinuity of* $f(x)$ *in a finite interval* $[a, b]$. *The integral* $\int_a^b f(x)dx$ *is convergent and has the value I if for any preassigned positive number* ε', *however small, there corresponds a positive number* δ *such that.*

$$\left|I - \int_{a+\varepsilon}^{b} f(x)\,dx\right| < \varepsilon' \quad 0 < \varepsilon < \delta$$

Also the Cauchy Criterion may be restated as:

Theorem 2.5.1. ***(Cauchy Criterion)***

A necessary and sufficient condition for the convergence of the integral $\int_a^b f(x)\,dx$ *is that for any preassigned positive number* ε', *however small, there corresponds a positive number* δ *such that*

$$\left|\int_{a+\varepsilon_1}^{a+\varepsilon_2} f(x)\,dx\right| < \varepsilon' \quad \textit{for} \quad 0 < \varepsilon_1 < \varepsilon_2 < \delta.$$

for all values x_1, x_2 *for which* $x_2 > x_1 > X$.

Clearly, a similar criterion holds for the existence of an improper integral when the singularity is at the lower limit.

Absolute and Conditional Convergence

Definition 2.5.2.

Let a be the only point of infinite discontinuity of a function $f(x)$ *ina finite interval* $[a,b]$. *The integral* $\int_a^b f(x)\,dx$ *is said to converge absolutely if* $f(x)$ *is bounded and integrable in the arbitrary interval* $[a+\varepsilon, b]$ *where* $0 < \varepsilon < b-a$ *and* $\int_a^b |f(x)|\,dx$ *converges. But if the first integral converges and the second diverges, then we say that* $\int_a^b f(x)\,dx$ *is conditionally convergent.*

Theorem 2.5.2.

An absolutely convergent improper integral $\int_a^b f(x)\,dx$, *where* $f(x)$ *has an infinite discintinuity at* $x = a$ *only, carries with it ordinary convergence, but the converse is not necessarily true.*

2.6. Tests for Convergence (Second Type)

(A) Comparison Test

Theorem 2.6.1.

If $f(x)$ *be a non-negative integrable function in* $a < x < b$ *and a be the only point of discontinuity of* $f(x)$ *in a finite interval* $[a,b]$ *and* $\int_{a+\varepsilon}^{b} f(x)dx$ *is bounded above for* $0 < \varepsilon < b-a$*; the integral* $\int_{a}^{b} f(x)dx$ *will converge, otherwise it will diverge to* ∞.

Proof. Since $f(x)$ *is non-negative in* $0 < x \le b$, *the integral.*

$$\int_{a+\varepsilon}^{b} f(x)dx$$

monotonically increase as ε decreases and will approach a finite limit if bounded above, but if unbounded, it will tend to ∞. □

Theorem 2.6.2.

Let a be the only point of infinite discontinuity. If $f(x)$ *and* $g(x)$ *is integrable functions in* $a < x \le b$ *such that* $0 \le f(x) \le g(x)$, *then*

(i) $\int_{a}^{b} f(x)dx$ *converges if* $\int_{a}^{b} g(x)dx$ *converges*

(ii) $\int_{a}^{b} g(x)dx$ *diverges if* $\int_{a}^{b} f(x)dx$ *diverges.*

(B) Limit Tests

Theorem 2.6.3.

Let $f(x)$ *and* $g(x)$ *be integrable function when* $a < x \le b$ *and* $g(x)$ *be positive.*

Then if $\lim_{x\to a+} \frac{f(x)}{g(x)} = \lambda \ne 0$,

the integrals $F=\int_{a}^{b} f(x)dx$ *and* $G=\int_{a}^{b} g(x)dx$ *both converges absolutely or both diverges.*

If $f/g \to 0$ *and G converges, then F converge absolutely. If* $f/g \to \pm\infty$ *and G diverges, then F diverges.*

Comparison Integral 3.

Show that the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ exists, if $\mu < 1$ and does not exist, if $\mu \geq 1$.

Proof. When $\mu \neq 1$, we have

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^\mu} = -\frac{1}{1-\mu}\left\{(b-a)^{1-\mu} - e^{1-\mu}\right\}$$

and when $\mu = 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon\,.$$

On letting $\varepsilon \to 0+$, we obtain

$$\int_a^b \frac{dx}{(x-a)^\mu} = \begin{cases} \dfrac{(b-a)^{1-\mu}}{1-\mu}, & \text{when } 0 < \mu < 1 \\ \infty, & \text{when } \mu \geq 1 \end{cases}$$

when however, $\mu \leq 1$, the integral is proper.

Comparison Integral 4.

The integral $\int_a^b \frac{dx}{(x-a)^\mu}$ converges if $(\mu < 1)$ and diverges if $\mu \geq 1$.

Theorem 2.6.4. ***(The*** μ***-test for convergence)***

Let $f(x)$ *be an integrable function in the arbitrary interval* $(a+\varepsilon, b)$, *where* $0 < \varepsilon < b-a$. *Then* $F = \int_a^b f(x)\,dx$ *converges absolutely if*

$$\lim_{x \to a+} (x-a)^\mu f(x) = \lambda, \quad \textit{for} \quad 0 < \mu < 1,$$

and F diverges if

$$\lim_{x \to a+} (x-a)^\mu f(x) = \lambda(\neq 0) \quad \textit{or} \ \pm\infty \quad \textit{for} \quad \mu \geq 1.$$

Illustrative Examples :

Example 2.6.1.

$\int_0^1 \frac{\log x}{(1+x)\sqrt{x}}$ *converges, since*

$$\lim_{x\to 0+}(x-0)^{\frac{1}{2}} = \lim_{x\to 0+}\frac{1}{1+x} = 1 \quad \textit{for} \quad \mu < 1.$$

Example 2.6.2.

$\int_0^1 \frac{\log x}{\sqrt{x}}dx$ *converges, since*

$$\lim_{x\to 0+}(x-0)^{\frac{3}{4}}\frac{\log x}{\sqrt{x}} = \lim_{x\to 0+}\frac{\log x}{x^{-\frac{1}{4}}} = \lim_{x\to 0+}\left(-4x^{\frac{1}{4}}\right) = 0 \quad \textit{for} \quad \mu < 1.$$

Example 2.6.3.

$\int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}}$ *converges, since*

$$\lim_{x\to 1-}(1-x)^{\frac{1}{2}}\cdot f(x) = 1 \quad \textit{for} \quad \mu < 1.$$

2.7. Convergence of the Integral of a Product

Theorem 2.7.1. ***(Test for absolute convergence. Type A.)***

Let $f(x)$ be a bounded and integrable function when $x \geq a$ and $\int_a^\infty \phi(x)dx$ converge absolutely, then $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent.

Proff. Since $f(x)$ is bounded when $x \geq a$

$$|f(x)| < M \text{ for every } x \geq a \qquad (2.7.1)$$

Where M is some definite positive constant. Again since $\int_a^\infty |\phi(x)|dx$ converges, there exists a positive number M' such that

$$\int_a^X |\phi(x)|dx < M' \qquad \text{for every} \qquad X \geq a \qquad (2.7.2)$$

Thus from (2.7.1) and (2.7.2), for every $X \geq a$

$$\int_a^X |f(x)\phi(x)|dx < M\int_a^X |\phi(x)|dx < MM'$$

so that $\int_a^\infty |f(x)\phi(x)|dx$ converges or, in other words $\int_a^\infty f(x)\phi(x)dx$ converge absolutely. Hence the theorem.

□

Theorem 2.7.2. ***(Test for absolute convergence. Type B.)***

Suppose a to be the only point of infinite discontinuity. Let $f(x)$ be a bounded and integrable function when $ax \geq b$ *and* $\int_a^b \phi(x)dx$ converge absolutely, then $\int_a^b f(x)\phi(x)dx$ is absolutely converges.

Theorem 2.7.3. ***(Abel's Test. Type A.)***

Let $f(x)$ be bounded and monotonic when $x \geq a$ and $\phi(x)$ be bounded and integrable on the arbitrary interval $a \leq x < B$ for every $B > a$ and also let $\int_a^\infty \phi(x)dx$ be convergent then $\int_a^\infty f(x)\phi(x)dx$ converges.

Proof. We have from second mean value theorem

$$\int_{x_1}^{x_2} f(x)\phi(x)dx = f(x_1)\int_{x_1}^{\xi}\phi(x)dx + f(x_2)\int_{\xi}^{x_2}\phi(x)dx \qquad (2.7.3)$$

for $a < x_1 \leq \xi \leq x_2$.

Since $f(x)$ is bounded, we can find a positive number M such that |f (x)|<M for every $x \geq a$. Thus in particular.

$$|f(x_1)| < M \text{ and } |f(x_2)| < M. \qquad (2.7.4)$$

Also since $\int_a^\infty \phi(x)dx$ is convergent, we can choose a positive number $X(\varepsilon)$ such that

$$\left|\int_{x_1}^{x_2}\phi(x)dx\right| < \frac{\varepsilon}{2M} \quad \text{for } x_2 > x_1 > X$$

We now suppose that in (2.7.3), x_1, x_2 are numbers greater than X so that

$$\left|\int_{x_1}^{\xi}\phi(x)\,dx\right|<\frac{\varepsilon}{2M} \quad \textit{and} \quad \left|\int_{\xi}^{x_2}\phi(x)\,dx\right|<\frac{\varepsilon}{2M}. \qquad (2.7.5)$$

It follows from (2.7.3), (2.7.4), (2.7.5) that

$$\left|\int_{x_1}^{x_2} f(x)\phi(x)\,dx\right| < M\cdot\frac{\varepsilon}{2M}+M.\frac{\varepsilon}{2M}=\varepsilon \quad \text{for } x_2>x_1>X(\varepsilon)$$ and the theorem follows. □

Theorem 2.7.4. ***(Dirichlet's Test.Type A.)***

Let $f(x)$ be bounded and nonotonic when $x \geq a$ and let $f(x) \to 0$ as $x \to \infty$. Also let $\phi(x)$ be bounded and integrable on the arbitrary interval $a \leq x < B$ for every $B > a$ and $\int_a^B \phi(x)\,dx$ be bounded when $B > a$. Then $\int_a^\infty f(x)\phi(x)\,dx$ converges.

Proof. We have from second mean value theorem

$$\int_{x_1}^{x_2} f(x)\phi(x)\,dx = f(x_1)\int_{x_1}^{\xi}\phi(x)\,dx + f(x_2)\int_{\xi}^{x_2}\phi(x)\,dx \qquad (2.7.6)$$

for $a < x_1 \leq \xi \leq x_2$.

Since $\int_a^B \phi(x)\,dx$ is bounded for $B > a$, there exists a positive number M such that $\left|\int_a^B \phi(x)\,dx\right| < M$ for every $B > a$. Therefore,

$$\left|\int_a^{x_1}\phi(x)\,dx\right| = \left|\int_a^{\xi}\phi(x)\,dx - \int_a^{\xi}\phi(x)\,dx\right|$$

$$\leq \left|\int_a^{\xi}\phi(x)\,dx\right| + \left|\int_a^{x_1}\phi(x)\,dx\right| < M + M$$

$$= 2M \cdot \qquad (2.7.7)$$

Similarly

$$\left|\int_{\xi}^{x_2}\phi(x)\,dx\right| < 2M. \qquad (2.7.8)$$

Next since $f(x) \to 0$ as $x \to \infty$, we can find a positive number $X(\varepsilon)$ such that $|f(x)| < \frac{\varepsilon}{4M}$ when $x > X$.

Next taking $x_2 > x_1 > X$, we have

$$|f(x_1)| < \frac{\varepsilon}{4M} \text{ and } |f(x_2)| < \frac{\varepsilon}{4M} \qquad (2.7.9)$$

Thus from (2.7.6)-(2.7.9) we have

$\left|\int_{x_1}^{x_2} f(x)\phi(x)dx\right| \le \frac{\varepsilon}{4M}2M + \frac{\varepsilon}{4M}2M = \varepsilon$ for $x_2 > x_1 > X(\varepsilon)$, and the theorem follows. □

2.8. Convergence of Gamma and Beta Functions

(1) Gamma Function :

Let us discuss the convergence of

$$\int_0^\infty e^{-x}x^{n-1}dx. \qquad (2.8.1)$$

We write, $f(x) = e^{-x}x^{n-1}, I_1 = \int_0^1 e^{-x}x^{n-1}dx \; I_2 = \int_1^\infty e^{-x}x^{n-1}dx$.

The part I_1 is proper when $n \ge 1$, improper but absolutely convergent when $0 < n < 1$; for as $x \to 0+, \left[e^{-x}x^{n-1} \to \infty \text{ as } x \to 0+\right]$ by $\mu-$ test

$$x^{1-n}f(x) = x^{1-n}e^{-x}x^{n-1} = e^{-x} \to 1$$

for $0 < \mu = 1-n < 1$, i.e., for $0 < n < 1$.

The part I_2 also converges absolutely for all values of n by $\mu-$test, for as $x \to \infty$,

$$x^2 f(x) = x^2.e^{-x}x^{n-1} = e^{-x}x^{n+1} \to 0$$

Thus (2.8.1) converges for $n > 0$. This is called Gamma function denoted by $\Gamma(n)$.

Hence

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, \quad n > 0$$

(2) Beta Function :

Next let us discuss the convergence of

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad (2.8.2)$$

This is a proper integral when $m, n > 1$ but is improper at the lower limit when

$m<1$, and at the upper limit when $n<1$. We, therefore, split it into two parts I_1+I_2 where

$$I_1=\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$$

and

$$I_2=\int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\,dx$$

Now I_1 converges for $0<m<1$, diverges when $m\le 0$, for as $x\to 0+$, by $\mu-$test

$$x^{1-m}f(x)=x(1-m)x^{m-1}(1-x)^{n-1}=(n-1)^{n-1}\to 1$$

for $=\mu=1\text{-m}$ and for convergence $0<\mu=1\text{- m}<1$ that is $0<\text{m}<1$ also.

$$\text{xf (x)}=\text{xx}^{\text{m-1}}(1\text{-x})^{\text{n-1}}=\text{x}^{\text{m}}(1\text{-x})^{\text{n-1}}\to\begin{cases}1 & \text{when m}=0\\ \alpha & \text{when m}<0\end{cases}$$

where $f(x)=x^{m-1}(1-x)^{n-1}(./\text{par})$ par Next if we change the variable $x=1-y$, the second integral reduces to the first with *m* and *n* interchanged. Hence we may draw the same calculation as before with *n* in place of *m*. Thus (2.8.2) converges for $m,n>0$. This is called the Beta function denoted by $B(m,n)$ or

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \text{ for } m,n>0.$$

Illustrative Examples :

Example 2.8.1.

Show that $\int_0^{\infty}\frac{\cos x}{\sqrt{1+x^3}}dx$ *converges absolutely.*

Solution :

since $\mu=\frac{5}{4}>1$ we have

$$\lim_{x\to\alpha} x^{\frac{5}{4}}\cdot\frac{\cos x}{\sqrt{1+x^3}}=\lim_{x\to\alpha}\frac{\cos x}{x^{\frac{1}{4}}\sqrt{1+x^{-3}}}=0.$$

Hence $\int_0^{\infty}\frac{\cos x}{\sqrt{1+x^3}}\,dx$ converges absolutely.

Example 2.8.2.

Show that $\int_0^\infty \frac{1-\cos x}{x^2}dx$ *converges.*

Solution :

By $\mu-$test, since at the lower limit $x=0$,

$$\lim_{x\to 0^+}\frac{1-\cos x}{x^2}=\frac{1}{2},$$

and at the upper limit, by $\mu-$test

$$\lim_{x\to\infty} x^{\frac{3}{2}}.\frac{1-\cos x}{x^2}=\lim_{\mu=\frac{3}{2}}\frac{1-\cos x}{\sqrt{x}}=0 \text{ for } \mu=\frac{3}{2>1}.$$

Hence $\int_0^\infty \frac{1-\cos x}{x^2}dx$ converges.

Example 2.8.3.

Prove that $\int_0^1 \frac{x^{m-1}}{1+x}dx$ *converges for* $m>0$.

Solution :

The integral is proper when $m\geq 1$ but is improper at the lower limit when $m<1$. Now the integral converges for $0<m<1$ by $\mu-$test, since as $x\to 0+$,

$$x^{1-m}\cdot\frac{x^{m-1}}{1+x}=\frac{1}{1+x}\to 1$$

and for convergence $0<\mu<1$, that is $0<1-m<1$, or $0<m<1$. Hence the integral is convergent for $m>0$.

Example 2.8.4.

Prove that $\int_1^\infty \frac{x^{m-1}}{1+x}dx$ *converges for* $m<1$.

Solution :

By $\mu-$test as $x\to\infty$

$$x^{2-m}\cdot\frac{x^{m-1}}{1+x}=\frac{x}{1+x}=\frac{1}{1+\frac{1}{x}}\to 1$$

and for convergence $\mu=2-m>1$, or $m<1$.

Hence the integral is convergent for $m<1$.

Example 2.8.5.

Discuss the convergence of $\int_0^{\frac{\pi}{2}} \log\sin x dx$.

Solution :

Note that the only singularity is at $x=0$. Also

$$\log \sin x = \log\left(x \cdot \frac{\sin x}{x}\right) = \log x + \log \frac{\sin x}{x}.$$

By $\mu-$test,

$$\lim_{x\to 0+0} x^{\mu} \log \sin x = \lim_{x\to 0+0}\left(x^{\mu}\log x + x^{\mu}\log\frac{\sin x}{x}\right) = 0$$

$$\left(\text{since} \lim_{x\to 0+0} x^{\mu} \log x = 0 \text{ if } \mu > 0 \text{ and } \lim_{x\to 0} \frac{\sin x}{x} = 1\right) \text{ for } \mu > 0.$$

See also that μ cannot be taken to be ≥ 1. Thus $0<\mu<1$. Hence the integral converges.

Example 2.8.6.

Show that $\int_0^{\frac{\pi}{2}} \sin x \log\sin x dx$ *converges and find its value.*

Solution :

The only singularity is at $x=0$. Now

$$\int_{\varepsilon}^{\frac{\pi}{2}} (\log\sin x)(\sin x)dx = \left[-\cos x\log\sin x\right]_{\varepsilon}^{\frac{\pi}{2}}$$

$$= \cos\varepsilon\log\sin\varepsilon + \left[\cos x + \log\tan\frac{x}{2}\right]_{\varepsilon}^{\frac{\pi}{2}}$$

$$= \cos\varepsilon\log\sin\varepsilon - \cos\varepsilon - \log\tan\frac{\varepsilon}{2}$$

$$\to \log 2 - 1 \quad \text{as} \quad \varepsilon \to 0+0$$

Since $\lim_{\varepsilon\to 0^+}\left(\cos\varepsilon\log\sin\varepsilon - \cos\varepsilon - \log\tan\frac{\varepsilon}{2}\right)$

$$\left[\text{Writing} \sin\varepsilon = 2\sin\frac{\varepsilon}{2}\cos\frac{\varepsilon}{2}, \tan\frac{\varepsilon}{2} = \sin\frac{\varepsilon}{2}/\cos\frac{\varepsilon}{2}\right]$$

$$= \lim_{\varepsilon\to 0+}\left\{(\cos\varepsilon-1)\log\sin\frac{\varepsilon}{2}+\cos\varepsilon\log 2\cos\frac{\varepsilon}{2}+\log\cos\frac{\varepsilon}{2}-\cos\varepsilon\right\}$$

and by L' Hospital's rule

$$\lim_{\varepsilon\to 0+}\left\{(\cos\varepsilon-1)\log\sin\frac{\varepsilon}{2}\lim_{\varepsilon\to 0+}\frac{\log\sin\frac{\varepsilon}{2}}{-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}}=\left(\frac{\infty}{\infty}\right)\right.$$

$$= \lim_{\varepsilon\to 0+}\frac{1}{2}\sin^2\frac{\varepsilon}{2}=0,$$

and also, $\lim_{\varepsilon\to 0+}\left\{\cos\varepsilon\log 2\cos\frac{\varepsilon}{2}+\log\cos\frac{\varepsilon}{2}-\cos\varepsilon\right\}=\log 2-1$.

Thus the integral converges and its value $=\log 2-1$.

Example 2.8.7.

Show that $\int_1^\infty \frac{\sin x}{x^p}dx$ *converges for* $p>0$.

Solution :

Apply Dirichlet's test, taking $f(x)=\frac{1}{x^p}, \phi(x)=\sin x$, $f(x)=\frac{1}{x^p}$ is bounded and monotone for $x\geq 1$ and $\to 0$ as $x\to\infty$ for $p>0$. Also $\int_1^B \phi(x)xdx=\int_1^B \sin dx$ is bounded for $B>1$, since

$\left|\int_1^B \sin xdx\right|=|\cos 1-\cos B|\leq|\cos 1|+|\cos B|\leq 2$ for $B>1$.

Therefore, by Dirichlet's test

$$\int_1^\infty \frac{\sin x}{x^p}dx \text{ converges for } p>0.$$

Example 2.8.8.

Show that $\int_a^\infty \frac{\cos x}{\log x}dx$ *for* $a>1$ *converges by Dirichlet's test.*

Solution :

Let $f(x)=\frac{1}{\log x}$ and $\phi(x)=\cos x$.

Then $f(x)$ is monotonic decreasing $\to 0$ as $x \to \infty$ and $\phi(x)$ is bounded in $[a,B]$, $B > a$.

Hence by Dirichlet's test, the integral converges for $a > 1$.

Example 2.8.9.

Examine the converges of the integral $\int_a^\infty e^{-a^2x^2} \frac{\sin 2bx}{x} dx$.

Solution :

See that

$$\int_0^\infty e^{-a^2x^2} \frac{\sin 2bx}{x} dx = 2b\int_0^\infty e^{-a^2x^2} \frac{\sin 2bx}{2bx} dx \qquad (2.8.3)$$

First part is a proper integral as $\lim_{x\to 0} \frac{\sin 2bx}{2bx} = 1$.

For the second part, let $f(x) = \frac{\sin 2bx}{2bx}$ and $\phi(x) = e^{-a^2x^2}$.

Then $\int_1^\infty f(x)dx = \int_1^\infty \frac{\sin 2bx}{2bx} dx$ is congergent by Dirichlet's test. Also $\phi(x)$ is bounded and monotonically decreasing in $[1, \infty)$.

Therefore , by Abel's test $\int_1^\infty f(x)\phi(x)dx = \int_1^\infty e^{-a^2x^2} \frac{\sin 2bx}{2bx} dx$ is convergent.

Hence from (2.8.3) it follows that the given integral; is convergent.

2.9 Summary

1. In Section 2.1 and 2.2 we have studied the types of Improper Integrals and a useful theorem in evaluating Improper Integrals with some examples.

2. In Section 2.3, 2.4, 2.5, and 2.6 we have discussed about necessary and sufficient condition for convergence of Improper integrals, Cauchy Criterion, Limit Test, Comparision Test, $\mu -$ test for convergence and two comparision Integrals.

3. In Section 2.7 and 2.8 we have discussed Abel's Test and Dirichlet's Test for absolute convergent and Convergence of Beta and Gamma function.

2.10. Exercise 2.

1. Show that the following integrals converge:

(i) $\int_0^1 \frac{dx}{(1+x)\sqrt{x}} dx$ (ii) $\int_0^1 \frac{\log x}{\sqrt{x}} dx$

(iii) $\int_0^\infty \frac{dx}{e^x+1}$ (iv) $\int_0^1 \frac{\log x}{(1-x)^{\frac{3}{2}}}dx$

(v) $\int_0^\infty \frac{x^2}{\sqrt{x^7+1}}dx$

2. Show that the following integrals are non-convergent :

(i) $\int_0^\infty \frac{dx}{\log x}dx$ (ii) $\int_0^\pi \frac{dx}{1-\cos x}$

(iii) $\int_0^\infty \frac{dx}{x\log x}$

3. Show that $\int_0^1 \frac{x^{m-1}}{1+x}dx$ converges for $m>0$ and $\int_1^\infty \frac{x^{m-1}}{1+x}dx$ converges for $m<1$ and hence $\int_0^\infty \frac{x^{m-1}}{1+x}dx$ converges for $0<m<1$.

4. Show that $\int_0^\infty \sin x^2 dx$ and $\int_0^\infty \cos x^2 dx$ converge.

5. Verify that $\int_0^\infty \frac{xdx}{1+x^6\sin^2 x}$ converges and $\int_0^\infty \frac{xdx}{1+x^2\sin^2 x}$ diverges.

6. Show that $\int_0^\infty \frac{xdx}{1+x^4\sin^2 x}$ converges but $\int_0^\infty \frac{dx}{1+x^2\sin^2 x}$ does not.

7. Prove that $\int_0^a \frac{1}{\sqrt{x}}\sin\left(\frac{1}{x}\right)dx$ converges for $0<a<\infty$.

8. Discuss the convergence of $\int_0^{\frac{\pi}{2}} \cos 2nx \log\sin x dx$ and evaluate it.

9. Show that the improper integral $\int_2^\infty \frac{\cos x}{\log x}dx$ is convergent but not absolutely convergent but not absolutely convergent.

10. Show that $\int_1^\infty \frac{\sqrt{x}}{(1+x)^2}dx$ converges to $\frac{1}{2}+\frac{\pi}{4}$

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Unit - 3

SEQUENCE AND SERIES OF FUNCTIONS

Unit - 3 : Sequence And Series Of Functions

Structure

3.0 Objectives

In this third unit we have focussed on Pointwise Convergent and Uniformly Convergent of Sequence and Series of functions, different tests for Uniformly Convergent and some theorems regarding Uniform Convergent and Continutity, Uniform Convergent and Integrability and Uniform Convergent and Differentiability. Also we have discussed on Limit superior and Limit inferior and eventually different tests for Convergent of Power Series.

3.1 Sequence of Functions

Let E be a subset of real number and for each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ be a function. Then $\{f_n\}$ is a sequence of functions on E to $\mathbb{R}$. E is said to be the domain of sequence of functions $\{f_n\}$.

To each $x_0 \in E$ the sequence $\{f_n\}$ gives rise to a sequence of real numbers $\{f_n(x_0)\}$, which is obtained by evaluating each f_n at x_0.

For sonme $x \in E$, the sequence $\{f_n(x)\}$ may converge to a limit and for some other x ε E, the sequence $\{f_n(x)\}$ not converge.

Pointwise Convergence

Let $E \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function. The sequence $\{f_n\}$ is said to be pointwise convergent on E if for each $x \in E$, the sequence $\{f_n(x)\}$ is convergent. And then it is natural to say that $\{f_n\}$, $n \in \mathbb{N}$ converges to the function f on E i.e.

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in E \tag{3.1.1}$$

If (3.1.1) holds we say that f is the limit or the limit function of $\{f_n\}$, $n \in \mathbb{N}$. For if (3.1.1) holds, then for every point x on E, the sequence $\{f_n(x)\}$, $n \in \mathbb{N}$ of real numbers converges to $f(x)$.

Definition 3.1.1.

The sequence of function $\{f_n\}$, $n \in \mathbb{N}$ *defined on a set E converges (pointwise) to f on E, if for each* $x \in E$ *and for a given* $\varepsilon > 0, \exists$ *a positive number N such that*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for} \quad n > N \tag{3.1.2}$$

In general the number N depends on both ε *and x.*

Example 3.1.1.

Let $f_n(x) = x^n, 0 \le x \le 1$. *We know that* $\{x^n\}, n \in \mathbb{N}$ *is*

$$\lim_{n\to\infty} x^n = 0 \quad \text{for} \quad 0 \le x < 1$$

$$= 1 \quad \text{for} \quad x = 1.$$

Hence $\{f_n\}, n \in \mathbb{N}$ *converges pointwise to f on* $[0,1]$ *i.e.,* $\lim_{n\to\infty} f_n(x) = f(x)$ *on* $[0,1]$ *where*

$$f(x) = 0 \text{ for } 0 \le x < 1$$
$$= 1 \text{ for } x = 1.$$

Example 3.1.2.

For each $n \in \mathbb{N}$, *let* $f_n : \mathbb{R} \to \mathbb{R}$ *be defined by* $f_n(x) = \frac{x}{n}$, $x \in \mathbb{R}$. *Then* $\{f_n\}$ *is a sequence of function on* $\mathbb{R}$. *For each* $x \in \mathbb{R}$, *the sequence* $\{f_n(x)\}$ *converges to 0.*

Therefore the sequence $\{f_n(x)\}$ *is pointwise convergent on* $\mathbb{R}$ *and the limit function f is defined by* $f(x) = 0,\ x \in \mathbb{R}$.

Example 3.1.3.

Let $f_n(x) = \frac{x}{1+nx}$, $0 \le x < \infty$. *When* $x > 0$, $0 < f_n(x) \le \frac{x}{nx} = \frac{1}{n}$, *whereby* $\lim_{n\to\infty} f_n(x) = 0,\ x > 0$.

Also, since $f_n(0) = 0$ *for each* $n \in \mathbb{N}$, *it is clear that* $\{f_n\}, n \in \mathbb{N}$ *converges pointwise to f on* $0 \le x < \infty$, *where* $f(x) = 0$.

Example 3.1.4.

Let $f_n(x) = \frac{nx}{1+n^2x^2}$ *where* $x \in \mathbb{R}$ *i.e.,* $-\infty < x < \infty$.

For $x > 0, f_n(x) = \frac{\frac{1}{nx}}{\frac{1}{n^2x^2}+1}$ *and hence* $\lim_{n\to\infty} f_n(x) = 0$.

Also $f_n(0) = 0$. *Again* $\lim_{n\to\infty} f_n(x) = 0$ *for* $x < 0$.

Thus $\lim_{n\to\infty} f_n(x)=0$ *for each* $x\in\mathbb{R}$ *and for each* $n\in\mathbb{N}$.

Therefore, $\{f_n\}, n\in\mathbb{N}$ *converges to* $f=0$ *in* $(-\infty,\infty)$, *whereby* $f(x)=0$.

Uniform Convergence

We have seen that $\{f_n\}, n\in\mathbb{N}$ converges pointwise to *f* on *E,* if for each $x\in E$ and for a given $\varepsilon>0, \exists$ a positive number *N* such that

$$|f_n(x)-f(x)|<\varepsilon \text{ for } n>N.$$

We have also onserved in the previous example 3.1.4 that it is not always possible to find an *N* such that (3.1.2) holds for all $x\in E$ simultaneously, if for each $\varepsilon>0$, it becomes possible to find a unique *N* such that (3.1.1) holds for all $x\in E$, then we say that $\{f_n(x)\}, n\in N$ converges uniformly to *f* on *E.*

Definition 3.1.2.

Let $\{f_n(x)\}, n\in\mathbb{N}$ *be a swquence of real-valued functions on a set E. We say that* $\{f_n(x),\}$ $n\in\mathbb{N}$ *converges uniformly to the function f on E if for any given* $\varepsilon>0, \exists$ *a positive integer N such that.*

$$|f_n(x)-f(x)|<\varepsilon \textit{ for } n>N \textit{ and for all } x\in E. \qquad (3.1.3)$$

Here *N* depends on ε alone but not on x. It automatically follows that the uniform convergence implies its pointwisre convergence.

But that the converse is not true is discussed in the following example.

Example 3.1.5.

In example 3.1.1 the sequence $\{f_n\}$ *converges on* $(-1,1]$ *to the function f where.*

$$f(x)=0 \quad \textit{for } 0\le x<1$$

$$=1 \quad \textit{for } x=1$$

Let us examine if the convergence of the sequence $\{f_n\}$ *is uniform on* $(0,1)$.

Let $c \in (0,1)$. *Then* $|f_n(c) - f(c)| = c^n$.

Let $0 < \varepsilon < 1$. *Then* $|f_n(c) - f(c)| < \varepsilon$ *whenever* $c^n < \varepsilon$,

i.e., whenever $n \log(1/c) > \log(1/\varepsilon)$,

i.e., whenever $n \log\left(\frac{1}{\varepsilon}\right) > \log(1/c)$.

Let $k = \left[\log\left(\frac{1}{\varepsilon}\right)\left(\frac{1}{c}\right)\right] + 1$. *Then k is a natural number and* $|f_n(c) - f(c)| < \varepsilon$ *for all* $n \geq k$.

Therefore for all $x \in (0,1), |f_n(c) - f(c)| < \varepsilon$ *for all* $n \geq k$,

where $k = [\log(1/\varepsilon)/\log(1/c)] + 1$.

This k depends on ε *as well as x. As* $x \to 1-, k \to \infty$. *It follows that there does not exist a natural number k such that for all* $x \in (0,1), |f_n(x) - f(x)| < \varepsilon$ *holds for all* $n \geq k$. *Consequently,* $\{f_n\}$ *is not uniformly convergent on* $(0,1)$.

Example 3.1.6.

Let $f_n(x) = \dfrac{x}{1+nx}, 0 \leq x < \infty$.

We have seen in Example 3.1.3 that $0 < f_n(x) \leq \dfrac{1}{n}$ *and* $f(x) = 0$ *on* $0 \leq x < \infty$.

Hence for any given $\varepsilon > 0$, *and for all x in* $0 \leq x < \infty$.

$$|f_n(x) - f(x)| = |f_n(x) - 0| \leq \frac{1}{n} < \varepsilon \quad \text{for } n > N$$

if we accept N = integral part of $\left(\dfrac{1}{\varepsilon}\right)$. *Thus N depends only on* ε *but not on x. Hence it is uniformly convergent in* $0 \leq x < \infty$.

Remark 3.1.1.

We have seen in Example 3.1.1 that $\{f_n\}$, $n \in \mathbb{N}$ *where* $f_n(x) = x^n, 0 \le x \le 1$ *converges pointwise but not uniformly convergent in the interval, whereas in Example 3.1.6* $\{f_n\}$, $n \in \mathbb{N}$ *where* $f_n(x) = \frac{x}{1+nx}, 0 \le x \le 1$ *converges uniformly and automatically converges pointwise.*

(A) The Cauchy condition for uniform convergence.

Theorem 3.1.1.

Let $\{f_n(x)\}$, $n \in N$ *be a sequence of real-valued functions on a set E. A necessary and sufficient condition for a sequence* $\{f_n(x)\}$, $n \in N$ *of functions defined on a set E to be uniformly convergent is that for each given* $\varepsilon > 0, \exists$ *a positive integer N such that for m.n>N*

Putting $p = m - n$,

$$|f_m(x) - f_n(x)| < \varepsilon \quad \textit{for } n > N,\ P = 1,2,3.....$$

$$\textit{and for all} \qquad (3.1.4)$$

Proof. The condition is necessary

Let the sequence $\{f_n\}$, n ε N be uniformly convergent sequence of functions over E, convergens to f on E. Then for a given ε (>0) ∃ a positive integer N such that.

$$|f_n(x) - f(x)| < \varepsilon \quad \textit{for all } x \in E \textit{ and for all } n > N$$

Thus if m,n>N, we have for all r ε E,

$$|f_m(x) - f_n(x)| = \left| \{f_m(x) - f(x)\} + \{f(x) - f_n(x)\} \right|$$

$$\le |f_m(x) - f(x)| + |f_n(x) - f(x)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The condition is sufficient.

Let $\{f_n\}$, $n \in \mathbb{N}$ be a sequence of real-valued functions on E such that, given a positive $\varepsilon > 0 \,\exists$ *a positive integer N such that.*

$|f_m(x) - f_n(x)| < \varepsilon$ *holds for* $m, n > N$ *and* $\forall x \varepsilon E$. (3.1.5)

We are to show that there is function f on E such that $\{f_n\}$, $n \in \mathbb{N}$ and converges uniformly to f on E.

From (3.1.5), it is clear that for each fixed $x \varepsilon E$, the sequence of real numbers $\{f_n(x)\}$, $n \in \mathbb{N}$ is a Cauchy sequence.

Hence $\lim\limits_{n\to\infty} f_n(x)$ exists for each x εE and Call $\lim\limits_{n\to\infty}$ for all $f_n(x) = f(x)$ *for all* x ε E.

Keep m fixed and let $n \to \infty$, then *from (3.1.5)*

$|f_m(x) - f_n(x)| = | \langle \varepsilon$ *for* $m \rangle N$ *and for all* $x \varepsilon E$

This show that $\{f_m\}$, $m \in \mathbb{N}$ converges uniformly to f on E the froof is complete.

Hence the theorem. □

(B) Test for Uniform Convergence.

Theorem 3.1.2. (M_n Test)

Let $\lim_{n\to\infty} f_n(x) = f(x)$, *for all* $x \in E$ *and* $M_n = \sup\limits_{x\in E} |f_n(x) - f(x)|$.

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof. The condition is necessary.

Given that $\{f_n(x)\}, n \in \mathbb{N}$ converges uniformly to $f(x)$ on E.

Then for a given $\varepsilon > 0, \exists$ a positive integer N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for } n > N \text{ and for all } x \in E.$$

$\therefore M_n = \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$ for $n > N$ whereby $M_n \to 0$ as $n \to \infty$.

The condition is sufficient.

Given $M_n \to 0$ as $n \to \infty$, then for any given $\varepsilon > 0, \exists$ a positive integer N such that $M_n < \varepsilon$ for $n > N$ and for all $x \in E$.

Then, $\sup_{x\in E}\left|f_n(x)-f(x)\right|<\varepsilon$ for $n>N$ and for all $x\in E$.

Thus for all $x\in E$,

$$\left|f_n(x)-f(x)\right|\le\sup_{x\in E}\left|f_n(x)-f(x)\right|<\varepsilon \text{ for all } n>N.$$

That means $\{f_n(x)\}$ converges uniformly to $f(x)$ on E.

This completes the proof. □

Example 3.1.7.

Prove that $\{f_n(x)\}=\left\{\frac{x}{nx+1}\right\}$ *converges uniformly to 0 on* $0\le x\le 1$.

Solution.

$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{x}{nx+1}=0=f(x)$, for all $xE\,[0,1]$. Or, in other words, $\{f_n(x)\}, n\in N$ converges pointwise to 0 on $[0,1]$. Next,

$$M_n=\sup_{x\in[0,1]}\left|f_n(x)-f(x)\right|\sup_{x\in[0,1]}\frac{x}{nx+1}.$$

Now to find the supremum of $\frac{x}{nx+1}$ for all $x\in[0,1]$. Let us call it $g_n(x)$, then

$$g_n(x)=\frac{x}{nx+1}.$$

$$\therefore g_n'(x)=\frac{nx+1-nx}{(nx+1)^2}=\frac{1}{(nx+1)^2}>0 \text{ for all } x\in[0,1].$$

Thus $g_n(x)$ is strictly increasing on $[0,1]$. Moreover $g_n(x)$ is continuous on $[0,1]$. Hence $g_n(x)$ assumes its maximum (supremum) value at $x=1$. Thus

$$M_n=\sup_{x\in[0,1]}\frac{x}{nx+1}=\frac{1}{n\cdot 1+1}=\frac{1}{n+1}\to 0 \text{ as } n\to\infty.$$

Hence $\{f_n(x)\}$ converges uniformly to 0 for all $x\in[0,1]$.

Example 3.1.8.

Show that $\{f_n(x)\} = \left\{\frac{nx}{1+n^2x^2}\right\}$ *is not uniformly convergent in any interval containing zero.*

Solution.

Here

$$f(x) = \lim_{n\to\infty} \int_n (x) = \lim_{n\to\infty} \frac{\text{nx}}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{\text{n}}+nx^2} = 0$$

for any $x \in [a,b]$ containing zero.

Now,

$$M_n = \sup_{x\in[a,b]} \left|f_n(x) - f(x)\right| = \sup_{x\in[a,b]} \left|\frac{nx}{1+n^2x^2}\right|.$$

To find maximum of $\frac{nx}{1+n^2x^2} = g_n(x)$, say.

Hence $g_n'(x) = \frac{n\left(1-n^2x^2\right)}{\left(1+n^2x^2\right)^2}$ is 0 at $x = \frac{1}{n}, -\frac{1}{n}$. Also,

$$g_n''(x) = \frac{\left(1+n^2x^2\right)^2\left(-2n^3x\right) - 2n\left(1-n^2x^2\right)\left(1+n^2x^2\right).2n^2x}{\left(1+n^2x^2\right)^4}$$

is negative when $x = \frac{1}{n}$ and is positive if $x = -\frac{1}{n}$.

Hence the maximum has been reached at $x = -\frac{1}{n}$ and max $g_n(x) = \frac{1}{2}$.

Therefore $M_n = \frac{1}{2}$ which does not $\to 0$ as $n \to \infty$.

Hence the result.

(C) Uniform Convergence and Continuity

Let us begin with an illustration to discuss the problem of interchange of the order of the limiting operations.

Example 3.1.9.

We start with the sequence of functions $\{f_n(x)\}$ *where*

$$f_n(x) = x^n \text{ on } 0 \le x \le 1 \text{ for } n \in \mathbb{N}.$$

Then $f_n(x)$ *is continuous on* $[0,1]$.

However $\{f_n\}$ *converges pointwise to f on* $[0,1]$, *where at* $x = 1$, $f_n(x) = 1$ *for all n but for all x in* $0 \le x < 1$. $f_n(x) = x^n \to 0$; *whereby*

$$f(x) = 0, \text{ when } 0 \le x < 1$$
$$= 1, \text{ when } x = 1$$

Clearly $f(x)$ *is discontinuous on* $0 \le x \le 1$ since $\lim_{x \to 1-0} f(x) = 0$, *but* $f(1) = 1$. *This shows that a sequence of continuous functions may converge pointwise to a discontinuous function.*

Remark 3.1.2.

This example shows that

$$\lim_{n \to \infty} \lim_{x \to 1-0} f_n(x) = 1$$

whereas,

$$\lim_{n \to 1-0} \lim_{x \to \infty} f_n(x) = 1$$

That is the two limits are not interchangeable as they would be if $f_n(x)$ *were continuous at* $x = 1$.

We should then like to know what conditions on $f_n(x)$ *will ensure that* $f(x)$ *will be continuous, if the approximating functions are themselves continuious. Uniform converges provides a sufficient condition to guarantee this result, We therefore, come to the following theorem.*

Theorem 3.1.3. (*Interchange of the order of the limiting operations.*)

A sequence $\{f_n(x)\}$, $n \in \mathbb{N}$ *of real-valued functions is defined on* $I : a \le x \le b$. *Suppose* $\{f_n(x)\}$ *converges uniformly to* $f(x)$ *on I.*

Let $x_0 \in [a,b]$ *and suppose that* $\lim_{x \to x_0} f_n(x) = a_n$, $n = 1, 2, 3, \ldots$

Then show that

(i) *the sequence* $\{a_n\}$ *of real constants converges, and*

(ii) $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$.

In other words,

$$\lim_{x \to x_0} \lim_{x \to \infty} f_n(x) = \lim_{x \to \infty} \lim_{x \to x_0} f_n(x) .$$

Proof. Let $\varepsilon > 0$ be given. By uniform convergence of $\{f_n(x)\}$ on $I : a \le x \le b$, for this ε, there exists a positive integer N such that for all $x \le\in I$ and $\forall m, n > N$,

$$|f_n(x) - f_m(x)| < \varepsilon \qquad (3.1.6)$$

Keep m, n fixed and let $x \to x_0$ then

$$|a_n - a_m| < \varepsilon, \text{ for } \forall\ m, n > N .$$

Therefore, by Cauchy's general principle of convergence $\{a_n\}$ of real constants becomes a Cauch sequence and $\{a_n\}$ converges, say to A, i.e.,

$$\lim_{x \to \infty} a_n = A$$

Which proves (i).

Next since $\{a_n\}$ converges to A as $n \to \infty$ and $\{f_n(x)\}$ converges uniformly to $f(x)$ on I, then for any $\varepsilon > 0$, $\exists$ a suitable positive integer N such that

$$|a_n - A| < \frac{\varepsilon}{3} \text{ for } n > N$$

and

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for } n > N$$

and for all $x \in [a, b]$.

Again by the given condition, $\lim_{x \to x_0} f_n(x) = a_n$ for all n, and hence for the same $\varepsilon > 0$, $\exists\ \delta > 0$ such that for all $x \in |x - x_0| < \delta$,

$$|f_n(x) - a_n| < \frac{\varepsilon}{3} \text{ for all } n.$$

Thus for all $n > N$ and for $x \in |x - x_0| < \delta$, we have

$$|f(x)-A| \le |f(x)-f_n(x)-a_n+a_n-A|$$
$$\le |f(x)-f_n(x)|+|f_n(x)-a_n|+|a_n-A|$$
$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

whereby

$$\lim_{x\to x_0} f(x) = A = \lim_{x\to\infty} a_n$$

or,

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)$$

which proves (*ii*).

Hence the theorem. □

Theorem 3.1.4. (*Continuity of the limit function.*)

If $\{f_n\}, n \in \mathbb{N}$ *be a sequence of continuous functions on an interval I, and if* $f_n \to f$ *uniformly on I, then f is continuous on I.*

Proof. Let x_1 be any arbitrary point on I. Then we are required to show that for each $\varepsilon > 0$, there corresponds a $\delta(\varepsilon)$ such that

$$|f(x)-f(x_1)| < \varepsilon \text{ for } |x-x_1| < \delta$$

Now for any n,

$$|f(x)-f(x_1)| = |f(x)-f_n(x)+f_n(x)-f_n(x_1)+f_n(x_1)-f(x_1)|$$
$$\le |f(x)-f_n(x)|+|f_n(x)-f_n(x_1)|+|f_n(x_1)-f(x_1)| \qquad (3.1.7)$$

Since the given sequence is uniformly convergent, there exists an $N(\varepsilon)$ independent of x, for which

$$|f_x(x)-f(x)| < \frac{\varepsilon}{3} \text{ if } n > N \text{ and for all } x \text{ in } I.$$

Therefore

$$|f_n(x_1)-f(x_1)| < \frac{\varepsilon}{3} \text{ if } n > N$$

Again $f_n(x)$ being a continuous function, there exists a $\delta(\varepsilon)$ for a fixed $n > N$, for which

$$|f_n(x)-f_n(x_1)|<\frac{\varepsilon}{3} \text{ whenever } |x-x_1|<\delta$$

Thus (3.1.7) gives

$$|f(x)-f(x_1)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \text{ for } |x-x_1|<\delta.$$

Hence the theorem. □

(D) Uniform Convergence and Integration

Let us begin with the illustration

Example 3.1.10.

Let $f_n(x)=nxe^{-nx^2}$, $n\in\mathbb{N}$ *and* $0\le x\le 1$.

Determine whether

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\int_0^1\left\{\lim_{n\to\infty}f_n(x)\right\}dx.$$

Now

$$\int_0^1 f_n(x)\,dx=\int_0^1 nxe^{-nx^2}\,dx=\left[-\frac{1}{2}e^{-nx^2}\right]_{x=0}^{x=1}=\frac{1}{2}\left(1-e^{-n}\right),$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\lim_{n\to\infty}\frac{1}{2}\left(1-e^{-n}\right)=\frac{1}{2}.$$

$f(x)=\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}nxe^{-nx^2}=0$, *whether* $x=0$ *or,* $0<x\le 1$.

Then $\int_0^1 f(x)\,dx=0$.

It follows that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx\ne\int_0^1\left\{\lim_{n\to\infty}f_n(x)\,dx\right\}$$

i.3., limit cannot be taken under the integral sign. The reason is that although the sequence $\{f_n(x)\}$ *converges to 0, it does not converge uniformly to 0.*

So we come to the theorem.

Theorem 3.1.5.

Suppose the sequence of the functions $\{f_n\}, n \in \mathbb{N}$ *be R– integrable on an interval* $I : a \le x \le b$ *and suppose* $f_n \to f$ *uniformly on I which is also R– integrable on I, then*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \left\{\lim_{n\to\infty} f_n(x)\right\}dx = \int_a^b f(x)\,dx.$$

Proof. Let $\varepsilon > 0$ be given. Science $\{f_n\}$ converges uniformly to f on I, we can find a positive integer $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{(b-a)},\ \text{for all } n \in \mathbb{N} \text{ and for all } x \in I.$$

Thus for $n > N$, we have

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b \{f_n(x) - f(x)\}\,dx\right|$$

$$\le \int_a^b |f_n(x) - f(x)|\,dx$$

$$\le \int_a^b \frac{\varepsilon}{b-a}\,dx = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

Thus

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \left\{\lim_{n\to\infty} f_n(x)\right\}dx = \int_a^b f(x)\,dx.$$

Hence the theorem □

(E) Uniform Convergence and Differentiation

Theorem 3.1.6.

Let $\{f_n\}, n \in \mathbb{N}$ *be a sequence of functions, differentiable on* $I : [a,b]$ *and such that* $\{f_n(x)\}$ *converges for each* $x \in [a,b]$. *If each* f_n *has a continuous derivative* f_n' *on* $[a,b]$ *and if* $\{f_n'\}$ *converges uniformly on [a,b], then if* $\{f_n\}$ *conveges unifomly to a function* f, *on* $[a,b]$ *then.*

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} = \lim_{n\to\infty}\left\{\frac{d}{dx} f_n(x)\right\}$$

$$or,\ f'(x) = \lim_{n\to\infty} f_n'(x), \qquad a \le x \le b.$$

Proof. Since each $f_n'(x)$ is continuous and $\{f_n'\}$ converges uniformly to $F(x)$, say on $[a,b]$, $F(x)$ is continuous on $[a,b]$, by Theorem 3.1.4

That is $\lim_{n\to\infty} f_n'(x) = F(x)$, since they are continuous.

Also $f_n'(x)$ and F (x) are R-integrable on $[a,b]$, then by the fundamental theorem on integrals and by Theorem 3.1.3, for $a \le x \le b$

$$\int_a^x \left\{\lim_{n\to\infty} f_n'(t)\right\} dt = \lim_{n\to\infty}\int_a^x f_n'(t)\,dt$$

$$\text{or, } \int_a^x F(t)\,dt = \lim_{n\to\infty}\int_a^x f_n'(t)\,dt = \lim_{n\to\infty}\left\{f_n(x) - f_n(a)\right\}$$

$$= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) = f(x) - f(a),$$

Since $f_n(x)$ coverges uniformly to $f(x)$ on I.

Hence $\dfrac{d}{dx}\displaystyle\int_a^x F(t)\,dt = \dfrac{d}{dx}\{f(x) - f(a)\} = f'(x)$

And since $F(x)$ is continuous, we have $F(x) = f'(x)$

And the theorem is established . □

3.2 Illustrative Examples

Examples 3.2.1.

Show that the limit function of the sequence of functions $\{f_n(x)\}$, *where* $f_n(x) = 1 - \left|1 - x^2\right|^n$, *in the domain* $\left(x : \left|1 - x^2\right| \le 1\right)$ *is not continuous at* $x = 0$.

Solution.

When $\left|1 - x^2\right| < 1, \lim_{n\to\infty} f_n(x) = 1 - 0 = 1$

When $\left|1-x^2\right|=1$, i.e., $x=0$ or, $x=\pm\sqrt{2}$

$\therefore \lim_{n\to\infty} f_n(x)=1-1=0$, when $x=0,\sqrt{2},-\sqrt{2}$

Therefore limit function $f(x)$ of the given sequence of functions is

$f(x)=1$, if $\left|1-x^2\right|<1$ i.e., $-\sqrt{2}<x<0$ and $0<x<\sqrt{2}$

$=0$, if $x=0,-\sqrt{2},\sqrt{2}$

Now $\lim_{x\to 0} f(x)=1$ but $f(0)=0$

So the limit function of the given sequence of functions is not continuous at $x=0$.

Example 3.2.2.

Show that the sequence of functions $\{f_n\}$, where $f_n(x)=\frac{x}{n}$ *does not converge uniformly on* $[0,\infty)$.

Solution.

Here $f_n(x)=\frac{x}{n}\forall\ n\in\mathbb{N}$ and $x\in[0,\infty)$

Now $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\infty\frac{x}{n}=x\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$

Therefore limit function $f(x)$ on $0\le x<\infty$, is $f(x)=0\ \forall\ 0\le x<\infty$.

Let $\varepsilon>0$ be given. Now

$$\left|f_n(x)-f(x)\right|=\left|\frac{x}{n}-0\right|=\frac{x}{n}<\varepsilon \text{ if } n>\frac{x}{\varepsilon}$$

But as $0\le x<\infty$, then $\frac{x}{\varepsilon}$ can have sufficiently large value (however large), so that is not possible to choose any number m such that $\forall\ n\ge m$ and $x\in[0,\infty)$, such that

$$\left|f_n(x)-f(x)\right|<\varepsilon.$$

That means $\{f_n(x)\}$ dies not converge uniformly on $[0,\infty)$.

Remark 3.2.1.

But the sequence of functions converges uniformly on $[0,k]$, *whatever k may be.*

Example 3.2.3.

Test Uniform convergence of trhe sequence of functions $\left\{\frac{nx^2}{nx+1}\right\}$ *on* $[0,1]$.

Solution.

Here $f_n(x)=\frac{nx^2}{nx+1}$ on $[0,1]$. Now

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{nx^2}{nx+1}=\lim_{x\to\infty}\frac{x^2}{x+\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} f_n(x)=x, \text{ if } 0<x\le 1$$

$$=0, \text{ if } x=0.$$

Therefore the limit function $f(x)$ on $[0,1]$ is

$$f(x)=x \text{ for } 0\le x\le 1$$

Let

$$M_n=\sup_{x\in[0,1]}\left|f_n(x)-f(x)\right|=\sup_{x\in[0,1]}\left|\frac{nx^2}{nx+1}-x\right|$$

$$=\sup_{x\in[0,1]}\left|\frac{-x}{nx+1}\right|=\sup_{x\in[0,1]}\frac{x}{nx+1}[\because x\ge 0]$$

Now $\frac{x}{nx+1}<\frac{x}{nx}=\frac{1}{n}\left[\text{if } 0<x\le 1, \text{ in particular if } x=0, \text{ then } \frac{x}{nx+1}=0\right]$.

$$\therefore M_n=\sup_{x\in[0,1]}\frac{x}{nx+1}<\frac{1}{n}$$

$$\therefore \lim_{n\to\infty} M_n=0\left[\because \lim_{n\to\infty}\frac{1}{n}=0 \text{ and } M_n\ge 0\right]$$

Hence the sequence of functions $\left\{\frac{nx^2}{nx+1}\right\}$ is uniformly convergent on $[0,1]$.

3.3 Exercise 1

1. *(a)* Show that uniform convergence of a sequence of functions on a set E implies its pointwise convergence on E.

(b) Hence show that $\{f_n(x)\}, n \in \mathbb{N}$ where $f_n = \frac{x}{1+nx}, 0 \le x < \infty$ converges uniformly to 0 and necessarily converges pointwise on $0 \le x < \infty$.

2. *(a)* State and prove the Cauchy condition (necessary and sufficient) uniform convergence of a sequence of functions over a set E.

(b) State and prove M_n-test for the uniform convergence of a sequence of functions on a set E.

3. Test the sequence $\{f_n(x)\}$ for uniform convergence, where $f_n(x) = 1 - \frac{x^n}{n}, x \in \mathbb{N}[0,1]$.

4. For each $n \in \mathbb{N}$, define $g_n : [0,1] \to \mathbb{R}$ by $g_n(x) = \frac{\sin nx}{\sqrt{n}}$. Show that $\{g_n\}$ converges pointwise on $[0,1]$ to a differentiable function g, but $\{g'_n(0)\}$ does not converge to $g'(0)$.

5. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \tan^{-1}(nx) \forall x \in [0,1]$ and for every positive integer n. Find the limit function $f(x)$ of the sequence of the functions $\{f_n(x)\}$ in $[0,1]$. Is $\{f_n(x)\}$ uniformly convergent on $[0,1]$? Justify your answer.

6. Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x) = \frac{n+\cos x}{2n+\sin^2 x}$, $x \in \mathbb{R}, n \in \mathbb{N}$, is uniformly convergent on $\mathbb{R}$.

7. Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x) = \frac{k^2}{k^2+n^2x^2}$, is uniformly convergent on $[a,b]$, where $0 < a < b$, but the convergence in not uniform on $[-1,1]$.

3.4 Series of Functions

Let $E \subset \mathbb{R}$. Let $\{f_n(x)\}$be a sequence of functions on E to $\mathbb{R}$. Then $u_1(x)+u_2(x)+u_3(x)+..u_n(x)+...$ is said to be a series of functions on E. The infinite series is denoted by $\sum f_n\left(orby\sum_{n=1}^{\infty} f_n(x)\right)$. The sequence $S_n(x)$ where

$$S_n(x)=u_1(x)+u_2(x)+u_3(x)+...+u_n(x)$$

is said to be the partial suns of the infinite series $\sum f_n$.

Pointwise convergence

Let $u_1(x), u_2(x), u_3(x)$, ..., $u_n(x)$,... be real-valued functions of x, each defined on a set E. Then the series (infinite series)

$$\sum_{n=1}^{\infty} = u_1(x)+u_2(x)+u_3(x)+ ...+u_n(x)+ ...$$

Coverges pointwise to the functions $S(x)$ on E, if the sequence $\{S_n(x)\}$ converges to $S(x)$ on E where $S_n(x)=u_1(x)+u_2(x)+u_3(x)+...+u_n(x)$.

Definition 3.4.1.

The series of functions $\sum_{n=1}^{\infty} u_n$ converges pointwise to $S(x)$ on E, if for each $x \in E$, given $\varepsilon > 0$, $\exists$ a positive integer N such that

$$|S_n(x)-S(x)| < \varepsilon \qquad \textit{for} \quad n > N \tag{3.4.1}$$

As usual, N, in general depends on both ε and x.

Uniform Convergence

Let $u_1(x), u_2(x), u_3(x),..., u_n(x)$, ... be real-valued functions of x, each difined on a set E. Then the series (infinite series)

$$\sum_{n=1}^{\infty} = u_1(x)+u_2(x)+u_3(x)+ ...+u_n(x)+ ...$$

Converges uniformly to the function $S(x)$ on E, if the sequence of functions $\{S_n(x)\}$ converges uniformly to $S(x)$ on E where

$S_n(x) = u_1(x) + u_2(x) + u_3(x) + ... + u_n(x)$.

Definition 3.4.2.

The series of functions $\sum_{n=1}^{\infty} u_n$ *converges uniformly to* $S(x)$ *on E, if for any given* $\varepsilon > 0$, $\exists$ *a positive integer N such that*

$$|S_n(x) - S(x)| < \varepsilon \textit{ for } n > N \textit{ and for all } x \in E. \qquad (3.4.2)$$

(A) The Cauchy Condition for Uniform Convergence of Series.

Theorem 3.4.1. [*Necessary and sufficient condition for uniform convergence.*]

The series of functions $\sum_{n=1}^{\infty} u_n$ *converges uniformly on a set E, if the sequence of functions* $\{S_n(x)\}$ *where where* $S_n(x) = u_1(x) + u_2(x) + u_3(x) + ... + u_n(x)$

defined on E converges uniformly on E. Thus $\sum_{n=1}^{\infty} u_n$ *converges uniformly on E if and only if for every given* $\varepsilon > 0$, $\exists$ *a positive integer N such that for* $m, n > N$,

$$|S_m(x) - S_n(x)| < \varepsilon \quad \textit{for all } x \in E. \qquad (3.4.3)$$

Putting $p = m - n$,

$|S_{n+p}(x) - S_n(x)| < \varepsilon$ *for* $n > N, p = 1, 2, 3, ...$ *and for all* $x \in E$.

Proof. The condition is necessary.

Let $\sum_{n=1}^{\infty} u_n(x)$ converge uniformly, that is, the sequence of functions $\{S_n(x)\}$ converge uniformly to $S(x)$ on E. Then for any given $\varepsilon(>0), \exists$ a positive integer N such that

$|S_m(x) - S(x)| < \varepsilon$ for all $x \in E$ and for all $n > N$.

Thus if $m, n > N$, we have for all $x \in E$,

$$|S_m(x) - S_n(x)| \le |S_m(x) - S(x)| + |S_n(x) - S(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The condition is sufficient.

Let $\{S_n(x)\}$, $n \in \mathbb{N}$ be any sequence of real-valued functions on E such that, given $\varepsilon(>0), \exists$ a positive integer N such that

$|S_m(x) - S_n(x)| < \varepsilon$ holds for $m, n > N$ an for all $x \in E$.

Now it is clear that for each fixed $x \in E$, the sequence of real numbers $\{S_n(x)\}$, $n \in E$ is a Cauchy sequence. Hence

$$\lim_{n\to\infty} S_n(x) \text{ exists for each } x \in E.$$

$$\text{Call } \lim_{n\to\infty} S_n(x) = S(x) \text{ for all } x \in E.$$

Kep m fixed and let $n \to \infty$, then

$$|S_n(x) - (x)| < \varepsilon \text{ for } n > N \text{ and for all } x \in E.$$

This shows that $\{S_m(x)\}$, $m \in \mathbb{N}$ converges uniformly to $S(x)$ on E and the theorem is established. □

(B) Test for Uniform Convergence of Series.

Theorem 3.4.2. [*Weierstrass's M-test*]

The series $\sum_{n=1}^{\infty} u_n(x)$ of real-valued functions defined on a set E converges uniformly and absolutly on a set E, if each term satisfies $|u_n(x)| \le M_n$ for all $x \in E$ and $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive terms.

Proof. Let $S_n(x) = \sum_{r=1}^{n} u_r(x)$. Since $|u_n(x)| \le M_n$, we have

$$\sum_{n=1}^{\infty} |u_n(x)| \le \sum_{n=1}^{\infty} M_n$$

And hence $\sum_{n=1}^{\infty} u_n(x)$ is absolutely convergent.

Second part :

Since $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive terms, then for a given $\varepsilon > 0, \exists$

a positive integer N such that for all $n > m > N$ (Cauchy criterion)

$$M_{m+1} + M_{m+2} + M_{m+3} + \ldots + M_n < \varepsilon. \tag{3.4.4}$$

Therefore, for $n > m > N$,

$$|S_n(x) - S_m(x)| = |u_{m+1}(x) + u_{m+2}(x) + \ldots + u_m(x)|$$

$$\leq M_{m+1} + M_{m+2} + M_{m+3} + \ldots + M_n < \varepsilon.$$

Hence by Cauchy criterion $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on a set E and the theorem is established. □

Example 3.4.1.

Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ *and* $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ *are uniformly convergent for all x.*

Solution :

See that $\left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$ *for all x and that* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges being a p-series with* $p>1$.

Hence by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ *converges uniformly for all x.*

Similarly we can show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ *converges uniformly for all x.*

(B) Some Useful Theorems on Uniformly Convergent of Series of Functions :

Theorem 3.4.3. [*Continuity in the Sum*]

If the series $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to* $S(x)$ *on an interval I and if each term* $u_n(x)$ *of the series be continuous on the interval, then the sum functions* $S(x)$ *is continuous on I.*

Proof. Take $\sum_{n=1}^{\infty} u_n(x) = S_n(x)$. Since $\sum_{n=1}^{\infty} u_n(x)$ converge uniformly to $S(x)$ on *I*, then for any given $\varepsilon > 0$, $\exists$ a positive integers $S(\varepsilon)$ such that for all $x \in I$,

$$|S_n(x) - S(x)| \leq \frac{\varepsilon}{3}, \text{ for all } n > N. \qquad (3.4.5)$$

Let now x_0 be a point in I, then from (3.4.5)

$$|S_n(x_0) - S(x_0)| \leq \frac{\varepsilon}{3}, \text{ for all } n > N \qquad (3.4.6)$$

Again since each u_n (x) is continuous at evey point of I, it is necessaily continuous at x= x_0 and therefore S_n (x) which is the sum of a finite number of continuous functions must be continuous at x = x_0.

Hence of the same positive number ε, ∃ a positive number δ such that for all n,

$$|Sn(x) - S(x_0)| \leq \frac{\varepsilon}{3}, \text{ for all x - } x_0 \qquad (3.4.7)$$

Thus for all $n > N$ and for all $x \in |x - x_0| < \delta$, we have

$$|S(x) - S(x_0)| = |S(x) - S_n(x) + S_n(x) - S_n(x_0) + S_n(x_0) - S(x_0)|$$

$$\leq |S(x) - S_n(x)| + |S_n(x) - S_n(x_0)| + |S_n(x_0) - S(x_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $S(x)$ is continuous at x_0. But x_0 is arbitrary, whereby $S(x)$ is continuous on I. Hence the theorem. □

Remark 3.4.1.

If for each $n \in \mathbb{N}$, $S_n(x)$ is continuous on I and the sum function $S(x)$ of the series $\sum_{n=1}^{\infty} u_n(x)$ is not continuous on I, then it follows from the theorem that the series $\sum_{n=1}^{\infty} u_n(x)$ is not uniformly continuous on I.

Remark 3.4.2.

If each $u_n(x)$ be continuous on I, then the condition of uniform convergence of the series $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for continuity of the sum function $S(x)$ on I.

Example 3.4.2.

We consider the series of functions $\sum_{n=1}^{\infty} u_n(x)$,

where $u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0,1]$

Then $S_n(x) = u_1(x) + u_2(x) + ... + u_n(x) = \frac{nx}{1+n^2x^2} \forall x \in [0,1]$.

The sequence $\{S_n(x)\}$ *converges to the* $S(x)$, *where* $S(x) = 0$ *for all* $x \in [0,1]$

$\left[\textit{Since } \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 \ \forall x \in [0,1]\right]$

Here each $u_n(x)$ *is continuous on* $[0,1]$ *and* $S(x)$ *is also continuous on* $[0,1]$, *but in, examnple 3.1.8 we see that the sequence* $\{u_m(x)\}$ i.e., $\left\{\frac{nx}{1+n^2x^2}\right\}$ *is not uniformly convergent on* $[0,1]$.

This proves that, for a convergent series of continuous function, The uniform convergence of the series is not necessary for continuity of the sum function.

Theorem 3.4.4. [*Term by Term Integration*]

If a series $\sum_{n=1}^{\infty} u_n(x)$ *of Riemann integrable functions converges uniformly to* $S(x)$ *on* $I : a \le x \le b$ *which itself is* $\mathbb{R}$*-integrable on I, then*

$$\int_a^b S(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + ... + \int_a^b u_n(x)dx + ...$$

Proof. Since the given series is uniformly convergent on $[a,b]$, then to an arbitrary $\varepsilon > 0$, there corresponds a positive integer *N* independent of *x* in $[a,b]$ such that for $n > N$,

$$|S_n(x) - S(x)| < \frac{\varepsilon}{b-a}, \quad a \le x \le b.$$

Thus for $n > N$, we have since $S(x)$ is integrable on *I* as also $S_n(x)$ being the sum of a finite number of $\mathbb{R}$-integrable functions is also integrable on *I*,

$$\left|\int_a^b S_n(x)dx - \int_a^b S(x)dx\right| = \left|\int_a^b \{S_n(x) - S(x)\}dx\right|$$

$$\leq \int_a^b \left|S_n(x) - (x)\right| dx$$

$$\leq \int_a^b \frac{\varepsilon}{b-a} = \frac{\varepsilon}{b-a}.(b-a) = \varepsilon,$$

and the theorem is established. □

Remark 3.4.3.

The result $\int_a^b S(x)dx = \int_a^b u_1(x)dx + \int_a^b u_2(x)dx + ... + \int_a^b u_n(x)dx + ...$ *implies that the series of functions can be integrated term by term on* $[a,b]$ *if the series of functions* $\sum_{n=1}^{\infty} u_n(x)$ *is uniformly convergent.*

Remark 3.4.4.

If each $u_n(x)$ *be integrable on* $[a,b]$, *then the uniform convergence of the series* $\sum_{n=1}^{\infty} u_n(x)$ *is only a sufficient but not a necessary condition for the integrability of the sum function* $S(x)$ *on* $[a,b]$.

Example 3.4.3.

Let us cosider the series of function $\sum_{n=1}^{\infty} u_n(x)$,

where $u_n(x) = \frac{n^2 x}{1+n^4x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}$, $x \in [0,1]$.

Let $S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$, $x \in [0,1]$

Now $S_n(x) = \frac{n^2 x}{1+n^4x^2}$, $x \in [0,1]$.

Clearly $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{n^2 x}{1+n^4x^2} = \lim_{n\to\infty} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2} = 0$.

Therefore the sequence $\{S_m(x)\}$ converges pointwise to the function $S(x)$, *where* $S(x)=0$ *Similarly as example 3.1.8 we can show that the sequence* $\{S_m(x)\}$ *does not converge uniformly on* $[0,1]$.

Now it is clear that the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *is such that each* $u_n(x)$ *is integrable on* $[0,1]$ *and it converges to the sum functions* $S(x)$ *which is also integrable on* $[0,1]$, *but the convergence of the series is not uniform on* $[0,1]$.

This result proves that for a series of integrabl;e functions the uniform convergence of the series is not necessary for integrability of the sum function.

Remark 3.4.5.

If the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *converges to a function* $S(x)$ *on* $[a,b]$ *which is integrable on* $[a,b]$ *and each* $u_n(x)$ *is also integrable on* $[a,b]$, *then the uniform convergence of the series is only a sufficient but not necessary condition for term by term integration of the series on* $[a,b]$.

Example 3.4.4.

We take same example 3.4.2

i.e., $u_n(x)=\dfrac{n^2x}{1+n^2x^2}-\dfrac{(n-1)x}{1+(n-1)^2x^2},\ x\in[0,1]$.

We have already seen in example 3.4.2 *that the series of functions* $\sum_{n=1}^{\infty} u_n(x)$ *converges to* $S(x)$ *on* $[0,1]$, *where* $S(x)=0, x\in[0,1]$ *but the series is not uniformly convergent on* $[0,1]$.

Now $\int_0^1 S(x)dx=0\left[\because S(x)=0,\quad 0\le x\le 1\right]$.

And $\int_0^1 u_1(x)dx=\int_0^1 \dfrac{x}{1+x^2}dx+\dfrac{1}{2}0\left[\log\left(1+x^2\right)\right]_0^1=\dfrac{1}{2}\log 2$.

For $n\ge 2$,

$$\int_0^1 u_n(x)dx = \int_0^1 \frac{nx}{1+n^2x^2}dx - \int_0^1 \frac{(n-1)x}{1+(n-1)^2x^2}dx$$

$$= \frac{1}{2n}\log\left(1+n^2\right) - \frac{1}{2(n-1)}\log\left\{1+(n-1)^2\right\}.$$

Let

$$t_n = \int_0^1 u_1(x)dx + \int_0^1 u_2(x)dx + \ldots + \int_0^1 u_n(x)dx$$

$$= \frac{1}{2}\log 2 + \left[\frac{1}{4}\log 5 - \frac{1}{2}\log 2\right] + \ldots$$

$$+ \left[\frac{1}{2n}\log\left(1+n^2\right) - \frac{1}{2(n-1)}\log\left\{1+(n-1)^2\right\}\right]$$

$$= \frac{\log\left(1+n^2\right)}{2n}.$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\log\left(1+n^2\right)}{2n}\left[\frac{\infty}{\infty}\right]$$

$$= \lim_{n\to\infty} \frac{2n}{\left(1+n^2\right)2} \quad \textit{[L' Hospital's rule]}$$

$$= 0 \cdot$$

Therefore

$$\int_0^1 S(x)dx = 0 = \lim_{n\to\infty} t_n$$

or, $$\int_0^1 S(x)dx = \int_0^1 u_1(x)dx + \int_0^1 u_2(x)dx + \ldots + \int_0^1 u_n(x)dx + \ldots$$

Thus the series can be integrated term by term on $[0,1]$, *although the series is not uniformly convergent on* $[0,1]$.

So for term by term integration of a series of functions $\sum_{n=1}^{\infty} u_n(x)$ *on an interval* $[a,b]$, *the condition of uniform convergence of the series is sufficient but not necessary.*

Theorem 3.4.5. [*Term by Term Differentiation*]

If the series $\sum_{n=1}^{\infty} u_n(x)$ *converges to* $S(x)$ *on* $a \le x \le b$ *and if the derivative of each* $u_n(x)$ *be continuous on* $[a,b]$ *and the series derivatives* $\sum_{n=1}^{\infty} u'_n(x)$ *be uniformly convergent on a* $a \le x \le b$, *then the series of derivatives converges to* $S'(x)$.

Proof. Let us denote $\sum_{n=1}^{\infty} u_n(x)$ by $f(x)$. Then since the series of derivatives is assumed to be uniformly convergent on $a \le x \le b$. By theorem 3.4.4 this series to be integrated term by term, so that for $a \le x \le b$

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^x u'_n(x)dx = \sum_{n=1}^{\infty} \{u_n(x) - u_n(a)\} = S(x) - S(a).$$

Hence

$$\frac{d}{dx}\int_a^x f(x)dx = \frac{d}{dx}\{S(x) - S(a)\} = S'(x).$$

and since $f(x)$ is a continuous function, we obtain

$$f(x) = S'(x)$$

Hence the theorem. □

Remark 3.4.6.

Only the uniform convergence of the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *on* $[a,b]$ *is not sufficient to ensure validity of term by term differentiation of the series on* $[a,b]$.

Example 3.4.5.

Let us consider the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *on* $[0,1]$,

where $u_n(x) = \frac{x}{1+nx^2} - \frac{x}{1+(n-1)x^2} \forall n \ge 2$ *and* $u_1(x) = \frac{x}{1+x^2}$.

Now

$$S_n(x) = \sum_{r=1}^{n} u_r(x) = u_1(x) + u_2(x) + ... + u_n(x),\ x \in [0,1] = \frac{x}{1+nx^2}$$

Now $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \forall x \in [0,1]$

Therefore the sequence $\{S_n(x)\}$ *converges pointwise to the function* $S(x)$, *where* $S(x) = 0,\ 0 \le x \le 1$.

Let

$$M_n = \sup_{x\in[0,1]} |S_n(x) - S(x)| = \sup_{x\in[0,1]} \frac{x}{1+nx^2}$$

$$\left[\because S(x) = 0,\ \textit{and}\ x \ge 0, 1+nx^2 > 0\right]$$

Now for $x > 0$,

$$\frac{\frac{1}{x} + nx}{2} \ge \sqrt{\frac{1}{x} \cdot nx} \qquad (3.4.8)$$

the equality occurs when $nx = \frac{1}{x}$ *i.e.,* $x = \frac{1}{\sqrt{n}} [\because x > 0]$

From (3.4.8)

$$\frac{1+nx^2}{2x} \ge \sqrt{n} \quad \textit{as} \quad \frac{x}{1+nx^2} \le \frac{1}{2\sqrt{n}}$$

and

$$\frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}} \quad \textit{at}\ x = \frac{1}{\sqrt{n}}$$

For $x = 0$, $\frac{x}{1+nx^2} = 0$

Therefore $\displaystyle \mathop{Sup}_{x\in[0,1]} \frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$

$$Mn = \sup_{x\in[0,1]} |S_n(x) - S(x)| = \frac{1}{2\sqrt{n}}$$

Now $\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$.

Which implies that the sequence $\{S_n(x)\}$ *is uniformly convergent on* $[0,1]$.

Hence the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to the function* $S(x)$ *on* $[0,1]$. *Now*

$$\frac{d}{dx}\{S_n(x)\} = \frac{(1+nx^2)\cdot 1 - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$$\therefore \lim_{x\to\infty} S_n'(x) = 0, \quad \text{if } 0 < x \leq 1$$

$$= 1, \quad \text{if } x = 0.$$

Therefore the series $\sum_{n=1}^{\infty} u_n'(x)$ *converges to the function* $g(x)$, *where*

$$g(x) = 0, \quad \text{if } 0 < x \leq 1$$

$$= 1, \quad \text{if } x = 0.$$

Now $\frac{d}{dx}\left[u_1(x) + u_2(x) + \ldots\right] = 0 = u_1'(x) + u_2'(x) + \ldots$ *when* $0 < x \leq 1$.

But $\frac{d}{dx}\left[u_1(x) + u_2(x) + \ldots\right] \neq u_1'(x) + u_2'(x) + \ldots$ *when* $x = 0$.

Hence only the uniform convergence of the series of functions is not sufficient to ensure validity of term by term differentiation of the series.

Remark 3.4.7.

If the series of functions $\sum_{n=1}^{\infty} u_n(x)$ *be convergent on* $[a,b]$, *then the uniform convergence of the series* $u_1'(x) + u_2'(x) + \ldots$ *on* $[a,b]$ *is only a sufficient but not necessary condition for the validity of term by term differentiation of the series* $\sum_{n=1}^{\infty} u_n(x)$.

3.5 Illustrative Examples

Example 3.5.1.

Prove that the series $\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2}$ *is uniformly convergent for all real x.*

Solution :

Here $f_n(x)\dfrac{x}{n+n^2x^2}$.

When $x=0,\ u_n(x)=0$

When $x\neq 0$, we take two numbers $\dfrac{n}{|x|}$ and $n^2|x|$ and get

$$\frac{\frac{n}{|x|}+n^2|x|}{2}\geq\sqrt{\frac{n}{|x|}\cdot n^2|x|}$$

the equality occurs when $\dfrac{n}{|x|}=n^2|x|$ or $|x|=\dfrac{1}{\sqrt{n}}$

or, $\dfrac{n}{|x|}=n^2|x|\geq 2n^{\frac{3}{2}}$

or, $\dfrac{|x|}{n+n^2x^2}\leq\dfrac{1}{2n^{\frac{3}{2}}}$

Then $|u_n(x)|=\dfrac{|x|}{n+n^2x^2}\leq\dfrac{1}{2n^{\frac{3}{2}}}$, then equality occurs when $|x|=\dfrac{1}{\sqrt{n}}$.

i.e., we can say $|u_n(x)|\leq=\dfrac{1}{2n^{\frac{3}{2}}}\forall x\in\mathbb{R}$ and $\forall n\in\mathbb{N}$.

If we take $M_n=\dfrac{1}{2n^{\frac{3}{2}}}$, then for all real $x,|u_n(n)|\leq M_n\ \ \forall n\in\mathbb{N}$.

The series $\sum_{n=1}^{\infty}M_n=\sum_{n=1}^{\infty}\dfrac{1}{2n^{\frac{3}{2}}}$ is a convergent series of positive real numbers.

Then by Weierstrass's M-test, $\sum_{n=1}^{\infty}\dfrac{x}{n+n^2x^2}$ is uniformly convergent for all real x.

Example 3.5.2.

Show that the series $\sum_{n=1}^{\infty}\dfrac{(n+1)^2}{n^5}\left(\dfrac{x}{3}\right)^n$ *is uniformly convergent on* $[-3,3]$.

Solution :

Here $u_n(x)=\frac{(n+1)^2}{n^5}\left(\frac{x}{3}\right)^n$

Now $|u_n(x)|=\left|\frac{(n+1)^2}{n^5}\left(\frac{x}{3}\right)^n\right|\le\frac{(n+1)^2}{n^5}$ for $\left|\frac{x}{3}\right|\le 1$ i.e., $-3\le x\le 3$.

Now $|u_n(x)|\le\left(\frac{1}{n^3}+\frac{2}{n^4}+\frac{1}{n^5}\right)$ $\forall n\in\mathbb{N}$ and $x\in[-3,3]$.

We take $M_n=\left(\frac{1}{n^3}+\frac{2}{n^4}+\frac{1}{n^5}\right.$, therefore $|u_n(x)|\le M_n$ $\forall_n\in\mathbb{N}$ and $x\in[-3,3]$.

We know $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent when $p>1$.

$\therefore\sum_{n=1}^{\infty}M_n$ is a sum of three convergent sequence.

Therefore $\therefore\sum_{n=1}^{\infty}M_n$ is convergent.

Therefore by Weierstrass's *M*-test, we conclude that $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^5}\left(\frac{x}{3}\right)^n$ is uniformly convergent on $[-3,3]$.

Example 3.5.3.

Prove that the series $x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\ldots$ *is not uniformly convergent on* $[0,1]$.

Solution :

Let $u_n(x)=\frac{x^2}{(1+x^2)^{n-1}}$ and

$$S_n(x)=u_1(x)+u_2(x)+\ldots u_n(x)$$

$$=x^2\left[1+\frac{1}{1+x^2}+\frac{1}{(1+x^2)^2}+\ldots+\frac{1}{(1+x^2)^{n-1}}\right]$$

When $x=0, S_n(x)=0$.

When $0<x\le 1$,

$$S_n(x)=x^2.\frac{\left[1-\frac{1}{\left(1+x^2\right)^n}\right]}{1-\frac{1}{1+x^2}}=\left(1+x^2\right)\left[1-\frac{1}{\left(1+x^2\right)^n}\right]$$

Now

$$\lim_{x\to\infty} S_n(x) = 0, \quad \text{if } x=0$$

$$=\left(1+x^2\right), \quad \text{if } 0<x\le 1.$$

Hence the sequence $\{S_n(x)\}$ converges pointwise to the limit function $S(x)$, where

$$S(x) = 0, \quad \text{for } x=0,$$

$$=\frac{1}{1+x^2} \quad \text{for } 0<x\le 1.$$

Now $S(x)$ is not continuous at $x=0$. So $S(x)$ is not continuous on $[0,1]$. Now each $S_n(x)$ is continuous on $[0,1]$. Therefore the sequence $\{S_n(x)\}$ is not uniformly convergent on $[0,1]$. Consequently, the series $x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\ldots$ is not uniformly convergent on $[0,1]$.

3.6 Exercise 2

1. Find the pointwise limit of the following series of functions $\sum_{n=1}^{\infty} x^n\left(1-x^n\right)$ on $[0,1]$.

2. Find the sum function of the series $\sum_{n=0}^{\infty}(1-x)x^n$, $0\le x\le 1$. Hence state with reason weather the series is uniformly convergent on $[0,1]$ or not.

3. State Cauchy's principle of convergence of a series of functions defined on an interval.

Prove that the series $\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}$ converges uniformly in $-\mathbb{R}\le x\le\mathbb{R}$, where $\mathbb{R}$ is a real number.

4. State Weierstrass's M-test and apply this to prove the series $\sum_{n=1}^{\infty}\frac{\cos x^2}{5n^6}$ is uniformly convergent for all $x\in[-3,2]$.

5. Find the sum function of the series $x^4+\frac{x^4}{1+x^4}+\frac{x^4}{\left(1+x^4\right)^2}+\frac{x^4}{\left(1+x^4\right)^3}+\ldots,$ $0\le x\le 1$.

Hence examine weather the series is uniformly convergent on $[0,1]$.

6. Show that the series of functions $\sum_{n=1}^{\infty}\frac{1}{n^2+\sin^2 x}$ is uniformly convergent on $\mathbb{R}$.

7. Using Weierstrass's M-test show that the series $\sum_{n=1}^{\infty}\frac{x}{n\left(1+nx^2\right)}$ is uniformly convergent on any interval $[a,b]$.

Answer :

1. $f(x)=0, 0\le x\le 1$

2. $S(x)=1,\ 0\le x<1$ and $f(x)=0,\ x=1$, Not uniformly convergent on $[0,1]$.

5. $S(x)=0,\ x=0$ and $S(x)=1+x^4,\ 0<x\le 1$. It is not uniformly convergent on $[0,1]$.

3.7 Limit Superior and Limit Inferior

We know that every convergent sequence is bounded but not every bounded sequence is convergent. With every bounded sequence, however, we associate two real numbers upper limit or limit superior (μ) and lower limit or limit inferior (λ). A bounded sequence converges only when $\lambda=\mu$.

(A) For a Bounded sequence $\{x_n\}$ of Real Numbers:

Definition 3.7.1. [*First Definition of* μ *and* λ]

1. *A real number* μ *is said to be the limit superior of the bounded sequence* $\{x_n\}$ *if* μ *satisfies the following two conditions :*

(i) For every $\varepsilon > 0, \exists$ *only a finite number of members of* $\{x_n\}$ *which are* $> \mu + \varepsilon$, *i.e., for every* $\varepsilon > 0, \exists$ *a positive integer N such that*

$$n > N \Rightarrow x_n \leq \mu + \varepsilon.$$

(ii) Given $\varepsilon > 0$ *and given positive number* $m, \exists$ *a positive integer* $n > m$ *such that* $x_n > \mu - \varepsilon$. *We write*

$$\mu = \lim\ \sup x_n \text{ or } \overline{\lim} x_n.$$

2. *A real number* λ *is said to be the limit inferior of the bounded sequence* $\{x_n\}$ *if* λ *satisfies the following tow conditions :*

(i) For every $\varepsilon > 0, \exists$ *only a finite number of members of* $\{x_n\}$ *less than* $\lambda - \varepsilon$, *i.e., for every* $\varepsilon > 0, \exists$ *a positive integer* n_0 *such that*

$$n > n_0 \Rightarrow x_n \geq \lambda - \varepsilon.$$

(ii) Given $\varepsilon > 0$ *and given positive number* $m, \exists$ *a positive integer* $n > m$ *such that* $x_n < \lambda + \varepsilon$. *We write*

$$\lambda = \lim\inf x_n \text{ or } \underline{\lim} x_n$$

Definition 3.7.2. [*Second Definition of* μ *and* λ]

Let $\{x_n\}$ *be a bounded sequence of real numbers. We collect all the subsequential limits of* $\{x_n\}$. *This collection is denoted by E, i.e., each member of E is the limit of a convergent sub-sequence. The set E is clearly non-empty and bounded; hence E has a glb and lub. We now define*

(i) The lub or supremum of E as the limit superior μ *of the sequence* $\{x_n\}$ *; we write* $\mu = \lim\ \sup x_n$ or $\overline{\lim} x_n$.

(ii) The glb or infimum of E is called the limit inferior λ *of the sequence* $\{x_n\}$. *Then we write* $\lambda = \lim\inf x_n$ *or* $\underline{\lim} x_n$.

B. For unbounded Sequence $\{x_n\}$ of Real Numbers :

(i) If $\{x_n\}$ is unbounded above, then $\lim\sup x_n = +\infty$.

(ii) If $\{x_n\}$ is unbounded below, then $\lim\inf x_n = -\infty$.

(iii) If $\{x_n\}$ is unbounded below, and there is no other sub-sequential limit then its limit inferior is also $-\infty$, so that

$$\overline{\lim} x_n = \underline{\lim} x_n - \alpha$$

(iv) If $\{x_n\}$ is unbounded above, and there is no other sub-sequential limit then its limit inferior is also $+\infty$, so that

$$\overline{\lim} x_n = \underline{\lim} x_n + \alpha .$$

Example 3.7.1.

The bounded sequence $\left\{(-1)^n\right\} = \{-1, +1, -1, +1, ...\}$ *has limit inferior* $\lambda = -1$ *and limit superior* $\mu = +1$. *There are only two sub-sequence limits* $\{-1, 1\}$ *for the sequence* $\left\{(-1)^n\right\}$ *Here* $\lambda \neq \mu$, *the sequence oscillates finitely between* -1 *and* $+1$.

Example 3.7.2.

The sequence $\left\{-2, 2 - \frac{3}{2}, \frac{3}{2}, -\frac{4}{3}, \frac{4}{3}, ...\right\}$ *has* $\lambda = -1$ *and* $\mu = 1$.

Remark 3.7.1.

Observe that none of the terms of the sequence lies between -1 *and* $+1$, *i.e.,* -1 *is not a lower bounded and* $+1$ *is not an upper bound.*

Remark 3.7.2.

The lower limit of a sequence is not necessarily the glb of the sequence, nor is the upper limit the lub. In fact, the lower limit may not even be alower bound, as we can be seen from example 3.7.2 *above.*

Example 3.7.3.

Let us consider the sequence $\{x_n\}$ *where* $x_n = \sin\frac{n\pi}{3}(n = 1, 2, 3, 4, ...)$.

The sequence $\{x_n\}$ *is*

$$\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, ...\right\}$$

The sub-sequential limits are $\frac{\sqrt{3}}{2}$, *0 and* $-\frac{\sqrt{3}}{2}$ *as those terms occur infinitely many times in the sequence.*

$$\underline{\lim} x_n = -\frac{\sqrt{3}}{2} \text{ and } \overline{\lim}\, x_n = +\frac{\sqrt{3}}{2}$$

Important Inequalities:

Theorem 3.7.1.

If $\{x_n\}$ *and* $\{y_n\}$ *are bounded sequence of real numbers, then*

(i) $\overline{\lim}(x_n + y_n) \le \overline{\lim} x_n + \overline{\lim} y_n$.

(ii) $\underline{\lim}(x_n + y_n) \ge \underline{\lim} x_n + \underline{\lim} y_n$.

Proof. Let $\overline{\lim} x_n \mu_1, \overline{\lim} y_n = \mu_2, \overline{\lim}(x_n + y_n) = \mu$

To prove

$$\mu \le \mu_1 + \mu_2 .$$

Let ε be any arbitrary bpositive number ; $\exists$ positive integers m_1 and m_2 such that

$$x_n < \mu_1 + \frac{1}{2}\varepsilon, \quad \forall n \ge m_1$$

$$y_n < \mu_2 + \frac{1}{2}\varepsilon, \quad \forall n \ge m_2 .$$

Hence, $x_n + y_n < (\mu_1 + \mu_2) + \varepsilon, \quad \forall n \ge m$, where $m = \max(m_1, m_2)$.

This proves that $\overline{\lim}(x_n + y_n)$ cannot exceed $\mu_1 + \mu_2$ (ε being arbitrary) or,

$$\overline{\lim}\left(x_n + y_n\right) \leq \mu_1 + \mu_2$$

i.e.,

$$\overline{\lim}\left(x_n + y_n\right) \leq \overline{\lim} x_n + \overline{\lim} y_n$$

Proof of (*ii*) is similar to (*i*). □

3.8 Power Series

We have studied sequence and series of functions in the previous sections. Here we will study a special type of series of functions, which are called Power Series. We will first define power series and then discuss about its convergency and some important properties.

Definition 3.8.1.

A series of the form $\sum_{n=0}^{\infty} a_n\left(x - x_0\right)^n = a_0 + a_1\left(x - x_0\right) + \ldots + a_n\left(x - x_0\right)^n + \ldots$

whose terms are powers of $x - x_0$ *multiplied by constants is called a power series. To study power series, it is sufficient to assume that* $x_0 = 0$, *since the substitution* $x - x_0 = x'$ *transforms the series into the form* $\sum_{n=0}^{\infty} a_n x'^n$. *Hence let us take*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

as the general type of power series in x.

Example 3.8.1.

$1 + 2x + 3x^2 + \ldots\infty$ *is a power series which can be written as* $\sum_{n=0}^{\infty} a_n x^n$, *where* $a_n = n + 1$.

Convergence of Power Series

A power series $\sum_{n=0}^{\infty} a_n x^n$ is said to be convergent (non-cnvergent) at $x = c$, according as the infinite series $\sum_{n=0}^{\infty} a_n c^n$ is convergent (non-cnvergent) . Clearly any power series $\sum_{n=0}^{\infty} a_n x^n$ is always convergent at $x = 0$.

There are three types of power series:

(i) If a power series is convergent for all $x \in \mathbb{R}$, then it is called everywhere convergent power series.

The power series $1+x+\frac{x^2}{2!}+\frac{x^3}{2!}+...+\frac{x^n}{n!}+...$ is an example of everywhere convergent power series.

(ii) If a power series does not converge for any real $x(\neq 0)$, then it is called nowhere convergent power series.

The power series $1+x+2!x^2+3!x^3+...$ is an example of nowhere convergent power series.

(iii) There are another type of power series, which are convergent for some values at x and non-convergent for other values of x.

The power series $\sum_{n=0}^{\infty} x^n$ i.e., $1+x+x^2+...$ is convergent when $|x| \geq 1$.

Theorem 3.8.1.

If $\sum_{n=0}^{\infty} a_n x^n$ *is convergent for* $x = a$, *it is absolutely convergent for every value of x such that* $|x| < |a|$. *If it diverges for* $x = b$, *it is devergent for all values of x such that* $|x| < |b|$.

Proof. since the series is convergent for $x = a$, the sequence $\{a_n x^n\}$ converges to zero and hence there exists a positive number M such that

$$|a_n x^n| < M \text{ for every value of } n \geq 0.$$

Now, if $|x| < |a|$, i.e., if $|x/a| = x < 1$,

$$|a_n x^n| = |a_n x^n| \times |x/a|^n < M\,k^n.$$

Hence the series converges by comparison with the convergent geometrical series with $k < 1$.

For the proof of the second part of the theorem let if possible the power series be convergent for $x = c$, where $|c| > |b|$ Since the series is convergent fo x = c, then by first part of this theorem the series would be convergent for $x = b$, which is a contradiction.

Hence the series is divergent for all real x stisfying $|x| > |b|$. □

Theorem 3.8.2. [*Fundamental Theorem*]

If $\sum_{n=0}^{\infty} a_n x^n$ *be any power series which does not merely converge everywhere or nowhere, then a definite positive integer R (called the radius of convergence of the power series) Exists such that* $\sum_{n=0}^{\infty} a_n x^n$ *converges for every* $|x| < R$ *(in fact absolutely), but diverges for* $|x| > R$. *In the two extreme cases, we write* $R = 0$ *if the series converges only at* $x = 0$ *(nowhere convergent) and* $R = +\infty$, *when the series converges for all x (everywhere convergent).*

Proof. Suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ is not merely everywhere convergent or no where convergent, so that there exist at least one point of convergence, a positive x_0 and one point of divergence, a positive y_0. Hence clearly $x_0 < y_0$. We call $[x_0, y_0]$ by I_0. Divide I_0 into two equal parts and denote by I_1, the left or right half of I_0 according as the series diverges or converges at the middle of I_0.

By a continuation of the process, the intervals of the nest (I_n) all have the property that $\sum_{n=0}^{\infty} a_n x^n$ converges at their left end point, say, x_n and diverges at their right enc point, say, y_n. The number R (necessarily positive) which the nest determines is the required number of the theorem.

In fact, if E be the set of x's for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges then

$$R = \sup_E |x|.$$

Now let any x in $|x| < R$ be given. Then there exists an x_0 such that

$$|x| < |x_0| < R$$

for which the series converges (this is by the difinition of supremum). By the previous theorem, It converges absolutely at x. Hence it converges for all such x that is all x for which $|x| < R$.

Suppose now that the series does not diverge for some x_0 where $|x_0| > R$. This means that it converges for x_0. But then we have found a member of the set E larger than the supremum. This is a contradiction. When $R = 0$ or $R = +\infty$, we have nothing to prove. Hence the theorem.

Definition 3.8.2.

Let for a given power series $\sum_{n=0}^{\infty} a_n x^n$, *E be the set of these real x's for which the series converges. Then the number R, defined below, will be called the radius of convergence of* $\sum_{n=0}^{\infty} a_n x^n$ *if*

(*i*) $R = 0$, *if* $\sum_{n=0}^{\infty} a_n x^n$ *converges only for* $x = 0$, *i.e.,* $E = \{0\}$.

(*ii*) $R = \infty$, *if* $\sum_{n=0}^{\infty} a_n x^n$ *converges only for real* x *i.e.,* $E = \mathbb{R} =$ *set of all real numbers.*

(*iii*) $R = \sup_{x \in E} |x|$, *if* $\sum_{n=0}^{\infty} a_n x^n$ *converges for some x's and diverges for others.*

The interval $-R < x < R$ *is called the interval of convergence of the power series* $\sum_{n=0}^{\infty} a_n x^n$.

Determination of the Radius of Convergence :

Theorem 3.8.3. [*First Method: Use of Ratio Test*]

If R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ *and if* $\lim_{n\to\infty} \left|\frac{a_n + 1}{a_n}\right| = q(q \neq 0)$ *then* $R = \frac{1}{q}$.

Proof. By D'Alembert's ratio-test, since

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1} + x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| = q\,|x|,$$

the series $\sum_{n=0}^{\infty} a_n x^n$ converges if $q|x| < 1$ i.e., $|x| < 1/q$ and diverges when $q|x| > 1$, i.e., $|x| > 1/q$. Thus $R = 1/q$. Hence the theorem. □

Theorem 3.8.4. [*Second Method: Cauchy-Hadamard Formula*]

For the power series $\sum_{n=0}^{\infty} a_n x^n$, *if*

$$q = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

then the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n x^n$ *is given by* $R = 0$ *if* $q = \infty$, $R = \infty$ *if* $q = 0$ *and* $R = 1/q$ *if* $0 < q < \infty$.

Also $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely if* $|x| < R$, *and diverges if* $|x| > R$.

Proof. As a consequence of Cauchy's root test,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x| = q|x|.$$

Now for convergence $q|x| < 1$, which is true for all values of x if $q = 0$. If $q = +\infty$, it holds only for $x = 0$; if $0 < q < \infty$, it holds only if $|x| < \frac{1}{q}$. Hence the theorem. □

Example 3.8.2.

Find the radiul of convegence of

$$x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - .... = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}$$

and discuss its convergence at each end of the interval.

Solution :

By ratio-test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|-\frac{n}{n+1}\right| = \frac{n}{n+1} \to 1 = q, \text{ as } n \to \infty.$$

Hence $R = 1/q = 1$. Thus the series is absolutely convergent for $|x| < 1$.

Also at $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - ...$ which is seen to convergent and at $x = -1$, the series takes the form $-\left(1 + \frac{1}{2} + \frac{1}{3} + ...\right)$ which is clearly divergent. Thus the series convergens in $-1 < x \le 1$.

Example 3.8.3.

Find the interval of convergence of

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}.$$

Solution

Here

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|-\frac{(2n-1)!}{(2n+1)!}x^2\right|=\frac{x^2}{2n(2n+1)}\to 0<1, \text{ as } n\to\infty.$$

So the series is absolutely convergent for all values of x.

Example 3.8.4.

Find the radius of convergence of the power series

$$1+\frac{x}{3}+\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^3+\ldots$$

Solution :

Here the given power series is $\sum_{n=0}^{\infty}a_nx^n$, where $a_n=\left(\frac{x}{3}\right)^n$.

If R be the radius of convergence, then by Cauchy-Hadamard formula we have

$$\frac{1}{R}=\limsup_{n\to\infty}\sqrt[n]{\left(\frac{1}{3}\right)^n}=\frac{1}{3}.$$

Hence $R=3$ is the radius of convergence of the given power series.

Some Properties of Power Series :

Theorem 3.8.5.

Let $\sum_{n=0}^{\infty}a_nx^n$, *be a power series with radius of convergence* $R(>0)$. *Then the series is uniformly convergent on* $[-s,s]$, *where* $0<s<R$.

Proof. Let $u_n(x)=a_nx^n,\ n\geq 0$

Since R is the radius of convergence of the power series, the series is absolutely convergent for all real x satisfying $|x|<R$.

Sience $0 < s < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x satisfying $|x| \leq s < R$.

Therefore the series $\sum_{n=0}^{\infty} |a_n s^n|$ is convergent.

Now $|u_n(x)| = |a_n x^n| \leq |a_n| s^n$ for all real x satisfying $|x| \leq s$.

Let $M_n = |a_n| s^n$ for all $n \in \mathbb{N}$.

Then $\sum_{n=0}^{\infty} M_n$ is a convergent series of positive real numbers and for all $n \in \mathbb{N}$, $|u_n(x)| \leq M_n$ for all $x \in [-s, s]$.

By Weierstrass's M-test, the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on $[-s, s]$.

Consequently, the series $\sum_{n=1}^{\infty} u_n(x)$, i.e., the power series $\sum_{n=1}^{\infty} a_n x^n$ is uniformly convergent on $[-s, s]$. □

Theorem 3.8.6. [*Continuity in the Sum.*]

A power series represents a continuous sum-function within its interval of convergence. That is, suppose the power series

$$\sum_{n=0}^{\infty} a_n x^n \quad \textit{converges for } |x| < R \qquad (3.8.1)$$

and define

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \textit{for} \quad |x| < R.$$

Then the functions S(x) is continuous on $[a, b]$ where $-R < a < b < R$.

Proof. Let x_1 be any point on $[a, b]$. For continuity of $S(x)$ at x_1, we are to show that for any given $\varepsilon > 0$, there must correspond a $\delta(\varepsilon)$ such that

$$|S(x) - S(x_1)| < \varepsilon \text{ for } |x - x_1| < \delta.$$

Now for any n, $\left(\text{taking } S_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n\right)$

$$|S(x)-S(x_1)|=|S(x)-S_n(x)+S_n(x)-S_n(x_1)+S_n(x_1)-S(x_1)|$$

$$\leq|S(x)-S_n(x)|+|S_n(x)-S_n(x_1)|+|S_n(x_1)-S(x_1)|. \qquad (3.8.2)$$

Again (3.8.2) being uniformly convergent to $S(x)$ on $[a,b]$, for any given $\varepsilon>0$ there exists a positive integer N (independent of x) such that

$|S_n(x)-S(x)|<\frac{\varepsilon}{3}$ for $n>N$ and for all x on $[a,b]$.

Since x_1 is any point on $[a,b]$,

$$|S_n(x_1)-S(x_1)|<\frac{\varepsilon}{3}, \text{ for } n>N.$$

Next, $S_n(x)=a_0+a_1x+a_2x^2+...+a_nx^n$ is a continuous function and hence there must exist a $\delta(\varepsilon)$ for any given $\varepsilon>0$ and for a fixed $n>N$ such that

$$|S_n(x_1)-S(x_1)|<\frac{\varepsilon}{3}, \text{ whenever } |x-x_1|<\delta.$$

Thus (3.8.2) gives,

$$|S(x)-S(x_1)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \text{ for } |x-x_1|<\delta.$$

Since x_1 is an arbitrary point on $[a,b]$, the theorem extablished. □

Theorem 3.8.7. [*Integration of Power Series.*]

A power series may be integrated term by term in any closed interval which lies entirely within its interval of convergence. That is, suppose the power series

$$\sum_{n=0}^{\infty}a_nx^n \qquad \textit{convrges for } |x|<R \qquad (3.8.3)$$

and define

$$S(x)=\sum_{n=0}^{\infty}a_nx^n \qquad \textit{for } |x|<R.$$

Then for $-R<a<b<R$

$$\int_a^b S(x)dx=\int_a^b a_0dx+\int_a^b a_1xdzx+...+\int_a^b a_nx^ndx+...$$

Proof. Let $S_n(x)=a_0+a_1x+a_2x^2+...+a_nx^n$. Now (3.8.3) converges uniformly to

$S(x)$ on $[a,b]$ and, therefore, for a given $\varepsilon > 0$, there exists a positive integer N (independent of x) such that

$$|S_n(x) - S(x)| < \frac{\varepsilon}{b-a} \text{ for } n > N \text{ and for all } x \text{ on } [a,b] \quad (3.8.4)$$

Again since (3.8.3) represents a continuous sum-function $S(x)$ on $[a,b]$, and hence integrable on $[a,b]$, we have for $n > N$ and with the help of (3.8.4),

$$\left|\int_a^b S_n(x)dx - \int_a^b S(x)dx\right| = \left|\int_a^b \{S_n(x) - S(x)\}dx\right|$$

$$\leq \int_a^b |S_n(x) - S(x)|dx < \int_a^b \frac{\varepsilon}{b-a}dx = \frac{\varepsilon}{b-a} \times (b-a) = \varepsilon$$

and the theorem follows.

□

Theorem 3.8.8. [*Differentiation of Power Series*]

A power series may be differentiated term by term in any closed interval which lies entirely within its interval of convergence. That is, suppose the power series

$$\sum_{n=0}^{\infty} a_n x^n \quad \textit{converges for } |x| < R \qquad (3.8.5)$$

and define

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \textit{ for } |x| < R. \qquad (3.8.6)$$

Then the function $S(x)$ is differentiable on $[a,b]$ where $-R < a < b < R$, and

$$S'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \textit{in} \quad a \leq x \leq b. \qquad (3.8.7)$$

Proof. Let $S_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$. Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$,

$$\limsup_{n\to\infty} \sqrt[n]{|na_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

so that the series (3.8.6) and (3.8.7) have the same interval of convergence $(-R, R)$. Since (3.8.7) is a power series, it converges uniformly to a function $\phi(x)$, say, on $[a,b]$, where $-R<a<b<R$. Integrating term by term for any x on $[a,b]$,

$$\int_a^x \phi(x)\,dx = \sum_{n=1}^{\infty}\int_a^x na_n x^{n-1}\,dx = \sum_{n=1}^{\infty}\{a_n x^n - a_n a^n\}$$

$$= \sum_{n=0}^{\infty}\{a_n x^n - a_n a^n\} = S(x) - S(a).$$

Hence,

$$\frac{d}{dx}\int_a^x \phi(x)\,dx = \frac{d}{dx}\{S(x)-S(a)\} = S'(x)$$

Since $\phi(x)$ is continuous on $[a,b]$, we have $\phi(x)=S'(x)$. Hence the theorem.

□

Theorem 3.8.9. [***Abel's Theorem (Limit Form).***]

If $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with finite radius of convergence R, and let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad -R<x<R.$$

If the series $\sum_{n=0}^{\infty} a_n x^n$ *converges, then*

$$\lim_{x\to R-0} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

Proof. Let us first show that there is no loss of generality in taking $R=1$. Put $x=Ry$, so that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n y^n = \sum_{n=0}^{\infty} b_n y^n, \qquad \text{where } b_n = a_n R^n.$$

It is a power series with radius of convergence R', where $R' = \frac{1}{\overline{\lim}\left|a_n R^n\right|^{\frac{1}{n}}}$. Thus it is sufficient to prove the following:

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with unit radius of convergence and let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad -1<x<1.$$

If the series $\sum_{n=0}^{\infty} a_n$ converges, then

$$\lim_{x\to -0} f(x) = \sum_{n=0}^{\infty} a_n$$

Let $S_n = a_0 + a_1 + a_2 + ... + a_n$. $S_{-1} = 0$, and let $\sum_{n=0}^{\infty} a_n = S$, then

$$\sum_{n=0}^{m} a_n x^n = \sum_{n=0}^{m} (S_n - S_{n-1}) x^n = \sum_{n=0}^{m-1} S_n x^n + S_m x^m - \sum_{n=0}^{m} S_{n-1} x^n$$

$$= \sum_{n=0}^{m-1} S_n x^n - x \sum_{n=0}^{m} S_{n-1} x^{n-1} + S_m x^m$$

$$= (1-x) \sum_{n=0}^{m-1} S_n x^n + S_m x^m$$

For $|x| < 1$, when $m \to \infty$, since $S_m \to S$, and $x^m \to 0$, we get

$$f(x) = (1-x) \sum_{n=0}^{\infty} S_n x^n, \qquad \text{for } 0 \le x \le 1. \qquad (3.8.8)$$

Again, since $S_n \to S$, for $\varepsilon > 0$, there exists N such that

$$|S_n - S| < \frac{\varepsilon}{2}, \quad \text{for all } n \ge N \qquad (3.8.9)$$

Also

$$(1-x) \sum_{n=0} S_n x^n = 1, \quad |x| < 1 \qquad (3.8.10)$$

Hence for $n \ge N$, and using (3.8.8) and (3.8.10), we have for $0 < x < 1$,

$$|f(x) - S| = \left| (1-x) \sum_{n=0}^{\infty} S_n x^n - S \right|$$

$$= \left| (1-x) \sum_{n=0}^{\infty} (S_n - S) x^n \right|$$

$$\le (1-x) \sum_{n=0}^{N} |S_n x^n - S| x^n + \frac{\varepsilon}{2} (1-x) \sum_{n=N+1}^{\infty} x^n$$

$$\le (1-x) \sum_{n=0}^{N} |S_n x^n - S| x^n + \frac{\varepsilon}{2}$$

But for a fixed $N, (1-x)\sum_{n=0}^{N}|S_n x^n - S|x^n$ is a positive continuous function of x, having zero value at $x=1$. Therefore there exists $\delta>0$, such that for $1-\delta<x<1, (1-x)\sum_{n=0}^{N}|S_n - S|x^n < \frac{\varepsilon}{2}$.

$$\therefore |f(x)-S| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \qquad \text{when } 1-\delta<x<1$$

Hence

$$\lim_{x\to 1-0} f(x) = S = \sum_{n=0}^{\infty} a_n$$

□

Theorem 3.8.10. [*Weierstrass Approximation Theorem.*]

If f is a real continuous function defined on a closed interval $[a,b]$ then there exixts a sequence of real polynomials $\{P_n\}$ which converges uniformly to $f(x)$ on $[a,b]$, i.e., $\lim_{n\to\infty} P_n(x = f(x)$, uniformly on $[a,b]$.

Proof. If $a=b$, the conclusion follows by taking $P_n(x)$ to be a constant polynomial, defined by $P_n(x)=f(a)$, for all n.

We may thus assume that $a<b$.

We next observe that a linear transformation $x' = \frac{(x-a)}{(b-a)}$ is a continuous mapping of $[a,b]$ onto $[0,1]$. Accordingly, we assume without loss of generality that $a=0,\ b=1$.

Consider

$$F(x)=f(x)-f(0)-x[f(1)-f(0)], \qquad \text{for } 0\le x\le 1.$$

Here $F(0)=0=F(1)$, and if F can be expressed as a limit of a uniformly convergent sequence of polynomials, then the same is true for f, since $f-F$ is a polynomial. So we may assume that $f(1)=f(0)=0$.

Let us further define $f(x)$ to be zero for *x* outside $[0,1]$. Thus *f* is now uniformly continuous on the whole real line.

Let us consider the polynomial (non-negative for $|x| \le 1$).

$$B_n(x) = C_n\left(1-x^2\right)^n, \quad n = 1,2,3,\ldots \qquad (3.8.11)$$

Where C_n independent of *x*, is so chosen that

$$\int_{-1}^{1} B_n(x)\,dx = 1, \quad \text{for } n = 1,2,3,\ldots \qquad (3.8.12)$$

$$I = \int_{-1}^{1} C_n\left(1-x^2\right)^n dx = 2C_n\int_0^1 \left(1-x^2\right)^n dx$$

$$\ge 2C_n\int_{-1}^{\frac{1}{\sqrt{n}}} \left(1-x^2\right)^n dx$$

$$\ge 2C_n\int_0^{\frac{1}{\sqrt{n}}} \left(1-nx^2\right) dx = \frac{4C_n}{3\sqrt{n}} > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow C_n < \sqrt{n} \qquad (3.8.13)$$

Which gives some information about the order of magnitude of C_n.

Therefore, for any $\delta > 0$, (3.8.13) gives

$$B_n(x) \le \sqrt{n}\left(1-\delta^2\right)^n, \quad \text{when } \delta \le |x| \le 1 \qquad (3.8.14)$$

So that $B_n \to 0$ uniformly, for $\delta \le |x| \le 1$.

Again, let

$$P_n(x) = \int_{-1}^{1} f(x+t)B_n(t)\,dt, \qquad 0 \le x \le 1$$

$$= \int_{-1}^{-x} f(x+t)B_n(t)\,dt + \int_{-x}^{1-x} f(x+t)B_n(t)\,dt + \int_{1-x}^{1} f(x+t)B_n(t)\,dt$$

For $|x| \le 1,\ 1+x \le x+t \le 0$, for $-1 \le t \le -x$, so that $x+t$ lies outside $[0,1]$ and therefore $f(x+t) = 0$, and hence the first integral on the right vanishes. Similarly the third integral is also equal to zero. Hence

$$P_n(x) = \int_{-1}^{1-x} f(x+t)B_n(t)\,dt$$

$$= \int_0^1 f(t)B_n(t-x)\,dt$$

which clearly is real [polynomial.

We now proceed to show that the sequence $\{P_n(x)\}$ converges uniformly to *f* on $[0,1]$.

Continuity of *f* on the closed interval $[0,1]$ implies that *f* is bounded and uniformly continuous on $[0,1]$.

Therefore there exists M such that

$$M = \sup|x| \qquad (3.8.15)$$

and for any $\varepsilon > 0$, we can choose $\delta > 0$ such that for any two points x_1, x_2 in $[0,1]$.

$$|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}, \qquad \text{when } |x_1 - x_2| < \delta \le 1 \qquad (3.8.16)$$

For $0 \le x \le 1$, we have

$$|P_n(x) - f(x)| = \left|\int_{-1}^{1} f(x+t)B_n(t)\,dt - f(x)\right|$$

$$= \left|\int_{-1}^{1} \{f(x+t) - f(x)\}B_n(t)\,dt\right| \qquad \text{[using (3.8.13)]}$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)|B_n(t)\,dt \qquad (\because B_n(t) \ge 0)$$

$$\le \int_{-1}^{-\delta} |f(x+t) - f(x)|B_n(t)\,dt$$

$$+\int_{-\delta}^{\delta} |f(x+t) - f(x)|B_n(t)\,dt + \int_{\delta}^{1} |f(x+t) - f(x)|B_n(t)\,dt$$

$$\le 2M\int_{-1}^{-\delta} B_n(t)\,dt + \frac{\varepsilon}{2}\int_{-\delta}^{-\delta} B_n(t)\,dt + 2M\int_{\delta}^{1} B_n(t)\,dt$$

[Using (3.8.15) and (3.8.16)]

$$\le 2M\sqrt{n}\left(1-\delta^2\right)^n \left\{\int_{-1}^{-\delta} dt + \int_{\delta}^{1} dt\right\} + \frac{\varepsilon}{2}$$

[using (3.8.13) and (3.8.15)]

$$\le 4M\sqrt{n}\left(1-\delta^2\right)^n + \frac{\varepsilon}{2}$$

$< \varepsilon$, for larg values of n.

Thus for $\varepsilon > 0$, there exists N (independent of x) such that

$$|P_n(x) - f(x)| < \varepsilon, \qquad \forall \geq N$$

$\Rightarrow \lim_{n\to\infty} P_n(x) = f(x)$, uniformly on $[0,1]$. □

3.9 Illustrrative Examples

Example 3.9.1.

Find the radius of convergence of the power series

$$1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots$$

Solution:

Let $\sum_{n=0}^{\infty} a_n x^n$ be the given power series.

Then $a_0 = 1$ and $a_n = \dfrac{1\cdot 3\cdot 5\cdot \ldots (2n-1)}{2\cdot 4\cdot 6\cdot \ldots 2n}$ for all $n \geq 1$.

Now

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdot \ldots\cdot(2n)(2n+2)} \times \frac{2\cdot 4\cdot 6\cdot \ldots\cdot 2n}{1\cdot 3\cdot 5\cdot \ldots\cdot(2n-1)}$$

$$= \lim_{n\to\infty}\frac{2n+1}{2n+2} = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{2+\frac{2}{n}} = \frac{2}{2} = 1.$$

Hence the radius of convergence of the given power series is 1.

Example 3.9.2.

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. *Show that the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n \cdot \dfrac{x^{n+1}}{n+1}$ *will also be R.*

Solution :

Since the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R, so by Cauchy-Hadamard formula,

$$\frac{1}{R}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} \text{ i.e., } R=\frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}}.$$

The series is nowhere or everywhere convergent according as $\overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}=\infty$ or zero.

Now for the second series

$$\overline{\lim_{n\to\infty}}\left|\frac{a_n}{n+1}\right|^{\frac{1}{n}}=\overline{\lim_{n\to\infty}}\frac{1}{(n+1)^{\frac{1}{n}}}\sqrt[n]{|a_n|}$$

$$=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}\quad\left[\because \lim_{nx\to\infty}\frac{1}{(n+1)^{\frac{1}{n}}}=1\right]$$

[using L'Hospital rule]

$$=\frac{1}{R}$$

Therefore it is proved that the radii of convergence of the power series $\sum_{n=0}^{\infty}a_nx^n$ and $\sum_{n=0}^{\infty}a_n\cdot\frac{x^{n+1}}{n+1}$ are same.

Example 3.9.3.

Starting form the power series expansion for $\log_e(1+x)$, *show that the power series representing* $(1+x)\log_e(1+x)=x+\frac{x^2}{1\cdot2}-\frac{x^3}{2\cdot3}+\frac{x^4}{3\cdot4}-\ldots$ *Find its radius of convergence. Deduce that* $\frac{1}{1\cdot2}-\frac{1}{2\cdot3}+\frac{1}{3\cdot4}-\ldots\infty=2\log_e 2-1$.

Solution:

We know that the power series expansion of $\log_e(1+x)$ is

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+\frac{(-1)^{n-1}x^n}{n}+\ldots,\quad -1<x\le1$$

This series is uniformly convergent and term by term integrable within its interval of convergence, i.e., $(-1,1]$. So

$$\int_0^1 \log(1+x)\,dx = \frac{x^2}{2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \ldots + \frac{(-1)^{n-1}x^{n+1}}{n(n+1)} + \ldots$$

or, $$(1+x)\log_e(1+x) - x = \frac{x^2}{2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \ldots + \frac{(-1)^{n-1}x^{n+1}}{n(n+1)} + \ldots$$

or, $$(1+x)\log_e(1+x) = x + \frac{x^2}{2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \ldots + \frac{(-1)^{n-1}x^{n+1}}{n(n+1)} + \ldots$$

Let $\sum_{n=0}^{\infty} a_n x^n$ be the form of the above power series, where $a_n = \frac{(-1)^{n-2}}{(n-1)n}, n \geq 2,\ a_1 = 1$. Now

$$\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n-1}}{n(n+1)}}{\frac{(-1)^{n-2}}{(n-1)n}}\right| = \lim_{n\to\infty}\frac{n-1}{n+1} = 1.$$

So the radius of convergence is 1.

When $x=1$, the series becomes

$$1 + \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \ldots + \frac{(-1)^{n-1}}{(n+1)^n} + \ldots$$

which is an althernating series and it is convergent by Leibnitz's test.

Hence by Abel's theorem

$$2\log_e 2 = 1 + \frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \ldots + \frac{(-1)^{n-1}}{(n+1)^n} + \ldots$$

Therefore $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \ldots\infty = 2\log_e 2 - 1$.

3.10 Summary

1. In Section 3.1 we have focussed on pointwise convergence and uniform convergence of sequence of functions, Cauchy Condition for Uniform Convergence, Test for uniform convergence and different theorems on uniform convergence and continuity, uniform convergence and integrability and uniform convergence and differentialbility.
2. In Section 3.4 we have focussed on pointwise convergence and uniform convergence of series of functions, Cauchy Condition for Uniform Convergence, Test for uniform convergence and different theorems on uniform convergence and continuity, uniform convergence and integrability and uniform convergence and differentiability.
3. In Section 3.7 we have discussed difinition some theorems on Limit superior and Limit Inferior.
4. In Section 3.8 we have studied convergence and absolute convergence of power series and discussed some tests (Abel's Test) for convergence of power series and different theorems regarding term by term integration and term by term differentiation of power series. Also we focussed on Weierstrass's Approximation Theorem.

3.11 Exercise 3

1. Verify the interval of convergence of the following series :

(*i*) $1+x+x^2+x^3+...$ for $-1<x<1$

(*ii*) $1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+...$ for $-1<x<1$

(*iii*) $1+2x+3x+4x+...$ for $-1<x<1$

(*iv*) $1+\frac{x}{2^2}+\frac{x^2}{2^4}+\frac{x^3}{2^6}+...$ for $-4<x<4$

(*v*) $x+\frac{x^2}{2\cdot 10}+\frac{x^3}{3\cdot 10^2}+...+\frac{x^n}{n\cdot 10^{n-1}}+...$ for $-10<x<10$

2. Calculate the radii of convergence of the following series:

(*i*) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ (*ii*) $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$

(*iii*) $\sum_{n=1}^{\infty}\left(n\frac{1}{3}+1\right)^n x^n$ (*iv*) $\sum_{n=1}^{\infty}\frac{(-1)^n n!}{n^n}x^n$

3. Give example to show that a power series $\sum_{n=0}^{\infty} a_n x^n$ may be convergent for all values of *x*, for a certain region of values of *x* or for no value of *x* except $x = 0$.

4. Prove Abel's theorem in the form: If the power series $\sum_{n=0}^{\infty} a_n x^n$ has a finite non-zero radius of convergence R and if it converge at $x = R$, then it converges uniformly in $-R < x < 0$.

5. Find the series for $\log_e(1+x)$ by integration and use Abel's theorem to show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \log 2$.

6. Find by integration or otherwise the power series for $\tan^{-1} x$ in the form

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ..., \quad -1 \le x \le 1$$

and show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

7. Without finding the sum $f(x)$ of the series

$$1 + \frac{x^2}{1!} + \frac{x^4}{2!} + ... + \frac{x^2 n}{n!} + ..., \quad -\infty \le x \le \infty,$$

Show that $f'(x) = 2x f(x)$ in $-\infty < x < \infty$.

8. Prove the following :
   (*i*) A power series can be integrated term by term in any closed interval wich lies entirely within its interval of convergence.
   (*ii*) A power series may be Differentiated term by term in any closed interval wich lies entirely within its interval of convergence.

Answer :

2. (*i*) 1 (*ii*) 1

(*iii*) $\frac{1}{2}$ (*iv*) *e*.

3.12 References

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