PREFACE

With its grounding in the "guiding pillars of Access, Equity, Equality, Affordability and Accountability," the New Education Policy (NEP 2020) envisions flexible curricular structures and creative combinations for studies across disciplines. Accordingly, the UGC has revised the CBCS with a new Curriculum and Credit Framework for Undergraduate Programmes (CCFUP) to further empower the flexible choice based credit system with a multidisciplinary approach and multiple/ lateral entry-exit options. It is held that this entire exercise shall leverage the potential of higher education in three-fold ways - learner's personal enlightenment; her/his constructive public engagement; productive social contribution. Cumulatively therefore, all academic endeavours taken up under the NEP 2020 framework are aimed at synergising individual attainments towards the enhancement of our national goals.

In this epochal moment of a paradigmatic transformation in the higher education scenario, the role of an Open University is crucial, not just in terms of improving the Gross Enrolment Ratio (GER) but also in upholding the qualitative parameters. It is time to acknowledge that the implementation of the National Higher Education Qualifications Framework (NHEQF) and its syncing with the National Skills Qualification Framework (NSQF) are best optimised in the arena of Open and Distance Learning that is truly seamless in its horizons. As one of the largest Open Universities in Eastern India that has been accredited with 'A' grade by NAAC in 2021, has ranked second among Open Universities in the NIRF in 2024, and attained the much required UGC 12B status, Netaji Subhas Open University is committed to both quantity and quality in its mission to spread higher education. It was therefore imperative upon us to embrace NEP 2020, bring in dynamic revisions to our Undergraduate syllabi, and formulate these Self Learning Materials anew. Our new offering is synchronised with the CCFUP in integrating domain specific knowledge with multidisciplinary fields, honing of skills that are relevant to each domain, enhancement of abilities, and of course deep-diving into Indian Knowledge Systems.

Self Learning Materials (SLM's) are the mainstay of Student Support Services (SSS) of an Open University. It is with a futuristic thought that we now offer our learners the choice of print or e-slm's. From our mandate of offering quality higher education in the mother tongue, and from the logistic viewpoint of balancing scholastic needs, we strive to bring out learning materials in Bengali and English. All our faculty members are constantly engaged in this academic exercise that combines subject specific academic research with educational pedagogy. We are privileged in that the expertise of academics across institutions on a national level also comes together to augment our own faculty strength in developing these learning materials. We look forward to proactive feedback from all stakeholders whose participatory zeal in the teaching-learning process based on these study materials will enable us to only get better. On the whole it has been a very challenging task, and I congratulate everyone in the preparation of these SLM's.

I wish the venture all success.

Professor. Indrajit Lahiri

Vice-Chancellor

Netaji Subhas Open University

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Ananya Mitra

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UG : Mathematics
(NMT)

Course Title: Introduction to Calculus
Corse Code: NEC-MT-03

Unit-1 ❑ Hyperbolic Functions

Structure

1.1 Objectives

We all know about trigonometric functions. In this lesson we will know about hyerbolic functions and their relations to the trigonometric functions. After studying this chapter the learner should be :

- understanding the meaning of hyperbolic functions and inverse hyperbolic functions.
- able to derive the hyperbolic functions in terms of exponential functions.
- mighty to realize the infinite series expansion of hyperbolic functions.
- aware of some important identities.
- skilled to express the inverse hyperbolic functions in terms of logarithmic functions.

1.2 Introduction

We have seen the shape of a heavy cable suspended between pylons. Is there any mathematical function corresponding the shape of the cable ? Yes, there is a function named hyperbolic function which has great applications in mathematics, physics and engineering. The hyperbolic functions have similar names to the trigonometric functions but they are defined in the forms of exponential functions. In this chapter we will derive the expressions of hyperbolic functions and express them in logarithmic forms. The first systematic development of hyperbolic functions was implemented by Swiss Mathematician Johann Heinrich Lambert (1728-1777).

1.3 Derivation of Hyperbolic Functions

Hyperbolic functions can be derived mathematically in various ways. We will derive the hyperbolic functions graphically. In this method an analogous relation can be found between the circular (trigonometric) functions and the hyperbolic functions. We elaborate this by starting first with the unit circle $u^2 + v^2 = 1$. Consider x as an angle forming a circular sector MOP of area C (see Fig. 1.1). Now the area C of this circular sector MOP is $\frac{1}{2}x$. Then twice C (the area of the circular sector MOP) is equal to circular angle x in radians.

For the unit circle $u^2 + v^2 = 1$, where $OM = 1$, we see that sin $x = v/OM = v$, and cos $x = u/OM = u$.

We can now develop analogously for the hyperbolic functinos. Suppose that H is the area of hyperbolic sector MOP (see Fig. 1.2) of the unit rectangular (equilateral) hyperbola $u^2 - v^2 = 1$, or $v = \sqrt{u^2 - 1}$.

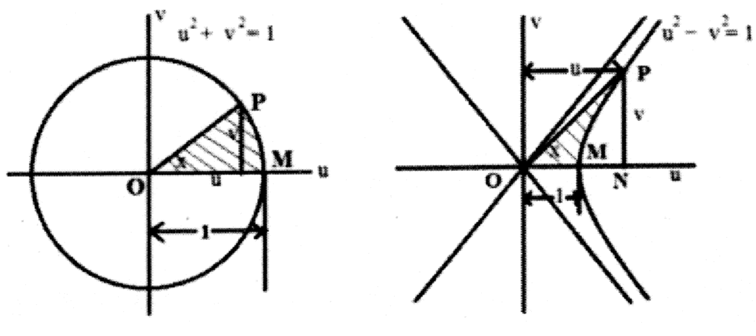

Fig. 1.1 Fig. 1.2

Then twice H (the area of the hyperbolic sector MOP) is equal to the hyperbolic angle x in radians. Now, from Fig. 1.2 we see that H (the area of the hyperbolic sector MOP) is the area of NOP less the area NMP, where area $NOP = \frac{1}{2}uv$ and area

$NMP = \int_1^u v du$. Then we obtain.

$$x = 2H \text{ (twice the area of MOP) } = 2\left[\frac{1}{2}uv - \int_1^u v du\right]$$

$$= \left[uv - 2\int_1^u \sqrt{u^2-1}\, du\right]$$

$$= \left[uv - u\sqrt{u^2-1} + \log\left(u + \sqrt{u^2-1}\right)\right]$$

$$= \left[uv - uv + \log\left(u + \sqrt{u^2-1}\right)\right]$$

$$= \log\left(u + \sqrt{u^2-1}\right).$$

From this we have

$$\left(u + \sqrt{u^2-1}\right) = e^x \text{ and } \left(u - \sqrt{u^2-1}\right) = e^{-x}$$

From this we get $u = \frac{e^x + e^{-x}}{2}$ and $v = \frac{e^x - e^{-x}}{2}$. These last two expressions are the familiar formulas "hyperbolic cosine" and "hyperbolic sine" denoted by cosh x and sinh x respectively. So we have

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ and } \sinh x = \frac{e^x - e^{-x}}{2},$$

where x is the twice of the area of the hyperbolic sector.

Remark : We need to point out that graphically it is not possible to draw the hyperbolic angle x in the same way that the circular angle x is drawn, for x has no such reality. It only exists as a function of the hyperbolic sector area H. It is important to avoid attempting to interpret x as an angle meeting at a point on the hyperbola.

1.3.1. Infinite Series Expansion of Hyperbolic Functions

Expanding e^x and e^{-x}, we get the expansions of cosh x and sinh x as

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

1.3.2. Periodicity of Hyperbolic Functions

Using the definition of hyperbolic functions we can easily prove that sinh $(2n\pi i + x)$ = sinh x, cosh $(2n\pi i + x)$ = cosh x, tanh $(2n\pi i + x)$ = tanh x. Thus hyperbolic functions are periodic functions of imaginary periods.

1.3.3. Some Important Identities

(i) $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1.$

(ii) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$

(iii) $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$

(iv) $\operatorname{sech}^2 x + \tanh^2 x = \left(\frac{2}{e^x + e^{-x}}\right)^2 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$

(v) $\coth^2 x - \operatorname{cosech}^2 x = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - \left(\frac{2}{e^x - e^{-x}}\right)^2 = 1.$

(vi) cosh $(-x)$ = cosh x. sinh $(-x)$ = $-$ sinh x.

(vii) tanh $(-x)$ = $-$ tanh x, coth $(-x)$ = $-$ coth x.

(viii) sech $(-x)$ = sech x, cosech $(-x)$ = $-$ cosech x.

(ix) cos (ix) = cosh x, sin (ix) = i sinh x.

(x) tan (ix) = i tanh x, cot (ix) = $-$ i sinh x.

(xi) cosh 0 = 1, sinh 0 = 0, tanh 0 = 0.

Example 1.3.1 Show that cosh $(x - y)$ = cosh x cosh y $-$ sinh x sinh y.

Solution : From the definition of hyperbolic functions we have

cosh x cosh y $-$ sinh x sinh y

$$= \frac{1}{4}\left\{\left(e^x + e^{-x}\right)\left(e^y + e^{-y}\right) - \left(e^x - e^{-x}\right)\left(e^y - e^{-y}\right)\right\}$$

$$= \frac{1}{4}\left\{\left(e^{x+y} + e^{x-y} + e^{-(x-y)} + e^{-(x+y)}\right) - \left(e^{x+y} - e^{x-y} - e^{-(x-y)} + e^{-(x+y)}\right)\right\}$$

$$= \frac{1}{2}\left\{e^{x-y} + e^{-(x-y)}\right\}$$

= cosh $(x - y)$.

Example 1.3.2 Show the equation sinh $x = \frac{3}{4}$.

Solution : $\sinh x = \frac{3}{4}$

$$\Rightarrow \frac{e^x - e^{-x}}{2} = \frac{3}{4}$$

$$\Rightarrow 2e^x - 3 - 2e^{-x} = 0$$

$$\Rightarrow 2e^{2x} - 3e^x - 2 = 0$$

$$\Rightarrow \left(e^x - 2\right)\left(2e^x + 1\right) = 0$$

$$\Rightarrow e^x = 2 \quad or\ 2e^x = -1.$$

But e^x is always positive, so $e^x = 2 \Rightarrow x = \log 2$.

1.4 Inverse Hyperbolic Functions

The inverse hyperbolic function $\sinh^{-1} x$, $\cosh^{-1} x$ are written as

$y = \sinh^{-1} x \Rightarrow \sinh y = x$ with $x \in (-\infty, \infty)$.

$y = \cosh^{-1} x \Rightarrow \cosh y = x$ with $x \geq 1$ and $y \geq 0$.

$y = \tanh^{-1} x \Rightarrow \tanh y = x$ with $|x| < 1$ and $y\ (-\infty, \infty)$.

1.4.1. Logarithmic Interpretation of Inverse Hyperbolic functions

Suppose

$$y = \sinh^{-1} x$$

$$\Rightarrow x = \sinh y$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \text{ as } e^y > 0$$

$$\Rightarrow y = \log\left(x + \sqrt{x^2 + 1}\right).$$

Thus $y = \sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right)$.

Similarly we see that

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right).$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

1.5 Summary

In this unit we derived hyperbolic functions in terms of exponential functions and their important identities. We also defined Inverse hyperbolic functions and their logarithmic expressions. We learned about the relations between hyperbolic functions and trigonometric functions. We expressed the infinite series expansion of hyperbolic functions and discussed the periodicity property of this functions.

1.6 Exercises

1. Prove the following identities.

(i) $\sinh(x + y) = \sinh x \cosh y + \cosh y \sinh x$

(ii) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 - \tanh x \tanh y}$

(iii) $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(iv) $\cosh 2x = 1 + 2\sinh^2 x$.

2. Solve the following equations.

(i) $2\cosh 2x + 10\sinh 2x = 5$ (ii) $\sinh x = \dfrac{5}{12}$

(iii) $4\cosh x + \sinh x = 4$ (iv) $\tanh x = \dfrac{1}{2}$

(v) $9\cosh x - 5\sinh x = 15$ (vi) $3\cosh^2 x + 11\sinh x = 17$.

3. Express the followings in logarithmic form

(i) $\sinh^{-1}\dfrac{3}{4}$ (ii) $\operatorname{sech}^{-1} x$ (iii) $\tanh^{-1} x$.

1.7 References

1. J.G. Chakravorty, P.R. Ghosh : Advanced Higher Algebra, U.N. Dhar & Sons Private Limited.
2. W. K. Robinson, Slide Rules with Hyperbolic Function Scales, The Journal of the Oughtred Society, Vol. 14, No. 1, 2005.

Unit-2 ❑ Higher Order Derivatives

Structure

2.1 Objectives

In this unit the readers will learn the followings.

- The definition of higher order derivatives.
- Leibnitz's formula.
- The differentiation of power functions.
- The higher order derivatives of product functions and quotient functions.

2.2 Introduction

The derivative is the first of the two main tools of calculus which was discover independently by Issach Newton and Gottfried Leibniz in the mid 17th century. The derivative of a function of real variable measures the rate of change of function value with respect to the change of independent variable. In this chapter we will see how to compute higher order derivatives and will explore some of their applications.

2.3 Higher Order Derivatives

Successive differentiation is the process of differentiating a function repeatedly n times and the results of such differentiation are called successive derivatives. The higher order derivatives have most importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its first derivative be $f'(x)$. If $f'(x)$ itself differentiable, we denote the derivate of $f'(x)$ by $f''(x)$ and call $f''(x)$, the second order derivative of $f(x)$. Continuing in this manner, we obtain the functions $f'(x)$, $f''(x)$, $f'''(x)$, $f^{iv}(x)$, ... $f^{(n)}(x)$, each of which is the derivative of previous one. We call $f^{(n)}(x)$, the nth derivative of $f(x)$ or the derivative of order n of the function $f(x)$.

2.3.1. Notations of Higher Order Derivatives

1st order derivative : $f'(x)$ or $\dfrac{dy}{dx}$ or Dy or y_1 or y'

2nd order derivative : $f''(x)$ or $\dfrac{d^2y}{dx^2}$ or D^2y or y_2 or y''

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n^{th} order derivative : $f^{(n)}(x)$ or $\dfrac{d^ny}{dx^n}$ or D^ny or y_n or $y^{(n)}$

Clearly $\dfrac{d^ny}{dx^n}=\dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)=\dfrac{d^2}{dx^2}\left(\dfrac{d^{n-2}y}{dx^{n-2}}\right)=\ldots,$

and so on.

2.4 Calculation of n^{th} Order Derivatives

2.4.1. n^{th} derivative of x^k

Let $y = x^k$, k being any real number.

Then $y_1 = kx^{k-1}$

$y_2 = k(k-1)x^{k-2}$

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$y_n = k(k-1)(k-2)...(k-n+1)x^{k-n}$, for all positive integer n.

If k be positive integer, then $y_k = k(k-1)(k-2)...(k-k+1)x^{k-k} = k!$

If k be positive integer but n is positive integer greater than k, then $y_n = 0$.

If $y = x^{-k}$, k being positive real number, then

$$y_n = -k(-k-1)(-k-2)...(-k-n+1)x^{-k-n}, \text{ for all positive integer } n$$

$$= (-1)^n \frac{k(k+1)(k+2)...(k+n-1)}{x^{n+k}}$$

$$= (-1)^n \frac{(k+n-1)!}{(k-1)!x^{n+k}}.$$

e.g., $y = x^{-1}$, $y_n = (-1)^n \frac{n!}{x^{n+1}}$.

$y = x^{-2}$, $y_n = (-1)^n \frac{(n+1)!}{x^{n+2}}$.

$y = \log x \ (x > 0)$, $y_1 = \frac{1}{x}$.

Hence y_n = nth derivative of $\log x$

$= (n-1)$th derivative of $\frac{1}{x}$

$$= (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

2.4.2. n^{th} derivative of e^{ax}.

Let $y = e^{ax}$.

Then $y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

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$y_n = a^ne^{ax}$.

2.4.3. n^{th} derivative of $\sin(ax + b)$

Let $y = \sin(ax + b)$.

Then $y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$

$$y_2 = a^2 \cos\left(ax+b+\frac{\pi}{2}\right) = a^2 \sin\left(ax+b+\frac{2\pi}{2}\right)$$

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$$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

$$y_{2n} = a^{2n} \sin\left(ax+b+\frac{2n\pi}{2}\right) = a^{2n}(-1)^n \sin(ax+b) = (-a^2)^n \sin(ax+b).$$

Similarly, if $y = \cos(ax+b)$,

$$y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

$$y_{2n} = (-a^2)^n \cos(ax+b).$$

2.4.4. n^{th} derivative of $e^{ax} \sin(bx + c)$

Let

$$y = e^{ax}\sin(bx+c).$$

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

$$= e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$$

$$= e^{ax}\left[r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)\right]$$

[Putting $a = r\cos\alpha$, $b = r\sin\alpha$]

$$= re^{ax}\sin(bx+c+\alpha), \text{ where } r^2 = a^2+b^2, \tan\alpha = \frac{b}{a}.$$

Similarly, $y_2 = r^2e^{ax}\sin(bx+c+2\alpha)$

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$$y_n = r^ne^{ax}\sin(bx+c+n\alpha).$$

Similarly, if $y = e^{ax}\cos(bx+c)$,

$$y_n = r^ne^{ax}\cos(bx+c+n\alpha).$$

Example 2.4.1 If $y = \frac{1}{x^2 - a^2}$, then find y_n.

Solution : Here

$$y = \frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

Thus, $$y_n = \frac{1}{2a}(-1)^n n!\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right].$$

Example 2.4.2 If $y = \sin^3 x$, then find y_n.

Solution : We know that $\sin 3x = 3\sin x - 4\sin^3 x$. Hence

$$y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

and $$y_n = \frac{1}{4}\left[3\sin\left(x + \frac{n\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right].$$

Example 2.4.3 If $y = \sin 3x \cos 2x$, then find y_n.

Solution : $$y = \sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x).$$

Therefore, $$y_n = \frac{1}{2}\left[5^n \sin\left(5x + \frac{n\pi}{2}\right) + \sin\left(x + \frac{n\pi}{2}\right)\right].$$

Example 2.4.4 If $y = \sqrt{x}$, then find y_n.

Solution : Here

$$y = \sqrt{x} = x^{1/2}.$$

Thus, $$y_1 = \frac{1}{2}x^{-1/2}$$

$$y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$$

$$y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} = (-1)^2 \frac{1.3}{2^3}\frac{1}{x^{3-1/2}}.$$

Differentiating continuously, we get

$$y_n = (-1)^{n-1}\frac{1.3.5.....(2n-3)}{2^n}\frac{1}{x^{n-1/2}}.$$

2.5 Leibnitz's Theorem on Successive Differentiation

If u and v are two functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(uv)_n = {}^nC_o u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n,$$

where u_r and v_r *represent* r^{th} derivatives of u and v respectively.

Proof :

By differentiating directly, we get

$$(uv)_1 = u_1 v + uv_1$$

$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2) = u_2 v + 2u_1 v_1 + uv_2$$

$$= {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 uv_2.$$

Thus, the theorem is true for $n = 1$ and $n = 2$.

Now we assume that the theorem is true for a certain positive integer m ($m < n$).

Then $(uv)_m = {}^mC_o u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots + {}^mC_r u_{m-r} v_r + \ldots + {}^mC_m u v_m.$

Differentiating both sides once more, we obtain

$$\begin{aligned}(uv)_{m+1} &= {}^mC_0(u_{m+1}v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1}v_2) + \ldots \\ &\quad + {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \ldots + {}^mC_m(u_1 v_m + uv_{m+1}) \\ &= {}^mC_0 u_{m+1} v + ({}^mC_0 + {}^mC_1)u_m v_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2 \\ &\quad + \ldots + ({}^mC_{r-1} + {}^mC_r)u_{m-r+1}v_r + \ldots \\ &\quad + ({}^mC_{m-1} + {}^mC_m)u_1 v_m + {}^mC_m uv_{m+1} \\ &= {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + \ldots + {}^{m+1}C_r u_{m-r+1} v_r \\ &\quad + \ldots + {}^{m+1}C_{m+1} uv_{m+1} \quad (\text{using } {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r).\end{aligned}$$

Thus, the theorem is true for $n = m + 1$ if it is true for $n = m$. Thus, by Mathematical induction the theorem is true for all positive integers n.

2.5.1. n^{th} derivative of $e^{ax+b} \sin x$

Let $y = e^{ax+b} \sin x = uv$, where $u = e^{ax+b}$ and $v = \sin x$.

Then $u_k = a^k e^{ax+b}$ and $v_k = \sin\left(x + \frac{k\pi}{2}\right)$.

Thus, by Leibnitz's rule

$$y_n = a^n e^{ax+b}\sin x + {}^nC_1 a^{n-1} e^{ax+b}\sin\left(x+\frac{\pi}{2}\right) + \ldots + e^{ax+b}\sin\left(x+\frac{n\pi}{2}\right)$$

$$= e^{ax+b}\left\{a^n \sin x + {}^nC_1 a^{n-1}\sin\left(x+\frac{\pi}{2}\right) + \ldots + \sin\left(x+\frac{n\pi}{2}\right)\right\}.$$

Similarly, if $y = e^{ax+b}\cos x$, then

$$y_n = e^{ax+b}\left\{a^n \cos x + {}^nC_1 a^{n-1}\cos\left(x+\frac{\pi}{2}\right) + \ldots + \cos\left(x+\frac{n\pi}{2}\right)\right\}.$$

2.5.2. n^{th} derivative of $(ax + b)^n \sin x$.

Let $y = (ax + b)^n \sin x = uv$, where $u = (ax + b)^n$ and $v = \sin x$.

Then $u_k = a^k \dfrac{n!}{(n-k)!}(ax+b)^{n-k}$ and $v_k = \sin\left(x+\dfrac{k\pi}{2}\right)$.

Thus, by Leibnitz's theorem, we obtain

$$y_n = \frac{n!}{0!}a^n \sin x + \frac{{}^nC_1 n!}{1!}a^{n-1}(ax+b)\sin\left(x+\frac{\pi}{2}\right)$$

$$+\ldots+(ax+b)^n \sin\left(x+\frac{n\pi}{2}\right)$$

$$= n!\left\{\frac{1}{0!}a^n \sin x + \frac{{}^nC_1}{1!}a^{n-1}(ax+b)\sin\left(x+\frac{\pi}{2}\right)\right.$$

$$\left.+\ldots+\frac{1}{n!}(ax+b)^n \sin\left(x+\frac{n\pi}{2}\right)\right\}.$$

Similarly, if $y = (ax + b)^n \cos x$, then

$$y_n = n!\left\{\frac{1}{0!}a^n \cos x + \frac{{}^nC_1}{1!}a^{n-1}(ax+b)\cos\left(x+\frac{\pi}{2}\right)\right.$$

$$\left.+\ldots+\frac{1}{n!}(ax+b)^n \cos\left(x+\frac{n\pi}{2}\right)\right\}.$$

Example 2.5.1 Find the n^{th} derivative of $y = x^3 \log x$.

Solution : Let $u = \log x$ and $v = x^3$.

Then $u_k = \dfrac{(-1)^{k-1}(k-1)!}{x^k}$ and $v_k = 0$ for $k \geq 4$.

By Leibnitz's theorem, we have

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n.$$

Thus,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}.x^3 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.3x^2$$

$$+\frac{n(n-1)}{2!}.\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}.6x + \frac{n(n-1)(n-2)}{3!}.\frac{(-1)^{n-4}(n-4)!}{x^{n-3}}.6$$

$$= (-1)^n \frac{6(n-4)!}{x^{n-3}}.$$

Example 2.5.2 Find the n^{th} derivative of $y = x^2 e^{3x} \sin 4x$.

Solution : Let $u = e^{3x} \sin 4x$ and $v = x^2$.

Then $u_k = e^{3x} 5^k \sin\left(4x + k\tan^{-1}\frac{4}{3}\right)$ and $v_k = 0$ for $k \geq 3$.

By Leibnitz's theorem, we have

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n.$$

Thus,

$$y_n = e^{3x} 5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right).x^2 + n.e^{3x} 5^{n-1} \sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right).2x$$

$$+\frac{n(n-1)}{2!}.e^{3x} 5^{n-2} \sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right).2$$

$$= e^{3x} 5^n \left\{x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right)\right.$$

$$\left. + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right\}.$$

Example 2.5.3 If $y = \sin (m \sin^{-1} x)$, then show that

$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n.$$

Also find $y_n(0)$.

Solution : $y = \sin (m \sin^{-1} x)$ (2.5.1)

$$\Rightarrow \quad y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x) \tag{2.5.2}$$

$$\Rightarrow \quad (1-x^2)y_1^2 = m^2\cos^2(m\ \sin^{-1}x)$$

$$\Rightarrow \quad (1-x^2)y_1^2 = m^2(1-y^2)$$

$$\Rightarrow \quad (1-x^2)y_1^2 + m^2y^2 = m^2.$$

Differentating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + y_1^2(-2x) + m^2 2yy_1 = 0$$

$$\Rightarrow \left(1-x^2\right)y_2 - xy_1 + m^2y = 0. \tag{2.5.3}$$

Using Leibnitz's theorem, we get

$$\left[y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) + {}^nC_2y_n(-2)\right] - \left(y_{n+1}x + {}^nC_1y_n.1\right) + m^2y_n = 0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow \left(1-x^2\right)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n. \tag{2.5.4}$$

Putting $x = 0$ in (2.5.1), (2.5.2) and (2.5.3), we get

$$y(0) = 0,\ y_1(0) = m,\ \ and\ \ y_2(0) = 0.$$

Putting $x = 0$ in (2.5.4), we have

$$y_{n+2}(0) = (n^2-m^2)y_n(0).$$

Putting $n = 1, 2, 3 \ldots$ in the above equation we have

$$y_3(0) = \left(1^2-m^2\right)y_1(0)$$

$$= \left(1^2-m^2\right)m$$

$$y_4(0) = \left(2^2-m^2\right)y_2(0)$$

$$= \left(2^2-m^2\right).0$$

$$= 0$$

$$y_5(0) = \left(3^2-m^2\right)y_3(0)$$

$$= m\left(1^2-m^2\right)\left(3^2-m^2\right).$$

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Therefore

$$y_n(0)=\begin{cases}0, \text{ if } n \text{ is even}\\ m\left(1^2-m^2\right)\left(3^2-m^2\right)\ldots\left((n-2)^2-m^2\right), \text{ if } n \text{ is odd.}\end{cases}$$

2.6 Summary

After studying this unit we have seen that we can derive a general formula of nth order derivative of a function without computing intermediate derivatives or by Leibnitz's Rule. To derive a general formula of nth order derivative of a function, it is better to differentiate again and again until it is clear.

2.7 Exercises

1. Find n^{th} order derivative of the following functions :

(i) $e^x \sin x \sin 2x$ (ii) $\tan^{-1}\dfrac{x}{a}$ (iii) $\dfrac{x^2}{x-1}$

(iv) $\dfrac{a-x}{a+x}$ (v) $\sin x \sin 2x \sin 3x$ (vi) $\tan^{-1}\dfrac{1+x}{1-x}$.

2. Use Leibnitz's formula to find the n^{th} derivative of the following functions :

(i) $e^x \log x$ (ii) $x^2 \tan^{-1} x$

(iii) $\log(ax + x^2)$ (iv) $x^3 \sin x$.

3. If $y = e^m \sin^{-1} x$, then show that

$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+m^2\right)y_n=0$. Also find $y_n(0)$.

4. If $y = \tan^{-1} x$, then show that

$\left(1-x^2\right)y_{n+2}+2(n+1)xy_{n+1}+n\left(n+1\right)y_n=0$. Also find $y_n(0)$.

5. If $y = (\sin^{-1} x)^2$, then show that

$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$. Also find $y_n(0)$.

2.8 References

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Unit-3 ❑ Curvature

Structure

3.1 Objectives

After going through this lesson the readers will learn :

- the definition of curvature.
- to derive the formula of radius of curvature.
- to find out the centre of curvature.

3.2 Introduction

In this chapter we will describe the nature of bending of a curve at a particular point and its numerical measurement. The curvature measures how fast a curve is changing direction at a given point.

3.3 Definitions

Suppose the tangents at two points P and Q on a curve make angles ψ and $\psi + \Delta\psi$ with positive x-axis. Suppose that are $AP = s$, arc $AQ = s + \Delta s$ so that are $PQ = \Delta s$, A being fixed point on the curve from which the length of ares are measured. We then construct the following definitions.

The angle $\Delta\psi$ between the tangents at P and Q is called the total curvature of the arc PQ.

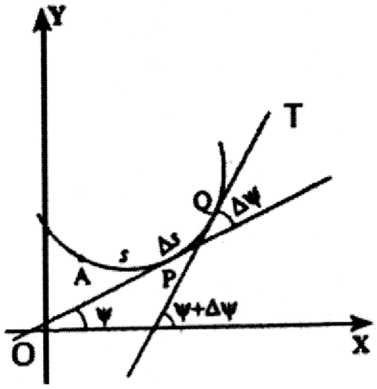

Fig. 3.1

The mean or average curvature of the arc PQ is defined as the ratio $\dfrac{\Delta\psi}{\Delta s}$.

The curvature (k) at a point P of the curve is defined as the limiting value of mean curvature when the arc $\Delta s \to 0$: that is

$$\text{Curvature } (k) \text{ at } P = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}.$$

Thus the curvature is the rate fo change of direction of the curve with respect to the arc length.

Suppose that $k \neq 0$ and $\rho = \dfrac{1}{k} = \dfrac{ds}{d\psi}$. Now construct a circle of radius ρ and a center C so that the circle and the curve Γ have the same tangent at P. The circle is drawn in such a way that it lies on the same side of the tangent as the curve. This circle has the same curvature as the given curve at P. We call this circle as the circle of curvature at P; its centre C is the center of curvature for the curve at P and its radius ρ, normal to curve at P is the radius of curvture of the curve at P. Thus the reciprocal of the curvature at any point P is called the radius of curvature at P, and is denoted by $\rho = \dfrac{1}{k} = \dfrac{ds}{d\psi}$.

3.4 Formulae for the Radius of Curvature

3.4.1. For the Intrinsic Equation $s = f(\psi)$.

The relation between the length of the arc (s) of a given curve, measured from a given fixed point on the curve and the angle between the tangents at its end (ψ)

is called the intrinsic equation of the curve and the formula of radius of curvature for this equation is

$$\rho = \frac{ds}{d\psi}.$$

For example, the intrinsic equation of Catenary is $s = c \tan \psi$ and $\rho = \frac{ds}{d\psi} = c \sec^2 \psi.$

3.4.2. For the Cartesian Equation (Explicit Function) $y = f(x)$ or $x = f(y)$.

In a rectangular Cartesian co-ordinates system, we have

$$\tan \Psi = \frac{dy}{dx} = y_1.$$

Therefore

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan \psi) = \frac{d}{d\psi}(\tan \psi).\frac{d\psi}{dx}$$

$$= \sec^2 \psi.\frac{d\psi}{ds}.\frac{ds}{dx}$$

$$= \sec^2 \psi.\frac{d\psi}{ds} \quad \left[\text{since } \frac{ds}{dx} = \sec\psi\right].$$

Since sec $\psi = \left(1+\tan^2\psi\right)^{\frac{1}{2}} = \left(1+y_1^2\right)^{\frac{1}{2}}$, we have

$$\rho = \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}, \text{ where } y_2 \neq 0. \qquad (3.4.1)$$

Similarly, for the equation $x = f(y)$,

$$\rho = \frac{(1+x_1^2)^{\frac{3}{2}}}{x_2}, \; [x_2 \neq 0]$$

where x_1 and x_2 denote $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$ respectively.

Note 3.4.1 Since ρ is always positive the root of numerator will be taken positive when y_2 is positive and negative when y_2 is negative.

Example 3.4.1 Find the radius of curvature of the parabola $y^2 = 4ax$ at the point $(a, 2a)$.

Solution : Here $y^2 = 4ax$.

Differentiating w.r.t x, we obtain

$$2yy_1 = 4a.$$

Again differentiating we have

$$2y_1^2 + 2yy_2 = 0.$$

Thus, at $(a, 2a)$, $y_1 = 1$ and $y_2 = -\frac{1}{2a}$.

Hence the required radius of curvature at $(a, 2a)$ is

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{-(1+1)^{\frac{3}{2}}}{-1/2a} = 4\sqrt{2}a.$$

3.4.3. For the Cartesian Equation (Implicit Function) $f(x, y) = 0$.

For the implicit equation, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}\left(f_y \neq 0\right),$$

$$\text{i.e., } f_x + f_y\frac{dy}{dx} = 0.$$

Differentiating again, we have

$$f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\frac{d^2y}{dx^2} = 0 \text{ [taking } f_{xy} = f_{yx}\text{]} \quad (3.4.2)$$

Putting the value of $\frac{dy}{dx}$ in (3.4.2), we get

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{f_y^3}.$$

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (3.4.1), we obtain

$$\rho = -\frac{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}, \quad (3.4.3)$$

where denominator is not equal to zero.

Example 3.4.2. Find the radius of curvature of the ellipse $9x^2 + 4y^2 = 36x$ at the point (2, 3).

Solution : We have $f(x, y) = 9x^2 + 4y^2 - 36x.$

Differentation $f(x, y)$ partially w.r.t. x, we get $f_x = 18x - 36$.

Similarly, we obtain

$$f_y = 8y,\ f_{xx} = 18,\ f_{yy} = 8,\ f_{xy} = 0.$$

Now at (2, 3),

$$f_x = 0,\ f_y = 24,\ f_{xx} = 18,\ f_{yy} = 8,\ f_{xy} = 0.$$

Thus, using the formula (3.4.3), we get

$$\rho = -\frac{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

$$= -\frac{\left[0 + (24)^2\right]^{\frac{3}{2}}}{18.(24)^2 - 0 + 0} = \frac{4}{3}$$

which is the required radius of curvature.

3.4.4. For the Parametric Equation $x = f(t)$, $y = \phi(t)$.

Here

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{y'}{x'} \quad (x' \neq 0)$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

$$= \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{x'^2}\frac{1}{x'}$$

Thus, using the formula (3.4.1), we get

$$\rho = \frac{\left(1 + \left(\frac{y'}{x'}\right)^2\right)^{\frac{3}{2}}}{\frac{x'y'' - y'x''}{x'^3}}$$

$$= \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''},\ x'y'' - y'x'' \neq 0. \tag{3.4.4}$$

Example 3.4.3 Find the radius of curvature of $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0$.

Solution : We have

$$x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta).$$

Differentiating w.r.t. θ, we obtain

$$x' = a(1 + \cos\theta),\ \ y' = a\sin\theta.$$

Again differentiating w.r.t. θ, we obtain

$$x'' = -a\sin\theta, \quad y'' = a\cos\theta$$

Now at $\theta = 0$, we have

$$x' = 2a, \quad x'' = 0, \quad y' = 0, \quad y'' = a.$$

Thus, using the formula (3.4.4), we get the radius of curvature at $\theta = 0$ as

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''} = \frac{\left[(2a)^2 + 0\right]^{\frac{3}{2}}}{2a.a - 0} = 4a.$$

3.4.5. For the Polar Equation $r = f(\theta)$.

We know $\psi = \theta + \phi = \theta + \tan^{-1}\frac{r}{r_1}$, where $r_1 = \frac{dr}{d\theta}$.

Thus,

$$\frac{d\psi}{d\theta} = 1 + \frac{1}{1 + \frac{r^2}{r_1^2}} . \frac{r_1^2 - rr_2}{r_1^2}$$

$$= \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}.$$

Again

$$\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}.$$

Thus

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\frac{d\theta}{d\psi} = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}. \tag{3.4.5}$$

Corollary 3.4.1 For the polar equation $u = f(\theta)$, where $u = \frac{1}{r}$.

Since $r = \frac{1}{u}$, we have

$$r_1 = \frac{dr}{d\theta} = -\frac{1}{u^2}.\frac{du}{d\theta} = -\frac{u_1}{u^2},\ r_2 = -\frac{uu_2 - 2u_1^2}{u^3},$$

where u_1 and u_2 denote $\frac{du}{d\theta}$ and $\frac{d^2u}{d\theta^2}$ respectively.

Thus

$$\rho = \frac{\left(u^2 + u_1^2\right)^{\frac{3}{2}}}{u^3\left(u + u_2\right)}. \qquad (3.4.6)$$

Example 3.4.4 Find the radius of curvature of the curve $r = a(1 - \cos\theta)$ at the point $(r,\ \theta)$.

Solution : We have $r = a(1 - \cos\theta)$.

Differentiating w.r.t. θ, we get $r_1 = a\sin\theta$. Again differentiating w.r.t. θ, we have $r_2 = a\cos\theta$.

Thus the radius of curvature is

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} = \frac{\left[a^2\left(1-\cos\theta\right)^2 + a^2\sin^2\theta\right]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta).a\cos\theta}$$

$$= \frac{a^3(2-2\cos\theta)^{\frac{3}{2}}}{a^2(3-3\cos\theta)} = \frac{2\sqrt{2a}}{3}\sqrt{1-\cos\theta}.$$

3.4.6. For the Pedal Equation $p = f(r)$.

We know the pedal equation as $p = r\sin\phi$.

Differentiating w.r.t. r, we get

$$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr} \quad \left[\tan\phi = r\frac{d\theta}{dr}\right]$$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{ds}$$

$$= r\frac{d}{ds}(\theta + \phi)$$

$$= r\frac{d\psi}{ds}.$$

Thus

$$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp}.$$

Example 3.4.5 Find the radius of curvature of the ellipse $\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2.$

Solution : We have $\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2.$

Differentiating w.r.t. p, we get $\frac{-2a^2b^2}{p^3} = -2r\frac{dr}{dp}.$

Therefore

$$\rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3}.$$

3.4.7. For the Tangential Polar Equation $p = f(\psi)$.

We have

$$\frac{dp}{d\psi} = \frac{dp}{dr}\frac{dr}{ds}\frac{ds}{d\psi} = \frac{dp}{dr}\cos\phi . r\frac{dr}{dp} = r\cos\phi$$

Thus,

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2.$$

Differentiating w.r.t. p, we get

$$2p + 2\frac{dp}{d\psi}\frac{d^2p}{d\psi^2}\frac{d\psi}{dp} = 2r\frac{dr}{dp}.$$

Thus,

$$\rho = p + \frac{d^2p}{d\psi^2}. \qquad (3.4.8)$$

Example 3.4.6 Find the radius of curvature of the epicycloid $p = a \sin b\psi$.

Solution : We have $p = a \sin b\psi$.

Differentiating w.r.t. ψ, we get

$$\frac{dp}{d\psi} = ab\cos\psi, \quad \frac{d^2p}{d\psi^2} = -ab^2 \sin b\psi = -b^2 p.$$

Thus, the radius of curvature is

$$\rho = p + \frac{d^2p}{d\psi^2} = p - b^2 p = p(1 - b^2).$$

3.5 Radius of Curvature : Newton's Approach

I. If a curve passes through the origin and the axis of x is tangent at the origin, then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

gives the radius of curvature at the origin.

II. If a curve passes through the origin and the axis of y is tangent at the origin, then

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

gives the radius of curvature at the origin.

III. If a curve passes through the origin and $ax + by = 0$ be the tangent at the origin, then

$$\frac{\sqrt{a^2+b^2}}{2} \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2+y^2}{ax+by}$$

gives the radius of curvature at the origin.

Example 3.5.1 Find the radius of curvature at the origin for the curve
$x^2 + y^3 - 2x^2 + 6y = 0$.

Solution : Here $y = 0$, the x-axis is the tangent at the origin. Thus by Netwon's formula, the radius of curvature is given by $\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$, i.e., $2\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{y}$.

Now dividing the given equation by y and making $x \to 0$ and $y \to 0$, we get

$$0.\rho + 0 - 4\rho + 6 = 0.$$

Therefore

$$\rho = \frac{3}{2},$$

which is the required radius of curvature.

3.6 Centre of Curvature

The co-ordinates $C(\bar{x}, \bar{y})$ of centre of curvature at a point $P(x, y)$ of a curve $y = f(x)$ is given by

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}, \ (y_2 \neq 0). \qquad (3.6.1)$$

Proof : We use the fact that the centre of curvature at P is the limit point of intersection of the normal at P and the normal at a neighbouring point Q when $Q \to P$ along the curve. The equation of the normal at P is

$$(Y - y)\phi(x) + (X - x) = 0, \qquad (3.6.2)$$

where the slope of the tangent at $P(x, y)$ is $y_1 = \frac{dy}{dx} = \phi(x)$ and X, Y are the current co-ordinate of any point on the normal.

The normal at a neighbouring point $Q(x + h, y + k)$ is

$$(Y - y - k)\phi(x+h) + (X - x - h) = 0. \qquad (3.6.3)$$

At their point of intersection, the ordinate is given by,

$$(Y - y)\{\phi(x+h) - \phi(x)\} - k\phi(x+h) - h = 0. \qquad (3.6.4)$$

Dividing by h and making $h \to 0$, we get

$$(\bar{y} - y)\left\{\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}\right\} - \left\{\lim_{h\to 0}\frac{k}{h}\right\}\lim_{h\to 0}\phi(x+h) - 1 = 0.$$

$$\text{or,} \quad (\bar{y} - y)\phi'(x) - \phi(x)\phi(x) - 1 = 0,$$

$$\text{or,} \quad \bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}.$$

As $(\bar{x}, \bar{y})$ is a point in (3.6.2), we get

$$(\bar{y} - y)\phi(x) + (\bar{x} - x) = 0$$

or, $$\frac{\left(1+y_1^2\right)}{y_2}.y_1+(\bar{x}-x)=0$$

or, $$\bar{x}=x-\frac{y_1\left(1+y_1^2\right)}{y_2}.$$

Example 3.6.1 Find the centre of curvature at any point (x, y) on the parabola y^2=4ax.

Solution : Here y^2 = 4ax. Differentiating w.r.t. x,

$$2yy_1 = 4a.$$

Thus,

$$y_1=\frac{2a}{y}=\sqrt{\frac{a}{x}}.$$

Again differentiating we get.

$$yy_2+y_1^2=0.$$

Thus,

$$y_2=-\frac{y_1^2}{y}=-\frac{1}{2}\frac{\sqrt{a}}{x\sqrt{x}}.$$

Hence

$$y_1\left(1+y_1^2\right)=\sqrt{\frac{a}{x}}\left(1+\frac{a}{x}\right)=\frac{\sqrt{a}(x+a)}{x\sqrt{x}}.$$

Thus,

$$\bar{x}=x-\frac{y_1\left(1+y_1^2\right)}{y_2}=x+2(x+a)=3x+2a$$

$$\bar{y}=y+\frac{1+y_1^2}{y_2}=y-\frac{2\sqrt{x}(x+a)}{\sqrt{a}}=-\frac{2x\sqrt{x}}{\sqrt{a}}.$$

3.7 Summary

In this chapter we have discussed the curvature of a smooth curve. We have also derived several formula for determining the radius of curvature for a curve and centre of curvature.

3.8 Exercises :

1. Find the radius of curvature at any point (s, ψ) on the following curves:

(i) $s = 8a \sin^2 \frac{1}{6}\psi$ (ii) $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$

(iii) $s = c \log \sec \psi$ (iv) $s = a\left(e^{mx} - 1\right)$.

2. Find the radius of curvature at any point (x, y) on the following curves :

(i) $e^{y/a} = \sec(x/a)$ (ii) $xy = c^2$

(iii) $x^{2/3} + y^{2/3} = a^{2/3}$ (iv) $x^3 + y^3 = 3axy$

(v) $y = 4\sin x - \sin 2x$ (vi) $\sqrt{x} + \sqrt{a} = \sqrt{a}$.

3. Find the radius of curvature at any point t on the following curves :

(i) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$

(ii) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

(iii) $x = a \sin 2t(1 + \cos 2t)$, $y = a \cos 2t(1 - \cos 2t)$

(iv) $x = at^2$, $y = 2at$

(v) $x = ae^t(\sin t - \cos t)$, $y = ae^t(\sin t + \cos t)$.

4. Find the radius of curvature at any point (r, θ) on the following curves :

(i) $r^2 = a^2 \cos 2\theta$ (ii) $r = ae^{\theta \cot \alpha}$

(iii) $r^2 \cos 2\theta = a^2$ (iv) $r = a \sin n\theta$.

5. Find the radius of curvature at any point on the following curves :

(i) $p = a(1 + \sin \psi)$ (ii) $p^2 + a^2 \cos 2\psi = 0$

(iii) $p = r \sin \alpha$ (iv) $p^2 = ar$.

6. Find the radius of curvature at the origin on the following curves :

(i) $y = x^4 - 4x^3 - 18x^2$ (ii) $3x^2 + xy + y^2 - 4x = 0$

(iii) $x^2 + 6y^2 + 2x - y = 0$ (iv) $x^4 + y^2 = 6a(x + y)$

(v) $3x^4 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$.

7. Find the centre of curvature of the following curves :

(i) $a^2y = x^3$ (ii) $2y = a(e^{x/a} + e^{-x/a})$

(iii) $y = x^3 + 2x^2 + x + 1$ at (0, 1) (iv) $y = \sin^2 x$ at (0, 0)

(v) $x^2 = 4ay$ (vi) $xy = x^2 + 4$ at (2, 4).

8. Find the radius of curvature of the curve $r = \dfrac{1}{1+e\cos\theta}$ **at** $\theta = \pi$.

9. Find the centre of curvature of the curve $x^3 + 2x^2 + x + 1$ **at (0, 1).**

3.9 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental differential Calculus, Santra Publication Pvt. Ltd., India, 2018.
4. D. Sengupta, Application of Calcus, Books and Allied (P) Ltd., Kolkata, India, 2012.

Unit-4 ❑ Concavity, Convexity and Points of Inflection

Structure

4.1 Objectives

We all have intuitive concepts of concavity and convexity. After reading this lesson the students will learn :

- the definition of concavity and convexity.
- the criterion for convexity and concavity.
- the meaning of points of inflection.
- to determine the points of inflection.

4.2 Introduction

In this unit we shall discuss about the sense of concavity and convexity at a special point of a curve $y = f(x)$. This special point is called a point of inflection.

4.3 Concavity and Convexity w.r.t. to a Line and Points of Inflection

Let P be a point on a plane curve. Let l be a straight line not passing through P. Then the curve is

(i) concave at P w.r.t. the line l if a sufficiently small arc containing P lies within the acute angle formed by l and the tangent to the curve at P.

(ii) convex at P w.r.t. the line l if a sufficiently small arc containing P lies outside the acute angle formed by l and the tangent to the curve at P.

On the other hand, if the curve is concave on one side of P and convex on other side w.r.t. l, then evidently the curve crosses its tangent at P. This point P is called a point of inflection.

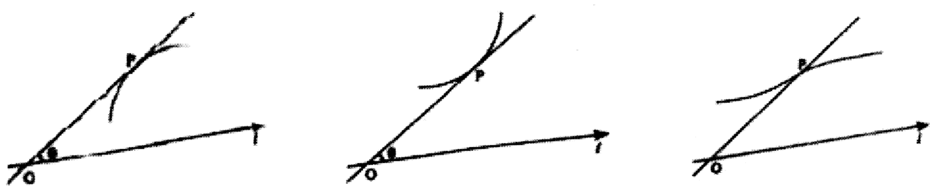

Fig. 4.1

4.4 Criterion of Concavity or Convexity w.r.t. *x*-axis

Let $P(x, y)$ be a point of a curve $y = f(x)$ and $Q(x+\Delta x,\ y+\Delta y)$ be a neighbouring point of P. Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R. The equation of PT is

$$Y - y = f'(x)(X - x).$$

Since the abscissa X of R is $x + \Delta x$, its ordinate is

$$RM = Y = y + \Delta x f'(x).$$

Also the ordinate of Q is

$$QM = f(x+\Delta x)$$

$$= f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x+\theta\Delta x),\ 0<\theta<1.$$

[Using Taylor's theorem]

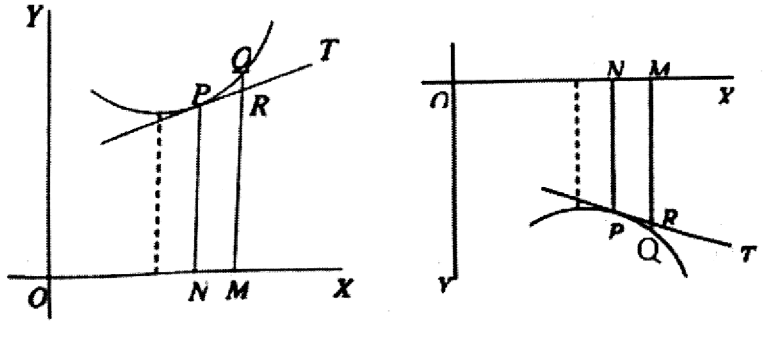

Fig. 4.2

Therefore

$$QM - RM = \frac{(\Delta x)^2}{2!} f''(x + \theta\Delta x).$$

If $f''(x)$ does not vanish and is continuous at x, $f''(x + \theta\Delta x)$ has the same sign as that of $f''(x)$ when $|\Delta x|$ is sufficiently small. Hence $QM - RM$ has the same sign as $f''(x)$ for sufficiently small values of $|\Delta x|$.

Thus, if $f''(x)$ is positive, then $QM - RM$ is positive for sufficiently small values of $|\Delta x|$ i.e., the small arc of the curve in a small neighborhood of P will be situated outside the acute angle formed by the tangent at P to the x-axis if the curve lie in the upper-side of x-axis.

Again if $f''(x)$ is negative, then $QM - RM$ is negative for sufficiently small values of $|\Delta x|$ i.e., the small arc of the curve in a small neighborhood of P will be situated outside the acute angle formed by the tangent at P to the x-axis if the curve lie in the lower-side of x-axis.

Combining this two situation we can say that the curve is convex at P to the x-axis if $yf''(x)$ at P is positive.

Analogously, if $yf''(x)$ at P is negative, then the curve at P is concave to the x-axis.

Note 4.4.1. The curve at P is convex or concave w.r.t. the y-axis according as $xf''(y)$ is positive or negative at P.

Example 4.4.1 Prove that the curve $y = e^{2x}$ is convex to the x-axis at every point.

Solution : Here

$$\frac{dy}{dx} = 2e^{2x}$$

$$\frac{d^2y}{dx^2} = 4e^{2x}.$$

Hence

$$y\frac{d^2y}{dx^2} = 4e^{4x} > 0, \text{ for all values of x.}$$

Thus, the curve is convex to the x-axis at every point.

4.5 Criterion for Points of Inflection

We have defined a point of inflection on the curve $y = f(x)$ as a point where the curve crosses its tangent. We have shown that such a point can only exist if $f'(x)$ = 0. The abscissa of the points of inflection are therefore the roots of the equation

$$f''(x) = 0.$$

But the converse is not true.

From the previous discussion we see that if $f''(x) = 0$ and $f'''(x) \neq 0$.

$$QM - RM = \frac{(\Delta x)^3}{3!} f'''(x + \theta \Delta x).$$

This gives opposite sign for positive and negative value of Δx. Hence in order that the abscissa x corresponds to a point of inflection.

$$f''(x) = 0 \text{ and } f'''(x) \neq 0.$$

More general form :

Suppose that at P, $f''(x) = f'''(x) = \ldots = f^{(n-1)}(x) = 0$ *and* $f^{(n)}(x) \neq 0$.

Then by Taylors theorem,

$$QM - RM = \frac{(\Delta x)^n}{n!} f^{(n)}(x + \theta \Delta x).$$

If n is even, then the curve is convex at P to the the x-axis when $yf^{(n)}(x)$ at P is positive and concave at P to the the x-axis when $f^{(n)}(x)$ at P is negative.

If n is odd, then the point of inflection are the roots of the equation $f^{(n)}(x) = 0$.

Example 4.5.1 Show that the curve $y = x^3$ has a point of inflection at $x = 0$.

Solution : Here $\frac{dy}{dx} = 3x^2$ and $\frac{d^2y}{dx^2} = 6x$.

At $x = 0$, $\frac{d^2y}{dx^2} = 0$.

When $x < 0$ (sufficiently near to zero) $\frac{d^2y}{dx^2}$ remains negative so that the curve is concave downwards there. But when $x > 0$ (sufficiently near to zero) $\frac{d^2y}{dx^2}$ becomes positive so that the curve is concave upwards there. Hence $x = 0$ is a point of inflection.

Example 4.5.2 Examine the curve $y = \sin x$ regarding its concavity or convexity to the x-axis, and determine its point of inflection, if any.

Solution : Here $\frac{dy}{dx} = \cos x$ and $\frac{d^2y}{dx^2} = -\sin x$.

Hence $y\frac{d^2y}{dx^2} = -\sin^2 x$ which is negative for all values of x excepting those which make $\sin x = 0$, i.e., for $x = k\pi$, k being any integer.

Thus the curve is concave w.r.t. x-axis at every point except at points where the curve crosses the x-axis.

Hence these points given by $x = k\pi$, where $\frac{d^2y}{dx^2} = 0$, crosses the x-axis are points of inflection.

4.6 Summary

After reading this lesson we came to know a very important significance of second derivatives. It's change of values determines the concavity, convexity and point of inflection of a curve at a point.

4.7 Exercises

1. **Find the points of inflection, if any on the following curves.**

   (i) $y = \frac{x}{(x+1)^2+1}$ (ii) $y^2 = x(x+1)^2$

   (iii) $x = 3y^4 - 4y^3 + 5$ (iv) $y(x-a)^2 = a^2x$.

2. Prove that the curve $y = \cos^{-1} x$ is everywhere convex to the y-axis excepting where it crosses the y-axis.
3. Show that the curve $(y - a)^3 = a^3 - 2a^2x + ax^2$, $(a > 0)$ is concave to the x-axis.
4. Show that the curve $y = \log x$ is convex everywhere to the y-axis.

4.8 References

1. B.C. Das, B.N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India. 1949.
2. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
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Unit-5 ❑ Asymptotes

Structure

5.1 Objectives

In this chapter the students will learn the followings :

- definition of asymptote.
- type of asymptotes.
- the method for finding asymptotes of a curve.

5.2 Introduction

The concept that a curve may come arbitrary close to a line may introduce the word 'Asymptote' which was introduced by Apollonius of Perga in his work on conic sections. Asymptotes of a curve are very important to sketch its graph.

5.3 Definition

A point P with co-ordinate (x, y) on an infinite branch of a curve is said to tend to infinity ($P \to \infty$) along the curve if either x or y or both tend to $\pm\infty$ as P traverses along the branch of the curve.

5.4 Asymptotes

A straight line is said to be a rectilinear asymptote of an infinite branch of a curve if as a point P of the curve tends to infinity along the branch, the perpendicular distance of P from the straight line tends to zero.

Example 5.4.1 For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$ the straight lines $y = \pm x$ are two asymptotes as shown in the Fig. 5.1.

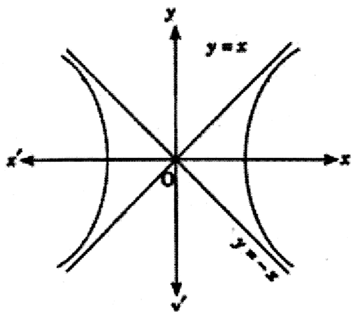

Fig. 5.1

5.5 Asymptotes Parallel to Axes

5.5.1. Asymptotes parallel to y-axis for the curve $y = f(x)$:

Theorem 5.5.1 A necessary and sufficient condition that the line $x = a$ may be an asymptote to the curve $y = f(x)$ is that $|f(x)| \to \infty$ as $x \to a+0$ or $x \to a - 0$ or $x \to a$.

Proof : First suppose that $x \to a - 0$. Let $P(x, y)$ be a point on an infinite branch of the curve $y = f(x)$. As $|f(x)| \to \infty$, i.e., $y \to +\infty$ or $-\infty$ for $x \to a-0$, it immediately follows that $P \to \infty$. As $x \to a - 0$, the perpendicular distance PT of P from the line $x = a$ is $|x - a|$ which tends to zero. Hence $x = a$ is an asymptote.

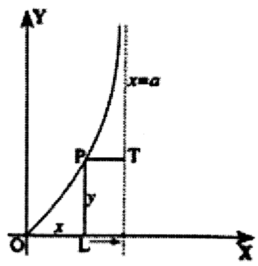

Fig. 5.2

Similar arguments follows for the cases $x \to a + 0$ or $x \to a$.

Conversely, let $x = a$ is an asymptote. Now as $x \to a + 0$ or $x \to a - 0$ or $x \to 0$, we must say $|f(x)| \to \infty$, otherwise P can not tend to ∞ which is essential for obtaining the asymptote.

In the same ideas we obtain the following :

5.5.2. Asymptotes parallel to x-axis for the curve $x = g(y)$

Theorem 5.5.2 A necessary and sufficient condition that the line $y = b$ may be an asymptote to the curve $x = g(y)$ is that $|g(y)| \to \infty$ as $y \to b + 0$ or $y \to b - 0$ or $y \to b$.

Example 5.5.1 Find the asymptotes parallel to the axes of the following curves :

(i) $y = f(x) = \dfrac{3x}{x-5}$, (ii) $y = f(x) = xe^{\frac{1}{x}}$, (iii) $x = \dfrac{t^2+1}{t^2-1}$, $y = \dfrac{t^2}{t-1}$.

Solution : (i) Here

$$\lim_{x \to 5+0} f(x) = \lim_{x \to 5+0} \frac{3x}{x-5} = +\infty$$

$$\lim_{x \to 5-0} f(x) = \lim_{x \to 5-0} \frac{3x}{x-5} = -\infty.$$

Hence $x = 5$ is the asymptote parallel to y-axis of the given curve.

(ii) Here

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = +\infty.$$

Hence the curve has a vertical symptote $x = 0$.

(iii) As $t \to 1+0, x \to \infty, y \to \infty$ *and as* $t \to -1-0, x \to \infty, y \to -\frac{1}{2}$.

Hence $y = -\frac{1}{2}$ is an asymptote parallel to x-axis.

5.5.3. Asymptotes parallel to the axes for the rational algebraic curve $f(x, y) = 0$

Let the equation $f(x, y) = 0$, when arranged in descending powers of y be represented by

$$f(x, y) = y^n\phi_0(x) + y^{n-1}\phi_1(x) + y^{n-2}\phi_2(x) + \ldots + \phi_n(x) = 0, \qquad (5.5.1)$$

where $\phi_0(x), \phi_1(x), \phi_2(x), \ldots\ldots, \phi_n(x)$ are polynomials in x.

Rearranging (5.5.1), we get

$$\phi_0(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots + \frac{1}{y^n}\phi_n(x) = 0. \qquad (5.5.2)$$

If now there exists an asymptote parallel to y-axis, say $x = \lambda$, (a finite real number), then as $x \to \lambda$, $y \to \infty$ and so (5.5.2) gives

$$\phi_0(\lambda) = 0 \qquad (5.5.3)$$

i.e., λ is a root of the equation

$$\phi_0(x) = 0, \qquad (5.5.4)$$

where $\phi_0(x)$ is the coefficient of the highest degree terms in $f(x, y) = 0$.

If it so happens that $\lambda_1, \lambda_2, \ldots$ are the real roots of $\phi_0(x) = 0$, then $x = \lambda_1$, $x = \lambda_2, \ldots$ are the asymptotes parallel to y-axis provided the infinite branches of the curve corresponding to the asymptotes actually exist.

We now from algebra if $\lambda_1, \lambda_2, \ldots\ldots$ are the real roots of $\phi_0(x) = 0$, then $\phi_0(x) = (x - \lambda_1)(x - \lambda_1)(x - \lambda_2) \ldots\ldots$ Hence we have the following rules :

I. The asymptotes parallel to y-axis are determined by equating to zero the real linear factors in the coefficient of the highest power of y present in the equation $f(x, y) = 0$.

Note 5.5.1 No such vertical asymptotes exist if the coefficient of the highest power of y is a constant or not resolvable into real linear factors.

In similar manner we have the following rule :

II. The asymptotes parallel to x-axis are determined by equating to zero the real linear factors in the coefficient of the highest power of x present in the equation $f(x, y) = 0$.

Note 5.5.2 No such horizontal asymptotes exist if the coefficient of the highest power of x is a constant or not resolvable into real linear factors.

Example 5.5.2 Find the asymptotes of the curve

$$x^2y^2 - a^2(x^2+y^2) - a^3(x+y) + a^4 = 0,$$

which are parallel to axes.

Solution : This equation is in algebraic form. Here the highest of x is x^2 and its coefficient is $(y^2 - a^2)$. Hence the asymptotes parallel to x-axis are $y = a$, $y = -a$.

Similarly, we see that the asymptotes parallel to y-axis are $x = a$, $x = -a$.

Example 5.5.3 Find the asymptotes, if any, parallel to the co-ordinate axes to the curve $x^3 - 2x^2y + xy^2 + x - xy + 2 = 0$.

Solution : The coefficient of highest degree of x i.e., of x^3 is constant. Hence there is no asymptote parallel to x-axis.

The highest degree term in y is y^2 and its coefficient is x. Hence the asymptote parallel to y-axis is $x = 0$.

5.6 Oblique Asymptotes

Theorem 5.6.1 If an infinite branch of a curve possesses an asymptote $y = mx + c$, (m and c being finite), then

$$m = \lim_{|x|\to\infty} \frac{y}{x}; \quad c = \lim_{|x|\to\infty}(y - mx)$$

and conversely.

Proof Let $P(x, y)$ be a point on an infinite branch of a curve. The perpendicular distance of P from the line $y = mx + c$ is

$$d = \left|\frac{y - mx - c}{\sqrt{1+m^2}}\right|.$$

If the line $y = mx + c$ is an asymptote then d should tend to zero as $P \to \infty$,

$$i.e., \ d = \left|\frac{y - mx - c}{\sqrt{1+m^2}}\right| \to 0 \text{ as } |x| \to \infty,$$

$$\text{i.e., } |y - mx - c| \to 0 \text{ as } |x| \to \infty.$$

Hence

$$c = \lim_{|x|\to\infty}(y - mx).$$

Again

$$\lim_{|x|\to\infty}\left(\frac{y}{x} - m\right) = \lim_{|x|\to\infty}(y - mx).\lim_{|x|\to\infty}\frac{1}{x} = c.0 = 0.$$

Therefore

$$m = \lim_{|x|\to\infty}\frac{y}{x}.$$

Conversely, if the given condition holds, them $y - mx - c \to 0$ as $P \to \infty$ which means that $d \to 0$ as $P \to \infty$. Hence $y = mx + c$ is an asymptote.

Example 5.6.1 Examine the asymptotes of the curve $y = \frac{3x}{x-1} + 3x.$

Solution : Since

$$\lim_{x\to 1+0} y = \lim_{x\to 1+0}\left(\frac{3x}{x-1} + 3x\right) = +\infty$$

$$\lim_{x\to 1-0} y = \lim_{x\to 1-0}\left(\frac{3x}{x-1} + 3x\right) = -\infty,$$

the curve has a vertical asymptote $x = 1$.

Moreover for the oblique asymptotes

$$m = \lim_{|x|\to\infty}\frac{y}{x} == \lim_{|x|\to\infty}\left(\frac{3}{x-1} + 3\right) = 3,$$

$$c = \lim_{|x|\to\infty}(y - mx) == \lim_{|x|\to\infty}\left(\frac{3x}{x-1} + 3x - 3x\right) = 3.$$

Therefore the straight line $y = 3x + 3$ is an oblique asymptote.

5.7 Asymptotes Non-parallel to *y*-axis of the Rational Algebraic Curve *f*(*x*, *y*) = 0

Let the equation of the curve $f(x, y) = 0$ be arranged in groups of homogeneous terms as

$$f(x,y) = \left(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n\right)$$
$$+\left(b_1x^{n-1} + b_2x^{n-2}y + b_3x^{n-3}y^2 + \ldots + b_ny^{n-1}\right)$$
$$+ \ldots + \left(l_{n-1}x + l_ny\right) + k_n = 0,$$

which may be written as

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+\ldots+x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0, \qquad (5.7.1)$$

where $\phi_r\left(\frac{y}{x}\right)$ denotes an algebraic polynomial in $\left(\frac{y}{x}\right)$ of degree r.

Dividing by x^n and then make $|x| \to \infty$, we suppose that $\lim_{|x|\to\infty}\frac{y}{x}=m.$ We then obtain

$$\phi_n(m)=0,$$

which gives the slopes of asymptotes correponding to different branches of the curve. To get c corresponding to m, we put $y - mx = k$, where $k \to c$ as $|x| \to \infty$.

Putting $\frac{y}{x}=m+\frac{k}{x}$ in (5.7.1) we obtain

$$x^n\phi_n\left(m+\frac{k}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{k}{x}\right)$$

$$+\ldots\ldots+x\phi_1\left(m+\frac{k}{x}\right)+\phi_0\left(m+\frac{k}{x}\right)=0. \qquad (5.7.2)$$

Expanding Taylor's expansion, we have

$$x^n\left\{\phi_n(m)+\frac{k}{x}\phi_n'(m)+\frac{k^2}{2x^2}\phi_n''(m)+\ldots\right\}$$

$$+x^{n-1}\left\{\phi_{n-1}(m)+\frac{k}{x}\phi_{n-1}'(m)+\frac{k^2}{2x^2}\phi_{n-1}''(m)+\ldots\right\}$$

$$= \ . \ . \ . = 0 \qquad (5.7.3)$$

Arranging (5.7.3) we get

$$x^n\phi_n(m)+x^{n-1}\left\{k\phi_n'(m)+\phi_{n-1}(m)\right\}$$

$$+x^{n-2}\left\{\frac{k^2}{2}\phi_n''(m)+k\phi_{n-1}'(m)+\phi_{n-2}(m)\right\}$$

$$+ \ . \ . \ . = 0 \qquad (5.7.4)$$

Since $\phi_n(m)=0$, dividing (5.7.4) by x^{n-1} and making $|x| \to \infty$, we get

$$c\phi_n'(m)+\phi_{n-1}(m)=0, \ \text{[since } k\to c \text{ as } |x|\to\infty] \qquad (5.7.5)$$

or, $c = -\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$ provided $\phi'_n(m) \neq 0.$ (5.7.6)

Thus $y = mx - \dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$

is the asymptote corresponding to the slope m, provided $\phi'_n(m) \neq 0.$

Note 5.7.1 If $\phi'_n(m) = 0$ but $\phi_{n-1}(m) \neq 0,$ then there is no asymptote corresponding to the slope m.

Note 5.7.2 If $\phi'_n(m) = 0 = \phi_{n-1}(m),$ then (5.7.5) becomes an identity and then dividing (5.7.4) by x^{n-2} and making $|x| \to \infty$, we have

$$\frac{c^2}{2}\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2} = 0, \text{ [since} \to c \text{ as} |x| \to \infty] \qquad (5.7.7)$$

which gives two values of c, in general, provided $\phi''_n(m) \neq 0.$ Thus, we have two parallel asymptotes.

Note 5.7.3 The cases explained in Note 5.7.1 and Note 5.7.2. be treated similar manner for the next terms in the equation (5.7.4).

Remark 5.7.1 The polynomial $\phi_n(m)$ can be obtained by putting $x = 1$ and $y = m$ in the n^{th} degree homogeneous polymonial of $f(x, y)$.

Example 5.7.1 Find the asymptotes of the curve $xy^2 - y^2 - x^3 = 0$.

Solution : The coefficient of highest power of y is $(x - 1)$. Hence $x = 1$ is a vertical asymptote of the curve. The cofficient of highest power of x is constant. Hence there are no horizontal asymptotes.

Putting $x = 1$ and $y = m$ in highest degree (third degree) terms $(xy^2 - x^3)$ to get

$$\phi_3(m) = m^2 - 1; \ \ \phi'_3(m) = 2m$$

$$\phi_3(m) = 0 \ \text{ gives } \ m = 1, -1.$$

Putting $x = 1$ and $y = m$ in the terms of 2nd degree $(-y^2)$, we have

$$\phi_2(m) = -m^2.$$

Now for $m = 1, c = -\frac{\phi_2(1)}{\phi_3'(1)} = \frac{1}{2}.$ Hence $y = x + \frac{1}{2}$ is an asymptote of the curve.

For $m = -1, c = -\frac{\phi_2(-1)}{\phi_3'(-1)} = -\frac{1}{2}.$ Hence $y = -x - \frac{1}{2}$ is another asymptote of the curve.

Example 5.7.2 Find the asymptotes of the curve $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 = 2.$

Solution : The coefficients of highest power of x and y are constants. Hence there are no horizontal and vertical asymptotes.

Putting $x = 1$ and $y = m$ in the highest degree (third degree) terms $\left(x^3 + x^2y - xy^2 - y^3\right)$ to get

$$\phi_3(m) = 1 + m - m^2 - m^3; \quad \phi_3'(m) = 1 - 2m - 3m^2$$

$$\phi_3(m) = 0 \text{ gives } m = 1, -1, -1.$$

Putting $x = 1$ and $y = m$ in the terms of 2nd degree $(x^2 - y^2)$, we have

$$\phi_2(m) = 1 - m^2.$$

Now for $m = 1, c = -\frac{\phi_2(1)}{\phi_3'(1)} = 0.$ Hence $y = x$ is an asymptote of the curve.

For $m = -1$, since $\phi_3'(-1) = 0 = \phi_2(-1),$ the value of c can be obtained from

$$\frac{c^2}{2}\phi_3''(-1) + c\phi_2'(-1) + \phi_1(-1) = 0$$

$$\text{or, } \frac{c^2}{2}.4 + c.2 + 0 = 0.$$

Thus, $c = 0, -1.$

Hence $y = -x$ and $y = -x - 1$ are two parallel asymptotes of the curve.

5.8 An Alternative Method of Finding Asymptotes of Algebraic Curves

Let the equation of an rational algebraic curve of n^{th} degree be represented by

$$P_n + Q_{n-1} = 0, \tag{5.8.1}$$

where P_n is homogeneous polynomials in x and y of degree n and Q_{n-1} contain the terms of degree not higher thaan $n - 1$.

I. Let $y - m_1x$ be a non-repeated factor of P_n. Then the equation (5.8.1) can be written as

$$(y - m_1x)F_{n-1} + Q_{n-1} = 0, \tag{5.8.2}$$

where F_{n-1} is homogeneous polynomials in x and y of degree $n - 1$.

Clearly m_1 is a root of $\phi_n(m) = 0$. Hence there exist an asymptote $y = m_1x + c_1$ provided we can determine the value of c_1. Using art. 5.6 and the equation (5.8.2) we obtain

$$c_1 = \lim_{|x|\to\infty}(y - m_1x) = -\lim_{|x|\to\infty}\frac{Q_{n-1}}{F_{n-1}},$$

where to determine the limiting value, we use $\lim_{|x|\to\infty}\frac{y}{x} = m_1$. Thus, the asymptote under this discussion is

$$y = m_1x - \lim_{|x|\to\infty}\frac{Q_{n-1}}{F_{n-1}}.$$

For each non-repeated linear factor of the n^{th} degree homogeneous terms we may proceed in a similar manner.

II. If the n^{th} degree homogeneous terms in the equation of the curve contain $(y - m_1x)^2$ as a factor and $(n - 1)^{th}$ degree homogeneous terms do not contain the factor $y - m_1x$, then there is no asymptote corresponding to the slope m_1.

III. On the other hand, we could write the equation of the curve in the form

$$(y - m_1x)^2F_{n-2} + (y - m_1x)P_{n-2} + Q_{n-2} = 0, \tag{5.8.3}$$

where F_{n-2} contain the terms of degree not higher than $n - 3$. Then on similar arguments as in case I,

$$(y - m_1x)^2 + (y - m_1x).\lim_{|x|\to\infty}\frac{P_{n-2}}{F_{n-2}} + \lim_{|x|\to\infty}\frac{Q_{n-2}}{F_{n-2}} = 0$$

will be the pair of parallel asymptotes.

IV. We can proceed exactly in a similar manner if the n^{th} degree terms contain $(y - m_1x)^3$ or higher power of $(y - m_1x)$ as factor.

V. If in I, we have the factor $(ax + by + c)$ instead of $(y - m_1x)$, then the asymptote will be

$$ax + by + c + \lim_{|x|\to\infty} \frac{Q_{n-1}}{F_{n-1}} = 0,$$

where $\lim\limits_{|x|\to\infty} \frac{y}{x} = -\frac{a}{b}$.

Example 5.8.1 Find all the asymptotes of $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$.

Solution : The coefficient of highest power of y is x. Hence the asymptote parallel to y-axis is $x = 0$.

The coefficient of highest power of x is constant. Hence there is no asymptote parallel to x-axis.

Factorizing the terms of third degree, the given equation becomes

$$x(y - x)^2 - x(y - x) + 2 = 0.$$

Hence the parallel asymptotes will be given by

$$(y - x)^2 - (y - x)\lim_{|x|\to\infty} \frac{x}{x} + \lim_{|x|\to\infty} \frac{2}{x} = 0,$$

i.e., $(y - x)^2 - (y - x) = 0,$

or, $y - x = 0;\ y - x - 1 = 0.$

Thus, the three asymptotes are $x = 0;\ y - x = 0;\ y - x - 1 = 0.$

5.9 Asymptotes by Inspection

If the equation of a curve be of the form

$$F_n + F_{n-2} = 0,$$

where F_n is a polynomial of degree n and F_{n-2} is a polynomial of degree $(n - 2)$ at the most and if F_n can be broken up into n distinct linear factors so that when equated to zero they represent n straight lines, no two of which are parallel, then all the asymptotes of the curve are given by $F_n = 0$,

e.g., the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ have asymptotes $\frac{x}{a} \pm \frac{y}{b} = 0$.

5.10 Summary

Throughout this unit we have learnt the meaning of asymptotes of a curve. We also studied several types of asymptotes and discussed the method to find out the equation of asymptote of a curve.

5.11 Exercises

Find the asymptotes of the following curves :

(i) $y^3 - x^2y - 2xy^2 - 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

(ii) $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y = 3$

(iii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

(iv) $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + 5x - 6y + 7 = 0$

(v) $y^3 = x(a^2 - x^2)$

(vi) $4x^3 - 3x^2y - y^3 + 2x^2 - xy - y^2 - 1 = 0$

(vii) $y(x - y)^3 - y(x - y) - 2 = 0$

(viii) $(x^2 - y^2)(x^2 - 9y^2) + 3xy - 6x - 5y + 2 = 0$

(ix) $(y + x + 1)(y + 2x + 2)(y + 3x + 3)(y - x) + x^2 + y^2 - 8 = 0$

(x) $(x + y)^2(x + 2y + 2) = x + 9y + 2.$

5.12 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.

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Unit-6 ❑ Envelope

Structure

6.1 Objectives

After going through this unit, the learners will be able to :

- understand the definition of envelopes.
- determine the envelopes of family of curves.

6.2 Introduction

A curve which touches each member of a given family of curves is called envelope of that family. In this chapter we shall study the idea of envelope and its determination.

6.3 Family of Curves

Let $(x-\alpha)^2+y^2=a^2$, where a and α are fixed in a certain moment, but if we allow α to take a series of values keeping a fixed, then we have a series of cricles of equal radii a. A system of curves fromed in this way is called family of curves and the quantity which takes a series of values is called a parameter. We write the equation of a one parameter family of curves by a symbol

$$f(x,y,\alpha)=0.$$

We may think a two or three or more parameter family of curves. The equation of a two parameter family of curves is of the form

$$f(x, y, \alpha, \beta) = 0,$$

where α and β are arbitrary parameters : e.g., $(x-\alpha)^2+(y-\beta)^2=1$ is a two parameter family of circles of radii 1. Also $(x-\alpha)^2+(y-\beta)^2=c^2$ gives the three parameter family of circles with center at any point of the plane and with any radius, i.e., the family of all circles on the plane.

6.4 Definitions

Definition 6.4.1 A point $P(a,\ b)$ is a singular point of a curve

$$f(x, y, \alpha) = 0 \ \ (\alpha \text{ is fixed}),$$

if it satisfies the curve as well as the two equations

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0.$$

The point P is said to be ordinary point if at least one of the two partial derivatives f_x, f_y is not zero at $(a,\ b)$.

Definition 6.4.2 The characteristic points of a family of curves

$$f(x,\ y,\ \alpha) = 0 \ (\alpha \text{ is arbitrary})$$

are those points of the family where the two equations

$$f(x, y, \alpha) = 0 \ \ \textit{and} \ \ \frac{\partial f}{\partial \alpha}(x,\ y,\ \alpha) = 0$$

simultaneously hold.

Note 6.4.1 If $f(x,\ y,\ \alpha) = 0$ and $\frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$ both hold for a point where $f_x = 0$ and $f_y = 0$ then the point is a singular point and therefore not a characteristic point.

Example 6.4.1 Find the characteristic points of the family of circles

$$(x-\alpha)^2 + y^2 = a^2 \ \ (\alpha \text{ is arbitrary}).$$

Solution : Solving the equations

$$f(x,\ y,\ \alpha) \equiv (x-\alpha)^2 + y^2 - a^2 = 0$$

$$\text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) \equiv -2(x-\alpha) = 0,$$

we get the points $(\alpha, \pm a)$. It can be easily shown that these points do not satisfy $f_x = f_y = 0$. Hence $(\alpha, \pm a)$ are the characteristic point of the family.

Definition 6.4.3 The envelope of a family of curves $f(x, y, \alpha) = 0$ (α is arbitrary) is the locus of their isolated characteristic points.

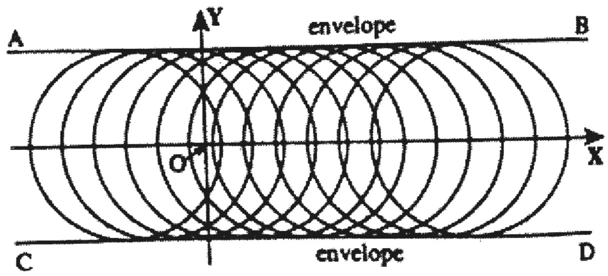

Fig. 6.1

Thus, envelope of a family of curves $f(x, y, \alpha) = 0$, (α being parameter) is a curve which touches every member of the family i.e., each point on the curve is touched by some member of the family.

6.5 Envelopes of One Parameter Family of Curves

If there exists an envelope of a family of curves, its equation may be obtained in either of the following ways :

I. Eliminate α between

$$f(x,y,\alpha) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad (6.5.1)$$

The elimination (an expression in x and y) is the envelope.

II. Solve for x and y in terms of α from the equation (6.5.1). It will give the parametric representation of the envelope.

III. For an algebraic curve, the equation of envelope obtained by eliminating α between

$$f(x,y,\alpha) = 0 \quad \text{and} \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0. \qquad (6.5.2)$$

is exactly the condition that the relation $f(x, y, \alpha) = 0$, considered as an equation in α, has a repeated root. Thus if

$$f(x,y,\alpha) = A(x,y)\alpha^2 + B(x,y)\alpha + C(x,y) = 0,$$

then the envelope is given by $B^2 - 4AC = 0$.

Example 6.5.1 Obtain the envelope of the family of ellipses $\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1$, α being the parameter.

Solution : We have

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1. \qquad (6.5.3)$$

Differentiating w.r.t. α, we obtain

$$\frac{-2x^2}{\alpha^3} + \frac{2y^2}{(a-\alpha)^3} = 0.$$

Thus,

$$\frac{x^2}{\alpha^3} = \frac{y^2}{(a-\alpha)^3}.$$

Therefore

$$\frac{\frac{x^2}{\alpha^2}}{\alpha} = \frac{\frac{y^2}{(a-\alpha)^2}}{(a-\alpha)} = \frac{\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2}}{a} = \frac{1}{a} \text{ [by (6.5.3)]}$$

Hence

$$\alpha = a^{\frac{1}{3}}x^{\frac{2}{3}};\ (a-\alpha) = a^{\frac{1}{3}}y^{\frac{2}{3}}.$$

Putting this values in (6.5.3), we obtain

$$\frac{x^2}{a^{2/3}x^{4/3}} + \frac{y^2}{a^{2/3}y^{4/3}} = 1$$

i.e., $$x^{2/3} + y^{2/3} = a^{2/3}.$$

This is the required envelope.

Example 6.5.2 Find the envelope of the system of parabolas $\lambda x^2 + \lambda^2 y = 1$, being parameter.

Solution : Since the equation of family is $\lambda^2 y + \lambda x^2 - 1 = 0,$ the quadratic form of parameter, the equation of envelope of the family is $x^4 + 4y = 0.$

6.6 Envelopes of Two Parameter Family of Curves

Let

$$f(x, y, \alpha, \beta) = 0 \tag{6.6.1}$$

be the family of curves involving two parameters α and β connected by the relation

$$\phi(\alpha, \beta) = 0. \tag{6.6.2}$$

We can find out the envelope by two methods.

I. First we express β in terms of α from the equation (6.6.2) and then substituting it in the equation (6.6.1) to obtain the equation (6.6.1) in one parameter family of curves. Hence as before the envelope of the family of curves will be determined.

II. Differentiating both of (6.6.1) and (6.6.2) partially with respect to α (α being regarded as independent variable whereas β is dependent variable) we obtain

$$\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta}\frac{d\beta}{d\alpha} = 0 \ \ and \ \ \frac{\partial \phi}{\partial \alpha} + \frac{\partial \phi}{\partial \beta}\frac{d\beta}{d\alpha} = 0.$$

Eliminating $\frac{d\beta}{d\alpha}$ from the above equations we get

$$\frac{\partial f}{\partial \alpha}\bigg/\frac{\partial \phi}{\partial \alpha} = \frac{\partial f}{\partial \beta}\bigg/\frac{\partial \phi}{\partial \beta}. \tag{6.6.3}$$

Now eliminating α and β from (6.6.1), (6.6.2) and (6.6.3) we get the required envelope of the family of curves.

Example 6.6.1 Find the envelope of the family of ellipes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ where the parameters are connected by $a + b = c$, c being constant.

Solution : Let the family of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6.6.4}$$

where parameters are connected by the relation

$$a + b = c, \ \ c \text{ being constant.} \tag{6.6.5}$$

Differentiating (6.6.4) and (6.6.5) w.r.t. a, we get

$$x^2(-2/a^3)+y^2(-2/b^3)\frac{db}{da}=0$$

and $$1+\frac{db}{da}=0.$$

Eliminating $\frac{db}{da}$, we obtain

$$\frac{x^2}{a^3}=\frac{y^2}{b^3}$$

i.e., $$\frac{x^2/a^2}{a}=\frac{y^2/b^2}{b}=\frac{x^2/a^2+y^2/b^2}{a+b}=\frac{1}{c}.$$ [from (6.6.4) and (6.6.5)]

Therefore $a=(cx^2)^{1/3}$ and $b=(cy^2)^{1/3}$. Hence from (6.6.5), we get

$$(cx^2)^{1/3}+(cy^2)^{1/3}=c$$

i.e., $$x^{2/3}+y^{2/3}=c^{2/3},$$

which is the required envelope.

Example 6.6.2 Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters are connected by $a + b = c$, c being constant.

Solution : Let the family of lines be

$$\frac{x}{a}+\frac{y}{b}=1, \qquad (6.6.6)$$

where parameters are connected by the relation

$$a + b = c,\ c \text{ being constant.} \qquad (6.6.7)$$

From (6.6.7), we get $b = c - a$. Then the family of curves becomes

$$\frac{x}{a}+\frac{y}{c-a}=1, \qquad (6.6.8)$$

where a is only one parameter.

From (6.6.8), we get

$$x(c-a)+ya-a(c-a)=0$$

or, $$a^2+a(y-x-c)+cx=0,$$

which is a quadratic equation of the parameter a, Hence the required envelope is

$$(y-x-c)^2=4cx.$$

6.7 Summary

In the unit we discuss the definition of envelope of a family of curves and the method to work out the equation of envelope of the family of curves.

6.8 Exercises

Find the envelopes of the following families of curves :

(i) $y = mx + am^3$, m being parameter

(ii) $(x-a)^2 + (y-a)^2 = 2a$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^n + b^n = c^n$, c is constant

(iv) $\frac{x}{a} + \frac{y}{b} = 1$, where $\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1$, l, m are constants

(v) $y = mx + \sqrt{a^2m^2 + b^2}$

(vi) $x \cos \alpha + y \sin \alpha = 4$.

6.9 References

1. B. C. Das, B. N. Mukherjee, Differential Calculus, U. N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
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Unit-7 ❑ Curve Tracing

Structure

7.2 Objectives

After reading this chapter, the learners will be able to :

- draw graph of a curve.
- know various properties of a curve.

7.1 Introduction

In this unit we study the systematic procedure to draw or sketch the graphs of curves given by the equations which are either in Cartesian Co-ordinates system or in Polar Co-ordinates system. With the knowledge of tangents, normals, curvature, asymptotes, singular points, extreme points, symmetry of curves etc., we may obtain a good idea to trace the shape of a curve.

7.3 Procedure of Curve Tracing in Cartesian Co-ordinate System

To draw a curve we need the following observations :

I. Symmetry : A curve is symmetric w.r.t.

(i) the x-axis, if its equation contains only even power of y and hence remains unchanged if y is replaced by $-y$.

(ii) the y-axis, if its equation contains only even power of x and hence remains unchanged if x is replaced by $-x$.

(iii) the line $y = x$, if its equation remains unchanged when x and y are interchanged.

(iv) the line $y = -x$, if its equation remains same when (x, y) is replaced by $(-y, -x)$.

(v) the origin, if its equation remains unchanged when (x, y) is replaced by $(-x, -y)$.

II. Intercepts : To obtain the points where the curve intersects the co-ordinate axes

(i) put $y = 0$ in the equation to get x intercepts.

(ii) put $x = 0$ in the equation to get y intercepts.

III. Passes through origin : The curve passes through the origin if its equation satisfy $x = 0$ and $y = 0$ simultaneously. If the curve passes through the origin, write down the equation of tangents at origin. If the origin is singular point, find the nature of singularity, cusp of various species, node, or isolated. Also check the origin is whether multiple point of higher order than two or not.

IV. Concavity, convexity or point of inflection : We determine the points where the curve has concavity, convexity or point of inflection.

V. Extreme points : We determine the points where the curve has extermum. We also determine the intervals where the curve increases or decreases. In fact, y increases or decreases for those values of x where $\frac{dy}{dx}$ is positive or negative respectively.

VI. Region of existence : By solving the equation of curve for one variable in terms of other and thereby we can find out the set of values of one variable which make the imaginary. In this way we can find the region of existence of the concerned curve to be traced.

VII. Asymptotes : We determine the vertical or horizontal or oblique asymptotes if, any, to the curves having infinite branches. Also we determine the points where these meet the curve and the sides of the curves towards which this lie.

VIII. Periodicity : For trigonometric functions we check it is whether periodic or not. It will enable us greatly to sketch the graph of the curve.

Example 7.3.1. Trace the curve $y^2(x-1) = x^3$.

Solution : The curve is symmetric with respect to x-axis. The intercepts are $x = 0$ and $y = 0$. The curve exists in the range $-\infty < x \leq 0$ and $x > 1$ and for all values of y.

For the branch $y = x\sqrt{\frac{x}{x-1}}$, the point $x = 3/2$ gives the minimum point.

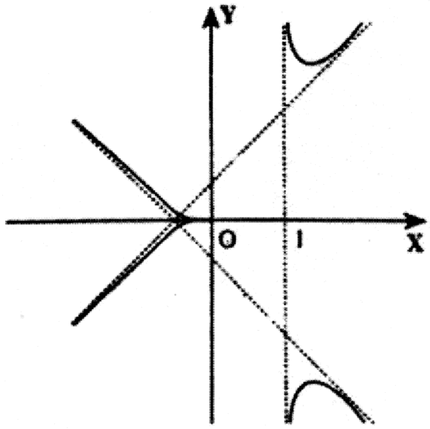

Fig. 7.1

There is no point fo inflection. This branch is convex to the x-axis. The lines $x = 1$, $y = x + \frac{1}{2}$, $y = -x - \frac{1}{2}$ are asymptotes. The origin is a cusp of first species. $y = 0$ is the cuspidal tangent. Thus, the graph of the given curve is shown in Fig. 7.1.

7.4 Procedure of Curve Tracing in Polar Coordinate System

We also observe the following characteristics in tracing a curve $r = f(\theta)$ or $f(r, \theta) = 0$.

I. The curve passes through the pole if $r = 0$ for some values of θ.

II. If the values of r does not exist or becomes imaginary for some values of θ, say θ_1, θ_2 where $\theta_1 < \theta < \theta_2$, then the curve has no portion between the lines $\theta = \theta_1$ and $\theta = \theta_2$.

III. If a and b are respectively the minimum and maximum values of r, then the curve lies wholly within the circles $r = a$ and $r = b$.

IV. Observe the variation of the values of r for increasing and decreasing values of

θ from 0° in the anticlockwise and clockwise senses. In fact, if $\frac{dr}{d\theta} > 0$, r increases as θ increases and if $\frac{dr}{d\theta} > 0$, r decreases as θ increases.

V. When θ is replaced by $-\theta$, if it is observed that the equation remains unchanged, we say that the curve is symmetric about the line $\theta = 0$. If θ is replaced by $\pi - \theta$, and the equation remains unchanged, the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

Also the curve is symmetric about the pole if the equation of the curve does not change when θ is replaced by $\pi + \theta$.

VI. Let ϕ be the angle between the radius vector and the tangent to the curve at a point (r, θ). Then we know that

$$\tan\phi = r\frac{d\theta}{dr}.$$

If $\phi = 0$ for some values of θ, say θ_1, then the line $\theta = \theta_1$ is a tangent to curve at $\theta = \theta_1$ and if $\phi = \pi/2$ for some values of θ, say θ_2, then at the point $\theta = \theta_2$, the tangent is perpendicular to the line $\theta = \theta_2$.

Example 7.4.1 Trace the curve $r = a \sin 3\theta$, $a > 0$ (Rose-petal).

Solution : We observed the followings :

(i) $r = 0$ for $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

Here sin 3θ is periodic function of period 2π, hence we consider only those values of θ ranges from 0 to 2π, whreas the remaining values of θ yield no new branches of the curve.

(ii) We also observe that the curve passes through the pole.

(iii) Here $\tan \phi = r\frac{d\theta}{dr} = \frac{r}{3a\cos 3\theta}$.

Hence $\phi = 0$ for $r = 0$ and the corresponding values of θ are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Thus $\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \pi, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}$ are the tangents to the curve at pole.

(iv) Replacing θ by π – θ, it follows that the equation of the curve remains unchanged and hence the curve is symmetric about the line $\theta = \frac{\pi}{2}$.

(v) As $-1 \le \sin 3\theta \le 1$, the maximum value fo r is a. Consequently, the curve lies wholly within a circle of radius a.

(vi) Table of variation of the values of r and θ :

θ :	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r :	a	0	$-a$	0	a	0	$-a$

Thus r increases from 0 to a when θ increases from 0 to π/6; r then decreases from a to 0 when θ increases from π/6 to π/3 and r increases numerically from 0 to a, when θ increases from π/3 to π/2, the portion of which lies in the third quadrant.

The curve being symmetric about the line θ = π/2, we consider the table of variations for r as θ increases from – π/2 to π/2.

With all these facts which we trace the curve as given in Fig. 7.2.

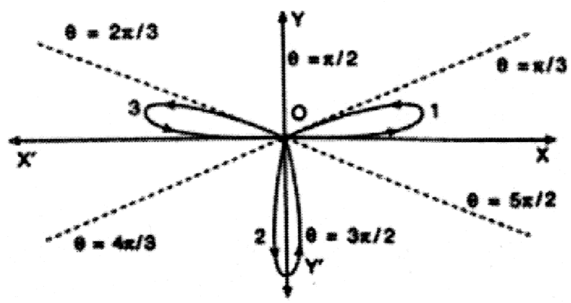

Fig. 7.2

Note 7.4.1 The curve $r = a \sin n\theta$, $a > 0$, n = an integer (rose-petals), traces is similar loops as given in Fig. 7.2 lying wholly within a circle of radius a with center at the origin and are symmetric about the pole. In case n be odd, there are n-loops and if n be even, the number of loops are $2n$.

The order in which loops occurs as θ increases from 0 to 2π are mentioned in the figure by numbers.

In the following Fig. 7.3 we trace the curve $r = a$ ain 2θ, $a > 0$.

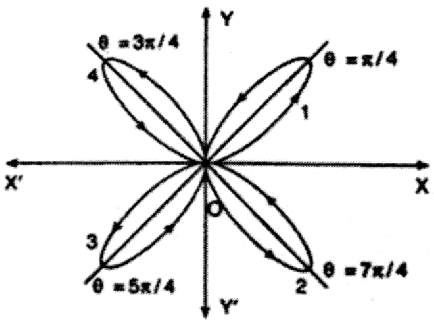

Fig. 7.3

Example 7.4.2 Trace the curve $r^2 = a^2 \cos 2\theta$ (Lemniscate of Bernouli).

Solution : Here the curve satisfies the following conditions :

(i) Replacing θ by $-\theta$ and $\pi - \theta$, it is observed that the equation remains unchanged. Hence the curve is symmetric about the intial line and the line $\theta = \pi/2$.

(ii) When $r = 0$, $\theta = \pm\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$.

(iii) Table of variations fo r and θ :

θ :	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	π
r :	0	$\pm\frac{a}{\sqrt{2}}$	$\pm a$	$\pm\frac{a}{\sqrt{2}}$	0	$\pm a$

(iv) As maximum of $\cos 2\theta$ is 1, maximum value of r is a and the curve lies wholly within the circle fo radius a with center at the pole.

(v) Here $\cos 2\theta$ is positive for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$ and we get real values of r there at. But for θ satisfying $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $\cos 2\theta$ becomes negative and as such r becomes imaginary.

Also r increases for $-\frac{\pi}{4} < \theta < 0$ and r decreases for $0 < \theta < \frac{\pi}{4}$.

Thus the curve has two similar loops and we trace the curve as given in the Fig. 7.4.

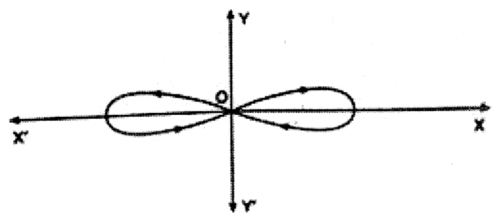

Fig. 7.4

7.5 Some Well Known Curves

1. Cycloid : $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

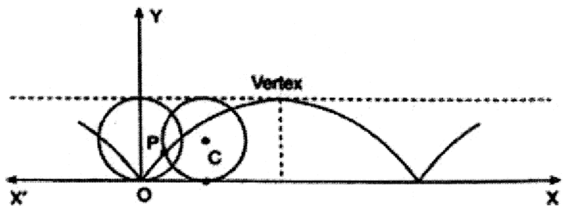

Fig. 7.5

2. Astroid : $x = a\cos^3\theta$, $y = a\sin^3\theta$ $\left(x^{2/3} + y^{2/3} = a^{2/3}\right)$.

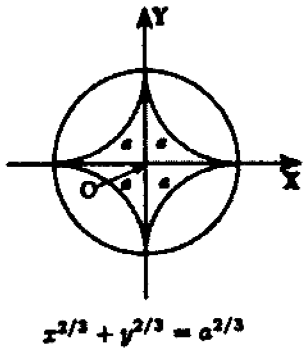

Fig. 7.6

3. Cardioid : $r = a(1 - \cos\theta)$.

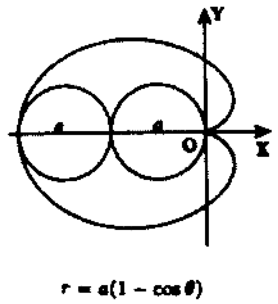

Fig. 7.7

4. Follium of Descartes : $x^3 + y^3 = 3axy$.

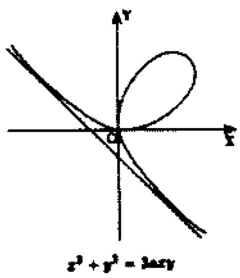

Fig. 7.8

5. Equiangular Spiral : $r = ae^{\theta \cot \alpha}$.

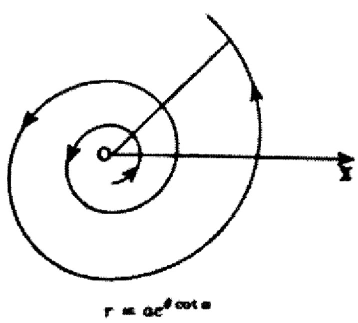

Fig. 7.9

7.6 Summary

In this unit we discuss the procedure to draw a graph of a given curve in Cartesian or Polar co-ordinate system. We draw the graph of curves using their properties. We sketched also some well known curves.

7.7 Exercises

Trace the following curves :

(i) $x^3 + y^3 = 3axy$ (Folium of Descartes)

(ii) $\left(a^2 + x^2\right)y = a^2x$

(iii) $x^5 + y^5 = 5a^2x^2y$

(iv) $x = a(t + \sin t),\ y = a(1 - \cos t)$ (Cycloid)

(v) $r = a + b \cos \theta$, $(a < b)$

(vi) $y = \cosh x/c$

(vii) $r = a\sin \theta \tan \theta$.

7.8 References

1. B.C. Das, B.N. Mukherjee, Differential Calculus, U.N. Dhur & Sons Private Ltd., Kolkata, India, 1949.
2. R.K. Ghosh, K.C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.
3. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.
4. D. Sengupta, Application of Calculus, Books and Allied (P) Ltd. Kolkata, India, 2012.

Unit-8 ❑ L'Hospital's Rule

Structure

8.1 Objectives

After going through this unit, the learners will be able to :

- understand the L'Hospital's Rule.
- determine the limits of indeterminate forms.

8.1 Introduction

In this unit we investigated a very important application of mean value theorem.

In the case $\lim\limits_{x\to a}\dfrac{\phi(x)}{\psi(x)}=\dfrac{\lim\limits_{x\to a}\phi(x)}{\lim\limits_{x\to a}\psi(x)}$, if both the limits $\lim\limits_{x\to a}\phi(x)$ and $\lim\limits_{x\to a}\psi(x)$ are zero then we face with the problem like 0/0 which is meaningless. Such a case is known as Indeterminate form.

Other indeterminate forms are ∞/∞, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. For evaluation of indeterminate forms ∞/∞ or 0/0 we shall use a particular device known as L'Hospital's Rule.

8.3 L'Hospital's Rule

8.3.1. 0/0 form :

The quotient $\dfrac{f(x)}{g(x)}$ of two functions $f(x)$ and $g(x)$ is undefined at $x = a$ if $g(a) = 0$. But if $f(a) = 0 = g(a)$, then the ratio is of the indeterminate form 0/0 and we can determine the limit of the ratio $\frac{f(x)}{g(x)}$ at $x = a$ by the conception of derivatives. In this connection we state a basic theorem known as L'Hospital's Rule.

*** L' Hospital's rule :** If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a + h]$,

(ii) derivable in the open interval $(a, a + h)$ and

(iii) $\lim_{x \to a+0} f(x) = 0 = \lim_{x \to a+0} g(x)$, $h > 0$ is a suitably small number, then,

$$\lim_{x \to a+0} \frac{f(x)}{g(x)} = \lim_{x \to a+0} \frac{f'(x)}{g'(x)},$$

provided the limit of the right hand side exists.

Proof : From Cauchy mean value theorem we obtain

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}, \; a < c < x < a+h.$$

$$\text{i.e.,} \quad \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \text{ as } f(a) = 0 = g(a).$$

Therefore $\lim_{x \to a+0} \frac{f(x)}{g(x)} = \lim_{x \to a+0} \frac{f'(c)}{g'(c)}$.

Again since $a < c < x$, $c \to a + 0$ when $x \to a + 0$, we get

$$\lim_{x \to a+0} \frac{f(x)}{g(x)} = \lim_{x \to a+0} \frac{f'(c)}{g'(c)} = \lim_{x \to a+0} \frac{f'(x)}{g'(x)}.$$

Note 8.3.1 It can be similarly shown that

$$\lim_{x \to a-0} \frac{f(x)}{g(x)} = \lim_{x \to a-0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note 8.3.2 If $\lim_{x \to a+0} \frac{f'(x)}{g'(x)}$ does not exist, then we should not conclude that $\lim_{x \to a+0} \frac{f(x)}{g(x)}$ does not exist.

Note 8.3.3 L'Hospital's Rule also holds when $a = \infty$.

***Generalization of L'Hospital's Rule :** If also $\lim_{x\to a} f'(x) = 0 = \lim_{x\to a} g'(x)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)},$$

provided the last limit exists.

We continue in this manner until one of the derivative $g^{(r)}(a) \neq 0$.

Example 8.3.1 Evaluate the limit $\lim_{x\to 0}\frac{\log(1+x)}{x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\log(1+x)}{x}\left[\frac{0}{0}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{1}{1+x}$$

$$= 1.$$

Example 8.3.2 Evaluate the limit $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\tan x - x}{x - \sin x}\left[\frac{0}{0}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x}\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{\sin x}$$

$$= \lim_{x\to 0} 2\sec^3 x$$

$$= 2.$$

8.3.2. ∞/∞ form :

If two functions $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a+h]$,

(ii) derivable in the open interval $(a, a+h)$ and

(iii) $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$, where $h > 0$ is a suitably small number,

then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit of the right hand side exists.

Example 8.3.3 Evaluate the limit $\lim_{x\to 0}\frac{\log x^2}{\log \cot^2 x}$.

Solution : Here

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\log x^2}{\log\cot^2 x}\left[\frac{\infty}{\infty}\text{ form}\right].$$

Thus, by L'Hospital's Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{x^2}.2x}{\frac{1}{\cot^2 x}.2\cot x(-\operatorname{cosec}^2 x)}$$

$$= -\lim_{x\to 0}\frac{\sin x\cos x}{x}\left[\frac{0}{0}\text{ form}\right]$$

$$= -\lim_{x\to 0}\frac{\cos^2 x - \sin^2 x}{1}$$

$$= -1.$$

8.3.3. ∞ – ∞ from :

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$. To find $\lim_{x\to a}\{f(x)-g(x)\}$, we take

$$f(x)-g(x)=\frac{1/g(x)-1/f(x)}{1/\{f(x)g(x)\}},$$

which is of the form 0/0 and can be evaluated by the method discussed in 8.3.1.

Example 8.3.4 Evaluate the limit $\lim_{x\to 0}\left\{\frac{1}{x}-\frac{2}{x(e^x+1)}\right\}$.

Solution : Here

$$\lim_{x\to 0}\left\{\frac{1}{x}-\frac{2}{x(e^x+1)}\right\} \text{ [}\infty-\infty\text{ form]}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x(e^x+1)} \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0}\frac{e^x}{e^x+1+xe^x} \text{ [using L' Hospital's Rule]}$$

$$= \frac{1}{2}.$$

8.3.4. 0 × ∞ form :

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to\alpha} f(x)=0$ and $\lim_{x\to\alpha} g(x)=\infty$.

To find $\lim_{x\to\alpha}\{f(x)g(x)\}$, we take

$$f(x)g(x)=\frac{f(x)}{1/\{g(x)\}},$$

which is of the form 0/0 and can be evaluated by the method discussed in 8.3.1.

Example 8.3.5 Evaluate $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$.

Solution : Here

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} \quad [0\times\infty \text{ form}]$$

$$= \lim_{x\to 1}\frac{1-x}{\cot\frac{\pi x}{2}} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}} \qquad \left[\text{by L' Hospital's Rule}\right]$$

$$= \frac{2}{\pi}.$$

8.3.5. 0^0, ∞^0, $1^{\pm\infty}$ forms :

The three exponential forms 0^0, ∞^0, $1^{\pm\infty}$ are dealt with by taking their logarithms and each of the forms is reduced to the form $0 \times \infty$ already discussed in 8.3.4.

Example 8.3.6 Find the limit $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$.

Solution : Here the limit is of the form 1^∞.

Let $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. Then $\log\, y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$. Thus

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\text{ form}\right].$$

Using L'Hospital's Rule we get

$\lim_{x\to 0}\log y$

$$= \lim_{x\to 0}\frac{\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{x^2}}{2x} = \lim_{x\to 0}\frac{x - \frac{1}{2}\sin 2x}{x^2\sin 2x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{2x\sin 2x + 2x^2\cos 2x} \quad \left[\frac{0}{0}\text{ form}\right] \text{ [using L'Hospital's Rule]}$$

$$= \lim_{x\to 0}\frac{2\sin 2x}{2\sin 2x + 4x\cos 2x + 4x\cos 2x - 4x^2\sin 2x} \quad \text{[using L'Hospital's Rule]}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{\sin 2x + 4x\cos 2x - 2x^2\sin 2x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2x}{2\cos 2x + 4\cos 2x - 8x\sin 2x - 4x\sin 2x - 4x^2\cos 2x}$$

[using L'Hospital's Rule]

$$= \frac{2}{2+4} = \frac{1}{3}.$$

Thus, $\lim_{x\to 0}\log y = \frac{1}{3}$. We know that $\lim_{x\to 0}\log y = \log\left(\lim_{x\to 0} y\right)$. Therefore, $\log\left(\lim_{x\to 0} y\right) = \frac{1}{3}$.

Hence $\lim_{x\to 0} y = e^{1/3}$ *or*, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{1/3}$.

8.4 Summary

In this unit we have learnt a very important technique to evaluate the limits in the indeterminate forms. We gave some examples to understand the technique.

8.5 Exercises

Evaluate the following limits :

(i) $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x}$ (ii) $\lim_{x\to 0}\frac{e^{2x}-1}{\log(1+x)}$

(iii) $\lim_{x\to\infty}\frac{x+\log x}{x\log x}$ (iv) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$

(v) $\lim_{x\to 0}\left(\frac{1}{x^2}\right)^{\tan x}$ (vi) $\lim_{x\to 0}\left(\frac{x-\sin x}{\tan^3 x}\right)$

(vii) $\lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2\log(1+x)}{x\sin x}\right)$ (viii) $\lim_{x\to 1}\left(\frac{\log(1-x)}{\cos(\pi x)}\right)$.

8.6 References

1. R. K. Ghosh, K. C. Maity, An introduction to analysis : Differential Calculus [Part I], NCBA, India, 1960.

2. B. Pal, S. Raychaudhuri, S. Jana, Fundamental Differential Calculus, Santra Publication Pvt. Ltd., India, 2018.

Unit-9 ❑ Applications in Business Economics and Life Sciences

Structure

9.1 Objectives

After studying this lesson the readers should be able to :

- understand the concepts related to business economics and life sciences with the applications of calculus.
- apply the techniques of differentiation to solve business economics and life science problems.

9.2 Introduction

Calculus is a very important part of our real life. But many of us ask how calculus help us in real life. In this unit we are going to discuss how it is useful in our real life. Calculus is used to determine the right time for buying and selling of products. It helps economists to grow up their business economics.

Biologists also make use of calculus to determine the growth rate of bacteria, modeling population growth and so on. In medical field calculus is also useful.

Calculus is required by architects, engineers to build roads, bridges, tunnels, building etc. and without the use of calculus our real life is unsafe.

9.3 Definitions Related to Business Economics

Total Cost (TC) : Total cost is the combinations of fixed cost and variable cost of output. If the production increases, only variable cost will increase in direct proportion but the fixed cost will remain unchanged within a relevant range.

Total Revenue (TR) : Total revenue is the product of price/ demand functions and output.

Profit (P) : Profit are defined as the excess total revenue over total cost. Symbolically it can be expressed as

$$P = TR - TC.$$

The rules for finding a maximum point tell us that P is maximized when the derivative of the profit function is equal to zero and te second derivative is negtive. If we denote the derivatives of the total revenue and total cost functions by dTR and dTC, we have P will be maximum when $dTR - dTC = 0$.

Hence the derivative of the total revenue function must be equal to the derivative of the total cost function for profit to be maximized. Hence

$$\text{Profit maximizing output} = \frac{d(\text{profit function})}{dx}.$$

Therefore in case of maximization, the conditions are $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0$.

Similarly, we have

$$\text{Cost minimizing output} = \frac{d(\text{total cost function})}{dx},$$

and the conditions of minimizing are $\frac{dTC}{dx} = 0$ and $\frac{d^2TC}{dx^2} > 0$.

Merginal Cost (MC) : Marginal cost is the extra cost for producing one additional unit when the total cost at certain level of output is known. Hence it is the rate of change in total cost with respect to the level of output at the point where the total cost is known. Therefore we have $MC = \frac{dTC}{dx}$.

Marginal Production (MP) : Marginial production is the incremental production i.e., the additional production added to the total production (TP), i.e.,

$$MP = \frac{dTP}{dx}.$$

Marginal Revenue (MR) : Marginl revenue is defined as the change in the total revenue for the sale of an extra unit. Hence it is the rate of change in total revenue with respect to the quantity demanded at the point where total revenue is known. Therefore we have

$$MR = \frac{dTR}{dx},$$

where total revenue is the function of x, the quantity demanded.

Example 9.3.1 Let the profit function of a company is given by $P = f(x) = x - 0.00001x^2$, where x is units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

Solution : Here $P = f(x) = x - 0.00001x^2$. The profit will be maximum if $\frac{dP}{dx} = 0$ and $\frac{d^2P}{dx^2} < 0.$

Now

$$\frac{dP}{dx} = 1 - 0.00002x.$$

Hence

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 1 - 0.00002x = 0$$

$$\Rightarrow \quad x = 50000 \text{ units.}$$

Also $\frac{d^2P}{dx^2} = -0.00002 < 0.$ Hence The profit will be optimum for the sales of 50000 units and the profit will be

$$P = 50000 - 0.00001.(50000)^2$$

$$= 25000 \text{ currency units.}$$

9.4 Applications in Life Sciences

9.4.1. Density Dependent (Logistic) Growth in a Population

Biologists have seen that the growth rate of a population depends not only on the size of the population but also on how crowded it is. Constant growth is not sustainable.

When individuals have to compete for resources, nesting sites, mates, or food, they cannot invest time or energy in reproduction, leading to a decline in the rate of growth of the population. Such population growth is called density depndent growth.

We suppose that the growth rate of the population is G, which depends on the density of the population, N as follows :

$$G(N) = rN\left(\frac{K-N}{K}\right).$$

Here N is the independent variable and $G(N)$ is the function of interest. All other quantities are constant :

- $r > 0$ is a constant, called the intrinsic growth rate.
- $K > 0$ is a constant, called carrying capacity. It represents the population density that a given environment can sustain.

Example 9.4.1 (i) Find the population density N that leads to the maximal growth rate $G(N)$.

(ii) Find the value of the maximum growth in terms of r and K.

(iii) For what population size is the growth rate zero ?

Solution : We can rewrite $G(N)$ as

$$G(N) = rN\left(\frac{K-N}{K}\right) = rN - \frac{r}{K}N^2,$$

from which it is apparent that $G(N)$ is a polynomial in powers of N, with constant coefficients r and r/K.

(i) To find critical points of $G(N)$, we find N such that $G(N) = 0$, and then test for maxima :

$$G'(N) = r - 2\frac{r}{K}N = 0 \quad \Rightarrow N = \frac{K}{2}.$$

Hence $N = \frac{K}{2}$ is a critical point, but is it a maximum ? We check this as follows :

$$G''(N) = -2\frac{r}{K} < 0.$$

Thus $N = \frac{K}{2}$ is the maximum point. Therefore the population density with the greatest growth rate is $K/2$.

(ii) The maximal growth is give by

$$G(K/2) = r.K/2 - r/K.(K/2)^2 = \frac{rK}{2}.$$

(iii) To find out the population size at which the growth rate is zero, we solve the equation

$$G(N) = rN\left(\frac{K-N}{K}\right) = rN - \frac{r}{K}N^2 = 0.$$

There are two solution, one is $N = 0$ and other is $N = K$. The solution $N = 0$ is biologically interesting in the sense that life can arise on its own. So no population arises to logistic growth. The solution $N = K$ means that the population is at its carrying capacity.

9.4.2 Cell Size for Maximum Nutrient Accumulation Rate

The nutrient absorption and consumption rates, $A(r)$ and $C(r)$, of a simple spherical cell of radius r are

$$A(r) = k_1 S = 4k_1\pi r^2,\ C(r) = k_2 V = \frac{4}{3}\pi k_2 r^3,$$

for k_1, $k_2 > 0$ constants,

The net rate of increase of nutrients, which is the difference of the two is

$$N(r) = A(r) - C(r) = 4k_1\pi r^2 - \frac{4}{3}\pi k_2 r^3.$$

This quantity is the function of radius r of the cell.

Example 9.4.2 Determine the radius of the cell for which the net rate of increase of nutrients $N(r)$ is largest.

Solution : We know

$$N(r) = A(r) - C(r) = 4k_1\pi r^2 - \frac{4}{3}\pi k_2 r^3.$$

Differentiating w.r.t. r we get

$$N'(r) = 8k_1\pi r - 4\pi k_2 r^2.$$

To find the larget nutrients rate the condition of critical points is $N'(r) = 0$. Hence $8k_1\pi r - 4\pi k_2 r^2 = 0 \Rightarrow r = 0,\ 2\frac{k_1}{k_2}$. To test the critical points for extereme we differentiate again to have

$$N''(r) = 8k_1\pi - 8\pi k_2 r.$$

Now at $r = 2\frac{k_1}{k_2}$,

$$N''\left(2\frac{k_1}{k_2}\right) = 8k_1\pi - 8\pi k_2.2\frac{k_1}{k_2} = -8\pi k_1 < 0.$$

Hence the net rate of increase of nutrients $N(r)$ is largest for $r = 2\frac{k_1}{k_2}$.

9.5 Summary

Calculus was invented from the visions of master minds. It took little time to break through the bridge of theoretical inquiry to practical skills of human activities. The application of the novel methods of calculus enabled to determine the timing of buying, selling the products and to help us to know how much units should be sold to maximize profit. Calculus also determines the activities in our human body.

9.6 Exercises

1. If the total cost y of manufacturing x units of a production is given by $y = 20x + 5000$, then

(i) What is the variable cost per unit ?

(ii) What is the fixed cost ?

(iii) What is the total cost of manufacturing 4000 units ?

(iv) What is the marginal cost of producing 2000 units ?

2. The total cost of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ and market demand is $P = 2530 - 5x$, where x is the no. of units of production. Find the profit maximizing price.

9.7 References

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Unit-10 ❑ Reduction Formula

Structure

10.2 Objectives

After going through this chapter, the learners will be able to :

- derive the reduction formula of some standard integral problems.
- understand the technique of integration to derive the reduction formula.

10.1 Introduction

A Reduction formula is one that enables us to solve an integral problem by reducing it to an easier integral problem, and then reducing that to the more easier integral problem, and so on. Reduction formulae are mostly obtained by the process of integration by parts.

10.3 Derivation of Reduction Formulae

10.3.1. Reduction Formula for $\int x^n e^{ax} dx$, n being a positive integer :

Let

$$I_n = \int x^n e^{ax} dx$$

$$= x^n . \frac{e^{ax}}{a} - \int n x^{n-1} . \frac{e^{ax}}{a} dx. \quad \text{[Int. by parts]}$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx.$$

Thus,

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}.$$

Example 10.3.1 Find $\int x^4 e^{ax} dx.$

Solution : Applying reduction formula, we have

$$I_4 = \frac{x^4 e^{ax}}{a} - \frac{4}{a} I_3, \quad I_3 = \frac{x^3 e^{ax}}{a} - \frac{3}{a} I_2$$

$$I_2 = \frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1, \quad I_1 = \frac{x e^{ax}}{a} - \frac{1}{a} I_0$$

and $I_0 = \int x^0 e^{ax} dx = \frac{e^{ax}}{a}.$

Hence

$$I_4 = \frac{x^4 e^{ax}}{a} - \frac{4x^3 e^{ax}}{a^2} + \frac{12x^2 e^{ax}}{a^3} - \frac{24x e^{ax}}{a^4} + \frac{24 e^{ax}}{a^5} + c.$$

10.3.2. Reduction Formula for $\int \sin^n x \, dx$ and $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$, n being a positive integer greater than 1

Here

$$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

$$= \sin^{n-1} x(-\cos x) - \int (n-1) \sin^{n-2} x \cos x(-\cos x) dx \quad \text{[Int. by parts]}$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x(1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n.$$

Simplifying the above we get the reduction formula

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}.$$

Furthermore, we take

$$J_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} J_{n-2}.$$

Thus, in this case we have the reduction formula as

$$J_n = \frac{n-1}{n} J_{n-2}.$$

Example 10.3.2 Find $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx.$

Solution : Applying reduction formula, we have

$$J_5 = \frac{4}{5} J_3, \quad J_3 = \frac{2}{3} J_1$$

and $\quad J_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = -[\cos x]_0^{\frac{\pi}{2}} = 1.$

Hence

$$J_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{15}.$$

10.3.3. Reduction Formula for $\int \cos^n x\,dx$ and $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$, n being a positive integer greater than 1

Taking $I_n = \int \cos^n dx = \int \cos^{n-1} x.\cos x\,dx$ and proceeding as in the previous article, we may find

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{and} \quad J_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n} J_{n-2}.$$

10.3.4. Reduction Formula for $\int \sin^m x \cos^n x\,dx$, m, n being positive integers greater than 1

Let

$$I_{m,n} = \int \sin^m x \cos^n x\,dx = \int \cos^{n-1} x (\sin^m x \cos x) dx$$

$$= \cos^{n-1} x. \frac{\sin^{m+1} x}{m+1} - \int (n-1) \cos^{n-2} x(-\sin x). \frac{\sin^{m+1}}{m+1} dx$$

[Int. by parts, taking $u = \cos^{n-1} x$, $dv = \sin^m x \cos x\,dx$]

$$=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\cos^{n-2}x\sin^{m}x\sin^{2}x\,dx$$

$$=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}\int\cos^{n-2}x\sin^{m}x\left(1-\cos^{2}x\right)dx$$

$$=\frac{\sin^{m+1}x\cos^{n-1}x}{m+1}+\frac{n-1}{m+1}I_{m,n-2}-\frac{n-1}{m+1}I_{m,\,n}.$$

Simplifying we obtain

$$I_{m,n}=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}I_{m,n-2}. \qquad (10.3.1)$$

In the similar manner, if we take

$$I_{m,n}=\int\sin^{m}x\cos^{n}x\,dx=\int\sin^{m-1}x(\cos^{n}\sin x)dx$$

integrating by parts, taking $u=\sin^{m-1}x,\ dv=\cos^{n}x\sin x\,dx,$ we obtain

$$I_{m,n}=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}I_{m-2,n}. \qquad (10.3.2)$$

10.3.5. Reduction Formula for $\int_0^{\frac{\pi}{2}}\sin^{m}x\cos^{n}x\,dx,\ m,\ n$ being positive integers greater than 1

Take

$$J_{m,n}=\int_0^{\frac{\pi}{2}}\sin^{m}x\cos^{n}x\,dx$$

$$=\left[\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}\right]_0^{\frac{\pi}{2}}+\frac{n-1}{m+n}J_{m,n-2}=\frac{n-1}{m+n}J_{m,n-2}$$

$$=-\left[\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}\right]_0^{\frac{\pi}{2}}+\frac{m-1}{m+n}J_{m-2,n}=\frac{m-1}{m+n}J_{m-2,n}.$$

Therefore

$$J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} = \frac{m-1}{m+n} J_{m-2,n}. \qquad (10.3.3)$$

Example 10.3.3 Find $\int \sin^4 x \cos^2 x\, dx.$

Solution : Applying reduction formula, we have

$$I_{4,2} = -\frac{\sin^3 x\cos^3 x}{6} + \frac{3}{6} I_{2,2} \quad \text{[from (10.3.2)]}$$

$$= -\frac{\sin^3 x\cos^3 x}{6} + \frac{1}{2}\left(-\frac{\sin x\cos^3 x}{4} + \frac{1}{4} I_{0,2}\right) \quad \text{[from (10.3.2)]}$$

$$= -\frac{\sin^3 x\cos^3 x}{6} - \frac{\sin x\cos^3 x}{8} + \frac{1}{8}\left(\frac{\sin x\cos x}{2} + \frac{1}{2} I_{0,0}\right) \quad \text{[from (10.3.1)]}$$

Also

$$I_{0,0} = \int \sin^0 x\cos^0 x\, dx = x.$$

Therefore

$$I_{4,2} = -\frac{\sin^3 x\cos^3 x}{6} - \frac{\sin x\cos^3 x}{8} + \frac{\sin x\cos x}{16} + \frac{x}{16} + c.$$

Example 10.3.4 Find $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^8 x\, dx.$

solution : By (10.3.3), we get

$$I_{4,8} = \frac{7}{12} I_{4,6} = \frac{7}{12}\cdot\frac{5}{10} I_{4,4} = \frac{7}{12}\cdot\frac{5}{10}\cdot\frac{3}{8} I_{4,2} = \frac{7}{12}\cdot\frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1}{6} I_{4,0}$$

$$= \frac{7}{12}\cdot\frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1}{6}\cdot\frac{3}{4} I_{2,0} = \frac{7}{12}\cdot\frac{5}{10}\cdot\frac{3}{8}\cdot\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2} I_{0,0} = \frac{7}{1024} I_{0,0}.$$

Also

$$I_{0,0} = \int_0^{\frac{\pi}{2}} \sin^0 x\cos^0 x\, dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

Therefore

$$I_{4,8} = \frac{7\pi}{2048}.$$

10.3.6. Reduction Formula for $\int \sin^m x \cos^n x\, dx$, where either *m* or *n* or both are negative integers

Let

$$I_{m,n} = \int \sin^m x \cos^n x\, dx.$$

We have from (10.3.1),

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$$

Changing n to $n + 2$,

$$I_{m,n+2} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n+2} + \frac{n+1}{m+n+2} I_{m,n}$$

and transposing,

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2}\ (n+1 \neq 0). \qquad (10.3.4)$$

Similarly, from (10.3.2), we can find

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}\ (m+1 \neq 0). \qquad (10.3.5)$$

Example 10.3.5 Find $\int \frac{\cos^4 x}{\sin^2 x} dx.$

Solution : Applying (10.3.5), with $m = -2$, $n = 4$, we get

$$I_{-2,4} = \frac{(\sin x)^{-1} x \cos^5 x}{-1} + \frac{4}{-1} I_{0,4}.$$

Now applying (10.3.1) with $m = 0$, $n = 4$, we obtain

$$I_{-2,4} = -\frac{\cos^5 x}{\sin x} - 4\left(\frac{\sin x \cos^3 x}{4} + \frac{3}{4} I_{0,2}\right)$$

$$= -\frac{\cos^5 x}{\sin x} - \sin x \cos^3 x - 3\left(\frac{\sin x \cos x}{2} + \frac{1}{2} I_{0,0}\right)$$

Also

$$I_{0,0} = \int \frac{\cos^0 x}{\sin^0 x} dx = \int dx = x.$$

Therefore

$$\int \frac{\cos^4 x}{\sin^2 x} dx = -\frac{\cos^5 x}{\sin x} - \sin x \cos^3 x - \frac{3}{2}\sin x \cos x - \frac{3}{2}x + c.$$

10.3.7. Reduction Formula for $\int \tan^n x dx$ and $\int_0^{\frac{\pi}{4}} \tan^n x\, dx$, n being a positive integer greater than 1

Let

$$I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} \text{ [substituting } \tan x = z].$$

Thus,

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}. \tag{10.3.6}$$

Using (10.3.6) and taking $J_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, we get

$$J_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - J_{n-2}$$

$$= \frac{1}{n-1} - J_{n-2}.$$

Therefore

$$J_n = \frac{1}{n-1} - J_{n-2}. \tag{10.3.7}$$

Example 10.3.6 Find $\int \tan^4 x\, dx$.

Solution : Applying (10.3.6), we get

$$I_4 = \frac{\tan^3 x}{3} - I_2 = \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - I_0\right)$$

where $I_0 = \int \tan^0 x\, dx = x.$

Therefore

$$I_4 = \frac{\tan^3 x}{3} - \tan x + x + c.$$

Example 10.3.7 Find $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx.$

Solution : Applying (10.3.7), we get

$$J_6 = \frac{1}{5} - J_4 = \frac{1}{5} - \frac{1}{3} + J_2$$

$$= \frac{1}{5} - \frac{1}{3} + 1 - J_0,$$

where $J_o = \int_0^{\frac{\pi}{4}} \tan^0 x\, dx = [x]_0^{\pi/4} = \frac{\pi}{4}.$

Thus

$$J_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}.$$

10.3.8. Reduction Formula for $\int \cot^n x\, dx$ and $\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cot^n x dx$, *n* being a positive integer greater than 1

Proceeding similar as in the art. 10.3.7 and expressing $\cot^n x = \cot^{n-2} x(\operatorname{cosec}^2 x - 1)$, we see that

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2} \text{ and } J_n = -\frac{1}{n-1} - J_{n-2}.$$

10.3.9. Reduction Formula for $\int \sec^n x\, dx$, *n* being a positive integer greater than 1

Let

$$I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$$

$$= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan x \tan x\, dx \quad \text{[Int. by parts]}$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)(I_n - I_{n-2}),$$

and transposing

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

10.3.10. Reduction Formula for $\int \operatorname{cosec}^n x\,dx$, n being a positive integer greater than 1

As in art. 10.3.9 we may find

$$I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

Example 10.3.8 Find $\int \sec^4 x\,dx$.

Solution : Here

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\left(\frac{\tan x}{1} + 0\right)$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + c.$$

10.3.11. Reduction formula for $\int x^m (\log x)^n dx$, n being a positive integer

Let

$$I_{m,n} = \int x^m (\log x)^n dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1} dx \quad \text{[Int. by parts]}$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}.$$

Thus, the reduction formula is

$$I_{m,n} = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1} I_{m,n-1}.$$

10.3.12. Reduction formula for $\int \cos^m x \sin nx\, dx$, *m, n* being a positive integer

Let

$$I_{m,\,n} = \int \cos^m x \sin nx\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx\, dx \quad [\text{int. by parts}]$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin nx \cos x\, dx$$

$$+\frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

[since cos nx sin x = sin nx cos x – sin $(n - 1)x$]

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,\,n-1}.$$

Thus, the reduction formula is

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}.$$

10.4 Summary

In this unit we have learnt the reduction formula of several functions. These fomula give us to find out the integrals easily.

10.5 Exercises

1. Evaluate the following integrals :

(i) $\int x^3 e^a x\, dx$ (ii) $\int x^3 (\log x)^2 dx$

(iii) $\int \sin^8 x\, dx$ (iv) $\int \sin^8 x \cos^2 x\, dx$

(v) $\int \frac{\sin^5 x}{\cos^4 x} dx$ (vi) $\int \tan^5 x\, dx$

(vii) $\int \sec^5 x\, dx$ (viii) $\int \mathrm{cosec}^3 x\, dx$

(ix) $\int \tan^5 x\, dx$ (x) $\int \cot^3 2x\, dx$.

2. Find the value of the following ingegrals

(i) $\int_0^{\frac{\pi}{2}} \sin^7 x\, dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^6 x\, dx$

(iii) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 3x\, dx$.

10.6 References

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Unit-11 ❑ Arc Length

Structure

11.1 Objectives

After going through this chapter, the readers will be able to :

- understand the formula of arc length.
- determine the length of an arc of a curve.

11.2 Introduction

In this unit, we use definite integral to find the arc length of a curve.

11.3 Length of an Arc of a Curve

Let the given arc AB of a curve $y = f(x)$ between $x = a$ and $x = b$ be divided into n parts by points P_1, P_2, . . . , P_{r-1}, P_r, . . . , P_{n-1} as shown in the Fig. 11.1. Suppose that the corredponding abscissae of these points are

$$x_1, x_2, .., x_{r-1}, x_r, ..., x_{n-1}$$

and ordinates are

$$y_1, y_2, .., y_{r-1}, y_r, ..., y_{n-1}.$$

We draw chords $AP_1, P_1P_2, \ldots, P_{r-1}P_r, \ldots, P_{n-1}B$ through consecutive points. Then the sum of lengths of these chords is

$$AP_1 + P_1P_2 + \ldots + P_{r-1}P_r + \ldots + P_{n-1}B.$$

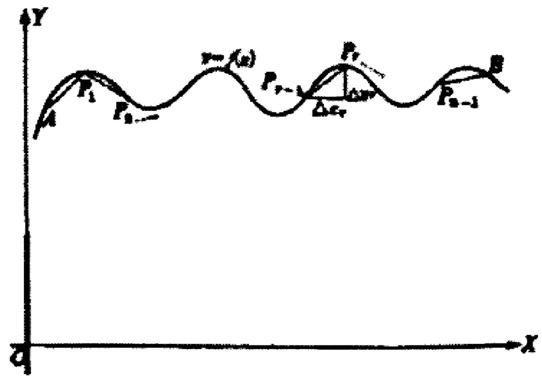


Fig. 11.1

When n becomes infinite, the length of each chord tends to zero. Hence the limiting value of the sum of the length of chords will be the length of the given are AB.

Now

$$\text{length of chord } P_{r-1}P_r = \sqrt{(\Delta x_r)^2 + (\Delta y_r)^2} = \sqrt{1+\left(\frac{\Delta y_r}{\Delta x_r}\right)^2}\,\Delta x_r.$$

Now if the curve be continuous and derivable at every point on $[a, b]$, from the mean value theorem of the differential calculus, there exists at least one point, say $x = \xi_r$ on the are $P_{r-1}P_r$ at which the slope of the tangent, $f'(\xi_r)$ is equal to $\frac{\Delta y_r}{\Delta x_r}$.

Thus the length of the chord $P_{r-1}\,P_r$ becomes $\sqrt{1+\{f'(\xi_r)\}^2}\Delta x_r$ and, consequently, by the fundamental theorem of integration, the total length (s) of the are AB is

$$s = \lim_{n\to\infty}\sum_{r=1}^{n}\sqrt{1+\{f'(\xi_r)\}^2}\Delta x_r = \int_a^b \sqrt{1+\{f'(x)\}^2}dx,$$

where a, b are respectively the abscissae of A, B.

I. Therefore the length of the arc of the courve $y = f(x)$ between the points whose abscissae are a and b is given by

$$\int_a^b \sqrt{1+\{f'(x)\}^2}dx \quad \text{or} \quad \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

II. If the curve be given in the form $x = f(y)$, the length of the arc between two points (a, c) and (b, d), may similarly be given by

$$\int_c^d \sqrt{1+\{f'(y)\}^2}\,dy \quad \text{or} \quad \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

III. If a curve be given in the parametric from $x = f(t)$ and $y = \phi(t)$ and if t_1 and t_2 be the corresponding points of a and b respectively, then the length of the arc of the curve be derived from I as

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

IV. If the polar equation of the curve be $r = f(\theta)$, then as $x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$, the length of the arc between two points whose vectorial angles are θ_1 and θ_2 is given by

$$\int_{\theta_1}^{\theta_2} \sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2}\,d\theta \quad \text{or} \quad \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

V. If the pedal equation of a curve be $p = f(r)$, then the length of an arc of the curve from $r = r_1$ to $r = r_2$ may similarly be given by

$$\int_{r_1}^{r_2} \frac{rdr}{\sqrt{r^2-p^2}} \quad \left[\frac{ds}{dr} = \sec\phi = \frac{1}{\cos\phi} = \frac{r}{\sqrt{r^2-p^2}}\right]$$

Example 11.3.1 Find the length of the perimeter of the circle $x^2 + y^2 = 25$.

Solution : Using the formula I and Fig. 11.2, we see that

the perimeter of the circle = 4 × the perimeter of the circle in the first quadrant

$$= 4\int_0^5 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^5 \frac{5}{\sqrt{25-x^2}}\,dx$$

$$= 20\left[\sin^{-1}\frac{x}{5}\right]_0^5 = 10\pi.$$

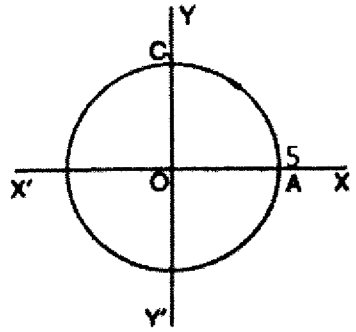


Fig. 11.2

Example 11.3.2 Determine the length of one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

Solution : Referring to Fig. 11.3 and applying formula III, we see that the length of on arch of the cycloid

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} 2a \sin\frac{\theta}{2} d\theta = 8a.$$

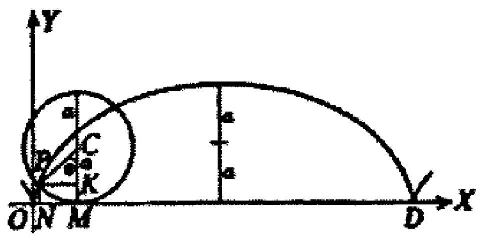


Fig. 11.3

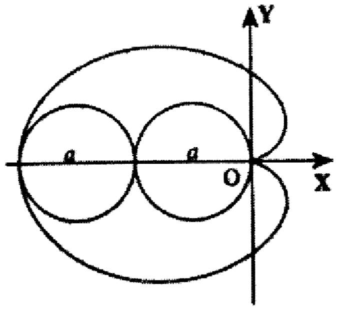
Example 11.3.3. Determine the perimeter of the cardioid $r = a(1 - \cos\theta)$.

Solution : Referring to Fig. 11.4 and applying formula IV, we see that the total length of the cardioid

$$= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\; d\theta$$

$$= 4a\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 8a.$$



$r = a(1 - \cos\theta)$

Fig. 11.4

11.4 Summary

In this unit we see that the formula of arc length comes from the approximating the curve by straight lines connecting successive points on the curve using Pythagorean theorem. An integral fromula is developed to compute the arc length of a curve.

11.5 Exercises

1. Find the length of the followings :

(i) $y = \log(1 - x^2)$ between $x = 0$ and $x = \frac{1}{3}$.

(ii) $ay^2 = x^3$ from $x = 0$ to $x = 5a$.

(iii) $r = \theta^2$; $\theta = 0$ and $\theta = \sqrt{5}$.

(iv) $x = e^{\theta} \sin\theta$, $y = e^{\theta}$ cos; $\theta = 0$ and $\theta = \frac{\pi}{2}$.

(v) the perimeter of the astroid $x^{2/3} + y^{3/2} = a^{3/2}$.

2. Find the length of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$.

3. Find the length of the arc of the parabola $x^2 = 4y$ from the vertex to the point where $x = 2$.

11.6 References

1. D. Chatterjee, B. K. Pal, Integral Calculus and Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 2019.

2. D. C. Das, B. N. Mukherjee, Integral Calculus : Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 1938.

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Unit-12 ❑ Volume and Area of urface of Revolution

Structure

12.1 Objectives

After reading this text, the students should be able to :

- find the volume of solid by revolving a curve around a line.
- determine the area of surface of revolution.

12.2 Introduction

In this unit we shall discuss a very important process to find out the volume of solid and area of surface of revolution. The method of definite integration enabled us to find these. The process of finding the area of plane figure will be extended to determine the volume of solid and area of surface.

12.3 Volume of Solid of Revolution

Let V be the volume formed when an area $ABCD$ in Fig. 12.1, under the curve $y = f(x)$ between $A(x = a)$ and $B(x = b)$ is revolved about the x-axis. We divide the interval $[a, b]$ into n parts by means of the arbitrary set of points

$$x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}.$$

Let

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_r, \ldots, \Delta x_n$$

be the length of respective sub-intervals into which $[a, b]$ is subdivided. As the entire area about the x-axis being perpendicular to it, a general infinitesimal strip of area *PQRS*, of base Δx_r likewise revolves and generates an infinitesimal disc of volume ΔV_r (say). The entire volume can now be thought of as composed of the set of discs generated by the revolution of the various strips of the area *ABCD*.

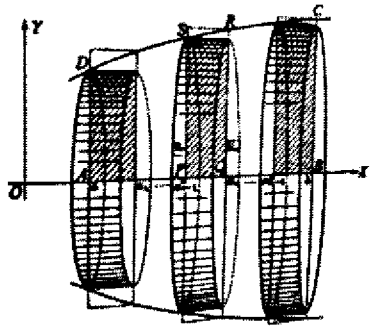

Fig. 12.1

Next let $\underline{y_r}$ and $\overline{y_r}$ denote respectively the least and greatest value of $y = f(x)$ in Δx_r. Then since volume = face area × thickness, we have

$$\pi \underline{y_r}^2 \Delta x_r \le \Delta V_r \le \pi \overline{y_r}^2 \Delta x_r.$$

By addition and nothing that $\sum_{r=1}^{n} V_r = V$, we have

$$\sum_{r=1}^{n} \pi \underline{y_r}^2 \Delta x_r \le V \le \sum_{r=1}^{n} \pi \overline{y_r}^2 \Delta x_r.$$

Now if the manner of subdivision of $[a, b]$ be such that the greatest of $\Delta x_r \to 0$ as $n \to \infty$, both sums approach the same limit. Hence

$$V = \lim_{n\to\infty} \sum_{r=1}^{n} \pi y_r^2 \Delta x_r = \lim_{n\to\infty} \sum_{r=1}^{n} \pi \{f(\xi_r)\}^2 \Delta x_r$$

in which $y_r = f(\xi_r)$ is the ordinate of an arbitrary point $x = \xi_r$, in Δx_r. Applying the fundamental theorem to the last sum

$$V = \pi \int_a^b y^2 dx = \pi \int_a^b \{f(x)\}^2 dx.$$

I. Thus the volume generated by revolving an area bounded by the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is expressed by the integral

$$V = \pi\int_a^b y^2 dx = \pi\int_a^b \{f(x)\}^2 dx.$$

II. If however the curve be expressed by $x = f(t)$, $y = \phi(t)$

$$V = \pi\int_a^b y^2 dx = \pi\int_{t_1}^{t_2} \{\phi(t)\}^2 f'(t)dt,$$

where t_1, t_2 are values of t that correspond to $x = a$ and $x = b$ respectively.

III. If again the curve $x = \phi(y)$ bounded by $y = c$ and $y = d$ be revolved about the y-axis, the volume is given by

$$V = \pi\int_c^d x^2 dy = \pi\int_c^d \{\phi(y)\}^2 dy.$$

Example 12.3.1 Find the volume of a sphere of radius a.

Solution : Let the equation of the circle in Fig. 12.2 be $x^2 + y^2 = a^2$. The center is at the origin and radius $OA = a$. Let the quadrant OAB be rotated about OX. Then a hemisphere will be created.

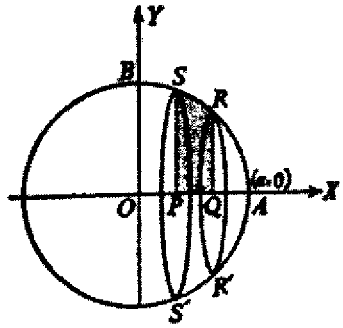

Fig. 12.2

Thus, the volume of sphere will be

$$V = 2\pi\int_0^a y^2 dx$$

$$= 2\pi\int_0^a (a^2 - x^2)dx = \frac{4}{3}\pi a^3.$$

Example 12.3.2 Find the solid formed by the rotation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution :

(i) About the major axis :

Referring to Fig 12.3 and rotating about x-axis, we see that the volume obtained by the rotation of the upper half of the ellipse

= 2 × volume obtained by the rotation of the quadrant OAB

$$=2\times\int_0^a \pi y^2 dx$$

$$=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx=\frac{4}{3}\pi ab^2.$$

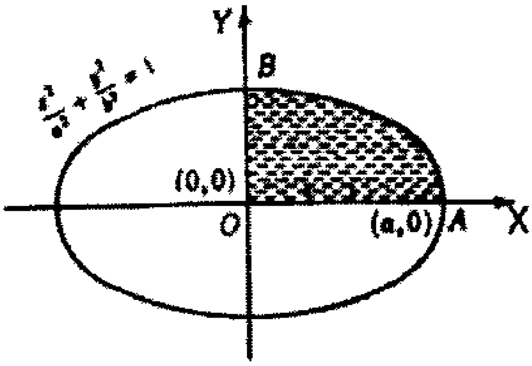

Fig. 12.3

(ii) About the minor axis :

The rotation being about y-axis, the volume of the whole ellipsoid

$$=2\times\int_0^b \pi x^2 dy$$

$$=2\pi\int_0^b \frac{a^2}{b^2}(b^2-y^2)dx=\frac{4}{3}\pi a^2 b.$$

Example 12.3.3 Find the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.

Solution : Referring to Fig 12.4 and applying Rule II we find the required volume

= 2 × volume generated by half of the arch

$$=2\times\int_0^\pi \pi a^3(1-\cos\theta)^3 d\theta$$

$$=2\pi a^3\int_0^\pi 8\sin^6\frac{\theta}{2}d\theta$$

$$=5\pi^2 a^3.$$

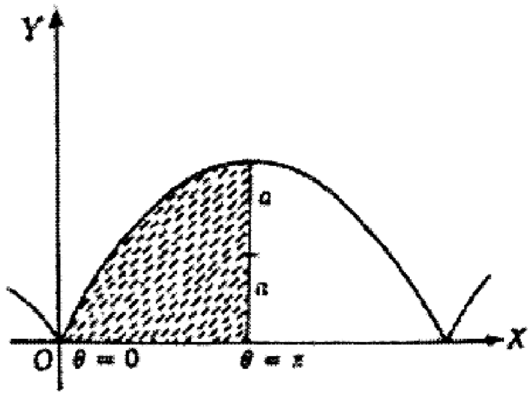

Fig. 12.4

Example 12.4 Find the volume of the solid generated by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line.

Solution : Referring to Fig 12.5 the required volume by rule I is $\pi\int y^2dx$.

Now changing the variables from x and y to r and θ, we observed that

$x = r\cos\theta = a(1 - \cos\theta)\cos\theta$, thereby $dx = a(-\sin\theta + 2\sin\theta\cos\theta)d\theta$

$y = r\sin\theta = a(1 - \cos\theta)\sin\theta$.

and the volume becomes

$$\pi\int_{\pi}^{0} a^2(1-\cos\theta)^2\sin^2\theta . a\sin\theta(2\cos\theta-1)d\theta$$

$$= \pi a^3\int_{-1}^{1}(1-z)^2(1-z^2)(1-2z)dz \quad [\text{putting } \cos\theta = z]$$

$$= \frac{8}{3}\pi a^3.$$

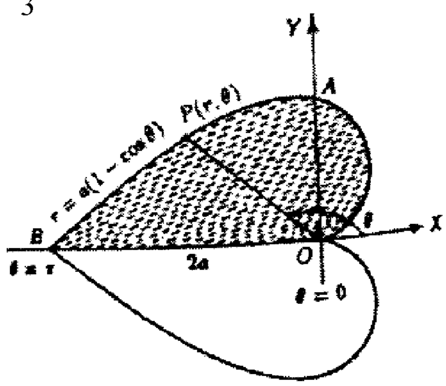

Fig. 12.5

12.4 Area of Surface of Revolution

When an arc of a plane curve is revolved about an axis in its plane, a surface of revolution is generated. The area of such a surface is defined and calculated as follows.

We can derive the formula of the area of surface from the formula of arc length. Let's look at rotating the continuous function $y = f(x)$ in the interval $[a, b]$ about the x-axis.

Let the generating arc be the portion DC of the curve $y = f(x)$ between $x = a$ and $x = b$ and let the axis of revolution be the x-axis. We divide the interval $[a, b]$ into n parts by means of the arbitrary set of points

$$x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_{n-1}$$

and let

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_r, \ldots, \Delta x_n$$

be the length of respective sub-intervals into which $[a, b]$ is subdivided. Let the arc DC be divided into n parts by means of the arbitrary set of points

$$P_1, P_2, \ldots, P_{r-1}, P_r, \ldots, P_{n-1},$$

the corresponding ordinates being

$$y_1, y_2, \ldots, y_{r-1}, y_r, \ldots, y_{n-1},$$

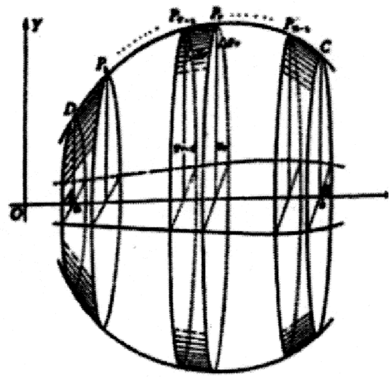

Fig. 12.6

We next draw chords through consecutive points

$$DP_1, P_1P_2, \ldots, P_{r-1}P_r, \ldots, P_{n-1}C$$

and consider a typical one $P_{r-1}P_r$ corresponding to Δx_r and $\Delta y_r (= y_r - y_{r-1})$ revolving about the x-axis. This chord generates and infinitesimal frustum of a cone, whose surface are ΔS_r (say) is given by

$$\Delta S_r = \pi \times \text{sum of the radii of the two bases} \times \text{slant height}$$

$$= \pi(y_{r-1} + y_r) \times P_{r-1}P_r = \pi(y_{r-1} + y_r)\sqrt{(\Delta x_r)^2 + (\Delta y_r)^2}$$

$$= 2\pi \frac{y_{r-1} + y_r}{2} \sqrt{1 + \left(\frac{\Delta y_r}{\Delta x_r}\right)^2} .\Delta x_r.$$

Now if the curve be continuous and has a derivative at every point, then by the mean value theorem fo the differential calculus, there exists at least one point on the arc $P_{r-1}P_r$, at which the slope of the tangent $f'(\xi_r)$ is equal to the slope of the secant $\frac{\Delta y_r}{\Delta x_r}$. Moreover $\frac{1}{2}(y_{r-1} + y_r)$ is just the average height, or height at the middle point of the chord $P_{r-1}P_r$ and from the continuity of the curve $y = f(x)$, there must exist at least one point between P_{r-1} and P_r say the point $x = \eta_r$ at which the ordinate is equal to the average height.

Hence

$$\Delta S_r = 2\pi f(\eta_r)\sqrt{1 + \{f'(\xi_r)\}^2}\,\Delta x_r.$$

Defining the area of the entire surface to be the limit of the sum of this typical areas when $n \to \infty$ in such a way that the length of each chord approaches zero, we have

$$S = \lim_{n\to\infty} \sum_{r=1}^{n} S_r = \lim_{n\to\infty} 2\pi \sum_{r=1}^{n} f(\eta_r)\sqrt{1 + \{f'(\xi_r)\}^2}\,\Delta x_r$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + \{f'(x)\}^2}\,dx$$

$$= 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

or briefly,

$$S = 2\pi \int_a^b y \frac{ds}{dx} dx,$$

in which y and ds are to be replaced by their equals in terms of x.

Cor. 1. In the case of the curve $x = f(t),\ y = \phi(t)$

$$S = 2\pi\int_{t_1}^{t_2} y\frac{ds}{dt}dt = 2\pi\int_{t_1}^{t_2} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Cor. 2. When the axis of revolution is the y-axis the corresponding formula will be (taking $y = c$ and $y = d$)

$$S = 2\pi\int_{c}^{d} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi\int_{c}^{d} x\frac{ds}{dy}dy.$$

Cor. 3. The area may also be found out in terms of polar coordinates when an equation is given in the polar form by the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

Example 12.4.1 Find the surface of a sphere generated by the circle $x^2 + y^2 = a^2$ about the x-axis.

Solution : To find out the area of surface of sphere we consider to Fig 12.7 and apply the result of art. 12.4. Since $x^2 + y^2 = a^2$, $\frac{dy}{dx} = -\frac{x}{y}$ and $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{y^2} = \frac{a^2}{y^2}$. Therefore

$$\begin{aligned} S &= 2\times \text{ surface area generated by arc } AB \\ &= 2\times 2\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 4\pi\int_0^a a\,dx = 4\pi a^2. \end{aligned}$$

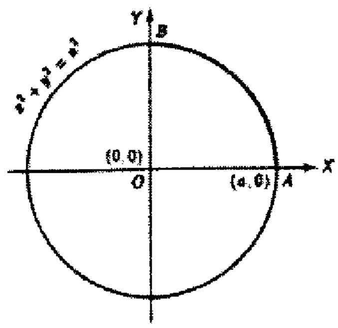

Fig. 12.7

Example 12.4.2 Find the area of the surface generated by revolving about y-axis that part of the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$, that lies in the first quadrant.

Solution : Using Cor. 1, of art. 12.4, with

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

we have from Fig 12.8,

$$S = 2\pi\int_0^{\pi/2} x\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2} 3a\sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta}.\ a\cos^3\theta d\theta$$

$$= 6\pi a^2\int_0^{\pi/2} \sin\theta\cos^4\theta d\theta = \frac{6}{5}\pi a^2.$$

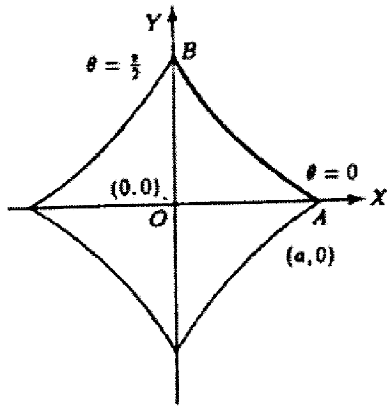

Fig. 12.8

Example 12.4.3 What is the area of the entire surface formed when the cardioid $r = a(1 + \cos\theta)$ is revolved about the initial line ?

Solution : Using Cor. 3 of art. 12.4, with

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta} = 2a\cos\frac{\theta}{2},$$

we have from Fig. 12.9,

$$S = 2\pi\int_0^{\pi} y\frac{ds}{d\theta}d\theta$$

$$= 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta . 2a\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2\int_0^{\pi} \sin\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta = \frac{32}{5}\pi a^2.$$

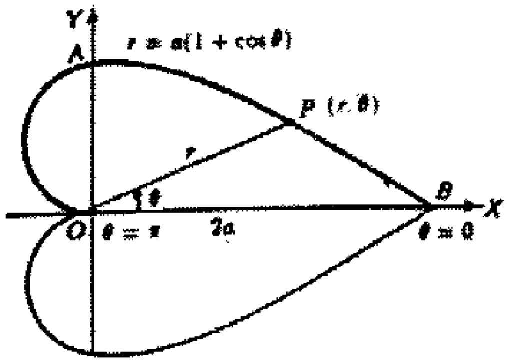

Fig. 12.9

12.5 Summary

In this unit we have learnt how a solid is formed by revolution of arc of a curve about a line and determined the formula of volume and the area of surface of that solid.

12.6 Exercises

1. Find the volume of solid generated by revolving the following curve :

(i) $y = \sqrt{x}$ between the lines $x = 1$ and $x = 4$ about x-axis,

(ii) $y = 5x - x^2$ between the lines $x = 0$ and $x = 5$ about x-axis,

(iii) $y^2 = 4ax$ between the lines $x = 0$ and $x = 2a$ about x-axis,

(iv) $x^2 - y^2 = a^2$ between the lines $x = 0$ and $x = 2a$ about x-axis,

(v) $xy = 2$ between the lines $y = 1$ and $y = 4$ about y-axis.

2. Find the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2(2 - x)$.

3. Find the volume of the solid generated by the revolution of the loop of the curve $y^2(a + x) = x^2(a - x)$ about the x-axis.

4. Find the area of surface generated by revolving the following curve :

(i) $y = x^2$ between the lines $x = 0$ and $x = \sqrt{2}$ about y-axis,

(ii) $r = 2a \cos \theta$ about the initial line,

(iii) an arc of $y = \sin x$ about x-axis,

(iv) $2y = x + 1$ between the lines $x = 1$ and $x = 3$ about x-axis.

5. Find the area of surface generated by revolving the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.

12.7 References

1. R. K. Ghosh, K. C. Maity, An introduction to analysis : Integral Calculus, NCBA, India, 1956.
2. D. Chatterjee, B. K. Pal, Integral Calculus and Differential Equations, U. N. Dhur & Sons Private Ltd. Kolkata., India, 2019.
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Unit- 13 □ Limits and Continuity of Functions

Structure

13.1 Objectives

This unit gives

- Various types of functions and their classification
- Sequence of real number and its convergence
- Concept of limit of a real function
- Various properties of limit of a function such as algebric operation on limits, sandwich property, etc.

13.2 Introduction

The limit of a function is a fundamental concept in analysis concerning the behaviour of that function near a particular point. Although implicit in the development of calculus of the 17th & 18th centuries, the modern idea of the limit of a function goes back to Bolzano who, in 1817, introduced the basic of the epsilon-delta technique to define limit of functions. The motion of a limit has many applications in modern Calculus. In particular, the many definitions of continuity employ the limit. It also appears in the definition of the derivative.

13.3 Pre requisites

(or Recapitulation of prior elementary ideas that are needed to introduce the concept of limit):

A. Functions

(i) Let A and B be two non-void subsets $\mathbb{R}$ & $f: A \to B$ is a rule of correspondence that assigns to each $x \in A$, a uniquely determined $y \in B$ or $y = f(x)$.

The set of values of x for which f can be defined is known as **Domain** of f, denoted by D_f and the corresponding collection of y's (as mentioned above) is known as Range set of f generally denoted by R_f.

A few examples of *f*, D_f and R_f :

(i) $$f(x) = \sqrt{\left[\log_c \frac{5x - x^2}{4}\right]}$$

f can be defined for those x for which $\frac{5x - x^2}{4} \geq 1$ and this gives $1 \leq x \leq 4$

so $D_f \equiv [1, 4]$

(ii) $f(x) = \sqrt{\left(x - \frac{x}{1-x}\right)}$

f can be defined only when $x - \frac{x}{1-x} \geq 0 \Rightarrow 1 < x < \infty$ & $D_f = (1, \infty)$

(iii) $f(x) = \cos^{-1} \frac{3}{4 + 2\sin x}$. Here we must have $-1 \leq \frac{3}{4 + 2\sin x} \leq 1$

& for this $D_f \equiv \left[-\frac{\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi\right]$ where $k = 0, \pm 1, \pm 2, ...$

Note that D_f may be a closed and bounded interval, may be an open interval (bounded or unbounded), union of intervals and so on.

(Readers are requested to verify the validity of D_f as mentioned in above examples and as well as to look for other functions and their domain).

(i) Consider the function $f:[-1,\ 1]\to\mathbb{R}$ defined by $f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Here D_f is an interval $[-1,\ 1]$ but $R_f=\{-1,0,1\}$ which is not an interval.

(ii) Consider the function $f:(-1,\ 1)\to\mathbb{R}$ defined by $f(x)=\frac{1}{x^2+1}\ \forall\ x\in(-1,1)$

$$R_f=\left(\frac{1}{2},\ 1\right] \text{ or } \frac{1}{2}<x\leq 1.$$

Note that in D_f, -1 and $+1$ are not included but 1 is included as right hand end point in R_f.

We are interested to learn the reason for such differences of nature of D_f & R_f.

Equal functions : $f,\ g : D\to\mathbb{R}$ are same (or equal) when $f(x)=g(x)$ for each $x\in D$.

Note that x and $\frac{x^2}{x}$ are not same.

Operations on Functions : Let f and g be two functions having domain $D_f(\subset\mathbb{R})$ and $D_g(\subset\mathbb{R})$ respectively. If $D_f\cap D_g\neq\phi$, then $f\pm g, fg$ can be defined on $D_f\cap D_g$ by

(i) $(f\pm g)x=f(x)\pm g(x)\ \ \forall x\in D_f\cap D_g$ and

$(fg)(x) = f(x)\ g(x) \quad \forall x \in D_f \cap D_g$

Again deleting those points of D_g (if any) for which $g(x) = 0$, we can define $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $x \in D_f \cap D_g \setminus \{x : g(x) = 0\}$.

Composition of functions : Let f and g be two functions such that

$x \in D_f \Rightarrow f(x) \in D_g$. In other words $R_f \subset D_g$. Then we can define

$(g \circ f)(x) = g[f(x)] \forall x \in D_f$.

$g \circ f$ is called the composite of two functions *f* and *g*.

Similarly, we can define $(f \circ g)(x)$ with appropriate restrictions.

In general $(f \circ g)(x) \neq (g \circ f)(x)$. For example, $f(x) = x^2, g(x) = \sin x$

Then $(g \circ f)(x) = g(f(x)) = \sin x^2$ & $(f \circ g)(x) = f(g(x)) = f(\sin x) = \sin^2 x$.

Injective (one-one), Surjective (onto) and Bijective functions :

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

If for $x, y \in D$, $f(x) = f(y) \Rightarrow x = y$, *f* is called injective or one-one function

$f(x) = 3x + 4,\ x \in \mathbb{R}$ is Injective but $g(x) = |x|,\ x \in \mathbb{R}$ is not Injective.

Let $f : D \to E$ where $D, E \subset \mathbb{R}$, obviously $f(D) \subseteq E$. But if $f(D) = E$, we say that *f* is surjective or onto function. $f : [1, 2] \to [2, 3]$ defined by $f(x) = x + 1$ is onto function.

But $f : [1, 2] \to [2,\ 4]$, $f(x) = x + 1$ is not so,

$$\frac{7}{2} \in [2,\ 4] \text{ and } \frac{7}{2} = x + 1 \Rightarrow x = \frac{5}{2} \notin [1,\ 2].$$

f is bijective if it is both injective and surjective.

Invertible functions : Let $f: X \to Y$ where $X, Y \subset \mathbb{R}$ be such that for each $y \in Y$, there exists a single value of x such that $f(x) = y$. Then this correspondence defines a function $x = g(y)$. We say that f is invertible and $x = g(y)$ is the inverse function. Note that if f be bijective, then f is invertible.

For example, if $y = \log_a\left(x + \sqrt{x^2 + 1}\right)$, $a > 0, a \neq 1$, then

$$x = \frac{1}{2}\left(a^y - a^{-y}\right) \text{ or } \sinh(y \ln a)$$

Increasing function & Decreasing function :

Let $f: D \to \mathbb{R}$ where $D \subset \mathbb{R}$. If for each pair $x, y \in D$,

$x > y \Rightarrow f(x) \geq f(y)$ or $f(x) > f(y)$, we say that f is increasing function.

But if $x > y \Rightarrow f(x) \leq f(y)$ or $f(x) < f(y)$, we say that f is decreasing function.

$f(x) = \sin x$ is increasing in $\left[0, \frac{\pi}{2}\right]$ but is decreasing in $\left[\frac{\pi}{2}, \pi\right]$

Periodic function :

A function $f: D \to \mathbb{R}(D \subset \mathbb{R})$ is periodic if there exists a number p such that $f(x + p) = f(x) \forall x \in D$.

The smallest positive p for which $f(x + p) = f(x) \forall x$ holds, is called the period of f.

Bounded and unbounded functions :

$f: D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be bounded above if there exists $\lambda \in \mathbb{R}$ such that $f(x) \leq \lambda \forall x \in D$, we say that f is bounded above (by λ). If there exists $\mu \in \mathbb{R}$ such that $f(x) \geq \mu \forall x \in D$, we say that f is bounded below (by μ). If f be both bounded above & bounded below, then f is bounded on $D_f\,(\equiv D)$. In other words, If there exists $K \in \mathbb{R}$ such that $|f(x)| \le K$ for all $x \in D$, we say that f is bounded on D. For future course of discussion the following concepts are useful.

Let $f : D \to \mathbb{R}\,(D \subset \mathbb{R})$ be bounded above.

Then $\lambda(\in \mathbb{R})$ is said to be the least upper bound or supremum of f in D if $\exists\ \lambda \in \mathbb{R}$ such that (i) $f(x) \le \lambda \forall x \in D$ and (ii) for any $\varepsilon > 0, \exists\ y \in D$ such that $f(y) > \lambda - \varepsilon$ (or in other words, no real $< \lambda$ is an upper bound of f) this $\lambda =$ sup f. If f be bounded above, then sup $f\,(\in \mathbb{R})$ exists.

If f is unbounded above we say that sup $f = \infty$

Let $f : D \to \mathbb{R}$ be bounded below. Then $\mu(\in \mathbb{R})$ is greatest lower bound or infimum of f in D if

(i) $f(x) \ge \mu$ for all $x \in D$ & (ii) if for any $\varepsilon > 0, \exists\ y \in D$ such that $f(y) < \mu + \varepsilon$, then $\mu =$ inf f (in other words, no real $> \mu$ is lower bound of f). Then $\mu =$ inf f. If f be bounded below, then inf $f\,(\in \mathbb{R})$ exists.

If f be unbounded below, we write inf $f = -\infty$

Sup f – inf f is known as oscillation of function f on D.

13.4 Sequences in $\mathbb{R}$

(i) A function $f : \mathbb{N} \to \mathbb{R}$ is known as a sequence (note that $\mathbb{N}$ is the set of natural numbers).

Examples : $\left\{(-1)^n\right\}_n, \left\{\frac{1}{n}\right\}_n, \left\{\frac{4n+3}{3n+4}\right\}, \left\{n^2\right\}_n$ etc.

Symbolically, $\{a_n\}_n\,(n \to a_n)$. Note that the range set of $\left\{(-1)^n\right\}_n$ is the set $\{-1, 1\}$ where as the range sets of the next three are infinite sets.

A sequence $\{a_n\}_n$ is bounded if its range set is bounded.

Range sets of $\{(-1)^n\}_n, \left\{\frac{1}{n}\right\}_n, \left\{\frac{4n+3}{3n+4}\right\}_n$ are bounded but range set of $\{n^2\}_n$ is not bounded.

(ii) Note that $\mathbb{N}$ is unbounded above, as there is no real $\lambda \in \mathbb{R}$ for which $n \le \lambda \quad \forall n \in \mathbb{N}$.

So an interesting question is that when n becomes arbitrary large without any bound, then what will be the fate of $\{a_n\}_n$?

Consider the above examples : As n becomes larger and larger, $\frac{1}{n}$ becomes smaller & smaller we say that, the difference between $\frac{1}{n}$ and 0 decreases steadily. Neither $\frac{1}{n}$ coincides with zero nor it goes to the left side of 0. We say $\frac{1}{n} \to 0$ (tends to zero) as $n \to \infty$. But note that as n becomes arbitrarily large. n^2 increases more rapidly & we say that $n^2 \to \infty$ as $n \to \infty$. In case of $\{(-1)^n\}_n$, it is either +1 or -1.

Limit of a sequence in $\mathbb{R}$: A sequence $\{a_n\}_n$ is said to converge to a limit $l(\in \mathbb{R})$ if for arbitrary $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $|a_n - l| < \varepsilon$ for all $n \ge m$.

$\lim_{n\to\infty} a_n = \infty$ if for all $G > 0$ there exists $m \in \mathbb{N}$ such that $a_n > G \quad \forall n \ge m$. We say that $\{a_n\}_n$ diverges to ∞.

To explain this definition, we take $a_n = \frac{1}{n}$ as mentioned earlier. We have seen that $\frac{1}{n} \to 0$ as $n \to \infty$.

let $\varepsilon = \frac{7}{1000}$. Then $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}$, if $n > \frac{1000}{7}\left(= 142\frac{6}{7}\right)$

so m = 143 & for this $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}$, $n \geq 143$

Let us change $\varepsilon = \frac{8}{3439}$. Then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n > \frac{3439}{8}\left(= 429\frac{7}{8}\right)$

So m = 430 & then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n \geq 430$

These two simple examples exhibit the dependence of m on the arbitrary positive value of ε.

We state the following results without proof at this stage :

(a) A Convergent sequence in $\mathbb{R}$ is necessarily bounded but a bounded sequence may not be convergent $\left(Ex. \left\{(-1)^n\right\}_n\right)$.

(b) Limit of a sequence, if exists, is unique.

(c) Cauchy's general principle of convergence : A necessary & sufficient condition for the convergence of $\{a_n\}_n$ is that given $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $\left|a_{n+p} - a_n\right| < \varepsilon \;\; \forall\, n \geq m, \;\; p \in \mathbb{N}$.

(d) Sandwich rule : Let $a_n < b_n < c_n$ for all $n \geq m$ (or for all n) and $\{a_n\}_n, \{c_n\}_n$ both converge to same limit $l(\in \mathbb{R})$. Then $\lim\limits_{n\to\infty} b_n$ exists & = l.

(iii) Monotonic sequences in $\mathbb{R}$

A sequence $\{a_n\}_n$ in $\mathbb{R}$ is said to be monotonic increasing if $a_{n+1} \geq a_n$ for all n, but if $a_{n+1} \leq a_n$ for all n, $\{a_n\}_n$ is said to be monotonic decreasing sequence in $\mathbb{R}$.

We state the following results without proof :

(a) A monotonic increasing sequence $\{a_n\}_n$ in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded above and $a_n \to \sup a_n$. If $\{a_n\}_n$ be unbounded above, then $\lim\limits_{n\to\infty} a_n = \infty$ (diverges to ∞)

(b) A monotonic decreasing sequence in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded below and and $a_n \to \inf\limits_n a_n$. If $\{a_n\}_n$ be unbounded below, then $\lim\limits_{n\to\infty} a_n = -\infty$ (diverges to $-\infty$)

(iv) The following results are easily deducible following definition and basic results :

If $\lim\limits_{n\to\infty} a_n = l(\in \mathbb{R})$, $\lim\limits_{n\to\infty} b_n = m(\in \mathbb{R})$, then

$\lim\limits_{n\to\infty}(a_n \pm b_n), = l \pm m$, $\lim\limits_{n\to\infty}(a_n b_n) = lm$,

$\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{l}{m}$ provided $b_n \neq 0\ \forall n$ and $m \neq 0$.

(c) Accumulation point (or limit point) of a set

Let $S(\subset \mathbb{R})$ be a set and $\xi \in \mathbb{R}$. ξ is said to be an accumulation point (or limit point) of S if there exists a sequence of distinct elements $\{x_n\}_n$ of S such that $x_n \to \xi$ as $n \to \infty$. '0' is limit point of $S = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$. 1 is limit point of $T\left\{1 + \dfrac{1}{n} : n \in \mathbb{N}\right\}$ etc. Note that $0 \notin S, 1 \notin T$.

Note that a finite set has no accumulation point. The set $U = \{n^2; n \in \mathbb{N}\}$ has no accumulation point in $\mathbb{R}$.

(D) Neighbourhood of a point & Interior point of a set :

(i) Let $x \in \mathbb{R}$. By a δ-neighbourhood of x, we mean the interval $(x-\delta, x+\delta)$ where $\delta > 0$. This is denoted by $N(x,\delta)$ or $N_\delta(x)$.

The set $N(x,\delta)-\{x\}$ is called the deleted δ-neighbourhood $(\text{or } \delta - nbd)$ of x, denoted by $N'(x,\delta)$ or $N'_\delta(x)$. $\cup(\subset \mathbb{R})$ is nbd of $x \in \mathbb{R}$ if $\exists$ an open interval I such that $x \in I \subset U$ for example, $\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ is a neighbourhood of 1. The set $\mathbb{R}$ (of all real numbers) is a neighbourhood of each of its points. The situation is different in case of Q, the set of rational numbers for if $\xi \in Q$, then every $(\xi-\delta,\ \xi+\delta)$ contains rational as well as irrational points also. So Q is not a neighbourhood of its points.

(ii) Let $D \subset \mathbb{R}$. We say that $x \in D$ is interior point of D if there exists a neighbourhood of x, say $(x-\delta,\ x+\delta)$, which is contained in D.

For example consider $[a,b] = \{x : a \le x \le b\}$

Let $a < c < b$. we take $0 < \delta < \min\{c-a, b-c\}$ & so $(c-\delta, c+\delta) \subset (a,b)$, so c is interior point of the set but a, b are not interior points of it.

Accumulation point can also be defined as follows :

Let $S \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. If every deleted neighbourhood of ξ, $N'(\xi,\delta) \cap S \ne \phi$, then ξ is accumulation point of S.

This can be shown that $N'(\xi,\delta) \cap S$ is an infinite set. On the basis of this approach, it obviously follows that a finite set $(\subset \mathbb{R})$ has no accumulation point.

On the basis of these pre-requisites, we are now in a position to introduce the concept of limit of a function.

13.5 Limit of function

Let $f : D(\subset \mathbb{R}) \to \mathbb{R}$ and p be an accumulation point of D.

(A) Sequential approach : $\lim_{x\to p} f(x) = l(\in \mathbb{R})$ if for every sequence $\{x_n\}_n$, $x_n \in D$ for all n, $x_i \neq x_j$ if $i \neq j$, $x_n \neq p$, converging to p, the sequences $\{f(x_n)\}_n$ converge to same limit $l(\in \mathbb{R})$.

If on the otherhand, $\{f(x_n)\}_n$ converge to different limits for different $\{x_n\}_n$'s we say that the limit does not exist.

To explain the matter, let us consider the following examples :

Example :

(i) $\lim_{x\to 0} \sin\frac{1}{x}$: Note that the sequences $\left\{\frac{2}{2n\pi}\right\}_n$ and $\left\{\frac{2}{(2n+1)\pi}\right\}_n$ both converge to zero. But $\{\sin n\pi\}_n$ converges to zero whereas $\left\{\sin\left(n\pi+\frac{\pi}{2}\right)\right\}_n$ is not convergent, (n even and n odd give different limits). So by above definition, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

(2) $\lim_{x\to 0} \frac{1}{x} \sin\frac{1}{x}$

For $x_n = 1/n\pi (\to 0)$, $\frac{1}{x_n}\sin\frac{1}{x_n} \to 0$ but for $y_n = \frac{1}{\left(2n+\frac{1}{2}\right)\pi} \to 0$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}\left(2n+\frac{1}{2}\right)\pi \sin\left(2n+\frac{1}{2}\right)\pi = \infty$$

So $\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$ does not exist.

(B) ($\varepsilon-\delta$ approach) let $\varepsilon>0$ be any number. If corresponding to such ε, there exists $\delta>0$ such that $|f(x)-l|<\varepsilon$ whenever $x\in\mathbb{N}'(p,\delta)\cap D$. we say that $\lim_{x\to p}f(x)$ exists and $=l(\in\mathbb{R})$.

Here $x\in N'(p,\delta)\cap D$ can be written as $0<|x-p|<\delta$ or $p-\delta<x<p$, $p<x<p+\delta$, $x\in D$.

(C) The two definitions stated in (A) and (B) are equivalent :

Proof : Let $\lim_{x\to p}f(x)=l(\in\mathbb{R})$ in the sense of $\varepsilon-\delta$ definition.

Then for arbitrary $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(x)-l|<\varepsilon \text{ wherever } 0<|x-p|<\delta \quad \text{(i)}$$

As p is accumulation point of D, so there exists a sequence

$\{x_n\}_n\left(x_n\in D\ \forall n,\ x_i\neq x_j \text{ if } i\neq j,\ x_n\neq p \text{ for all } n\right)$ which converges to p. Hence corresponding to above $\delta>0$, there exists natural number m such that

$0<|x_n-p|<\delta$ for all $n\geq m$ (2)

Combining (1) & (2), $|f(x_n)-l|<\varepsilon$ for all $n\geq m$

Note that m depends on ε (as m depends on δ & δ depends on ε).

So $\lim_{n\to\infty}f(x_n)=l(\in\mathbb{R})$ and $\{f(x_n)\}_n$ converges to $l(\in\mathbb{R})$.

Next let $\lim_{x\to p}f(x)=l(\in\mathbb{R})$ following sequential criterion.

If possible let $\lim\limits_{x \to p} f(x) = l$ does not hold in the sense of $\varepsilon - \delta$ definition.

Then for some number $\varepsilon > 0$, the corresponding δ does not exist. That indicates, however small $\delta > 0$ may be, there exists always at least $x'(\neq p)$ for which $0 < |x' - p| < \delta$ nonetheless $|f(x') - l| \geq \varepsilon$.

Let us consider a decreasing positive termed sequence $\{\delta_n\}_n$ converging to zero (in particular, $\delta_n = \frac{1}{n}$ for all $n \in \mathbb{N}$). Then for every δ_n, x'_n can be found such that $0 < |x'_n - p| < \delta_n$ nonetheless $|f(x'_n) - l| \geq \varepsilon$. $\delta_n \to 0 \Rightarrow x'_n \to p$ by Sandwich rule. By assumption, $\{f(x'_n)\}_n$ converges to *l*. But $|f(x'_n) - l| \geq \varepsilon$.

Thus we arrive at a contradiction. So $\varepsilon - \delta$ definition follows from that of sequential approach. Thus the two definitions are equivalent.

(D) One sided limits

(i) Let p be an accumulation point of D from the left (i.e. $x_n \to p, x_n < p \; \forall n, x_n \in D$ etc) or f has been defined in some left-deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$, we say that $\lim\limits_{x \to p-} f(x)$ $\left(\text{or } \lim\limits_{x \to p-0} f(x)\right)$ exists and $= l_1 (\in \mathbb{R})$. This is commonly known as left hand limit of $f(x)$ as $x \to p$.

(ii) Let p be an accumulation point of D from the right (i.e $x_n \to p,\ x_n > p \;\; \forall n$, $x_n \in D$ etc.) or f has been defined in some right deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l_2| < \varepsilon$ whenever $p < x < p + \delta$, we say that $\lim\limits_{x \to p+} f(x)$ $\left(\text{or } \lim\limits_{x \to p+0} f(x)\right)$ exists and $= l_2 (\in \mathbb{R})$. This is commonly known as right hand limit of $f(x)$ as $x \to p$.

(E) In this connection, the following result is useful in determining the existence of limit. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and let p be (both sided) accumulation point of D (or f has been defined in both sided deleted neighbourhood of p).

Then $\lim\limits_{x \to p} f(x) = l(\in \mathbb{R})$ if and only if $\lim\limits_{x \to p-0} f(x) = \lim\limits_{x \to p+0} f(x) = l$

Proof : let $\lim\limits_{x \to p} f(x) = l(\in \mathbb{R})$

Following $\varepsilon - \delta$ definition, corresponding to arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - p| < \delta, x \in D$

$\Rightarrow |f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$ as well as $p < x < p + \delta$.

$\Rightarrow \lim\limits_{x \to p-} f(x) = l = \lim\limits_{x \to p+} f(x)$

Converse let $\lim\limits_{x \to p-} f(x) = l = \lim\limits_{x \to p+} f(x)$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta_1 < x < p$ & $|f(x) - l| < \varepsilon$ whenever $p < x < p + \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - p| < \delta$, $|f(x) - l| < \varepsilon \Rightarrow \lim\limits_{x \to p} f(x) = l$

Examples (i) $f(x) = \begin{cases} 3x + 7, & x < 1 \\ 2x + 11, & x > 1 \end{cases}$

Here $\lim\limits_{x \to 1-} f(x) = 10$, $\lim\limits_{x \to 1+} f(x) = 13$ & so $\lim\limits_{x \to 1} f(x)$ does not exist.

(2) $f(x) = \begin{cases} 7x + 3, & x < 2 \\ 8x + 1, & x > 2 \end{cases}$

Here $\lim\limits_{x\to 2-} f(x)=17,\ \lim\limits_{x\to 2+} f(x)=17$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|7x+3-17| < \varepsilon$ i.e. $|x-2| < \frac{\varepsilon}{7}$ whenever $2-\delta_1 < x < 2$ & so $\delta_1 = \frac{\varepsilon}{7}$ is admissible & $|8x+1-17| < \varepsilon$ i.e. $|x-2| < \frac{\varepsilon}{8}$ whenever $2 < x < 2+\delta_2$ & so $\delta_2 = \frac{\varepsilon}{8}$ is admissible.
Taking $\delta = \min\{\delta_1, \delta_2\}$, we get $\lim\limits_{x\to 2} f(x)=17$

(F) Cauchy Criterion for the existence of limit

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and p be an accumulation point of D.

A necessary and sufficient condition for the existence of $\lim\limits_{x\to p} f(x)$ is that given $\varepsilon > 0$, there exists a deleted neighbourhood of $p, \mathbb{N}'(p,\delta)$ such that

$|f(x)-f(y)| < \varepsilon$ whenever $x, y \in \mathbb{N}'(p,\delta) \cap D$

Proof : Let $\lim\limits_{x\to p} f(x) = l\,(\in \mathbb{R})$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists a deleted neighbouhood $\mathbb{N}'(p,\delta)$ such that $|f(x)-l| < \varepsilon/2$ whenever $x \in \mathbb{N}'(p,\delta) \cap D$

If moreover $y \in \mathbb{N}'(p,\delta) \cap D, |f(y)-l| < \varepsilon/2$ As a result,

$|f(x)-f(y)| \le |f(x)-l| + |f(y)-l| < \varepsilon$ holds.

Converse : Let for given $\varepsilon > 0$, there exists a deleted neighbourhood $\mathbb{N}'(p,\delta)$ such that $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in \mathbb{N}'(p,\delta) \cap D$

Let p be accumulation point of D. So there exists $\{x_n\}_n (x_n \in D \, \forall_n, x_i \neq x_j$ if $i \neq j, \; x_n \neq p)$ which converges to p. Hence corresponding to above $\delta(>0)$, there exists $m \in \mathbb{N}$ such that $x_n \in \mathbb{N}'(p, \delta) \cap D$ for all $n \geq m$.

Therefore, $|f(x_n) - f(x_k)| < \varepsilon$ for all $n, \; k \geq m$.

So by Cauchy's general principle of convergence of a sequence, $\{f(x_n)\}_n$ is convergent and so $\lim\limits_{x \to p} f(x)$ exists.

Illustration : Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is irrational} \end{cases}$

Let $a \in (0, 1)$. Note that for any $\delta > 0$, $N'(a, \delta) \cap (0, 1)$ contains both rational as well as irrational points. If such rational be x & such irrational be y, then $|f(x) - f(y)| = |1 - (-1)| = 2 \nless$ arbitrary $\varepsilon > 0$.

So by Cauchy Criterion, $\lim\limits_{x \to a} f(x)$ does not exist.

(G) Infinite limits and Limit at infinity

(i) Infinite limits :

Let $f : D \to \mathbb{R}$ and p be an accumulation point of $D(\subset \mathbb{R})$. Then $f(x)$ is said to be tend to ∞ as $x \to p$, if given any $G > 0$ (as large as we please), there exists $\delta > 0$ such that

$f(x) > G$ whenever $x \in N'(p, \delta) \cap D$.

If we opt for sequential approach, if for $\{x_n\}_n (x_n \in D \, \forall n, \, x_i \neq x_j$ if $i \neq j, x_n \neq p)$ converges to p, $\{f(x_n)\}_n$ diverges to ∞, we say that $\lim\limits_{x \to p} f(x) = \infty$

Illustration : $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$

For any $G > 0, \frac{1}{x} > G$ if $x < \frac{1}{G}(\to 0 \text{ as } G \to \infty)$.

If for given $G > 0$ (as large as we please), there exists $\delta > 0$ such that $f(x) < -G$ whenever $x \in N'(p, \delta) \cap D$, we say that $\lim_{x \to p} f(x) = -\infty$

(ii) Limit at infinity

Let $f : D \to \mathbb{R}$ where D is unbounded above.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (G, \infty) \cap D$,

We say that $\lim_{x \to \infty} f(x) = l(\in \mathbb{R})$ $ex \cdot \lim_{x \to \infty} \frac{1}{x} = 0$.

Next let $f : D \to \mathbb{R}$ where D is unbounded below.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (-\infty, G)$, we say that $\lim_{x \to -\infty} f(x) = l(\in \mathbb{R})$

Illustration (1) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, x \in \mathbb{R}$

To solve this, we will assume the very standard limit of sequence $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e(\in \mathbb{R})$.

We can take $x > 1$. There exists natural number n such that

$n \le x < n+1$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)\right\} = e \text{ and}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = e$$

So, $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x} = e$

(2) $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x} = e$

We take $x=-y$ and So $y\to\infty$ and x $\to-\infty$

$$\left(1+\frac{1}{x}\right)^{x} = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y} = \left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)\to e \text{ as } y\to\infty.$$

In this connection, we state the following result :

Let $f:(a,\infty)\to\mathbb{R}$, Then $\lim\limits_{x\to\infty}f(x)$ exists if and only if for every $\varepsilon>0$, there exists $X(>a)$ such that $|f(x)-f(y)|<\varepsilon\,\forall\, x, y>X$.

(iii) Infinite limits at infinity

Let $f:D\to\mathbb{R}$ where $D(\subset\mathbb{R})$ is unbounded above.

Let $G>0$ be any number, as large as we please.

Corresponding to G, there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x>K$, we say that $\lim\limits_{x\to\infty}f(x)=\infty$.

Let D be unbounded below, if corresponding to $G>0$ (as large as we please), there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x<K$, we say that $\lim\limits_{x\to-\infty}f(x)=\infty$.

But if $f(x)<-G$ for all $x<K$, we say $\lim\limits_{x\to-\infty}f(x)=-\infty$.

Example : $\lim_{x\to\infty} \log_a x = \infty,\ a > 1$

Let $G > 0$ be any arbitrary number. If we take $a^G = M$, then

$x > M \Rightarrow \log_a x > G$. Hence $\lim_{x\to\infty} \log_a x = \infty$.

(H) Some standard limits :

(i) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \dfrac{\log_a (1+x)}{x} = \log_a e$ where $a > 0, a \neq 1$

(iii) $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \ln a, a > 0$

(iv) $\lim_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

(I) Algebra of limits :

Let $g, f : D \to \mathbb{R}$ when $D \subset \mathbb{R}$ and p be an accumulation point of D.

Let $\lim_{x\to p} f(x) = l(\in \mathbb{R}),\ \lim_{x\to p} g(x) = m(\in \mathbb{R})$.

Then (i) $\lim_{x\to p} \{f(x) \pm g(x)\} = l \pm m$

(ii) $\lim_{x\to p} \{f(x) g(x)\} = lm$

(iii) $\lim_{x\to p} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$ where $g(x) \neq 0$ and $m \neq 0$.

Proof : (i) Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0,\ \delta_2 > 0$ such that $|f(x) - l| < \dfrac{\varepsilon}{2}$ whenever $0 < |x - p| < \delta_1,\ x \in D$ and $|g(x) - m| < \dfrac{\varepsilon}{2}$ whenever

$0<|x-p|<\delta_2, x \in D$.

Let $\delta=\min\{\delta_1, \delta_2\}$. So for $0<|x-p|<\delta$, $x \in D$ both hold.

Hence $|\{f(x) \pm g(x)\}-\{l \pm m\}| \leq |f(x)-l|+|g(x)-m|<\varepsilon$

whenever $0<|x-p|<\delta$, $x \in D$

$\Rightarrow \lim\limits_{x \to p}\{f(x) \pm g(x)\}=l \pm m=\lim\limits_{x \to p} f(x) \pm \lim\limits_{x \to p} g(x)$

Note : (1) This result can be generalised for finite number of functions.

(2) The converse of the result is not true, in general

Let $f(x)=\begin{cases}1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational}\end{cases}$ and $g(x)=\begin{cases}0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational}\end{cases}$

Let $p \in \mathbb{R}$. Every deleted *nbd* of *p* contains both rational (say *a*) and irrational *b* (say) points. Then in case of both *f* and *g*, $|f(a)-f(b)|$ or $|g(a)-g(b)|=1 \nless$ arb ε.

So neither $\lim\limits_{x \to p} f(x)$ nor $\lim\limits_{x \to p} g(x)$ exists. But $f(x)+g(x)=1$ and

$\lim\limits_{x \to p}\{f(x)+g(x)\}=1$.

(ii) To establish it we will first show that as $\lim\limits_{x \to p} g(x)$ exists, so there exists a deleted neighbourhood of *p*, in which *g* is bounded.

There exists $\delta_1>0$ such that $|g(x)-m|<1$ where $0<|x-p|<\delta_1$, $x \in D$ $(\text{or } x \in N'(p, \delta) \cap D)$

$\Rightarrow |g(x)|<1+|m|$ in $N'(p, \delta_1) \cap D$

$\Rightarrow g$ is bounded in $N'(p, \delta_1) \cap D$

$$|f(x)g(x)-lm| = |g(x)\{f(x)-l\}+l(g(x)-m)| \le |g(x)||f(x)-l|+|l|\,|g(x)-m|$$

.....(1)

As $\lim_{x\to p} g(x)$ exists, so there exists $\delta_1 > 0$ such that $|g(x)| < \lambda$ for some $\lambda \in \mathbb{R}^+$ in $N'(p, \delta_1) \cap D$ .....(1)

Let $\varepsilon > 0$ be any number, corresponding to ε, there exists $\delta_2 > 0, \delta_3 > 0$ such that

$|f(x)-l| < \frac{\varepsilon}{2\lambda}$ whenever $x \in N'(p, \delta_2) \cap D$ ....(2)

and $|g(x)-m| < \frac{\varepsilon}{2(|l|+1)}$ whenever $x \in N'(p, \delta_3) \cap D$ ....(3)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then in $N'(p, \delta) \cap D$, by (1), (2), (3)

$$|f(x)g(x)-lm| < \lambda \cdot \frac{\varepsilon}{2\lambda} + |l| \cdot \frac{\varepsilon}{2(|l|+1)}$$

$$\Rightarrow |f(x)g(x)-lm| < \varepsilon \text{ in } N'(p, \delta) \cap D$$

$$\Rightarrow \lim_{x\to p} f(x)g(x) = lm = \left(\lim_{x\to p} f(x)\right)\left(\lim_{x\to p} f(g)\right)$$

Note : (1) This result can be generalised for finite number of functions.

(2) $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist but $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

Let $\varepsilon > 0$ be any number $\left|x\sin\frac{1}{x} - 0\right| \le |x| < \varepsilon$ whenever $x \in N'(0, \delta) \cap D_f$ where $\delta \equiv \delta(\varepsilon)$.

(3) If $g(x)$ be bounded on D and $\lim_{x\to p} f(x) = 0$, then $\lim_{x\to p} f(x)g(x)$ exists $= 0$.

(iii) $\left|\dfrac{f(x)}{g(x)}-\dfrac{l}{m}\right|=\left|\dfrac{m\{f(x)-l\}-l(g(x)-m)}{m\,g(x)}\right|$

$$\leq \frac{|m|\,|f(x)-l|+|l|\,|g(x)-m|}{|m|\,|g(x)|} \quad \text{.... (1)}$$

As $\lim_{x\to p} g(x)=m(\neq 0)$, there exists $\delta_1>0$ such that

$|g(x)-m|<\dfrac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$ .... (2)

$\Rightarrow |g(x)|>\dfrac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$

Let $\varepsilon>0$ be any number.

As $\lim_{x\to p} f(x)=l$, corresponding to ε, there exits $\delta_2>0$ such that

$|f(x)-l|<\dfrac{\varepsilon|m|}{4}$ whenever $x\in N'(p,\ \delta_2)\cap D$ ... (3)

As $\lim_{x\to p} g(x)=m$, corresponding to ε, there exists $\delta_3>0$ such that

$|g(x)-m|<\dfrac{\varepsilon|m|^2}{4(|l|+1)}$ whenever $x\in N'(p,\ \delta_3)\cap D$ ... (4)

Let $\delta=\min\{\delta_1,\ \delta_2,\ \delta_3\}$. So whenever $x\in N'(p,\ \delta)\cap D$, (2), (3) (4) hold.

Recalling 1. $\left|\dfrac{f(x)}{g(x)}-\dfrac{l}{m}\right|<\dfrac{2}{|m|^2}\left|\left\{\dfrac{\varepsilon.|m|^2}{4}+\dfrac{|l|.\varepsilon|m|^2}{4(|l|+1)}\right\}\right|$

$$\Rightarrow \left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| < \varepsilon \text{ whenever } x \in N'(p, \delta) \cap D$$

$$\Rightarrow \lim_{x \to p} \frac{f(x)}{g(x)} = \frac{l}{m} = \frac{\lim\limits_{x \to p} f(x)}{\lim\limits_{x \to p} g(x)}$$

Note : Neither $\lim\limits_{x \to 0} \frac{1}{x}$ nor $\lim\limits_{x \to 0} \sin \frac{1}{x}$ exists, but $\lim\limits_{x \to 0} x \sin \frac{1}{x}$ exists & = 0.

So the Converse of (iii) is not, in general, true.

Illustration : Evaluate (1) $\lim\limits_{x \to 0} \frac{(e^x - 1)\tan^2 x}{x^3}$ (2) $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

(3) $\lim\limits_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\left(\sqrt{3} - 2\cos x\right)}$

(1) $= \lim\limits_{x \to 0} \left\{ \frac{e^x - 1}{x} \cdot \left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{1}{\cos x}\right)^2 \right\}$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x \to 0} \left(\frac{1}{\cos^2 x}\right) = 1 \text{ (As all exist)}$$

So limit is 1.

(2) $\lim\limits_{x \to 0} \frac{\sin x(1 - \cos x)}{x^3 \cos x} = \lim\limits_{x \to 0} \left\{ \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{2\sin^2 \frac{x}{2}}{x^2} \right\}$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}.\frac{1}{\cos x}.\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\right\}\frac{1}{2} = 1.1.1.\frac{1}{2} = \frac{1}{2}$$

(3) (Method of substitution) Put $x - \frac{\pi}{6} = t$ and so $x \to \frac{\pi}{6} \Leftrightarrow t \to 0$.

$$\text{Given limit} = \lim_{t\to 0}\frac{\sin t}{\sqrt{3} - 2\cos\left(t + \frac{\pi}{6}\right)} = \lim_{t\to 0}\frac{\sin t}{\sqrt{3} + \sqrt{3}\cos t + \sin t}$$

$$= \lim_{t\to 0}\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{\sqrt{3}\left(2\sin^2\frac{t}{2}\right) + 2\sin\frac{t}{2}\cos\frac{t}{2}} = \lim_{t\to 0}\frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2} + \cos\frac{t}{2}} = 1$$

(J) Neighbourhood properties :

(a) Let $f : D \to \mathbb{R}, D \subset \mathbb{R}$ and p be an accumulation point of D. Let $\lim_{x\to p} f(x) = l(\in \mathbb{R})$. Then

(i) f is bounded in some deleted nbd of p

(ii) If l be greater than some real number K, then there exists a deleted nbd of p in which $f(x) > K$.

(iii) If l be less than some real number μ, then there exists a deleted nbd of p in which $f(x) < \mu$.

Proof : (i) Proved earlier in I(ii)

(ii) Let $0 < \varepsilon < l - K$. Corresponding to this ε, then exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in N'(p, \delta) \cap D$

$\Rightarrow l-\varepsilon < f(x) < l+\varepsilon \forall x \in N'(p,\delta)\cap D$

Considering the above choice of $\varepsilon, f(x) > K$ in $N'(p,\delta)\cap D$

(iii) As in (ii), taking $0 < \varepsilon < \mu - l$.

(b) Let $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ & p be an accumulation point of D.

Let $\lim_{x\to p} f(x) = A(\in \mathbb{R}), \lim_{x\to p} g(x) = B(\in \mathbb{R})$.

If $A < B$, then there exists a deleted neighbourhood of p in which $f(x) < g(x)$.

Proof : Let $A < C < B$.

As $\lim_{x\to p} f(x) = A$, there exists $\delta_1 > 0$ such that $|f(x) - A| < C - A$ for all $x \in N'(p,\delta_1)\cap D$.

As $\lim_{x\to p} g(x) = B$, there exists $\delta_2 > 0$ such that $|g(x) - B| < B - C$ for all $x \in N'(p,\delta_2)\cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta)\cap D$, both hold.

In $N'(p,\delta)\cap D$, $f(x) < C - A + A = C = B - (B - C) < g(x)$ holds.

(c) Sandwich property :

Let $f, g, h : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$. Let $f(x) < g(x) < h(x)$ for all $x \in D$ & let p be an accumulation point of D. Given that $\lim_{x\to p} f(x) = l$, $\lim_{x\to p} h(x) = l (l \in \mathbb{R})$.

Then $\lim_{x\to p} g(x) = l$.

Proof : Let $\varepsilon > 0$ be any number. Corresponding to this ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ in $N'(p,\delta_1)\cap D$ & $|h(x) - l| < \varepsilon$ in $N'(p,\delta_2)\cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta)\cap D$,

$l - \varepsilon < f(x) < g(x) < h(x) < l + \varepsilon \Rightarrow |g(x) - l| < \varepsilon$ in $N'(p,\delta)\cap D$

So $\lim_{x\to p} g(x) = l$.

(d) $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$, p be accumulation point of D and

let $\lim_{x\to p} f(x) = l(\in \mathbb{R})$, $\lim_{x\to p} g(x) = m(\in \mathbb{R})$. If $f(x) < g(x)$ in D, then $l \le m$.

Proof : If possible let $l > m$ & let $o < \varepsilon < \frac{l-m}{10}$, corresponding to such ε, there exists $\delta_1, \delta_2 >$ such that $|f(x) - 1| < \varepsilon$ in $N'(p,\delta_1)\cap D$ & $|g(x) - m| < \varepsilon$ in $N'(p,\delta_2)\cap D$.

If $\delta = \min\{\delta_1, \delta_2\}$, then in $N'(p,\delta)\cap D$, both hold.

In $N'(p,\delta)\cap D$, $l - \varepsilon < f(x) < g(x) < m + \varepsilon \Rightarrow l - m < 2\varepsilon \Rightarrow 10\varepsilon < 2\varepsilon$ — absurd as $\varepsilon > 0$.

So $l \le m$.

[You can take $f(x) = 1 - x$, $g(x) = 1 + x$ where $x > 0$. $f(x) < g(x)$ for all x and $\lim_{x\to 0+} f(x) = 1 = \lim_{x\to 0+} g(x)$.]

K. Infinitesimal :

(a) $f : D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be infinitesimal as $x \to a$ if $\lim_{x\to a} f(x) = 0$.

(b) If $f, g : D \to \mathbb{R}$ are infinitesimals, then $f \pm g$, fg are also so.

(c) If $f : D \to \mathbb{R}$ be infinitesimal as $x \to a$ and $g : D \to \mathbb{R}$ be bounded, then fg is infinitesimal.

(d) We say $f = o(g)$ (or f is of little –*oh* of g over D) if

$f(x)=\alpha(x)g(x)$ where $\alpha(x)$ is infinitesimal.

(e) We say $f=O(g)$ (or f is of big *–oh* of g over D) if $f(x)=\beta(x)g(x)$ where $\beta(x)$ is bounded on D.

(f) The functions f and g are of same order over $D(\subset\mathbb{R})$, if $f=O(g)$ and $g=O(f)$ simultaneously.

13.6 Exercise I

1. Find the limits (if exist)

(a) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

(b) $\lim\limits_{x\to 0}\dfrac{\left(2x^2+|x|\right)}{x}$

(c) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+3\right)$, $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+1\right)$ and $\lim\limits_{x\to 0}\left\{\left(\dfrac{1}{x^2}+3\right)-\left(\dfrac{1}{x^2}+1\right)\right\}$

(d) $\lim\limits_{x\to 3}\dfrac{\sqrt{(3x)}-3}{\sqrt{2x-4}-\sqrt{2}}$

(e) Apply Cauchy's principle for the existence of limit to evaluate $\lim\limits_{x\to 0}\dfrac{1+x}{1-x}$.

2. Choose the correct one : $\lim_{x\to 0}\frac{\sin[x]}{[x]}$

(a) the limit exists and is 1

(b) the limit does not exist.

(c) if at $x = 0,\ f(0) = 0$, the limit will exist

(d) if at $x = 0,\ f(0) = 1$, the limit will exist.

13.7 Definition of Continuity

I. (a) A function $f: D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be continuous at $p \in D$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - f(p)| < \varepsilon$ or $f(x) \in N(f(p), \varepsilon)$ whenever $x \in N(p, \delta) \cap D$.

If *f* is not continuous at *p*, then *f* is discontinuous at *p*.

(b) Let $f: D \to \mathbb{R}(D \subset \mathbb{R})$ and $p \in D$.

(i) If *p* is an isolated point of *D* (i.e. not a limit point of *D*), then *f* is continuous at *p* (ii) if *p* be limit point of *D* i.e. $p \in D \cap D'$ (D' is the collection of limit points of *D*) and if $\lim_{x \to p} f(x) = f(p)$, then *f* is continuous at *p*.

(c) Continuity in an interval $[a, b]$ or in $\{x : a \le x \le b\}$.

f is continuous in $[a, b]$ if (i) $\lim_{x \to a+0} f(x) = f(a)$ (ii) $\lim_{x \to b-0} f(x) = f(b)$ and (iii) if $a < c < b$, then $\lim_{x \to c-0} f(x) = \lim_{x \to c+0} f(x) = f(c)$.

Examples. 1. Let $f(x) = \begin{cases} x, \text{ for } x \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} \\ 1, \text{ for } x = 1 \end{cases}$

be defined on $S = \left\{1 - \frac{1}{n} \middle| n \in \mathbb{N}\right\} \cup \{1\}$.

The only accumulation point of S is 1 and all other points of S are its isolated points. Here $\lim_{x\to 1} f(x) = f(1) = 1 \Rightarrow f$ is continuous at 1, f is also continuous at the isolated points $1-\frac{1}{n} : n \in \mathbb{N}$. Hence f is continuous on S.

2. Let $f(x) = \dfrac{1}{(1-x)}$, $x \neq 1$. Find the points of discontinuity of $y = f\left[f\left(f(x)\right)\right]$.

$x = 1$ is a point of discontinuity of $f(x)$.

If $x \neq 1$, $f\left[f(x)\right] = \dfrac{1}{1-\dfrac{1}{(1-x)}} = \dfrac{x-1}{x}$, $x \neq 0 \Rightarrow x = 0$ is a point of discontinuity of $f\left[f(x)\right]$.

If $x \neq 0$, $x \neq 1$, $y = \dfrac{1}{1-\dfrac{x-1}{x}} = x$ is continuous everywhere.

So points of discontinuity of the given composite function are $x = 0,\ x = 1$.

(3) Let $E = \left\{1-\frac{1}{n} \middle| n \in \mathbb{N}\right\} \cup [1,\ 2]$ and $f : E \to \mathbb{R}$ be defined by $f(x) = x^2$.

Each $1-\frac{1}{n}$ is isolated point of E and so by definition, f is continuous at all such points.

Let $p \in [1,\ 2]$. Then $p \in E \cap E'$ (E' derived set of E) and then $x^2 \to p^2$ or $f(x) \to f(p)$. So f is continuous at p.

Thus f is continuous on E.

(Continuation of definiton (d)) $f : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and $p \in D \cap D'$.

f is continuous at p if for every sequence

$\{x_n\}_n \left(x_n \in D \ \forall n,\ x_i \neq x_j \text{ if } i \neq j,\ x_n \neq p\right)$ converging to p, $\{f(x_n)\}_n$ converges to $f(p)$.

Examples (1) Let $A = \{x \in \mathbb{R} | x > 0\}$ and let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is irrational} \\ \dfrac{1}{n}, \text{ if } x = \dfrac{m}{n} \text{ where } m, n \in \mathbb{N} \text{ and } (m, n) = 1 \end{cases}$$

To examine the continuity of f in A.

We require the following lemma :

Let i be any irrational number between 0 and 1.

Let p, q, n be any positive integers such that $p < q \leq n$ and n is fixed. Then there exists a neighbourhood of i which has the property that no rational number of the form $\dfrac{p}{q}$ belongs to it.

Proof of lemma : Let d be the least of the differences $\left| i - \dfrac{p}{q} \right|$ for all p, q such that $p < q \leq n$. Let δ be chosen so that $0 < \delta < d$. Then $(i - \delta,\ i + \delta)$, a nbd of i, which has the property stated above.

Let us now examine the continuity of f.

Let b be any irrational number and let $\varepsilon > 0$.

Now there exists $n_0 \in \mathbb{N}$ such that $n_0 \varepsilon > 1$ (known as Archimedean property of real numbers). By above lemma, $\delta > 0$ can be chosen so small that the nbd $(b - \delta,\ b + \delta)$ contains no rational number with denominator $< n_0$.

If then follows that for $|x-b|<\delta$, $x \in A$, we have

$$|f(x)-f(b)| = |f(x)| \le \frac{1}{n_0} < \varepsilon \Rightarrow f \text{ is continuous at irrational point } b.$$

Let $a \in A$ be any rational point. Let $\{x_n\}_n$ be any sequence of irrational numbers in A that converges to a. Then $\lim_{n\to\infty} f(x_n)=0$ where as $f(a)>0$. Hence f is discontinuous at all rational points.

(2) (Dirichlet's function) $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} 1, \text{ if } x \text{ be rational} \\ 0, \text{ if } x \text{ be irrational} \end{cases}$$

Applying sequential approach, it can be shown that f is discontinuous everywhere.

(3) Let $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$

To investigate the continuity of f on $\mathbb{R}$.

Let $\varepsilon>0$ be any number.

$$\left|f(x)-f\left(\frac{1}{2}\right)\right| = \begin{cases} \left|x-\frac{1}{2}\right|, \text{ if } x \text{ is rational} \\ \left|1-x-\frac{1}{2}\right| = \left|x-\frac{1}{2}\right|, \text{ if } x \text{ is irrational} \end{cases}$$

So $\left|f(x)-f\left(\frac{1}{2}\right)\right| = \left|x-\frac{1}{2}\right| < \varepsilon$ whenever $\left|x-\frac{1}{2}\right| < \delta(=\varepsilon)$.

f is continuous at $x=\frac{1}{2}$.

Next let $x \neq \frac{1}{2}$ and x is rational. Let $\{x_n\}_n$ be a sequence of irrationals such that $\lim\limits_{n\to\infty} x_n = x$. So $f(x_n) = 1 - x_n \to 1 - x$ as $n \to \infty$.

As $x \neq \frac{1}{2}$, so $x \neq 1 - x$ and f is discontinuous on $Q - \left\{\frac{1}{2}\right\}$.

Next let x be irrational number and let $\{y_n\}_n$ be a sequence of rational numbers such that $\lim\limits_{n\to\infty} y_n = x$. Here $f(y_n) = y_n \to x$ as $n \to \infty$. But $f(x) = 1 - x$.

So $\lim\limits_{n\to\infty} f(y_n) \neq f\left(\lim\limits_{n\to\infty} y_n\right) \Rightarrow f$ is discontinuous at all irrational points.

Consequently f is continuous only at $x = \frac{1}{2}$.

Classification of discontinuities :

Let f be not continuous at $p(\in D_f)$. This discontinuity of f at p may be due to different reasons which may be classified into two types / kinds.

Definition : (a) Let f be defined in both-sided neighbourhood of point $p(\in D_f)$.

Let $\lim\limits_{x\to p+} f(x)$ and $\lim\limits_{x\to p-} f(x)$ both exist finitely but are unequal, then $x = p$ is known as jump discontinuity of f.

$f(p+o) - f(p-o)$ is known as height of the jump. If f has jump discontiuity on the right at a, the height of jump is $f(a+o) - f(a)$ and similarly at b, it is $f(b) - f(b-o)$, if it is left discontinuous at b.

Example : let $:[0,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}0, & \text{if } x=0\\ \frac{1}{2}-x, & \text{if } 0<x<\frac{1}{2}\\ \frac{1}{2}, & \text{if } x=\frac{1}{2}\\ \frac{3}{2}-x, & \text{if } \frac{1}{2}<x<1\\ 1, & \text{if } x=1\end{cases}$$

$f(0+)=\frac{1}{2}\neq f(0)$, $f\left(\frac{1}{2}-0\right)=0$, $f\left(\frac{1}{2}+0\right)=1$ so $f\left(\frac{1}{2}-0\right)\neq f\left(\frac{1}{2}+0\right)$

$f(1-0)=\frac{1}{2}\neq 1$ so, 0, $\frac{1}{2}$, 1 are points of jump discontinuity of f.

If $f(p-0), f(p+0)$ both exist and are equal but $\neq f(p)$,

then p is removable discontinuity of $f\left(\text{i.e} \lim_{x\to p} f(x)\neq f(p)\right)$

Example : $f(x)=\begin{cases}5x+7, & x<2\\ 13, & x=2\\ 4x+9, & x>2\end{cases}$

$f(2-0)=17=f(2+0)$ but $f(2)=13$

$x=2$ is removable discontinuity. These two types of discontinuity are known as discontinuity of first kind or ordinary discontinuity.

(b) (i) If f is defined in both sided nbd of p including p and at least one of $f(p-0)$ & $f(p+0)$ fails to exist finitely though f is bounded in some deleted neighbourhood of p, then p is discontinutiy of second kind with finite oscillation.

Example : $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Neither $\lim_{x\to 0+} f(x)$ nor $\lim_{x\to 0-} f(x)$ exists & but f is bounded in *nbd* of 0.

(ii) f is unbounded in every nbd p and $\lim_{x\to P+0} f(x)$ or $\lim_{x\to p-0} f(x)$ is $+\infty$ or $-\infty$. Such a discontinuity is known as infinte discontinuity.

Example : $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2, & x = 0 \end{cases}$

13.8 Neighbourhood properties

Let $f : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and p be an accumulation point of D as well as an element of D. Let f be continuous at p.

Then the following results hold :

(i) There exists a neighbourhood of p in which f is bounded.

(ii) If $f(p) \neq 0$, there exists a neighbourhood of p in which $f(x)$ & $f(p)$ have the same sign.

(iii) If in every neighbourhood of $p, f(x)$ assumes both positive & negative values, then $f(p) = 0$

The first two properties follow from the neighbourhood properties for the existence of limit.

For (iii) if $f(p) > 0$, by (ii) there exists *nbd* of p in which $f(x) > 0$ for all $x \in \mathbb{N}(p,\delta) \cap D$. But $f(x)$ have both signs in every *nbd* of p & so $f(p) \not> 0$. By similar logic, $f(p) \not< 0$. Hence $f(p) = 0$.

The converse of (iii) is not true. For example, $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity of some special types of functions

(i) Let $f : D \to \mathbb{R}$ be monotone function (increasing or decreasing). Then at every point c of D, both $f(c+0)$ & $f(c-0)$ exist. So if c be any point of discontinuity, then that discontinuity is of first kind. In other words a monotone function can not have any discontinuity of second kind (for proof, see Apendix).

(ii) Polynomial function $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n (a_i \in \mathbb{R} \ \forall \ i, a_0 \neq 0)$ is continuous on $\mathbb{R}$. Rational functions $\frac{p(x)}{q(x)}$ are continuous for all $x \in \mathbb{R}$ for which the functions can be defined.

$\sin x$ and $\cos x$ are continuous an $\mathbb{R}$. $\tan x$ & $\sec x$ are continuous for all $x \neq (2n+1)\frac{\pi}{2}$ and $\cot x$, $\operatorname{cosec} x$ are continuous for all $x \neq n\pi$ (n is integer in both cases)

(iii) $a^x, a > 0,$ is continuous for all $x \in \mathbb{R}$. $\log x, x > 0$ is continuous for all $x > 0$.

(iv) For even positive integer n, the function $g : x \to \sqrt[n]{x}$ is continuous for all $x \in [0, \infty)$ and for an odd positive integer n, g is continuous for all $x \in (-\infty, \infty)$.

(v) Limit of composite function :

Let $f : (a, b) \to \mathbb{R}$ be continuous at $c \in (a,b)$. Suppose that $g : I \to (a,b)$ where I is an open interval and $x_\circ \in I$. If $\lim\limits_{x \to x_\circ} g(x)$ exists and is equal to c. then $\lim\limits_{x \to x_\circ} f(g(x)) = f(c)$.

Proof : Continuity of f at c implies that for each pre-assigned $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(c)| < \varepsilon$ whenever $|y - c| < \delta,\ (y \in (a, b))$ ......(1)

As $\lim\limits_{x \to x_\circ} g(x) = c$, so corresponding to above δ, we can find $\eta > 0$ such that $|g(x) - c| < \delta$ for $0 < |x - x_\circ| < \eta$ ....(2)

By (1) and (2) for $0<|x-x_\circ|<\eta$, we have $|f(g(x))-f(c)|<\varepsilon,\ 0<|x-x_\circ|<\eta$

Hence $\lim\limits_{x\to x_\circ} f(g(x))=f(c)$ follows.

Corollary : Let I, J be open intervals, $g: I\to J$ be continuous at $x_0\in I$. If $f: J\to\mathbb{R}$ is continuous at $g(x_0)\in J$ then $f\circ g: I\to\mathbb{R}$ is continuous at x_0. In other words, the composition of two continuous functions is continuous.

Note : Continuity of f at c in (v) is needed.

Let $f, g:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(y)=\begin{cases}3, & y=1\\ 4, & y\neq 1\end{cases}\quad g(x)=1 \text{ for all } x.$$

Note that as $y\to 1, f(y)\to 4$ & $g(x)\to 1$ as $x\to 0$

For all x, $f(g(x))=f(1)=3$ & so it is not true that $f(g(x))\to 4$ as $x\to 0$

Illustration :

To evaluate $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$

Let $f(x)=\dfrac{1+x}{2+x},\ g(x)=\dfrac{1-\sqrt{x}}{1-x}$

$\lim\limits_{x\to 1} f(x)=\dfrac{2}{3}$ (f is continuous at x =1) & $\lim\limits_{x\to 1} g(x)=\lim\limits_{x\to 1}\dfrac{1}{1+\sqrt{x}}=\dfrac{1}{2}$

Hence $\lim\limits_{x\to 1}\left[f(x)\right]^{g(x)}=\left(\dfrac{2}{3}\right)^{1/2}$

(Note that $\lim\limits_{x\to 1}\left[f(x)\right]^{g(x)}=e^{\lim\limits_{x\to 1} g(x)\ln f(x)}=e^{B\ln A}=A^B$ if

$\lim\limits_{x\to 1} f(x)=A>0$ and $\lim\limits_{x\to 1} g(x)=B$)

(vi) Piecewise Continuous function :

Let $f:[a,b]\to\mathbb{R}$ be such that it is continuous in $[a,b]$ except for a finite number of points, at each of which f has jump discontinuity. Then f is said to be piecewise continuous function in $[a,b]$

Illustration : Let $f:[0,3]\to\mathbb{R}$ be defined by $f(x)=[x]$

$$\text{Then } f(x)=\begin{cases}0, & 0\le x<1\\ 1, & 1\le x<2\\ 2, & 2\le x<3\\ 3, & x=3\end{cases}$$

Note that f has jump discontinuity at 1, 2, 3 only & is continuous in (0, 1), (1, 2) and (2, 3)

Example : Let $f(x)=[x],\ x\in\mathbb{R}^+$

Then f is not continuous at any point of $\mathbb{Z}$ but is continuous on $\mathbb{R}^+\setminus\mathbb{Z}$.

(i) Let $C\in\mathbb{Z}$.

Note that $C-\frac{1}{n}\to C$ as $n\to\infty$. $f\left(C-\frac{1}{n}\right)=C-1$ for all $n\in\mathbb{Z}$. But $f(C)=C$. So $\lim\limits_{n\to\infty} f\left(C-\frac{1}{n}\right)\neq f\left[\lim\limits_{n\to\infty}\left(C-\frac{1}{n}\right)\right]\Rightarrow f$ is not continuous at any point of $\mathbb{Z}$.

(ii) Let $C\in\mathbb{R}^+\setminus\mathbb{Z}$

We take $0<\varepsilon<\min\{C-[C],[C]+1-C\}$

Let $\lim\limits_{n\to\infty} x_n=C$. So corresponding to above ε, $\exists\, n_\circ\in\mathbb{Z}$ such that

$|x_n-c|<\varepsilon$ whenever $n\ge n_0$

Above choice of ε implies $[C] < x_n < [C]+1$ for all $n \ge n_\circ$

Then $f(x) = [x_n] = [C]$ for all $n \ge n_\circ$

Therefore $f(x_n) \to f(C)$ as $n \to \infty$. Hence the result follows.

Examples of piecewise continuous functions

(i) $f(x) = x - [x],\ x \in [0,4]$

(ii) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}+5}{x^{2n}+1},\ x \in [-2,2]$

(iii) $f(x) = \begin{cases} 2x+1, & 0 \le x < 1 \\ 5, & x = 1 \\ 3x+2, & 1 < x \le 2 \\ 7, & x = 2 \end{cases}$

13.9 Properties of functions continuous in a closed and bounded interval [a, b]

Theorem (1) : Let $f:[a,b] \to \mathbb{R}$ be continuous in the closed and bounded interval [a, b] & $f(a)f(b) < 0$. Then there exists at least one point $c \in (a,b)$ such that $f(c) = 0$.

[To prove this, we require the following result, known as Nested interval property :

If $\{[a_n, b_n]\}_n$ be a sequence of closed and bounded intervals such that each is contained in the preceeding. Then $\bigcap\limits_n [a_n, b_n] \ne \phi$

If more over $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ then if $p \in \bigcap\limits_n [a_n, b_n]$, p is unique.]

Also $\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} b_n$

Proof : We assume that $f(a) < 0, f(b) > 0$

For the sake of convenience, let $[a, b] = [a_1, b_1] \equiv I_1$

Let us bisect I_1 at $c_1 = \dfrac{a_1 + b_1}{2}$, If $f(c_1) = 0$ the result is proved

If $f(c_1) \neq 0$, either $f(c_1) > 0$ or $f(c_1) < 0$

If $f(c_1) > 0$ we take $[a_1, c_1]$ as I_2 so that $f(a_1) f(c_1) < 0$

& if $f(c_1) < 0$, we take $[c_1, b_1]$ as I_2. $I_2 \equiv [a_2, b_2]$

Let us bisect $[a_2, b_2]$ at $c_2 = \dfrac{a_2 + b_2}{2}$ If $f(c_2) = 0$ the result is proved. Otherwise, we assume that sub-interval as $[a_3, b_3] = I_3$ for which $f(a_3) f(b_3) < 0$ This process is continued indefinitely & we get a sequence $\{I_n\}_n$ of closed & bounded intervals $[a_n, b_n]$ for which

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \dfrac{b-a}{2^{n-1}} = 0$

Also $f(a_n) f(b_n) < 0$ for all $n \in \mathbb{N}$

By Nested interval property, $\bigcap_n I_n = \{c\}$ Also $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$

By construction, $f(a_n) < 0$ and $f(b_n) > 0$ for all n

By continuity of f, $\lim_{n\to\infty} f(a_n) \leq 0$ & $\lim_{n\to\infty} f(b_n) \geq 0$

$\Rightarrow f\left(\lim_{n\to\infty} a_n\right) \leq 0$ & $f\left(\lim_{n\to\infty} b_n\right) \geq 0$

$\Rightarrow f(c) \leq 0$ & $f(c) \geq 0 \Rightarrow f(c) = 0$

Note : This theorem is due to B. P. J. N Bolzano (1781-1848)

Theorem (2) : Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$ and $f(a)\neq f(b)$. If k be any real number such that $f(a)<k<f(b)$ then there exists $c\in(a,b)$ such that $f(c)=k$.

Proof : Let $\phi:[a,b]\to\mathbb{R}$ be defined by $\phi(x)=f(x)-k$

Continuity of f in $[a,b]\Rightarrow$ continuity of ϕ in $[a,b]$

$\phi(a)\,\phi(b)=\{f(a)-k\}\{f(b)-k\}<0$

Then by Bolzano's theorem, there exists $c\in(a,b)$ such that $\phi(c)=0$ i.e $f(c)=k$

Note : (i) This property is known as Intermediate value (I.V.) property of f in $[a, b]$

(ii) I. V. property does not hold in case of functions defined on a set.

Let $S=[0,1]\cup[2,3]$ & $f:S\to\mathbb{R}$ be defined by $f(x)=x$

f is continuous on S but f does not attain the value $\frac{3}{2}$ on S.

(iii) Continunity of f in $[a,b]\Rightarrow$ validity of I V property by f on $[a, b]$ but the converse is not true

Example : $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0, & x=0\\ \frac{1}{2}-x, & 0<x<\frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ \frac{3}{2}-x, & \frac{1}{2}<x<1\\ 1, & x=1\end{cases}$

f assumes every value between $f(0)$ & $f(1)$, f is not continuous in $[0,1]$ & so the validity of I. V property by a function in a closed & bounded interval does not characterise the continuity of the function. In this connection, we state the following two important results :

(1) Let $f:[a,b]\to\mathbb{R}$ obey the Intermediate value property in $[a,b]$ & let f be monotonic in $[a,b]$. Then f is continuous on $[a,b]$.

(2) Let f be stictly monotonic function in the interval $[a,b]$. If $f([a,b])$, the range set is an interval, then f is continous on $[a,b]$.

Theorem (3) : Let $f:[a,b]\to\mathbb{R}$ be continuous and assume each value between $f(a)$ and $f(b)$ just once. Then f is strictly monotonic in $[a,b]$.

Proof : Let $f(a)<f(b)$. We propose to show that f is strictly increasing function.

Let $a<x_1<b$. As $f(x)$ assumes each value between $f(a)$ and $f(b)$ just once, so $f(x_1)=f(a)$ or, $f(x_1)=f(b)$ is not possible. ....(1)

If $f(x_1)<f(a)(<f(b))$, then by I. V. property $f(x)$ must assume the value $f(a)$ for some $x\in(x_1,b)$. As a result $f(x)=f(a)$, once at $x=a$ & for some $x\in(x_1,b)$. This contradicts the hypothesis that $f(x)$ assumes each value between $f(a)$ & $f(b)$ just once. So $f(x_1)<f(a)$ is not possible. ....(2)

By similar logic, $f(x_1)>f(b)$ is not possible. ....(3)

In that case, $f(x)$ assumes the value $f(b)$ at least twice — once at b & another in (a,x_1) by I. V. property.

By (1), (2) & (3), $f(a)<f(x_1)<f(b)$

This leads to the conclusion that if $a<x_1<x_2<b$ then

$f(a)<f(x_1)<f(x_2)<f(b)$

$\Rightarrow f$ is strictly monotonic increasing in $[a, b]$

If at the outset, we assume that $f(a) > f(b)$, then arguing in a similar way f is strictly monotonic decreasing in $[a, b]$.

Examples : $f:[0, 2] \to \mathbb{R}$ be defined by $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n+2} - \cos x}{1 + x^{2n}}$

Show that $f(0)f(2) < 0$ but $f(x)$ is never zero in $(0, 2)$. Explain why.

When 0 <x <1, $x^{2n} \to 0$ & when $1 < x < 2,\ x^{2n} \to \infty$

Here $f(0) = -1$. When $0 < x < 1,\ f(x) = -\cos x$

$f(1) = \dfrac{1}{2}[1 - \cos 1]$. When $1 < x \le 2,\ f(x) = \lim\limits_{n\to\infty} \dfrac{x^2 - \dfrac{\cos x}{x^{2n}}}{1 + \dfrac{1}{x^{2n}}} = x^2$

$$\text{So } f(x) = \begin{cases} -1, & x = 0 \\ -\cos x, & 0 < x < 1 \\ \dfrac{1}{2}(1 - \cos 1), & x = 1 \\ x^2, & 1 < x \le 2 \end{cases}$$

So $f(0)f(2) = -4 < 0$, but $f(x)$ is never zero in $(0, 2)$. The reason is that f is not continuous in $[0, 2]$ & I. V. Property is not applicable.

(2) Let $f:\left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \log(2 + x), & 0 \le x < 1 \\ \dfrac{1}{2}(\log 3 - \sin 1), & x = 1 \\ -\sin x, & 1 < x \le \pi/2 \end{cases}$$

Here $f(0)\ f\left(\frac{\pi}{2}\right)=(\log 2)(-1)<0$ but $f(x)$ is never zero in $\left(0, \pi/2\right)$. The reason is $f(x)$ is not continuous in $\left(0, \frac{\pi}{2}\right)$ & so I. V. property is not applicable here.

(3) Let $f:[0,1]\to\mathbb{R}$ be continuous function and assume only rational values in the entire interval. If $f(x)=5$ at $x=\frac{2}{3}$, show that $f(x)=5$ everywhere.

If possible, let there exist $c\in[0,1]$, $c\neq\frac{2}{3}$ and $f(c)=K\in\mathbb{R}$.

If $K\neq 5$, then by I. V. property of continuous function, $f(x)$ must assume every value between K & 5. Between K & 5, there are rational as well as irrational points also. But $f(x)$ assumes rational values only. So $f(x)=5$ throughout $[0,1]$.

(4) Let $f:[0,1]\to\mathbb{R}$ be continuous function and $f(0)=f(1)$. Show that there exists $y\in[0,1]$, such that $|x-y|=\frac{1}{2}$ and $f(x)=f(y)$.

Let us consider the function $g:\left[0,\frac{1}{2}\right]\to\mathbb{R}$ defined by $g(x)=f\left(x+\frac{1}{2}\right)-f(x)$

Continuity of f in $[0,1]\Rightarrow$ continuity of g in $\left[0,\frac{1}{2}\right]$.

$$g(0)\,g\left(\frac{1}{2}\right)=\left(f\left(\frac{1}{2}\right)-f(0)\right)\left(f(1)-f\left(\frac{1}{2}\right)\right)<0$$

By Bolzano's theorem on continuous function, there exists $c\in\left(0,\frac{1}{2}\right)$ such that

$g(c)=0 \Rightarrow f\left(c+\frac{1}{2}\right)=f(c)$ we get $x,\ y \in [0,1],\ \ |x-y|=\frac{1}{2}$ for which

$f(x)=f(y)$.

(5) (Fixed point property) Let $f:[a,b]\to[a,b]$ be continuous function. Show that for some $\xi\in[a,\ b],\ f(\xi)=\xi$ holds.

If $f(a)=a$ or $f(b)=b$, the result is established.

We take $f(a)>a,\ f(b)<b.$ $\left(\text{as } f:[a,b]\to[a,b]\right)$

Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x)=f(x)-x$

Continuity of f in $[a,b]\Rightarrow$ continuity of g in $[a,b]$.

$g(a)\,g(b)=\{f(a)-a\}\{f(b)-b\}<0$. So by Bolzano's theorem, there exists $\xi\in(a,\ b)$ such that $g(\xi)=0$ or $f(\xi)=\xi$.

Notes : (i) The condition of continuity of f can not be dropped

$f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}1-x, & 0\le x\le\frac{1}{2}\\ \frac{1}{2}-\frac{x}{2}, & \frac{1}{2}<x\le 1\end{cases}$

(ii) The result may fail if the interval be not closed and bounded :

(a) $f:[0,1)\to\mathbb{R}$ be defined by $f(x)=\frac{1+x}{2}$

(b) $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x)=x+\frac{1}{x}$

(iii) f must be defined on some interval $(\subset\mathbb{R})$

$f:S\to\mathbb{R}$ be defined by $f(x)=-x$ where $x\in S\left(\equiv[-2,-1]\cup[1,2]\right)$

Also $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+1$

Exercise :

1. Show that $x.2^x = 1$ has a solution in $[0, 1]$.

2. Let $f : [a, b] \to \mathbb{R}$ be continuous function & the equation $f(x) = 0$ have finite number of roots in $[a, b]$ & arranging them in the ascending order, these are

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots x_{n-1} < b$

Prove that in each of (x_{r-1}, x_r) $f(x)$ must have the same sign.

3. If $f : [a, b] \to \mathbb{R}$ be a continuous function & $f(x)$ be always a rational number, then $f(x)$ is a constant function.

4. Examine for the continuity of $f : f(x) = \begin{cases} x^2 - 2x, & \text{when } x \text{ is rational} \\ 3x - 6, & \text{when } x \text{ is irrational} \end{cases}$

5. Does the equation $\sin \text{x} - x + 1 = 0$ have a root ?

6. Does the equation $f(x) = \frac{x^3}{4} - \sin \pi x + 3$ take on the value $2\frac{1}{3}$ within the interval $[-2, 2]$?

7. Show that there exists $x \in \left(0, \frac{\pi}{2}\right)$ such that $x = \cos x$

Theroem (4) : Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Then f is bounded in $[a, b]$ & attains its bounds in $[a, b]$.

Proof : If possible let f be not bounded in $[a, b]$. So corresponding to $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$.

All such x_n's are in $[a, b]$. So we get a sequence $\{x_n\}_n$ in $[a, b]$. Hence $\{x_n\}_n$ is bounded in $[a, b]$.

By Bolzano-Weierstrass theorem on subsequence, there exists a convergent sub sequence $\{x_{r_n}\}_n$ (say) of $\{x_n\}_n$, which converges to $l(\in\mathbb{R})$. This $l\in[a,b]$ as $[a,b]$ is closed. Due to continuity of f, $\{f(x_{r_n})\}_n$ should converge to $f(x)$. Every convergent sequence is necessarily bounded. So $\{f(x_{r_n})\}_n$ is bounded. But by construction, $|f(x_{r_n})|\geq r_n$ & as $\{r_n\}_n$ is strictly increasing sequence of natural numbers, so $r_n \geq n$. Conequenctly, $|f(x_{r_n})|\geq n$

This contradicts $\{f(x_{r_n})\}_n$ is bounded.

This f is bounded on $[a,b]$

Let $M=\sup\limits_{[a,b]} f,\ m=\inf\limits_{[a,b]} f$

If possible, let there be no point x in $[a,b]$ at which $f(x)=M$. So $f(x)<M$ in $[a,b]$.

We construct $\phi:[a,b]\to\mathbb{R}$ defined by $\phi(x)=\dfrac{1}{M-f(x)}$ for all $x\in[a,b]$.

Continuity of f in $[a,b]\Rightarrow$ Continuity of ϕ in $[a,b]$. So ϕ is bounded in $[a,b]$. Let $G>0$ be any number, as large as we please.

As $M=\sup\limits_{[a,b]} f$, there exists at least one point $\xi\in[a,\ b]$ such that

$$f(\xi)>M-\frac{1}{G}$$

$\Rightarrow \dfrac{1}{M-f(\xi)}>G \Rightarrow \phi(\xi)>G$. This contradicts the fact that ϕ is bounded in $[a,b]$

So there exists a point in $[a,b]$ at which $f(x)=M$.

Similarly, it can be shown that there exists a point in $[a, b]$ at which $f(x) = m$ holds.

Corollaries : (i) If $f : [a, b] \to \mathbb{R}$ be a non-constant continuous function, then $f(x)$ assumes every value between its infimum & supremum.

By above theorem, there are points $\xi, \eta \in [a, b]$ such that $f(\xi) = M$, $f(\eta) = m$. By I. V. property of continuous function, applied to f in $[\xi, \eta]$ (or $[\eta, \xi]$) the result follows.

(ii) Let $I (\subset \mathbb{R})$ be a closed and bounded interval & let $f : I \to \mathbb{R}$ be non constant continuous function in I.

Then the set $f(I) = \{f(x) : x \in I\}$ is a closed & bounded interval.

If $M = \sup\limits_{[a, b]} f$, $m = \inf\limits_{[a, b]} f$, then $m \le f(x) \le M$ for all $x \in I$

$\Rightarrow f(I) \subseteq [m, M]$ ...1

Let k be any element of [m, M]. Then by Corollany 1, there exists $c \in I$ such that $f(c) = k \in f(I)$

So $[m, M] \subseteq f(I)$ ... (2)

By (1) and (2), $f(I) = [m, M]$

Note : The result fails if the condition of continuity be dropped.

$f : I \equiv [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f(I)$ is not an interval.

2. The continuous image of an open interval may not be open.

Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2 + 1}$

Here $f(I)=(\frac{1}{2},1]$ which is not open interval

3. The continuous image of an unbounded closed interval may not be closed.

Let $f: I \equiv [0,\infty) \to \mathbb{R}$ be defined by $f(x)=\dfrac{1}{x^2+1}$

Here $f(I)=(0,1]$ which is not closed.

Example (1) : Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$

Let $x_1\, x_2, ...x_n \in [a,b]$. Show that there exists a point ξ in $[a,b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

As f is continuous in $[a,b]$, there are points $\alpha,\beta\in[a,b]$ such that

$f(\alpha)\le f(x)\le f(\beta)$ for all $x\in[a,b]$

$$\Rightarrow nf(\alpha)\le\sum_{i=1}^{n} f(x_i)\le nf(\beta)$$

$$\Rightarrow f(\alpha)\le\frac{1}{n}\sum_{i=1}^{n} f(x_i)\le f(\beta)$$

By I.V. property of continuous functions, there exists $\xi\in[a,\ b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

(2) Let $f, g:\mathbb{R}\to\mathbb{R}$ are continuous on $\mathbb{R}$. Show that

$A=\{x\in\mathbb{R}\,|\,f(x)>g(x)\}$, $B=\{x\in\mathbb{R}\,|\,f(x)<g(x)\}$, $C=\{x\in\mathbb{R}\,|\,f(x)\ne g(x)\}$

are open sets in $\mathbb{R}$ whereas $D=\{x\in\mathbb{R}\,|\,f(x)\equiv g(x)\}$ is a closed set in $\mathbb{R}$.

Let $\phi(x)=f(x)-g(x)$, $x\in\mathbb{R}$. As f, g are continuous, so $\phi(x)$ is continuous in $\mathbb{R}$.

(i) $A=\{x\in\mathbb{R}\,|\,\phi(x)>0\}$

Case I : If $\phi(x) \le 0$ in $\mathbb{R}$. Then $A = \phi$ & So A is open set in $\mathbb{R}$.

Case II : If $\phi(x) > 0$ in $\mathbb{R}$. So $A \equiv \mathbb{R}$ & $\mathbb{R}$ being open set, A is open set in $\mathbb{R}$.

Case III : Let $A \subset \mathbb{R}$.

Let $p \in A$, So $\phi(p) > 0$ & by neighbourhood property of continuous function, there exists $\delta > 0$ such that $x \in \mathbb{N}(p, \delta) \Rightarrow \phi(x) > 0$

Thus $\mathbb{N}(p, \delta) \subset A$ & so p is interior point of A. This is true for all $p \in A$. Consequently A is open set in $\mathbb{R}$.

Arguing in a similar way, B is open set in $\mathbb{R}$.

Set $C = A \bigcup B$ so C is union of two open sets in $\mathbb{R}$ & so C is open set in $\mathbb{R}$. D is the complement of open set C & hence D is closed.

(3) Let $I\,(\subset \mathbb{R})$ be a given open interval. Let $f : I \to \mathbb{R}$ be continuous on *I*. Let α be an arbitrary real constant.

Then $I[f < \alpha] \equiv \{x \in I : f(x) < \alpha\}$ and $J[f > \alpha] = \{x \in I : f(x) > \alpha\}$ are open sets.

If $f(x) = \alpha$ for all *x*, *I* and *J* are void sets & so are open sets in $\mathbb{R}$.

Next let $I\left[f(x) < \alpha\right] \neq \phi$

So there exists $p \in I$ i.e. $f(p) < \alpha$. Let $0 < \varepsilon < \frac{1}{2}\left[\alpha - f(p)\right]$.

Continuity of f at $p \Rightarrow$ corresponding to above chosen ε, there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $x \in N(p, \delta) \bigcap I$ ...(1)

By hypothesis, *I* is open set & *p* is interior point of *I*. By definition of interior point, there exists $r, 0 < r < \delta$, such that $N(p, r) \subset I$ ...(2)

By (1) & (2), $f(x) < f(p) + \varepsilon < f(p) + \frac{1}{2}(\alpha - f(p))$

$\Rightarrow f(x) < \alpha$ where $x \in N(p,r) \Rightarrow N(p,r) \subset I[f < \alpha]$

$\Rightarrow I[f < \alpha]$ is an open set in $\mathbb{R}$.

Following similar argument, $J[f > \alpha]$ is also open set in $\mathbb{R}$.

(iv) Let $f, g : [0,1] \to [0,\infty)$ be continuous functions satisfying

$$\sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$$

Show that there exists $c \in [0,1]$ such that $f(c) = g(c)$

Continuity of f, in $[0,1] \Rightarrow$ boundedness & their attainment of bounds in [0,1].

Let $M = \sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$

If both f & g attain M at the same point, the result is established.

Otherwise : Let $f(\xi) = M$ and $g(\eta) = M$ for some $\xi, \eta \in [0,1]$,

So $g(\xi) < M$, $f(\eta) < M$.

We construct $h : [0,1] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. Then h is continuous in [0, 1] & by above $h(\xi) = f(\xi) - g(\xi) = M - g(\xi) > 0$ and

$h(\eta) = f(\eta) - g(\eta) = f(\eta) - M < 0$. So $h(\xi)\,h(\eta) < 0$.

$\Rightarrow$ By Bolzano's theorem, there exists $c \in (\xi, \eta) \subset (0,1)$ such that $h(c) = 0$

or in other words, $f(c) = g(c)$.

Continuity of Inverse function :

Theorem : Let $f : [a,b] \to \mathbb{R}$ be strictly monotonic and continuous on the closed and bounded interval $[a,b]$. Then there exists an inverse function $g : f[a,b] \to \mathbb{R}$ such that (i) g is strictly monotonic in $f[a,b]$ and (ii) g is continuous in $f[a,b]$

Proof : Let f be strictly increasing in [a, b] ..... (1)

Continuity of f in $[a, b] \Rightarrow$ boundedness of f in $[a, b]$ & attainment of bounds in [a, b]. So $\sup_{[a,b]} f = f(b)$, $\inf_{[a,b]} f = f(a)$.

Therefore, here $f([a, b]) = [f(a), f(b)]$... (1)

As f is strictly increasing, so for any distinct pair of points $x_1, x_2 \in [a, b]$, $f(x_1) \neq f(x_2) <\Rightarrow x_1 \neq x_2$. So f is injective. ... (2)

Consequently by (1) & (2) f is bijective. So $f^{-1} = g$ exists where $g : f([a, b]) \to [a, b]$. where $f(x) = y \Rightarrow x = g(y), x \in [a, b], y \in f[a, b]$

Let $y_1, y_2 \in f[(a,b)]$. So there are $x_1, x_2 \in [a, b]$ such that

$y_1 = f(x_1), y_2 = f(x_2)$

f being strictly increasing in $[a, b], y_1 < y_2 \Rightarrow x_1 < x_2$

As a result, $y_1 < y_2 \Rightarrow g(y_1) < g(y_2) \Rightarrow g$ is strictly increasing in $f([a, b])$.

Let y_0 be any point between $f(a)$ and $f(b)$ & x_0 be the corresponding value of x.

Let $\varepsilon > 0$ be arbitrary number such that $x_0 - \varepsilon$, $x_0 + \varepsilon$ are in $[a, b]$. Let $g(y_0 - \eta_1) = x_0 - \varepsilon$ and $g(y_0 + \eta_2) = x_0 + \varepsilon$ such that $\eta_1, \eta_2 > 0$ exist by above.

Let η be such that $0 < \eta < \min\{\eta_1, \eta_2\}$. Then

$|x - x_0| < \varepsilon$ whenever $|y - y_0| < \eta, \eta$ depends on ε.

So $g(y)$ is continuous at y_0. and this is true for all $y_0 \in [f(a), f(b)]$

Hence the result follows :

Note (i) Continuity of Inverse function is preserved only when the domain is closed and bounded.

Let $A = [0, 1) \cup [2, 3]$ and $f : A \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases} x, & x\in[0,1) \\ x-1, & x\in[2,3] \end{cases}$$

$$f^{-1}(x)=\begin{cases} x, & x\in[0,1) \\ x+1, & x\in[1,2] \end{cases} \Rightarrow f^{-1} \text{ is discontinuous at } x = 1.$$

Theorem : If $f:[a,b]\to\mathbb{R}$ be continuous, injective function, then *f* is strictly monotone function.

If possible, let f be not strictly monotone function in $[a,b]$ though f is continuous & injective in $[a,b]$. So we say that there are three points $p,q,r\in[a,b]$ where $p<q<r$ nonetheless $f(q)$ does not lie between $f(p)$ and $f(r)$. Consequently, either $f(r)$ lies between $f(p)$ and $f(q)$ or $f(p)$ lies between $f(q)$ and $f(r)$. For definiteness, let $f(p)$ be between $f(q)$ and $f(r)$.

By hypothesis, *f* is continuous in $[q,r]\subset[a,b]$. By I. V. property, there exists $s\in(q,r)$ such that $f(s)=f(p)$.

So $p<s$ but $f(p)=f(s)$. This contradicts the injectivity of *f*.

Similarly if we assume that $f(r)$ lies between $f(p)$ and $f(q)$, we would arrive at same type of contradiction. So *f* is strictly monotone.

Corollary : A continuous function $f:[a,b]\to\mathbb{R}$ is injective if and only if *f* is strictly monotone in $[a,b]$.

Example : Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(f(x))=f^2(x)=-x$ for all $x\in\mathbb{R}$.

Then *f* can not be continuous.

First we propose to show that *f* is injective.

$f(x_1)=f(x_2)\Rightarrow f^2(x_1)=f^2(x_2)\Rightarrow -x_1=-x_2\Rightarrow x_1=x_2$

If f be continuous then it would be either strictly increasing or strictly decreasing. In both cases, f^2 would be increasing.

For if $p<q$, then $f(p)<f(q)$ (in case f is increasing) & $f(p)>f(q)$ (in case f is decreasing). In the first case, $f(f(p))<f(f(q))$ & in the second case $f(f(p))<f(f(q))$. So in any case, $f^2(p)<f^2(q)$

$\Rightarrow -p<-q$ absurd as $p<q$

So f can not be continuous.

Exercise :

1. Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}2x-1, \text{ if } x\in(0,1)\\ 0, \quad \text{if } x=0 \text{ or } 1\end{cases}$

Choose the correct answer :

(a) f is unbounded function (b) f is bounded function and attains its bounds there in (c) f is bounded function but does not attain its bounds.

2. Let $f(x)=\begin{cases}2^x+1, & \text{for } -1\le x<0\\ 2^x, & \text{for } x=0\\ 2^x-1, & \text{for } 0<x\le 1\end{cases}$

Choose the correct answer :

(a) f is bounded in $[-1,1]$

(b) f is unbounded in $[-1,1]$

(c) f is continuous in $[-1,1]$

(d) f has jump discontinuity in $[-1,1]$

13.10 Summary

In this unit, we have defined the terms continuity and discontinuity and given various examples. We have studied various types of discontinuities and their properties. We have explained the most important properties of functions continuous in a closed and bounded interval [a, b], such as, Intermediate value property, Fixed point property. We have also shown the relation between continuity and monotonicity. We have further study the maximum-minimum property. We have introduced the notion of uniform continuity and shown that in a closed and bounded interval [a, b] this concept is same with the concept of continuity. We also studied the uniform continuity on an open interval (a, b), and give an important non-uniform continuity criteria. We have also shown that every uniformly continuous function maps cauchy sequence into a cauchy sequence.

13.11 Exercise

1. Prove or disprove : If $f:S\to\mathbb{R},\ g:T\to\mathbb{R}\,(S,T\subset\mathbb{R})$ are uniformly continuous and $f(S)\subset T$, then the composite function $g\circ f:S\to\mathbb{R}$ is uniformly continuous on S.

2. Show that $e^x\cos\frac{1}{x}$ is not uniformly continuous on $(0,1)$.

(Hints : You can consider the sequences $\left\{\frac{1}{2n\pi}\right\}_n$ & $\left\{\frac{1}{(2n+1)\pi}\right\}_n$)

3. Let $f(x)=\sqrt{x},\ x\in[0,2]$

Choose the correct answer(s) :

(i) f is Lipschitz function in $[0,2]$

(ii) f is not Lipschitz function in $[0,2]$

(iii) f is uniformly continuous in $[0, 2]$

(iv) f is not uniformly continuous in $[0, 2]$

4. Correct or justify : $x \sin^2 x$ is uniformly continuous on $\mathbb{R}$.

5. Let $f:[0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \cos \dfrac{\pi}{2x}, & x \neq 0 \\ 0, x = 0 \end{cases}$

Examine whether f is uniformly continuous on $[0, 1]$.

6. Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a, b]$ and let the equation $f(x) = 0$ have finite number of roots in $[a, b]$. Arrange them in the ascending order.

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots . < x_n < b$

Prove that in each of the intervals $(a_1, x_1), (x_1, x_2), (x_{r-1}, x_r)(x_n, b)$ the function *f(x)* retains the same sign.

Unit-14 □ Differentiation of Functions

Structure

13.0 Objectives

This unit gives

- The concept of differentiation of a function
- Algebric operation of differentiable function
- Rolle's theorem and some application
- Expansion of a differentiable function in series form

13.1 Introduction

The problem of finding tangent lines and the seemingly unrelated problem of finding maximum or minimum values were first seen to have a connection by Fermat in the 1630s. And the relation between tangent lines to curve and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Newton's theory of 'fluxions' which was based on an intuitive idea of limit. But the vital observation, made by Newton and, independently, by Gottfried Leibniz in the 1680s, was that areas under curves could be calculated by reversing the differentiation process. In this chapter we will develop the theory of differentiation.

13.2 Differentiation of Functions

The derivative : Let $f : D \to \mathbb{R}(D \subset \mathbb{R})$ be a given function, c be a point of D as well as an accumulation point of D. So the function $x \to \dfrac{f(x)-f(c)}{x-c}$ is defined on $D-\{c\}$.

If $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists finitely and be $=L(\in \mathbb{R})$ then we say that f is derivable at c, $f'(c)$ exists and = L

If $f:[a,b]\to\mathbb{R}$ then $f'(a)$ is in fact $Rf'(a)=\lim\limits_{x\to a+}\dfrac{f(x)-f(a)}{x-a}$ provided the limit exists and $f'(b)$ is in fact $Lf'(b)=\lim\limits_{x\to b-}\dfrac{f(x)-f(b)}{x-b}$, provided it exists.

If c be interior point of [a, b], then $f'(c)$ exists provided

$\lim\limits_{x\to c-0}\dfrac{f(x)-f(c)}{x-c}=Lf'(c)$ exists, $\lim\limits_{x\to c+0}\dfrac{f(x)-f(c)}{x-c}=Rf'(c)$ exists and $Lf'(c)=Rf'(c)(\in\mathbb{R})$

Notes : If $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ does not exist finitely, we say that f' does not exist at c.

2. $f:D\to\mathbb{R}$ is said to be differentiable on a set $D_0\subset D$, if the restriction of f to D_0 is differentiable at every point of D_0.

Result : 1. Let $f:D\to\mathbb{R}$ be differentiable at $p\in D\cap D'$, then there exists $\delta>0$ and a constant $M>0$ such that

$|f(x)-f(p)|\le M|x-p|$ for every $x\in D\cap N(p,\delta)$

Proof : By hypothesis $\lim\limits_{x\to p}\frac{f(x)-f(p)}{x-p}$ exists & $=f'(p)$

Let $\varepsilon>0$ be given, corresponding to this ε, there exists $\delta>0$ such that $\left|\frac{f(x)-f(p)}{x-p}-f'(p)\right|<\varepsilon$ whenever $x\in N'(p,\delta)\cap D$

$\Rightarrow\left|\frac{f(x)-f(p)}{x-p}\right|<\varepsilon+|f'(p)|=M(\text{say})$, M is a positive constant,

Hence $|f(x)-f(p)|<M|x-p|,\ x\in N(p,\delta)\cap D$

Note : Instead of ε in above, you can choose any fixed positive number

Corollary : If we take $\delta=\frac{\varepsilon}{M}$, then from above result

$|f(x)-f(p)|<M\frac{\varepsilon}{M}$ wherever $|x-p|<\frac{\varepsilon}{M}$

$\Rightarrow f$ is continuous at p

So derivability at a point $\Rightarrow$ continuity at that point

Note : converse is not true.

$$f(x)=\begin{cases} x\sin\frac{1}{x},\ x\neq 0\\ 0,\ x=0\end{cases}$$

As $\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist so f' does not exist at $x=0$.

But f is continuous at $x=0$.

Examples :

1. Let $f:(a,b)\to\mathbb{R}$ be differentiable at $x\in(a,b)$. Let $\{\alpha_n\}_n$ & $\{\beta_n\}_n$ be sequences such that $a<\alpha_n<x<\beta_n<b,\ \alpha_n\to x, \beta_n\to x$

Then show that $\lim\limits_{n\to\infty}\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(x)$

Let $\lambda_n = \dfrac{\beta_n - x}{\beta_n - \alpha_n}$. Then $0 < \lambda_n < 1$

$$\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} - f'(x) = \lambda_n \left\{ \frac{f(\beta_n)-f(x)}{\beta_n - x} - f'(x) \right\}$$

$$+(1-\lambda_n)\left\{ \frac{f(\alpha_n)-f(x)}{\alpha_n - x} - f'(x) \right\}$$

By hypothesis, $f'(x)$ exists & so the expressions within the brackets both tend to zero as $n \to \infty, \{\lambda_n\}_n$ & $\{1-\lambda_n\}_n$ are both bounded.

Hence $\lim\limits_{n\to\infty} \dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$ exists & is equal to $f'(x)$.

Note : If $x < \alpha_n < \beta_n$, the result may fail.

Let $\beta_n = \dfrac{1}{n}(n \in \mathbb{N})$ and let $\{\alpha_n\}_n$ be a sequence such that $\beta_{n+1} < \alpha_n < \beta_n$

Let $f:[-1,1] \to \mathbb{R}$ be a piecewise linear function such that $f\left(\dfrac{1}{n}\right) = \dfrac{1}{n^2}, f(\alpha_n) = 0, f(x) = 0$ for $-1 \le x \le 0$

We choose α_n nearer to β_n. Let $\alpha_n = \dfrac{1}{2}\left(\dfrac{1}{n} + \dfrac{1}{n+1}\right)$

Then $\lim\limits_{n\to\infty} \dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} = \lim\limits_{n\to\infty} \dfrac{\dfrac{1}{n^2}-0}{\dfrac{1}{n} - \dfrac{1}{2}\left(\dfrac{1}{n}+\dfrac{1}{n+1}\right)} = \dfrac{2(n+1)}{n} > 1$

But $f'(0) = 0$ & so the conclusion mentioned in the problem, fails.

2. If the function $xf(x)$ has a derivative at a given point $x_0 \neq 0$ and if $f(x)$ is continuous there, show that $f(x)$ has a derivative there.

$$\frac{xf(x)-x_0f(x_0)}{x-x_0}-f(x)=x_0\left\{\frac{f(x)-f(x_0)}{x-x_0}\right\}\text{.......}(1)$$

By hypothesis, $xf(x)$ has a derivative at $x_0 \neq 0$ and $f(x)$ is continuous at x_0.

So as $x \to x_0$, L.H.S. of (1) $\to \frac{d}{dx}(x\,f(x))\big|_{x=x_0} - f(x_0)$

Hence $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ exists and is equal to

$$\frac{1}{x_0}\left[\frac{d}{dx}(xf(x))\Big|_{x=x_0}-f(x_0)\right]$$

3. $f:(-1,1)\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x^{\alpha}\sin\frac{1}{x^{\beta}},\ x\neq 0 \\ 0,\ x=0\end{cases}$

Show that (i) if $\alpha=1, \beta>0, f'$ does not exist at $x = 0$ but if $\alpha>1, \beta>0, f'$ exists at $x = 0$.

(ii) if $0<\beta<\alpha-1, f'$ is continuous at 0

(iii) if $0<\alpha-1\leq\beta, f'$ is discontinuous at 0.

$$\frac{f(x)-f(0)}{x-0}=x^{\alpha-1}\sin\frac{1}{x^{\beta}},\ x\neq 0$$

(i) if $\alpha=1, \beta>0, x^{\alpha-1}\sin\frac{1}{x^{\beta}}=\sin\frac{1}{x^{\beta}}$. As $\lim\limits_{x\to 0}\sin\frac{1}{x^{\beta}}(\beta>0)$ does not exist.

so in this case, f' does not exist at $x = 0$.

(ii) Let $\alpha>1, \beta>0$

Let $\varepsilon>0$ be any number.

$$\left|\frac{f(x)-f(0)}{x-0}-0\right|=\left|x^{\alpha-1}\sin\frac{1}{x^{\beta}}\right|\le|x|^{\alpha-1}<\varepsilon \text{ wherever } |x-0|<\delta=\varepsilon^{1/(\alpha-1)}$$

So $\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}$ exists i.e. $f'(0)$ exists.

So for $0<\beta<\alpha-1, f'(0)=0$

if $x\ne0, f'(x)=\alpha x^{\alpha-1}\sin\dfrac{1}{x^{\beta}}-\beta x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}$

As $\alpha>1,\beta>0, \lim\limits_{x\to0}x^{\alpha-1}\sin\dfrac{1}{x^{\beta}}=0$

and as $\alpha-1-\beta>0, \lim\limits_{x\to0}x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}=0$

Hence $\lim\limits_{x\to0}f'(x)=f'(0)$ & f' is continuous at $x=0$.

(iii) Let $0<\alpha-1\le\beta$

Then $\alpha-1-\beta\le0$, $\lim\limits_{x\to0}x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}$ does not exist

Hence f' is discontinuous at 0. (Nature of this discontinuity is that of second kind)

4. Consider a polynomial $f(x)$ with real coefficients having the property that $f[g(x)]=g[f(x)]$ for every polynomial $f(x)$ with real coefficients.

Show that $f(x)=x$

Let us take $g(x)=x+h, h\in\mathbb{R}$

So $f(x+h)=f(x)+h$ as $f[g(x)]=g[f(x)]$ by hypothesis.

$\Rightarrow\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}=1\Rightarrow f(x)=x+\lambda$ where λ is real constant.

Let $g(x)=0$, then $f[g(0)]=g[f(0)]\Rightarrow0=0+\lambda\Rightarrow f(x)=x$

5. If $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $c\in\mathbb{R}$, show that

$$f'(c)=\lim_{n\to\infty}\left\{n\left[f\left(c+\frac{1}{n}\right)-f(c)\right]\right\}$$

Hence show that if *f* is the derivative of a function *g*, then *f* is the limit of a sequence of continuous functions.

As $f'(c)$ exists, so $\lim_{h\to 0}\dfrac{f(c+h)-f(c)}{h}=f'(c)$

$\lim_{n\to\infty}\dfrac{1}{n}=0$, so replacing h by $\dfrac{1}{n}$, $\lim_{n\to\infty}\dfrac{f\left(c+\frac{1}{n}\right)-f(c)}{1/n}=f'(c)$

So the first part follows.

By hypothesis, *g* is derivable & so *g* is continuous function.

We define $g_n(x)=g\left(x+\dfrac{1}{n}\right)$ & so these g_n's are continuous.

Also $n[g_n-g]$ are continuous.

By hypothesis, *f* is derivative of *g*, it follows that

$$f=\lim_{n\to\infty}n\{g_n-g\}=\lim_{n\to\infty}n\left\{g\left(x+\frac{1}{n}\right)-g(x)\right\}$$

So *f* is the limit of a sequence of continuous functions.

Sign of the derivative at a point.

Theorem : Let $f:I\to\mathbb{R}$ and let *c* be an interior point of interval *I*.

Let $f'(c)$ exist and $f'(c)\neq 0$

(a) If $f'(c)>0$, there exists a neighbourhood of '*c*' in which *f* is increasing function. In other words, there exists $\delta>0$ such that

$f(x)>f(c)$ for all $x\in(c,c+\delta)\cap I$ and

$f(x)<f(c)$ for all *x* in $(c-\delta,c)\cap I$

(b) If $f'(c)<0$, there exists a neighbourhood of '*c*' in which *f* is decreasing function. In other words, there exists $\delta>0$ such that

$f(x)<f(c)$ for all $x\in(c,c+\delta)\cap I$

$f(x)>f(c)$ for all $x\in(c-\delta,c)\cap I$

Proof : Let $0<\varepsilon<\frac{|f'(c)|}{2}$ be any number.

Corresponding to such ε, there exists $\delta>0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\frac{|f'(c)|}{2} \text{ whenever } 0<|x-c|<\delta(x\in I)$$

Case I : Let $f'(c)>0$. Then $\frac{f'(c)}{2}<\frac{f(x)-f(c)}{x-c}<\frac{3f'(c)}{2}$

whenever $c-\delta<x<c,\ c<x<c+\delta$

If $c-\delta<x<c,\ f(x)-f(c)<\frac{(x-c)f'(c)}{2}<0\Rightarrow f(x)<f(c)$ in $c-\delta<x<c$

If $c<x<c+\delta, f(x)-f(c)>\frac{(x-c)f'(c)}{2}>0\Rightarrow f(x)>f(c)$ in $c<x<c+\delta$

Consequently, *f* is increasing in the neighbourhood of *c*.

Case II : Let $f'(c)<0$. Then $\varepsilon=-\frac{f'(c)}{2}$, so we have

$$\frac{3f'(c)}{2}<\frac{f(x)-f(c)}{x-c}<\frac{f'(c)}{2}, 0<|x-c|<\delta.$$

If $c-\delta<x<c, f(x)-f(c)>\frac{(x-c)f'(c)}{2}>0$

$\Rightarrow f(x)>f(c)$ in $c-\delta<x<c$

If $c < x < c+\delta, f(x) - f(c) < \dfrac{3(x-c)f'(c)}{2} < 0$

$\Rightarrow f(x) < f(c)$ in $c < x < c+\delta$

So f is decreasing in the $\delta-$ neighbourhood of c.

Note : If $f'(c) = 0$, no conclusion can be drawn.

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$

In $0 < x < 0+\delta$ & in $0-\delta < x < 0, f(x) - f(0) < 0$

So f is increasing in $N'(0, \delta) \cap D_f$

Let $f(x) = x^2$. Then $f'(x) = 2x$ & $f'(0) = 0$

$f(x) - f(0) > 0$ in both $0-\delta < x < 0$ & $0 < x < 0+\delta$

$f(x)$ is neither increasing nor decreasing in any δ-neighbourhood of 0.

13.3 Algebra of differentiable functions

Let f and g are two functions differentiable at $c\left(\in D_f \cap D_g\right)$, then

(i) $\alpha f(x)$ is differentiable at c where $\alpha \in \mathbb{R}$

(ii) $f \pm g$ are differentiable at c & $(f \pm g)'(c) = f'(c) \pm g'(c)$

(iii) $f\,g$ is differentiable at c & $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$

(iv) if $g(c) \neq 0, \dfrac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\{g(c)\}^2}$$

(v) $|f|$ is differentiable at c, $f(c) \neq 0$

Proof : Deduction of (i) and (ii) are simple and follow straight way from the definition of derivative.

(iii) $\dfrac{(fg)(x)-(fg)c}{x-c}=f(x)\left\{\dfrac{g(x)-g(c)}{x-c}\right\}+g(c)\left\{\dfrac{f(x)-f(c)}{x-c}\right\}$

Existence of f', g' at $c \Rightarrow \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c)$, $\lim\limits_{x\to c}\dfrac{g(x)-g(c)}{x-c}=g'(c)$

Due to continuity of *f*, *g* at *c*,

$$\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}=f(c)g'(c)+g(c)f'(c)$$

(iv) Given $g(c)\neq 0$, due to continuity of g at c, there exists *nbd* of *c*

or, interval *I* having c as its interior point such that $g(x)\neq 0$ in *I*

Let $x\in D\left(\dfrac{f}{g}\right)\cap I$

$$\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)c}{x-c}=\frac{1}{g(x)g(c)}\left\{g(c).\frac{f(x)-f(c)}{x-c}-f(c).\frac{g(x)-g(c)}{x-c}\right\}$$

By hypothesis, $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c), \lim\limits_{x\to c}\dfrac{g(x)-g(c)}{x-c}=g'(c)$

Due to continuity of g at c, $\lim\limits_{x\to c}g(x)=g(c)$

So $\lim\limits_{x\to c}\dfrac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(c)}{x-c}=\dfrac{1}{\{g(c)\}^2}\left[f'(c)g(c)-f(c)g'(c)\right]$

(v) $\left|\dfrac{|f|(x)-|f|(c)}{x-c}\right|\leq\left|\dfrac{f(x)-f(c)}{x-c}\right|$

As the limit of RHS as $x \to c$ exists finitely, so the limit $\lim\limits_{x \to c} \frac{|f|(x) - |f|(c)}{x - c}$ exists finitely.

Note The condition $f(c) \neq 0$ is required. otherwise the result may fail.

For example, $f(x) = x$ and $c = 0$.

Derivative of composite function (Chain Rule)

Theorem : If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)).f'(c)$$

Note that c is interior point of domain of $g \circ f$.

Let us consider the function $h : D_g \to \mathbb{R}$ as follows :

$$h(y) = \begin{cases} \dfrac{g(y) - g(f(c))}{y - f(c)}, & y \neq f(c) \\ g'(f(c)), & y = f(c) \end{cases}$$

Then $\lim\limits_{y \to f(c)} h(y) = g'(f(c)) = h(f(c))$ & so h is continuous at $f(c)$.

Again $g(y) - g(f(c)) = (y - f(c))h(y)$ for all $y \in D_g$ (by construction of h)

Hence for $x \in D_{g \circ f}$

$$(g \circ f)(x) - (g \circ f)(c) = h(f(x))(f(x) - f(c))$$

$\Rightarrow$ for $x \in D_{g \circ f}$ & $x \neq c$

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = h(f(x)) . \frac{f(x) - f(c)}{x - c}$$

Continuity of f at c & continuity of h at $f(c) \Rightarrow$ $h \circ f$ is continuous at c.

As $x \to c, RHS \to h(f(c))$

So $\lim\limits_{x\to c}\dfrac{(g\circ f)(x)-(g\circ f)(c)}{x-c}$ exists and is $h(f(c)).f'(c)$

$\Rightarrow g\circ f$ is differentiable at c and

$(g\circ f)'(c)=g'(f(c)).f'(c)$

Derivative of inverse function

Let f be strictly monotone and continuous in an interval $I(\subset\mathbb{R})$

and let x_0 be an interior point of I at which f has a derivative $f'(x_0)\neq 0$.

Then f^{-1} has a derivative at this point $y_0=f(x_0)$, equal to $1/f'(x_0)$

Proof : Here domain of f^{-1} is an interval J (say).

By hypothesis, x_0 is an interior point of I. By definition of interior point, there exists points $p, q\in I$ such that $p<x_0<q$ and then $f(x_0)$ is interior point of the closed interval J_1 = [$f(p)$, $f(q)$] as f is strictly monotone.

f is continuous on $[p, q]\subset I$, the interval $J_1\subset J$ (by I.V. property of continuous function), so $y_0=f(x_0)$ is interior point of J.

Now $\lim\limits_{y\to y_0}\left\{\dfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\right\}=\lim\limits_{y\to y_0}\left\{\dfrac{f^{-1}(y)-f^{-1}(y_0)}{f(f^{-1}(y))-f(f^{-1}(y_0))}\right\}$.

Due to continuity of f^{-1}, f^{-1} is continuous at y_0, so that

$\lim\limits_{y\to y_0} f^{-1}(y)=f^{-1}(y_0)=x_0$

Following substitution rule for composite function

$$\lim_{y\to y_0}\left\{\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\right\}=\lim_{x\to x_0}\left\{\frac{x-x_0}{f(x)-f(x_0)}\right\}$$

since $f'(x_0)\neq 0$, we get

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

consequenctly, $\lim_{y \to y_0} \left\{ \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \right\} = \frac{1}{f'(x_0)}$

Note : Alternative proofs of the last two results follow from concept of differentiability, discussed subsequently.

Diffentiability and differential

$f(x)$ is said to be differentiable at a point of its domain if

$f(x + \Delta x) - f(x) = A.\Delta x + \varepsilon.\Delta x$ where A is independent of Δx and $\varepsilon \to 0$ as $\Delta x \to 0$.

Let f be differentiable at x

From above definition, $\frac{f(x + \Delta x) - f(x)}{\Delta x} = A + \varepsilon$

Taking $\Delta x \to 0$, $RHS \to A$ so $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists & $\left(= f'(x)\right) = A$

So differentiabity at a point of its domain $\Rightarrow$ existence of first order derivative at that point.

Converse let $f'(x)$ exist. so $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists & $= f'(x)$

Let $\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x) = \varepsilon$ & so $\varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow f(x + \Delta x) - f(x) = f'(x)\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow f$ is differentiable at x and hence differentiability $\Leftrightarrow$ existence of derivative at that point.

Note : 1. This result is of importance in the sense that the result differes for functions $f : \mathbb{R}^2 \to \mathbb{R}$

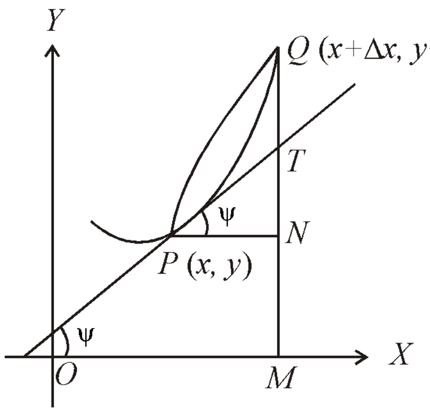
2. If $y = f(x)$ & $y + \Delta y = f(x + \Delta x)$, then

$\Delta y = f(x + \Delta x) - f(x) = f'(x).\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

Δy is increment of y for the increment Δx of x.

$f'(x)$. Δx is known as the differential of y, denoted by dy.

This $dy \neq \Delta y$ but $dx = \Delta x$ (taking $f(x) = x$, it is evident)



$P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ are two neighbouring points on the curve.

$$\tan\psi = \frac{dy}{dx} = f'(x) = \frac{TN}{PN} \Rightarrow TN = f'(x)\Delta x$$

$\Rightarrow dy = TN$

So $dy = TN$ but $\Delta y = PN$

& So $dy \neq \Delta y$

Alternative proof for diffentiability of composite function and Chain Rule.

Let f be differentiable at $x(\in D_f)$ & g be differentiable at $f(x)(\in D_g)$.

Here we assume that the composite function $g \circ f$ can be defined in the sense that $(g \circ f)(x) = g(f(x))$, $x \in D_f$.

Here $\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$ (taking $y = f(x)$.......(1)

Taking $x = g(t)$

$\Delta x = g(t+\Delta t) - g(t) = g'(t)\Delta t + \eta.\ \Delta t$ where $\eta \to 0$ as $\Delta t \to 0$ ....(1)

$\Delta y = \left(f'(x)+\varepsilon\right)\left(g'(t)\Delta t + \eta\Delta t\right)$ where $\eta \to 0$ as $\Delta t \to 0$ ....(2)

$\Delta y = \left(f'(x)+\varepsilon\right)\left(g'(t)\Delta t + \eta\Delta t\right)$

$= f'(x)g'(t)\Delta t + \left(f'(x).\eta + \varepsilon.g'(t) + \varepsilon\eta\right)\Delta t$ (3)

As $\Delta t \to 0, \Rightarrow \Delta x \to 0$ due to the continuity of g.

As $\Delta x \to 0, \varepsilon \to 0$. Consequently, $f'(x).\eta + \varepsilon.\ j'(t) + \varepsilon\eta \to 0$ as $\Delta t \to 0$

Recalling (3), $\Delta y = f'(x)g'(t)dt + \tau.\Delta t\left(\tau = f'(x).\ \eta + \varepsilon.\ g'(t) + \varepsilon\eta\right)$

where $\tau \to 0$ as $\Delta t \to 0$

$\Rightarrow y$ is a differentiable function of t and $\dfrac{dy}{dt} = \dfrac{dy}{dx}.\dfrac{dx}{dt}$

Note Similarly the differentiability of inverse function can be discussed.

Theorem (Darboux Theorem) : Let $f:[a,b] \to \mathbb{R}$ be derivable in the closed and bounded interval $[a, b]$ and $f'(a)f'(b) < 0$. Then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.

Proof : Let $f'(a) > 0,\ f'(b) < 0$

f, being derivable in $[a, b]$ is continuous in $[a, b]$. So f is bounded in $[a, b]$ and attains its bounds in $[a, b]$. So there are points $\mathrm{c, d} \in [\mathrm{a,b}]$ such that $\sup\limits_{[a,b]} f = M = f(c)$ and $\inf\limits_{[a,b]} f = m = f(d)$.

As $f'(a) > 0$, So $f(x)$ is increasing in some neighbourhood of a and hence there exists $\delta > 0$ that $f(x) > f(a)$ in $a < x < a + \delta$.

If $c = a$, then $f(x) > M$ in $a < x < a + \delta$ which is absurd. So $c \neq a$.

As $f'(b) < 0,\ f(x)$ is decreasing in some neighbourhood of b & so there exists $\eta > 0$ such that $f(x) > f(b)$ in $b - \eta < x < b$.

If $c = b$, then $f(x) > M$ in $b - \eta < x < b$ which is also absurd. So $c \neq b$.

So $c \in (a, b)$. By hypothesis, $f'(c)$ exists. We propose to show that $f'(c) = 0$.

If possible, let $f'(c) > 0$. Then there exists $\delta' > 0$ such that $f(x) > f(c)(= M)$ in $(c, c + \delta')$ $(\subset [a, b])$ this is absurd, so $f'(c) \not> 0$.

If possible, let $f'(c) < 0$. Then there exists $\eta' > 0$ such that $f(x) > f(c)(= M)$ in $(c - \eta', c)(\subset [a, b])$ & this is absurd, so $f'(c) \not< 0$. Hence $f'(c) = 0$.

Corollaries (1) : Let $f : [a, b] \to \mathbb{R}$ be derivable in [a, b] & $f'(a) \neq f'(b)$. If k be any number between $f'(a)$ and $f'(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = k$.

Let us construct $\varphi : [a, b] \to \mathbb{R}$ defined by $\varphi(x) = f(x) - kx$.

Derivability of f in $[a, b] \Rightarrow$ derivability of φ in $[a, b]$, $\varphi'(x) = f'(x) - k$

$\varphi'(a)\varphi'(b) = (f'(a) - k)(f'(b) - k) < 0$

So by Darboux Theorem, there exists $c \in (a, b)$ for which $\phi'(c) = 0$ i.e. $f'(c) = k$.

(2) If f be derivable in a closed and bounded interval I, then the range set of f' on I is either a singleton or an interval.

If two distinct members $p_1, p_2 \in J$, there exists distinct elements $x_1, x_2 \in I$ such that $f'(x_i) = p_i$ for i = 1, 2. Let $x_1 < x_2$. So $[x_1, x_2] \subset I$.

If $p_1 < p < p_2$, by Darboux's theorem on derivative, there exists $c \in (x_1, x_2)$

such that $f'(c) = p$. So $p \in J$. But p is arbitrary point between p_1 & p_2. This shows that if $p_1, p_2, \in J$, then every element between p_1 & p_2 belongs to J. So J is an interval in $\mathbb{R}$.

(3) Let $f:[a, b]\to \mathbb{R}$ be derivable on [a, b]. Then f' can not have any jump discontinuity on [a, b]

Let $c\in(a,b)$. We propose to show that

(i) if $\lim_{x\to c-} f'(x)$ exists, then it is $f'(c)$, $c\in(a,b]$

(ii) if $\lim_{x\to d+} f'(x)$ exists, then it is $f'(d)$, $d\in[a,b)$

(i) Let $a<c\le b$ & $\lim_{x\to c-} f'(x)=l(\in\mathbb{R})$. We have to show that $l=f'(c)$

Let $l<f'(c)$. Let $0<\varepsilon<f'(c)-l$.

As $\lim_{x\to c-0} f'(x)=l$, corresponding to above chosen $\varepsilon>0$, there exists $\delta>0$ such that $|f'(x)-l|<\varepsilon$ whenever $x\in(c-\delta,c)\cap[a,b]$

So if $p\in(c-\delta,c)\cap[a,b]$, then $l-\varepsilon<f'(p)<l+\varepsilon<f'(c)$ (by above ε)

So by Darboux theorem, there exists point $\xi\in(p,\ c)$ such that $f'(\xi)=l+\varepsilon$.

Now $\xi\in(p,c)\Rightarrow\xi\in(c-\delta,c)\cap[a,b]$ & so by above $f'(\xi)<l+\varepsilon$. Thus we arrive at a contradiction. So $l\nless f'(c)$.

If possible let $l>f'(c)$. We choose ε such that $0<\varepsilon<l-f'(c)$

$\lim_{x\to c-0} f'(x)=l\Rightarrow$ Corresponding to above ε, there exists $\eta>0$ such that

$l-\varepsilon<f'(x)<l+\varepsilon$ whenever $x\in(c-\eta,c)\cap[a,b]$.

Let $q\in(c-\eta,c)\cap[a,b]$. Then $f'(c)<l-\varepsilon<f'(q)$.

Again by Darboux theorem on derivative. there exists point τ in (q,c)

such that $f'(\tau)=l-\varepsilon$.

But $\tau\in(q,c)\Rightarrow\tau\in(c-\eta,c)\cap[a,b]$ and hence $f'(\tau)>l-\varepsilon$.

We arrive at a contradiction. So $l \not> f'(c)$.

As a result $\lim\limits_{x \to c-0} f'(x) = f'(c),\ a < c < b$

Similarly it can be shown that $\lim\limits_{x \to c+0} f'(x)$ exists $= f'(c)$

So a derived function on an interval $[a, b](\subset \mathbb{R})$ can have a discontinuity of second kind only

(3) Let $f'(x)$ exist and be monotone on an open interval (a, b). Then f' is continuous on (a, b).

If possible, let f' have a discontinuity at some point $c \in (a, b)$.

c is interior point of (a, b) & we have a closed sub interval $[\alpha, \beta]$ of (a, b) which contains c in its interior.

By hypothesis f' is monotone in $[\alpha, \beta]$ & so the discontinuity at c must be a jump discontinuity. But a derived function can not have any jump discontinuity . So f' is continuous on (a, b).

13.4 Theorem (Rolle's theorem)

Let $f : [a, b] \to \mathbb{R}$ be

(i) continuous in $[a, b]$ (ii) derivable in (a, b) (iii) $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof : Continuity of f in $[a, b]$ ensures the boundedness of f in $[a, b]$ & attainment of bounds in $[a, b]$. Let $M = \sup\limits_{[a,b]} f$ and $m = \inf\limits_{[a,b]} f$.

There are points $c, d \in [a, b]$ such that $f(c) = M$, $f(d) = m$.

Case I : Let $M = m$. Then $f(x)$ is constant function in $[a, b]$ & so $f'(x) = 0$ in (a, b).

Case II : Let $M \neq m$. As $f(a) = f(b)$, So at least one of M and m is different from $f(a)$ and $f(b)$. So $c \neq a, c \neq b$ (if M be different from $f(a)$ and $f(b)$)

Hence $c \in (a, b)$ & by hypothesis (ii) $f'(c)$ exists.

If $f'(c) > 0$, there exists $\delta > 0$ such that $f(x) > f(c)$ in $c < x < c+\delta$ where $(c, c+\delta) \subset (a, b)$. So $f(x) > M$ in $(c, c+\delta)$ which is absurd. So $f'(c) \not> 0$.

If $f'(c) < 0$, then there exists $\eta > 0$ such that $f(x) > f(c)$ in $(c-\eta, c) \subset (a, b)$.

Again $f(x) > M$ in $(c-\eta, c)$ which is absurd. thus $f'(c) \not< 0$.

Consequence $f'(c) = 0$.

Note : The above theorem gives a set of sufficient conditions for the vanishing of f' at an interior point of D_f. The conditions are not necessary. For example,

$$f(x) = \frac{1}{x-1} + \frac{1}{2-x}, \quad 1 < x < 2$$

$f'(x) = 0$ at $x = \frac{3}{2}$ but f does not obey the conditions of Rolle's theorem in [1, 2].

Geometrical interpretation of Rolle's Theorem

If the two end points of the graph of $y = f(x)$ be on the same horizontal line (i.e. on a line parallel to x-axis) and if the graph be continuous throughout the interval and if the curve has a tangent at every point on it except possibly the two end points, then there must exist at least one point on the curve at which the tangent is parallel to x-axis.

Examples :

(1) Let $f, g, h : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and be derivable in (a, b), then show that there exists $c \in (a, b)$ for which

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Let us construct $F:[a,b]\to\mathbb{R}$ as $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b)\end{vmatrix}$

Continuity of f, g, h in $[a,b]\Rightarrow$ continuity of F in $[a, b]$.

$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b)\end{vmatrix}$ exists in (a, b), as f', g', h' exist in (a, b)

Also $F(a) = 0 = F(b)$. So $F(x)$ satisfies all the conditions of Rolle's Theorem in $[a, b]$. Therefore by Rolle's theorem, there exists $c\in(a,b)$ s.t. $F'(c)=0$

Hence the result follows.

(2) Let f, g be differentiable on the inerval I. Let $a,b\in I$ and $a<b$ and $f(\text{a}) = 0 = f(b)$. Show that there exists $c\in(a,b)$ such that

$f'(c)+f(c)g'(c)=0$

We construct the function $h:[a,b]\to\mathbb{R}$ as $h(x)=f(x).e^{g(x)}$

Continuity of f & g in $[a,b]\Rightarrow$ continuity of h in $[a, b]$.

$h'(x)=f'(x)\,e^{g(x)}+f(x)\,e^{g(x)}.\,g'(x)$ exists in (a, b) as f, g, are derivable in (a, b)

$h(a)=0=h(b)$ by given condition. So by Rolle's theorem, there exists

$c\in(a,b)$ such that $h'(c)=0\Rightarrow e^{g(c)}\{f'(c)+f(c)g'(c)\}=0$.

As $e^{g(c)}\neq 0$, so $f'(c)+f(c)g'(c)=0$ for some $c\in(a,b)$

Particular Case $f'(c)+\lambda f(c)=0\,(\lambda\in\mathbb{R})$ under the same set of conditions mentioned above.

(3) $x^4+2x^2-6x+2=0$ has

(A) 4 real roots (B) exactly two real roots (C) no real root (D) one pair of equal roots.

$f(0) = 2, f(1) = -1, f(2) = 14$. $f(x)$ is continuous function &

$f(0)f(1)<0, \quad f(1)f(2)<0$.

By Bolzano's theorem on continuous function, $f(x)$ must vanish at least once in (0, 1) & at least once in (1, 2).

If possible, let it have more than two real roots. Then by Rolle's Theorem, $f'(x)$ must vanish at least twice & $f''(x)$ must vanish at least once. But $f''(x)=12x^2+4>0$ for all x. (B) is true.

(4) If $a<c<b$ and $f''(x)$ exists finitely in $[a, b]$, then there exists $k\in(a,b)$ such that

$$\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=\frac{1}{2}f''(k)$$

Let us construct the function $\phi:[a, b]\to\mathbb{R}$ as follows.

$$\phi(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a)+\frac{(x-c)(x-a)}{(b-c)(b-a)}f(b)+\frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)-f(x)$$

Continuity of f in $[a,b]\Rightarrow$ Continuity of ϕ in $[a, b]$.

$$\phi'(x)=\frac{2x-(b+c)}{(a-b)(a-c)}f(a)+\frac{2x-(a+c)}{(b-c)(b-a)}f(b)+\frac{2x-(a+b)}{(c-a)(c-b)}f(c)-f'(x)$$

& as $f'(x)$ exists in (a, b), ϕ' exists in (a, b). Also $\phi(a)=\phi(b)=\phi(c)=0$

Given that $a < c < b$, so ϕ' satisfies the conditions of Rolle's theorem in both $[a, c]$ & $[c, b]$.

By Rolle's theorem, there exists $\xi_1\in(a,c)$ & $\xi_2\in(c,b)$ such that

$\phi'(\xi_1)=0=\phi'(\xi_2)$

As $f''(x)$ exists in (a, b)

$$\varphi''(x)=\frac{2}{(a-b)(a-c)}f(a)+\frac{2}{(b-c)(b-a)}f(b)+\frac{2}{(c-a)(c-b)}f(c)-f''(x)$$

exists in (ξ_1,ξ_2). Applying Rolle's theorem to ϕ' in $[\xi_1,\xi_2]$, there exists $k\in(\xi_1,\xi_2)\subset(a,b)$ such that $\phi''(k)=0$

$$\Rightarrow\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=\frac{1}{2}f''(k).$$

(5) let $f,g:[a,b]\to\mathbb{R}$ be such that each is derivable in (a, b), each is continuous at a & b. Then there exists $c\in(a,b)$ such that

$$f'(c)\{g(b)-g(a)\}=g'(c)\{f(b)-f(a)\}$$

We construct $h:[a,b]\to\mathbb{R}$ as follows :

$h(x)=f(x)\{g(b)-g(a)\}-g(x)\{f(b)-f(a)\}$ for all $x\in[a,b]$

Continuity of f & g in $[a,b]\Rightarrow$ Continuity of h in [a, b].

$h'(x)=f'(x)\{g(b)-g(a)\}-g'(x)\{f(b)-f(a)\}$ exists in (a, b) as f',g' exist in (a, b).

$h(a)=f(a)g(b)-f(b)g(a)$, $h(b)=-f(b)g(a)+f(a)g(b)$

and so $h(a)=h(b)$

So h satisfies all the conditions of Rolle's theorem in [a, b]. By Rolle's theorem, there exists $c\in(a,b)$ such that $h'(c)=0$.

$$\Rightarrow f'(c)\{g(b)-g(a)\}=g'(c)\{f(b)-f(a)\}$$

(6) Show that between any two real roots of $e^x\sin x=1$, there is at least one real root of $e^x\cos x+1=0$.

Let $f(x)=e^x\sin x-1$ and a, b be two real roots of $f(x)=0$

Let $g(x)=e^{-x}-\sin x,\ \ a\le x\le b$.

g is continuous in $[a, b]$ and $g'(x)=-e^{-x}-\cos x$ exists in (a, b)

Also $g(a)=g(b)=0$ by above hypothesis.

g satisfies all the conditions of Rolle's theorem in $[a, b]$ & so by Rolle's theorem, there exists $c\in(a, b)$ such that $g'(c)=0$ i. e. $e^{-c}+\cos c=0$ or $1+e^{c}\cos c=0$.

$\Rightarrow c$ is root of $e^{x}\cos x+1=0$

(7) Let $f:[a,b]\rightarrow\mathbb{R}$ be continuous in $[a,b]$, differentiable in (a,b) and be nowhere zero in (a,b). Show that there exists $\theta\in(a,b)$ such that

$$\frac{f'(\theta)}{f(\theta)}=\frac{1}{a-\theta}+\frac{1}{b-\theta}$$

We construct $g:[a,b]\rightarrow\mathbb{R}$ as follows :

$g(x)=(x-a)(x-b)f(x)$ for all $x\in[a,b]$

Continuity of f in $[a,b]\Rightarrow$ continuity of g in $[a,b]$

$g'(x)=(x-b)f(x)+(x-a)f(x)+(x-a)(x-b)f'(x)$ exists in (a, b) as f' exists in (a, b). Also $g(a)=0=g(b)$. Applying Rolle's theorem to g in $[a,b]$, there exists at least one $\theta\in(a,b)$ such that $g'(\theta)=0$.

$$\Rightarrow(\theta-b)f(\theta)+(\theta-a)f(\theta)+(\theta-a)(\theta-b)f'(\theta)=0$$

$$\Rightarrow\frac{f'(\theta)}{f(\theta)}=\frac{1}{a-\theta}+\frac{1}{b-\theta}$$

(8) Show that the equation $x\log x=3-x$ has at least one root in $(1, 3)$.

Let $f:[1, 3]\rightarrow\mathbb{R}$ be defined as follows : $f(x)=(x-3)\log x$.

f is continuous in $[1, 3]$, $f'(x) = \log x + \dfrac{x-3}{x}$ exists in $(1, 3)$ & $f(1) = 0 = f(3)$

Applying Rolle's theorem to f in $[1, 3]$, there exists $c \in (1, 3)$ such that $f'(c) = 0$

$\Rightarrow \log c + 1 - \dfrac{3}{c} = 0$ i.e. c is root of $x \log x + x = 3$

(9) If f', g' exist in $[a, b]$ & $g'(x) \neq 0$ in (a, b), show that there exists $c \in (a, b)$

such that $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$

We construct $h : [a, b] \to \mathbb{R}$ as follows :

$h(x) = f(x) g(x) - f(a) g(x) - g(b) f(x)$ for all $x \in [a, b]$

Existence of f', g' in $[a, b] \Rightarrow$ continuity & derivability of h in $[a, b]$

Also $h(a) = -g(b) f(a) = h(b)$. Applying Rolle's theorem to h in $[a, b]$, there exists $c \in (a, b)$ such that

$$h'(c) = 0 \Rightarrow f'(c)\, g(c) + f(c) g'(c) - f(a) g'(c) - g(b) f'(c) = 0$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(c) - f(a)}{-g(c) + g(b)}$$

EXERCISE

1. If f', g' are continuous in $[a-h,\ a+h]$, derivable in $(a-h,\ a+h)$, $g''(x) \neq 0$, show that there exists $d \in (a-h, a+h)$ such that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{g(a+h) - 2g(a) + g(a-h)} = \frac{f''(d)}{g''(d)}$$

2. Let $f, g : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Assume that g, g' are nowhere zero in $[a, b]$ & (a, b) respectively..

Let $\dfrac{f(a)}{g(a)} = \dfrac{f(b)}{g(b)}$. Show that there exists $c \in (a, b)$ such that $\dfrac{f(c)}{g(c)} = \dfrac{f'(c)}{g'(c)}$

3. Let $\sum_{k=0}^{n} \dfrac{C_k}{k+1} = 0$ where $C_k \in \mathbb{R}$ for all k. Show that the equation

$C_0 + C_1 x + ... + C_n x^n = 0$ has at least one root in $(0, 1)$

4. Let $u(x), v(x), u'(x), v'(x)$ are all continuous on $\mathbb{R}$ and $uv' - u'v \neq 0$ in $\mathbb{R}$. Prove that between any two real roots of $u(x) = 0$, there lies one root of $v(x) = 0$.

5. Examine whether the equation $x^3 - 3x + k = 0, k \in \mathbb{R}$, has two distinct roots in $(0, 1)$.

6. Correct or justify the statement : Rolle's Theorem is not applicable to $|x|$ in any interval $[a, b] \subset \mathbb{R}$.

7. Using Rolle's theorem, show that the derivative $f'(x)$ of the function

$$f(x) = \begin{cases} x \sin \dfrac{\pi}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

vanishes on an infinite set of points of the interval $(0, 1)$.

8. Let $f, g : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$, f'', g'' exist in (a, b), f & g vanish at end points a and b, $g''(x) \neq 0$ in (a, b).

If $a < c < b$ & $g(c) \neq 0$, show that there exists $\xi \in (a, b)$ such that

$$\frac{f(c)}{g(c)} = \frac{f''(\xi)}{g''(\xi)}$$

(Hints : Contruct $F:[a,b]\to\mathbb{R}$ as $F(x)=f(c)g(x)-g(c)f(x)$ for all $x\in[a,b]$)

9. let $f:\mathbb{R}\to\mathbb{R}$ be differentiable upto any number of times and let

for some $n\in\mathbb{N},\ f(0)=f'(0)=...=f^{n}=0$

Show that $f^{(n+1)}(x)=0$ for some $x\in(0,1)$

10. Show that each of the equations

(i) $\sin(\cos x)=x$ (ii) $\cos(\sin x)=x$ has exactly one root in $\left(0,\pi/2\right)$

Lagrange's Mean value theroem or first mean value theorem of Differential calculus.

Let $f:[a,b]\to\mathbb{R}$ be (i) continuous in [a, b] (ii) derivable in (a, b). Then there exists at least one point $c\in(a,b)$ such that $f(b)-f(a)=(b-a)f'(c)$.

Proof : We construct $F:[a,b]\to\mathbb{R}$ as follows.

$F(x)=f(x)+Ax$ where the constant A is to be determined from the functional relation $F(b)=F(a)$

Continuity of f in $[a,b]\Rightarrow$ continuity of F in $[a,b]$ &

derivability of f in $(a,b)\Rightarrow$ derivability of F in (a, b). By construction, $F(b)=F(a)$

So F satisfies all the conditions of Rolle's theorem in $[a,b]$. Hence by Rolle's theorem, there exists $c\in(a,b)$ such that $F'(c)=0$.

So $F'(c)=f'(c)+A=0\to A=-f'(c)$ & $F(b)=F(a)\Rightarrow -A=\dfrac{f(b)-f(a)}{b-a}$

Therefore $\dfrac{f(b)-f(a)}{b-a}=f'(c)$

Note : 1. Conditions stated above are sufficient but not necessary.

Consider the function $f:[0,3]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & 0\le x<\frac{1}{2}\\ x+\frac{1}{2}, & \frac{1}{2}\le x\le\frac{3}{2}\\ \frac{2}{3}x+1, & \frac{3}{2}<x\le 3\end{cases}$$

f does not sastisfy the conditions stated above but

$$\frac{f(3)-f(0)}{3-0}=1=f'\left(\frac{3}{4}\right)$$

2. Geometrical Interpretation : If the graph of a function be a continuous curve having tangent at every point on it except possibly the two end points, then there is at least one point on the curve at which the tangent is parallel to the chord joining the end points.

Examples : (i) Let f have the property that $|f'(x)|<1$ for all x in $(0,1)$ and let f be continuous at $x=0,1$. Show tht the sequence $\left\{f\left(\frac{1}{n}\right)\right\}_n$ is convergent.

We note that LMV theorem is applicable to f in any interval $\subset\mathbb{R}$

Let $\varepsilon>0$ be given. By Archimedean property of $\mathbb{R}$, there exists natural number K such that $K\varepsilon>2$. Let $m,n\in\mathbb{N}$ be such that $m,n>K$.

By L M V theorem, there exists at least one point $c\in\left(\frac{1}{m},\frac{1}{n}\right)\left(\text{or}, c\in\left(\frac{1}{n},\frac{1}{m}\right)\right)$

Such that $\left|f\left(\frac{1}{m}\right)-f\left(\frac{1}{n}\right)\right|=\left|\frac{1}{m}-\frac{1}{n}\right||f'(c)|<\left|\frac{1}{m}-\frac{1}{n}\right|$ (by hypothesis)

$\Rightarrow\left|f\left(\frac{1}{m}\right)-f\left(\frac{1}{n}\right)\right|<\frac{2}{K}<\varepsilon$ for $m,n,>K$

$\Rightarrow \left\{ f\left(\frac{1}{n}\right) \right\}_n$ is cauchy sequence in $\mathbb{R}$ & so $\left\{ f\left(\frac{1}{n}\right) \right\}_n$ is convergent sequence in $\mathbb{R}$. Hence $\left\{ f\left(\frac{1}{n}\right) \right\}_n$ has a limit in $\mathbb{R}$

2. Find real solutions of $2^x + 5^x = 3^x + 4^x$

The equation can be written as $5^x - 4^x = 3^x - 2^x$

We consider the function $f(t) = t^x$ in (i) [4, 5] (ii) [2, 3]

f is continuous in both the intervals & is derivable in both, Applying L M V theorem to t^x in both [4, 5] and [2, 3], we see that there are points $t_1 \in (4, 5)$ & $t_2 \in (2, 3)$ so that

$5^x - 4^x = xt_1^{x-1}$ & $3^x - 2^x = xt_2^{x-1}$

Therefore $xt_1^{x-1} = xt_2^{x-1} \Rightarrow \left(\frac{t_1}{t_2}\right)^{x-1} = 1 \Rightarrow x - 1 = 0$ as t_1, t_2 belong to different sub intervals & so $t_1 \neq t_2$

Hence, $x = 1$

Hence $x = 0, 1$ are only solutions.

3. $f : [0, 2] \to \mathbb{R}$ is differentiable & $f(0) = 0, f(1) = 2, f(2) = 1$. Show that there exists $c \in (0, 2)$ such that $f'(c) = 0$

Applying LMV theorem to *f* in [0, 1], there exists $\xi \in (0, 1)$ such that

$f(1) - f(0) = (1 - 0) f'(\xi) \Rightarrow f'(\xi) = 2$

Applying LMV theorem to *f* in $[1, 2]$, there exists $\eta \in (1, 2)$ such that

$f(2) - f(1) = (2 - 1) f'(\eta) \Rightarrow f'(\eta) = 1 - 2 = -1$

So $f'(\xi) f'(\eta) < 0$. By Intermediate value theorem on derivative. there exists $c \in (\xi, \eta) \subset (0, 2)$ such that $f'(c) = 0$

4. If $\phi''(x) \geq 0$ for all $x \in (a, b)$, show that

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\{\phi(x_1) + \phi(x_2)\}$$

for every pair of points x_1, x_2 in (a, b)

Let $x_2 > x_1$ & so $x_1 < \frac{x_1 + x_2}{2} < x_2$

$$\phi(x_1) + \phi(x_2) - 2\phi\left(\frac{x_1 + x_2}{2}\right) = \left\{\phi(x_2) - \phi\left(\frac{x_1 + x_2}{2}\right)\right\} - \left\{\phi\left(\frac{x_1 + x_2}{2}\right) - \phi(x_1)\right\}$$

$= \frac{1}{2}(x_2 - x_1)\{\phi'(\xi) - \phi'(\eta)\}$ for some

$\xi \in \left(x_1, \frac{x_1 + x_2}{2}\right) \& \eta \in \left(\frac{x_1 + x_2}{2}, x_2\right)$ by L M V theorem ... (1)

Again by hypothesis, $\phi''(x)$ exists, so by applying L M V theorem to ϕ' in $[\xi, \eta]$, there exists $c \in (\xi, \eta)$

$\phi'(\xi) - \phi'(\eta) = (\xi - \eta)\ \phi''(c) \geq 0$ by hypothesis ...... (2)

By (1) & (2), $\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\{\phi(x_1) + \phi(x_2)\}$

Note : Converse is not true. $f(x) = |x|$ fulfils the given result but $|x|$ is not derivable at 0.

5. Let f be a function such that $f(x) > 0$ for all x & $f'(x)$ be continuous at every real x. If $f'(t) \geq \sqrt{f(t)}$ for all t, show that

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x - 1) \text{ for all } x \geq 1$$

By hypothesis, $\phi(x) = \sqrt{f(x)}$ is derivable for all $x \geq 1$. By L M V theorem there exists $\xi \in (1, x)$ such that

$\phi(x)-\phi(1)=(x-1)\,\phi'(\xi)$

$\Rightarrow \sqrt{f(x)}-\sqrt{f(1)}=(x-1).\dfrac{f'(\xi)}{2\sqrt{f(\xi)}}\geq \dfrac{1}{2}(x-1)\left(\text{as } f'(t)\geq\sqrt{f(t)}\right)$

6. On the curve $y=x^3$, find the point at which the tangent line is parallel to the chord joining the points $A(-1,-1)$ and $B(2,8)$

Let us refer to the geometrical interpretation of L M V theorem.

By L M V theorem, there exists $\xi\in(-1,2)$ such that

$f(2)-f(-1)=(2+1)\,f'(\xi)$ for some $\xi\in(-1,2)$

(taking $f(x)=x^3$ in $[-1,2]$)

$\Rightarrow 9=3\,\xi^2.\,3\Rightarrow \xi=\pm 1$. Here -1 is not interior point of $[-1,2]$ or $-1\notin(-1,2)$

So $\xi=1$ i.e (1, 1) is the point at which the tangent is paralled to AB.

7. Apply mean value theorem to find derivative of a function, assuming that the derivatives which occur are continuous.

Let $F\{f(x)\}$ be the composite function

Mean value theorem is applicable to $f(x)$ and there exists $\xi\in(x,x+h)$ or $(x+h,x)$ for which

$f(x+h)=f(x)+hf'(\xi)=u+k$ say that $u=f(x)\,\&\,k=h\,f'(\xi)$

Mean value theorem is applicable to $F(u)$ and there exists $\eta\in(u,u+k)$ or $(u+k,u)$ for which $F(u+k)=F(u)+k\,F'(\eta)$

As $h\to 0$, $\xi\to x$ Also $k=h\,f'(\xi)\to 0$. Further as $k\to 0$, $\eta\to u$. Therefore

$$\lim_{h\to 0}\frac{F\{f(x+h)\}-F\{f(x)\}}{h}=\lim_{h\to 0}\frac{F(u+k)-F(u)}{h}$$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{k\,F'(\eta)}{h}=\lim\limits_{h\to 0}\,f'(\xi)\;F'(\eta)$ exists & is

$f'(x)F'(u) = f'(x)F'\{f(x)\}$

8. If f be continuous at c and $\lim_{x \to c} f'(x)$ exists finitely, then show that f' is also continuous at c.

Let $\lim_{x \to c} f'(x) = l \,(\in \mathbb{R})$

Hence there exists an interval $(c, c+h], h > 0$ at every point of which f' exists & so f is continuous in $(c, c+h]$. Given that f is continuous at c. So f is continuous in $[c, c+h]$ & f' exists in $(c, c+h)$. By L M V theorem, there exists

$\xi, c < \xi < x < c+h$

such that $f(x) - f(c) = (x-c)f'(\xi)$

As $\lim_{x \to c} f'(x) = l$, so

$$\lim_{x \to c+0} f'(\xi) = \lim_{\xi \to c+0} f'(\xi) = l \Rightarrow \lim_{x \to c+} \frac{f(x) - f(c)}{x-c} = Rf'(c) = l$$

Similarly, considering $[c-h, c)$ & arguing as in the previous case, $Lf'(c) = l$

so, $f'(c) = l \Rightarrow \lim_{x \to c} f'(x) = f'(c)$ & therefore f' is continuous at c.

Increasing & decreasing nature of function in an interval :

Result : If $f(x)$ is continuous in $[a, b]$ and $f'(x) > 0$ (or < 0) in (a, b), then $f(x)$ is increasing (or decreasing) function in $[a, b]$. If $f'(x) = 0$ in $(a, b), f(x)$ is constant in the interval.

Proof : We choose x_1, x_2 so that $a \le x_1 < x_2 \le b$. Applying L M V theorem to f in $[x_1, x_2]$

there exists $\xi \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(\xi)$

So $f'(\xi) > 0 \Rightarrow f(x_2) > f(x_1)$

$\left(f'(\xi)<0 \Rightarrow f(x_2)<f(x_1)\right)$

It is true for every pair of points x_1, x_2 of $[a, b]$. So if $f'(x)>0$ for all x. f is increasing in $[a, b]$ & if $f'(x)<0$ for all x, f is decreasing in $[a, b]$

If $f'(x)=0$ then $f(x_1)=f(x_2)$ & so $f(x)$ is constant in $[a, b]$

Examples : 1. If $0<x<1,\ 2x<\log\frac{1+x}{1-x}<\frac{2x}{1-x}$

Let $f(x)=\log\frac{1+x}{1-x}-2x, 0\le x<1$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}-2=\frac{2x^2}{1-x^2}>0 \text{ for all } x\in(0,1)$$

Next let $g(t)=\log t,\ t\in[1-x, 1+x]$, $0<x<1$. Applying LMV theorem to $g(t)$ in [1–x, 1+x], there exists $\xi\in(1-x, 1+x)$

for which $g(1+x)-g(1-x)=2x\,g'(\xi)=\frac{2x}{\xi}$

$$\Rightarrow g(1+x)-g(1-x)=\log(1+x)-\log(1-x)<\frac{2x}{1-x}, 0<x<1$$

2. Show that $\frac{2}{\pi}<\frac{\sin x}{x}<1$ when $0<x<\frac{\pi}{2}$

Let us construct $g:\left[0, \frac{\pi}{2}\right]\to\mathbb{R}$ as follows :

$$g(x)=\begin{cases}\frac{\sin x}{x}, & 0<x\le\frac{\pi}{2}\\ 1 & x=0\end{cases}$$

So g is continuous in $\left[0, \frac{\pi}{2}\right]$ & $g'(x)=\frac{x\cos x-\sin x}{x^2}$ exists in $\left(0, \frac{\pi}{2}\right)$

Let $t(x) = x\cos x - \sin x$ in $\left[0, \frac{\pi}{2}\right]$ & so

$t'(x) = -x\sin x + \cos x - \cos x < 0$ in $\left(0, \frac{\pi}{2}\right)$

So $t(x) < t(0)$ or $x\cos x - \sin x < 0$. Hence $g'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow g\left(\frac{\pi}{2}\right) < g(x) < g(0)$ & hence $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ in $\left(0, \frac{\pi}{2}\right)$

3. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

Let $f(x) = \tan x \sin x - x^2, 0 \le x < \frac{\pi}{2}$

f is continuous in $[0, p]\left(p < \frac{\pi}{2}\right)$ &

$f'(x) = \sec^2 x \sin x + \sin x - 2x = t(x)$ in $[0, p]$

$t'(x) = 2\sec^2 x \tan x.\sin x + \sec^2 x.\cos x + \cos x - 2$

$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x \sec^3 x$

$t(x)$ is continuous in $[0, p]$ & $t'(x)$ exists in $(0, p)$. Also $t'(x) > 0$ in $(0, p)$

$\Rightarrow t(x) > t(0), x > 0$ & so $f'(x) > 0 \Rightarrow f(x) > f(0), 0 < x < \frac{\pi}{2}$

Consequently, $\frac{\tan x}{x} > \frac{x}{\sin x}, \ 0 < x < \frac{\pi}{2}$

4. Let $\phi(x) = f(x) + f(1-x)$ & $f''(x) < 0$ in $[0, 1]$. Show that $\phi(x)$ is monotonic increasing in $\left[0, \frac{1}{2}\right]$ and monotonic decreasing in $\left[\frac{1}{2}, 1\right]$

By hypothesis, $\phi'(x) = f'(x) - f'(1-x)$

Applying L M V theorem to f' in $[x, 1-x]$ or in $[1-x, x]$, there exists $\xi \in (x, 1-x)$ or $(1-x, x)$ such that

$f'(x) - f'(1-x) = (2x-1) f''(\xi)$

By hypothesis $f''(\xi) < 0$ & so $f'(x) - f'(1-x) \begin{cases} \geq 0 & , 0 \leq x \leq \frac{1}{2} \\ \leq 0 & , \frac{1}{2} \leq x \leq 1 \end{cases}$

So $\phi(x)$ is increasing in $\left[0, \frac{1}{2}\right]$ & is decreasing in $\left[\frac{1}{2}, 1\right]$

5. Show that $\cos x + x \sin x > 1,\ x \in \left(0, \frac{\pi}{2}\right)$

Let $f(x) = \cos x + x \sin x,\ 0 \leq x \leq \frac{\pi}{2}$

$f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, $f'(x) = -\sin x + \sin x + x \cos x$ exists in $\left(0, \frac{\pi}{2}\right)$

Also $f'(x) > 0$ in $\left(0, \frac{\pi}{2}\right)$. So $f(x) > f(0),\ 0 < x < \frac{\pi}{2}$

$\Rightarrow \cos x + x \sin x > 1,\ 0 < x < \frac{\pi}{2}$

6. Show that $f(x) = \tan^{-1} x$ defined on $(-\infty, \infty)$ is uniformly continuous & f' is also uniformly continuous.

We note that $f'(x) = \frac{1}{1+x^2}$ exists for all $x \in \mathbb{R}$

Let us consider any pair of points x, y of $\mathbb{R}$. By L M V theorem, there exists $\xi \in (x, y)$ such that

$|f(x) - f(y)| = |x-y| \frac{1}{1+\xi^2} < |x-y| < \delta$ for any pair of points $x, y \in \mathbb{R}$ satisfying $|x-y| < \delta$, δ depends only on ε. So f is uniformly continuous on $\mathbb{R}$.

Again $|f'(x)-f'(y)| = |x-y|\, f''(\eta)$ for some $\eta \in (x, y)$ (by L M V theorem) ...(1)

$f''(x) = \dfrac{-2x}{1+x^2}$ & so $|f''(x)| < 2$ for all x ...(2)

For if $|x|<1$, $\dfrac{x^2+1}{2} \geq |x| \Rightarrow |f''(x)| < 2$

& if $|x|>1$, $|f''(x)| < \left|\dfrac{2x}{x^4}\right| < 2$

Recalling (2), $|f'(x)-f'(y)| < 2|x-y| < \varepsilon$ whenever $|x-y| < \delta$, $\delta = \dfrac{\varepsilon}{2}$

for any pair of points x, y of $\mathbb{R}$

Hence f' is uniformly continuous on $\mathbb{R}$

7. Find all possible positive solutions of $x^2+y^2 = u^2+v^2$, $x^3+y^3 = u^3+v^3$ where u, v be fixed positive constants.

Obiviously $x = u, y = v$ and $x = v, y = u$ are two solutions of the system.

Let $x_1 = x^3$, $y_1 = y^3$, $u_1 = u^3$, $v_1 = v^3$

We consider the function $f(t) = t^{2/3}$ in (i) $[u_1, x_1]$ (ii) $[v_1, y_1]$

By L M V theorem, there exists $t_1 \in (u_1, x_1)$ & $t_2 \in (v_1, y_1)$ such that

$$x_1^{2/3} - u_1^{2/3} = (x_1 - u_1)\frac{2}{3}t_1^{-1/3} \;\;\&\;\; v_1^{2/3} - y_1^{2/3} = (v_1 - y_1)\frac{2}{3}t_2^{-1/3} \;\; ...(1)$$

Given $x_1 + y_1 = u_1 + v_1$ & $x_1^{2/3} + y_1^{2/3} = u_1^{2/3} + v_1^{2/3}$ so (1) $\Rightarrow t_1 = t_2$

But $t_1 \in (u_1, x_1)$ & $t_2 \in (v_1, y_1)$. So $t_1 \neq t_2$. So $x = u$, $y = v$ & $x = v$, $y = u$ are only solutions.

Exercise :

1. Show that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$, $x > 0$

2. Show that $0 < x^{-1}\log\left(\frac{e^x - 1}{x}\right) < 1,\ x > 0$

3. Let $f:[1,3] \to \mathbb{R}$ be a continuous function that is derivable in (1, 3) with derivative $f'(x) = |f(x)|^2 + 4$ for all $x \in (1,3)$

State with reasons, whether $f(3) - f(1) = 5$ is true or false.

4. If $f''(x)$ exists in $[a, b]$ and $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$ for some $c \in (a, b)$, show that there exists at least one point $\xi \in (a, b)$ for which $f''(\xi) = 0$.

5. If $f'(x)$ exists for $a < x \le b$ and $|f(x)| \to \infty$ as $x \to a$, then show that $|f'(x)| \to \infty$ as $x \to a$.

(Hints Apply LMV Theorem to f in [x, b]).

6. Let f be continuous in [0,1] and differentiable in (0,1). If f' be monotonic increasing in (0,1), prove that $F(x) = \frac{f(x)}{x}$ is monotonic increasing in (0,1).

7. Show that $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$ where $x_2 > x_1$

8. Determine the intervals of monotonicity for the following functions :

(i) $f(x) = 2x^3 - 9x^2 - 24x + 7$

(ii) $f(x) = 4x^3 - 21x^2 + 18x + 20$

(iii) $f(x) = \sin x + \cos x \text{ in } [0, 2\pi]$

9. Show that :

(a) $x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6},\quad 0 < x \le 1$

(b) $x - \frac{x^3}{6} < \sin x < x,\ x > 0$

10. Prove that for $0 \le p \le 1$ and for any positive a and b the inequality $(a+b)^p \le a^p + b^p$ is valid.

(Hints : Take $f(x) = 1 + x^p - (1+x)^p$, $x \ge 0$ & then $x = \frac{a}{b}$)

11. At what value (s) of b, does the function $f(x) = \sin x - bx + c$ decrease along the entire number scale ?

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and that $f(0) = 0;\ f(4) = 2,\ f(6) = 2$ show that

(i) there exists $x \in (0, 4)$ such that $f'(x) = \frac{1}{2}$

(ii) there exists $x \in (0, 6)$ such that $f'(x) = \frac{2}{5}$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $f'(x) > f(x)$ for all $x \in \mathbb{R}$ & $f(x_\circ) = 0$. Show that $f(x) > 0$ for all $x > x_\circ$

$\langle$Hints : Let $g(x) = e^{-x} f(x)$ & consider the sign of $g'(x)\rangle$

14. Let $a > b > 0$. Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$ for all $n \ge 2$

(Consider $f(x) = x^{1/n} - (x-1)^{1/n}$, $x \ge 1$; sign of $f'(x)$ & then put $x = \frac{a}{b}$)

15. Justify the following :

(a) if $x > 0,\ x > \frac{5\sin x}{4 + \cos x}$

(b) if $0 < x < \pi/2,\ 0 < x \sin x - \frac{1}{2}\sin^2 x < \frac{1}{2}(\pi - 1)$

(c) $\sqrt{1+x} < 4 + \frac{x-15}{x}$, if $x > 15$

(d) $\tan^{-1} x < \frac{\pi}{4} + \frac{x-1}{2}$, if $x > 1$

(e) $p(x-1) < x^p - 1 < px^{p-1}(x-1),\ x > 1,\ p > 1$

Cauchy's Mean value theorem :

Let $f, g : [a, b] \to \mathbb{R}$ be such that (i) both are continuous in $[a, b]$ (ii) both are derivable in (a, b) (iii) $g'(x) \neq 0$ in (a, b), then there exists at least one point $c \in (a,b)$ for which

$$\frac{f(b) - f(a)}{g(b) - f(a)} = \frac{f'(c)}{g'(c)}$$

Proof : We construct the function $F(x) = f(x) + A.g(x)$ where the constant A is to be determined from the functional relation $F(a) = F(b)$.

$F(a) = F(b) \Rightarrow -A = \frac{f(b) - f(a)}{g(b) - g(a)}$. In this connection, it is to be noted that $g(b) \neq g(a)$, for if $g(b) = g(a)$, then g would satisfy all the conditons of Rolle's theorem in $[a, b]$ and so $g'(x)$ must vanish at least once in (a,b). But condition (iii) tells otherwise. So $g(b) \neq g(a)$ & $-A$ is well-defined,

Here $F'(x) = f'(x) + Ag'(x)$ exists in (a,b) by condition (ii). Also F is continuous in $[a,b]$ by hypothesis. By construction, $F(a) = F(b)$. So F satisfies all the conditions of Rolle's theorem in $[a,b]$. By Rolle's theorem, there exists at least one point $c \in (a,b)$ for which $F'(c) = 0$.

So $F'(c) = 0 \Rightarrow -A = \frac{f'(c)}{g'(c)}$. Hence $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

Note : Putting $g(x) = x$ in $[a,b]$, we get LMV theorem.

Examples : (1) If f' exists in [0,1], show that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in (0,1).

We take $g(x)=x^2$ in [0,1]. Both *f*, *g* are continuous in [0,1], are derivable in (0,1) & $g'(x)\neq 0$ in (0,1). By Cauchy's mean value theorem, there exists at least one point $c\in(0,1)$ such that $\frac{f(1)-f(0)}{g(1)-g(0)}=\frac{f'(c)}{g'(c)}\Rightarrow f(1)-f(0)=\frac{f'(c)}{2c}$ & so c is a solution of $f(1)-f(0)=\frac{f'(x)}{2x}$.

2. Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$, derivable in (a,b) where $0<a<b$. Show that for some $c\in(a,b)$

$$f(b)-f(a)=cf'(c)\log\left(\frac{b}{a}\right)$$

We take $g(x)=\log x$ in $[a,b]$, $0<a<b$. Applying CMV theorem to *f*, *g*, in [*a*, *b*] there exists $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{f(b)-f(a)}{\log\left(\frac{b}{a}\right)}=cf'(c)$$

3. Let *f*, *g* be differentiable on [0,2] such that $f(0)=2, f(2)=5$, $g(2)\neq 0, g(0)=0, f'(x)=g'(x)(\neq 0)$ in (0, 2). Find $g(2)$

By C M V theorem, there exists $c\in(0,2)$ such that $\frac{f(2)-f(0)}{g(2)-g(0)}=\frac{f'(c)}{g'(c)}$

$$\Rightarrow\frac{5-0}{g(2)-0}=1\Rightarrow g(2)=5$$

13.5 Taylor's Theorem

Let $f:[a,b]\to\mathbb{R}$ be such that (i) $f^{(n-1)}(x)$ is continuous in $[a,b]$ (ii) $f^{(n)}(x)$ exists in (a,b).

Then there exists $\theta \in (0,1)$ such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

where

$$R_n=\begin{cases}\dfrac{(b-a)^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n\left[a+\theta(b-a)\right],\ p\in\mathbb{N}\ (\text{schlomilch}-\text{Röche'sform})\\ \dfrac{(b-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n\left[a+\theta(b-a)\right](\text{Cauchy's form})\\ \dfrac{(b-a)^n}{n!}\qquad f^n\left[a+\theta(b-a)\right](\text{Lagrange's form})\end{cases}$$

Proof : Continuity of f^{n-1} in $[a,b]$ implies the existence and continuity of $f, f', f'', \ldots f^{n-2}, f^{n-1}$ in $[a,b]$

We construct $\phi:[a,b]\to\mathbb{R}$ as follows :

$$\phi(x)=f(x)+(b-x)f'(x)+\frac{(b-x)^2}{2!}f''(x)+\ldots+\frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)+\lambda(b-x)^p$$

where λ is a constant to be determined from the functional relation $\phi(b)=\phi(a)$.

ϕ is continuous in $[a,b]$ by hypothesis (i)

$$\phi'(x)=f'(x)-f'(x)+(b-x)f''(x)-\ldots+\frac{(b-x)^{n-1}}{(n-1)!}f^n(x)-\lambda p(b-x)^{p-1}$$

exists in $(a,\ b)$

By construction, $\phi(a)=\phi(b)$. So $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a,b]$.

Therefore, by Rolle's theorem, there exists $c\in(a,\ b)$ such that $\phi'(c)=0$

$$\Rightarrow\frac{(b-c)^{n-1}}{(n-1)!}f^n(c)=\lambda p(b-c)^{p-1}\Rightarrow\lambda=\frac{(b-c)^{n-p}}{(n-1)!\,p}f^n(c)$$

Therefore $\phi(b)=\phi(a)$ implies

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots$$

$$+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(b-c)^{n-p}(b-a)^p}{(n-1)!\,p}f^n(c)$$

As $a<c<b$ we can write $c=a+\theta(b-a)$ for some $\theta\in(0,1)$.

We write $R_n=\frac{(b-a)^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n\left[a+\theta(b-a)\right]$ (schlomilch & Röche's form)

For $p=1$, $R_n=\frac{(b-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n\left[a+\theta(b-a)\right]$ (Cauchy's form)

For $p=n$, $R_n=\frac{(b-a)^n}{n!}f^n\left[a+\theta(b-a)\right]$ (Lagrange's form)

Note : 1. The relevance of these forms by taking $p=n$ & $p=1$ will be discussed in the subsequent results.

2. The readers should note the particular forms of this theorem (also known as Generalised mean value theorem) by taking $n=2,3$ etc for solution of problems.

3. Taking $b=a+h$,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h)$$

for some $\theta\in(0,1)$

Problems : (i) Let $f^{n+1}(x)$ be continuous and $\neq 0$, the number θ which occurs in the Lagrange's form of remainder of Taylor's theorem, viz, $\frac{h^n}{n!}f^n(a+\theta h)$ tends to $\frac{1}{n+1}$ as $h\to o+$.

We know that—

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{h^n}{n!}f^n(x+\theta h)$$ for some $\theta\in(0,1)$.

By hypothesis $f^{n+1}(x)$ exists & so

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+...+\frac{h^n}{n!}f^{(n)}(x)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\theta' h)$$

for some $\theta' \in (0,1)$

These two imply $f^n(x+\theta h)=f^n(x)+\frac{h}{n+1}f^{(n+1)}(x+\theta' h)$

By LMV theorem $f^n(x+\theta h)-f^n(x)=\theta h f^{(n+1)}(x+\theta\theta'' h)$ for some $\theta'' \in (0,1)$

So $\theta f^{(n+1)}(x+\theta\theta'' h)=\frac{f^{(n+1)}(x+\theta' h)}{n+1}$

As $f^{n+1}(x)$ is continuous by hypothesis & as $f^{n+1}(x)\neq 0$ we get $\lim_{h\to 0}\theta=\frac{1}{n+1}$

2. For $x>0$, show that $0\le \sin x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\right)\le\frac{x^9}{9!}$

Let $f(x)=\sin x$. Then $f^{(n)}(x)=\sin(x+\frac{n\pi}{2})$ for all $x\in\mathbb{R},\ n\in\mathbb{N}$

These $f^{(n)}(x)$'s are continuous for all x.

For $x>0$ we have

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$

for some $\theta\in(0,1)$ (taking a = 0, b = x in Taylor's theorem with Lagrange's form of remainder)

We take $n=7$ & $n=9$ respectively :

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cos(\theta_1 x) \text{ for some } \theta_1\in(0,1)$$

$$\& \ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\cos(\theta_2 x) \text{ for some } \theta_2 \in (0,1)$$

As $-1 \le \cos(\theta_k x) \le +1$ for $k = 1,2$ (here), so for $x > 0$, we get

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

Expansion of functions :

Taylor's infinite series suppose f possesses continuous derivatives of every order in $[a, a+h]$

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$

Then $f(a+h) = S_n + R_n$

If now it is given that $R_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(a+h)$

Hence under the condition that $\lim_{n\to\infty} R_n = 0$, the infinite series

$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^n}{n!}f^n(a) + ...$ converges to $f(a+h)$

This result can be stated in the following way also.

Let f be defined in some open interval $I(\subset \mathbb{R})$ containing 'a' and that derivatives of every order of f exist & be throughout I. Let there exist $M \in \mathbb{R}^+$ such that $|f^n(t)| \le M$ for all $t \in I$ and for all $n \in \mathbb{N}$, then following Lagrange's form of remainder,

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1} f^{(n+1)}(c)}{(n+1)!} \text{ for some } c \in I.$$

As $n \to \infty$, the upper bound in RHS tends to zero. So taking $n \to \infty$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^n(a)$$

Maclaurin's infinite series :

If f possess continuous derivatives of every order in $[0,h]$ and $x \in [0,h]$ and if further $\lim_{n\to\infty} R_n = 0$, then $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots\infty$

Expansion of some elementary functions :

I. $f(x) = \sin x, x \in \mathbb{R}$

For all $n \in \mathbb{N}$, $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ & these derivatives are continuous

R_n in Lagrange's form $R_n\big|_L = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{nl}\sin\left(\frac{n\pi}{2} + \theta x\right)$ for some $\theta \in (0,1)$

$\Rightarrow |R_n| \le \frac{x^n}{n!}$. As $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so $\lim_{n\to\infty} R_n = 0$

Hence Maclaurins expansion is valid here & so

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots,\ x \in \mathbb{R}$$

II. $f(x) = e^x,\ x \in \mathbb{R}$

Here $f^{(n)}(x) = e^x$ continuous for all $x \in \mathbb{R}$

$$R_n\big|_L = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}e^{\theta x},\ \theta \in (0,1)$$

$e^{\theta x} < e^x$ and $\lim_{n+\infty}\frac{x^n}{n!} = 0 \Rightarrow \lim_{n+\infty} R_n = 0$

so $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,\ x \in \mathbb{R}$$

III. $f(x) = \log(1+x), \quad -1 < x \le 1$

$f^{n}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, $x > -1$ & these f_n's are continuous, $-1 < x \le 1$

To consider $\lim\limits_{n\to\infty} R_n$

Case I : Let $0 < x < 1$

$R_n\big|_L = \dfrac{x^n}{n!} f^{(n)}(\theta x)$ for $\theta \in (0,1)$

$= \dfrac{(-1)^{n-1}}{n}\left(\dfrac{x}{1+\theta x}\right)^n$ for some $\theta \in (0,1)$

Here $0 < \dfrac{x}{1+\theta x} < 1$, so $\lim\limits_{n\to\infty}\left(\dfrac{x}{1+\theta x}\right)^n = 0$. Also $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$

$\Rightarrow \lim\limits_{n\to\infty} R_n = 0$

Case II $-1 < x < 0$

(In this case, it is not possible to ascertain $\lim\limits_{n\to\infty} R_n$ if R_n be considered in Lagrange's form

To substantiate this claim, let $x = \dfrac{-3}{4}$, $0 < \theta < \dfrac{1}{3}$. We see that $R_n \not\to 0$ as $n \to \infty$)

We take R_n in Cauchy's form = $R_n\big|_c = \dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$ for some $\theta \in (0,1)$

Here $R_n\big|_C = (-1)^{n-1} x^n \dfrac{1}{1+\theta x}\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}$

Also $|x| < | \Rightarrow \lim\limits_{n\to\infty} x^n = 0$. Again $\dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$

consequently $\lim\limits_{n\to\infty} R_n = 0$

Thus the conditions for Maclaurin's expansion of $\log(1+x)$ are satisfied.

Conseqently $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$

& $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+\frac{(-1)^{n-1}x^n}{n}+...$

If $x=1$ the series in RHS is $1-\frac{1}{2}+\frac{1}{3}...+\frac{(-1)^{n-1}}{n}+...$ which is Alternating series & is convergent by Leibnitz lest.

So the region of validity of above expansion of $\log(1+x)$ is $-1<x\le 1$

Iv. $f(x)=(1+x)^m$ where m is any real number other than positive integer.

(If $n\in\mathbb{N}$, the series will be finite series expansion having $(n+1)$ terms)

Here $f^{(n)}(x)=m(m-1)(m-2)...(m-n+1)(1+x)^{m-n}$

We take $|x|<1$ & f^n's are continuous in $-1<x<1$.

$R_n|_C=\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$ for some $\theta\in(0,1)$.

$=\frac{m(m-1)(m-2)...(m-n+1)}{(n-1)!}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},\ (1+\theta x)^{m-1}$

We know that $\lim_{n\to\infty}\frac{m(m-1)...(m-n+1)}{(n-1)!}x^n=0(\text{as }|x|<1)$

As $-1<x<1,\ 0<\theta<1$, we have $0<\frac{1-\theta}{1+\theta x}<1$ & so $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\to 0$ as $n\to\infty$

If $(m-1)$ be positive, $0<(1+\theta x)^{m-1}<2^{m-1}$ & if $(m-1)$ be negative, $(1+\theta x)^{m-1}<(1-|x|)^{m-1}$. As result $R_n|_C\to 0$ as $n\to\infty$

The conditions for the Maclaurin's expansion of $(1+x)^m$ are satisfied & so

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots$$

$$\Rightarrow (1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n\ldots, |x|<1.$$

Particular case $\frac{1}{ax+b}=\left(1+\frac{ax}{b}\right)^{-1}\cdot\frac{1}{b}\ (b\neq 0,\ a\neq 0)$

So the series can be deduced from above.

Application to approximate Calculations :

Examples :

1. Compute the approximate value of $\sqrt[4]{83}$ accurate to six decimal places.

We note that $\sqrt[4]{83}=\sqrt[4]{81+2}=3\left(1+\frac{2}{81}\right)^{1/4}$

By the expansion of $(1+x)^m$, taking $\frac{2}{81}$ in place of x & $\frac{1}{4}$ in place of m.

the expansion is

$$3\left[1+\frac{1}{4}\frac{2}{81}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{2}{81}\right)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(\frac{2}{81}\right)^3+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)}{4!}\left(\frac{2}{81}\right)^4+\ldots\right]$$

$$=3\left[1+\frac{1}{162}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2}\frac{2^2}{(81)^2}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{6}\frac{2^3}{(81)^3}+\ldots\right]$$ & this can be computed.

2. Compute the approximate value of $\cos 5^\circ$

As in case of sinx, Maclaurin's expansion for $\cos x, x\in\mathbb{R}$, can be deduced as follows :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$x = 5^\circ = \pi/36$ & putting $x = \pi/36$ & confining upto 2nd order terms,

$$\cos x \approx 1 - \frac{x^2}{2} \approx 1 - \left(\frac{\pi}{36}\right)^2 \text{ then}$$

$$R_4(x) = \left|\frac{\cos\theta x}{4!} x^4\right| \leq \frac{x^4}{4!}\left(= \frac{1}{4!}\left(\frac{\pi}{36}\right)^4\right) \text{ etc.}$$

13.6 Summary

In this unit, we have examined the concepts of derivative, differentiability and differential. We have also studied the Rolle's theorem, Lagrange's Mean Value Theorem, Cauchy's Mean Value Theorem, Taylor's Theorem. We have further developed the Maclaurin's infinite series to expansion of some elementary functions such as e^x $\sin x$, $\log(1+x)$, $(1+x)^m$, etc. We have explained the Young's form of Taylor's Theorem.

13.7 Exercise

1. Expand $f(x) = \sin^2 x - x^2 e^{-x}$ in positive integral powers of x upto the terms of fourth order.

2. Expand $f(x) = \ln(1+\sin x)$ upto the fourth order terms.

3. Show that $\sin(\alpha + h)$ differs from $\sin\alpha + h\cos\alpha$ by not more than $\frac{h^2}{2}$

4. Expand $\ln\cos x$ upto the term containing x^4

5. If $p(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$, show that

$$p(x) = 3(x-1) + 3(x-1)^4 + (x-1)^5$$

Young's form of Taylor's theorem :

(Note : This form of Taylor's theorem has very important & useful application in the theory of maxima-minima...)

If a function f be such that $f^{(n)}(a)$ exists and M is defined by the equation

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}M$$

$M \to f^n(a)$ as $h \to 0$

(In equivalent form, if we write the last term as $\frac{h^n}{n!}\left[f^n(a)+\varepsilon\right]$, then $\varepsilon \to 0$ as $h \to 0$)

Proof : Existence of $f^{(n)}(a)$ implies the existence of $f, f', ... f^{(n-1)}$ in

$N(a,\delta) \equiv (a-\delta, a+\delta)$ for some $\delta > 0$

Let $\varepsilon > 0$ be any number. First we take $h > 0$. We define a function ϕ as follows.

$$\phi(h)=f(a)+hf'(a)+...+\frac{h^{n-1}}{(n-1)}f^{n-1}(a)+\frac{h^n}{n!}\left[f^n(a)+\varepsilon\right]-f(a+h),\ 0<h<\delta$$

Here $\phi(0)=\phi'(0)=...=\phi^{n-1}(0)=0$ and $\phi^n(0)=\varepsilon>0$

Since $\phi^n(0)>0$ & $\phi^{n-1}(0)=0$, we see that there exists $\delta_1, 0<\delta_1<\delta$, such that $\phi^{(n-1)}(h)>0$ when $0<h<\delta_1$

Again $\phi^{n-1}(h)>0$ in $0<h<\delta_1$, $\Rightarrow \phi^{n-2}(h)>0, 0<h<\delta_1$

Proceeding in this way, we get $\phi(h)>0$ when $0<h<\delta_1$

Thus when $0<h<\delta_1$, we get

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)}f^{(n-1)}(a)+\frac{h}{n!}\left[f^n(a)+\varepsilon\right]-f(a+h)>0 \quad ...(1)$$

Similarly, we can show that there exists $0<\delta_2<\delta$ such that for $0<h<\delta_2$.

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}\left[f^n(a)-\varepsilon\right]-f(a+h)<0 \quad ...(2)$$

Let $\eta=\min\{\delta_1,\delta_2\}$, so for $0<h<\eta$, both (1) & (2) hold

Taking into account the given relation, we get

$f^{(n)}(a)-\varepsilon<M<f^{(n)}(a)+\varepsilon$ when $0<h<\eta$

$\Rightarrow \lim_{h\to0+} M=f^n(a)$

Taking $h<0$ arguing as before, we can get $\lim_{h\to0} M=f^n(a)$

Combining $\lim_{h\to0} M=f^n(a)$.

Unit- 15 □ Maxima-Minima of a Function

Structure

14.0 Objectives

This unit gives

- The concept of maxima-minima of a function
- Test of maxima and minima of a function using first derivation test
- Some miscellaneous exercise will also be introduced of the end of this unit

14.1 Introduction

The maxima and minima of a function, known collectively as extrema, are the largest and smallest value of the function. In this chapter we have shown how differentiation can be used to find the extrema values of a function.

14.2 Maxima-Minima of a function

Let $f : I \to \mathbb{R}$ where I denote any interval $\subset \mathbb{R}$.

f is said to have a relative maximum (relative minimum) at $c \in I$ if there exists a neighbourhood V of c such that $f(x) \leq f(c)\left(f(x) \geq f(c)\right)$ for all x in $V \cap I$. If f has either relative maximum or relative minimum at c, we say that f has a relative extremum at c.

Interior extremum Theorem :

Let c be an interior point of interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f exists at c. then $f'(c) = 0$

If possible, let $f'(c) > 0$. Then there exists a neighbouhood $V(\subset I)$ of c such that

$$\frac{f(x)-f(c)}{x-c} > 0(x \in V, x \neq c)$$

So if $x \in V,\ x > c,\ \ f(x)-f(c) > 0$ in V and if for $x \in V,\ x < c$

$f(x)-f(c) < 0$. As a result $f(x)-f(c)$ does not maintain the same sign througout the both-sided neighbourhood of c. As a result f has no extremum at c. Thus we arrive at a contradiction. So $f'(c) \not< 0$. As a result, $f'(c) = 0$

Note : f may not be derivable at an extremum. For example $f(x) = |x|$ has minimum at $x = 0$ but f' does not exist at $x = 0$

2. At a point of domain of f, $f'(x) = 0$ does, not ensure the existence of extrenum at that point.

For example, $f(x) = x^{2n+1}, n \in \mathbb{N}$

Note that $f(x) = 0$ at $x = 0$. But $f(x)-f(0) > 0$ if $x > 0,\ f(x)-f(0) < 0$ if $x < 0$ i.e $f(x)-f(0)$ does not maintain the same sign in both sided neighbourhood of 0.

Sufficient condition for maximum/minimum of function.

Let c be an interior point of the domain I of f

Let (i) $f^{(n)}(c)$ exist and $f^{(n)}(c) \neq 0$

(ii) $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$

Then if n is odd, f has no extremum at c.

But if n be even, f has an extremum at c and $f(c)$ is maximum or minimum at c according as $f^{(n)}(c) < 0$ or $f^{(n)}(c) > 0$

Proof : Recalling Young's form of Taylor's theorem,

$$f(c+h)-f(c)=hf'(c)+\frac{h^2}{2!}f''(c)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(c)+\frac{h^n}{n!}M \Rightarrow$$

$$M \to f^{(n)}(c) \text{ as } h \to 0$$

By (ii) $f(c+h)-f(c)=\frac{h^n}{n!}M$ ....(1)

Since $M \to f^{(n)}(c)$ as $h \to 0$, there exists $\delta > 0$ such that for $0<|h|<\delta$, M and $f^{(n)}(c)$ have the same sign.

So (1) $\Rightarrow$ when n is even, $f(c+h)-f(c)$ and M have the same sign. If $f^{(n)}(c)>0$, M & hence $f(c+h)-f(c)>0$ which implies that $f(c)$ is minimum. If $f^n(c)<0$, $f(c+h)-f(c)<0$, meaning thereby that $f(c)$ is maximum. If n be odd, $M<0$ & as a result, $f(c+h)-f(c)$ changes sign with the change in the sign of h. So if n be odd, $f(c)$ is not an extreme value.

14.3 First derivative test

Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$. Let $a<c<b$ and let f be differentiable in both (a,c) and (c,b). Then

(i) if there exists $\delta>0$ such that $f'(x)\geq 0$ in $(c-\delta,c)$ and $f'(x)\leq 0$ in $(c,c+\delta)$, then f has a local maximum at c.

(ii) if there exists $\delta>0$ such that $f'(x)\leq 0$ in $(c-\delta,c)$ and $f'(x)\geq 0$ in $(c,c+\delta)$, then f has a local minimum at c.

(iii) if $f'(x)$ maintains the same sign in both $(c-\delta,c)$ & $(c,c+\delta)$, then f has no extremum at c.

Proof : By hypothesis, f satisfies the conditions of L M V theorem in both $[c-\delta, c]$ & in $[c, c+\delta]$. So by L M V theorem, there exists $\xi \in (x,c)(\subset(c-\delta, c))$ and $\eta \in (c,x)(\subset(c, c+\delta))$ for which

$f(c)-f(x)=(c-x)f'(\xi)$ & $f(x)-f(c)=(x-c)f'(\eta)$

(i) given that $f'(x)\ge 0$ in $(c-\delta,c)$ & so $f(c)-f(x)\ge 0$

and $f'(x)\le 0$ in $(c,c+\delta)\Rightarrow f(x)-f(c)\le 0$

so in both cases, $f(c)\ge f(x)$ in $N(c,\delta)\cap[a,b]$

$\Rightarrow f$ has a local maximum at c.

(ii) given that $f'(x)\le 0$ in $(c-\delta, c)$ & so $f(c)-f(x)\le 0$

& $f'(x)\ge 0$ in $(c,c+\delta)\Rightarrow f(x)-f(c)\ge 0$ n $(c,c+\delta)$

In both cases, $f(x)-f(c)\ge 0$ or $f(x)\ge f(c)$ in $N(c,\delta)\cap[a,b]$

$\Rightarrow f$ has a local minima at c.

(iii) if $f'(x)$keeps same sign in both $(c-\delta,c)$ & in $(c,c+\delta)$, $f(x)-f(c)$ does not maintain the same sign & meaning thereby that f has no extremum at c.

Note : The conditions are sufficient but not necessary for the existence of extremum.

Let $f(x)=\begin{cases} 2x^2+x^2\sin\frac{1}{x}, & x\ne 0 \\ 0, & x=0 \end{cases}$

Here $x^2\le f(x)\le 3x^2$ & so f has a strict local minimum at $x=0$ but $f'(x)=4x+2x\sin\frac{1}{x}-\cos\frac{1}{x}$ is not of constant sign in any deleted neighbourhood of $x=0$.

Problems on Maxima-Minima :

1. Let $f(x)=1-\sqrt{(x^2)}$ where the square root is to be taken positive. Test for the existence of maximum/minimum

Here $f(x)=\begin{cases}1-x, & if\ x\geq 0\\ 1+x, & if\ x<0\end{cases}$

In $0<x<0+\delta$, $f(x)-f(0)=1-x-1=-x<0$ & in

$0-\delta<x<0,\ f(x)-f(0)=x<0$

So in any case, $f(x)-f(0)<0$ meaning f has a maximum at $x=0$

Note : $\lim\limits_{x\to 0+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0+}\dfrac{1-x-1}{x}=-1<0\Rightarrow R\,f'(0)<0$

& $\lim\limits_{x\to 0-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0-}\dfrac{1+x-1}{x}=1>0\ L.f'(0)>0$

Hence f' does not exist at $x=0$)

2. P is any point on the curve $y=f(x)$ & C is a fixed point not on the curve. If the length PC is either maximum or minimum, show that the line PC is perpendicular to the tangent at P.

Let $P(x_1, y_1)$ be any point on curve $y=f(x)$ & fixed point C be (α,β).

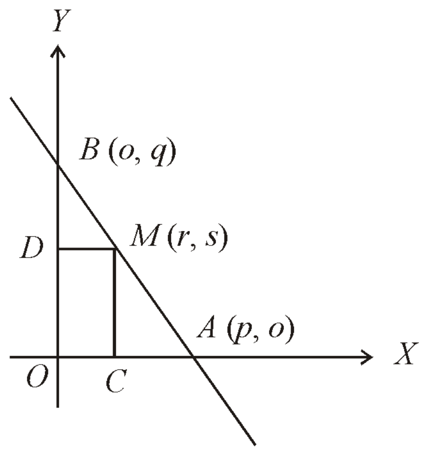
So $PC=\sqrt{\left[(x_1-\alpha)^2+(y_1-\beta)^2\right]}=\sqrt{\left[(x_1-\alpha)^2+(f(x_1)-\beta)^2\right]}$

$$\frac{d(PC)}{dx_1}=0\Rightarrow 2(x_1-\alpha)+2\left[f(x_1)-\beta\right]f'(x_1)=0\Rightarrow f'(x_1)=-\frac{x_1-\alpha}{f(x_1)-\beta}=m_1,$$

the slope of tangent at P.

Slope of $PC=m_2=\dfrac{f(x_1)-\beta}{x_1-\alpha}$ and hence $m_1m_2=-1$. Therefore for extremum of PC, PC is perpendicular to the tangent at P.

3. A rectangle is inscribed in a right-angled triangle so as to have one angle coincident with the right angle. Prove that its area is maximum when the opposite corner bisects the hypotenuse.



We take x-axis and y-axis as along the base & the perpendicular line of the given triangle, OCMD is the rectangle.

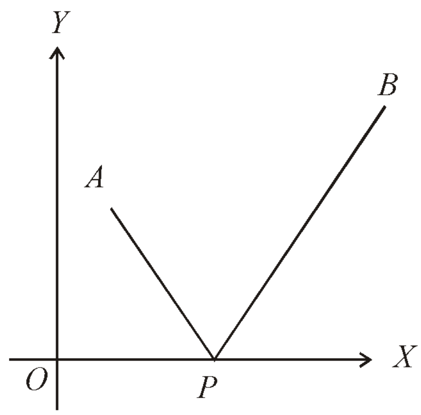
AB is $\frac{x}{p}+\frac{y}{q}=1$ & $M(r,s)$ is on $AB \Rightarrow \frac{r}{p}+\frac{s}{q}=1$

Area of rectangle = $rs = r\left(1-\frac{r}{p}\right)q = f(r)$

$f'(r)=q\left(1-\frac{2r}{p}\right)=0 \Rightarrow r = p/2$ so $f''(r)=\frac{-2q}{p}<0$

$f(r)$ is maximum when $r=\frac{p}{2},\ s=\frac{q}{2}$. Hence M is midpoint of AB.

4. Find a point on a given straight line such that the sum of its distances from two given points on the same side of the line is a minimum.



$A(a_1, b_1)$ and $B(a_2, b_2)$ are the given points. With reference to the given line as x-axis & line perpendicular to it as y-axis.

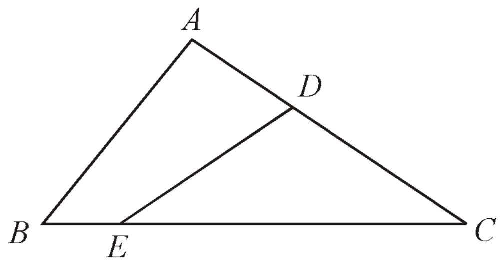
Let $f(x) = \sqrt{\left[(x-a_1)^2 + b_1^2\right]} + \sqrt{\left[(a_2 - x)^2 + b_2^2\right]}$

$f'(x) = 0 \Rightarrow x = \dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}$

Note that $f''\left(\dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}\right) > 0 \Rightarrow f(x)$ is minimum when $x = \dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}$

Consequently we conclude that when $f(x)$ is minimum, the x-co-ordinate of the point on the fixed line is same as the x-co-ordinate of the point which divides AB internally in the ratio $b_1 : b_2$.

5. A person wishes to divide a triangular field by a straight fence into two equal parts. Show how it is to be done so that the fence may be of minimum length.



Let the fence be DE.

Length $|DC| = y$, length $|EC| = x$

Length $|DE| = z$

So $z^2 = x^2 + y^2 - 2xy \cos C$ ...(1)

By hypothesis, $\dfrac{1}{2} xy \sin C = \dfrac{1}{2} \cdot \dfrac{1}{2} ab \sin C \Rightarrow y = \dfrac{ab}{2x}$ ...(2)

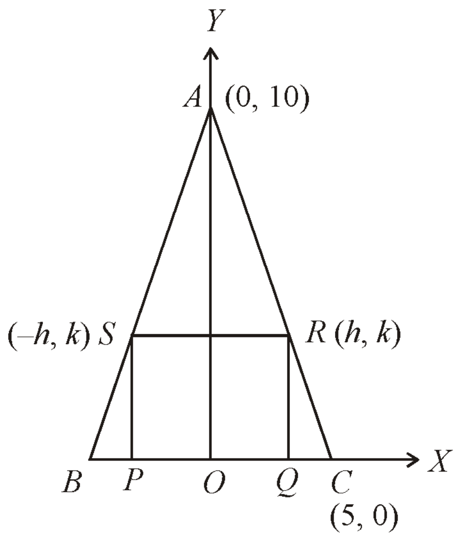
So $z^2 = x^2 + \dfrac{a^2 b^2}{4x^2} - ab \cos C = f(x)$ (say)

$$f'(x)=2x-\frac{a^2b^2}{2x^3}\Rightarrow x=\sqrt{\frac{ab}{2}}$$

$$f''(x)=2+\frac{3a^2b^2}{4}\Rightarrow f''\left(\sqrt{\frac{ab}{2}}\right)>0\Rightarrow \text{ minimum for } x=\sqrt{\frac{ab}{2}}$$

So z is minimum for $x=y=\sqrt{\frac{ab}{2}}$

6. Find the dimensions of the largest rectangle which can be inscribed in an isosceles triangle of base 10 cm & altitude 10 cm.



As $\Delta\, ABC$ is isosceles $(AB=AC)$, so median from A on BC is perpendicular on BC. We take mid point O of BC. as origin, positive side of x-axis along OC & positive side of y-axis along OA. Refering to the figure, area of rectangle A = 2hk (unit). AC is $\frac{x}{5}+\frac{y}{10}=1$ & so $2h+k=10\Rightarrow k=10-2h$

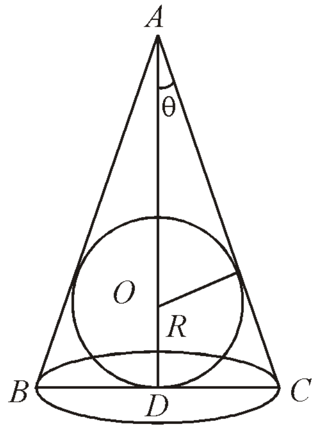
As $A=2h(10-2h)=f(h)$, $f'(h)=20-4.2h$

$$f'(h)=0\Rightarrow h=\frac{20}{8}=\frac{5}{2} \text{ (unit) \& } k=5 \text{ (unit)}$$

$f''(h) = -8 < 0$. So A is maximum for $h = \frac{5}{2}$, $k = 5$

These give the dimensions of the largest rectangle.

7. A cone is circumscribed about a sphere of radius R. Show that when the volume of the cone is minimum, its altitude is $4R$ and its semivertical angle is $\sin^{-1}\left(\frac{1}{3}\right)$.



Let the radius of the base be x (unit) and the height of the cone be z (unit). By property of elementary geometry,

A, O, D are collinear, $BD = DC$ & $AB = AC$

From the figure $\sin\theta = \frac{R}{Z-R} = \frac{x}{\sqrt{(x^2+z^2)}} \Rightarrow x^2 = \frac{R^2 z}{z-2R}$

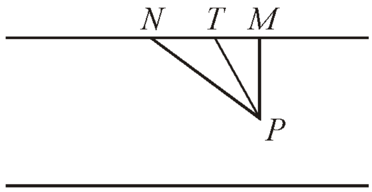
Volume $V = \frac{1}{3}\pi\, x^2 z = \frac{\pi R^2}{3} . \frac{z^2}{z-2R} = f(z)$

For extremum $f'(z) = 0$ and $\Rightarrow z = 4R$ and $f''(4R) > 0$

So V is minimum for $z = 4R$ and so $\sin\theta = \frac{1}{3}$ i.e. $\theta = \sin^{-1}\frac{1}{3}$.

8. A man in a boat $\frac{\sqrt{3}}{2}$ miles from the bank wishes to reach a village that is $5\frac{1}{2}$ miles distant along the bank from the point nearest to him. He can walk 4 m.p.h. & row 2 m.p.h. Where should he land in order to reach the village in the least time ?

Find also the time.



Let P be the position of the man & let he land at T. Let $MT = x$ miles.

$$PT = \frac{\sqrt{(4x^2+3)}}{2} \text{ \& } NT = \frac{11}{2} - x$$

Let t be the total time to reach N then

$$t = \frac{\sqrt{(4x^2+3)}}{4} + \frac{11}{8} - \frac{x}{4} = f(x) \text{ (say)}$$

For extremum $f'(x) = 0 \Rightarrow x = \frac{1}{2}$

Here $f''\left(\frac{1}{2}\right) > 0$ & so t is least for $x = \frac{1}{2}$. Then $t = 1\frac{3}{4}$ hours.

9. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

Let the cone be of semi-vertical angle α & radius of its base be r (unit). Then volume $V = \frac{1}{3}\pi r^3 \cot\alpha$ & surface area $S = \pi r^2 \operatorname{cosec}\alpha$

$$\frac{dV}{d\alpha} = 0 \text{ gives } \frac{1}{3}\pi\left[3r^2 \cot\alpha \frac{dr}{d\alpha} - r^3 \operatorname{cosec}^2\alpha\right] = 0$$

$$\Rightarrow \frac{dr}{d\alpha} = \frac{r^3 \operatorname{cosec}^2\alpha}{3r^2 \cot\alpha} = \frac{r \operatorname{cosec}^2\alpha}{3 \cot\alpha}$$

Also $\dfrac{dS}{d\alpha} = \pi\left[2r\ \text{cosec}\alpha \dfrac{dr}{d\alpha} - r^2 \text{cosec}\ \alpha\ \cot\alpha\right]$

putting the expression for $\dfrac{dr}{d\alpha}$, we get $\dfrac{dS}{d\alpha}$

$$= \pi\left[2r\ \text{cosec}\,\alpha. \frac{r\,\text{cosec}^2\alpha}{3\cot\alpha} - r^2\ \text{cosec}\,\alpha\cot\alpha\right]$$

$$\frac{dS}{d\alpha} = 0 \Rightarrow \frac{2r^2\,\text{cosec}^3\alpha}{3\cot\alpha} - r^2\ \text{cosec}\alpha\ \cot\alpha \Rightarrow \cot\alpha = \sqrt{2} \text{ or } \alpha = \cot^{-1}\sqrt{2}$$

As α passes through the value $\cot^{-1}\sqrt{2}, \dfrac{dS}{d\alpha}$ changes its sign from negative to positive & by first derivative test, S is minimum for $\alpha = \cot^{-1}\sqrt{2}$.

Then height = $r\cot\alpha = r\sqrt{2} = \sqrt{2}\times$ radius of the base.

14.4 Exercise-I

1. Prove that the greatest acute angle at which the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be cut by a concentric circle is $\tan^{-1}\left[\dfrac{a^2 - b^2}{2ab}\right]$

2. Show that the maximum & minimum values of $r^2 = x^2 + y^2$ where $ax^2 + 2hxy + by^2 = 1$ are given by the quadratic $\left(a - \dfrac{1}{r^2}\right)\left(b - \dfrac{1}{r^2}\right) = h^2$

[**Hints :** $x = r\cos\theta,\ y = r\sin\theta \Rightarrow \dfrac{1}{r^2} = a\cos^2\theta + h\sin 2\theta + b\sin^2\theta$

$\dfrac{dr}{d\theta} = 0 \Rightarrow \dfrac{\sin 2\theta}{2h} = \dfrac{\cos 2\theta}{a-b} = \dfrac{1}{\sqrt{\left[(a-b)^2 + 4h^2\right]}} = \dfrac{1}{k}$ (say)]

3. Show that the maximum value of $\dfrac{\log x}{x}$ in $0 < x < \infty$ is $\dfrac{1}{e}$.

4. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'a' is $2a/\sqrt{3}$.

5. If $f(x)=(x-a)^{2n}(x-b)^{2m+1}\,(m,n\in\mathbb{N})$, test for the existence of extremum.

6. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius 'a'.

7. A rectangle is drawn inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having sides parallel to axes of the ellipse. Show that when the rectangle is greatest, the diagonals of the rectangle will be along the conjugate diameters of the ellipse.

8. Of all triangles with the given base $2a$ unit & given area ah square unit, find that with the least perimeter.

9. At which point on the ellipse $\frac{x^2}{8}+\frac{y^2}{18}=1$ must a tangent be drawn such that the area of the triangle formed by the tangent & the co-ordinate axes is the smallest ?

10. Investigate for extremum :

(i) $f(x)=\begin{cases} -2x, & x<0 \\ 3x+5, & x\geq 0 \end{cases}$

(ii) $f(x)=\begin{cases} 2x^2+3, & x\neq 0 \\ 4, & x=0 \end{cases}$

4.5 Appendix

On monotonic functions :

In chapter II, we have just stated an important result on the continuity/discontinuity of monotone functions without giving the proof or any other property. Here we are going to discuss some properties of monotone functions :

Theorem : Let $f:[a,b]\to\mathbb{R}$ be monotonic increasing in $[a,b]$ & $a<x_\circ<b$.

Then $\lim\limits_{x\to x_0^-} f(x)$ (or $f(x_0-)$) and $\lim\limits_{x\to x_0+} f(x)$ (or $f(x_0+)$) both exist and

$$f(x_0-)\leq f(x_0)\leq f(x_0+)$$

Proof : Let $A=\{f(x)|a<x<x_0\}$. Since f is increasing function, the set A is bounded above by $f(x_0)$. By completeness axiom of $\mathbb{R}$, the set A has the least upper bound (Sup) C (say). Then $C \le f(x_0)$.

We propose to show that $f(x_0-)$ exists & that it equals C.

Let $\varepsilon>0$ be given. As $\sup A = C$, so corresponding to ε, there exists $\delta>0$ such that $a<x_0-\delta<x_0$ & $C-\varepsilon<f(x_0-\delta)\le C$.

Since f is monotonic increasing function, we have

$x_0-\delta<x<x_0 \Rightarrow f(x_0-\delta)\le f(x)\le C$.

So $C-\varepsilon<f(x)\le C$ for $x_0-\delta<x<x_0$

$\Rightarrow f(x_0-)=C\le f(x_0)$

Next let $B=\{f(x)|x_0<x<b|\}$, since f is increasing function, the set B is bounded below by $f(x_0)$. So the set B has the greatest lower bound (inf f) d (say). Then $f(x_0)\le d$.

We propose to show that $f(x_0+)$ exists & $f(x_0+)=d$

Let $\varepsilon>0$ be given. As $d=\inf B$, corresponding to ε, there exists $\delta>0$ such that $x_0<x_0+\delta<b \Rightarrow d\le f(x_0+\delta)<d+\varepsilon$

Since f is monotonic increasing function, we have

$x_0<x<x_0+\delta \Rightarrow d\le f(x)\le f(x_0+\delta)\Rightarrow d\le f(x)<d+\varepsilon,\ x_0<x<x_0+\delta$

$\Rightarrow f(x_0+)$ exists and $f(x_0+)=d\ge f(x_0)$

consequently $f(x_0-)\le f(x_0)\le f(x_0+)$.

Remark : At the end points, $f(a)\le f(a+), f(b-)\le f(b)$

Note : (1) Let $a<x<y<b$

Then $f(x+)=\inf\limits_{x<t<b} f(t)\le \inf\limits_{x<t<y} f(t)$

$f(y-)=\sup\limits_{a<t<y} f(t)\ge \sup\limits_{x<t<y} f(t)$

As $\inf_{x<t<y} f(t) \le \sup_{x<t<y} f(t)$, so $f(x+) \le f(y-)$

Also if $x_1, x_2 ..., x_n$ be n interior points of $(a,b), a < x_1 < x_2 < ... < x_n < b$, we have $\sum_{k=1}^{n}\left[f(x_k+) - f(x_k-)\right] \le f(b-) - f(a+)$

Note 2. Monotonic functions can have only discontinuity of first kind or in other words, monotonic functions can have no discontinuity of second kind.

Theorem : If $f:[a,b] \to \mathbb{R}$ be a monotonic, discontinuous function, the set E of points of discontinuity of f is atmost enumerable.

Proof : With every point x of E, we associate a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$

since $x_1 < x_2 \Rightarrow f(x_1+) \le f(x_2-)$ we see that $r(x_1) \ne r(x_2)$ if $x_1 \ne x_2$. We have thus established a one-one correspondence between the set E & a subset of the set of rational numbers. The set of rational numbers is atmost enumerable. Hence the result follows.

Note : 3. Jump of f at a point :

We know that the jump of function f at a point c is defined by

$j_f(c) = f(c+0) - f(c-0)$

Let $f:[a,b] \to \mathbb{R}$ be increasing function. Let us now consider jumps of f at distinct points.

Let $a < p < x < q < b$. f being increasing,

$f(a) \le f(p-0) \le f(p+0) \le f(x) \le f(q-0) \le f(q+0) \le f(b)$

$\Rightarrow j_f(p) + j_f(q) \le f(b) - f(a)$

$\Rightarrow$ for distinct points $p_1, p_2 ..., p_n$ in (a,b) we have

$j_f(p_1) + ... j_f(p_n) \le f(b) - f(a)$

Hence if there are k distinct points where the jump of f is at least t, then

$k \le \left[f(b) - f(a)\right]/t$

2. On second mean value theorem of differential calculus

Let $f:[a,b]\to\mathbb{R}$ be such that

(i) f, f' are continuous in $[a,b]$

(ii) f'' exists in (a,b)

Then there exists at least one point $c\in(a,b)$ such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(c)$$

Proof : Let us construct $F:[a,b]\to\mathbb{R}$ by

$F(x)=f(x)+(b-x)f'(x)+A(b-x)^2$ where the constant A is to be determined from $F(b)=F(a)\Rightarrow f(b)=f(a)+(b-a)f'(a)+A(b-a)^2$ ...(1)

Continuity of $f, f', (b-x)^2 \Rightarrow$ continuity of F in $[a,b]$

$F'(x)=f'(x)-f'(x)+(b-x)f''(x)-A.2(b-x)$ exists in (a,b) by hyp...(2)

By construction $F(b)=F(a)$. So F satisfies all the conditions of Rolle's theorem in $[a,b]$

Therefore, by Rolle's theorem, there exists $c\in(a,b)$ for which $F'(c)=0$

$\Rightarrow (b-c)f''(c)-2A(b-c)=0$

$\Rightarrow A=\frac{1}{2}f''(c)$

Putting in (1) $f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(c)$

Note this result is in fact Taylor's theorem for n = 2.

3. On convex function

Definition : Let $f:I\to\mathbb{R}$ where I is some open interval $\subset\mathbb{R}$. If for pair of points $x_1, x_2\in I$ and any number $\alpha_1, \alpha_2(\geq 0), \alpha_1+\alpha_2=1$, the inequality

$f(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f(x_1) + \alpha_2 f(x_2)$...(1) holds, then f is said to be a convex function or convex downward. If only < holds in (1), f is stricly convex on I. If the opposite equality holds $f(\alpha_1 x_1 + \alpha_2 x_2) \ge \alpha_1 f(x_1) + \alpha_2 f(x_2)$ for any pair x_1, x_2 as stated above, f is concave or convex upward on I.

Remarks : Taking $x = \alpha_1 x_1 + \alpha_2 x_2$ when $\alpha_1 + \alpha_2 = 1$ we have

$$\alpha_1 = \frac{x_2 - x}{x_2 - x_1}, \ \alpha_2 = \frac{x - x_1}{x_2 - x_1}$$

& hence $\dfrac{f(x) - f(x_1)}{x - x_1} \le \dfrac{f(x_2) - f(x)}{x_2 - x}$ for $x_1 < x < x_2$...(2)

Theorem : A necessary & sufficient condition for $f : I \to \mathbb{R}$ that is derivable on I to be convex (downward) on I is that its derivative f' to be increasing on I. (A strictly increasing f' corresponds to strictly convex function)

Proof : Let the convex function $f : I \to \mathbb{R}$ be differentiable on I.

In (2) taking x tends first to x_1 & then to x_2, we have

$$f'(x) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le f'(x_2)$$

Applying L M V theorem to f in $[x_1, x_2]$, there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi)$$

So $f'(x_1) \le f'(\xi) \le f'(x_2)$ & so the derivative of f is monotonic.

For a strictly convex function. $f'(x_1) < f'(\xi) < f'(x_2)$ & f' is strictly monotonic.

Converse : L M V theorem $\Rightarrow f(x) - f(x_1) = (x - x_1) f'(\xi_1)$ for some $\xi_1 \in (x_1, x)$

& $f(x_2) - f(x) = (x_2 - x) f'(\xi_2)$ for some $\xi_2 \in (x, x_2)$

If $f'(\xi_1) \le f'(\xi_2)$ then (2) follows & f is convex function.

Note : Let $f : I \to \mathbb{R}$ be twice differentiable on the open interval $I(\subset \mathbb{R})$

Then f is convex on I if $f''(x) > 0$ throughout I.

Theorem : Let f be convex on the open interval $I(\subset \mathbb{R})$. Then

(i) the limits $\lim\limits_{h\to 0-} \dfrac{f(x+h)-f(x)}{h}$ & $\lim\limits_{h\to 0+} \dfrac{f(x+h)-f(x)}{h}$ both exist for each $x \in I$

(ii) f is continuous on I.

Proof : As f is convex on I, by (2) for $x_1 < x_2 < x_3 (\text{all} \in I)$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

we take $x_1 = x,\ x_2 = x+h_1,\ x_3 = x+h_2$ where $0 < h_1 < h_2$

Then $\dfrac{f(x+h_1)-f(x)}{h_1} \le \dfrac{f(x+h_2)-f(x)}{h_2}$

so if $F(h) = \dfrac{f(x+h)-f(x)}{h}$, $h > 0$, then $F(h)$ increases in some interval $(0, \delta)$

so $\lim\limits_{h\to 0+} F(h)$ exists. Similarly $\lim\limits_{h\to 0-} F(h)$ exists

$$\lim_{h\to 0+} \{f(x+h)-f(x)\} = \lim_{h\to 0+} \left\{\frac{f(x+h)-f(x)}{h} h\right\} = 0$$

Similarly $\lim\limits_{h\to 0+} \{f(x+h)-f(x)\} = 0$. Hence f is continuous function on I.

Notes : The result may fail if I be not open.

$$f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 3, & x = 1 \end{cases}$$

Examples :

(1) Let $f(x)=x^{\alpha}, x>0, \alpha \in \mathbb{R}$

$$f''(x)=\alpha(\alpha-1)x^{\alpha-2}\begin{cases}>0, & \text{for } \alpha<0 \text{ or } \alpha>1\\ <0, & \text{for } 0<\alpha<1\end{cases}$$

So if $\alpha<0$ or $\alpha>1$ f is strictly convex & for $0<\alpha<1$, f is concave function

(2) $\sin x$ is strictly convex when $2k\pi<x<(2k+1)\pi$ & concave when $(2k-1)\pi<x<2k\pi$.

(3) $a^{x}(a>0, a\neq 1)$ is convex for $0<a<1$, $a>1$

4. On Periodic function :

A function $f:\mathbb{R}\to\mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number p such that $f(x+p)=f(x)$ for all x. The least positive value of p for which $f(x+p)=f(x)$ is known as the period of f or the primitive period of f. For example, sinx, cosx are periodic functions of period 2π.

Result : (i) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic with period 1. Then (i) f is bounded above & below and achieves its maximum & minimum values. (ii) there exists a real number $x_{\circ}$ such that $f(x_{\circ}+\pi)=f(x_{\circ})$.

Proof : Let f_1 be the restriction of f to $[0,2]$. As $f(x+1)=f(x)$ for all $x\in\mathbb{R}$. the ranges of f & f_1 are same. f is bounded & attains its maxima and minima there in. As f is continuous & periodic on $\mathbb{R}$. So f is bounded above & below and achieves its maximum and minimum. Let f attain its maximum & minimum at p and q respectively.

Hence $f(p+\pi)-f(p)\leq 0$ & $f(q+\pi)-f(q)\geq 0$

If the equality holds in the first case, p is desired $x_{\circ}$

If the equality holds in the second case, q is desired x_0.

Otherwise : Let $g(x)=f(x+\pi)-f(x), x\in\mathbb{R}$

So g is continuous & $g(p)g(q)<0$. Applying Bolzano's theorem on continuous function to g on $[p,q]$ (or on $[q,p]$), there exists $x_\circ \in \mathbb{R}$ for which $g(x_\circ)=0$ i.e $f(x_\circ+\pi)=f(x_\circ)$.

Result 2. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic & let $T(>0)$ be the period. Then f is uniformly continuous on $\mathbb{R}$

Proof : Continuity of f in $[-T, 2T]$ implies f is uniformly continuous in $[-T, 2T]$. For arbitrary $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any pair of points $x, y\in[-T, 2T]$ satisfying $|x-y|<\delta$, we have $|f(x)-f(y)|<\varepsilon$ (1)

We take $0<\delta<T$

Let $x, y, \in\mathbb{R}$ satisfying $|x-y|<\delta$

There exists $n\in\mathbb{Z}$ such that $nT\le x<(n+1)T$ & so $x-nT\in[0,T]$ & $y-nT\in[-T,2T]$.

Note that $|(x-nT)-(y-nT)|=|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$

$\Rightarrow$ Uniform Continuity of f on $\mathbb{R}$

Result 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic function. Show that if $\lim\limits_{x\to\infty} f(x)$ exists, then f is a constant function.

Proof : Let $\lim\limits_{x\to\infty} f(x)=l(\in\mathbb{R})$ and $T(>0)$ be the period of f. We propose to show that $f(x)=l$ for all x.

If not and if possible, let there exist $a\in\mathbb{R}$ such that $f(a)\neq l$.

Let $0<\varepsilon<\dfrac{|f(a)-l|}{10}$. As $\lim\limits_{x\to\infty} f(x)=l$, so corresponding to above ε, there exists $G\in\mathbb{R}$ such that

$|f(x)-l|<\varepsilon$ whenever $x>G$ ...(1)

By Archemedean property of real numbers, there exists $n\in\mathbb{N}$ such that $nT>G-a$ so $nT+a>G$ ...(2)

By (1) and (2) $|f(a+nT)-l|<\varepsilon \Rightarrow |f(a)-l|<\varepsilon$ (as T is the period of f)

$\Rightarrow 10\varepsilon<\varepsilon$ but this is absurd as $\varepsilon>0$

So $f(x)=l$ for all x & as a result f is constant function

Example : $\lim\limits_{x\to\infty}\sin x$ does not exist

14.6 Summary

In this unit we have defined the term extrema of a function and shown how the differentiation can be used to find the maxima & minima. We have also studied the first derivative test for extrema and formulated a sufficient condition for extrema of a function.

14.7 Miscellanous Exercise

1. Let $f,\ g:S\to\mathbb{R}\,(S\subset\mathbb{R})$ and p be an accumulation point of S.

Let $\lim\limits_{x\to p} f(x)=l(\in\mathbb{R})$ and $\lim\limits_{x\to p} g(x)=m(\in\mathbb{R})$

Test for the existence of

(i) $\lim\limits_{x\to p}\max\{f,g\}$ (ii) $\lim\limits_{x\to p}\min\{f,g\}$

2. Using the results (i) $\left\{\left(1+\frac{1}{n}\right)^n\right\}_n$ converges to e and (ii) for $x>1$ there exists $n\in\mathbb{N}$ such that $n\le x<n+1$, show that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

3. Let $f:[a,\infty)\to\mathbb{R}$. Then show that $\lim\limits_{x\to\infty}f(x)$ exists if and only if for every $\varepsilon>0$, there exists $X>a$ such that

$|f(x)-f(y)|<\varepsilon$ for all $x,y>X$

4. Let $n\in\mathbb{N}$ and $\lambda>0$. Show that there exists unique $y>0$ such that $y^n=\lambda$

5. Let $f(x)=\begin{cases} x^2-2x, \text{ when } x \text{ is rational} \\ 3x-6, \text{ when } x \text{ is irrational} \end{cases}$

If $a \in \mathbb{R}$, examine whether $\lim_{x\to a} f(x)$ exists.

6. Prove or disprove : If $f(x)=\begin{cases} x, & \text{when } x \text{ is rational} \\ 1-x, & \text{when } x \text{ is irrational} \end{cases}$ in $[0,1]$

then $g(x)=f(x)f(1-x)$ is continuous everywhere.

7. Let $f(x)=\lim_{n\to\infty}\frac{x^{2n}+3}{x^{2n}+1}$, $x\in[-2,2]$. Test for the continuity of f in $[-2,2]$.

8. A function $f:[0,1]\to\mathbb{R}$ is continuous on $[0,1]$ and f assumes only rational values on [0, 1]. Prove that f is constant.

9. $f:[0,2]\to\mathbb{R}$ be continuous on $[0,2]$ and $f(0)=f(2)$ prove there exists *a* point c in [0, 1] such that $f(c)=f(c+1)$

10. Prove that $\cos x = x^2$ for some $x\in\left(0,\frac{\pi}{2}\right)$

11. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} 0, & \text{if } x=0 \text{ or } x \text{ is irrational} \\ \frac{1}{q^3}, & \text{if } x=\frac{p}{q} \text{ where } p\in\mathbb{Z}, q\in\mathbb{N} \text{ and } gcd(p,q)=1 \end{cases}$$

show that f is differentiable at 0 and $f'(0)=0$

(Hints : For $x\neq 0$, $0\le\left|\frac{f(x)}{x}\right|\le|x|^2$)

12. $f:\mathbb{R}\to\mathbb{R}$ satisfies the condition $|f(x)-f(y)|\le|x-y|^{\alpha}$ when $\alpha>1$ for all $x,y\in\mathbb{R}$. Show that f is constant.

13. Prove that the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3=0$ has only one real root.

14. Prove that between any two real roots of $e^x \sin x+1=0$, there is at least one real root of $\tan x+1=0$.

15. Given that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ & $f''(x)$ exists for all $x\in(a,b)$. If $a<c<b$ and $f(a)=f(b)=0$, prove that there exists a point $\xi\in(a,b)$ such that $f(c)=\frac{1}{2}(c-a)(c-b)f''(\xi)$

16. If $a<c<b, f''$ exists in $(a,b), f$ and f' are continuous at the end points a and b, show that there exists $t\in(a,b)$ such that

$$f(c)=\frac{(b-c)f(a)+(c-a)f(b)}{b-a}+\frac{1}{2}(c-a)(c-b)f''(t)$$

(**Hints :** By second mean value theorem,

$$f(a)=f(c)+(a-c)f'(c)+\frac{1}{2}(a-c)^2 f''(\xi) \text{ for some } \xi\in(a,c) \text{ \&}$$

$$f(b)=f(c)+(b-c)f'(c)+\frac{1}{2}(b-c)^2 f''(\eta) \text{ for some } \eta\in(c,b)$$

For ex. 15, take $f(a)=0=f(b)$)

17. Let $f:[a,b]\to\mathbb{R}$ be continuous positive valued function, differentiable in (a,b). Show that there exists $c\in(a,b)$ such that

$$\frac{f(b)}{f(a)}=e^{(b-a).f'(c)/f(c)}$$

(**Hints :** Applying L M V theorem to $F(x)=\ln f(x)$ in [a, b])

18. Let f'' exist in $[0,a]$, $a>0$. If $f(0)=0$ and $0<x\le a$, show that there exists $\xi\in(0,x)$ such that $f'(x)-\frac{f(x)}{x}=\frac{1}{2}xf''(\xi)$

Hence show that $\frac{f(x)}{x}$ is increasing in the above interval if $f''(x) > 0$ & is decreasing if $f''(x) < 0$ for all x.

[Consider $\varphi : [0, a] \to \mathbb{R}$ defined by $\phi(x) = -f(x) + xf'(x) + \frac{1}{2}Ax^2$ where $\varphi(0) = \phi(a)$]

19. Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and be derivable in (a, b). If $f^2(b) - f^2(a) = b^2 - a^2$, show that the equation $f'(x).f(x) = x$ has at least one root in (a, b).

20. An open tank with a square base must have a capacity of v liters, what size will it be if the least amount of tin is used.

21. On the curve $y = \frac{1}{1 + x^2}$, find a point at which the tangent forms with the x-axis the greatest (in absolute value) angle ?

22. Test the following function for increase or decrease :

$$y = \frac{1}{5}x^5 - \frac{1}{3}x^3$$

23. What right triangle of given perimeter $2p$ has the greatest area ?

24. p is the length of perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the normal at a variable point on the ellipse. Show that the greatest value of p is $a - b$.

25. Find the relative extremum points of f defined by

$$f(x) = \frac{x^2}{(1 - x)^3}$$

14.8 Further Readings

1. Introduction to Mathematical Analysis — A gupta (Academic Publishers)
2. Mathematical Analysis — S. C. Malik & Arora (Wiley Eastern Limited)
3. Introduction to Real Analysis — S. K. Mapa (Sarat Book Distributors)
4. First Course in Real Analysis — S. K. Mukherjee (Academic Publishers) (second edition)
5. Mathematical Analysis — Shantinarayan (S. Chand & Co.)

Unit- 15 □ Miscellaneous Examples & Exercises

Structure

15.0 Objectives

The main objective of the unit is to prevent various Examples and exercises of unit **13 and 14.** Also the solutions of each problem have also been given.

15.1 Unit-13A

Problems :

1. Let $f(x) = \dfrac{1}{x^2},\ x \neq 0, x \in \mathbb{R}$

(a) Determine $f(E)$ where $E = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

(b) Determine $f^{-1}(G)$ where $G = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$

2. Let $g(x) = x^2$ and $f(x) = x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h = g \circ f$.

(a) Find $h(E)$ where $E = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

(b) Find $h^{-1}(G)$ where $G = \{x \in \mathbb{R} : 0 \leq x \leq 4\}$

3. Show that if $f : A \to B$ and E, F are subsets of A then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$

4. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

5. Show that the function f defined by $f(x) = \dfrac{x}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.

6. For $a, b \in \mathbb{R}$ with $a < b$, find an explicit bijection of $A = \{x : a < x < b\}$ onto $B = \{y : 0 < y < 1\}$

7. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Show that if $g \circ f$ is injective, f is injective

(b) Show that if $g \circ f$ is surjective, g is surjective.

8. Let f, g be functions such that $(g \circ f)(x) = x$ for all $x \in D(f)$ and $(f \circ g)(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

9. Suppose that f and g are real-valued functions with common domain $D(\subset \mathbb{R})$. Assume that f and g are bounded.

Then (a) if $f(x) \le g(x) \forall x \in D$, then sup $f(D)) \le$ sup g (D)

(b) if $f(x) \le g(y) \forall x, y \in D$, then sup $f(D) \le$ inf g (D)

10. Let X be a nonempty set, and let f and g be defined on X and bounded. Show that

$$\sup\{f(x) + g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \le \inf\{f(x) + g(x) : x \in X\}$

11. Let $X = Y = \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) = 2x + y$

(a) For each $x \in X$, find $f(x) = \sup\{h(x, y) : y \in Y\}$, then find $\inf\{f(x) : x \in X\}$

(b) For each $y \in Y$, find $g(y) = \inf\{h(x, y) : x \in X\}$ then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (*a*).

Solutions of problems :

2. $h = gof : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = (g \circ f)(x)$$

$$= g(f(x))$$

$$= g(x+2)$$

$$= (x+2)^2$$

$$= x^2 + 4x + 4$$

(a) $0 \le x \le 1$

$$\Rightarrow 2 \le x + 2 \le 3$$

$$\Rightarrow 4 \le (x+2)^2 \le 9$$

so $h(E) = \{y \in \mathbb{R} : 4 \le y \le 9\}$

(b) $0 \le (x+2)^2 \le 4$

$$\Rightarrow -2 \le (x+2) \le 2$$

$$\Rightarrow -4 \le x \le 0$$

so $h^{-1}(G) = \{x \in \mathbb{R} : -4 \le x \le 0\}$

where $G = \{x \in \mathbb{R} : 0 \le x \le 4\}$

5. For $x, y \in \mathbb{R}$, let $f(x) = f(y)$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}} = \frac{y}{\sqrt{1+y^2}}$$

$$\Rightarrow \frac{x^2}{1+x^2} = \frac{y^2}{1+y^2}$$

$$\Rightarrow x^2 + x^2y^2 = y^2 + x^2y^2$$

$$\Rightarrow (x+y)(x-y) = 0$$

Then $(x+y)(x-y)=0$. But $x+y \neq 0$ for then $y = -x$ & $\frac{x}{\sqrt{(1+x^2)}} = \frac{y}{\sqrt{(1+y^2)}}$ does not stand.

If possible let $x \neq y$

as $\sqrt{1+x^2}, \sqrt{1+y^2}$

both are positive and *x*, *y* both are not zero.

So we arrive at a contradiction.

$\therefore f$ is one to one

$$-1 < y < 1$$

$$\Rightarrow 0 \leq y^2 < 1$$

$$\Rightarrow 0 < 1 - y^2 \leq 1$$

For $x = \frac{y}{\sqrt{1-y^2}}$, $f(x) = \frac{\frac{y}{\sqrt{1-y^2}}}{\sqrt{1+\frac{y^2}{1-y^2}}} = \frac{\frac{y}{\sqrt{1-y^2}}}{\frac{1}{\sqrt{1-y^2}}} = y$

$\therefore f : \mathbb{R} \to \{y \in \mathbb{R} : -1 < y < 1\}$ is onto. Thus *f* is a bijecton of $\mathbb{R}$ onto $\{y \in \mathbb{R} : -1 < y < 1\}$.

and $f^{-1}(x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$

7. Let for $a, b \in A$ so that

$f(a) = f(b) \Rightarrow g(f(a)) = g(f(b))$

$\Rightarrow (g \circ f)(a) = (g \circ f)(b)$ (As $g \circ f$ is injective)

$\Rightarrow a = b$

Thus f is injective

Let $y \in C$. As $g \circ f : A \to C$ is surjective, there exists $x \in A$ such that $(g \circ f)(x) = y$ or $g(f(x)) = y$

$\Rightarrow$ corresponding to y of C, $\exists\ f(x) \in B \Rightarrow g$ is surjective.

8. Let $y \in D(g), f(g(y)) = (f \circ g)(y) = y$

$\therefore f : D(f) \to D(g)$ is bijective and $f(g(x)) = x \forall x \in D(g)$

therefore, $f^{-1} : D(g) \to D(f)$ is given by $f^{-1}(x) = g(x) \forall x \in D(g)$

Thus, $g = f^{-1}$.

9. (a) $f(x) \le g(x) \forall x \in D$ so, $f(x) \le g(x) \le \sup\{g(x)\}, x \in D$

Let $\lambda = \sup f(D)$, $\mu = \sup g(D)$

$\therefore f(x) \le \mu \forall x \in D$

If possible let $\lambda > \mu$ choose $\varepsilon = \dfrac{\lambda - \mu}{2}$.

Then there is $x \in D$ s.t. $f(x) > \lambda - \varepsilon$

$$= \lambda - \left(\frac{\lambda - \mu}{2}\right)$$

$$= \frac{2\lambda - \lambda + \mu}{2}$$

$$= \frac{\lambda + \mu}{2}$$

$$f(x)-\mu > \frac{\lambda+\mu-2\mu}{2} = \frac{\lambda-\mu}{2} > 0$$

$\Rightarrow f(x) > \mu$ contradiction. Therefore, $\sup f(D) \le \sup g(D)$

(b) $f(x) \le g(y) \forall x, y \in D$. Fix $a\ y_\circ \in D$. Then $f(x) \le g(y_\circ) \forall\ x \in D$

Now make $y_\circ$ arbitrary. If possible let Inf g (D) < sup f (D)

Take $\varepsilon = \frac{\sup f(D) - \inf g(D)}{2} > 0$

Then there exists $a, b \in D$ such that

$\sup f(D) - \varepsilon < f(a)$ and $g(b) < \inf g(D) + \varepsilon$

$$\Rightarrow \sup f(D) - \frac{\sup f(D) - \inf g(D)}{2} < f(a)$$

$$\Rightarrow \frac{\sup f(D) + \inf g(D)}{2} < f(a) \ldots(1)$$ and

$$g(b) < \frac{\inf g(D) + \sup f(D)}{2} \ldots(2)$$

But $f(a) \le g(b) \ldots(3)$

From (1), (2), (3) we have

$$\frac{\sup f(D) + \inf g(D)}{2} < \frac{\inf g(D) + \sup f(D)}{2}$$

which is a contradiction.

Some solved problems on Limits.

1. Show that for $f(x) = [x]$, $\lim_{x \to 0} [x]$ does not exist.

In an arbitrary neighbourhood of 0, say $N(0,1)$

$$f(x) = -1 \quad \text{if } -1 < x < 0$$
$$= 0 \quad \text{if } 0 \le x < 1$$

Let us consider two sequences $\{x_n\}_n$ and $\{y_n\}_n$ in $N(0,1)$ defined by $x_n = \frac{1}{n+1}, y_n = \frac{-1}{n+1}, n \in \mathbb{N}$. Then the sequences $\{x_n\}_n$ and $\{y_n\}_n$. converge to 0. But the sequence $\{f(x_n)\}$ is {0, 0, 0,...} This converges to 0, and the sequence $\{f(y_n)\}_n$ is {–1, –1, –1,...}. This converges to –1.

$\therefore \lim\limits_{x\to 0}[x]$ does not exist.

2. $\lim\limits_{x\to 0}$ sgn x does not exist.

Let $f(x) = \operatorname{sgn} x$.

Then $f(x) = 1$ for $x > 0$

$= 0$ for $x = 0$

$= -1$ for $x < 0$

Domain D of f is $\mathbb{R}$. 0 is an accumulation point of D. Let us consider two sequences $\{x_n\}_n$ in $\mathbb{R}$ and $\{y_n\}_n$ in $\mathbb{R}$ defined by $x_n = \frac{1}{n}$, $y_n = \frac{-1}{n}$, $n \in \mathbb{N}$.

Then $\lim\limits_{n\to\infty} x_n = 0 = \lim\limits_{n\to\infty} y_n$

Also, $f(x_n) = 1 \;\&\; f(y_n) = -1 \;\forall\; n \in \mathbb{N}$

Therefore, $\lim\limits_{n\to\infty} f(x_n) = 1$, $\lim\limits_{n\to\infty} f(y_n) = -1$ which are different

$\therefore \lim\limits_{x\to 0}$ sgn x does not exist.

3. Show that the following limits do not exist :

(a) $\lim\limits_{x\to 0} \frac{1}{x^2} (x > 0)$

(b) $\lim\limits_{x\to 0} (x + \operatorname{sgn}(x))$

4. Use either $\varepsilon-\delta$ definition of limit or the sequential criterion for limits, to establish.

(a) $\lim\limits_{x\to 1}\frac{x}{1+x}=\frac{1}{2}$

(b) $\lim\limits_{x\to 0}\frac{x^2}{|x|}=0$

5. $f(x)=\operatorname{sgn} x$ Examine if $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R}$

Let $D_1=D\cap(0,\infty)$ and $D_2=D\cap(-\infty,0)$. 0 is an accumulation point of both D_1 and D_2.

$f(x)=1\ \forall\ x\in D_1$ and $f(x)=-1\ \forall\ x\in D_2$

Therefore $\lim\limits_{x\to 0+} f(x)=1$ and $\lim\limits_{x\to 0-} f(x)=-1$

6. $f(x)=e^{\frac{1}{x}}$.

Examine if $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ exist.

Here the domain D of f is $R\setminus\{0\}$. Let $D_1=D\cap(0,\infty)$ and $D_2=D\cap(-\infty,0)$. 0 is an accumulation point of D_1 and D_2. f is unbounded on $N(0)\cap D_1$ for any neighbourhood $N(0)$ of 0. Therefore $\lim\limits_{x\to 0+} f(x)$ does not exist.

We have $e^t>t>0\ \forall\ t>0$. Take $t=-\frac{1}{x}, x<0$ we have $e^{-\frac{1}{x}}>-\frac{1}{x}>0$, and this imples $0<e^{-\frac{1}{x}}<-x\ \forall x<0$

By Sandwich Theorem, $\lim\limits_{x\to 0-} f(x)=0$

7. Show that $\lim\limits_{x\to 0} f(x)=\infty$, where $f(x)=\frac{1}{x^2}$

In every neighbouhood of 0. f is unbounded above. Let us choose $G > 0$. Then $f(x) > G \ \forall\ x$ satisfying $x < \frac{1}{\sqrt{G}},\ x \neq 0$

That is, $f(x) > G \ \forall\ x \in N'(0, \delta)$ where $\delta = \frac{1}{\sqrt{G}}$.

Therefore $\lim\limits_{x \to 0} f(x) = \infty$

8. Examine if $\lim\limits_{x \to \frac{\pi}{2}} \tan x$ exists.

Let $f(x) = \tan x$. The domain D of f is $\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\} = \mathbb{R}^*$

$D_1 = D \cap \left(\frac{\pi}{2}, \infty\right)$, $D_2 = D \cap \left(-\infty, \frac{\pi}{2}\right)$ $D_1 \neq \phi$, $D_2 \neq \phi$. Also $\frac{\pi}{2}$ is an acumulation point of both D_1 and D_2. In $\frac{\pi}{2} < x < \pi$, f is monotonic decreasing function unbounded below. Therefore, $\lim\limits_{x \to \frac{\pi}{2}+} f(x) = -\infty$.

In $0 < x < \frac{\pi}{2}$, f is a monotonic increasing function unbounded above. Therefore, $\lim\limits_{x \to \frac{\pi}{2}-} f(x) = \infty$. We conclude that $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exist.

9. Using Cauchy's principle, prove that $\lim\limits_{x \to \infty} \cos x$ does not exist.

Let $f(x) = \cos x, x \in \mathbb{R}$. Here the domain of f is $\mathbb{R}$. Let us choose $\varepsilon = \frac{1}{2}$. In order that $\lim\limits_{x \to \infty} f(x)$ should exist, it is necessary that there exists a positive G such that $|f(a) - f(b)| < \frac{1}{2}$ for every pair of points $a, b > G$.

For a given positive real number G. We can find a natural number K such that $2K\pi > G$

Let $a=(2k+1)\pi$, $b=2k\pi$. Then $a,b>G$ and $f(a)=-1$, $f(b)=1$. Therefore, $|f(a)-f(b)|\not<\varepsilon$ for some pair of points $a,b>G$. This shows that cauchy's condition for the existence of $\lim\limits_{x\to\infty}\cos x$ is not satisfied. Therefore $\lim\limits_{x\to\infty}\cos x$ does not exist.

10. Prove that $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$.

We have $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=e$. Let $y=\frac{1}{x}$. As $x\to\infty$, $y\to 0+$

and $x\to-\infty$, $y\to 0-$

Then $e=\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=\lim\limits_{y\to 0+}(1+y)^{1/y}$...(1)

$e=\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=\lim\limits_{y\to 0-}(1+y)^{1/y}$....(2)

From (1) & (2), $\lim\limits_{y\to 0}(1+y)^{1/y}=e$

Thus, $\lim\limits_{x\to 0}(1+x)^{1/x}=e$

1. Use sequential criterion for limits to show that following limits do not exist.

(i) $\lim\limits_{x\to 0}\cos\frac{1}{x^2}$

(ii) $\lim\limits_{x\to\infty}x^{1+\sin x}$

2. $f(x)=x, \quad x\in Q$

$\quad =2-x, \; x\in\mathbb{R}\setminus Q$

Show that (i) $\lim\limits_{x\to 1}f(x)=1$, (ii) $\lim\limits_{x\to c}f(x)$ does not exist, if $c\neq 1$.

3. Show that the following limits do not exist

(i) $\lim\limits_{x\to 0} \dfrac{1}{1+e^{1/x}}$

(ii) $\lim\limits_{x\to 0} \dfrac{2x+|x|}{2x-|x|}$

4. Evaluate the limits

(i) $\lim\limits_{x\to 0+} \sqrt{x-[x]}$, $\lim\limits_{x\to 0-} \sqrt{x-[x]}$

(ii) $\lim\limits_{x\to 0+} x\left[\dfrac{1}{x}\right]$, $\lim\limits_{x\to 0-} x\left[\dfrac{1}{x}\right]$

(iii) $\lim\limits_{x\to 0+}\left[\dfrac{\sin x}{x}\right]$, $\lim\limits_{x\to 0-}\left[\dfrac{\sin x}{x}\right]$

5. Evaluate the limits

(i) $\lim\limits_{x\to\infty} \dfrac{x^2+3x}{x^2+x+1}$

(ii) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x+\cos x}$

(iii) $\lim\limits_{x\to\infty}\left(\sqrt[3]{x+1}-\sqrt[3]{x}\right)$

(iv) $\lim\limits_{x\to 3}\left([x]-\left[\dfrac{x}{3}\right]\right)$

15.2 Unit- 13B

1. A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y)=f(x)+f(y)\ \forall\ x, y\in\mathbb{R}$. If $f(1)=k$, prove that $f(x)=kx\ \ \forall\ x\in\mathbb{R}$. Also show that f is uniformly continuous on $\mathbb{R}$.

Take $x=y=0$, We have $f(0)=2f(0)\Rightarrow f(0)=0$.....(i)

Take $y=-x$, $f(x)+f(-x)=0 \Rightarrow f(-x)=-f(x)$.......$(ii)$

Let x be a positive integer.

$$\text{Then } f(x) = f(1+1+....+1)$$
$$= f(1)+f(1)+.....+f(1) \ (x \text{ times})$$
$$= x\,f(1)$$
$$= kx \text{ if } x \text{ be a positive integer......(iii)}$$

Let x be a negative integer,

let $x=-y, y>0$

$f(x)=f(-y)=-f(y)(by \text{ ii})=-ky$

$\Rightarrow f(x)=kx$, if x be negative integer.

So, $f(x)=kx$ if x is a negative integer......(iv)

From (i), (iii) and (iv) if follows that $f(x)=k\,x$ if x is an integer.... (v)

let $x \in Q$, $x=\dfrac{p}{q}$, 'say' $p \in \mathbb{Z}$, $q \in \mathbb{N}$

$f(qx)=f(p)=kp$ by (v)

$$f(qx)=f(x+.......+x)$$
$$=f(x)+f(x)+......+f(x) \ [q \text{ times}]$$
$$=q\,f(x)$$

Therefore $q\,f(x)=k\,p$

or, $f(x)=\dfrac{k\,p}{q}=k\,x$

So, $f(x)=k\,x$ if x is a rational number ......(vi)

Let $\alpha \in \mathbb{R}\setminus Q$. Let us consider a sequence of rational points $\{x_n\}_n$ converging to α. Since f is continuous at α, $\lim\limits_{n\to\infty} f(x_n)=f(\alpha)$.

But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} kx_n$, since $x_n \in Q$

As $\lim kx_n = k\alpha$, it follows that $f(\alpha) = k\,\alpha$

So, $f(x) = k\,x$ if x is an irrational number .... (vii)

From (v), (vi), (vii) $f(x) = kx \;\forall\; x \in \mathbb{R}$

Let $\varepsilon > 0$

and let x_1, x_2 be any two points in $\mathbb{R}$. Choose $\delta = \dfrac{\varepsilon}{|k|+1}$. Such δ depends only on ε.

Then $|f(x_1) - f(x_2)| = |k||x_1 - x_2| < \varepsilon$. This prove that f is uniformly continuous on $\mathbb{R}$.

2. A function f is defined on $\mathbb{R}$ by $f(x) = \cos\dfrac{1}{x}$, $x \neq 0$

$= 0$, $x = 0$

Prove that f is not continuous at 0.

Let us consider a sequence $\{x_n\}_n$ where $x_n = \dfrac{1}{2\pi n}$, $n \in \mathbb{N}$. Then

$\lim x_n = 0$, $f(x_n) = 1 \quad \forall n \in \mathbb{N}$. Therefore, $\lim f(x_n) = 1$.

We have a sequence $\{x_n\}_n$ in $\mathbb{R}$ that converges to 0 but $\lim f(x_n) \neq f(0)$, proving that f is not continuous at 0.

3. $\lim_{x\to 0+} \sqrt{1+\sqrt{x}} = 1$

Let $f(x) = 1+\sqrt{x}$, $x \geq 0$, $g(x) = \sqrt{x}$, $x \geq 0$

Let $A = \{x \in \mathbb{R} : x \geq 0\}$, $f(A) \subset D(g)$

$(g \circ f)(x) = g(f(x)) = \sqrt{1+\sqrt{x}}$, $x \geq 0$, $0 \in A'$ and $\lim_{x\to 0} f(x) = 1$,

$1 \in D(g)$ and g is continuous at 1.

Therefore $\lim\limits_{x\to 0+} \sqrt{1+\sqrt{x}} = \lim\limits_{x\to 0} (g\circ f)(x) = g(1) = 1$

4. $f(x) = \frac{1}{x}\sin\frac{1}{x},\ x > 0$

$= 0,\ x = 0$

$\lim\limits_{x\to 0+} f(x) = \infty,\ \lim\limits_{x\to 0-} f(x) = -\infty$. f is discontinuous at 0, 0 is a point of infinite oscillatory discontinuity.

5. $f(x) = x - [x].\ 0 < x < 2$

$f(x) = x,\ 0 < x < 1$

$= x - 1,\ 1 \le x < 2$

Here $\lim\limits_{x\to 1-} f(x) = 1,\ \lim\limits_{x\to 1+} f(x) = 0,\ f(1) = 0$. f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 0 - 1 = -1$

Problems on Chapter - 2

1. Determine the points of continuity of the functions

(a) $g(x) = x[x]$

(b) $k(x) = \left[\frac{1}{x}\right],\ x \ne 0$

2. $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$ and let $S = \{x\in\mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If $\{x_n\}$ is in S and $x = \lim x_n$, show that $x \in S$.

3. Suppose that $f(r) = 0\ \ \forall\ r\in Q$. Prove that $f(x) = 0\ \forall\ x\in\mathbb{R}$

4. Define $g:\mathbb{R}\to\mathbb{R}$ by $g(x) = 2x$ for $x\in Q$, $g(x) = x+3,\ x\in\mathbb{R}\setminus Q$. Find points at which g is continuous.

5. Let $g:\mathbb{R}\to\mathbb{R}$ satisfy the relation $g(x+y) = g(x)\,g(y)\ \forall\ x, y\in\mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a\in\mathbb{R}$, then $g(x) = 0\ \forall\ x\in\mathbb{R}$. Also show that if $g(x)\ne 0$ for any x, then $g(x) = a^x$ where $a > 0, a \ne 1$.

6. Let f be defined by $f(x)=\sin\frac{1}{x}, x\neq 0$ and $f(0)=0$. Prove that f has the intermediate value property on $[-1,1]$.

7. Let f be defined on an interval I and suppose that f is one-to-one on I.

(a) Give an example to show that f may not be monotone on I.

(b) Give an example to show that f may not be monotone on any subinterval of I.

(c) Suppose that f is continuous on I. Prove that f is monotone on I.

(d) Suppose that f has the intermediate value property on I. Prove that f is monotone on I.

8. Find the point of discontinuity of the functions.

(i) $f(x)=[\sin x],\ x\in\mathbb{R}$

(ii) $f(x)=(-1)^{[x]},\ x\in\mathbb{R}$

9. Examine the nature of discontinuity of f at 0.

(i) $f(x)=\frac{1}{\sqrt{x}},\ x>0$

$=0 \quad x=0$

(ii) $f(x)=\frac{\sin x}{\sqrt{x}},\ x\neq 0$

$=0 \quad x=0$

(iii) $f(x)=\cos\frac{1}{x},\ x\neq 0$

$=0 \quad x=0$

(iv) $f(x)=\frac{1}{x}\ \sin\frac{1}{x}, x\neq 0$

$=0, \quad x=0$

10. Show that f is piecewise continuous on the interval I

(i) $f(x)=[x], I=[0,3]$

(ii) $f(x) = \text{sgn}\, x, I = [-2, 2]$

(iii) $f(x) = x - [x], I = [0, 3]$

11. Prove that the following functions are uniformly continuous on the indicated interval.

(i) $f(x) = \sqrt{x} \quad on\,[1, \infty)$

(ii) $f(x) = \dfrac{1}{1+x^2}, \quad x \in \mathbb{R}$

(iii) $f(x) = x \sin\dfrac{1}{x}, \quad x \neq 0$

$= 0, \qquad x = 0$ on $[-1, 1]$

(iv) $f(x) = \tan x$ on $[a, b]$

where $-\dfrac{\pi}{2} < a < b < \dfrac{\pi}{2}$

12. $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $f(a) < g(a), f(b) > g(b)$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

15.3 Unit- 14

Some solved problems on unit 14.

1. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$f(x) = x, \quad 0 \le x \le 1$

$= 2 - x^2, \quad 1 < x < 2$

$= x - x^2, \quad 2 \le x \le 3$. Find the derivative function f' and its domain.

$f'(x) = 1$ for $x \in (0, 1)$

$= -2x$ for $x \in (1, 2)$

$=1-2x$ for $x \in (2,\ 3)$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x}{x} = 1,\ Rf'(0)=1$$

Hence f is differentiable at 0 and $f'(0)=1$

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x-1}{x-1} = 1,\ Lf'(1)=1$$

$$\lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{(2-x^2)-1}{x-1} = -\ 2,\ Rf'(1)=-2 \text{ so } Lf'(1) \neq Rf'(1)$$

Hence f is not differentiable at 1.

$$\lim_{x\to 2-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2-} \frac{2-x^2-(-2)}{x-2} = \lim_{x\to 2-} -(x+2) = -\ 4$$

$Lf'(2) = -\ 4$

$$\lim_{x\to 2+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2+} \frac{x-x^2+2}{x-2} = \lim_{x\to 2+} -(x+1) = -\ 3$$

$\therefore Rf'(2) = -3$ so $Lf'(1) \neq Rf'(2)$

Hence f is not differentiable at 2.

$$\lim_{x\to 3-} \frac{f(x)-f(3)}{x-3} = \lim_{x\to 3-} \frac{x-x^2-(-6)}{x-3} = \lim_{x\to 3-} \frac{x-x^2+6}{x-3}$$

$$= \lim_{x\to 3-} -(x+2) = -5 \Rightarrow Lf'(3) = -5$$

Hence f is differentiable at 3 and $f'(3) = -\ 5$

The derived function f' is defined by

$$\begin{aligned} f'(x) &= 1, && 0 \le x < 1 \\ &= -\ 2x, && 1 < x < 2 \\ &= 1-2x, && 2 < x \le 3 \end{aligned}$$

2. $f(x)=x^2, x\in[0,\infty)$. f is strictly increasing and continuous on $[0,\infty)$.

Let $I=[0,\ \infty)$. Then $f(I)=[0,\infty)$

The inverse function g defined by $g(y)=\sqrt{y},\quad y\in[0,\infty)$ is continuous in $[0,\infty)$

f is differentiable on $[0,\infty)$ and $f'(x)=2x,\ x\in[0,\ \infty)$

$f'(x)\neq 0$ *on* $(0,\ \infty)$. Let $I_1=(0,\infty)$. Then $f(I_1)=(0,\ \infty)$.

Hence $g'(y)$ exists $\forall\ y\in(0,\ \infty)$ and $g'(y)=\dfrac{1}{f'(x)}=\dfrac{1}{2x}=\dfrac{1}{2g(y)}$

$=\dfrac{1}{2\sqrt{y}},\ y\in(0,\infty)$

3. A function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(0)=0$ and $f(x)=0$ if $x\in\mathbb{R}\setminus Q$

$f(x)=\dfrac{1}{q},\ x=\dfrac{p}{q}$ where $p\in\mathbb{Z},\ q\in\mathbb{N}$ and $\gcd(p,q)=1$

Show that f is not differentiable at 0.

$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}$. Let $\phi(x)=\dfrac{f(x)}{x}$. Let $\{x_n\}_n$ be the sequence of rational points converging to 0 where $x_n=\dfrac{1}{n},\ n\in\mathbb{N}$. Then $\lim\limits_{n\to\infty}\phi(x_n)=\lim\limits_{n\to\infty}n.\dfrac{1}{n}=1$

Let $\{y_n\}_n$ be a sequence of irrational points converging to 0.

$\lim\limits_{n\to\infty}\phi(y_n)=\lim\limits_{n\to\infty}\dfrac{f(y_n)}{y_n}=0$, since $f(y_n)=0\ \forall\ n\in\mathbb{N}$.

Therefore, $\lim\limits_{x\to 0}\phi(x)$ does not exist, since for two sequences $\{x_n\}_n$ and $\{y_n\}_n$ both converging to 0, the sequences $\{\phi(x_n)\}_n$ and $\{\phi(y_n)\}_n$ converge to two different limits.

$\therefore f$ is not differentiable at 0.

4. Let $f:[-1,1]\to\mathbb{R}$ be defined by $f(x)=0, x\in[-1,0]$, $f(x)=1, x\in(0,1]$

Does there exist a function g such that $g'(x)=f(x), x\in[-1,1]$?

If possible, let there exist a function $g:[-1,1]\to\mathbb{R}$ such that

$g'(x)=f(x), x\in[-1,1]$.

Then g is differentiable on $[-1,1]$ and $g'(x)=0, x\in[-1,0]$ and $=1,\ x\in(0,1]$

Since g is differentiable on $[-1,1]$ and $g'(-1)\neq g'(1)$, by Darboux' s theorem g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i. e. between 0 and 1 on $[-1,1]$. But this is not so and therefore g does not exist.

5. Show that functions $\tan^{-1}x=f(x), -\infty<x<\infty,$ is uniformly continuous there in and $f'(x)$ is also so.

Let x, y be any pair of points in $(-\infty,\infty)$. By LMV theorem, $\exists\xi\in(x,y)$ such that $f(y)-f(x)=(y-x)f'(\xi)\Rightarrow|f(y)-f(x)|=|y-x|\dfrac{1}{1+\xi^2}\leq|y-x|$. Let $\varepsilon>0$ be any number. So $|f(y)-f(x)|\leq|y-x|<\varepsilon$ whenever $|y-x|<\delta,\ \delta=\varepsilon$.

$\Rightarrow f$ is uniformly continuous on $(-\infty,\infty)$.

Again $|f'(x)-f'(y)|=|x-y|f''(\xi)$ for some $\xi\in(x,y)$ (by LMV theorem)

$$|f''(\xi)|=\left|\frac{2\xi}{\left(\xi^2+1\right)^2}\right|.$$

Note : That $\dfrac{1+\xi^2}{2}\geq|\xi|\Rightarrow\dfrac{2|\xi|}{1+\xi^2}\leq 1$. Also $\dfrac{1}{1+\xi^2}\leq 1$

consequently, $|f'(x)-f'(y)|\leq|x-y|<\varepsilon$ whenever $|x-y|<\delta$

$\Rightarrow f'$ is uniformly continuous on $(-\infty,\infty)$.

Indeterminate form

In the process of examining the existence of limit of functions in $\mathbb{R}$ and in its determination we are very often faced with limits of following forms :

$$\frac{0}{0},\frac{\infty}{\infty},0\times\infty,\infty-\infty,\ 0^{0},\ \infty^{0},1^{\infty}.$$

These forms are generally known as 'Indeterminate forms'. Usually all the above forms can be reformulated to give rise to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

French mathematician G. F. L. Hopital (1661-1704) gave a method for computing such limits (provided they exist).

L' Hopital's Rule,

Result-1 : Let $f,g:[a,b]\to\mathbb{R}$ be both continuous in [a, b], differentiable in (a,b) such that $f(a)=0=g(a)$ & $g(x)\neq 0,\ g'(x)\neq 0$ in $a<x<b$. Then,

(i) if $\lim\limits_{x\to a+}\frac{f'(x)}{g'(x)}=l(\in R)$, then $\lim\limits_{x\to a+}\frac{f(x)}{g(x)}=l$

(ii) if $\lim\limits_{x\to a+}\frac{f'(x)}{g'(x)}=\infty$, then $\lim\limits_{x\to a+}\frac{f(x)}{g(x)}=\infty$.

Proof of (i) $\lim\limits_{x\to a+}\frac{f'(x)}{g'(x)}=l\Rightarrow$ corresponding to arbitrary $\varepsilon>0,\exists\,\delta>0,$ $0<\delta<b-a$, such that

$$\left|\frac{f'(x)}{g'(x)}-l\right|<\varepsilon \text{ whenever } a<x<a+\delta \qquad \ldots (1)$$

By hypothesis, Cauchy's M. V. theorem is applicable to $f\,\&\,g$ in $[a,x]$ where $a<x<a+\delta.\ \exists\ \xi_x, a<\xi_x<x,$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(\xi_x)}{g'(\xi_x)}\ \&\text{ so } \frac{f(x)}{g(x)}=\frac{f'(\xi_x)}{g'(\xi_x)} \qquad \ldots(2)$$

(1) & (2) $\Rightarrow\left|\frac{f(x)}{g(x)}-l\right|<\varepsilon$ whenever $a<x<a+\delta$

$$\Rightarrow\lim_{x+a+}\frac{f(x)}{g(x)}=l$$

Proof of (ii) : $\lim_{x\to a+}\frac{f'(x)}{g'(x)}=\infty \Rightarrow$ corresponding to any $G>0$, as large as we please, $\exists\, \delta, 0<\delta<b-a,$ such that

$$\frac{f'(x)}{g'(x)}>G \text{ whenever } a<x<a+\delta \qquad \ldots (1)$$

Again Cauchy's M V theorem is applicable to *f, g* in [*a, x*], $a<x<a+\delta$.

so $\exists\, \eta_x, a<\eta_x<x,$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(\eta_x)}{g'(\eta_x)}\Rightarrow \frac{f(x)}{g(x)}=\frac{f'(\eta_x)}{g'(\eta_x)} \quad \ldots (2)$$

By (1) & (2), $\frac{f(x)}{g(x)}>G$ whenever $a<x<a+\delta$

$$\Rightarrow \lim_{x\to a+0}\frac{f(x)}{g(x)}=\infty.$$

Simple Ilustrations :

(1) $\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}\left(\frac{0}{0}\right)=\lim_{x\to 0+}\frac{\cos x}{\frac{1}{2\sqrt{x}}}=\lim_{x\to 0+}2\sqrt{x}\cos=0$

(2) $\lim_{x\to 0}\frac{1-\cos x}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}.$

(3) Evaluate $\lim_{x\to\infty}\left[x-\sqrt[n]{(x-a_1)......(x-a_n)}\right]$ where a_i's are positive rationals.

The limit is of form $\infty-\infty$.

We take $x=\frac{1}{t}$. so $x\to\infty \Leftrightarrow t\to 0+$

The limit is $\lim_{t\to 0}\frac{1-\sqrt[n]{(1-a_1t)(1-a_2t)...(1-a_nt)}}{t}\left(\frac{0}{0}\right)$

$$=\lim_{t\to 0}\frac{-\frac{d}{dt}\left\{\sqrt[n]{(1-a_1t)...(1-a_nt)}\right.}{1}$$

$$= \lim_{t\to 0} \frac{\frac{f(t)}{n}\left[\frac{a_1}{1-a_1t}+\frac{a_2}{1-a_2t}+\ldots+\frac{a_n}{1-a_nt}\right]}{1}$$

where $f(t)=\sqrt[n]{\left[(1-a_1t)(1-a_2t)\ldots(1-a_nt)\right]}$

So the required limit is $\frac{1}{n}(a_1+a_2+\ldots+a_n)$.

(4) Evaluate $\lim_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$.

The limit is of form 1^∞

Let $y=\left(e^x+x\right)^{\frac{1}{x}}$, so $\log y=\frac{1}{x}\log(e^x+x)$.

Then $\lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log(e^x+x)}{x}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{1\left(e^x+1\right)}{e^x+x}=2$$

So $\lim_{x\to 0} y=e^2$.

Note : Standard limits like $\lim_{x\to 0}\frac{\sin x}{x}$, $\lim_{x\to 0}\frac{\log(1+x)}{x}$, $\lim_{x\to 0}\frac{e^x-1}{x}$, $\lim_{x\to a}\frac{x^n-a^n}{x-a}$ etc can not be evaluated by L. Hopital's rule.

The reason is that if you apply the above rule to find $\lim_{x\to 0}\frac{\sin x}{x}$, then you are differentiating $\sin x$ w.r.t. x & in order to do the same, you are using the limit $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Result II : Let f, g are differentiable in $[a,b]$, $\lim_{x\to a+} f(x)=\infty=\lim_{x\to a+} g(x)$, $g(x)\neq 0$, $g'(x)\neq 0$ in $a<x<b$, then

(i) if $\lim_{x\to a+}\frac{f'(x)}{g'(x)}=l(\in\mathbb{R})$, then $\lim_{x\to a+}\frac{f(x)}{g(x)}=l$

(ii) if $\lim_{x\to a+}\frac{f'(x)}{g'(x)}=\infty$, then $\lim_{x\to a+}\frac{f(x)}{g(x)}=\infty$

Proof : Let $a < x < c < b$.

Applying C M V theorem to f & g in $[a, c]$, $\exists \xi \in (a,c)$ such that

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(\xi)}{g'(\xi)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}, \frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} \qquad \ldots (1)$$

Let $0 < \varepsilon < 1$ be any number. Then

$\exists\, \delta,\ 0 < \delta < b-a$ such that $\left|\dfrac{f'(x)}{g'(x)} - l\right| < \varepsilon,\ a < x < a+\delta$

we can write $\dfrac{f'(\xi)}{g'(\xi)} = l + \delta_1$ where $|\delta_1| < l < 1$

Keeping chosen c fixed. $\displaystyle\lim_{x\to a+} \frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} = 1$.

Choosing x nearer to a.

$$\frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} = 1+\delta_2 \text{ where } |\delta_2| < \begin{cases} \varepsilon, \text{if } |l| < 1 \\ \dfrac{\varepsilon}{|l|}, \text{if } |l| > 1 \end{cases}$$

Then (i) $\Rightarrow \dfrac{f(x)}{g(x)} = (l+\delta_1)(1+\delta_2) = l + \delta_1 + l\delta_2 + \delta_1\delta_2$

$\Rightarrow \left|\dfrac{f(x)}{g(x)} - l\right| < 3\varepsilon$ in above neighbourhood of 'a'

$\Rightarrow \displaystyle\lim_{x\to a+} \frac{f(x)}{g(x)} = l$.

(ii) $\displaystyle\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \infty \Rightarrow$ corresponding to arbitrary large positive number G, $\exists\, \delta$,

$0 < \delta < b-a$, such that $\dfrac{f'(x)}{g'(x)} > G,\ a < x < a+\delta$.

Choosing $c \in (a, a+\delta)$

$$\frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} > 1-\frac{1}{G}\left(\text{as } \frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} \to 1\right)$$

Then (1) $\Rightarrow \dfrac{f(x)}{g(x)} > G\left(1-\dfrac{1}{G}\right)(=G-1) \Rightarrow \lim\limits_{x\to a+}\dfrac{f(x)}{g(g)} = \infty$.

Simple Illustrations:

(1) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{1}{x} = 0$

(2) $\lim\limits_{x\to\infty}\left(e^{-x}.\, x^2\right)\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{2x}{e^x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{2}{e^x} = 0$

(3) $\lim\limits_{x\to\frac{\pi}{2}+}\dfrac{\log\left(x-\dfrac{\pi}{2}\right)}{\tan x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\frac{\pi}{2}+}\dfrac{\cos^2 x}{x-\dfrac{\pi}{2}}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to\frac{\pi}{2}+}\frac{-\sin 2x}{1} = -\sin\pi = 0.$$

(4) $\lim\limits_{x\to 0+} x^{\sin x}(0^\circ)$

let $y = x^{\sin x}$. So $\lim\limits_{x\to 0+}\log y = \lim\limits_{x\to 0+}\sin x.\log x\ (0\times -\infty)$

$$= \lim_{x\to 0+}\frac{\log x}{\operatorname{cosec} x}\left(\frac{-\infty}{\infty}\right) = \lim_{x\to 0+}\frac{\dfrac{1}{x}}{-\operatorname{cosec} x\,\cot x}$$

$$= \lim_{x\to 0+}\frac{\sin 2x}{\cos x - x\sin x} = 0 \Rightarrow \lim_{x\to 0+} y = e^\circ = 1.$$

(5) $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}\right)^{\tan x}(\infty^\circ)$

let $y = \left(\frac{1}{x}\right)^{\tan x} \Rightarrow \log y = \tan x \log\left(\frac{1}{x}\right)$

$\Rightarrow \lim_{x\to 0+} \log y = \lim_{x\to 0+} \tan x\,(-\log x) \quad (0\times\infty)$

$$= \lim_{x\to 0+} \frac{-\log x}{\cot x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = \lim_{x\to 0+} \frac{\sin^2 x}{x} = \lim_{x\to 0+} \sin 2x = 0$$

so $\lim_{x\to 0+} y = e^{\circ} = 1$.

(6) $\lim_{x\to\infty}\left[\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{}\right]^{nx}$, a_i's are positive rationals.

let y be the expression mentioned above

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\log\left[\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{n}\right]}{\frac{1}{n_x}}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\left\{\frac{\frac{n}{a_1^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}\cdot\frac{1}{n}\left[a_1^{\frac{1}{x}}\log_e a_1 + \ldots + a_n^{\frac{1}{x}}\log_e a_n\right]\left(\frac{-1}{x^2}\right)}{\frac{-1}{nx^2}}\right\}$$

$$= \lim_{x\to\infty}\left[\frac{n}{a_1^{\frac{1}{x}} + \ldots. + a_n^{\frac{1}{x}}}\right]\left[a_1^{\frac{1}{x}}\log_e a_1 + \ldots. + a_n^{\frac{1}{x}}\log_e a_n\right]$$

$= \log_e(a_1 a_2 \ldots a_n)$

So, $\lim_{x\to\infty} y = a_1 a_2 \ldots a_n$.

Problems on Indeterminate form : Evaluate the limits in (1) to (5) :

(1) $\lim\limits_{x \to 0+} \left(x \log \sin^2 x \right)$

(2) $\lim\limits_{x \to 0+} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$

(3) $\lim\limits_{x \to 0} \frac{3^x - 2^x}{4^x - 3^x}$

(4) $\lim\limits_{x \to a} \left(2 - \frac{x}{a} \right)^{\tan \frac{\pi x}{2a}}$

(5) $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

(6) Determine the constants *a, b, c* such that

$$\lim_{x \to 0} \frac{x(a + b\cos x) + c\sin x}{x^5} = \frac{1}{60}.$$

(7) If $\lim\limits_{x \to 0} \left(\frac{1 + Cx}{1 - Cx} \right)^{1/x} = 4$, find $\lim\limits_{x \to 0} \left(\frac{1 - 2Cx}{1 + 2Cx} \right)^{1/x}$.

15.4 Summary

In this unit, we have given various problems and solution of the units 1, 2 and 3.

Unit- 17 □ Limit and continuity for function of two variables

Structure

16.0 Objectives

This unit gives

- Some preliminary notion of the distance in $\mathbb{R}^2$, diameter of a set, open and closed sts in $\mathbb{R}^2$.
- Convergence of sequence in $\mathbb{R}^2$.
- The concept of limit and continuity of two variable function.
- Continuity of a function in $\mathbb{R}^2$.

16.1 Introduction

This unit concerned with the calculus of functions whose domains are subsets of $\mathbb{R}^2$. Such functions are frequently called by the name "functions of several variables". The concept extends the idea of a function of a real variable to several variables. There are so many applications of this several variables' functions in geometry, applied mathematics, engineering, natural sciences and economics.

1 6.2 Preliminaries

[1] The set $\mathbb{R}^2$ and the distance on it

$\mathbb{R}^2$ denotes the set of ordered 2-tuples (x^1, x^2) (or ordered pairs) of real numbers $x^i \in \mathbb{R}$ for $i = 1, 2$, where the notion of distance between the points

$$x_1 = (x_1^1, x_1^2) \; and \; x_2 = (x_2^1, x_2^2)$$

is defined by $d\,(x_1, x_2) = \sqrt{\left[\sum_{i=1}^{2}\left\{(x_1^i - x_2^i)^2\right\}\right]}$ ....(1)

The function d : defined by (1), obeys the following properties :

(i) $d(x_1, x_2) > 0$ (ii) $d(x_1, x_2) = 0 \iff x_1 = x_2$ (iii) $d\,(x_1, x_2) = d(x_2, x_1)$

(iv) $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$.

[2] Diameter of a set :

The diameter of a set $E \subset \mathbb{R}^2$ is the quantity

$$d\,(E) = \underset{x_1, x_2 \in E}{Sup}\, d(x_1, x_2)$$

[3] Bounded set in $\mathbb{R}^2$:

A set $E \subset \mathbb{R}^2$ is bounded if its diameter is finite.

[4] Open and closed sets in $\mathbb{R}^2$:

Definition (1) : For $\delta > 0$, the set $B(a, \delta) = \left\{x \in \mathbb{R}^2 \mid d(a, x) < \delta\right\}$ is called the ball with centre $a \in \mathbb{R}^2$ of radius δ or the δ-neighbourhood of the point 'a' in $\mathbb{R}^2$.

In particular, if $(a, b) \in \mathbb{R}^2$ and $\delta > 0$, the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < \delta\right\}$$

is called an open disc of radius δ with centre at (a, b) & is denoted by $N(a, b, \delta)$.

The set $N'\left((a, b), \delta\right) \equiv N((a, b), \delta) - \left\{(a, b)\right\}$ is called the deleted δ-neighbourhood of (a, b), denoted by

$N'\left((a, b), \delta\right)$ as mentioned above.

The set $M \subset \mathbb{R}^2$ is a neighbourhood of (a, b) if and only if there exists $\delta > 0$ such that

$$N((a,b),\delta) \subset M$$

Definition (2) : A set $G\ (\subset \mathbb{R}^2)$ is open in $\mathbb{R}^2$ if every point $x \in G$, there exists a ball $B(x, \delta) \subset G$.

In other words, a set $G\ (\subset \mathbb{R}^2)$ is said to be open if it is a neighbourhood of each of its points.

Examples : (i) $\mathbb{R}^2$ is an open set (ii) Void set is open set (iii) A ball $B(a, b)$ is an open set in $\mathbb{R}^2$

Definition (3) : The set $F \subset \mathbb{R}^2$ is closed in $\mathbb{R}^2$ if its complement

$G = \mathbb{R}^2 \backslash F$ is open in $\mathbb{R}^2$.

Definition (4) : Let $E \subset \mathbb{R}^2$. A point x is interior point of E if some neighbourhood of it is contained in E.

On the otherhand, x is exterior point of E if it is an interior point of the complement of E in $\mathbb{R}^2$.

x is boundary point of E if it is neither an interior point of E nor an exterior point of E.

Definition (5) : A point $a \in \mathbb{R}^2$ is a limit point (accumulation point) of $E \subset \mathbb{R}^2$ if for any neighbourhood $N(a)$ of 'a' the intersection $E \cap N(a)$ is an infinite set.

The union of set E and all its limit points in $\mathbb{R}^2$, is the closure of E in $\mathbb{R}^2$, denoted by $\bar{E}$.

Results : We state the following results without proof :

(1) Intersection of any two open discs is an open set.

(2) The union of any number of open subsets of $\mathbb{R}^2$ is open.

(3) The intersection $\bigcap_{i=1}^{n} G_i$ of a finite number of open sets in $\mathbb{R}^2$ is an open set.

(4) The intersection $\bigcap_{\alpha \in \wedge} F_\alpha$ of the sets of any system $\{F_\alpha : \alpha \in \wedge\}$ of closed sets F_α in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$. (^ : Index set).

(5) The union of finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

(6) Every bounded infinite set S of points in a plane has at least one accumulation point.

Sequence of points : Convergence

Let us consider infinite sequence of points $P_n(x_n, y_n)$ in the plane. The sequence is bounded if a disc can be found containing all the points P_n *i.e.* if there is a point Q and a number M such that the distance $|P_nQ| < M$ for all $n \in \mathbb{N}$.

Examples : The sequence $P_n = \{(-1)^n + \frac{1}{n}, \frac{3}{n^2}\}_n$, $P_n^1 = \left\{3, \left(\frac{-2}{5}\right)^n\right\}_n$ are bounded but the sequence $\{(n^2, n^3)\}_n$ is not bounded. The sequence $\{P_n\}_n$ converges to a point Q $(or, \lim_{n\to\infty} P_n = Q)$ if the sequence of distances $\{\overline{P_nQ}\}_n$ converges to zero. For every $\varepsilon > 0$, there exists positive integer m such that P_n's lie in the ε-neighbourhood of Q for all n > m.

The sequence of points $\{(x_n, y_n)\}_n$ converges to (a, b) if and only if the sequences $\{x_n\}_n \& \{y_n\}_n$ converge to a and b respectively (Co-ordinatewise convergence).

16.3 Limit and continuity for function of two variables

Definition :

$u(x, y)$ is a function of independent variables x and y whenever some law f assigns a unique value of u, the dependent variable to each pair of values (x, y) belonging to a certain specified set, the domain of the function. A function $u(x, y)$ thus defines a mapping of a set of points in the xy-plane, the domain of f, on to a certain set of points on the u-axis, the range of f.

Geometrically, a function of two variables represents a surface.

Examples :

[1] Domain of $u(x, y) = \text{Sin}^{-1}\frac{x}{3} + \sqrt{xy}$ is $S_1 \cup S_2$ where

$$S_1 = \{(x, y) \in \mathbb{R}^2 \mid -3 \le x \le 0, y \le 0\} \ \& \ S_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 3, y \ge 0\}$$

[2] Domain of $u(x,y)=\sqrt{1-x^2}+\sqrt{1-y^2}$ is the square formed by the segments of the lines $x=\pm 1, y=\pm 1,$ including its sides $|x|\le 1, |y|\le 1$.

[3] Domain of $u(x,y)=\sqrt{(x^2-4)}+\sqrt{(4-y^2)}$ is the two strips $x\ge 2, -2\le y\le 2$ and $x\le -2, -2\le y\le 2$.

[4] Domain of $u(x,y)=\sqrt{y\sin x}$ is the strips $2n\Pi\le x\le(2n+1)\Pi, y\ge 0$ and $(2n+1)\Pi\le x\le(2n+2)\Pi, y\le 0$ (where n is integer).

[5] Domain of u(x, y) = ln (x^2 + y) is that part of the plane located above the parabola $y = -x^2$.

Definition (Limit of function of two variables)

Let $f:S\to\mathbb{R}$ when $S\subset\mathbb{R}^2$. Let (a, b) be an accumulation point of S. We say that $\lim\limits_{(x,y)\to(a,b)} f(x,y)=l(\in\mathbb{R})$ if for any number $\varepsilon>0$, there exists δ > 0 such that $|f(x, y) - l| < \varepsilon$ whenever $(x, y)\in N'((a,b),\delta)\cap S$.

This limit if exists is known as simultaneous limit or double limit.

Sequential approach : $\lim\limits_{(x,y)\to(a,b)} f(x,y)=l(\in\mathbb{R})$ if and only if for every sequence of points $(x_n, y_n)\to(a,b),$ we have $\lim\limits_{n\to\infty} f(x_n, y_n)=l(\in IR)$.

These two definitions (ε –δ approach & sequential approach) are equivalent.

Examples :

[1] $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

The sequences $\left\{\left(\frac{1}{n},\frac{1}{n}\right)\right\}_n$ & $\left\{\left(\frac{2}{n},\frac{1}{n}\right)\right\}_n$ both approach to (0, 0)

$f\left(\frac{1}{n},\frac{1}{n}\right)=\frac{1/n^2}{2/n^2}=\frac{1}{2}$ & $f\left(\frac{2}{n},\frac{1}{n}\right)=\frac{2}{5}$. These two are different.

So the limit does not exist.

[2] $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{(x^2+y^2)}}$

Refer to polar co-ordinates $x = r\cos\theta,\ y = r\sin\theta$. Then,

$$\frac{xy}{\sqrt{(x^2+y^2)}} = \frac{1}{2}r\ \sin\ 2\theta.$$

We note that $|f(x,y)-0| \le \frac{1}{2}r\left(=\frac{1}{2}\sqrt{(x^2+y^2)}\right) < \frac{\varepsilon}{2}$ whenever

$0 < \sqrt{(x^2+y^2)} < \delta \ \&\ \delta = \frac{\varepsilon}{2},\ \varepsilon > 0$ is arbitrary number.

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Theorem : (Necessary condition for the existence of double limit)

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and $(a,\ b)$ be an accumulation point of S.

If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L(\in \mathbb{R})$, then $f(x,\ \varphi\ (x)) \to L$ as $x \to a$, where φ is a real valued function of one variable x such that $(x,\ \varphi(x) \in S$

$x \in D_\varphi$ and $\varphi(x) \to b\ as\ x \to a$.

Note that in a plane $(x,\ y)$ may approach to $(a,\ b)$ through infinitely many paths, strictly within the domain. The genesis of the above theorem is that limit φ is independent of all such paths leading to $(a,\ b)$.

Proof : Given $\lim\limits_{(x,\ y)\to(a,\ b)} f(x,\ y) = L(\in \mathbb{R})$, Let $\varepsilon > 0$ be any number.

Corresponding to ε, there exists $\delta > 0$ such that

$|f(x,y)-L| < \varepsilon$ whenever $0 < |x-a| < \delta,\ 0 < |y-b| < \delta$ (1)

Again $\lim\limits_{x\to a}\varphi(x) = b$, corresponding to above δ, there exists $\eta > 0$

such that $|\varphi(x)-b| < \delta$ whenever $0 < |x-a| < \eta$ (2)

Let $\rho = \min\{\delta,\ \eta\}$.

Hence by (1) and (2), we have $|f(x, \varphi(x)) - L| < \varepsilon$ wherever $0 < |x - a| < \rho$

Consequently, $\lim_{x \to a} f(x, \varphi(x)) = L$

Remarks : If there be two functions $\varphi_1(x) \& \varphi_2(x)$ such that

$\lim_{x \to a} f(x, \varphi_1(x)) \neq \lim_{x \to a} f(x, \varphi_2(x))$

where $(x, \varphi_1(x)) \& (x, \varphi_2(x)) \in S$ for each $x \in D_{\varphi_i} (i = 1, 2)$ & as x → a,

$\varphi_i(x) \to b \ (i = 1, 2)$, then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.

Examples : $f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, x \neq y \\ 0 \ , \ x = y. \end{cases}$

Let $(x, y) \to (0, 0)$ along the path $x - y = mx^3$. Then

$\dfrac{x^3 + y^3}{x - y} = \dfrac{1 + (1 - mx^2)^3}{m} \to \dfrac{2}{m}$ as $x \to 0, \dfrac{2}{m}$ is different for different m.

So, $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Repeated limits :

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and (a, b) be an accumulation point of S.

Let $\lim_{(x \to a)} f(x, y)$ exist, then it is function of y, say $\varphi(y)$.

Let $\lim_{y \to b} \varphi(y)$ exist & $= \lambda (\in \mathbb{R})$, then $\lim_{y \to b} \lim_{x \to a} f(x, y) = \lambda$ ....(1)

Let $\lim_{y \to b} f(x, y)$ exist & it is then function of x, say $\psi(x)$.

Let $\lim_{x \to a} \psi(x)$ exist & $= \mu (\in \mathbb{R})$, then $\lim_{x \to a} \lim_{y \to b} f(x, y) = \mu$ ....(2)

(1) and (2) are known as 'Repeated limits'

So questions arise regarding the existence of Repeated limits, whether their existence ensure the existence of double limit & conversely etc.

In this connection, let us consider first the following examples :

(1) Let $f(x, y) = \begin{cases} xSin\frac{1}{y} + ySin\frac{1}{x}, xy \neq 0 \end{cases}$

Let ε > 0 be any number,

$$|f(x,y)-0| = |xSin\frac{1}{y}+ySin\frac{1}{x}| \leq |x|+|y| < \varepsilon \text{ whenever}$$

$$0 < |x-0| < \delta,\ 0 < |y-0| < \delta,\ \delta \text{ correspond to } \varepsilon, \text{ say } \frac{\varepsilon}{2}$$

So, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists & is 0.

But $\lim\limits_{t\to 0} Sin\frac{1}{t}$ does not exist, so neither $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ nor $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists.

This example illustrates that only the existence of double limit at a point does not ensure the existence of repeated limits.

(2) $f(x,y) = \dfrac{\text{Sin } x + \text{Sin } 2y}{\tan 2x + \tan y}$

Keeping y fixed, let $x \to 0$, then we take $y \to 0$.

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \frac{1}{2}$

On the other hand, first keeping x fixed, let $y \to 0$. Then we take $x \to 0$.

We get $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \frac{1}{2}$

So both repeated limits exist though they are unequal.

For consideration of double limit, let $(x, y) \to (0, 0)$ along $y = x$.

$$\frac{\sin x + \sin 2y}{\tan 2x + \tan y} = \frac{\sin x + \sin 2x}{\tan 2x + \tan x} = \frac{2\sin\frac{3x}{2}\cos\frac{x}{2}\cos x\cos 2x}{2\sin\frac{3x}{2}\cos\frac{3x}{2}}$$

$$= \frac{\cos\frac{x}{2}\cos x\cos 2x}{\cos\frac{3x}{2}} \to 1 \text{ as } x \to 0.$$

Next we consider the path y = 2x.

$$\frac{\sin 2x + \sin 2y}{\tan 2x + \tan y} = \frac{\sin x + \sin 4x}{2\tan 2x} = \frac{2\sin\frac{5x}{2}\cos\frac{3x}{2}\cos 2x}{2\sin 2x}$$

$$= \frac{1}{2}.\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}.\frac{x}{\sin x}.\frac{5}{2}.\cos\frac{3x}{2}.\cos 2x.\frac{1}{\cos x} = \frac{5}{4} \text{ as } x \to 0$$

So the double limit does not exist.

So existence of repeated limits $\not\Rightarrow$ existence of double limit.

Also if g(x,y) = $\frac{xy}{x^2+y^2}$, $x^2+y^2 \neq 0$, we note that both the repeated limits exist and are equal but the double limit does not exist.

So a question arises : whether there is any relation between the existence of repeated limits and that of double limit. In this connection, the following theorem is relevant :

Theorem : Let the double limit $\lim\limits_{(x,y)\to(a,b)}$ f(x,y) exist and be equal to A ($\in \mathbb{R}$).

Let the limit $\lim\limits_{x\to a} f(x,y)$ exist for each fixed value of y in the neighbourhood of *b* and like wise let the limit $\lim\limits_{y\to b} f(x,y)$ exist for each fixed value of *x* in the neighbourhood of '*a*'.

Then $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y) = A = \lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y).$

Proof : Let $\lim\limits_{(x\to a)} f(x,y) = F(y)$ (by hypothesis, it exists)

Let ε > 0 be given let 0 < |y – b| < δ_0

Corresponding to ε, there exists $\delta_1 > 0$ such that

$|f(x,y) - F(y)| < \frac{\varepsilon}{2}$ ...(1) for all x satisfying

$0 < |x - a| < \delta_1$. Also $0 < |y - b| < \delta_0$. ...(1)

Also due to the existence of double limit, corresponding to above ε, there exists $\delta_2 > 0$ such that

$|f(x,y) - A| < \frac{\varepsilon}{2}$.....(1) for all x,y satisfying $0 < |x - a| < \delta_2, 0 < |y - b| < \delta_2$ ...(2)

Let $\eta = \min\{\delta_{0,}\delta_{1,}\delta_2\}$

So $|F(y) - A| \leq |f(x,y) - F(y)| + |f(x,y) - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ whenever

$0 < |y - b| < \eta$. So, $\lim\limits_{y \to b} F(y) = A$.

Consequently $\lim\limits_{y \to b} \lim\limits_{x \to a} f(x,y) = A$

Similarly for the other part.

Illustration : Let $f(x,y) = \frac{(x-y)^2}{1 - 2xy + y^2}, (x,y) \neq (1,1)$

Examine the existence of $\lim\limits_{(x,y) \to (1,1)} f(x,y)$

If possible, let the double limit exist & be equal to A

For $y \neq 1, \lim\limits_{x \to 1} f(x,y) = \frac{(1-y)^2}{1 - 2y + y^2} = 1 \Rightarrow \lim\limits_{y \to 1} \lim\limits_{x \to 1} f(x,y) = 1$

For $x \neq 1, \lim\limits_{y \to 1} f(x,y) = \frac{(1-x)^2}{2(1-x)} = \frac{1-x}{2} \Rightarrow \lim\limits_{x \to 1} \lim\limits_{y \to 1} f(x,y) = 0$

So by above theorem ; the double limit does not exist.

16.4 Continuity at a point

Let f : $S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and let $(a, b) \in$ S.

(a) If (a, b) be an accumulation point of S &

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(b) Or If (a, b) be an isolated point of S,

then f is continuous at the point (a, b)

If for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < \varepsilon \text{ wherever } \sqrt{[(x-a)^2 + (y-b)^2]} < \delta \text{ or}$$

$$(x,y) \in N((a,b)\delta) \cap S,$$

then f is continuous at (a, b)

Note : Let $g(x)$ = f (x, b). Then by above, $|g(x) - g(a)| < \varepsilon$ wherever $|x-a|\delta \Rightarrow g(x)$ is continuous at 'a'.

Similarly if $h(y) = f(a,y)$, then it is continuous at $y = b$.

But continuity of g(x) at a & that of $h(y)$ at b |⇒ continuity of f (x, y) at (a, b)

Let f(x,y) = $\begin{cases} 0, \text{ if } xy \neq 0 \\ 1, \text{ if } xy = 0 \end{cases}$

Here $g(x) = f(x, 0) = 1$ for all $x \in \mathbb{R}$ & $h(y) = f(0,y) = 1$ for all $y \in \mathbb{R} \Rightarrow g(x), h(y)$ are continuous at $x = 0$, $y = 0$ respectively.

If possible, let $f(x,y)$ be continuous at (0, 0). Then for $\varepsilon = 1/2$ there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \varepsilon$ whenever $(x,y) \in N((0,0)\delta) \cap \mathbb{R}$

$$\Rightarrow \left| f\left(\frac{\delta}{2}, \frac{\delta}{2}\right) - 1 \right| < \varepsilon \Rightarrow |1| < \varepsilon\left(= \frac{1}{2}\right): \text{ Absurd.}$$

Thus f is not continuous at (0, 0),

Examples : (1) $f(x,y) = \begin{cases} xy\log(x^2+y^2), x^2+y^2 \neq 0 \\ 0, \ x^2+y^2 = 0 \end{cases}$

For $0 < x^2 + y^2 < 1$, $\log(x^2 + y^2) < 0$ & so we have

$$|f(x,y) - f(0,0)| = -|xy|\log(x^2 + y^2) \le -\frac{1}{2}(x^2 + y^2)\log(x^2 + y^2) \quad [AM \ge GM]$$

If we have $x^2 + y^2 = t$, by L Hospital's rule, $\lim_{t \to 0^+} t \log t = 0$

So given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|t \log t - 0| < 2\varepsilon$ whenever $0 < t < \delta$

$\Rightarrow |(x^2 + y^2)\log(x^2 + y^2)| < 2\varepsilon$ whenever $0 < t < \delta$

For $\eta = \min\{1, \delta\}$, we have for $0 < x^2 + y^2 < \eta$,

$|f(x,y) - f(0,0)| < \varepsilon \Rightarrow f$ is continuous at (0, 0).

(2) Let $f(x,y) = \begin{cases} 0, \text{ if } (x,y) = (2y, y) \\ \exp\{-|x - 2y|/(x^2 - 4xy + 4y^2)\}, (x,y) \ne (2y, y) \end{cases}$

we note that $\dfrac{|x - 2y|}{x^2 - 4xy + 4y^2} = \dfrac{1}{|x - 2y|}$

Let $0 < \varepsilon < 1$ and $\varepsilon_1 = \dfrac{-1}{\log \varepsilon}$

$|x - 2y| \le |x| + 2|y| < \dfrac{\varepsilon_1}{4} + \dfrac{2\varepsilon_1}{4}$ whenever $|x - 0| < \dfrac{\varepsilon_1}{4}, |y - 0| < \dfrac{\varepsilon_1}{4} \Rightarrow |x - 2y| < \varepsilon_1$

$$\Rightarrow \{|x - 2y|/(x^2 - 4xy + 4y^2)\} > \frac{1}{\varepsilon_1} \Rightarrow \frac{-|x - 2y|}{(x^2 - 4xy + 4y^2)} < -\frac{1}{\varepsilon_1} = \log \varepsilon$$

$$\Rightarrow \exp\left\{\frac{-|x - 2y|}{x^2 - 4xy + 4y^2}\right\} < \varepsilon$$

$\Rightarrow |f(x,y) - f(0,0)| < \varepsilon$ whenever $|x - 0| < \delta, |y - 0| < \delta, \delta \equiv \delta(\varepsilon)$.

Thus f is continuous at (0, 0).

(3) Let f and g be two functions of one variable which are continuous on $[a-\delta_1, a+\delta_1]$ & $[b-\delta_2, b+\delta_2]$ respectively, $\delta_1 > 0, \delta_2 > 0$.

If $h(x,y) = \max\{f(x), g(y)\}$, then h is continuous on

$\left[a-\delta_1, a+\delta_1; b-\delta_2, b+\delta_2\right]$

Here, $h(x,y) = \frac{1}{2}\left[f(x)+g(y)+|f(x)-g(y)|\right]$

Let (x', y') $\in [a-\delta_1, a+\delta_1; b-\delta_2, b+\delta_2]$

Let $\varepsilon > 0$ be any number

As f is continuous at x', so corresponding to above ε, there exists δ,

$0 < \delta < \delta_1$, such that $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $|x - x'| < \delta$. As g is continuous at y', corresponding to above ε there exists δ′,

$0 < \delta' < \delta_2$, such that $|g(y) - g(y')| < \frac{\varepsilon}{2}$ whenever $|y - y'| < \delta'$.

Let $\eta = \min\{\delta, \delta'\}$

So $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $|x - x'| < \eta$ & $|g(y) - g(y')| < \frac{\varepsilon}{2}$ whenever $|y - y'| < \eta$

Consequently $|\{f(x) \pm g(y)\} - \{f(x') \pm g(y')\}|$

$\leq |f(x) - f(x')| + |g(y) - g(y')| < \varepsilon$ whenever $|x - x'| < \eta, |y - y'| < \eta$

Therefore $f(x) + g(y)$ and $f(x) - g(y)$ are continuous at (x', y'), $|f(x) - g(y)|$ continuous at (x', y'). So their sum, difference and scalar multiple are continuous. Hence $h(x,y)$ is continuous at (x', y').

(4) Let $f(x,y) = \begin{cases} \dfrac{x^\alpha y^\beta}{x^2 + xy + y^2}, (x,y) \neq (0,0) \\ 0, \ (x,y) = (0,0) \end{cases}$

Examine for the continuity of f(x, y) at (0, 0).

Let $(x, y) \to (0, 0)$ along the line $y = mx$.

$$\frac{x^\alpha y^\beta}{x^2+xy+y^2} = \frac{m^\beta}{1+m+m^2}.x^{\alpha+\beta-2}$$

If $\alpha+\beta=2$ we get $\dfrac{m^\beta}{1+m+m^2}$ & if $\alpha+\beta<2,$ the limit does not exist

So $\alpha+\beta \nleq 2$

Let us **consider the case** $\alpha+\beta>2$

We put $x = r\cos\theta, y = r\sin\theta,$ Then,

$$\frac{x^\alpha y^\beta}{x^2+xy+y^2} = r^{\alpha+\beta-2}\frac{(\cos\theta)^\alpha(\sin\theta)^\beta}{1+\sin\theta\cos\theta}$$

For any $\theta, \dfrac{1}{2} \le 1+\sin\theta\cos\theta \le 2.$

Then, $|f(x,y)| = \left|r^{\alpha+\beta-2}\dfrac{(\cos\theta)^\alpha(\sin\theta)^\beta}{1+\sin\theta\cos\theta}\right| \le 2r^{\alpha+\beta-2} \quad 2r^{\alpha+\beta+2} \to 0$ as $r \to 0$

provided $\alpha + \beta > 2$.

Therefore, when $\alpha + \beta > 2$, $|f(x,y)-f(0,0)| < \varepsilon$ whenever

$|x-0|<\delta, |y-0|<\delta, \delta \equiv \delta(\varepsilon) \Rightarrow f$ is continuous at (0, 0) only when $\alpha + \beta > 2$.

(5) Let f(x,y) = $\begin{cases} (ax+by)\sin\dfrac{x}{y}, y \ne 0 \ (a,b \in \mathbb{R}) \\ 0, y=0 \end{cases}$

Test for continuity of $f(x, y)$ at (0, 0)

Let $\varepsilon > 0$ be any number

$$|f(x,y)-f(0,0)| = \left|(ax+by)\sin\frac{x}{y} - 0\right| \le |ax+by| \le |a||x|+|b||y|$$

$< \dfrac{|a|.\varepsilon}{2(|a|+1)} + \dfrac{|b|.\varepsilon}{2(|b|+1)}$ whenever

$|x-a| < \delta_1 = \dfrac{\varepsilon}{2(|a|+1)}, |y-b| < \delta_2 = \dfrac{\varepsilon}{2(|b|+1)}$

If $\delta = \min\{\delta_1, \delta_2\}$, we have

$|f(x,y) - f(0,0)| < \varepsilon$ whenever $|x-0| < \delta, |y-0| < \delta$; $\delta \equiv \delta(\varepsilon)$

So f is continuous at (0, 0).

(6) Let f(x, y) $= \begin{cases} \dfrac{x^6 - 2y^4}{x^2+y^2}, x^2+y^2 \neq 0 \\ 0, \; x^2+y^2=0 \end{cases}$

Test for the continuity of f(x, y) at (0, 0).

Put $x = r\cos\theta, y = r\sin\theta$

$\dfrac{x^6 - 2y^4}{x^2+y^2} = r^4\cos^6\theta - 2r^2\sin^4\theta.$ Let $\varepsilon > 0$ be any number

$|f(x,y) - f(0,0)| \leq r^4 + 2r^2 < \varepsilon$ whenever $x^2+y^2 < \delta \;\&\; \delta = \sqrt[4]{\dfrac{\varepsilon}{4}}$

$\Rightarrow f$ is continuous at (0, 0).

(7) $f(x,y) = \begin{cases} \dfrac{2x^3 - y^3}{x^2+y^2}, x^2+y^2 \neq 0 \\ 0, \; x^2+y^2=0 \end{cases}$

Put x = $r\cos\theta, y = r\sin\theta,$ Then $\dfrac{2x^3 - y^3}{x^2+y^2} = 2r\cos^3\theta - r\sin^3\theta.$

$|f(x,y) - f(0,0)| \leq 2r + r < \varepsilon$ whenever $r < \delta = \dfrac{\varepsilon}{3}(\varepsilon > 0$ is arbitrary)

$\Rightarrow f(x,y)$ is continuous at (0,0).

16.5 Neighbourhood Properties

(1) If $\lim\limits_{(x,y)\to(p,q)} f(x,y) = \lambda(\in \mathbb{R})$, there exists a deleted neighbourhood of (*p*, *q*) in which *f* is bounded.

We note that there exists $\delta > 0$ such that

$|f(x,y) - \lambda| < \frac{1}{2}$ whenever $(x, y) \in N'((p,q),\delta) \cap D_f$

$\Rightarrow |f(x,y)| \le |f(x,y) - \lambda| + |\lambda| < \frac{1}{2} + |\lambda|$ for all $(x,y) \in N'((p,q),\delta) \cap D_f$

$\Rightarrow f(x,y)$ is bounded in $N'((p,q),\delta) \cap D_f$

(2) If $\lim_{(x,y)\to(p,q)} f(x,y) = \lambda,\ \lambda \in \mathbb{R} - \{0\}$, there exists a deleted neighbourhood of (p, q) in which $f(x, y)$ does not vanish.
There exists $\eta > 0$ such that

$$|f(x,y) - \lambda| < \frac{|\lambda|}{2} \text{ whenever } (x,y) \in N'((p,q),\eta) \cap D_f$$

$\Rightarrow \frac{1}{2}|\lambda| < |f(x,y)| < \frac{|\lambda|}{2} + |f(x,y)|,\ (x,y) \in N'((p,q),\eta) \cap D_f$

LHS of the last inequality implies that f(x,y) does not vanish in N' $((p,q),\eta) \cap D_f$

(3) Let f be defined and continuous in S ($\subset \mathbb{R}^2$). If at two points $M'(x', y')$ and M'' (x", y") of S, $M'M''$ lies entirely in S, the function takes values of distinct signs, say $f(x', y') < 0$, $f(x'', y'') > 0$, then there exists a point M_0 (x_0, y_0) in the domain at which $f(x_0, y_0) = 0$.
Let $x = x' + t(x'' - x'),\ y = y' + t(y'' - y')$ where $0 \le t \le 1$.
$f(x,y) = f(x' + t(x'' - x'), y' + t(y'' - y')) = F(t)$.
By hypothesis $F(t)$ is continuous in [0, 1]. $F(0)\ F(1) = f(x', y')\ f(x'', y'')$
i.e., $F(0)\ F(1) < 0$. By Bolzano's theorem on continuous function for a function of one variable, there exists a point $t_0 \in (0,1)$ for which $F(t_0) = 0$ => there is a point (x_0, y_0) such that $f(x_0, y_0) = 0$ where
$x_0 = x' + t_0\ (x'' - x'),\ y_0 = y' + t_0\ (y'' - y')$

Some important results :

We state the following results without proof

Results (1) : If functions $\phi_i(P)\ (i = 1,2)$ are continuous at the point P' (t_1', t_2') in $S(\subset \mathbb{R}^2)$ and the functions $f(M)$, $M(x_1, x_2)$, be continuous at the corresponding point $M'(x_1', x_2')$ where $x_1' = \varphi_1(t_1', t_2')$

$x_2' = \varphi_2(t_1', t_2')$ then the composite function

$u = f(\varphi_1(t_1, t_2), \varphi_2(t_1, t_2)) = f(\varphi_1(p), \varphi_2(p))$ is continuous at p'.

Result (2) : If f(x, y) is defined & continuous in a bounded closed domain $S(\subset \mathbb{R}^2)$, then f is bounded above and below in S and f attains its bounds in S.

Example : Let $f(x, y)$ be defined in the square $S = \{(x,y) |x| \le 1, |y| \le 1\}$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^4 + y^4}, & x^4 + y^4 \neq 0 \\ 0, & x^4+y^4=0 \end{cases}$$

Examine for continuity and boundedness of f on S.

It is evident that f is continuous in x for every y and f is continuous in y for every x. To discuss double limit, let $(x, y) \to (0, 0)$ along $y = mx^3$.

Then $f(x,y) = \dfrac{mx^4}{x^4 + m^4x^{12}} = \dfrac{m}{1+m^4x^8} \to m$ as $x \to 0$

So double limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist & f is not continuous at (0, 0).

If possible, let f be bounded. Then there would exist M > 0 such that

$|f(x,y)| < M$ for all (x, y) $\in S$.

For $x = y = \dfrac{1}{2\sqrt{M}}$, $f\left(\dfrac{1}{2\sqrt{M}}, \dfrac{1}{2\sqrt{M}}\right) = \dfrac{1}{4M} / \left(\dfrac{1}{16M^2} + \dfrac{1}{16M^2}\right) = 2M > M$

This indicates $f(x, y)$ is not bounded on S.

17.6 Summary

In this chapter we have introduced the concept of limit and continuity for function of two variable as a generalization of one variable. We also have examined condition for the existence of double limit. We have studied repeated limits and continuity at a point with some examples. We have further developed the neighbourhood properties.