PREFACE

With its grounding in the "guiding pillars of Access, Equity, Equality, Affordability and Accountability," the New Education Policy (NEP 2020) envisions flexible curricular structures and creative combinations for studies across disciplines. Accordingly, the UGC has revised the CBCS with a new Curriculum and Credit Framework for Undergraduate Programmes (CCFUP) to further empower the flexible choice based credit system with a multidisciplinary approach and multiple/ lateral entry-exit options. It is held that this entire exercise shall leverage the potential of higher education in three-fold ways – learner's personal enlightenment; her/his constructive public engagement; productive social contribution. Cumulatively therefore, all academic endeavours taken up under the NEP 2020 framework are aimed at synergising individual attainments towards the enhancement of our national goals.

In this epochal moment of a paradigmatic transformation in the higher education scenario, the role of an Open University is crucial, not just in terms of improving the Gross Enrolment Ratio (GER) but also in upholding the qualitative parameters. It is time to acknowledge that the implementation of the National Higher Education Qualifications Framework (NHEQF) and its syncing with the National Skills Qualification Framework (NSQF) are best optimised in the arena of Open and Distance Learning that is truly seamless in its horizons. As one of the largest Open Universities in Eastern India that has been accredited with 'A' grade by NAAC in 2021, has ranked second among Open Universities in the NIRF in 2024, and attained the much required UGC 12B status, Netaji Subhas Open University is committed to both quantity and quality in its mission to spread higher education. It was therefore imperative upon us to embrace NEP 2020, bring in dynamic revisions to our Undergraduate syllabi, and formulate these Self Learning Materials anew. Our new offering is synchronised with the CCFUP in integrating domain specific knowledge with multidisciplinary fields, honing of skills that are relevant to each domain, enhancement of abilities, and of course deep-diving into Indian Knowledge Systems.

Self Learning Materials (SLM's) are the mainstay of Student Support Services (SSS) of an Open University. It is with a futuristic thought that we now offer our learners the choice of print or e-slm's. From our mandate of offering quality higher education in the mother tongue, and from the logistic viewpoint of balancing scholastic needs, we strive to bring out learning materials in Bengali and English. All our faculty members are constantly engaged in this academic exercise that combines subject specific academic research with educational pedagogy. We are privileged in that the expertise of academics across institutions on a national level also comes together to augment our own faculty strength in developing these learning materials. We look forward to proactive feedback from all stakeholders whose participatory zeal in the teaching-learning process based on these study materials will enable us to only get better. On the whole it has been a very challenging task, and I congratulate everyone in the preparation of these SLM's.

I wish the venture all success.

Professor Indrajit Lahiri
Vice Chancellor

Netaji Subhas Open University

Four Year Undergraduate Degree Programme

Under National Higher Education Qualifications Framework (NHEQF) &

Curriculum and Credit Framework for Under Graduate Programmes

B. Sc. Mathematics (Hons.)

Programme Code : NMT

Course Type : Skill Enhancement Course (SEC

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Unit 1 ❐ Techniques for sketching parabola, ellipse and hyperbola and their reflection properties

Structure

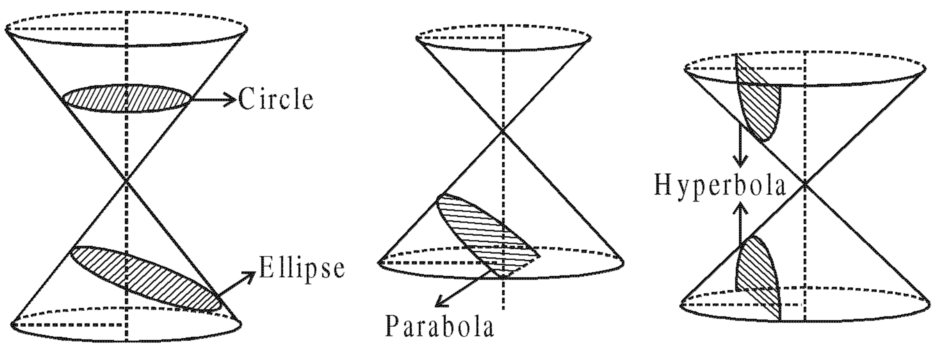
1.0 Objectives

The learners will be able to understand the conic sections, their basic property and sketch the graphs of

- parabola
- ellipse
- hyperbola

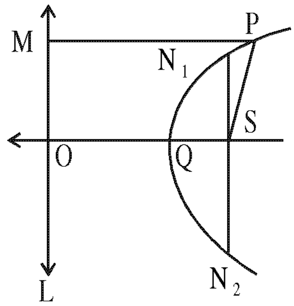
1.1 Introduction

By conic section or simply conics we mean the set of curves that can be formed by sectioning a double - napped right cone by a plane. Usually we use four curves, namely circle, parabola, ellipse and hyperbola the most in mathematics. The formation of these four standard curves can be done as,



1.2 Parabola

If a variable point on a plane satisfies the condition that it is equidistant from a fixed point and from a fixed straight line i.e., the ratio of the distance of the variable point from the fixed point and the distance of the variable point from the fixed straight line is 1, then the locus of the point is called a parabola.



In this picture P is the variable point, S is the fixed point called focus, L is the fixed straight line called directrix and the ratio $\frac{SP}{PM}$ is called eccentricity. We usually denote eccentricity by 'e'. Here e = 1.

The perpendicular drawn from focus S on the directrix L is called axis of the parabola. The axis is also called line of symmetry of the parabola. Here SO is the axis. The point at which the axis intersects the parabola is called vertex of the parabola. In this diagram Q is the vertex. Any chord which passes through S is called focal chord. The focal chord which is perpendicular on axis is called latus rectum. Here N_1N_2 is the Latus rectum.

Let the co-ordinate of the focus be $S(\alpha, \beta)$ and the equation of the directrix be ax + by + c = 0. Let the coordinate of the variable point P be (h, k). Then by definition,

$$\sqrt{(h-\alpha)^2+(k-\beta)^2} = \pm\frac{ah+bk+c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow (h-\alpha)^2+(k-\beta)^2 = \frac{(ah+bk+c)^2}{a^2+b^2}$$

$\therefore$ The locus of P is $(x-\alpha)^2+(y-\beta)^2 = \frac{(ax+by+c)^2}{a^2+b^2}$

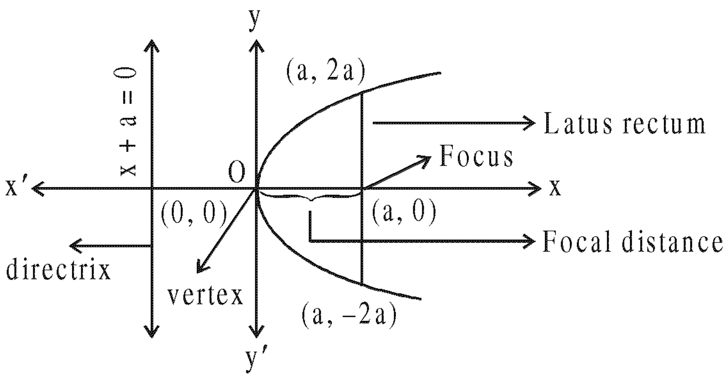
Now if we consider the focus at (a, 0) and the directrix at x + a = 0, then, $\alpha = a$, $\beta = 0$, a = 1, b = 0, c = a. Then the above equation reduces to,

$$(x-a)^2+y^2 = \frac{(x+a)^2}{1} \Rightarrow y^2 = 4ax$$

This is a standard equation of the parabola whose,

(1) vertex at origin (0, 0)
(2) axis is positive x-axis
(3) focal distance from vertex = a unit
(4) focus at (a, 0)
(5) directrix is x + a = 0
(6) length at latus rectum = 4a unit

The diagram of the paraboa $y^2 = 4ax$ is



Any parabola whose vertex is (α, β), axis is parallel to x-axis is of the form $(y - \beta)^2 = 4a(x - \alpha)$. It has the following properties,

(1) axis : $y = \beta$ (parallel to x-axis)

(2) vertex = (α, β)

(3) focus = $(a + \alpha, \beta)$

(4) directrix : $x + a = \alpha$

(5) Length of latus rectrum = 4a unit

(6) Tangent at vertex : $x = \alpha$

(7) Extremities of latus rectum = $(a + \alpha, 2a + \beta)$ and $(a + \alpha, -2a + \beta)$

Again any parabola whose vertex is at (α, β) and whose axis is parallel to y-axis is of the form $(x - \alpha)^2 = 4a(y - \beta)$.

Also it satisfies the following properties,

(1) axis : $x = \alpha$ (parallel to y-axis)

(2) vertex = (α, β)

(3) focus = $(\alpha, a + \beta)$

(4) directrix : $y + a = \beta$

(5) Length of latus rectum = 4a unit

(6) Tangent at vertex : $y = \beta$

(7) Extremities of latus rectum = $(\alpha + 2a, a + \beta)$ and $(\alpha - 2a, a + \beta)$

1.2.1 Tracing a parabola

To trace or arbitrary parabola we need to find,

(1) Axis/line of symmetry :

If the parabola is of the form $y = ax^2 + bx + c$ then the axis is $x = -b/2a$ and if the equation is given by $y = a(x - h)^2 + k$, then the axis is $x = h$.

(2) Points of intersection with the axis :

By putting $x = 0$ we find the points of intersection with y-axis and putting $y = 0$, we find them for y-axis. Obviously the parabola will pass through those points.

(3) Focus and directrix :

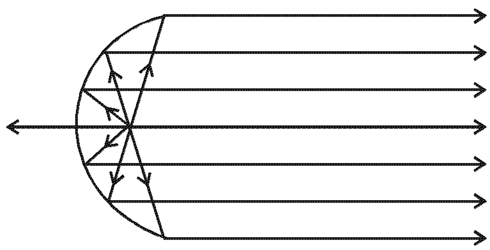
Focus will lie on the axis of the parabola and directrix will be perpendicular on the axis. Now, vertex of the parabola will be the mid point of the perpendicular line segment from focus on the directrix. Also any point on the parabola will be equidistant from focus and from directrix.

(4) Vertex :

Vertex is the point where the axis intersects with the parabola. Also the tangent at that point will be parallel to directrix.

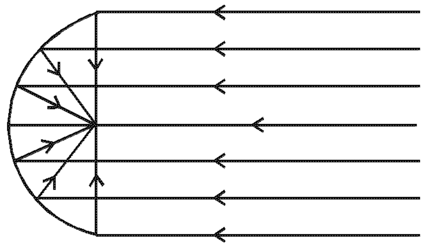
With the help of those we can trace any arbitrary parabola.

1.2.2 Reflection properties of parabola



If a light source is placed at the focus of a parabola, the result of reflection will be a focused beam of light emerging outward along the direction of the axis. We can express this using this diagram. This is how flash lights, headlight, search lights work.

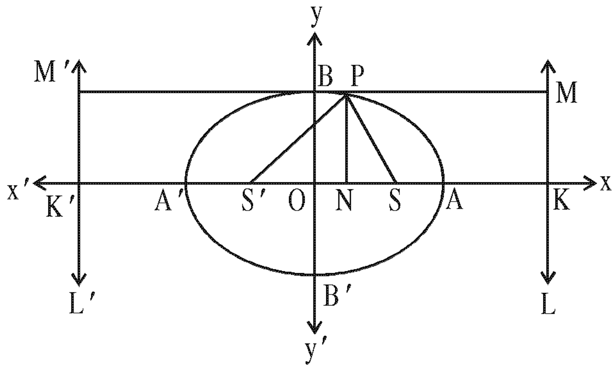
The opposite is also true. If a parallel beam of light incident on a parabola, then the lights will converge on the focus after being reflected on the parabola. This property is used in reflecting telescopes, satelight dishes etc.



1.3 Ellipse

If in a plane a variable point moves in such a way that its eccentricity i.e., the ratio of distances from a fixed point and a fixed straight line is constant which is < 1, then the locus of the variable point is called an ellipse.

Here in this picture S is the fixed point focus, L is the directrix. Again the straight line which is perpendicular to directrix and passes through focus is called axis (major). The point of intersection of the ellipse and the axis is called vertex. Here A is the vertex and $\frac{SA}{AK} = e$. Again A′ is another point which is point of external intersection of the ellipse such that $\frac{SA'}{A'K} = e$. Then in such manner we can always take internal and external ratios w.r.t. P. So, we get another fixed point and another directrix S′ and L′ respectively. So, in an ellipse we get two foci, two directrices.



The midpoint of those foci is called centre of the ellipse. And the line segment perpendicular to axis at centre and whose extremitics are points of intersection with ellipse is called minor axis. Here O is the centre and BB′ is the minor axis. Any chord of the ellipse which is perpendicular on foci is called latus rectum.

A standard equation of an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It satisfies the following properties (when $a^2 > b^2$).

(1) Major axis : x-axis (y = 0)

(2) Minor axis : y-axis (x = 0)

(3) Length of the major axis = 2a unit

(4) Length of the minor axis = 2b unit.

(5) Centre = (0, 0)

(6) Vertices = (a, 0), (–a, 0)

(7) eccentricity (e) = $\sqrt{1-\frac{b^2}{a^2}}$

(8) Foci = (ae, 0), (–ae, 0)

(9) Length of latus recturm = $2\frac{b^2}{a}$ unit

(10) Extremities of latus recturms = $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right), \left(-ae, \frac{b^2}{a}\right), \left(-ae, -\frac{b^2}{a}\right)$

(11) Equations of latus rectums : $x = \pm\, ae$

(12) Directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

Any ellipse whose centre is (α, β) and whose major axis is parallel to x-axis and minor axis is parallel to y-axis is of the form, $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$ $(a^2 > b^2)$

(1) Major axis : $y = \beta$ (parallel to x-axis)
(2) Minor axis : $x = \alpha$ (parallel to y-axis)
(3) Length of major axis : 2a unit
(4) Length of minor axis : 2b unit
(5) Centre : (α, β)
(6) Vertices : $(\alpha + a, \beta)$, $(\alpha - a, \beta)$
(7) Eccentricity (e) = $\sqrt{1 - \frac{b^2}{a^2}}$
(8) Foci : $(\alpha + ae, \beta)$, $(\alpha - ae, \beta)$
(9) Length of latus recturm : $2\frac{b^2}{a}$ unit
(10) Extremities of latus rectums : $\left(\alpha \pm ae,\ \beta \pm \frac{b^2}{a}\right)$
(11) Equations of latus rectums : $x = \alpha \pm ae$
(12) Directrices : $x = \alpha \pm \frac{a}{e}$

Any ellipse whose centre is (α, β) and whose major axis is parallel to y-axis and whose minor axis is parallel to x axis is of the form $\frac{(x-\alpha)^2}{b^2} + \frac{(y-\beta)^2}{a^2} = 1$ $(a^2 > b^2)$. It satisfies the following,

(1) Major axis : $x = \alpha$ (parallel to y-axis)

(2) Minor axis : $y = \beta$ (parallel to x-axis)

(3) Length of major axis = 2a unit

(4) Length of minor axis = 2b unit

(5) Centre = (α, β)

(6) Vertices = $(\alpha, \beta \pm a)$

(7) Eccentricity = $\sqrt{1-\frac{b^2}{a^2}}$

(8) Foci = $(\alpha, \beta \pm ae)$

(9) Length of latus rectum = $2\frac{b^2}{a}$ unit

(10) Extremities of latus rectums = $\left(\alpha \pm \frac{b^2}{a}, \beta \pm ae\right)$

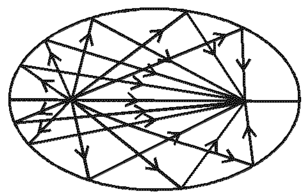
(11) Equations of latus rectums : $y = \beta \pm ae$

(12) Directrices : $y = \beta \pm \frac{a}{e}$

Any ellipse has two lines of symmetry one is major axis and another one is minor axis. The sum of disatance of a point on the ellipse from two foci is always constant. If we consider a circle whose centre is the centre of the ellipse and diameter is major axis is called auxiliary circle.

If two foci are given then we can sketch any ellipse with the help of the property that sum of distances from any point on ellipse to the foci is constant.

1.3.1 Reflection properties of an ellipse

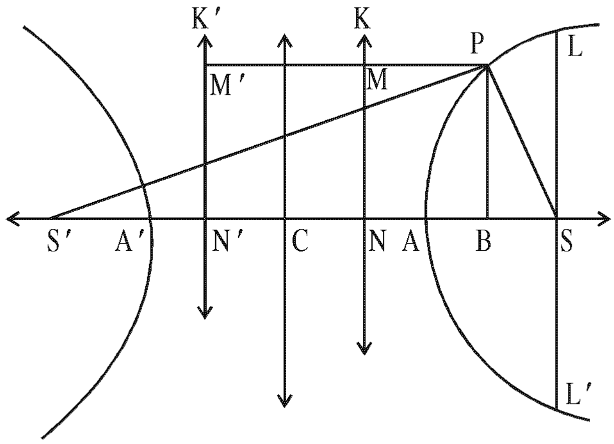


If we put a light source at one focus of an ellipse then the rays of light will converge to another focus after reflecting on the ellipse.

1.4 Hyperbola

If a variable point moves on a plane in such a way that its eccentricity $e > 1$. Then the locus of the point is called a hyperbola.

Here in this diagram P is the variable point and K is the directrix. By definition $\frac{PS}{PM} = e > 1$. The straight line which passes through S and perpendicular to directrix is called transverse axis. There are two points on the axis which intersects the curve internally and externally at a ratio e : 1, i.e., $\frac{AS}{AN} = e = \frac{A'S}{A'N}$. These two points A and A′ are called vertices of the hyperbola. The mid point of the two vertices is called centre of the hyperbola. Like ellipse we can also find two foci and two directrices in a hyperbola manely S, S′ and K, K′. The line which is perpendicular to the transverse axis on the centre is called conjugate axis. The focal chord which is perpendicular on transverse axis at focus is called latus rectum.



A standard equation of a hyperbola is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It statisfies the following properties,

(1) Transverse axis : y = 0 (x-axis)

(2) Conjugate axis : x = 0 (y-axis)

(3) Length of transverse axis : 2a unit

(4) Length of conjugate axis : 2b unit

(5) Centre : (0, 0)

(6) Vertices : (± a, 0)

(7) Eccentricity (e) : = $\sqrt{1+\frac{b^2}{a^2}}$

(8) Foci = (± ae, 0)

(9) Length of latus rectum = $2\frac{b^2}{a}$ unit

(10) Extremities of latus rectums = $\left(\pm ae, \pm\frac{b^2}{a}\right)$

(11) Equations of latus rectums : $x = \pm ae$

(12) Directrices : $x = \pm\frac{a}{e}$

A hyperbola whose centre is (α, β) and whose transverse axis is parallel to x-axis and conjugate axis is parallel to y-axis is given by $\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$. Whose,

(1) Transverse axis : $y = \beta$ (parallel to x-axis)

(2) Conjugate axis : $x = \alpha$ (parallel to y-axis)

(3) Length of transverse axis : 2a unit

(4) Length of conjuagate axis : 2b unit

(5) Centre : (α, β)

(6) Vertices : $(\alpha \pm a, \beta)$

(7) Eccentricity (e) : $\sqrt{1+\frac{b^2}{a^2}}$

(8) Foci : $(\alpha \pm ae, \beta)$

(9) Length of latus rectum : $2\frac{b^2}{a}$ unit

(10) Extremities of latus rectums = $\left(\alpha \pm ae, \beta\pm\frac{b^2}{a}\right)$

(11) Equations of latus rectums : $x = \alpha \pm ae$

(12) Directrices : $x = \alpha\pm\frac{a}{e}$

A hyperbola whose centre is (α, β) and whose transverse axis is parallel to y-axis and conjugate axis is parallel to x-axis given by, $\frac{(y-\beta)^2}{a^2}-\frac{(x-\alpha)^2}{b^2}=1$. Which satisfies the following properties,

(1) Transverse axis : $x = \alpha$ (parallel to y-axis)

(2) Conjugate axis : $y = \beta$ (parallel to x-axis)

(3) Length of transverse axis : 2a unit

(4) Length of conjugate axis : 2b unit

(5) Centre : (α, β)

(6) Vertices : $(\alpha, \beta \pm a)$

(7) Eccentricity : $\sqrt{1+\dfrac{b^2}{a^2}}$

(8) Foci : $(\alpha, \beta \pm ae)$

(9) Length of latus rectum : $2\dfrac{b^2}{a}$

(10) Extremities of latus rectums = $\left(\alpha \pm \dfrac{b^2}{a}, \beta \pm ae\right)$

(11) Equations of latus rectums : $y = \beta \pm ae$

(12) Directrices : $y = \beta \pm \dfrac{a}{e}$

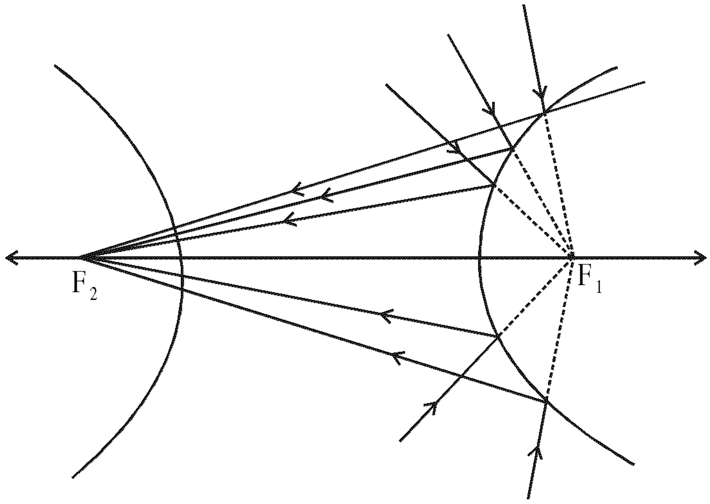
If the equation of a hyperbola is given by $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Then if we draw a perpendicular PB from P(x, y) [a point on the hyperbola] to the transverse axis, then, $\dfrac{PB^2}{AB.A'B} = \dfrac{b^2}{a^2}$ and $S'P - SP = 2a$, where A, A′ are the vertices of the hyperbola and S, S′ are the foci. The lengths of transverse axis and conjugate axis are 2a and 2b units respectively. Since the difference between the lengths two foci from any point on the hyperbola is constant, we can sketch a hyperbola using this property.

Let a hyperbola be given. Then if we construct another hyperbola such that its centre is the same and its transveric axis is the conjugate axis of the former and its conjugate axis is the transverse axis of the pre given hyperbola then the newly constructed hyperbola is called conjugate hyperbola of the former.

If the lengths of transverse axis and conjugate axis are same, then hyperbola is called rectangular hyperbola. A rectangular hyperbola is given by $x^2 - y^2 = a^2$ where the centre is (0, 0)

1.4.1 Reflection property of hyperbola

Rays directed towards the focus of the hyperbola are reflected at the hyperbola and then it passes through another focus. The given diagram ilustrates it,

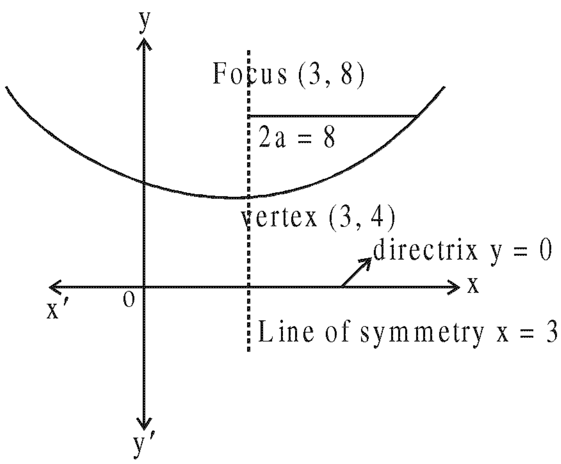


1.5 Worked out examples

Example 1 : Identify the vertex, axis of symmetry, focus and equation of directrix of the parabola $y - 4 = \frac{1}{16}(x-3)^2$, then sketch the parabola.

Solution : This equation is of the form $(x - h)^2 = \frac{1}{4a}(y-k)$. Thus the vertex is (3, 4) and the line of symmetry is x = 3. Since in the above equation a is the focal length and 4a = 16 ⇒ a = 4. Since the focal length is 4, the focus will be (3, 8). And the directrix will be at y = 0.

Sketch of the parabola :



Example 2 : Find a parabola whose vertex is (–2, 4) focus is (0, 4)

Solution : The line joining vertex and focus is y = 4. Since it is parallel to x-axis and it is axis/line of symmetry of the required parabola, the general form will be like.

$(y - k)^2 = 4a(x - h)$

where focus = (a + h, k) and vertex is (h, k). Comparing, h = –2, k = 4, a + h = 0

$\Rightarrow a = 2$.

$\therefore$ The required equation is $(y - 4)^2 = 8(x + 2)$.

Example 3 : Identify vertices, co-vertices, foci and then graph $9x^2 + 49y^2 = 441$.

Solution : $9x^2 + 49y^2 = 441 \Rightarrow \frac{x^2}{49} + \frac{y^2}{9} = 1$

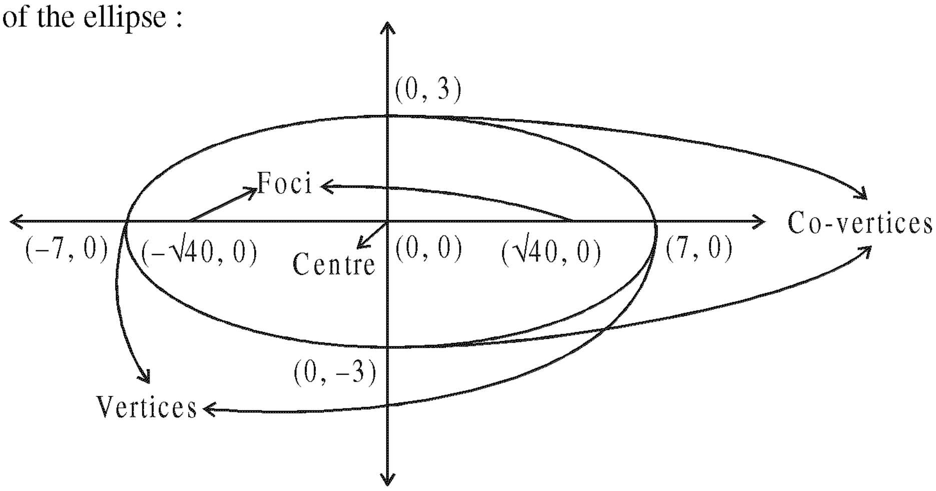
This is of the form $\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$. Where (α, β) is the centre. Here, $\alpha = \beta = 0$, a = 7, b = 3. Since $a^2 > b^2$, this is a horizontal ellipse with centre (0, 0).

Vertices are (±7, 0) and co-vertices are (0, ±3).

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{49}} = \frac{\sqrt{40}}{7}$$

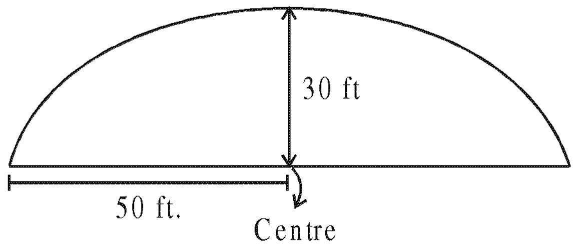
$\therefore$ Foci = $\left(\pm\sqrt{40}, 0\right)$

Sketch of the ellipse :



Example 4 : Two boys are standing in a whispering galary that is semi-elliptic in shape. The height at the arch in 30 ft. and width is 100 ft. How far from the centre of the room should whispering dishes be placed so that the boys can whisper to each other?

Solution : From the reflective property of an ellipse we know that the whispering dishes should be placed at foci. If we consider the centre to be at (0, 0) then the points (0, 50), (0, –50) and (30, 0) must be on the ellipse. Morover its length of major axis is 100 ft. and minor axis is 30 ft.



$\therefore$ The ellipse is of the form $\dfrac{x^2}{50^2}+\dfrac{y^2}{30^2}=1$

eccentricity (e) = $\sqrt{1-\dfrac{900}{2500}}=\dfrac{40}{50}$

$\therefore$ Foci = (±40, 0)

$\therefore$ Each boy should stand 40 ft from the centre of the room.

Example 5 : Two buildings in a shopping mall are shaped like a branch of hyperbola $729x^2 - 1024y^2 - 746496 = 0$, where x and y are in feet. How far apart are the buildings at their closest part?

Solution : $729x^2 - 1024y^2 - 746496 = 0$

$$\Rightarrow \frac{x^2}{1024}-\frac{y^2}{729}=1 \;\Rightarrow\; \frac{x^2}{32^2}-\frac{y^2}{27^2}=1$$

The vertices are (±32, 0). The distance between two vertices is the smallest distance. So, the required distance is 32 × 2 = 64 ft.

1.6 Summary

In this unit, we have learnt the basic conic concept of conic section and sketch the graphs of parabola, ellipse, hyperbola. Also we are now able to determine the crucial parameters and reflection properties of there conics.

1.7 Exercises

1. Identify the vertex, axis of symmetry, focus and equation of directrix of the following parabolas and sketch them,

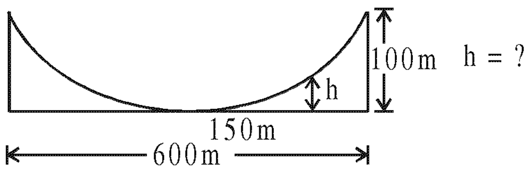
   (a) $(x - 5)^2 = 2(y + 1)$

   (b) $y^2 - 4y + 2x - 8 = 0$

2. Find a parabola with focus at (–2, 4) and a directrix y = 9.

3. Find the equation of a parabola with focus (–2, –7) that opens to the right and contains the point (6, –1).
4. The cables of the middle part of a bridge are in form of a parabola. The towers supporting the cable are 600m apart and 100 m high. What is the height of the cable at a point 150m from the centre of the bridge.

Hint :



Find the parabola containing (0, 0), (300, 100), (–300, 100). Then at x = 150, find y = ?

5. Identify vertices, co-vertices, foci of the following and then graph the ellipses.

   (a) $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{36}=1$

   (b) $4x^2 + y^2 + 24x + 2y + 33 = 0$
6. Find the equation of the ellipse whose focus at (–1, –3) and endpoints of major or minor axis at (–1, –6), (–1, –2).
7. An ice rink is shaped like an ellipse and is 150 ft. long and 75 ft. wide. What is the width of the rink 15 ft. from a vertex?

Answers :

1. (a) Vertex : (5, –1), axis of symmetry : x = 5, focus : $\left(5,-\frac{1}{2}\right)$, directrix : $y=-\frac{3}{2}$.

   (b) Vertex : (6, 2), axis of symmetry : y = 2, focus : $\left(\frac{11}{2},2\right)$ directrix : $x=\frac{13}{2}$.
2. $(x + 2)^2 = -10(y - 6.5)$
3. $(y + 7)^2 = 4(x + 3)$
4. 25 ft.
5. (a) Vertices : (–3, –4), (–3, 8), co-vertices : (–5, 2), (–1, 2) foci : $(-3, 2 \pm 4\sqrt{2})$.

   (b) Vertices : (–3, –3), (–3, 1), co-vertices : (–4, –1), (–2, –1), foci $(-3, -1 \pm \sqrt{3})$.
6. $\frac{(x+1)^2}{15}+\frac{(y+2)^2}{16}=1$
7. 45 ft.

Unit 2 ❐ Transformation of co-ordinates

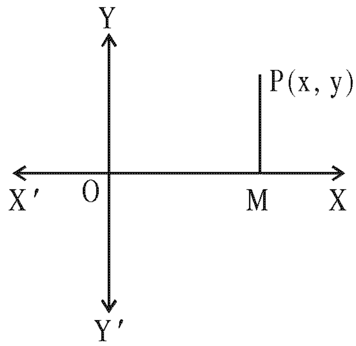
Structure

2.0 Objectives

We will be able to understand and apply the concepts of orthogonal transformation of co-ordinates.

2.1 Intruduction

Let XOX′ and YOY′ be two mutually perpendicular straight lines in the plane. The lines XOX′ and YOY′ are called x-axis and y-axis respectively. The two together is called axes of co-ordinates. The point 'O' is called origin of co-ordinates.

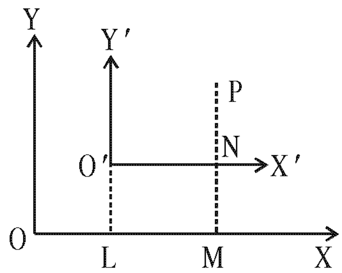


Let P be any point on the plane. Let us draw a perpendicular PM on XOX′. Then the lengths PM and OM with proper sign are called rectangular or orthogonal co-ordinates or simply co-ordinates of P. OM and PM are called abscissa or x-co-ordinate and ordinate or y-co-ordinate of P respectively. Let OM = x and PM = y, then we denote (x, y) as co-ordinates of P.

Usually we define co-ordinates of a point, equations of any curve and straight lines taking x-axis and y-axis as our reference axes. But if we take any other pair of mutually perpendicular straight lines as our reference axes, then the co-ordinates or equation will change accordingly. And this process is known as transformation of co-ordinates.

2.2 Change of origin without changing the directions of the axes (Translation)

Let (x, y) be the co-ordinates of any point P w.r.t. rectangular axes OX and OY. Let O′X′ and O′Y′ be two mutually perpendicular straight lines such that OX parallel to O′X′ and OY parallel to O′Y′. If we consider O′X′ and O′Y′ as our set of new co-ordinate axes then O′ will become our new origin. Let the co-ordinates of O′ be (α, β) w.r.t., OX and OY and the co-ordinates of P be (x', y') w.r.t. new coordinate axes O′X′, O′Y′. Let us draw a perpendicular O′L on OX and another perpendicular PM on OX which intersects O′X′ at N.



Then clearly, $OL = \alpha$, $O'L = \beta$, $OM = x$, $PM = y$, $LM = x'$, $PN = y'$. And,

$x = OM = OL + LM = \alpha + x'$

$y = PM = PN + NM = \beta + y'$.

Hence $x = \alpha + x'$ and $y = \beta + y'$ are the transformation formulae for translation. Thus equation of any curve or straight line given by $f(x, y) = 0$ will transform to $f(\alpha + x', \beta + y') = 0$ after translation.

Example 1 : Find the form of equation $x + 2y = 4$ due to change of origin to (3, –2) by translation.

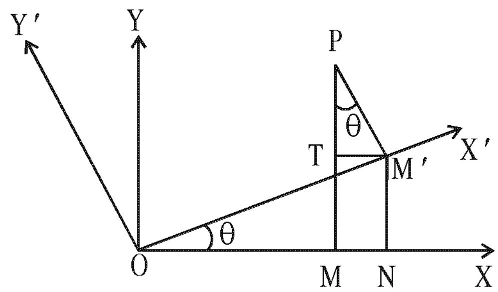
Solution : The transformed equation is,

$$(x' + 3) + 2(y' - 2) = 4$$

$$\Rightarrow x' + 2y' = 5$$

2.3 Transformation from one pair of rectangular axes to another with same origin (Rotation)

Let the original axes OX and OY be rotated through an angle θ in the anti-clckwise direction about origin O. Let us denote the new set of axes as OX′ and OY′. Let (x, y) and (x′, y′) be the co-ordinates of a point P w.r.t., OX, OY and OX′, OY′ respectively. Let us draw two perpendiculars from P on OX and OX′ and they intersects OX and OX′ at M and M′ respectively. Then we draw another perpendicular M′T from M′ to PM. Then finally we



draw a perpendicular from M′ to OX which intersects OX at N. Then, OM = x, PM = y, OM′ = x′, PM′ = y′. $\angle$TPM = θ.

And, x = OM = ON – NM = OM′cosθ – TM′ = x′cosθ – PM′sinθ = x′cosθ – y′sinθ.

y = PM = PT + TM = PM′cosθ + M′N = y′cosθ + OM′sinθ = y′cosθ + x′sinθ.

Hence x = x′cosθ – y′sinθ and y = x′sinθ + y′cosθ are the transformation formulae for rotation. This can also be expressed as,

	x′	y′
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Example 2 : Find the equation of the line y = 2x when the axes are rotated through an angle $\frac{\pi}{3}$.

Solution : The transformed equation is, $y'\cos\frac{\pi}{3} + x'\sin\frac{\pi}{3} = 2\left[x'\cos\frac{\pi}{3} - y'\sin\frac{\pi}{3}\right]$

$$\Rightarrow \frac{y'}{2} + \frac{\sqrt{3}x'}{2} = 2\left[\frac{x'}{2} - \frac{\sqrt{3}}{2}y'\right] \Rightarrow y' + \sqrt{3}x' = 2x' - 2\sqrt{3}y'$$

$$\Rightarrow (\sqrt{3} - 2)x' + (1 + 2\sqrt{3})y' = 0$$

2.4 Combination of translation and rotation

If the origin O of the set of rectangular axes OX and OY be shifted to (α, β) from (0, 0) without changing the direction of axes and then the axes be rotated through an angle θ in the anti-clockwise direction, then the total effect on co-ordinates is given by,

x = α + x′cosθ – y′sinθ

y = β + x′sinθ + y′cosθ

Where (x′, y′) is the new co-ordinates after the transformation of the point whose co-ordinates are (x, y).

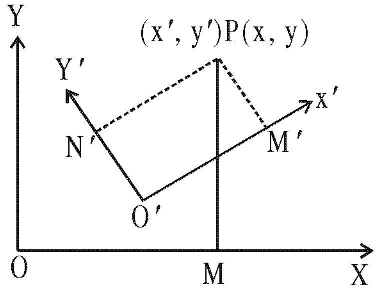
2.5 Transformation of co-ordinates when the equations of new axes are given (General orthogonal transformation)

Let (x, y) be the co-ordinates of a point P referred to rectangular axes OX and OY and (x′, y′) be the co-ordinates of the same point referred to a new set of rectangular axes

O′X′ and O′Y′ where the equations are given by ax + by + c = 0 and bx – ay + k = 0 w.r.t. OX and OY.

The perpendicular distance from P to ax + by + c = 0 is,

$$M'P = y' = \pm\frac{ax+by+c}{\sqrt{a^2+b^2}}$$



And, the perpendicular distance from P to bx – ay + k = 0 is,

$$N'P = x' = \pm\frac{bx-ay+k}{\sqrt{a^2+b^2}}.$$

This is the required transformation formalae.

Example 3 : Find the transformed equation of the curve (3x + 4y + 10)(4x – 3y + 2) = 10, when the axes are given by, 3x + 4y + 10 = 0 and 4x – 3y + 2 = 0.

Solution : If (x′, y′) be the co-ordinates of a point (x, y) w.r.t. new set of co-ordinates, then,

$$y' = \frac{3x+4y+10}{5} \text{ and } x' = \frac{4x-3y+2}{5}.$$

Then the equation can be written as,

$$5x'.5y' = 10$$

$$\Rightarrow 5x'y' = 2$$

$\therefore$ The transformed equation is $5x'y' = 2$

2.6 Worked out examples

Example 4 : Transform to parallel axes through the point (2, –3) the equation $2x^2 + 4xy + 3y^2 - 2x - 4y + 7 = 0$.

Solution : The transformation will effect as x = x′ + 2, y = y′ – 3, where (x′, y′) is the new co-ordinates after transformation of the point (x, y). The transformed equation is,

$$2(x' + 2)^2 + 4(x' + 2)(y' - 3) + 3(y' - 3)^2 - 2(x' + 2) - 4(y' - 3) + 7 = 0$$

$$\Rightarrow 2x'^2 + 4x'y' + 3y'^2 - 6x' - 14y' + 26 = 0$$

Example 5 : Transform to axes inclined at 30º to the original axes the equation,

$x^2 + 2\sqrt{3}\,xy - y^2 - 2 = 0$

Solution : The transformation formulae are,

$$x = x'\cos 30^\circ - y'\sin 30^\circ = \frac{1}{2}\left(x'\sqrt{3} - y'\right)$$

$$y = x'\sin 30^\circ + y'\cos 30^\circ = \frac{1}{2}\left(x' + y'\sqrt{3}\right)$$

The transformed equation is,

$$\left(x'\sqrt{3} - y'\right)^2 + 2\sqrt{3}\left(x'\sqrt{3} - y'\right)\left(x' + y'\sqrt{3}\right) - \left(x' + y'\sqrt{3}\right)^2 = 8$$

$$\Rightarrow \quad x'^2 - y'^2 = 1$$

Example 6 : How can the first degree term of the equation $x^2 + xy + 2y^2 - 7x - 5y + 12 = 0$ be removed?

Solution : To remove first degree terms from a equation we usually take suitable translation. Let the origin be translated to (α, β). Then, the transformation formulae will be,

$x = x' + \alpha$ and $y = y' + \beta$.

The transformed equation will be,

$$(x' + \alpha)^2 + (x' + \alpha)(y' + \beta) + 2(y' + \beta)^2 - 7(x' + \alpha) - 5(y' + \beta) + 12 = 0$$

$\Rightarrow x'^2 + x'y' + 2y'^2 + (2\alpha + \beta - 7)x' + (\alpha + 4\beta - 5)y' + (\alpha^2 + \alpha\beta + 2\beta^2 - 7\alpha - 5\beta + 12) = 0.$

We choose suitable α and β such that $\alpha + 4\beta - 5 = 0$ and $2\alpha + \beta - 7 = 0$

Solving we get, $\alpha = 3\frac{2}{7}$ and $\beta = \frac{3}{7}$

So, to remove the first degree terms we must shift the origin to $\left(3\frac{2}{7}, \frac{3}{7}\right)$ by translation.

Example 7 : Find the angle by which the axes should be rotated so that the equation $ax^2 + 2hxy + by^2 = 0$ becomes another equation in which xy term is absent.

In particular, find the angle through which the axes are to be rotated so that the equation $17x^2 + 18xy - 7y^2 = 1$ may be reduced to the form $Ax^2 + By^2 = 1, A > 0$. Also find A and B.

Solution : Let the axes be turned through an angle θ. Then after rotation the equation becomes,

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2 = 0$$

$$\Rightarrow (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2 + (-2a\sin\theta\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta$$
$$+ 2b\sin\theta\cos\theta)x'y' + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 = 0$$

The terms $x'y'$ will vanish if,

$$(b - a)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow -\frac{1}{2}(a-b)\sin 2\theta + h\cos 2\theta = 0$$

$$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

In this given case, $a = 17$, $h = 9$, $b = -7$

$$\therefore \theta = \frac{1}{2}\tan^{-1}\frac{18}{17+7} = \frac{1}{2}\tan^{-1}\frac{3}{4}$$

By invariant properties of transformation,

We have, $A + B = 17 - 7 = 10$ and $AB = 17(-7) - 9^2 = -200$

$$\therefore A - B = \sqrt{(A+B)^2 - 4AB} = \sqrt{100+800} = \pm 30$$

$\because A > 0, A = 20, B = -10$

$\therefore$ The transformed equation is $20x^2 - 10y^2 = 1$.

2.7 Summary

In this unit, we have learnt to determine the transformed equation i.e. the locus of a point when the co-ordinate axes are either translated or rotated or changed in combined way in two dimansional co-ordinate geometry.

2.8 Exercises

1. Reduce the following equations in the form $ax + by = 0$ by choice of new origin (a) on the x-axis, (b) on the y-axis without rotating the axes.

(i) $x + 2y - 4 = 0$

(ii) $y = mx + c$

(iii) $x\sin\alpha + y\cos\alpha = p$

2. Choose a new origin in such a way that the equations will reduce to the form $ax^2 + by^2 = 1$

(i) $5x^2 - 7y^2 + 2x - 3y = 0$

(ii) $4x^2 + 3y^2 - 2x - 3y - 7 = 0$

3. What does the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$ become when the axes are rotated through an angle $\frac{\pi}{6}$?

4. Find the angle of rotation of the axes for which the equation $x^2 - y^2 = a^2$ will reduce to $xy = c^2$. Determine c^2.

5. Show that if the origin is transferred to (0, 1) and the axes are rotated through 45°, the equation $5x^2 - 2xy + 5y^2 + 2x - 10y - 7 = 0$ referred to new axes becomes
$$\frac{x'^2}{3} + \frac{y'^2}{2} = 1.$$

6. Find the equations of the followings when $lx + my + n = 0$ and $mx - ly + p = 0$ are considered as axes of x and y respectively

(i) $(lx + my + n)^2 = l^2 + m^2$

(ii) $(lx + my + n)(mx - ly + p) = l^2 + m^2$

Answers :

1. (i) (4, 0), (0,2) (ii) $\left(-\frac{c}{m}, 0\right), (0, c)$ (iii) $\left(\frac{p}{\sin\alpha}, 0\right), \left(0, \frac{p}{\cos\alpha}\right)$

2. (i) $\frac{980}{17}y^2 - \frac{700}{17}x^2 = 1, \left(-\frac{1}{5}, -\frac{3}{14}\right)$ (ii) $\frac{x^2}{2} + \frac{3y^2}{8} = 1, \left(\frac{1}{4}, \frac{1}{2}\right)$

3. $5x^2 + y^2 = 1$ 4. $-\frac{\pi}{2}, \frac{a^2}{2}$

6. (i) $y^2 = 1$ (ii) $xy = 1$

Unit 3 ❒ General equation of second degree

Structure

3.0 Objectives

We will be able to

- Extract information about conic sections represented by general second degree equations.
- Characterize and clasify conic sections based on the general second degree equations.
- Reduce the general second degree equations to canonical forms.

3.1 Introduction

A general equation of second degree in x and y is of the form,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$

The curve represented by this equation is either a conic section or a pair of straight lines. The nature of the curve is usually determined by these quantities,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, D = ab - h^2 \text{ and } P = a + b$$

These quantities are invariant under transformation of co-ordinates.

(i) If $\Delta = 0$, the equation (1) represents a pair of straight lines.

(ii) If $a = b$ and $h = 0$, then the equation (1) represents a circle.

(iii) If $\Delta \neq 0$, then the equation (1) represents a proper conic. Then D determines the nature of it.

(a) If $D = 0$, then the conic is a parabola.

(b) If $D > 0$, then the conic is an ellipse.

(c) If $D < 0$, then the conic represents a hyperbola. Moreover, if $P = 0$ also holds, then is represents rectangular hyperbola.

3.2 Pair of straight lines

(1) A second degree homogeneous equation of the form, $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines, through the origin. If we consider the equation as a quadratic equation of x (given $a \neq 0$), then,

$$x = \frac{-2hy \pm \sqrt{4h^2y^2 - 4aby^2}}{2a}$$

$$\Rightarrow \; x = \frac{-h \pm \sqrt{h^2 - ab}}{a} y$$

$$\Rightarrow \; ax + (h \pm \sqrt{h^2 - ab})y = 0$$

This will represent,

(i) Two real and distinct lines through the origin if $h^2 - ab > 0$.

(ii) Two coincident lines through the origin if $h^2 - ab = 0$

(iii) Two imaginary lines if $h^2 - ab < 0$

The equations of the pair of lines are,

$ax + (h + \sqrt{h^2 - ab})y = 0$ and $ax + (h - \sqrt{h^2 - ab})y = 0$

This are of the form $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$

(2) Angle between the lines $ax^2 + 2hxy + by^2 = 0$:

Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$. Then comparing co-efficients we get,

$l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$

Let θ be the angle between $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Then,

$$\tan^2\theta = \frac{\left(\frac{l_1}{m_1} - \frac{l_2}{m_2}\right)^2}{\left(1 + \frac{l_1l_2}{m_1m_2}\right)^2} = \frac{(l_1m_2 - l_2m_1)^2 / m_1^2m_2^2}{(l_1l_2 + m_1m_2)^2 / m_1^2m_2^2}$$

$$= \frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{(l_1l_2 + m_1m_2)^2} = \frac{4h^2 - 4ab}{(a+b)^2}$$

$$\Rightarrow \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\Rightarrow \theta = \tan^{-1}\left(\pm\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

Note : The lines will be mutually perpendicular if $\theta = \pi/2$, i.e., $a + b = 0$. Again these lines will coincide if $\theta = 0$, i.e., $\sqrt{h^2 - ab} = 0$

(3) Bisectors of the angles between the lines $ax^2 + 2nxy + by^2 = 0$:

Let $ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$. Then,

$l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

$\therefore$ The equations of the bisecting lines are,

$$\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} = \pm\frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}}$$

$$\Rightarrow \left(\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} + \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}}\right)\left(\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} - \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}}\right) = 0$$

$$\Rightarrow \frac{(l_1x + m_1y)^2}{l_1^2 + m_1^2} - \frac{(l_2x + m_2y)^2}{l_2^2 + m_2^2} = 0$$

$$\Rightarrow (l_2^2 + m_2^2)(l_1x + m_1y)^2 - (l_1^2 + m_1^2)(l_2x + m_2y)^2 = 0$$

$$\Rightarrow (l_1^2m_2^2 - l_2^2m_1^2)(x^2 - y^2) = 2\,(l_1m_2 - l_2m_1)(l_1l_2 - m_1m_2)xy$$

$$\Rightarrow 2h(x^2 - y^2) = 2(a - b)xy$$

$$\Rightarrow \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

Theorem : The necessary and sufficient conditions for the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of real straight lines are.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ and } h^2 - ab \geq 0$$

Proof : (a) Conditions are necessary :

If the equation represents a pair of real straight lines, then we can assume,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$.

Comparing co-efficients, we have,

$l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$, $l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$, $n_1n_2 = c$.

Conditions are obtained by eliminating l_1, m_1, n_1, l_2, m_2, n_2 from the above relations. For elimination we consider the following product,

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 1 \\ m_2 & m_1 & 1 \\ n_2 & n_1 & 1 \end{vmatrix} = \begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_1m_2 + l_2m_1 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ l_1n_2 + l_2n_1 & m_1n_2 + m_2n_1 & 2n_1n_2 \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\because \begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Again, $4(h^2 - ab) = (l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2 = (l_1m_2 - l_2m_1)^2 \geq 0$

$\therefore\ h^2 - ab \geq 0$

(b) Conditions are sufficient :

Let $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 - ab \geq 0$ hold.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \Rightarrow (fh - bg)^2 = (h^2 - ab)(f^2 - bc)$$

Now, $ax^2 + 2hxy + by^2 + 2gx + 2fx + c$

$=$

$$= \frac{1}{b}[(by + hx + f)^2 + abx^2 + 2agx + bc - (hx + f)^2]$$

$$= \frac{1}{b}[(by + hx + f)^2 - \{(h^2 - ab)x^2 + 2(fh - bg)x + (f^2 - bc)\}]$$

$$= \frac{1}{b}[(by + hx + f)^2 - \{(h^2 - ab)x \pm 2\sqrt{h^2 - ab}.\sqrt{f^2 - bc}\,.x + (f^2 - bc)\}]$$

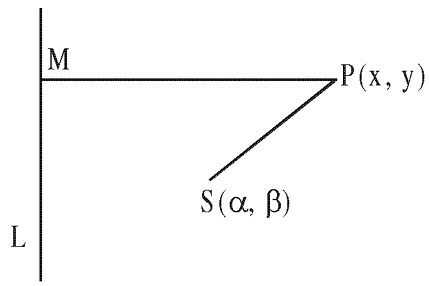
$$= \frac{1}{b}[(by + hx + f)^2 - (\sqrt{h^2 - ab}\,x \pm \sqrt{f^2 - bc})^2]$$

$$= \frac{1}{b}[\{by + (h + \sqrt{h^2 - ab})x + f \pm \sqrt{f^2 - bc}\} \times \{by + (h - \sqrt{h^2 - ab})x + f \mp \sqrt{f^2 - bc}\}]$$

Thus the equation represents two real straight lines when the conditions hold.

3.3 Conditions for proper conics

Let LM be the directrix whose equation is given by $lx + my + n = 0$, $S(\alpha, \beta)$ be the focus and $P(x, y)$ be any point on the conic whose eccentricity is e. If PM is perpenducular to the directrix then by definition of conic,



$$SP^2 = e^2.PM^2$$

$$\Rightarrow (x - \alpha)^2 + (y - \beta)^2 = e^2.\frac{(lx + my + n)^2}{l^2 + m^2}$$

$$\Rightarrow \{l^2(1 - e^2) + m^2\}x^2 - 2lme^2xy + \{l^2 + m^2(1 - c^2)\}y^2 - 2\{(l^2 + m^2)\alpha + lne^2\}x - 2\{(l^2 + m^2)\beta + mne^2\}y + (l^2 + m^2)(\alpha^2 + \beta^2) - n^2e^2 = 0$$

This equation represents equation (1) so,

$a = l^2(1 - e^2) + m^2$, $h = -lme^2$, $b = l^2 + m^2(1 - e^2)$,

$g = -\{(l^2 + m^2)\alpha + lne^2\}$, $f = -\{(l^2 + m^2)\beta + mne^2\}$, $c = (l^2 + m^2)(\alpha^2 + \beta^2) - n^2e^2$.

Now, $D = ab - h^2 = \{l^2(1 - e^2) + m^2\}\{l^2 + m^2(1 - e^2\} - l^2m^2e^4 = (l^2 + m^2)^2(1 - e^2)$

For parabola, $e = 1$, i.e., $D = 0$,

For ellipse, $e < 1$, i.e., $D > 0$,

For hyperbola, $e > 1$ i.e., $D < 0$.

3.4 Canonical form

We can reduce the general equation of second degree to standard equation of conic by suitable transformation of coordinates. This standard equation is also known as canonical equation or normal canonical form of the equation.

To find canonical equation from general second degree equation we take following transformations successively :

(1) The term xy is removed by taking suitable rotation.

(2) One or both (when possible) the terms of x and y are removed by translation.

(3) The constant is removed if possible.

3.4.1 Reduction to canonical form (Tracing of conics)

Let the general equation of 2nd degree is given by,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

Now if we give a anti-clockwise rotation θ. Then the co-ordinates will transform according to,

$$x = x'\cos\theta - y'\sin\theta,\ y = x'\sin\theta + y'\cos\theta$$

Substituting these in equation (1), we get,

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2 + 2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta) + c = 0$$

$$\Rightarrow (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)x'^2 + 2\{h(\cos^2\theta - \sin^2\theta) - (a - b)\sin\theta\cos\theta\}x'y' + (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)y'^2 + 2(g\cos\theta + f\sin\theta)x' + 2(f\cos\theta - g\sin\theta)y' + c = 0 \qquad (2)$$

We choose θ in such a way that the term x′y′ will vanish. So, we choose θ as,

$$h(\cos^2\theta - \sin^2\theta) - (a - b)\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad h\cos 2\theta - \frac{a-b}{2}\sin 2\theta = 0$$

$$\Rightarrow \quad \tan 2\theta = \frac{2h}{a-b}$$

$$\Rightarrow \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

After taking the rotation of $\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$, the equation will be of the form,

$$Ax'^2 + By'^2 + 2Gx' + 2Fy' + c = 0 \qquad (3)$$

By the property of invariants, $\Delta = \begin{vmatrix} A & 0 & G \\ O & B & F \\ G & F & C \end{vmatrix}$, D = AB and P = A + B

(a) If $\Delta \neq 0$ but D = 0, then the equation (1) represents a parabola :

There are 3 possibilities, (i) A = 0, B = 0,

(ii) A = 0, B $\neq$ 0,

(iii) A $\neq$ 0, B = 0.

If A = 0 and B = 0, then $\Delta = 0$. So, (i) is not possible

(ii) For A = 0, B $\neq$ 0 :

Clearly $\Delta \neq 0$ is possible if G $\neq$ 0. In this case the equation (3) reduces to,

$By'^2 + 2Gx' + 2Fy' + C = 0$ (4)

This is the equation of parabola, whose axis is parallel to new x-axis

From (4) we have,

$$B\left(y'^2 + \frac{2F}{B}y'\right) = -2Gx' - C$$

$$\Rightarrow \left(y' + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left(x' + \frac{BC - F^2}{2GB}\right)$$

Clearly the vertex of it is $\left(-\frac{BC - F^2}{2GB}, -\frac{F}{B}\right)$. Changing origin to that point we have,

$$y''^2 = -\frac{2G}{B}x''.$$

This is the canonical form of (1) when $\Delta \neq 0$, D = 0, and A = 0, B $\neq$ 0.

(iii) For A $\neq$ 0, B = 0

Similarly we can show that the required canonical form is, $x''^2 = -\frac{2F}{A}y''$.

(b) If $\Delta \neq 0$, D > 0, the equation (1) represents an ellipse :

We have D = AB. Since D > 0, then either both A and B are positive or both are negative. Without loss of generality we assume that both of them are positive. From (3) we can write,

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = k(\text{say})$$

By translation $x' = x'' - \frac{G}{A}$ and $y' = y'' - \frac{F}{B}$, the equation reduces to,

$$Ax''^2 + By''^2 = k$$

$$\Rightarrow \quad \frac{x''^2}{k/A} + \frac{y''^2}{k/B} = 1 \ (k \neq 0).$$

It is an equation of an ellipse in canonical form.

(c) When $\Delta \neq 0$, D < 0, the equation (1) represents a hyperbola :

If D < 0, then AB < 0. So, none of A and B is zero. Also if A > 0 then B < 0 and if A < 0 then B > 0. Without loss of generality let us assume A > 0, B < 0. Then we can write (3) as,

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C = K(\text{say}).$$

By translation $x' = x'' - \frac{G}{A}$ and $y' = y'' - \frac{F}{B}$, the above equation transforms to,

$$Ax''^2 + By''^2 = K$$

$$\Rightarrow \frac{x''^2}{K/A} + \frac{y''^2}{K/B} = 1. \ (K \neq 0)$$

If K > 0 we can express the above equation as, $\frac{x''^2}{\alpha^2} - \frac{y''^2}{\beta^2} = 1$

If K < 0, then this becomes, $\frac{y''^2}{\beta^2} - \frac{x''^2}{\alpha^2} = 1$

It is an equation of a hyperbola in canonical form. Again if $\alpha^2 = \beta^2$, i.e., A = B then this will represent a rectangular hyperbola.

3.5 Classification of second order curve

As rank of a second order curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

we mean rank of the matrix $\Delta = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

Since Δ is a 3×3 matrix the possible rank of Δ is ≤ 3. If rank $(\Delta) = 3$, then we call the curve non-singular or non-degenerate, and if rank $(\Delta) = 1$ or 2, then we call the curve singular or degenerate curve.

Δ	D	Cononical form	Name	Rank
$\Delta < 0$	$D > 0$	$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$	ellipse	3
$\Delta < 0$	$D > 0$	$x^2 + y^2 = \alpha^2$	circle	3
$\Delta > 0$	$D > 0$	$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=-1$	imaginary ellipse	3
$\Delta > 0$	$D < 0$	$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$	hyperbola	3
$\Delta < 0$	$D < 0$	$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=-1$	hyperbola	3
$\Delta \neq 0$	$D = 0$	$y^2 = 4\alpha x$ $x^2 = 4\beta y$	parabola	3
$\Delta = 0$	$D > 0$	$Ax^2 + By^2 = 0$	pair of imaginary straight lines or point ellipse	2
$\Delta = 0$	$D < 0$	$y^2 - k^2x^2 = 0$	pair of intersecting lines	2
$\Delta = 0$	$D = 0$	$y^2 = k^2$ $x^2 = l^2$	pair of parallel lines	2
$\Delta = 0$	$D = 0$	$x^2 = 0$ $y^2 = 0$	pair of coincident lines	1

3.6 Centre of the central conic

If any chord of a conic passing through a particular point is bisected by that point then the conic is said to be central and that particular point is called the centre of that conic.

Let (α, β) be the centre of the central conic whose equation is given by,

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \tag{1}$$

If we shift origin to (α, β) by translation, then the co-ordinates will transform according to, $x = x' + \alpha$, $y = y' + \beta$.

$$a(x' + \alpha)^2 + 2h(x' + \alpha)(y' + \beta) + b(y' + \beta)^2 + 2g(x' + \alpha) + 2f(y' + \beta) + c = 0$$

$$\Rightarrow ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y' + (a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c) = 0 \quad (2)$$

Since (α, β) is the centre, the co efficients of x' and y' in (2) must vanish.

$$\therefore\ a\alpha + h\beta + g = 0 \quad (3)$$

$$h\alpha + b\beta + f = 0 \quad (4)$$

from (3) and (4), we have, $\alpha = \dfrac{fh - bg}{ab - h^2}$, $\beta = \dfrac{gh - af}{ab - h^2}$.

So, centre of the conic is, $\left(\dfrac{fh - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

Again the constant term in (2) is,

$$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$$

$$= \alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c$$

$$= g\alpha + f\beta + c \text{ [using (3) and (4)]}$$

$$= \frac{g(fh - bg) + f(gh - af) + c(ab - h^2)}{ab - h^2}$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{D}$$

Thus the equation (2) can be written as, $ax'^2 + 2hx'y' + by'^2 + \dfrac{\Delta}{D} = 0$.

3.7 Worked out examples

Example 1 : Reduce the equation on $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ to the cononical form and determine the type of conic represented by it for different values of a.

Solution : Here,

$$\Delta = \begin{vmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & a \end{vmatrix} = 2\begin{vmatrix} 2 & 2 & -2 \\ 1 & 1 & -1 \\ -1 & -1 & a \end{vmatrix} = 0 \ [\because \text{ 1st and 2nd columns are indentical}]$$

$\therefore$ The equation represents a pair of straight lines. We can rewrite the equation as,

$(2x + y)^2 - 2(2x + y) + a = 0$

$\Rightarrow (2x + y - 1)^2 + a - 1 = 0$

$$\Rightarrow \left(\frac{2x+y-1}{\sqrt{5}}\right)^2 + \frac{a-1}{5} = 0$$

Putting $\frac{2x+y-1}{\sqrt{5}} = X$, the equation reduces to, $X^2 + \frac{a-1}{5} = 0$

It is the requried canonical form of the given equation.

(1) If $a < 1$, this represents two parallel lines.

(2) If $a = 1$, this represents two coincident lines.

(3) If $a > 1$, the given equation represents two imaginary lines.

Example 2 : Discuss the nature of the conic represented by $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$ and reduce it to its canonical form.

Solution : Here

$$\Delta = \begin{vmatrix} 9 & -12 & -9 \\ -12 & 16 & -\frac{101}{2} \\ -9 & -\frac{101}{2} & 19 \end{vmatrix} \neq 0, \ D = 9.16 - (-12)^2 = 0$$

$\therefore$ The given equation represents a parabola.

Let we rotate the axes anti-clockwise through an angle θ. Then the co-ordinates transform according to, $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$

And the equation will transform to,

$9(x'\cos\theta - y'\sin\theta)^2 - 24(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 16(x'\sin\theta + y'\cos\theta)^2 - 18(x'\cos\theta - y'\sin\theta) - 101(x'\sin\theta + y'\cos\theta) + 19 = 0$

$\Rightarrow (9\cos^2\theta - 24\sin\theta\cos\theta + 16\sin^2\theta)x'^2 - 2\{12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta\}$

$x'y' + (16\cos^2\theta + 24\sin\theta\cos\theta + 9\sin^2\theta)y'^2 - (18\cos\theta + 101\sin\theta)x'$

$+ (18\sin\theta - 101\cos\theta)y' + 19 = 0$

$\Rightarrow (3\cos\theta - 4\sin\theta)^2x'^2 - 2\{12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta\}x'y' + (4\cos\theta + 3\sin\theta)^2y'^2$

$- (18\cos\theta + 101\sin\theta)x' + (18\sin\theta - 101\cos\theta)y' + 19 = 0$

Let us choose θ in such a way that the term x′y′ vanishes

$\therefore$ $12(\cos^2\theta - \sin^2\theta) - 7\sin\theta\cos\theta = 0$

$\Rightarrow 12\tan^2\theta + 7\tan\theta - 12 = 0$

$\Rightarrow (4\tan\theta - 3)(3\tan\theta + 4) = 0$

$\Rightarrow \tan\theta = \frac{3}{4}, -\frac{4}{3}.$

Since θ is acute, $\tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

Substituting these values in the actual equation we get,

$$25y'^2 - 75x' - 70y' + 19 = 0$$

$$\Rightarrow \left(y' - \frac{7}{5}\right)^2 = 3\left(x' + \frac{2}{5}\right)$$

Changing the origin to $\left(-\frac{2}{5}, \frac{7}{5}\right)$, the equation further reduces to, $y''^2 = 3x''$. It is requied canonical form which represents a parabola.

Example 3 : Reduce the equation $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$ to cononical form.

Solution : $\Delta = \begin{vmatrix} 3 & 1 & -8 \\ 1 & 3 & 0 \\ -8 & 0 & 20 \end{vmatrix} = -32 \neq 0.\ D = 3.3 - 1 = 8 > 0$

$\therefore$ This equation represents an ellipse.

By rotating the axes anti-clockwise about origin at an angle θ, the equation transforms to,

$$3(x'\cos\theta - y'\sin\theta)^2 + 2(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 3(x'\sin\theta + y'\cos\theta)^2$$
$$- 16(x'\cos\theta - y'\sin\theta) + 20 = 0$$

$$\Rightarrow (3\cos^2\theta + 2\sin\theta\cos\theta + 3\sin^2\theta)x'^2 + 2(\cos^2\theta - \sin^2\theta)x'y'$$
$$+ (3\sin^2\theta - 2\sin\theta\cos\theta + 3\cos^2\theta)y'^2 - 16\cos\theta x' + 16\sin\theta y' + 20 = 0$$

$$\Rightarrow (3 + \sin2\theta)x'^2 + 2\cos2\theta x'y' + (3 - \sin2\theta)y'^2 - 16\cos\theta x' + 16\sin\theta y' + 20 = 0$$

We choose θ in such a way that the term x′y′ vanishes, so,

$$\cos2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

Putting $\theta = \dfrac{\pi}{4}$ in the above equation we get,

$$4x'^2 + 2y'^2 - 8\sqrt{2}x' + 8\sqrt{2}y' + 20 = 0$$

$$\Rightarrow 4(x' - \sqrt{2})^2 + 2(y' + 2\sqrt{2})^2 = 4$$

Changing origin to $(\sqrt{2}, -2\sqrt{2})$, the equation reduces to,

$4x''^2 + 2y''^2 = 4$

$$\Rightarrow x''^2 + \frac{y''^2}{2} = 1$$

It is rquired cononical form. The conic is an ellipse with semi axes 1 and $\sqrt{2}$.

Example 4 : Reduce the equation to its cononical form and state the nature of the conic.

$7x^2 - 48xy - 7y^2 - 20x + 140y + 300 = 0$

Solution : $\Delta = \begin{vmatrix} 7 & -24 & -10 \\ -24 & -7 & 70 \\ -10 & 70 & 300 \end{vmatrix} \neq 0$, $D = 7\cdot(-7) - (-24)^2 < 0$

$\therefore$ The given equation represents a hyperbola.

Let us take a rotation of axes about origin of an acute angle θ. Then after transformation, the equation becomes,

$$7(x'\cos\theta - y'\sin\theta)^2 - 48(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) - 7(x'\sin\theta + y'\cos\theta)^2$$
$$- 20(x'\cos\theta - y'\sin\theta) + 140(x'\sin\theta + y'\cos\theta) + 300 = 0$$

$$\Rightarrow 7(\cos^2\theta - 7\sin^2\theta - 48\sin\theta\cos\theta)x'^2 - \{48(\cos^2\theta - \sin^2\theta) + 28\sin\theta\cos\theta\}$$
$$x'y' - (7\cos^2\theta - 7\sin^2\theta - 48\sin\theta\cos\theta)y'^2 - 20(\cos\theta - 7\sin\theta)x' +$$
$$20(\sin\theta + 7\cos\theta)y' + 300 = 0$$

We choose θ in such a way that the term $x'y'$ vanishes,

$$\therefore \quad 48(\cos^2\theta - \sin^2\theta) + 28\sin\theta\cos\theta = 0$$

$$\Rightarrow \quad 12\tan^2\theta - 7\tan\theta - 12 = 0$$

$$\Rightarrow \quad \tan\theta = \frac{4}{3}, -\frac{3}{4}$$

Since θ is acute, $\tan\theta = \dfrac{4}{3} \Rightarrow \sin\theta = \dfrac{4}{5}$ and $\cos\theta = \dfrac{3}{5}$.

For these values of $\sin\theta$ and $\cos\theta$, the above equation becomes,

$$y'^2 - x'^2 + 4x' + 4y' + 12 = 0$$

$$\Rightarrow (x' - 2)^2 - (y' + 2)^2 = 12$$

By taking the translation $x' = x'' + 2$, $y' = y'' - 2$, the equation reduces to,

$$x''^2 - y''^2 = 12$$

$$\Rightarrow \frac{x''^2}{12} - \frac{y''^2}{12} = 1$$

which is the required canonical form of the hyperbola.

Example 5 : Determine the values of h and g so that the equation $x^2 - 2hxy + 4y^2 + 2gx - 12y + 9 = 0$ may represent :

(i) a conic with no centre.

(ii) a conic having infinitely many centres.

In the last case find the type of the conic.

Solution : The centre of the conic lies on

$$x - hy + g = 0 \text{ and } -hx + 4y - 6 = 0$$

By cross multiplication, $\dfrac{x}{6h - 4g} = \dfrac{y}{-gh + 6} = \dfrac{1}{4 - h^2}$

(i) The given equation will represent a conic with no centre if,

$$4 - h^2 = 0,\ 6h - 4g \neq 0,\ -gh + 6 \neq 0$$

$$\Rightarrow h = \pm 2,\ g \neq \pm 3.$$

(ii) The given equation will represent a conic with infinitely many centres if,

$$4 - h^2 = 0,\ 6h - 4g = 0,\ -gh + 6 = 0$$

$$\Rightarrow h = \pm 2,\ g = \pm 3$$

For this values of h and g, the equation will be,

$$x^2 \mp 4xy + 4y^2 \pm 6x - 12y + 9 = 0$$

$$\Rightarrow (x \mp 2y \pm 3)^2 = 0$$

$\therefore$ The equation represents a pair of coincident straight lines.

3.8 Summary

In this unit, we have learnt to identify few important parameters (Δ, D and P) from a general equation of second degree in x, y. These parameters help us to determine the conic. Also we have learnt to reduce these general equations to their canonical forms using suitable orthogonal transformations.

3.9 Exercises

1. Reduce the followings to the conical forms and state the types of the conics :
   (a) $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$
   (b) $8x^2 - 12xy + 17y^2 + 16x - 12y + 3 = 0$
   (c) $4x^2 - 3xy - 18 = 0$
   (d) $5x^2 - 20xy - 5y^2 - 16x + 8y - 7 = 0$
2. Determine whether the followings have a single centre or infinitely many centres or no centre.
   (a) $12x^2 + 4y^2 + 14xy - 2x - 3y + 7 = 0$
   (b) $4x^2 - 4xy + y^2 - 12x - 10y - 19 = 0$
   (c) $9x^2 - 6xy + y^2 + 24x - 8y + 16 = 0$
3. Reduce the equation $x^2 - 4xy + 4y^2 + 2x - 4y + c = 0$ to its canonical form and determine the type of conic represented by it for different values of c.
   [Hint : Proceed similarly like example - 1]
4. Find values of a and g for which the curve $ax^2 + 8xy + 4y^2 + 2gx + 4y + 1 = 0$ represents
   (a) a conic having no centre.
   (b) a conic having infinitely many centres.
   [Hint : Similar to example- 5]

Answers :

1. (a) $x''^2 - y''^2 = 0$, pair of intersecting lines.
   (b) $x''^2 + y''^2 = 1$, ellipse.
   (c) $x'^2 - 9y'^2 = -36$ or $9x'^2 - y'^2 = 36$, hyperbola.
   (d) $x''^2 - y''^2 = -\dfrac{351}{125\sqrt{5}}$, rectangular hyperbola.
2. (a) Single centre $\left(\dfrac{13}{2}, -11\right)$.
   (b) no centre.
   (c) infinitely many centres.
3. $y''^2 + \dfrac{c-1}{5} = 0$, parallel, coincident or imaginary lines according as c <, = or > 1.
4. (a) $a = 4,\ g \neq 2$.
   (b) $a = 4,\ g = 2$.

Unit 4 ❐ Tangent, Normal, Pole, Polar, Conjugate diameters

Structure

4.0 Objectives

We will be able to determine

- equations and properties of tangent and normal
- pair of tangents
- director circle
- chord of contact
- Pole and polar

of a given conic.

4.1 Tangent

If a line meets a curve at coincident point then the line is called the tangent to the curve at the meeting point and the point is called the point of contact.

Let, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

be the equation of the conic and (x_1, y_1) be a point in the plane of the conic. The equation of the line through (x_1, y_1) can be written as,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \text{ (say)} \qquad (2)$$

where l and m are the cosines of the angles made by the line with x-axis and y-axis respectively and r is the distance between (x, y) and (x_1, y_1) on the line.

From (2), $x = x_1 + lr$, $y = y_1 + mr$.

To find the point of intersection between (1) and (2), we have,

$$a(lr + x_1)^2 + 2h(lr + x_1)(mr + y_1) + b(mr + y_1)^2 + 2g(lr + x_1) + 2f(mr + y_1) + c = 0$$

$$\Rightarrow (al^2 + 2hlm + bm^2)r^2 + 2\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}r + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad (3)$$

Clearly this is a quadratic equation in r. Let the roots be r_1 and r_2, where r_1 and r_2 are the distance of the two points of intersection between (1) and (2) from (x_1, y_1). If (x_1, y_1) is on the conic and the line (2) is the tangent then, $r_1 = 0 = r_2$ and in this case,

$$r_1r_2 = 0 \text{ and } r_1 + r_2 = 0$$

$\therefore$ By (3), we have, the coefficients of r and the constant term are both 0.

$\therefore$ $(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0$ (4)

and, $ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$ (5)

Eliminating l and m from (2) and (4), we get,

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0$$

$$\Rightarrow axx_1 + h(xy_1 + yx_1) + byy_1 + gx + fy = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$$

$$= - gx_1 - fy_1 - c \text{ [By (5)]}$$

$$\Rightarrow axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

This is the standard equation of the tangent to the conic (1) at a point (x_1, y_1).

Note : To find the equation of a tangent at (x_1, y_1) to any conic, we simply replace x^2 by xx_1, y^2 by yy_1, 2x by $(x + x_1)$, 2y by $(y + y_1)$ and 2xy by $(xy_1 + x_1y)$.

4.1.1 Tangents of some standard equations

	Standard equation	Equation of tangent at (x_1, y_1)
1.	Circle : $x^2 + y^2 + 2gx + 2fy + c = 0$	$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
2.	Circle : $x^2 + y^2 = a^2$	$xx_1 + yy_1 = a^2$
3.	Parabola : $y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
4.	Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
5.	Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
6.	Rectangular hyperbola : $xy = k^2$	$xy_1 + x_1y = 2k^2$

4.1.2 Condition for a line to be tangent of a conic

Let the equations of the conic and a straight line are,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

$$lx + my + n = 0 \qquad (2)$$

From (2) we have, $y = -\frac{lx + n}{m}$

Putting the value of y in (1) we have,

$$x^2(am^2 - 2hlm + bl^2) - 2x(hmn - bln - gm^2 + flm) + bn^2 - 2fmn + cm^2 = 0$$

If the line given by the equation (2) is tangent to the conic (1), then the roots of the above equation must be equal. So,

$$(hmn - bln - gm^2 + flm)^2 - (am^2 - 2hlm + bl^2)(bn^2 - 2fmn + cm^2) = 0$$

$$\Rightarrow (bc - f^2)l^2 + (ca - g^2)m^2 + (ab - h^2)n^2 + 2(gh - af)mn + 2(hf - bg)nl + 2(fg - ch)lm = 0$$

This is the required condition.

Some conditions on some standard conics :

Here we shall deduce some conditions for the line $lx + my + n = 0$ to be tangent on some standard conics,

(1) Circle : $x^2 + y^2 = a^2$

Condition : $a^2(l^2 + m^2) - n^2 = 0$ i.e., $n = \pm a\sqrt{l^2 + m^2}$

(2) Parabola : $y^2 = 4ax$

Condition : $am^2 - ln = 0$ i.e., $n = \frac{am^2}{l}$

(3) Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Condition : $a^2l^2 + b^2m^2 - n^2 = 0$ i.e., $n = \pm\sqrt{a^2l^2+b^2m^2}$

(4) Hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Condition : $a^2l^2 - b^2m^2 - n^2 = 0$ i.e., $n = \pm\sqrt{a^2l^2-b^2m^2}$

4.2 Normal

The normal to a curve is the straight line which is perpendicular to the tangent at the point of contact.

Let the equation of the conic is given by,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Now, if we want to find the equation of the normal to the curve at the point (x_1, y_1), then we need to find the perpendicular straight line passing through (x_1y_1) on the tangent passing through (x_1, y_1). We know the equation of the tangent at (x_1, y_1) is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$\Rightarrow (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0$$

Since the normal is perpendicular to the above equation and passing through (x_1, y_1), the required equation is,

$$y - y_1 = \frac{hx_1+by_1+f}{ax_1+hy_1+g}(x - x_1)$$

4.2.1 Normals of some standard conics

Equation of standard conic	Normal on that conic at (x_1, y_1)
1. Parabola : $y^2 = 4ax$	$(y - y_1) = -\frac{y_1}{2a}(x - x_1)$
2. Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$(y - y_1) = \frac{a^2}{b^2}.\frac{y_1}{x_1}(x - x_1)$
3. Circle : $x^2 + y^2 = a^2$	$(y - y_1) = \frac{y_1}{x_1}(x - x_1)$
4. Hyperbola : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$(y - y_1) = -\frac{a^2}{b^2}.\frac{y_1}{x_1}(x - x_1)$

4.3 Some worked out examples

Example 1 : Find the equation of the tangent and the normal at (1, –1) to the conic $y^2 - xy - 2x^2 - 5y + x - 6 = 0$

Solution : The point (1, –1) lies on the conic, since, $(-1)^2 - 1(-1) - 2.1^2 - 5(-1) + 1 - 6 = 0$

The equation of the tangent at (1, –1) is

$$y(-1) - \frac{1}{2}(x(-1) + y.1) - 2x.1 - \frac{5}{2}(y - 1) + \frac{1}{2}(x + 1) - 6 = 0$$

$\Rightarrow x + 4y + 3 = 0$

Since the normal is perpendicular to $x + 4y + 3 = 0$ and passes through (1, –1), the required equation is, $y + 1 = 4(x - 1) \Rightarrow 4x - y - 5 = 0$

Example 2 : Find the equation of the tangents to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$ which are parallel to $x + 4y = 0$

Solution : Since the required tangent is parallel to the line $x + 4y = 0$, the required tangent is of the form,

$x + 4y + c = 0 \Rightarrow x = -(4y + c)$

Putting the value of x in the equation of the conic, we get,

$(4y + c)^2 - 4y(4y + c) + 3y^2 + 5(4y + c) - 6y + 3 = 0$

$\Rightarrow 3y^2 + 2(2c + 7)y + c^2 + 5c + 3 = 0$

Since the line is tangent to the conic, the above equation must have equal roots so,

$4(2c + 7)^2 - 4.3(c^2 + 5c + 3) = 0$

$\Rightarrow c^2 + 13c + 40 = 0$

$\Rightarrow (c + 5)(c + 8) = 0$

$\Rightarrow c = -5, -8$

$\therefore$ The required tangents are $x + 4y - 5 = 0$ and $x + 4y - 8 = 0$

Example 3 : Show that three normals can be drawn to a parabola from a given point and the sum of the ordinates of the feet of the normal is zero.

Solution : $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$. The equation of the normal at this point is,

$$y - 2at = -\frac{2at}{2a}(x - at^2)$$

$\Rightarrow y + tx = 2at + at^3$

If the normal passes through a fixed point (h, k) then,

$k + th = 2at + at^3$

$\Rightarrow at^3 + (2a - h)t - k = 0$

Since this is a cubic equation in t, it has 3 roots. Corresponding to each value of t, we get 3 normals. Passing through (h, k).

Let the roots of the equation are t_1, t_2, t_3 and the ordinates of the feet of the normals be y_1, y_2, y_3.

Then, $y_1 + y_2 + y_3 = 2a(t_1 + t_2 + t_3) = 0$ [$\because$ the coefficient of t^2 in the equation is 0].

Example 4 : Show that 4 normals can be drawn to an ellipse through a given point and the feet of the normals lie on a rectangular hyperbola.

Solution : We know the equation of the normal on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from (x_1, y_1) is,

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

If it passes through a given point (h, k) then, $\frac{h-x_1}{x_1/a^2}=\frac{k-y_1}{y_1/b^2}=\lambda(\text{say})$

$$\Rightarrow \frac{x_1}{a}=\frac{ah}{a^2+\lambda},\ \frac{y_1}{b}=\frac{bk}{b^2+\lambda}$$

Since (x_1, y_1) is on the ellipse,

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \frac{a^2h^2}{(a^2+\lambda)^2}+\frac{b^2k^2}{(b^2+\lambda)^2}=1$$

This is a biquadratic equation in λ and it has 4 roots in λ. This 4 values corresponds to 4 points on the ellipse and normals through these points pass through the given point.

Again $a^2 + \lambda = \frac{a^2h}{x_1}$, $b^2 + \lambda = \frac{b^2k}{y_1}$

$$\Rightarrow a^2 - b^2 = \frac{a^2h}{x_1}-\frac{b^2k}{y_1}$$

The point (x_1, y_1) is the foot of a normal. Hence the locus of the feet of the normals is,

$$a^2 - b^2 = \frac{a^2h}{x}-\frac{b^2k}{y}$$

$\Rightarrow (a^2 - b^2)xy = a^2hy - b^2kx$

Which is a rectangular hyperbola.

4.4 Pair of tangents : Director circle

A pair of tangents can be drawn to conic from a point not lying on the conic.

Let the equation of a conic be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

and (x_1, y_1) be a point not lying on the conic. Let a line through a point (x_1, y_1) touches the conic at (x_2, y_2). The equation of the tangent at (x_2, y_2) is,

$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0$

As it passes through (x_1, y_1), we have,

$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$ (2)

Again since (x_2, y_2) lies on the conic, so,

$ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0$ (3)

From (2) and (3), generally we can obtain 2 values of x_2 and y_2. Thus there will be two points of contact of tangents from (x_1, y_1) but they need not be real in all cases.

4.4.1 Equation of pair of tangents drawn from (x_1, y_1)

The equation of a line through (x_1, y_1) is,

$= r(\text{say})$ (4)

From (4), $x = lr + x_1$, $y = mr + y_1$. Putting these values in (1) we have,

$(al^2 + 2hlm + bm^2)r^2 + 2\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}r + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$ (5)

It is a quadratic equation in r. If the line (4) touches the conic (1), then the roots of (5) must be equal. So,

$\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}^2 = (al^2 + 2hlm + bm^2)$

$(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fx_1 + 2fy_1 + c)$

Eliminating l and m by (4),

$\{(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1)\}^2 = \{a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2\}(ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c)$ (6)

If we write,

$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$S_1 = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

and $T = axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$

Then the equation (6) can be written as,

$(T - S_1)^2 = (S + S_1 - 2T).S_1$

$\Rightarrow SS_1 = T^2.$

It is the required equation.

4.4.2 Director circle

If the locus of the points of intersection of pair of perpendicular tangents to a conic is a circle, then that circle is called the director circle of the conic.

4.4.2.1 Equations of directior circles of some standard conics

The equation of the pair of tangents from (x_1, y_1) to the circle is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$

If these lines are at right angles then, the coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow (x_1^2 + y_1^2 - a^2 - x_1^2) + (x_1^2 + y_1^2 - a^2 - y_1^2) = 0$

$\Rightarrow x_1^2 + y_1^2 = 2a^2$

Hence the locus of (x_1, y_1) i.e., equation of the director circle is $x^2 + y^2 = 2a^2$

Similarly one can check the director circle of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$ and the director circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.

But in case of parabola $y^2 = 4ax$, the locus of points of intersection of a pair of perpendicular tangents is $x + a = 0$, which is not a circle, but directrix of the parabola.

4.4.3 Chord of contact

If we draw two tangents to a conic from a point lying outside of the conic, then the cord joining two points of contact is called chord of contact.

Let, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be a conic and (x_1, y_1) be a point not lying on the conic, then we can draw two tangents on the conic from (x_1, y_1). Let (x_2, y_2) and (x_3, y_3) be the points of contact of those tangents.

The equations of the tangents through (x_2, y_2) and (x_3, y_3) are,

$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$

$axx_3 + h(xy_3 + x_3y) + byy_3 + g(x + x_3) + f(y + y_3) + c = 0.$

Since these lines passes through (x_1, y_1), we have,

$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$

$ax_1x_3 + h(x_1y_3 + x_3y_1) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0.$

These two conditions suggest that, the line,

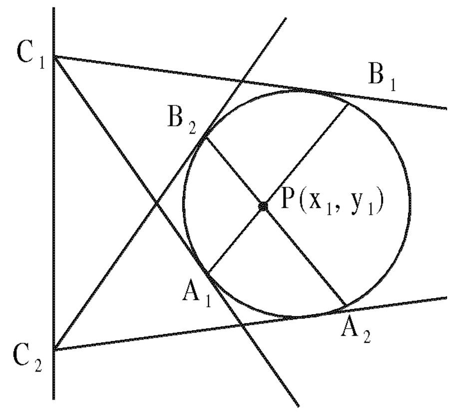
$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

passes through (x_2, y_2) and (x_3, y_3). Hence this is the equation of chord of contact of tangents through (x_1, y_1).

Note : This equation is identical with the equation of tangent passes through (x_1, y_1). But the point (x_1, y_1) lies outside of the conic.

4.5 Pole and polar

The polar of a point with respect to a conic is the locus of the points of intersection of tangents at the extremities of the chords through that point while the point itself is called the pole of its polar.



In this picture w.r.t. the circle P is the pole and C_1C_2 line is the polar.

Let a conic be given by, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and (x_1, y_1) be the pole. Let the tangent at the extremities of a chord through (x_1, y_1) meet at (x_2, y_2). According to definition of the polar, (x_2, y_2) lies on the polar of (x_1, y_1).

The chord of contact of (x_2, y_2) is,

$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$

As (x_1, y_1) lies on it, $ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$

This shows that, the locus of (x_2, y_2) is the line, $ax_1x + h(x_1y + y_1x) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

It is the required polar of (x_1, y_1).

Note : The polar of a point w.r.t., a conic coincides with the chord of contact of tangents from the point to the conic when the point does not lie on the conic.

4.5.1 Properties of pole and polar

1. If the polar of P passes through Q then the polar of Q will pass through P. These two points are called conjugate points.

Let a conic given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

Let the polar of $P(x_1, y_1)$ pass through $Q(x_2, y_2)$.

The polar of $P(x_1, y_1)$ is,

$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (2)

As it passes through $Q(x_2, y_2)$,

$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \quad (3)$$

Again the polar of $Q(x_2, y_2)$ is,

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad (4)$$

(3) implies that $P(x_1, y_1)$ lies on (4), which is polar of $Q(x_2, y_2)$. Hence proved.

2. If the pole of a line L w.r.t. a conic lies on another line L′, then the pole of L′ w.r.t. the conic lies on L. These two lines are called conjugate lines.

Let the pole of L_1 w.r.t., the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be (x_1, y_1) and that of L_2 be (x_2, y_2). then the equation of L_1 is,

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \quad (1)$$

The equation of L_2 is,

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad (2)$$

Since (x_1, y_1) lies on (2), so, $ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

This is the condition for (x_2, y_2) to lie on L_1. Hence proved.

3. If two lines L_1 and L_2 meet at C then the polar of C w.r.t., that conic passes through the poles of L_1 and L_2 w.r.t. that conic.

Let P and Q be poles of L_1 and L_2 w.r.t., a conic. Then the polars L_1 of P and L_2 of Q pass through C. By property-1, we can say that the polar of C passes through P and Q. Hence proved.

Note : If there points A, B, C are such that any two of them are conjugate points, then the triangle is called a self-conjugate (or self-polar) triangle. Since the polar of A passes through B and C, BC is a polar of A. Therefore each side of the triangle is the polar w.r.t., opposite vertex. Also any two lines are conjugate lines.

4.5.2 Pole of a given line w.r.t., a conic

Let the equation of the conic be,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

(x_1y_1) be pole of the line $lx + my + n = 0$. w.r.t. the conic. Then the equation of the polar of (x_1, y_1) w.r.t., the conic is,

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

$$\Rightarrow (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0$$

Comparing with the given line $lx + my + n = 0$ we get,

$$\frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n}.$$

By solving this, we can find the values of (x_1, y_1).

4.6 Illustrated examples

Example 5 : Show that the tangents at the extremities of a focal chord of a parabola meet at right angle on the diretcrix.

Solution : Let $y^2 = 4ax$ be the equation of the parabola and the ends of a focal chord be $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

The tangents at these points are

$yt = x + at^2$ and $-\frac{y}{t} = x + \frac{a}{t^2} \Rightarrow -ty = xt^2 + a$.

Clearly these two are at right angle. After adding them we get,

$(x + a)(1 + t^2) = 0 \Rightarrow x + a = 0$

$\therefore$ The tangents meet on the directrix $x + a = 0$ at right angle

Example 6 : Find the equation of the common tangents to the circle $x^2 + y^2 = 4ax$ and the parabola $y^2 = 4ax$.

Solution : Let the equation of the common tangent by $y = mx + c$.

It is tangent to $y^2 = 4ax$ then, $c = a/m$

$$y = mx + \frac{a}{m} \qquad (1)$$

Now let us find the point of intersection between (1) and the circle $x^2 + y^2 = 4ax$.

$$x^2 + \left(mx + \frac{a}{m}\right)^2 = 4ax$$

$$\Rightarrow (1 + m^2)x^2 - 2ax + \frac{a^2}{m^2} = 0$$

If the line (1) is tangent then the above equation must have equal roots.

$$\therefore\ 4a^2 - 4(1 + m^2).\frac{a^2}{m^2} = 0$$

$$\Rightarrow \frac{a^2}{m^2} = 0$$

Here $a \neq 0$, $\therefore$ $1/m = 0$

$\therefore$ From (1) we have, $\frac{y}{m} = x + \frac{a}{m^2} \Rightarrow x = 0$

$\therefore$ $x = 0$ is the common tangent.

Example 7 : Find the pole of the focal chord of the parabola $y^2 = 4ax$, passing through (9a, 6a).

Solution : The equation of the chord passing through the focus (a, 0) and (9a, 6a) is,

$$y = \frac{6a}{9a - a}(x - a)$$

$\Rightarrow 3x - 4y - 3a = 0$

Let (α, β) be the pole of it. The equation of polar of (α, β) is,

$$y\beta = 2a(x + \alpha)$$

$\Rightarrow 2ax - \beta y + 2a\alpha = 0$

This must be identical with $3x - 4y - 3a = 0$. So,

$$\frac{2a}{3} = \frac{\beta}{4} = \frac{2a\alpha}{-3a}$$

$$\Rightarrow \alpha = -a, \beta = \frac{8a}{3}$$

$\therefore$ The pole is $\left(-a, \frac{8a}{3}\right)$.

Example 8 : Show that the locus of the poles of tangents to the parabola $y^2 = 4ax$ w.r.t. the parabola $y^2 = 4bx$ is the parabola $y^2 = \frac{4b^2}{a}x$.

Solution : Let (α, β) be the pole. The polar of it w.r.t. $y^2 = 4bx$ is

$$y\beta = 2b(x + \alpha)$$

$$\Rightarrow y = \frac{2b}{\beta}x + \frac{2b\alpha}{\beta}$$

If it is tangent to the parabola $y^2 = 4ax$, then,

$$\frac{2b\alpha}{\beta} = \frac{a}{\frac{2b}{\beta}} \Rightarrow \beta^2 = \frac{4b^2}{a}\alpha.$$

Hence the required locus is $y^2 = \frac{4b^2}{a}x$.

Example 9 : Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ cut one another at an angle $\tan^{-1}\frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}$.

Solution : Let (x_1, y_1) be the point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$. Tangents at this point are,

$yy_1 = 2a(x + x_1)$ (1)

$xx_1 = 2b(y + y_1)$ (2)

Let θ be the angle between (1) and (2). Then,

$$\tan\theta = \frac{4ab - x_1y_1}{-2(ax_1 + by_1)} \quad (3)$$

Again we have $y_1^2 = 4ax_1$ and $x_1^2 = 4by_1$. Combining these two we have,

$y_1^4 = 64a^2by_1$

$\Rightarrow y_1 = 0$ or $y_1 = 4a^{2/3}\,b^{1/3}$

Similarly, $x_1 = 0$ or $x_1 = 4a^{1/3}\,b^{2/3}$

Putting $x_1 = 4a^{1/3}b^{2/3}$ and $y_1 = 4a^{2/3}\,b^{1/3}$ in (3),

$$\tan\theta = \frac{4ab - 16ab}{-8(a^{4/3}b^{2/3} + a^{2/3}b^{4/3})}$$

$$= \frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}$$

$$\therefore\ \theta = \tan^{-1}\frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}$$

For $x_1 = 0$, $y_1 = 0$, $\tan\theta = \infty \Rightarrow \theta = \pi/2$. So, the parabolas are at right angle at the origin.

Example 10 : Prove that the locus of the poles of the normal chords of the parabola $y^2 = 4ax$ is the curve $y^2(x + 2a) + 4a^3 = 0$.

Solution : The equation of the normal to the parabola at $(at^2, 2at)$ is,

$y + tx = 2at + at^3.$ (1)

Let the pole of the straight line (1) w.r.t. the parabola, $y^2 = 4ax$ be, (α, β). Then the polar of (α, β) w.r.t. the parabola is,

$\beta y = 2a(x + \alpha).$ (2)

Since (1) and (2) are identical, we must have,

$$\frac{1}{\beta} = \frac{t}{-2a} = \frac{2at + at^3}{2a\alpha}$$

$$\Rightarrow\ t = -\frac{2a}{\beta},\ 2at + at^3 = \frac{2a\alpha}{\beta}$$

Eliminating t, we have,

$$2a\left(-\frac{2a}{\beta}\right)+a\left(-\frac{2a}{\beta}\right)^3=\frac{2a\alpha}{\beta}$$

$\Rightarrow \beta^2(\alpha + 2a) + 4a^3 = 0$

$\therefore$ The locus of (α, β) is, $y^2(x + 2a) + 4a^3 = 0$.

Example 11 : Chords of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touch the circle $x^2 + y^2 = c^2$. Find the locus of their poles.

Solution : Let the pole w.r.t. one chord be (α, β). Its polar w.r.t. the ellipse is,

$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=1$$

Since it touches the circle, it is tangent to the circle. So, the normal distance of it from the origin (0, 0) is the radius.

$$\therefore \frac{1}{\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}}}=\pm c$$

$$\Rightarrow \frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}=\frac{1}{c^2}$$

$\therefore$ The required locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.

Example 12 : Show that the pole of any tangent to the hyperbola $xy = c^2$ w.r.t. the circle $x^2 + y^2 = a^2$ lies on concentric and similar hyperbola.

Solution : The equation of the tangent to $xy = c^2$ at $(ct, c/t)$ is,

$x + t^2y = 2ct$ (1)

Let (h, k) be the pole w.r.t. the circle. The equation of the polar of (h, k) w.r.t. the circle $x^2 + y^2 = a^2$ is

$xh + yk = a^2$ (2)

The equations (1) and (2) are identical so,

$$\frac{h}{1}=\frac{k}{t^2}=\frac{a^2}{2ct}$$

Eliminating t we have, $\frac{a^2}{2ch}=\frac{2ck}{a^2} \Rightarrow hk=\frac{a^4}{4c^2}$

Hence the locus of (h, k) is, $xy=\frac{a^4}{4c^2}$, which is concentric and similar to $xy = c^2$.

4.7 Summary

In this unit, we have learnt to find out equations of tangent, normal, pair of tangents. Also we are now able to determine the director circle, chord of contact, pole and polar as well as understand their various properties.

4.8 Exercises

1. Find the equations of tangents and normals to the following curves :
   (a) $x^2 + y^2 - 4x + 6y - 36 = 0$ at (2, 4)
   (b) $2x^2 + 5xy + 3y^2 + 4x - 10y - 4 = 0$ at (1, 1)
2. Find the points of contact and equations of tangents to
   (a) $x^2 + y^2 + 6x - 10y - 15 = 0$, which are parallel to x-axis.
   (b) $y^2 = 4ax$, which makes angle 60° with the x-axis.
3. Find the condition for which $lx + my + n = 0$ will be normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
4. Prove that if the straight line $ax + by + c = 0$ touches the parabola $y^2 - 4px + 4pq = 0$, then $a^2q + ac - pb^2 = 0$.
5. Find the equations of the tangents at the ends of the latus recta of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that they pass through the intersection of the axis and directrices.
6. If the tangents and normals to a rectangular hyperbola cut off interceps a_1 and a_2 on one axis and b_1 and b_2 on the other then, $a_1a_2 + b_1b_2 = 0$.
7. Prove that the locus of the foot of the perpendicular from the focus to any tangent to $y^2 = 4ax$ is the tangent at the vertex.
8. Find the poles of the lines,
   (a) $5x - 4y - 14 = 0$ w.r.t., $x^2 + y^2 + 6x - 10y - 15 = 0$.
   (b) $x + y - 1 = 0$ w.r.t., $3x^2 + 4xy - 2y^2 - 5x + 7y - 10 = 0$.
9. Show that the tangent formed by the straight lines, $5x = y + 4$, $x + 3y = 4$ and $x + y = 8$ is self polar w.r.t. the circle $x^2 + y^2 = 8$.
10. Show that the locus of the poles of the tangents to the circle $x^2 + y^2 = a^2$ w.r.t. the circle $x^2 + y^2 = 2ax$ is a parabola.
11. The polar of the point P w.r.t. the circle $x^2 + y^2 = a^2$ touches the circle $(x - \alpha)^2 + (y - \beta)^2 = b^2$. Show that the locus of P is the curve given by, $(\alpha x + \beta y - a^2)^2 = b^2 (x^2 + y^2)$.
12. Find the common tangents of the circles $x^2 + y^2 + 4x + 2y - 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$.

13. Find the condition that the coordinate axes may be conjugate lines w.r.t. the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

    [Hint : consider the pole of $y = 0$ as (x_1, y_1). Then the polar is identical with $y = 0$. Find x_1 and y_1. Again it also satisfies $x = 0$]

14. The polars of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ w.r.t., $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the conic in Q_1, R_1 and Q_2, R_2 respectively. Show that the six points lie on the conic,

    $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right).$$

    [Hint : let S : $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, L_1 : $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$ and L_2 : $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1=0$. $S-\lambda L_1L_2 = 0$ passes through Q_1, R_1, Q_2 and R_2. If this passes through P_1 and P_2 then,

    $\lambda=\dfrac{1}{\dfrac{x_1x_2}{a^2}+\dfrac{y_1y_2}{b^2}-1}$. Hence the result follows.]

Answers :

1. (a) $y - 4 = 0$, $x - 2 = 0$

   (b) $13x + y - 14 = 0$, $x - 13y + 12 = 0$

2. (a) $(-3, 12)$, $(-3, -2)$, $y - 12 = 0$, $y + 2 = 0$

   (b) $\left(\frac{a}{3},\frac{2a}{\sqrt{3}}\right)$, $y=\sqrt{3}x+\frac{a}{\sqrt{3}}$

3. $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$

8. (a) $(2, 1)$ (b) $\left(-\frac{34}{7},\frac{19}{7}\right)$

12. $x = 1$, $y = 2$, $4x - 3y - 10 = 0$, $3x + 4y - 5 = 0$

13. $ch = fg$

Unit 5 ❐ Equation of a chord of a conic in terms of its middle point

Structure

5.0 Objectives

We will be able to determine

- The chords of standard conics, when the middle point is given.
- diameter and conjugate diameters of standard conics and appreciate their various properties.

5.1 Introduction

Let (x_1, y_1) be the middle point of a chord of the conic S : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let the equation of the chord be,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=r(\text{say}) \qquad (1)$$

where l and m are the cosines of the angles made by the line with x-axis and y-axis respectively.

Then, $x = x_1 + lr$, $y = y_1 + mr$. Putting these values in the equation of the conic we get,

$(al^2 + 2hlm + bm^2)r^2 + 2\{ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}r + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$

It is quadratic equation in r. The roots are the distances of the points of intersection between the line (1) and the conic from (x_1, y_1). Since (x_1, y_1) is the mid point of the chord, the two roots are equal in magnitude but opposite in sign. So, the sum of two roots are zero. So,

$(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0$

Eliminating l and m by the help of (1) we get,

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0$$

$$\Rightarrow \quad axx_1 + h(xy_1 + yx_1) + byy_1 + gx + fy = ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$$

Adding $gx_1 + fy_1 + c$ to both sides, we have,

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow \quad T = S_1$

where, $T := axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$S_1 := ax_1^2 + 2hx_1y_1 + by_1{}^2 + 2gx_1 + 2fy_1 + c$

5.2 Chords of some standard conics, whose middle point is (x_1, y_1)

(1) Parabola : $y^2 = 4ax$

Chord : $yy_1 - 2ax = y_1^2 - 2ax_1$

(2) Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Chord : $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$

(3) Hyperbola : $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Chord : $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2}$

(4) Rectangular hyperbola : $xy = c^2$

Chord : $xy_1 + yx_1 = 2x_1y_1$

5.3 Diameter and conjugate diameter

The locus of the middle points of a series of parallel chords of a conic is called a diameter of the conic.

Let us consider all the chords of the conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which are parallel to the line $\frac{x}{l} = \frac{y}{m}$.

Let (x_1, y_1) be the middle point of one of these chords. Then the equation of the chord is,

$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y = ax_1^2 + 2hx_1y_1 + by_1{}^2 + gx_1 + fy_1$

Since it is parallel to $\frac{x}{l} = \frac{y}{m}$, $\frac{ax_1 + hy_1 + g}{m} = -\frac{hx_1 + by_1 + f}{l}$

So, locus of the middle point is,

$$\frac{ax + hy + g}{m} + \frac{hx + by + f}{l} = 0$$

$\Rightarrow (al + hm)x + (hl + bm)y + gl + fm = 0$

This is the equation of the required diameter.

5.3.1 Some examples of diameters of some standard conics

(1) Parabola : $y^2 = 4ax$

The equation of the chord whose middle point is (x_1, y_1) is $yy_1 - 2ax = y_1^2 - 2ax_1$

If it is parallel to $y = mx$ then $\frac{2a}{y_1} = m \Rightarrow y_1 = \frac{2a}{m}$

$\therefore$ The equation of diameter corresponding to this system of chords is $y = \frac{2a}{m}$. It is straight line parallel to the axis of the parabola.

(2) For ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it will be $y = -\frac{b^2}{a^2m}x$

(3) For hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, it will be $y = \frac{b^2}{a^2m}x$

(4) For rectangular hyperbola : $xy = c^2$, it will be $y = -mx$

Two diameters of a conic are said to be conjugate when each bisects the chords parallel to the other.

Let the diameter $(al + hm)x + (hl + bm)y + gl + fm = 0$ of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be parallel to $\frac{x}{l'} = \frac{y}{m'}$. Then,

$$l'(al + hm) + m'(hl + bm) = 0$$

$$\Rightarrow all' + h(lm' + l'm) + bmm' = 0$$

It is the condition for conjugate diameters corresponding to the chords parallel to $\frac{x}{l} = \frac{y}{m}$ and $\frac{x}{l'} = \frac{y}{m'}$.

We know that the diameter corresponding to the chords parallel to $y = mx$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = -\frac{b^2}{a^2m}x$. Let us consider the diameter corresponding to the chords parallel to $y = m'x$, where $m' = -\frac{b^2}{a^2m}$. The equation of this diameter is $y = mx$. It is a member of the system of chords for the diameter $y = m'x$.

Therefore $y = mx$ and $y = m'x$ are conjugate diameters. Moreover $mm' = -\frac{b^2}{a^2}$.

$\therefore$ $y = mx$ and $y = m'x$ will two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $mm' = -\frac{b^2}{a^2}$.

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. $y = mx$ and $y = m'x$ will be conjugate diameters if $mm' = \frac{b^2}{a^2}$. For rectangular hyperbola $xy = c^2$, the condition for conjugacy of two diameters $y = mx$ and $y = m'x$ is $m + m' = 0$.

5.3.2 Properties of diameters and conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(1) The tangent at the extremity of a diameter of parallel to the chords bisected by the diameter.

(2) The tangent at the extremities of any chord meet on the diameter which bisects this chord.

(3) The eccentric angles of the ends of a pairs of conjugate diameters differ by an odd multiple of $\pi/2$.

(4) The sum of squares of two conjugate semi-diameters (semi diameter = $\frac{1}{2}$ x diameter) is constant.

(5) The tangents at the ends of a pair of conjugate diameters form a parallelogram of constant area.

(6) The product of the perpendicular distances from the centre of the ellipse to the tangent at one end of a diameter and the semi-conjugate diameter is constant.

(7) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diametes parallel to the tangent at this point

Note : If two conjugate diameters of an ellipse are equal in length, then they are said to be equi-conjugate diameters.

5.3.3 Properties of diameters and conjugate diameters of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(1) If a pair of diameters be conjugate w.r.t. a hyperbola, then they will be also conjugate w.r.t., conjugate hyperbola.

[The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is called conjugate hyperbola of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.]

(2) If a diameter meets the hyperbola at real points, then it will meet its conjugate at imaginary points.

(3) Only one of a pair of conjugate diameters of a hyperbola meets the curve in real points.

(4) If a pair of conjugate diameters meets the hyperbola and its conjugate at P and Q, then $CP^2 - CQ^2 = a^2 - b^2$, where C is the centre of the hyperbola.

(5) The tangents at the points where a diameter and its conjugate meet a hyperbola and its conjugate form a parallelogram of constant area.

5.3.4 Conjugate diameters of a central conic

Theorem : The necessary and sufficient condition that the diameters $y = mx$ and $y = m'x$ are conjugate of the central conic $ax^2 + 2hxy + by^2 = 1$ is $a + h(m + m') + bmm' = 0$

Proof :

(A) The condition is necessary : Let (x_1, y_1) be the midpoint of a chord parallel to $y = mx$. The equation of the chord is,

$$axx_1 + h(xy_1 + yx_1) + byy_1 = ax_1^2 + 2hx_1y_1 + by_1^2.$$

$$\Rightarrow (ax_1 + hy_1)x + (hx_1 + by_1)y = ax_1^2 + 2hx_1y_1 + by_1^2$$

Since it is parallel to $y = mx_1$

$$(ax_1 + hy_1) + (hx_1 + by_1)m = 0$$

$\therefore$ The diameter corresponding to the chords parallel to $y = mx$ is, $(a + hm)x + (h + bm)y = 0$

Again if $y = m'x$ is conjugate to $y = mx$ then the above equation must be identical with $y = m'x$.

$$\therefore\ (a + hm) + (h + bm)m' = 0$$

$$\Rightarrow a + h(m + m') + bmm' = 0$$

(B) The condition is sufficient :

We have $a + h(m + m') + bmm' = 0$

The equation of the diameter corresponding to the system of chords parallel to $y = mx$ is,

$$(a + hm)x + (h + bm)y = 0$$

$$\Rightarrow -(hm' + bmm')x + (h + bm)y = 0$$

$$\Rightarrow y = m'x$$

$\therefore$ $y = mx$ and $y = m'x$ are conjugate diameters.

5.4 Illustrated examples

Example 1 : Find the equation of the chord of the parabola $y^2 = 8x$ which is bisected at the point $(2, -3)$.

Solution : The equation of the required chord is,

$$y(-3) -4(x + 2) = (-3)^2 - 8.2$$

$$\Rightarrow -4x - 3y - 8 = 9 - 16$$

$$\Rightarrow 4x + 3y + 1 = 0$$

Example 2 : Show that the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which passes through the point (p, q) is,

$$x^2 + y^2 = px + qy$$

Solution : Let (h, k) be the middle point of the chord, so that its equation is,

$$hx + ky = h^2 + k^2$$

Since it passes through the point (p, q), we have,

$$hp + kq = h^2 + k^2$$

$\therefore$ The locus of (h, k) is, $x^2 + y^2 = px + qy$.

Example 3 : Find the equation of the diameter of the ellipse $3x^2 + 4y^2 = 5$ conjugate to the diameter $y + 3x = 0$

Solution : The equation of the ellipse in standard form, $\frac{x^2}{5/3} + \frac{y^2}{5/4} = 1$

Since one diameter is given by $y = -3x$, the conjugate diameter be $y = mx$, such that,

$$m(-3) = -\frac{5/4}{5/3}$$

$$\Rightarrow -3m = -\frac{3}{4}$$

$$\Rightarrow m = \frac{1}{4}$$

$\therefore$ The required conjugate diameter is $y = \frac{x}{4} \Rightarrow 4y = x$.

Example 4 : Find the condition that the pair of straight lines $Ax^2 + 2Hxy + By^2 = 0$ may be conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : If the pair of straight lines be $y = m_1x$ and $y = m_2x$, then, $m_1m_2 = A/B$

Again, if two straight lines be conjugate diameters of the ellipse, then,

$$m_1m_2 = -\frac{b^2}{a^2}.$$

Hence the required condition is, $\frac{A}{B} = -\frac{b^2}{a^2} \Rightarrow Aa^2 + Bb^2 = 0$.

Example 5 : Any tangent to an ellipse with centre C meets the director circle in P and Q. Prove that CP and CQ are in the directions of the conjugate diameters of the ellipse.

Solution : For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have the equation of tangent as,

$$y = mx + \sqrt{a^2m^2+b^2} \qquad (1)$$

The equation of the director circle of the ellipse is,

$$x^2 + y^2 = a^2 + b^2 \qquad (2)$$

The joint equation of CP and CQ will be obtained by making (2) homogeneous with the help of (1), which is,

$$(a^2m^2 + b^2)(x^2 + y^2) = (a^2 + b^2)(y - mx)^2$$

$$\Rightarrow (b^2 - b^2m^2)x^2 + (a^2m^2 - a^2)y^2 + 2(a^2 + b^2)mxy = 0$$

$$\Rightarrow b^2(1 - m^2)x^2 + a^2(m^2 - 1)y^2 + 2(a^2 + b^2)mxy = 0 \qquad (3)$$

If the gradients of CP and CQ are respectively m_1 and m_2, then from (3),

$$m_1m_2 = \frac{b^2(1-m^2)}{a^2(m^2-1)} = -\frac{b^2}{a^2}.$$

Thus CP and CQ are in the directions of the conjugate diameters of the ellipse.

Example 6 : Lines drawn through the foci of an ellipse perpendicular respectively to a pair of conjugate diameters and intersect at Q. Show that the locus of Q is a concentric ellipse.

Solution : Let A(acosθ, bsinθ) and B(–asinθ, bcosθ) be two end points of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where centre C is (0, 0). The foci are (± ae, 0) where $b^2 = a^2(1 - e^2)$.

The slope of diameter CA is $\frac{b\sin\theta}{a\cos\theta}$ and that of CB is $\frac{b\cos\theta}{-a\sin\theta}$. The equation of the line through (ae, 0) and perpendicular to CA is $y = -\frac{a\cos\theta}{b\sin\theta}(x-ae)$ and that of the straight line through the point (–ae, 0) and perpendicular to CB is,

$$y = \frac{a\sin\theta}{b\cos\theta}(x+ae)$$

If these two straight lines meet at Q(α, β), then we have,

$$\cot\theta = \frac{b\beta}{a(ae-\alpha)} = \frac{a(\alpha+ae)}{b\beta}$$

Eliminating θ, we get,

$b^2\beta^2 = a^2(a^2e^2 - \alpha^2)$

$\Rightarrow a^2\alpha^2 + b^2\beta^2 = a^4e^2 = a^2(a^2 - b^2)$

$\Rightarrow \frac{\alpha^2}{b^2} + \frac{\beta^2}{a^2} = \frac{a^2 - b^2}{b^2}$

Hence the locus of Q is, $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2 - b^2}{b^2}$, which is an ellipse concentric with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

5.5 Summary

In this unit, we have learnt to find out equations of chords of standard conics whom the co-ordinates of the middle point of the chords are given. We also have learnt to determine of standard conics, conjugate diameters of central conics and identify their important properties.

5.6 Exercises

1. Find the equations of the chords of
   (a) $x^2 + y^2 = 81$, which is bisected at (–2, 3).
   (b) $y^2 = 4x + 5$, which is bisected at (1, 4).
2. Find the locus of the middle points of chords of
   (a) $y^2 = 16x$, which are parallel to $x - 3y = 5$.
   (b) $xy = c^2$, which are parallel to $2x + 3y = 7$.
3. Find the diameters of
   (a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$, which is conjugate to $5y - 4x = 0$.
   (b) $16x^2 - 9y^2 = 144$, which is conjugate to $x = 2y$.

4. Show that $25x^2 + kxy - 36y^2 = 0$ represents a pair of conjugate diameters of $25x^2 + 36y^2 = 900$ for any value of k.

5. Show that the conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are equally inclined to each of the axes are $bx + ay = 0$ and $bx - ay = 0$.

[Hints : Let $y = mx$ and $y = m'x$ be the conjugate diameters of the ellipse which are euqally inclined to each of the axes. Then $m = -m'$ and $mm' = -\frac{b^2}{a^2}$.

$\therefore\ m^2 = \frac{b^2}{a^2} \Rightarrow m = \pm\frac{b}{a}$]

6. If P and Q be the extrimities of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show that the locus of the middle point of PQ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$ and also PQ touches the ellipse.

7. The normal at a variable point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the diameter CD conjugate to CP at Q. Show that the locus of Q is, $\frac{a^2}{x^2}+\frac{b^2}{y^2}=\left(\frac{a^2-b^2}{x^2+y^2}\right)^2$.

8. Show that the locus of the middle points of the normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is, $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right) = (a^2-b^2)^2$.

Answers :

1. (a) $2x - 3y + 13 = 0$ (b) $x - 2y + 7 = 0$
2. (a) $y = 24$ (b) $2x = 3y$
3. (a) $4x + 5y = 0$ (b) $9y = 32x$

Unit 6 ❐ Polar Equations

Structure

6.0 Objectives

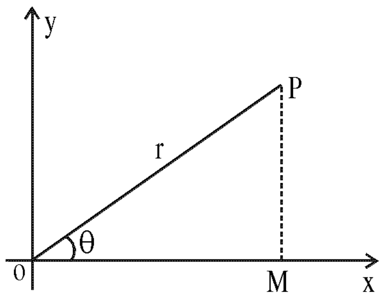
We will be introduced to polar coordinate system. We will also be able to determine

- the distance between two points using polar co-ordinates
- area of a triangle using polar co-ordinates
- polar equations of several two dimensional geometric entities.

6.1 Polar co-ordinates

Upto this point we have dealt with the cartesian (or rectangular or $x - y$) co-ordinate system. But this is not always the easiest co-ordinate system to work in. So, in this section we will start looking at another co-ordinate system, namely polar co-ordinate system.

Co-ordinate system are really nothing more than away to define a point in space. If P is a point with co-ordinates (x, y) in x – y plane, then we can also define it as (r, θ), where r is the distance of the point from origin and θ is the angle of the line segments joining origin and the point from the positive x-axis measured counter-clockwise. In this picture



OP = r, ∠POM = θ.

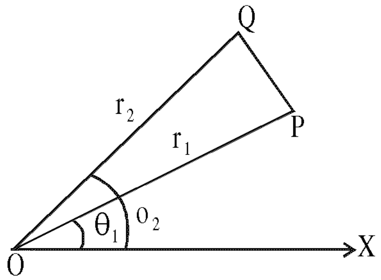
where PM is the perpendicular drawn from P on x-axis. Then from ΔPOM we have,

x = OM = r cosθ and y = PM = rsinθ

So, we can express any point P in (r, θ), this is polar co-ordinates of P.

6.2 Distance between two points

Let the polar co-ordinates of two points P and Q be (r_1, θ_1) and (r_2, θ_2) so that OQ = r_2 and OP = r_1, $\angle POX = \theta_1$, $\angle QOX = \theta_2$. By trigonometry, from ΔPOQ we have,



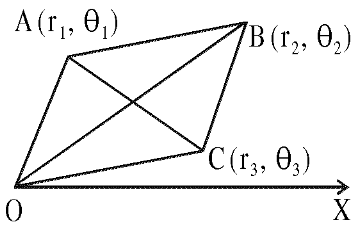
$$PQ^2 = OQ^2 + OP^2 - 2.OP.OQ.\cos\angle QOP$$

$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$$

$$\therefore\ PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

6.3 Area of a triangle

Let the polar coordinates of A, B and C be (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3) respectively w.r.t. pole O and initial line OX.



Then, Area of ΔABC = Area of ΔAOB + Area of ΔOBC – Area of ΔOCA.

$$= \frac{1}{2} OA.OB \sin\angle AOB + \frac{1}{2} OB.OC\sin\angle BOC$$

$$- \frac{1}{2} OC.OA.\sin\angle COA$$

$$= \frac{1}{2}[r_1r_2\sin(\theta_1 - \theta_2) + r_2r_3\sin(\theta_2 - \theta_3) - r_3r_1\sin(\theta_1 - \theta_3)]$$

$$= \frac{1}{2}[r_1r_2\sin(\theta_1-\theta_2)+r_2r_3\sin(\theta_2-\theta_3)+r_1r_3\sin(\theta_3-\theta_1)]$$

Note : If the points A, B, C are collinear, then,

$$r_1r_2\sin(\theta_1-\theta_2)+r_2r_3\sin(\theta_2-\theta_3)+r_1r_3\sin(\theta_3-\theta_1)=0$$

6.4 Polar equation of a straight line

Let the equation of a straight line in cartesian co-ordinate be

$$ax + by + c = 0 \qquad (1)$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$ in (1). Then,

$$ar\cos\theta + br\sin\theta + c = 0$$

$$\Rightarrow \frac{1}{r} = -\frac{a}{c}\cos\theta - \frac{b}{c}\sin\theta$$

$$= A\cos\theta + B\sin\theta$$

where $A = -\frac{a}{c}$, $B = -\frac{b}{c}$

which is the polar equation of a straight line.

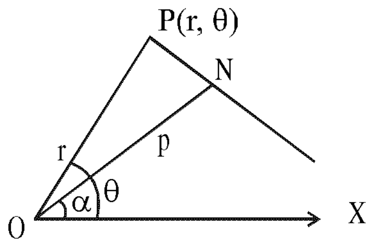
Another form :

Let (r, θ) be the co-ordinates of a point P on the line PN w.r.t. the pole O and the initial line OX. ON is the perpendicular to the line from origin.

Let $ON = p$ and $\angle XON = \alpha$

Now, $ON = OP\cos(\theta-\alpha)$

$$\Rightarrow p = r\cos(\theta-\alpha).$$



This is the polar equation of the straight line of which the perpendicular distance from the origin is p and the perpendicular from the origin makes an angle α with the initial line.

Corollary 1 : If $\alpha = 0$, then $p = r\cos\theta$ is the equation of the straight line perpendicular to the initial line OX.

Corollary 2 : If $\alpha = \frac{\pi}{2}$, then $p = r\sin\theta$ is the equation of the straight line which is parallel to the initial line.

Corollary 3 : If $p = 0$ then $r\cos(\theta-\alpha) = 0 \Rightarrow \cos(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = \frac{\pi}{2}$

$\therefore\ \theta = \alpha + \frac{\pi}{2}$ = constant.

This is the equation of the straight line passing through the pole (origin).

Corollary 4 : The polar equation of two parallel lines are of the form $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \alpha) = p'$.

The polar equations of two mutually perpendicular lines are of the form $r\cos(\theta - \alpha) = p$ and $r\cos\left(\theta - \alpha - \frac{\pi}{2}\right) = p'$ i.e., $r\sin(\theta - \alpha) = p'$.

Note : $A\cos\theta + B\sin\theta = \frac{k}{r}$ (1)

and $A\cos\theta + B\sin\theta = \frac{k_1}{r}$ (2)

are parallel lines.

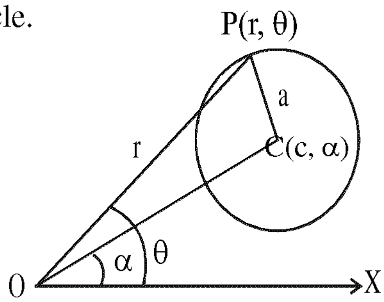
$$A\cos\left(\frac{\pi}{2} + \theta\right) + B\sin\left(\frac{\pi}{2} + \theta\right) = \frac{k_2}{r}$$

$$\Rightarrow\ -A\sin\theta + B\cos\theta = \frac{k_2}{r}$$

is perpendicular to (1) and (2)

6.5 Polar equation of a circle

Let $C(c, \alpha)$ be the polar co-ordinate of the centre C of the circle of radius 'a' w.r.t. pole 'O' and initial line OX. Let $P(r, \theta)$ be a point on the circle.



Now, from ΔCOP we have,

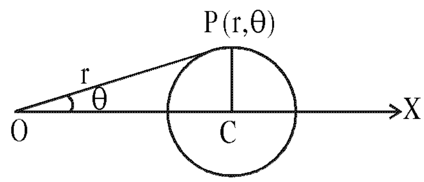
$CP^2 = OC^2 + OP^2 - 2OC.OP.\cos(\theta - \alpha)$

$\Rightarrow a^2 = c^2 + r^2 - 2cr\cos(\theta - \alpha)$

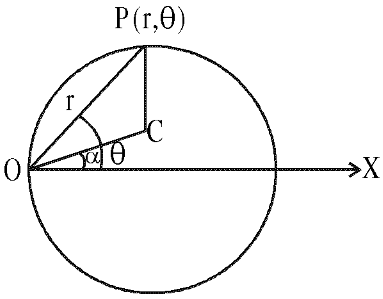
$\Rightarrow r^2 - 2cr\cos(\theta - \alpha) + c^2 - a^2 = 0$ (1)

This is required equation of a circle.

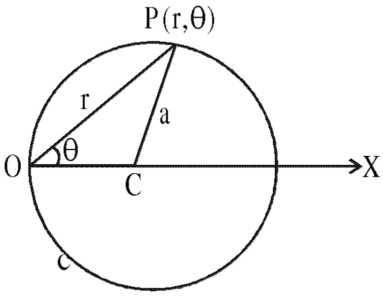
Corollary 1 : If the initial line passes through the centre of the circle, then $\alpha = 0$ and the equation becomes $r^2 - 2cr\cos\theta + c^2 - a^2 = 0$



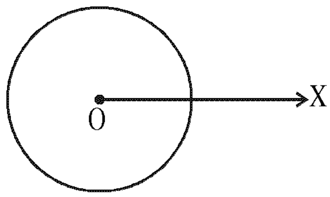
Corollary 2 : If the pole be taken on the circle, then c = a and the equation (1) reduces to, $r = 2a\cos(\theta - \alpha)$



Corollary 3 : If the pole is on the circle and the initial line passes through the centre, then $c = a$, $\alpha = 0$ then the equation of the circle is $r = 2a\cos\theta$.



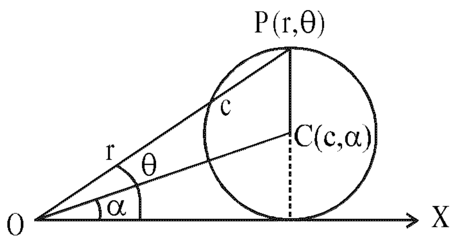
Corollary 4 : If the pole coincides with the centre then the equation of the circle is r = a



Corollary 5 : If the initial line touches the circle, then $c = a\operatorname{cosec}\alpha$ and the equation of the circle becomes

$$r^2 - 2r(a\operatorname{cosec}\alpha)\cos(\theta - \alpha) + a^2\operatorname{cosec}^2\alpha - a^2 = 0$$

$$\Rightarrow r^2 - 2ar\operatorname{cosec}\alpha\cos(\theta - \alpha) + a^2\cot^2\alpha = 0$$



Corollary 6 : If the initial line touches the circle at the pole, then $\alpha = 90^o$ and the above equation reduces to,

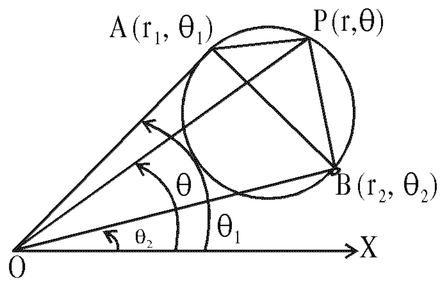
$$r^2 - 2ar\,\text{cosec}90^o\cos(\theta - 90^o) + a^2\cot^2 90^o = 0$$

$$\Rightarrow r = 2a\sin\theta$$

6.6.1 Equation of the circle when two ends of a diameter are given :

Let $p(r, \theta)$ be a point on the circle and the co-ordinates of the ends A and B of the diameter AB be (r_1, θ_1) and (r_2, θ_2) respectively w.r.t., the pole O and initial line OX.

From the figure,



For ΔOAP :

$$AP^2 = OA^2 + OP^2 - 2\,OA.OP.\cos\angle AOP$$

$$= r_1^2 + r^2 - 2r_1 r\cos(\theta_1 - \theta)$$

For ΔOBP :

$$BP^2 = OB^2 + OP^2$$

$$- 2OB.OP.\cos\angle BOP = r_2^2 + r^2 - 2rr_2\cos(\theta - \theta_2)$$

Now, $AB^2 = OA^2 + OB^2 - 2OA.OB.\cos\angle AOB$

$$= r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_1 - \theta_2)$$

Since AB is a diameter, $\angle APB$ is a right angle,

$$AP^2 + PB^2 = AB^2$$

$$\Rightarrow r_1^2 + r^2 - 2r_1r\cos(\theta_1 - \theta) + r_2^2 + r^2 - 2r_2r\cos(\theta - \theta_2) = r_2^{\,2} + r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$

$$\Rightarrow r^2 - r[r_1\cos(\theta - \theta_1) + r_2\cos(\theta - \theta_2)] + r_1r_2\cos(\theta_1 - \theta_2) = 0$$

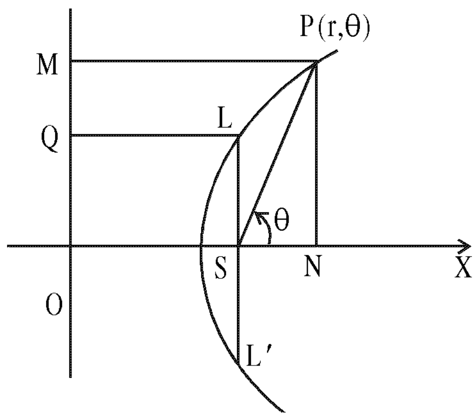
This is the required equation of the circle.

6.6 Polar equation of a conic with the focus as the pole

Let S be the focus, SX be the axis and OM the directrix of the conic in the figure.

Let $P(r, \theta)$ be a point on the the conic w.r.t. the pole S and the initial line SX. PM is perpendicular to the directrix. LSL′ is the latus rectum.

Now we draw perpendiculars PN and LQ to the initial line and the directrix respectively.



Now the eccentricity, $e = \dfrac{SP}{PM}$

$$\Rightarrow SP = e.PM = e.ON$$
$$= e(OS + SN)$$
$$= e(LQ + SN)$$
$$= e\left(\frac{SL}{e} + SN\right) \quad \left[\because \frac{SL}{LQ} = e\right]$$

$$\therefore\ SP = SL + e.SN$$
$$= SL. + e.SP.\cos\theta \quad \left[\because \frac{SN}{SP} = \cos\theta\right]$$

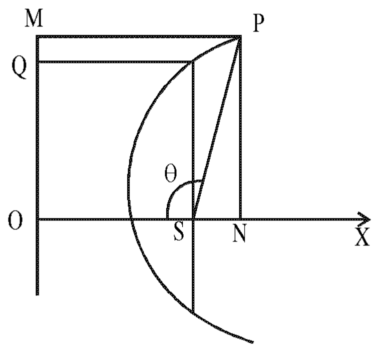
If l is the semi-latus recturm then,

$r = l + er\cos\theta.$

$$\Rightarrow \frac{l}{r} = 1 - e\cos\theta$$

It is required equation of the conic.

Note : If the initial line is taken in the direction so, then $\angle OSP = \theta$ and we have $\angle PSN = \pi - \theta$



$\therefore$ The equation of the conic will be,

$$\frac{l}{r} = 1 - e\cos(\pi - \theta)$$

$$\Rightarrow \frac{l}{r} = 1 + e\cos\theta$$

Corollary 1 : If the axis of the conic makes an angle α with the initial SX and $P(r, \theta)$ be any point on the conic then $\angle PSX = \theta - \alpha$, and $\angle PSO = \pi - (\theta - \alpha)$.

$\therefore$ The equation of the conic, taken as initial line SX is,

$$\frac{l}{r} = 1 - e\cos(\theta - \alpha).$$

and the equation of the conic taken as initial line SO is,

$$\frac{l}{r} = 1 - e\cos(\pi - (\theta - \alpha))$$

$$\Rightarrow \frac{l}{r} = 1 + e\cos(\theta - \alpha)$$

6.7 Polar equation of the directrices of the conic $\frac{l}{r} = 1 - e\cos\theta$

For the ellipse $\frac{l}{r} = 1 - e\cos\theta \quad (e < 1)$. Here g be the directrix corresponding to the focus S. Let Q be the extremity of the latus rectum and we draw perpendicular QM on g.

Let P(r, θ) be any point on the directrix g.

$\therefore \frac{r}{SZ} = \sec(\pi - \theta)$

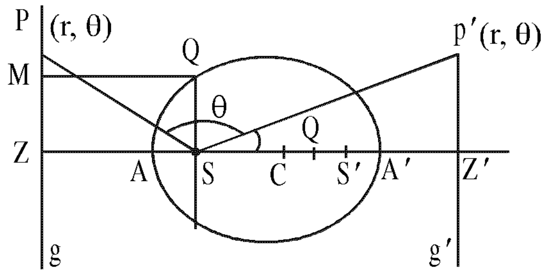
$\Rightarrow \frac{r}{SZ} = -\sec\theta$

$\Rightarrow r = -SZ \sec\theta$

$= -MQ \sec\theta$

$= -\frac{SQ}{e}\sec\theta \qquad \left[\because \frac{SQ}{MQ} = e\right]$

$= -\frac{l}{e}\sec\theta \qquad [\because SQ = l]$



Hence $\frac{l}{r} = -e\cos\theta$ is the equation of the directrix. Similarly other equation of the directrex is, $r = \frac{l}{e}.\frac{1+e^2}{1-e^2}\sec\theta$.

Corollary 1 : In case of the conic $\frac{l}{r} = 1 - e\cos\theta (e > 1)$, it can be shown that the equations of the directrices are,

$\frac{l}{r} = -e\cos\theta$ and $r = \frac{l}{e}.\frac{1+e^2}{1-e^2}\sec\theta$.

Corollary 2 : In case of parabola, $\frac{l}{r} = 1 - \cos\theta$, it can be shown that the equation of the directrix is,

$\frac{l}{r} = -\cos\theta$.

6.8 Equation of the chord of a conic (joining $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$)

Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ (1)

Let P and Q be two points on the conic and the respective vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$. Let the equation of the straight line PQ be,

$$\frac{l}{r} = A\cos\theta + B\sin\theta \qquad (2)$$

This equation contains two independent constants A and B which can be obtained from the condition that it passes through the two given points P and Q.

For the common points of (1) and (2) we have

$$A\cos\theta + B\sin\theta = 1 + e\cos\theta$$

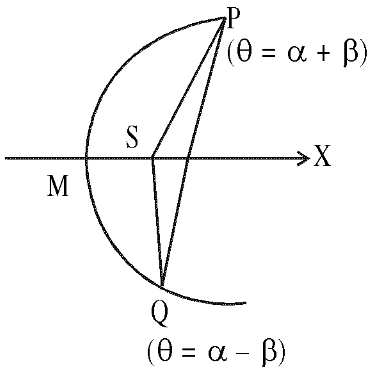
$$\Rightarrow (A - e)\cos\theta + B\sin\theta - 1 = 0 \qquad (3)$$

At the common points $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$, from (3) we get,

$$(A - e)\cos(\alpha - \beta) + B\sin(\alpha - \beta) - 1 = 0 \qquad (4)$$

and, $(A - e)\cos(\alpha + \beta) + B\sin(\alpha + \beta) - 1 = 0$ (5)

By cross-multiplication from (4) and (5) we get,



$$\frac{A-e}{-\sin(\alpha-\beta)+\sin(\alpha+\beta)} = \frac{B}{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{1}{\cos(\alpha-\beta)\sin(\alpha+\beta)-\sin(\alpha-\beta)\cos(\alpha+\beta)}$$

$$\Rightarrow \frac{A-e}{2\cos\alpha\sin\beta} = \frac{B}{2\sin\alpha\cos\beta} = \frac{1}{\sin 2\beta}$$

$$\Rightarrow A - e = \frac{\cos\alpha}{\cos\beta} \text{ and } B = \frac{\sin\alpha}{\cos\beta}$$

$$\Rightarrow A = \frac{\cos\alpha}{\cos\beta} + e \text{ and } B = \frac{\sin\alpha}{\cos\beta}$$

Putting values of A and B in (2) and we get the equation of the chord

$$\Rightarrow \frac{l}{r} = \sec\beta(\cos\alpha\cos\theta + \sin\theta\sin\alpha) + e\cos\theta$$

$$\Rightarrow \frac{l}{r} = \sec\beta\cos(\theta - \alpha) + e\cos\theta$$

which is polar equation of the chord joining the points at $\theta = \alpha - \beta$ and $\theta = \alpha + \beta$

Note 1 : If the equation of the conic be $\frac{l}{r} = 1 - e\cos\theta$, then the equation of the chord will be $\frac{l}{r} = \sec\beta\cos(\theta - \alpha) - e\cos\theta$.

Note 2 : Equation of the chord of the conic $\frac{l}{r} = 1 + e\cos\theta$, joining the two points whose vectorial angles are α and β is, $\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta$.

Note 3 : If the equation of the conic be $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$, then the chord joining the points whose vectorial angles are $(\alpha - \beta)$ and $(\alpha + \beta)$ is,

$$\frac{l}{r} = e\cos(\theta - \gamma) + \sec\beta\cos(\theta - \alpha).$$

6.9 The polar equation of tangent to a conic

The two points corresponding to vectorial angles $(\alpha - \beta)$ and $(\alpha + \beta)$ will coincide if $\beta = 0$. The point Q approaches to P along the curve, then the chord PQ will be tangent at α. Hence the equation of the tangent at the point whose vectorial angle is α,

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta$$

Note : The equation of the tangent at $\theta = \alpha$ of the conic $\frac{l}{r} = 1 - e\cos\theta$ is $\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta$.

6.10 The polar equation of normal to a conic

Let the equation of the normal at the point α be,

$$\frac{k}{r} = \cos\left(\frac{\pi}{2} + \theta - \alpha\right) + e\cos\left(\frac{\pi}{2} + \theta\right)$$

$$\Rightarrow \frac{k}{r} = -\sin(\theta - \alpha) - e\sin\theta \qquad (1)$$

It passes thorought the point of contact $\left(\dfrac{l}{1+e\cos\alpha}, \alpha\right)$

$$\therefore \frac{k(1+e\cos\alpha)}{l} = -\sin\theta - e\sin\alpha$$

$$\Rightarrow k = -\frac{le\sin\alpha}{1+e\cos\alpha}.$$

Putting the value of k in (1) we get,

$$-\frac{le\sin\alpha}{1+e\cos\alpha}.\frac{1}{r} = -\sin(\theta - \alpha) - e\sin\theta$$

$$\Rightarrow \frac{le\sin\alpha}{r(1+e\cos\alpha)} = \sin(\theta - \alpha) + e\sin\theta$$

This is the required polar equation of the normal at $\theta = \alpha$.

Note : If the equation of the conic is $\frac{l}{r} = 1 - e\cos\theta$, then the equation of the normal at $\theta = \alpha$ is,

$$\frac{le\sin\alpha}{r(1-e\cos\alpha)} = e\sin\theta - \sin(\theta - \alpha).$$

6.11 Polar equation of the chord of contact of tangent

Let the polar equation of the conic be,

$$\frac{l}{r} = 1 + e\cos\theta \qquad (1)$$

Let $(\alpha + \beta)$ and $(\alpha - \beta)$ be the vectorial angles of the two points of contact of the tangents from a given point (r_1, θ_1) to the conic

Then the equation of the chord is,

$$\frac{l}{r} = e\cos\theta + \sec\beta \cos(\theta - \alpha) \qquad (2)$$

Now the equations of the tangents at the points where vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$ are,

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \beta) \text{ and}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha + \beta).$$

Both of these tangents pass through (r_1, θ_1). So,

$$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha - \beta) \text{ and } \frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha + \beta) \quad (3)$$

Hence from (3) we have,

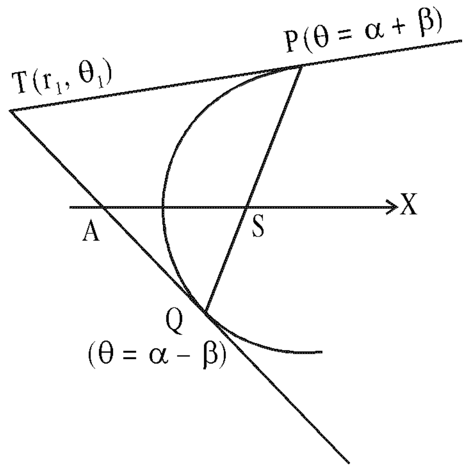
$\cos(\theta_1 - \alpha - \beta) = \cos(\theta_1 - \alpha + \beta)$

$\Rightarrow \theta_1 - \alpha - \beta = \pm (\theta_1 - \alpha + \beta)$.

If we take '+' sign, then we have $\beta = 0$, but $\beta \neq 0$, so, we neglect it.

Now taking '–' sign we have, $\theta_1 - \alpha - \beta = -\theta_1 + \alpha - \beta \Rightarrow \theta_1 = \alpha$

$\therefore$ From (3) we have, $\frac{l}{r_1} = e\cos\theta_1 + \cos\beta \quad (4)$



From (2) we get the required equation of the chord of contact of tangents from the point (r_1, θ_1) to the conic as,

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1)$$

Note : If the equation of the conic be $\frac{l}{r} = 1 - e\cos\theta$, then the equation of the chord of contact of the tangents to the conic from the point (r_1, θ_1) is,

$$\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r_1} + e\cos\theta_1\right) = \cos(\theta - \theta_1)$$

6.12 The equation of pair of tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$ from the point (r', θ')

The equation of the tangent to the conic $\frac{l}{r} = 1 - e\cos\theta$ at $\theta = \alpha$ is,

$$\frac{l}{r} = \cos(\theta - \alpha) - e\cos\theta \qquad (1)$$

If it passes through the point (r', θ'), then,

$$\frac{l}{r'} = \cos(\theta' - \alpha) - e\cos\theta' \qquad (2)$$

From (1), $\left(\frac{l}{r} + e\cos\theta\right)^2 = \cos^2(\theta - \alpha) = 1 - \sin^2(\theta - \alpha)$

$$\Rightarrow 1 - \left(\frac{l}{r} + e\cos\theta\right)^2 = \sin^2(\theta - \alpha) \qquad (3)$$

Similarly from (2) we have, $1 - \left(\frac{l}{r'} + e\cos\theta'\right)^2 = \sin^2(\theta' - \alpha) \qquad (4)$

Again from (1) and (2) we have,

$$\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r'} + e\cos\theta'\right) - \cos(\theta - \theta')$$

$$= \cos(\theta - \alpha)\cos(\theta' - \alpha) - \cos(\theta - \theta')$$

$$= \frac{1}{2}\{\cos(\theta + \theta' - 2\alpha) - \cos(\theta - \theta')\}$$

$$= -\sin(\theta - \alpha)\sin(\theta' - \alpha) \qquad (5)$$

Eliminating α from (3), (4), (5) we get,

$$\left\{1 - \left(\frac{l}{r} + e\cos\theta\right)^2\right\}\left\{1 - \left(\frac{l}{r'} + e\cos\theta'\right)^2\right\} = \sin^2(\theta - \alpha)\sin^2(\theta' - \alpha)$$

$$= \left\{\left(\frac{l}{r} + e\cos\theta\right)\left(\frac{l}{r'} + e\cos\theta'\right) - \cos(\theta - \theta')\right\}^2.$$

Thus the equation of pair of tangents to the conic $\frac{l}{r} = 1 - e\cos\theta$ from the point (r', θ') is,

$$\left\{1-\left(\frac{l}{r}+e\cos\theta\right)^2\right\}\left\{1-\left(\frac{l}{r'}+e\cos\theta'\right)^2\right\}$$

$$= \left\{\left(\frac{l}{r}+e\cos\theta\right)\left(\frac{l}{r'}+e\cos\theta'\right)-\cos(\theta-\theta')\right\}^2$$

6.13 Some worked out examples

Example 1 : Show that the locus of the point whose distance from the pole is equal to its distance from the straight line $r\cos\theta + k = 0$ is $2r\sin^2\frac{\theta}{2} = k$.

Solution : Let $P(r_1, \theta_1)$ be any point and O be the pole. Let the equation of the straight line AB be $r\cos\theta + k = 0$. Now PN be perpendicular from P on AB and OR be perpendicular from O on AB, where RO = k.

By the given condition, $OP = PN = r_1$

From ΔPOT, we get, $\frac{OT}{OP} = \cos\theta_1 \Rightarrow OT = OP\cos\theta_1 = r_1\cos\theta_1$

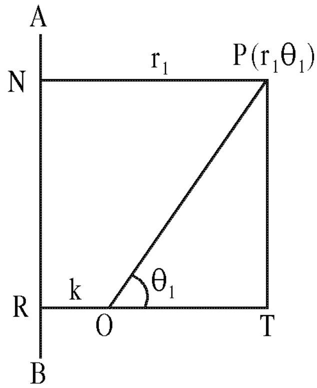
Now, NP = RT = RO + OT

$\Rightarrow r_1 = k + r_1\cos\theta_1$

$\Rightarrow r_1(1 - \cos\theta_1) = k$

$\Rightarrow 2r_1\sin^2\frac{\theta_1}{2} = k$

Hence the locus of the point (r_1, θ_1) is $2r\sin^2\frac{\theta}{2} = k$



Example 2 : If d is the diameter of the circle through the pole and the points (r_1, θ_1), (r_2, θ_2). Then show that, $d^2\sin^2(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$.

Solution : Let d be the diameter of the circle, then the radius of the circle is $\frac{d}{2}$.

$\therefore$ The equation of the circle through the pole is,

$$r = 2\frac{d}{2}\cos(\theta - \alpha)$$

$\Rightarrow r = d\cos(\theta - \alpha)$ (1)

Since the circle (1) passes through the points (r_1, θ_1) and (r_2, θ_2) then we have,

$r_1 = d\cos(\theta_1 - \alpha)$

$\Rightarrow d\cos\theta_1\cos\alpha + d\sin\theta_1\sin\alpha - r_1 = 0$ (2)

and $r_2 = d\cos(\theta_2 - \alpha)$

$\Rightarrow d\cos\theta_2\cos\alpha + d\sin\theta_2\sin\alpha - r_2 = 0$ (3)

By cross multiplicating (2) and (3) we get,

$$\frac{\cos\alpha}{d\{r_1\sin\theta_2 - r_2\sin\theta_1\}} = \frac{\sin\alpha}{d\{r_2\cos\theta_1 - r_1\cos\theta_2\}} = \frac{1}{d^2\sin(\theta_1 - \theta_2)}$$

$$\Rightarrow \cos\alpha = \frac{r_1\sin\theta_2 - r_2\sin\theta_1}{d\sin(\theta_1 - \theta_2)} \text{ and } \sin\alpha = \frac{r_2\cos\theta_1 - r_1\cos\theta_2}{d\sin(\theta_1 - \theta_2)}$$

Since $\sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow (r_1\sin\theta_2 - r_2\sin\theta_1)^2 + (r_2\cos\theta_1 - r_1\cos\theta_2)^2 = d^2\sin^2(\theta_1 - \theta_2)$

$\Rightarrow r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) = d^2\sin^2(\theta_1 - \theta_2)$ [Proved]

Example 3 : Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$, parallel to the tengent at $\theta = \alpha$ is given by, $l(e^2 + 2e\cos\alpha + 1) = r(e^2 - 1)\{\cos(\theta - \alpha) + e\cos\theta\}$

Solution : The given equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$ (1)

The equation of tangent to the conic (1) at $\theta = \alpha$ is,

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta$$

$$= (e + \cos\alpha)\cos\theta + \sin\theta\sin\alpha \quad (2)$$

Similarly, the equation of tangent to (1) at $\theta = \beta$ is,

$$\frac{l}{r} = (e + \cos\beta)\cos\theta + \sin\theta\sin\beta \quad (3)$$

Since (2) and (3) are parallel, so we have,

$$\frac{e+\cos\beta}{e+\cos\alpha} = \frac{\sin\beta}{\sin\alpha} = k \text{ (say)}$$

$\therefore$ $\sin\beta = k\sin\alpha$ (4)

and $\cos\beta = k(e + \cos\alpha) - e$ (5)

squaring and adding (4) and (5) we get,

$1 = k^2\sin^2\alpha + \{k(e + \cos\alpha) - e\}^2$

$\Rightarrow k^2(1 + 2e\cos\alpha + e^2) - 2ke(e + \cos\alpha) + e^2 - 1 = 0$

$\Rightarrow e^2(k^2 - 2k + 1) + e(2k^2 - 2k)\cos\alpha + k^2 - 1 = 0$

$\Rightarrow (k - 1)\{e^2(k - 1) + 2ke\cos\alpha + k + 1\} = 0$

$\Rightarrow (k - 1)\{(e^2 + 2e\cos\alpha + 1)k + 1 - e^2\} = 0$

If $k = 1$, then $\sin\beta = \sin\alpha$ and $\cos\beta = \cos\alpha$ which implies that $\beta = \alpha$, which is impossible

$\therefore$ $k \neq 1$, and then $k = \dfrac{e^2-1}{e^2+2e\cos\alpha+1}$

Putting the value of k in (4) and (5) we get,

$\sin\beta = \dfrac{(e^2-1)\sin\alpha}{e^2+2e\cos\alpha+1}$ and $\cos\beta = \dfrac{(e^2-1)(e+\cos\alpha)}{e^2+2e\cos\alpha+1} - e$

From (3) using (4) and (5) we get,

$$\frac{l}{r} = \cos\theta\{k(e + \cos\alpha) - e + e\} + \sin\theta(k\sin\alpha)$$

$$= k\cos\theta\cos\alpha + ke\cos\theta + k\sin\theta\sin\alpha$$

$$= k\cos(\theta - \alpha) + ke\cos\theta$$

$$= k[\cos(\theta - \alpha) + e\cos\theta]$$

$$\therefore \frac{l}{r} = \frac{e^2-1}{e^2+2e\cos\alpha+1}[\cos(\theta - \alpha) + e\cos\theta]$$

$\Rightarrow l(1 + 2e\cos\alpha + e^2) = r(e^2 - 1)[\cos(\theta - \alpha) + e\cos\theta]$ (proved)

Example 4 : Prove that the two conics $\frac{l_1}{r} = 1 - e_1\cos\theta$ and $\frac{l_2}{r} = 1 - e_2\cos(\theta - \alpha)$ will touch one another if, $l_1^2(1-e_2^2)+l_2^2(1-e_1^2) = 2l_1l_2(1-e_1e_2\cos\alpha)$

Solution : The given equation of the conics are,

$$\frac{l_1}{r} = 1 - e_1\cos\theta \qquad (1)$$

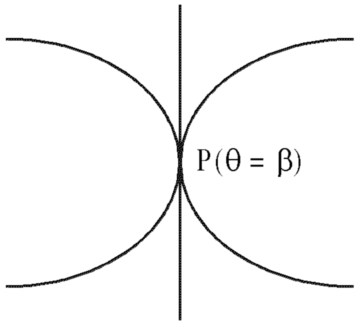
$$\frac{l_2}{r} = 1 - e_2\cos(\theta-\alpha) \qquad (2)$$

Let P(θ = β) be a point where (1) and (2) touch one another.

Now the equation of the tangent to the conic (1) at θ = β is,

$$\frac{l_1}{r} = \cos(\theta-\beta) - e_1\cos\theta$$

$$\Rightarrow \frac{l_1}{r} = (\cos\beta - e_1)\cos\theta + \sin\theta\sin\beta \qquad (3)$$



$\frac{l_1}{r} = \cos(\theta-\beta) - e_1\cos\theta$ $\quad \frac{l_2}{r} = 1 - e_2\cos(\theta-\alpha)$

Again the equation of the tangent to the conic (2) at θ = β is,

$$\frac{l_2}{r} = \cos(\theta-\beta) - e_2\cos(\theta-\alpha)$$

$$= (\cos\beta - e_2\cos\alpha)\cos\theta + (\sin\beta - e_2\sin\alpha)\sin\beta \qquad (4)$$

∵ The two tangents (3) and (4) are identical,

$$\frac{\cos\beta - e_1}{\cos\beta - e_2\cos\alpha} = \frac{\sin\beta}{\sin\beta - e_2\sin\alpha} = \frac{l_1}{l_2}$$

$\therefore\ l_2(\cos\beta - e_1) = l_1(\cos\beta - e_2\cos\alpha)$

$\therefore\ \cos\beta(l_2 - l_1) = e_1l_2 - l_1e_2\cos\alpha \qquad (5)$

and $l_2\sin\beta = l_1(\sin\beta - e_2\sin\alpha)$

$\Rightarrow (l_2 - l_1)\sin\beta = -e_2l_1\sin\alpha \qquad (6)$

From (5) and (6), we have,

$(l_2 - l_1)^2 = l_1^2e_2^2\cos^2\alpha + l_1^2(\sin\beta - e_2\sin\alpha)^2 + l_2^2e_1^2 - 2e_1e_2l_1l_2\cos\alpha$

$= l_2^2e_1^2 + l_1^2e_2^2 - 2e_1e_2l_1l_2\cos\alpha$

$\therefore\ l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1l_2(1 - e_1e_2\cos\alpha)$ [Proved]

Example 5 : Show that the equation of the circle which passes through the focus of the parabola $\frac{2a}{r} = 1 + \cos\theta$ and touches it at the point $\theta = \alpha$ is given by $r\cos^3\frac{\alpha}{2} = a\cos\left(\theta - \frac{3\alpha}{2}\right)$.

Solution : The equation of the parabola is, $\frac{2a}{r} = 1 + \cos\theta$ (1)

Let the equation of the circle be,

$r = 2b\cos(\theta - \gamma)$ (2)

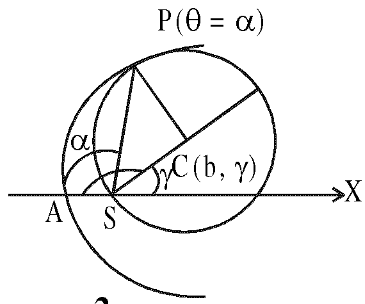
where 2b is the diameter and γ is the inclination of the diameter through the focus S with the initial line SX. Let C be the centre of the circle and it touches the parabola at $p(\theta = \alpha)$.

The equation of the normal at $P(\theta = \alpha)$ to the conic (1) is,

$$\frac{2a\sin\alpha}{r(1+\cos\alpha)} = \sin\theta + \sin(\theta - \alpha) \quad (3)$$

since the normal (3) passes through the centre $C(b, \gamma)$, then we have,

$$\frac{2a\sin\alpha}{b(1+\cos\alpha)} = \sin\gamma + \sin(\gamma - \alpha) \quad (4)$$



$\frac{2a}{r} = 1 + \cos\theta$

Since $P(\theta = \alpha)$ be the common point of the circle and the parabola.

$$\therefore \frac{2a}{1+\cos\alpha} = 2b\cos(\alpha - \gamma) \quad (5)$$

$$\Rightarrow \frac{2a\sin\alpha}{b(1+\cos\alpha)} = 2\cos(\alpha - \gamma)\sin\alpha$$

$\Rightarrow \sin\gamma + \sin(\gamma - \alpha) = 2\cos(\alpha - \gamma)\sin\alpha$ [using (4)]

$\Rightarrow \sin\gamma + \sin(\gamma - \alpha) = \sin(2\alpha - \gamma) + \sin\gamma$

$\Rightarrow \gamma - \alpha = 2\alpha - \gamma$

$$\Rightarrow \gamma = \frac{3\alpha}{2}$$

From (5),

$$b = \frac{a}{(1+\cos\alpha)\cos(\alpha-\gamma)} = \frac{a}{2\cos^2\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{a}{2\cos^3\frac{\alpha}{2}}$$

$\therefore$ Equation of the circle is $r = 2\dfrac{a}{2\cos^3\frac{\alpha}{2}}\cos\left(\theta-\dfrac{3\alpha}{2}\right)$

$\therefore$ $r\cos^3\dfrac{\alpha}{2} = a\cos\left(\theta-\dfrac{3\alpha}{2}\right)$.

Example 6 : (i) Show that the sum of the reciprocals of two perpendicular focal chords of a conic is constant. (ii) If PSP′ and QSQ′ be two perpendicular focal chord of a conic with focus S, then prove that

$$\frac{1}{SP.SP'}+\frac{1}{SQ.SQ'} = \text{constant}$$

Solution : Let the equation of the conic be

$$\frac{l}{r} = 1 - e\cos\theta \qquad (1)$$

Here let PSP′ and QSQ′ be two perpendicular focal chords. Let α be the vectorial angle of P. Then the vectorial angles of Q, P′, Q′ are respectively $\left(\alpha+\dfrac{\pi}{2}\right)$, $(\alpha+\pi)$ and $\left(\alpha+\dfrac{3\pi}{2}\right)$.

From (1) we have,

$$\frac{l}{SP} = 1 - e\cos\alpha \text{ and } \frac{l}{SP'} = 1 - e\cos(\pi+\alpha) = 1 + e\cos\alpha$$

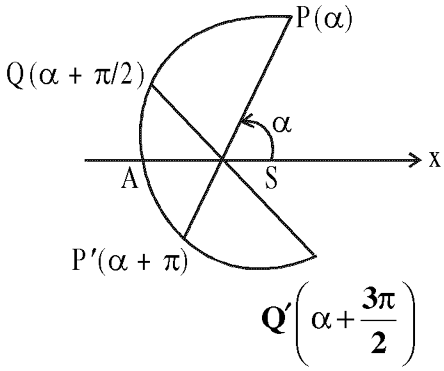
$$\frac{l}{SQ} = 1 - e\cos\left(\frac{\pi}{2}+\alpha\right) = 1 + e\sin\alpha \text{ and } \frac{l}{SQ'} = 1 - e\cos\left(\alpha+\frac{3\pi}{2}\right) = 1 - e\sin\alpha$$

(i) Now, PP′ = PS + SP′

$$= \frac{l}{1-e\cos\alpha}+\frac{l}{1+e\cos\alpha}$$

$$=$$

Again similarly, $QQ' = \dfrac{2l}{1-e^2\sin^2\alpha}$



∴ Sum of reciprocals of these two perpendicular focal chords,

$$\frac{1}{PP'}+\frac{1}{QQ'} = \frac{1-e^2\cos^2\alpha}{2l}+\frac{1-e^2\sin^2\alpha}{2l}$$

$$= \frac{2-e^2}{2l} = \text{constant (Proved)}$$

(ii) $SP.SP' = \dfrac{l}{1-e\cos\alpha}.\dfrac{l}{1+e\cos\alpha} = \dfrac{l^2}{1-e^2\cos^2\alpha}$

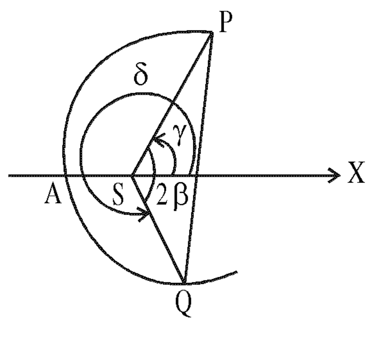
$$SQ.Q'S = \frac{l}{1-e\sin\alpha}.\frac{l}{1+e\sin\alpha} = \frac{l^2}{1-e^2\sin^2\alpha}$$

$$\frac{1}{SP.SP'}+\frac{1}{SQ.Q'S} = \frac{(1-e^2\cos^2\alpha)}{l^2}+\frac{(1-e^2\sin^2\alpha)}{l^2} = \frac{2-e^2}{l^2} = \text{constant (Proved)}$$

Example 7 : P, Q are two points on the conic $\dfrac{l}{r} = 1 - e\cos\theta$ with $(\alpha - \beta)$ and $(\alpha + \beta)$ as vectorial angles. Show that the locus of the foot of the perpendicular from the pole on the straight line PQ is $r^2(e^2 - \sec^2\beta) + 2ler\cos\theta + l^2 = 0$

or, If PQ be a variable chord of the conic $\dfrac{l}{r} = 1 - e\cos\theta$ subtending a constant angle 2β at the locus S, where S is the pole. Show that the locus of the foot of the perpendicular from S on PQ is the curve $r^2(e^2 - \sec^2\beta) + 2ler\cos\theta + l^2 = 0$

Solution : Let γ and δ be the vectorial angles of P and Q w.r.t., the pole S and initial line SX, as the axis of the conic,



$$\frac{l}{r} = 1 - e\cos\theta$$

The polar equation of the variable chord PQ is,

$$\frac{l}{r} = \sec\frac{\delta-\gamma}{2}\cos\left(\theta-\frac{\gamma+\delta}{2}\right)-e\cos\theta \qquad (1)$$

From the figure, $\delta - \gamma + 2\beta = 2\pi \Rightarrow \delta - \gamma = 2(\pi - \beta)$

The polar equation of the line perpendicular to (1) is of the form,

$$\frac{k}{r} = \sec(\pi - \beta)\cos\left(\frac{\pi}{2}+\theta-\frac{\gamma+\delta}{2}\right)-e\cos\left(\frac{\pi}{2}+\theta\right)$$

$$\Rightarrow \frac{k}{r} = \sec\beta\sin\left(\theta-\frac{\gamma+\delta}{2}\right)+e\sin\theta$$

If it passes through the focus, then k = 0

∴ The equation of the straight line passing through the focus S and perpendicular to the variable chord (1) is,

$$-e\sin\theta = \sec\beta\sin\left(\theta-\frac{\gamma+\delta}{2}\right) \qquad (2)$$

From (1), $\frac{l}{r}+e\cos\theta=\sec\beta\cos\left(\theta-\frac{\gamma+\delta}{2}\right)$ (3)

Squaring and adding (2) and (3) we get,

$$\left(\frac{l}{r}+e\cos\theta\right)^2 + e^2\sin^2\theta = \sec^2\beta.$$

$$\Rightarrow \frac{l^2}{r^2}+\frac{2le\cos\theta}{r}+e^2=\sec^2\beta$$

$$\Rightarrow r^2(e^2-\sec^2\beta)+2l\,er\cos\theta+l^2=0$$

It is the required locus.

6.14 Summary

In this unit, we have learnt about polar coordinate system and finding out distance between two points and area of a triangle using polar coordinates. Also we are now able to determine the polar equations of various two dimensioral geometric entities like straight lines, circles, conics and their directrices, chords and tangents etc.

6.15 Exercises

1. Find polar equation of a circle which passes through the pole and the points whose polar co-ordinates are (d, 0) and (2d, π/3). Find also radius of the circles.

2. Show that the triangle formed by the pole and the points of intersection of the circle $r = 4\cos\theta$ with the straight line $r\cos\theta = 3$ is equilateral.

3. Show that the point of intersection of the straight lines $r\cos(\theta - \alpha) = p$ and $r\cos(\theta - \beta) = p$ is $\left(p\sec\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$.

4. Find the polar equation of the straight line joining two points on the parabola $\frac{2a}{r} = 1 + \cos\theta$ with $(\alpha - \beta)$ and $(\alpha + \beta)$ as their vectorial angles.

5. Find the polar equation of the straight line joining two points on the conic $\frac{l}{r} = 1 - e\cos\theta$ whose vectorial angles are α and β.

6. Show that the condition that the straight line $\frac{l}{r} = a\cos\theta + b\sin\theta$ may touch the circle $r = 2k\cos\theta$ is $b^2k^2 + 2ak = 1$

   [Hint : The perpendicualr distance from the centre (k, 0) of the circle to the straight line must be radius = k.]

7. Show that the condition that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$ is, $A^2 + B^2 - 2e(A\cos\gamma + B\sin\gamma) + e^2 - 1 = 0$

8. Show that the straight line $\frac{l}{r} = A\cos\theta + B\sin\theta$ touches the conic $\frac{l}{r} = 1 + e\cos\theta$. if $(A - e)^2 + B^2 = 1$

9. If the straight line $r\cos(\theta - \alpha) = p$ touches the parabola $\frac{l}{r} = 1 + \cos\theta$, then show that $p = \frac{l}{2}\sec\alpha$.

10. Show that the equation of the circle which passes through the focus of the conic $\frac{l}{r} = 1 - e\cos\theta$ and touches it at $\theta = \alpha$ is, $r(1 - e\cos\alpha)^2 = l\cos(\theta - \alpha) - el\cos(\theta - 2\alpha)$.

    [Hint : proceed similarly like example - 5].

11. If P, Q be variable points on a conic $\frac{l}{r} = 1 - e\cos\theta$ with the vectorial angles α and β, where $\alpha - \beta = 2\gamma$, then show that the chord PQ touches the conic $\frac{l}{r}\cos\gamma = 1 - e\cos\gamma\cos\theta$ and the conic has the same directrix as the original one.
12. If r_1 and r_2 are two mutually perpendicular radius vectors of the ellipse $r^2 = \frac{b^2}{1-e^2\cos^2\theta}$, then show that $\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}+\frac{1}{b^2}$.
13. Prove that the lengths of the focal chords of the conic $\frac{l}{r}=1-e\cos\theta$, which is inclined to the initial line at an angle α is, $\frac{2l}{1-e^2\cos^2\alpha}$.
14. Prove that the locus of the middle point of any chord of the conic $\frac{l}{r}=1 - e\cos\theta$ is $r(1 - e^2\cos^2\theta) = el\cos\theta$.
    [Hint : The vectorial angles of the extremities of any chord are α and $\pi + \alpha$]
15. If a normal be down at one extremity $(l, \pi/2)$ of the latus rectum PSP′ of the conic $\frac{l}{r} = 1 + e\cos\theta$, where S is the pole, then show that the distance from the focus S of the other point in which the normal meets the conis is, $\frac{l(1+3e^2+e^4)}{1+e^2-e^4}$.
16. If the normals at the points of vertical angles $\alpha, \beta, \gamma, \delta$ on the conic $\frac{l}{r} = 1 + e\cos\theta$ meet a point, then prove that, $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}+\left(\frac{1+e}{1-e}\right)^2=0$.
17. Show that the auxiliary circle of the conic $\frac{l}{r} = 1 - e\cos\theta$ is $r^2(e^2 - 1) + 2ler\cos\theta + l^2 = 0$ or, prove that the locus of the foot of the perpendicular from a focus of the conic $\frac{l}{r} = 1 - e\cos\theta$ on a tangent to it is given by, $r^2(e^2 - 1) + 2ler\cos\theta + l^2 = 0$.

18. Show that the director circle of the conic $\frac{l}{r} = 1 + e\cos\theta$ is $r^2(e^2 - 1) - 2ler\cos\theta + 2l^2 = 0$.

    [Hint : director circle of a conic is the locus of the point of intersection of two perpendicular tangents drawn on a conic]

19. PSP′ is a focal chord of a conic. Prove that the angle between the tangents at P and P′ is $\tan^{-1}\frac{2e\sin\alpha}{1-e^2}$, where α is the angle between the chord and the major axis.

20. The tangents at the two points P and Q of a parabola meet at a point T, whose focus is S. Prove that SP.SQ = ST^2.

21. P, Q, R are three points on a parabola $\frac{l}{r} = 1 + \cos\theta$ with vectorial angles α, β, γ respectively. Show that the equation of the circle circumscribing the triangle formed by the tangents at these points to the parabola is

    $$r = \frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\cos\left(\theta - \frac{\alpha+\beta+\gamma}{2}\right).$$

    [Hint : Find point of intersections of three tangents. Then find the circle passing through those points]

Answers :

1. $r = 2d\cos(\theta - \pi/3)$, d

4. $\frac{2a}{r} = \sec\beta\cos(\theta - \alpha) + \cos\theta$

5. $\frac{l}{r} = \sec\frac{\beta-\alpha}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right) - e\cos\theta$

Unit 7 ❐ Introduction to three dimensional geometry

Structure

7.0 Objectives

We will be introduced to cartesian, cylindrical and spherical coordinate system in three dimensional geometry. We will also be able to understand and determine

- distance between two points in cartesian coordinate system
- coordinates of a point dividing the line joining two points in a ratio
- direction cosines and ratioes of a straight line
- projection of a line under certain condition
- angle between two straight lines

7.1 Introduction

In your previous classes three dimensional geometry has been introduced. Most of the objects in our daily life are three dimensional. That means for measurement of any quantity we need three variables. For example, in enboid we have length, breadth and height. For cylindrical objects we need a radius, angle and height etc. So at first we discuss about the three dimensional coordinate system in Cartesian form, cylindrical form and in spherical form.

7.2 Cartesian coordinate system in space

Let us consider three mutually perpendicular lines at a point O and name then anticlockwise as XOX′, YOY′ and ZOZ′. Hence OX, OY, OZ and in positive directions and OX′, OY′, OZ′ are in negative direction.

A point P is an arbitrary point in this space. So if we draw perpendicular from P on XOY plane at M and then two perpendiculars on OX and OY from M, we get ON = x′, ML = y′ and PM = z′. In that case we denote the coordinate of P as (x′, y′, z′). XOX′, YOY′ and ZOZ′ are called x-axis, y-axis and z-axis respectively. They are also called axes in altogether. O is called origin of the coordinate system and we can consider this origin is anywhere of the space.

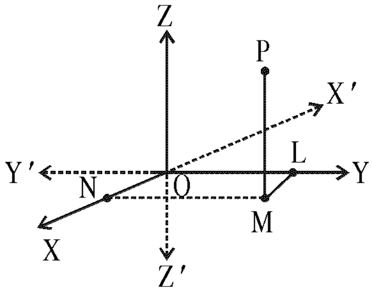


Fig. 7.2.1

The axes divide the space into eight parts where the coordinate of a point depends in which part it lies. Suppose a point is situated at positive parts of x-axis, positive part of y-axis but negative part of z-axis. So its coordinate will be P(x′, y′, –z′).

So depending upon octant the coordinate of a point may be (x′, y′, z′), (–x′, –y′, –z′), (x′, y′, –z′), (–x′, y′, z′), (x′, –y′, z′)(–x′, –y′, z′), (x′, –y′, –z′), (–x′, y′, –z′).

7.3 Cylindrical coordinate system in space

In cylindrical coordinate system we need three quantities to measure any object.

We need radius ‘r’ the angle θ in anticlockwise direction as positive sense and a height ‘z’ which is the same as the coordinate of z-axis.

So if P(x, y, z) be coordinate in cartesian system of a point on cylinder, then, in cylindrical system its coordinate will be P(r, θ, z), where $x = r\cos\theta$, $y = r\sin\theta$, where $o \le r < \propto$ and $o \le \theta \le 2\pi$, $-\infty < z < \infty$. So $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

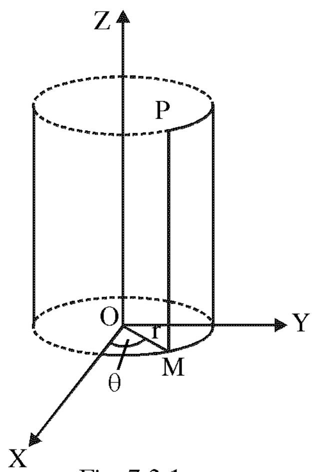


Fig. 7.3.1

Example : If a point P(2, 3, 4) has cartesian coordinate system then find its cylindrical coordinate system.

Solution : Here $x = 2$, $y = 3$, $z = 4$

$r\cos\theta = 2$, $r\sin\theta = 3$, $z = 4$

Squaing and adding we get $r^2(\cos^2\theta + \sin^2\theta) = 4 + 9 = 13$

$\therefore$ $r^2 = 13$ i.e. $r = \pm\sqrt{13}$. As r is always positive,
so $r = \sqrt{13}$.

And $\tan\theta = \frac{3}{2}$ i.e. $\theta = \tan^{-1}\frac{3}{2}$.

This in cylindrical coordinate system its coordinate will be $\left(\sqrt{13}, \tan^{-1}\frac{3}{2}, 4\right)$.

7.4 Spherical coordinate system in space

In spherical coordinate system we need three variables which one 'r' the radius vector, θ and ϕ are the angles which rotate accordingly $0 \le \theta \le \pi$ and $o \le \phi \le 2\pi$. If a point P(x, y, z) is the cartesian coordinate system then its spherical coordinate system, is $P_1(r, \theta, \phi)$ where

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

So $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$

and $\phi = \tan^{-1}\left(\frac{y}{x}\right)$.

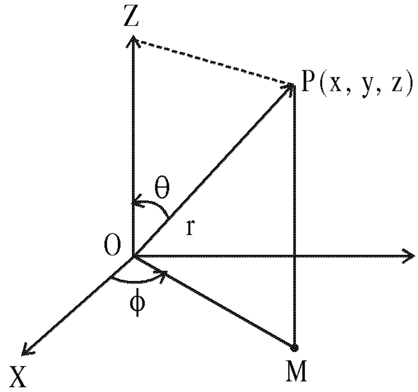


Fig. 7.4.1

Example : Find the spherical coordinate system of a point whose cartesian coordinate system is (2,3,4).

Solution : Here $r = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$

$\theta = \tan^{-1}\frac{\sqrt{13}}{4}$ and $\phi = \tan^{-1}\frac{3}{2}$.

So its spherical coordinate is $\left(\sqrt{29}, \tan^{-1}\frac{\sqrt{13}}{4}, \tan^{-1}\frac{3}{2}\right)$.

7.5 Distance between two points in cartesian coordinate system

If the coordinate of two points P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then distance between them is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ as PQ form the diagonal of a cuboid or rectangular parallelopiped and

$PQ^2 = PL^2 + LQ^2$

$= PM^2 + ML^2 + LQ^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

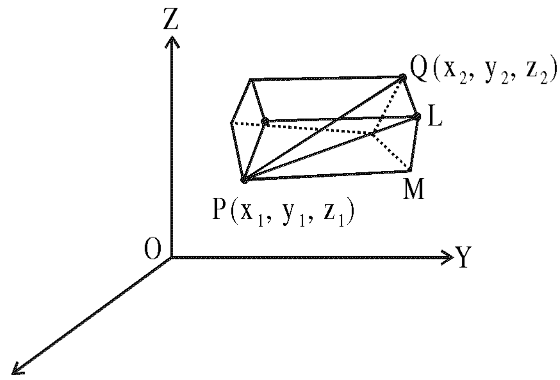


Fig. 7.5.1

Example : Find the distance between two points P(2, 4, –2) and Q(3, 6, 2).

Solution : $x_1 = 2$, $y_1 = 4$, $z_1 = -2$ and

$x_2 = 3$, $y_2 = 6$, $z_2 = 2$

$\therefore$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21} \text{ units.}$$

7.6 The coordinate of a point which divide the line joining two points in a ratio

If R is a point which divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio m : n then coodidnate of R will be, $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$ if R divides PQ internally in the ratio m : n. If R divides PQ externally in the ratio m : n where m > n, then coordinate of R will be

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right).$$

If R is the midpoint of PQ, then coordinate of R will be

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right).$$

7.7 Direction cosines of a straight line

If L is a straight line which makes angle with the axes as α, β and γ respectively then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the straight line L.

If L is an arbitray straight line then we consider its parallel line L′ which meets the origin. Then we denote $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

If P and Q are the two points on L, then

$$\frac{x_2 - x_1}{PQ} = \cos\alpha, \quad \frac{y_2 - y_1}{PQ} = \cos\beta \text{ and } \frac{z_2 - z_1}{PQ} = \cos\gamma$$

i.e., $l = \dfrac{x_2 - x_1}{PQ}$, $m = \dfrac{y_2 - y_1}{PQ}$ and $n = \dfrac{z_2 - z_1}{PQ}$.

Thus $l^2 + m^2 + n^2 =$

$$\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{PQ^2} = \frac{PQ^2}{PQ^2} = 1$$

So, $l^2 + m^2 + n^2 = 1$ is the relation between the direction cosines of a straight line.

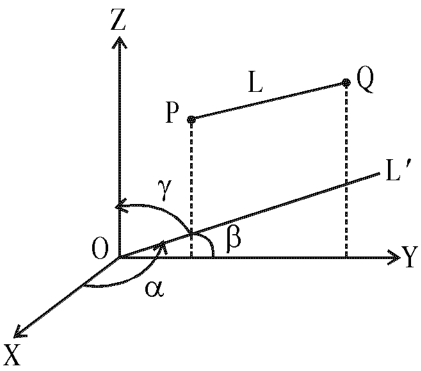


Fig. 7.7.1

Corollarly : The direction cosines of the x-aixs, y-axis and z-axis are (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively.

Example : Find the direction cosines of a straight line joining the points (2, 4, –2) and (3, 6, 2). In previous section we have the distance between these two points $PQ = \sqrt{21}$ units.

So, direction cosines of PQ will be,

$$l = \frac{x_2 - x_1}{PQ} = \frac{3-2}{\sqrt{21}} = \frac{1}{\sqrt{21}}, \; m = \frac{y_2 - y_1}{PQ} = \frac{6-4}{\sqrt{21}} = \frac{2}{\sqrt{21}}$$

$$\text{and } n = \frac{z_2 - z_1}{PQ} = \frac{2-(-2)}{\sqrt{21}} = \frac{4}{\sqrt{21}}.$$

So, direction cosines of PQ are $\left(\dfrac{1}{\sqrt{21}}, \dfrac{2}{\sqrt{21}}, \dfrac{4}{\sqrt{21}}\right)$.

7.8 Direction ratios of a straight line

The numbers which are proportional to the direction cosines are called direction ratios of the straight line. If a, b, c are any three numbers which satisfy the relation

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}},$$

then a, b, c are called direction ratios of the straight line which has l, m, n as direction cosines.

$$\therefore\; l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

'+' and '–' sign can be chosen as the direction of the straight line.

7.9 Projection of line joining two points on a given straight line

Let P and Q have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) and (l, m, n) are the direction cosines of a given straight line L.

Then projection of PQ on x-aixs $= x_2 - x_1$

Similarly projection of PQ on y-axis $= y_2 - y_1$ and projection of PQ on z-axis $= z_2 - z_1$.

So, length of projection MN of PQ on L will be

$MN = (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.

Note : If we consider $\overrightarrow{PQ}$ as vector $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ and $\vec{L} = l\hat{i} + m\hat{j} + n\hat{k}$, then $MN = \overrightarrow{PQ}.\vec{L} = (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$ is the length of projection of $\overrightarrow{PQ}$ on $\vec{L}$.

7.10 Angle between two straight lines having direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2)

If L_1 and L_2 are two straight lines with (l_1, m_1, n_1) and (l_2, m_2, n_2) as direction cosines, then in vector form we can write $\overrightarrow{L_1} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$ and $\overrightarrow{L_2} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$.

If θ is the angle between the two straight lines, then $\vec{L_1}.\vec{L_2} = |\vec{L_1}||\vec{L_2}|\cos\theta$

$\therefore\ l_1l_2 + m_1m_2 + n_1n_2 = \left(\sqrt{l_1^2 + m_1^2 + n_1^2}\right)\left(\sqrt{l_2^2 + m_2^2 + n_2^2}\right)\cos\theta$

$\therefore\ \cos\theta = (l_1l_2 + m_1m_2 + n_1n_2)$ and $\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$.

Corollary :

(1) If the two straight lines are perpendicular then $\theta = \pi/2$ and hence $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

(2) If the two straight lines are parallel then $\theta = 0^\circ$ or 180°, and using Lagrange's identity, we get $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

(3) If the two straight lines have direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively, then they are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ respectively.

Note : Lagrange's identity : $(l_2m_1 - m_2l_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0$.

7.11 Illustrated examples

1. Show that (3, 2, 2), (2, 1, 2) are equidistant from (1, 3, 4).

Solution : Let P : (3, 2, 2) and Q : (2, 1, 2) and A : (1, 3, 4).

$\therefore$ AP $=\sqrt{(3-1)^2+(2-3)^2+(2-4)^2}$ = 3 units

AQ = $\sqrt{(2-1)^2+(1-3)^2+(2-4)^2}$ = 3 units

$\therefore$ AP = AQ (proved).

2. Prove that the triangle whose vertices are (2, 3, 1)(–2, 2, 0) and (0, 1, – 1) is a right angled triangle. Find the other angles also.

Solution : Let A : (2, 3, 1), B : (–2, 2, 0) and C : (0, 1, –1)

Then direction cosines of BA, BC and CA are,

$$\left(\frac{4}{\sqrt{18}},\frac{1}{\sqrt{18}},\frac{1}{\sqrt{18}}\right);\left(\frac{2}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right);\left(\frac{2}{\sqrt{12}},\frac{2}{\sqrt{12}},\frac{2}{\sqrt{12}}\right).$$

Angle between BA and BC $=\cos^{-1}\left(\frac{8-1-1}{\sqrt{18}.\sqrt{6}}\right)=\cos^{-1}\frac{1}{\sqrt{3}}$.

Angle between BC and CA = $\cos^{-1}\left(\frac{4-2-2}{\sqrt{6}\sqrt{12}}\right)=\cos^{-1}(0)=90^{\circ}$,

So, the triangle is right angled triangle.

Angle between CA and BA = $\cos^{-1}\left(\frac{8+2+2}{\sqrt{18}\sqrt{12}}\right)=\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$.

3. Find the angle between the straight lines whose direction cosines are related by $3lm-4ln+mn=0$ and $l+2m+3n=0$.

Solution : $l+2m+3n=0$ i.e., $n=-\frac{(l+2m)}{3}$.

Putting the value of n in $3lm-4ln+mn=0$ we get

$$3lm+4l\left(\frac{l+2m}{3}\right)-m\left(\frac{l+2m}{3}\right)=0$$

or, $4l^2+16lm-2m^2=0$

or, $2\left(\frac{l}{m}\right)^2+8\left(\frac{l}{m}\right)-1=0$

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are direction cosines of the two straight lines, then the roots of the above equation are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

Thus $\frac{l_1}{m_1}.\frac{l_2}{m_2}=-\frac{1}{2}\Rightarrow\frac{l_1l_2}{-1}=\frac{m_1m_2}{2}$. .....(1)

Similarly, omitting l from the two equations we get $\left(\frac{m}{n}\right)^2 - 2 = 0$.

If $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are the two roots then $\frac{m_1}{n_1}.\frac{m_2}{n_2} = -2$

$\therefore \frac{m_1m_2}{2} = \frac{n_1n_2}{-1}$ .....(2)

From (1) and (2) we get $\frac{l_1l_2}{-1} = \frac{m_1m_2}{2} = \frac{n_1n_2}{-1} = k$ (say)

$\therefore l_1l_2 + m_1m_2 + n_1n_2 = -k + 2k - k = 0$

$\therefore \cos\theta = 0$, which implies the straight lines are perpendicular to each other.

7.12 Summary

In this unit, we have been introduced to the concepts of three dimensional geometry engaging cartesian, cylindrical and spherical coordinate system. We also now can determine distance between two points and the point dividing a line segment joining two given points using cartesian coordinate system in a given ratio. Moreover, we have learnt to determine direction cosines and ratios of a straight line angle between two straight lines, projection of a line joining two points on a given straight line.

7.13 Exercises

1. Find the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.
   [Hints : Use centroid divides the media in 2 : 1 ratio)
2. Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$, where α, β, γ are the angles made by a straight line on axes.
   [Hints : Use $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$)
3. Find the ratio in which xy-plane divides the line joining the points (2, –3, 5) and (7, 1, 3).
   [Hints : Use on xy-plane coordinate of z = 0]
4. Find the projection of P(2, 4, –7) and Q(5, 2, –4) on the line joining R(4, –1, 2) and S(2, 0, 3).
   [Hints : Find direction cosines of RS and then use the formula)
5. Find the angle between two straight lines, whose direction cosines are related by $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$
6. Direction ratios of two straight lines are (5, –12, 13) and (–3, 4, 5). Find the angle between the straight lines.

Answers :

1. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

3. $-\frac{5}{3}$

4. $\frac{7}{\sqrt{6}}$

5. $\cos^{-1}\left(\frac{1}{6}\right)$

6. $\cos^{-1}\left(\frac{1}{65}\right)$

Unit 8 ❐ Planes

Structure

8.0 Objectives

We will be able to determine

- equation of a plane in various forms
- perpendicular distance of a point from a plane
- angle between two planes
- position of a point with respect to a plane
- planes bisecting the angles between two planes
- planes through the common line of two planes
- volume of a tetrahedron.

8.1 Introduction

A plane is a three dimensional object which has no bending, i.e., whose curvature is zero. It is actually a surface with zero curvature in three dimensional space. So we can say that, a plane is a flat surface such that if any two points are taken on the surface and joined by a straight line, the straight line lies wholly on the surface.

8.2 Equation of a plane in certesian form

A first degree equation in x, y, z i.e every linear equation with three variables represents a plane.

The most general equation of the first degree in x, y, z is of the form

$ax + by + cz + d = 0$ .....(8.2.1)

where a, b, c, d are constants and they are not all zero.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the surface represented by the equation (8.2.1). Then we get

$ax_1 + by_1 + cz_1 + d = 0$ .....(8.2.2)

$ax_2 + by_2 + cz_2 + d = 0$ .....(8.2.3)

Multiplying (8.2.3) by λ and this adding to (8.2.2), we obtain

$a(x_1 + \lambda x_2) + b(y_1 + \lambda y_2) + c(z_1 + \lambda z_2) + d(1 + \lambda) = 0$

Now dividing by $(1 + \lambda)$, provided $\lambda \neq -1$,

$$a\frac{x_1+\lambda x_2}{1+\lambda}+b\frac{y_1+\lambda y_2}{1+\lambda}+c\frac{z_1+\lambda z_2}{1+\lambda}+d=0$$

It shows that the point $\left(\frac{x_1+\lambda x_2}{1+\lambda},\frac{y_1+\lambda y_2}{1+\lambda},\frac{z_1+\lambda z_2}{1+\lambda}\right)$ lies on the surface. Again this point divides this line PQ in the ratio 1 : λ, where λ may have any value. Therefore all the points on the line PQ lie on the surface, i.e., the line lies wholly on the surface. Hence equation (8.2.1) represents a plane.

Q. What happens when λ = –1?

Note : The general equation of a plane $ax + by + cz + d = 0$ can be written as

$\frac{a}{d}x+\frac{b}{d}y+\frac{c}{d}z+1=0$, having three arbitray constants $\frac{a}{d},\frac{b}{d},\frac{c}{d}$, for $d \neq 0$.

Corollary I : $ax + by + cz = 0$ represents a plane passing through the origin.

Corollary II : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ represents a plane passing through (x_1, y_1, z_1).

Corollary III : $x = 0$, $y = 0$, $z = 0$ represent the co-ordinate planes. i.e., yz-plane, zx-plane, xy-plane respectively.

Corollary IV : x = constant, y = constant, z = constant represent parallel planes to the co-ordinate planes. These planes are perpendicular to the axes and they are $by + cz + d = 0$, $ax + cz + d = 0$, $ax + by + d = 0$ respectively. These planes are perpendicular to the yz-plane, zx-plane and xy-plane respectively.

8.3 Equations of planes in different forms

8.3.1 Normal form

Let ABC be a plane in the coordinate system OX, OY and OZ. ON is perpendicular to the plane. ON is called the normal to the plane. Let l, m, n be the d.c.s of ON and ON = p. P(x, y, z) is an arbitray point on the plane. PM is perpendicular to XOY-plane. LM is parallel to OY.

$\therefore$ OL = x, LM = y MP = z

By the property of projection,

projection of OP on ON = projection of OL on ON + projection of LM on ON + projection of MP on ON.

Since ON is perpendicular to the plane ABC, ON is perpendicular to NP.

$\therefore$ Projection of OP on ON= ON = p

Again, projection of OL on ON = lx

projection of LM on ON = my

projection of MP on ON = nz

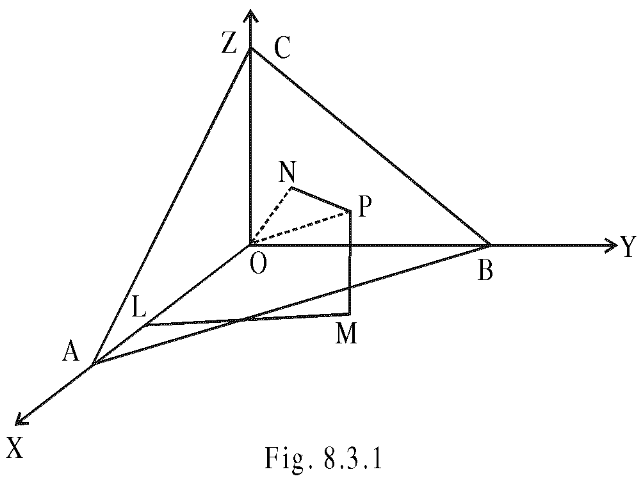


Fig. 8.3.1

$\therefore$ $lx + my + nz = p$ .....(8.3.1)

P is any point on the plane.

Hence it is the equation of the plane in normal form.

If the normal ON makes angles α, β, γ with the coordinates, then the equation of the plane is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

Corollary I : Comparing the general equation of the first degree ax + by + cz + d = 0 with the $lx + my + nz = p$, we see that

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d}$$

$$\therefore \quad l^2 + m^2 + n^2 = \frac{a^2+b^2+c^2}{d^2}p^2$$

$$\text{or,} \quad p^2\frac{a^2+b^2+c^2}{d^2} = 1$$

or $\quad p^2 = \dfrac{d^2}{a^2+b^2+c^2}$

or $\quad p = \pm\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

Since p is always positive, $p = \dfrac{d}{\sqrt{a^2+b^2+c^2}}$, when d is positive;

$p = -\dfrac{d}{\sqrt{a^2+b^2+c^2}}$, when d is negative.

Now, $l = \mp\dfrac{a}{\sqrt{a^2+b^2+c^2}}$; $m = \mp\dfrac{b}{\sqrt{a^2+b^2+c^2}}$; $n = \mp\dfrac{c}{\sqrt{a^2+b^2+c^2}}$

according as d is positive or negative.

So the normal form of the general equation is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \mp\frac{d}{\sqrt{a^2+b^2+c^2}}$$

according as d is positive or negative.

This is the reduced normal form of the general equation of the plane.

Corollary II : The equation $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ represents a plane through the point (x_1, y_1, z_1), where a, b, c are the direction ratios of the normal to the plane.

Note 1 : Perpendicular distance from the origin to $ax + by + cz + d = 0$ is $\dfrac{\pm d}{\sqrt{a^2+b^2+c^2}}$ according as d is positive or negative.

Note 2 : The direction cosines of the normal to the plane $ax + by + cz + d = 0$ are $\mp\left(\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}\right)$ according as d is positive or negative. a, b, c are direction ratios of the normal to the plane.

8.3.2 Intercept form

Let $ax + by + cz + d = 0$ be the equation of the plane which meets the coordinate axes at A, B, C and it makes interepts α, β, γ on the coordinate axes. Then the points $(\alpha, 0, 0)$, $(0, \beta, 0)(0, 0, \gamma)$ lie on the plane.

Then $a\alpha + d = 0$

or, $\alpha = -\dfrac{d}{a}$;

Similarly, $\beta = -\frac{d}{b}, \gamma = -\frac{d}{c}$.

Then the equation of the plane can be written as

$ax + by + cz = -d$

or $\frac{x}{-\frac{d}{a}} + \frac{y}{-\frac{d}{b}} + \frac{z}{-\frac{d}{c}} = 1$

or $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

It is called the equation of a plane in the intercept for m.

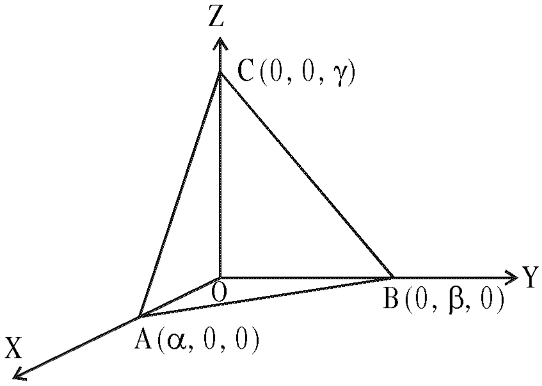


Fig. 8.3.2.1

Illustrated Examples :

Q1. Express the equation of the plane $2x + 6y - 3z + 5 = 0$ in the normal form and hence obtain the length of perpendicular from the origin upon the plane.

Solution : Here $d = 5, a = 2, b = 6, c = -3$

$\therefore \sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + 6^2 + (-3)^2} = 7$

Dividing each term of the given equation by -7, the required normal for m is

$-\frac{2}{7}x - \frac{6}{7}y + \frac{3}{7}z = \frac{5}{7}$ and the perpendicular distance from the origin = $\frac{5}{7}$.

Q2. Find the intercepts made by the plane $x - 2y + 3z - 18 = 0$ on the axes.

Solution : The equation of the plane is

$$x - 2y + 3z - 18 = 0$$

or, $$x - 2y + 3z = 18$$

or $$\frac{x}{18} + \frac{y}{-9} + \frac{z}{6} = 1.$$

So, the intercepts on the x, y, z axes are 18, –9, 6 respectively.

8.4 Equation of a plane passing through three given points

Let $ax + by + cz + d = 0$ (8.4.1)

represents a plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Therefore we get,

$$ax_1 + by_1 + cz_1 + d = 0 \quad (8.4.2)$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad (8.4.3)$$

$$ax_3 + by_3 + cz_3 + d = 0 \quad (8.4.4)$$

Eliminating a, b, c, d from these four equations (8.4.1), (8.4.2), (8.4.3), (8.4.4) we have,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & y_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \cdot$$

It is the required form.

Corollary I : The condition that the four points (x_i, y_i, z_i), i = 1, 2, 3, 4, no three of which are collinear, are coplanar iff $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & y_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \cdot$

Illustrated Example :

Q1. Find the equation of the plane passing through (0, 2, 4), (3, 1, 1) and (2, 0, –1).

Solution : The required equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & 4 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 0 & -1 & 1 \end{vmatrix} = 0$$

or, $x\begin{vmatrix} 2 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - y\begin{vmatrix} 0 & 4 & 1 \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} + z\begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 0$

or $x - 9y + 4z + 2 = 0$.

8.5 Perpendicular distance of a point from a plane

Let $ax + by + cz + d = 0$ ......(8.5.1)

be the equation of a plane and (x_1, y_1, z_1) be a given point. Then the equation of the plane parallel to (8.5.1) and passing through the point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

or, $ax + by + cz - (ax_1 + by_1 + cz_1) = 0$ ......(8.5.2)

The distance of (8.5.1) and (8.5.2) from the origin are

$\pm\dfrac{d}{\sqrt{a^2+b^2+c^2}}$ and $\mp\dfrac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}$ according as d is positive or negative.

$$\therefore \text{ The required distance } = \left|\pm\frac{d}{\sqrt{a^2+b^2+c^2}}-\left(\mp\frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}\right)\right|$$

$$=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Note : If d is positive, $ax_1+by_1+cz_1+d$ will be positive when (x_1, y_1, z_1) and origin are on the same side of the plane $ax+by+cz+d=0$ and negative when they are on opposite sides of the plane.

Illustrated Example :

Q1. Find the distance of the point (1, 2, –3) from the plane $5x-3y+z+5=0$.

Solution : The required distance $=\dfrac{|5.1-3.2+1.(-3)+5|}{\sqrt{5^2+(-3)^2+(1)^2}}=\dfrac{|5-6-3+5|}{\sqrt{25+9+1}}=\dfrac{1}{\sqrt{35}}$.

8.6 Angle between two planes

Angle between two planes is defined as the angle between the normals to the planes drawn through a point.

Let $a_1x+b_1y+c_1z+d_1=0$ .....(8.6.1)

and $a_2x+b_2y+c_2z+d_2=0$ .....(8.6.2)

be the equations of two planes. The direction ratios of the normals to the planes are a_1, b_1, c_1 and a_2, b_2, c_2. If θ be the angle between them, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,.\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Corollary I : Condition for perpendicular planes i.e. when θ = 90°, $a_1a_2+b_1b_2+c_1c_2=0$.

Corollary II : Condition for parallel planes $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

Illustrated Example :

Q1. Find the angle between the planes $2x - y + 3z + 7 = 0$ and $x - 2y - 3z + 8 = 0$.

Solution : Angle between the planes $= \cos^{-1} \dfrac{2.1+(-1).(-2)+3.(-3)}{\sqrt{2^2+(-1)^2+3^2}.\sqrt{1^2+(-2)^2+(-3)^2}}$

$$= \cos^{-1} \frac{2+2-9}{\sqrt{14}.\sqrt{14}}$$

$$= \cos^{-1}\left(\frac{-5}{14}\right).$$

8.7 Position of a point with respect to a plane

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, they do not lie on the plane $ax + by + cz + d = 0$, they lie on the same side or different sides of the given plane according as the two expressions $(ax_1 + by_1 + cz_1 + d)$ and $(ax_2 + by_2 + cz_2 + d)$ are of the same sign or of oppositie signs.

Suppose P and Q are of opposite sides of the given plane. Then PQ meets the plane internally at a point R where $PR : RQ = \lambda : \mu;(\lambda > 0, \mu > 0)$.

The coordinates of R $\left(\dfrac{\lambda x_2+\mu x_1}{\lambda+\mu}, \dfrac{\lambda y_2+\mu y_1}{\lambda+\mu}, \dfrac{\lambda z_2+\mu z_1}{\lambda+\mu}\right)$ and it is on the plane $ax + by + cz + d = 0$.

Therefore, $a\left(\dfrac{\lambda x_2+\mu x_1}{\lambda+\mu}\right)+b\left(\dfrac{\lambda y_2+\mu y_1}{\lambda+\mu}\right)+c\left(\dfrac{\lambda z_2+\lambda z_1}{\lambda+\mu}\right)+d=0$

or, $\lambda(ax_2 + by_2 + cz_2 + d) = -\mu(ax_1 + by_1 + cz_1 + d)$

or, $\dfrac{\lambda}{\mu} = -\dfrac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}$, which ensures that $(ax_1 + by_1 + cz_1 + d)$ and $(ax_2 + by_2 + cz_2 + d)$ are of opposite signs.

Again when P and Q be on the same side of the plane, then PQ meets the plane externally at the point R.

Therefore, for $PR : RQ = \lambda : \mu$, the coordinates of R are

$\left(\dfrac{\lambda x_2-\mu x_1}{\lambda-\mu}, \dfrac{\lambda y_2-\mu y_1}{\lambda-\mu}, \dfrac{\lambda z_2-\mu z_1}{\lambda-\mu}\right)$ and we have $\dfrac{\lambda}{\mu} = \dfrac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}$

i.e., $(ax_1 + by_1 + cz_1 + d)$ and $(ax_2 + by_2 + cz_2 + d)$ are of same sign.

8.8 Planes bisecting the angles between two planes

Let the equations of the planes be

$a_1x + b_1y + c_1z + d_1 = 0$ .....(8.8.1)

$a_2x + b_2y + c_2z + d_2 = 0$ .....(8.8.2)

Let $P(x_1, y_1, z_1)$ be any point on either of the two bisecting planes, then P is equidistant from the two planes;

i.e. $\dfrac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \dfrac{a_2x_1 + b_2y_1 + c_2z_1 + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

Hence the equation of plane which passing through P is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad .....(8.8.3)$$

Now if d_1 and d_2 are of same sign then the equation of plane bisecting that angle between the planes which contains the origin. If the origin lies in the acute angle, then the above plane bisects the acute angle, otherwise it will bisect the obtuse angle.

If $\cos\theta$ is negative, i.e., $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then the origin is within the acute angle between the planes. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then the origin is within the obtuse angle between the planes.

Illustrated Example :

Q.1. Find the equation of the plane which bisects the acute angle between planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$.

Solution : The equation of the given planes with constants of same sign are

$-x - 2y - 2z + 9 = 0$ .....(1)

$4x - 3y + 12z + 13 = 0$ .....(2)

Here $(-1).4 + (-2).(-3) + (-2).12 = 22 < 0$.

$\therefore$ The origin is within the acute angle between the planes. Hence the equation of the plane bisecting the acute angle between the planes is

$$\frac{-x - 2y - 2z + 9}{\sqrt{(-1)^2 + (-2)^2 + (-2)^2}} = \frac{4x - 3y + 12z + 13}{\sqrt{(4)^2 + (-3)^2 + (12)^2}}$$

or, $\dfrac{-x - 2y - 2z + 9}{3} = \dfrac{4x - 3y + 12z + 13}{13}$

or, $25x + 17y + 62z - 78 = 0$.

8.9 Equation of a plane passing through the common line of the two planes

Let the two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad .....(8.9.1)$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \qquad .....(8.9.2)$$

The equation $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$ (8.9.3)

represents a plane, since it is of first degree. Hence λ is a variable parameter and again any point satisfying the equation (8.9.1) and (8.9.2) must satisfy the equation (8.9.3). Therefore the equation (8.9.3) represents a plane passing through the line of intersection of the planes (8.9.1) and (8.9.2).

Note : A second degree equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes only if it can be resolved into two linear factors i.e. if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

The angle between these two planes is

$$\tan^{-1}\frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}.$$

Illustrated Example :

1. Find the equation of the plane passing through the point (–1, 0, 1) and the line of intersection of the planes $4x - 3y + 1 = 0$ and $y - 4z + 13 = 0$.

Solution : The equation of a plane passing through the line of intersection of the two planes can be written as

$$(4x - 3y + 1) + \lambda(y - 4z + 13) = 0, \text{ where } \lambda \text{ is a constant}$$

If passes through the point (–1, 0, 1), then

$$\{4(-1) - 3.0 + 1\} + \lambda(0 - 4.1 + 13) = 0$$

i.e. $\lambda = \frac{1}{3}$.

Therefore the required equation is

$$(4x - 3y + 1) + \frac{1}{3}(y - 4z + 13) = 0$$

or, $12x - 8y - 4z + 16 = 0$

or, $3x - 2y - z + 4 = 0$.

8.10 Volume of a tetrahedron

Let ABCD be a tetrahedron where $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are vertices.

Now equation of the plane passing through B, C, D is $\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$

$$\text{or, } x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}.$$

Now if the area of the triangle BCD is Δ and let P be the perpendicular distance from the point A to the BCD plane (or triangle), then the volume of the tetrahedron ABCD is

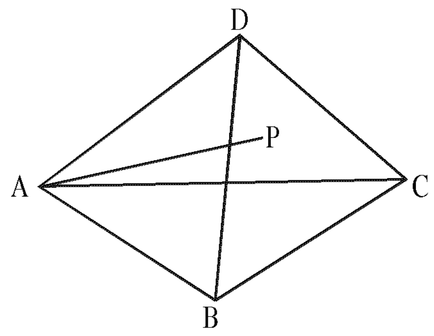


Fig. 8.9.1

$$\frac{1}{6}P\Delta = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

where $P = \dfrac{\text{Volume of ABCD}}{2.\Delta}$.

Illustrated Example

1. Find the volume of a tetrahedron ABCD whose vertices are A(2, 3, 0), B(–1, 2, 5), C(5, 0, 0), D(2, 0, 7).

Solution : Volume of the tetrahedron $= \dfrac{1}{6}\begin{vmatrix} 2 & 3 & 0 & 1 \\ -1 & 2 & 5 & 1 \\ 5 & 0 & 0 & 1 \\ 2 & 0 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{6}\left[2\begin{vmatrix}2 & 5 & 1\\0 & 0 & 1\\0 & 7 & 1\end{vmatrix}+3\begin{vmatrix}5 & -1 & 1\\0 & 5 & 1\\7 & 2 & 1\end{vmatrix}+0\begin{vmatrix}-1 & 2 & 1\\5 & 0 & 1\\2 & 0 & 1\end{vmatrix}-\begin{vmatrix}-1 & 2 & 5\\5 & 0 & 0\\2 & 0 & 7\end{vmatrix}\right]$$

$$= \frac{1}{6}[2.(-14) + 3(-27) + 0 + 70]$$

$$= \frac{1}{6}[-28 - 81 + 70]$$

$$= \frac{1}{6}[-109 + 70] = \frac{1}{6}[-39] = -\frac{13}{2} \text{ unit}^3.$$

8.11 Worked out Examples

1. Find the intercepts made on the coordinate axes by the plane $x + 2y - 2z = 6$. Find also the direction cosines of the normal to the plane and the length from the origin.

Solution : The given equation can be written in intercept form as $\frac{x}{6}+\frac{y}{3}+\frac{z}{-3}=1$.

Thus the intercepts on the coordinate axes are (6, 3, –3). Now we reduce the given equation of the plane in normal form. We divide the equation by $\sqrt{1^2+2^2+(-2)^2}$; i.e. 3 and we obtain, $\frac{1}{3}x+\frac{2}{3}x+\left(-\frac{2}{3}\right)z=2$.

Therefore the direction cosines of the normal to the plane are $\left(\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right)$.

The length of the perpendicular from the origin to the plane is 2 units.

2. Find the equation of the plane passing through the points (1, 1, 2) and (2, 4, 3) and perpendicular to the plane $x - 3y + 7z + 5 = 0$.

Solution : The equation of a plane passing through the point (1, 1, 2) is

$$a(x - 1) + b(y - 1) + c(z - 2) = 0 \qquad .....(1)$$

If it passes through the point (2, 4, 3), then

$$a(2 - 1) + b(4 - 1) + c(3 - 2) = 0$$

$$\text{or} \quad a + 3b + c = 0. \qquad .....(2)$$

Since the plane (1) is perpendicular to the plane $x - 3y + 7z + 5 = 0$, then we get

$$a - 3b + 7c = 0 \qquad .....(3)$$

Now eliminating a, b & c from (1), (2) and (3), we get the equation of the required plane as

$$\begin{vmatrix} x-1 & y-1 & z-2 \\ 1 & 3 & 1 \\ 1 & -3 & 7 \end{vmatrix} = 0$$

or, $4x - y - z - 1 = 0$ (Ans).

3. Find the equation of the plane passing through the point (2, 5, –8) and perpendicular to each of the planes $2x - 3y + 4z + 1 = 0$ and $4x + y - 2z + 6 = 0$.

Solution : Let the equation of the plane through the point (2, 5, –8) be $a(x - 2) + b(y - 5) + c(z + 8) = 0$.

Since it is perpendicular to each of the given planes.

$$2a - 3b + 4c = 0 \text{ and } 4a + b - 2c = 0.$$

Therefore, $\dfrac{a}{1} = \dfrac{b}{10} = \dfrac{c}{7} = k$ (say)

∴ The required equation of the plane is

$$k(x - 2) + 10k(y - 5) + 7k(z + 8) = 0$$

or, $x + 10y + 7z + 4 = 0$ (Ans.)

4. Show that the points (2, 3, –5) and (3, 4, 7) lie on the opposite sides of the plane $x + 2y - 2z = 9$.

Solution : Perpendicular distances from the given points to the plane are

$$\frac{2+2.3+2.5-9}{\sqrt{1^2+2^2-(-2)^2}} = 3 \text{ and } \frac{3+2.4-2.7-9}{\sqrt{1^2+2^2+(-2)^2}} = -4 .$$

Since these one of opposite signs the points lie on the opposite sides of the plane.

5. Find the equation of the plane parallel to the plane $2x - 2y - z - 3 = 0$ and situated at a distance of 7 units from it.

Solution : Let the equation of the plane parallel to the given plane be $2x - 2y - z + k = 0$.

Then the distance of the origin from these two planes are $\dfrac{-3}{\sqrt{2^2+(-2)^2+(-1)^2}}$ and

$$\frac{k}{\sqrt{2^2+(-2)^2+(-1)^2}} .$$

Now if the required plane be at a distance of 7 units from the given plane, then we have

$$\frac{k}{\sqrt{9}} \pm \left(\frac{-3}{\sqrt{9}}\right) = 7$$

or, $\frac{k}{3} \mp \left(\frac{3}{\sqrt{9}}\right) = 7$

or, k = 24 or, 18.

Hence the required planes are $2x - 2y - z + 24 = 0$ and $2x - 2y - z + 18 = 0$ (Ans.)

6. Find the distance between the two parallel planes $2x + 5y + 4z - 12 = 0$ and $2x + 5y + 4z + 6 = 0$.

Solution : The distance between two parallel planes is the difference between the distances of any point from the planes.

Let the point be the origin. The distance of the origin (0, 0, 0) from the planes $2x + 5y + 4z - 12 = 0$ and $2x + 5y + 4z + 6 = 0$ are respectively.

$$\frac{-12}{\sqrt{2^2 + 5^2 + 4^2}} = -\frac{12}{\sqrt{45}} \text{ and } \frac{6}{\sqrt{2^2 + 5^2 + 4^2}} = \frac{6}{\sqrt{45}}.$$

Hence the distance between the two given parallel planes is

$$\frac{6}{\sqrt{45}} - \left(-\frac{12}{\sqrt{45}}\right) = \frac{18}{\sqrt{45}} = \frac{6}{\sqrt{5}} = \frac{6}{5}\sqrt{5} \text{ units (Ans.)}$$

7. Find the equation of the plane which is perpendicular to the plane $x + 2y - z + 1 = 0$ and which contains the line of intersection of the plane $x + 2y + 3z - 4 = 0$ and $2x + y + z + 2 = 0$.

Solution : The equation of the plane through the line of intersection of the given planes can be written as

$(x + 2y + 3z - 4) + \lambda(2x + y + z + 2) = 0$

or, $(1 + 2\lambda)x + (2 + \lambda)y + (3 + \lambda)z + 2\lambda - 4 = 0$, where λ is an arbitray constant.

Since it is perpendicular to $x + 2y - z + 1 = 0$, then

$(1 + 2\lambda).1 + (2 + \lambda).2 + (3 + \lambda).(-1) = 0$

or $\lambda = -\frac{2}{3}$.

$\therefore$ The required equation is $(x + 2y + 3z - 4) - \frac{2}{3}(2x + y + z + 2) = 0$

or, $x - 4y - 7z + 16 = 0$ (Ans.)

8. Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represent a pair of planes. Find the angle between them.

Solution : The given equation being homogeneous and of second degree will represent two planes, if it can be resolved into two linear factors.

Therefore, from the given equation we obtain

$2x^2 + x(2z + y) - 6y^2 - 12z^2 + 18yz = 0$ as the equation for x.

or, $4x = -(y + 2z) \pm \sqrt{(y+2z)^2 - 4.2(-6y^2 - 12z^2 + 18yz)}$

or, $4x = -(y + 2z) \pm \sqrt{100z^2 - 140yz + 49y^2}$

or, $4x = -(y + 2z) \pm (10z - 7y)$

$= -8y + 8z$ or, $6y - 12z$

$\therefore$ $x + 2y - 2z = 0$ and $2x - 3y + 6z = 0$ are the pair of planes. (Ans.)

Now the angle θ between them is

$$\cos\theta = \frac{1.2 + 2.(-3) + (-2).6}{\sqrt{1^2 + 2^2 + (-2)^2}.\sqrt{2^2 + (-3)^2 + (6)^2}}$$

or, $\cos\theta = \dfrac{-16}{3.7}$

or, $\cos\theta = -\dfrac{16}{21}$

or, $\theta = \cos^{-1}\left(\dfrac{16}{21}\right)$ (Ans.)

9. Find the equation of the plane passing through the three points (2, 2, –1), (3, 4, 2) and (7, 0, 6).

Solution : The equation of the plane passing through the point (2, 2, –1) is

$a(x - 2) + b(y - 2) + c(z + 1) = 0$ .....(1)

It passes through the points (3, 4, 2) and (7, 0, 6) then we get

$a + 2b + 3c = 0$ .....(2)

and $5a - 2b + 7c = 0$ .....(3)

Eliminating a, b and c from (1), (2) and (3), we get the equation of the required plane as

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

or, $(x - 2).20 + (y - 2).8 + (z + 1).(-12) = 0$

or, $20x + 8y - 12y - 68 = 0$

or, $5x + 2y - 3z - 17 = 0$ (Ans.)

10. A variable plane at a constant distance p from the origin meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$.

Solution : Let the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Since its distance from the origin is p, then

$$p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

or, $$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \qquad .....(1)$$

The plane meets the axes at A(a, 0, 0), B(0, b, 0) and C(0, 0 c).

If (α, β, γ) be the centroid of the tetrahedron OABC, (O, the origin),

Then $\alpha=\frac{a}{4}, \beta=\frac{b}{4}, \gamma=\frac{c}{4}$.

From (1), $\frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}=\frac{1}{p^2}$

Hence the locus of (α, β, γ) is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$ (proved).

8.12 Summary

In this unit, we have learnt how to determine the equations of plane under various conditions including normal form, intercept form etc. Apart from this, we are now able to find out perpendicular distance of a point from a plane, angle between two planes, plane bisecting angles between to given planes. We can also determine the position of a point with respect to a given plane. The determination of planes through the common line of two planes and volume of a tetrahedron are also known to us.

8.13 Exercises

1. Find the intercepts made by the plane $x - y + z = 2$ on the coordinate axes. [Hints : see the worked out example - 1] [**Ans.** 2, – 2, 2]

2. Find the equation of the plane passing through the points (3, 5, 1) (2, 3, 0) and (0, 6, 0). [Hints : see the workedout example - 9] [**Ans.** $3x + 2y - 7z - 12 = 0$]

3. Find the equation of the plane passing through (4, 1, 1) and parallel to $3x - 4y + 7z + 5 = 0$. [Hints see the condition for parallel planes in Coro. II of 8.6]
[**Ans.** $3x - 4y + 7z - 15 = 0$]
4. Find the equation of plane through (2, 0, –1) and perpendicular to the line whose d.rs. are 3, 4, –2. [**Ans.** $3x + 4y - 2z - 8 = 0$]
5. Find the angle between the planes $2x + y + z = 6$ and $x - y + 2z = 3$. [Hints : see the illustrated example of 8.6] $\left[\textbf{Ans.} \frac{\pi}{3}\right]$
6. Find the distance between the parallel planes $x - 4y + 8z - 9 = 0$ and $x - 4y + 8z + 18 = 0$.
[Hints : distances of the planes from the origin are $\frac{-9}{9} = -1$ and $\frac{18}{9} = 2$. $\therefore$ The required distances = 2–(–1) = 3 units] [**Ans.** 3 units]
7. A variable plane ABC at a constant distance 3p from the origin meets the axes at A, B, C. Show that the locus of a centroid of the triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$. [Hints : See the worked out example - 10]
8. Find the value of a tetrahedron which creates a plane $\frac{x}{2} + \frac{y}{6} + \frac{z}{7} = 1$ with the axes.
[Hints : Vertices of tetrahedron are (0, 0, 0), (2, 0, 0), (0, 6, 0), (0, 0, 7)]
[**Ans.** 14 unit^3]
9. Find the equation of the plane passing through the intersection of the planes $2x + y + 2z = 9$ and $4x - 5y - 4z = 1$ and the point (3, 2, –1). [Hints : The required equation $(2x + y + 2z - 9) + \lambda(4x - 5y - 4z - 1) = 0$, then by putting (3, 2, –1), calculate λ] [**Ans.** $11x - 5y - z = 24$]
10. Show that $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes and find the angle between them. [Hints : see the workedout example - 8]
[**Ans.** $x + 2y - 2z = 0$; $2x - 3y + 6z = 0$ and $\cos^{-1}(16/21)$]
11. Show that the equation $2x^2 - y^2 + 3z^2 - xy + 7xz + 2yz = 0$ represents two planes and the angle between the two planes is $\tan^{-1}\left(\frac{5\sqrt{2}}{4}\right)$. [Hints : see the worked out example- 8 and see the note of 8.9][**Ans.** $2x + y + z = 0$; $x - y + 3z = 0$]

Unit 9 ❐ Straight lines

Structure

9.0 Objectives

We will be able to understand and determine several aspects of a straight line in three dimension including determiration of

- the straight line through given point with the given direction cosises
- the staight line of intersection of two planes in symetrical form
- intersection of straight line and a plane
- condition of coplanerity of two non-parallel straight lines
- perpendicular distance of a point from a line
- shortest distance between two skew lines
- intersection of three planes
- volume of a tetrahedron

9.1 Introduction

In three dimensional geometry the concept of straight line arises from the intersection of two planes. It is the line joining two points in space without any bent.

9.2 Equation of the straight line passing through a given point with the given direction cosines

Let P(x, y, z) be any point on the straight line and the given point is A(x_1, y_1, z_1) and the direction cosines of the straight line are l, m, n.

Let the distance from the given point A(x_1, y_1, z_1) to the points P(x, y, z) be r. Then $\frac{x-x_1}{r}=l$, $\frac{y-y_1}{r}=m$ and $\frac{z-z_1}{r}=n$. Then we obtain $x-x_1=lr$, $y-y_1=mr$, $z-z_1=nr$.

Therefore $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$

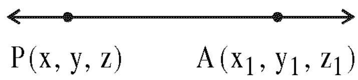


Fig. 9.1.1

This is known as the equation of a line in symmetrical form or the canonical equations or general form of a line.

We also have $x = lr + x_1$, $y = mr + y_1$, $z = nr + z_1$, these are called the parametric equations of the line.

Corallary - I : If the line passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2), then $\frac{x_2-x_1}{l}=\frac{y_2-y_1}{m}=\frac{z_2-z_1}{n}$.

Therefore, the equations of the line becomes $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.

Corallary - II : If the line passes through the origin, the equation of the line will be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

9.3 The equations of a straight line of intersection of two planes in the symmetrical form

Let the planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad .....(9.2.1)$$

and

$$a_2x + b_2y + c_2z + d_2 = 0 \qquad .....(9.2.2)$$

They together represent the straight line of intersection of the planes, since they are satisfied by any point on the line of intersection of the planes. It is the unsymmetrical general form of a line.

Since the line lies on both the planes, it must be perpendicular to the normals to the planes. If l, m, n be the d.c.s of the line, then

$la_1 + mb_1 + nc_1 = 0$ and $la_2 + mb_2 + nc_2 = 0$

From these two relations, $\dfrac{l}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$.

If the line meets the plane z = 0 at (x′, y′, 0), then

$$a_1x' + b_1y' + d_1 = 0 \text{ and } a_2x' + b_2y' + d_2 = 0$$

Solving these equations, we get

$$x' = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} \text{ and } y' = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}.$$

Thus one point on the straight line is $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$.

Hence the equation of the common line in symmetrical form is

$$\frac{x - \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

llustrated Examples

1. Find in the symmetrical form, the equations of the straight line of intersection of two planes x – 2y + 3z = 4, 2x – 3y + 4z = 5 and find its direction cosines.

Solution : Let the direction ratios of the straight line be l, m, n. Since the straight line is perpendicular to the normals of both the planes, we have

$$l - 2m + 3n = 0$$
$$2l - 3m + 4n = 0$$

Then we obtain, $\dfrac{l}{-8+9} = \dfrac{m}{6-4} = \dfrac{n}{-3+4}$

or, $\dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{1}$.

Hence the d.cs of the straight line are (1, 2, 1).

The point where the straight line meets the plane z = 0 is obtained by putting z = 0 in the equations of the straight line.

Thus we get x – 2y = 4 and 2x – 3y = 5,

so that x = –2, y = –3.

So, one point on the straight line is (–2, –3, 0).

Hence the equations of the straight line are $\dfrac{x+2}{1} = \dfrac{y+3}{2} = \dfrac{z}{1}$.

The direction cosines of the straight line are $\dfrac{\pm 1}{\sqrt{6}}, \pm\dfrac{2}{\sqrt{6}}, \pm\dfrac{1}{\sqrt{6}}$.

2. Find the point where the straight line joining the points (2, –3, 1) and (3, –4, –5) cuts the plane $3x + y + z = 8$.

Solution : The equations of the straight line are $\frac{x-2}{3-2} = \frac{y+3}{-4+3} = \frac{z-1}{-5-1}$

or, $\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6}$.

Any point on the straight line is given by

$(2 + r, -3-r, 1 - 6r)$.

If this point lies on the plane $3x + y + z = 8$, then we have $3(2 + r) - 3 - r + 1 - 6r = 8$

or, $r = -1$.

Hence the point is (1, –2, 7).

9.4 Intersection of a straight line and a plane

Let the equations of the plane and the straight line be

$$ax + by + cz + d = 0 \qquad .....(9.3.1)$$

and $$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}. \qquad .....(9.3.2)$$

Any point on the straight line is $(lr + x_1, mr + y_1, nr + z_1)$.

If the line meets the plane at this point, then

$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0 \qquad .....(9.3.3)$$

where r is proportional to the distance of the point from (x_1, y_1, z_1).

Therefore, $$r = -\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn} \qquad .....(9.3.4)$$

Deductions :

(i) The straight line will be parallel to the plane, if $al + bm + cn = 0$.

(ii) If the line lies on the plane, then $al + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d = 0$.

(iii) If the line is perpendicular to the plane, then it is parallel to the normal to the plane.

In this case $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

(iv) If θ is the anlge between the plane and the straight line, then $90^\circ - \theta$ is the angle between the line and the normal to the plane. Thus

$$\cos(90^\circ - \theta) = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

$$\text{i.e., } \theta = \sin^{-1}\left[\frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}\right].$$

Illustrated Example

1. Find the distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$.

Solution : Any point on the line is $(3r + 2, 4r - 1, 12r + 2)$.

If the line meets the plane at this point, then

$3r + 2 - 4r + 1 + 12r + 2 = 5$

or, $r = 0$.

Therefore the point of intersection of the line and the plane is $(2, -1, 2)$.

Now the distance between $(2, -1, 2)$ and $(-1, -5, -10)$ is

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{169} = 13.$$

9.5 Condition of coplanarity of two non-parallel straight lines

Let the lines be $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ .....(9.4.1)

and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$. .....(9.4.2)

If the two lines intersect, then they must lie on a plane and take the plane as $ax + by + cz + d = 0$.

$\therefore\ al_1 + bm_1 + cn_1 = 0$ .....(9.4.3)

$al_2 + bm_2 + cn_2 = 0$ .....(9.4.4)

$ax_1 + by_1 + cz_1 + d = 0$ .....(9.4.5)

and $ax_2 + by_2 + cz_2 + d = 0$ .....(9.4.6)

From (9.4.5) and (9.4.6),

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ .....(9.4.7)

Eliminating a, b, c from (9.4.7), (9.4.3) and (9.4.4), we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad .....(9.4.8)$$

This is the necessary condition of coplanarity of the two straight lines (9.4.1) and (9.4.2).

Again if the condition (9.4.7) holds, then the system of homogeneous equations

$(x_2 - x_1)x + (y_2 - y_1)y + (z_2 - z_1)z = 0$ .....(9.4.9)

$l_1x + m_1y + n_1z = 0$ .....(9.4.10)

$l_2x + m_2y + n_2z = 0$ .....(9.4.11)

has non-trivial solutions.

Let (α, β, γ) be a non-zero solution.

Then $(x_2 - x_1)\alpha + (y_2 - y_1)\beta + (z_2 - z_1)\gamma = 0$ .....(9.4.12)

$l_1\alpha + m_1\beta + n_1\gamma = 0$ .....(9.4.13)

and $l_2\alpha + m_2\beta + n_2\gamma = 0$ .....(9.4.14)

From (9.4.12), $\alpha x_1 + \beta y_1 + \gamma z_1 = \alpha x_2 + \beta y_2 + \gamma z_2 = k$ (say)

i.e. $\alpha x_1 + \beta y_1 + \gamma z_1 - k = 0$ .....(9.4.15)

and $\alpha x_2 + \beta y_2 + \gamma z_2 - k = 0$ .....(9.4.16)

The conditions (9.4.13) and (9.4.15) suggest that line (9.4.1) lies on

$\alpha x + \beta y + \gamma z - k = 0$ .....(9.4.17)

So by the conditions (9.4.14) and (9.4.16), the line (9.4.2) also lies on the plane (9.4.17).

Hence the lines are coplanar. This is the sufficient condition. Therefore the necessary and sufficient condition for the coplanarity of two given non-parallel straight lines is (9.4.8).

Corollary : The equation of the plane on which the lines (9.4.1) and (9.4.2) lie, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

provided the condition of coplanarity is satisfied.

Note : The condition of coplanarity is the necessary condition for the intersection of two lines but not sufficient. Conversely if two lines intersect, they must be coplanar.

Illustrated Example

1. Prove that the straight lines $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ and $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ are coplanar. Find also the equation of the plane.

Solution : Since $\begin{vmatrix} 4-1 & -3+1 & -1+10 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = \begin{vmatrix} +3 & -2 & 9 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$, the lines are coplaner.

The equation of the plane is $\begin{vmatrix} x-1 & y+1 & z+10 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$

or, $11x - 6y - 5z - 72 = 0$.

9.6 Perpendicular distance of a point from a line

Let $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be the equations of the line AN and P(α, β, γ) be a given point.

Let PN be perpendicular to the line AN and the coordinates of A be (x_1, y_1, z_1).

We have $PN^2 = AP^2 - AN^2$

Now $AP^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2$.

Again, AN, the projection of AP on the straight line is $= (x_1 - \alpha)l + (y_1 - \beta)m + (z_1 - \gamma)n$, where l, m, n are the direction cosines of the straight line.

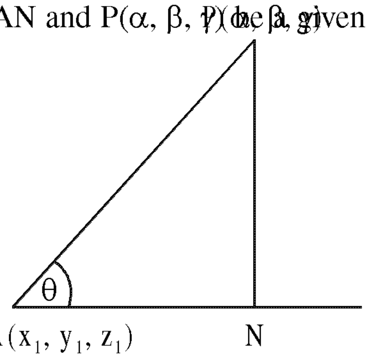


Fig. 9.6.1

Therefore, $PN^2 = \{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\} - \{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\}^2$

$$= \begin{vmatrix} x_1-\alpha & y_1-\beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y_1-\beta & z_1-\gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z_1-\gamma & x_1-\alpha \\ n & l \end{vmatrix}^2,$$

by Lagrange's identity, this PN is the perpendicular distance from the point P to the given straight line.

llustrated Example

1. Find the distance of the point (3, 2, 1) from the line $\frac{x-1}{3}=\frac{y}{4}=\frac{z-2}{1}$.

Solution : The required distance

$$= \sqrt{(3-1)^2 + (2-0)^2 + (1-2)^2 - \frac{\{(3-1).3 + (2-0).4 + (1-2).1\}^2}{3^2 + 4^2 + 1^2}}$$

$$= \sqrt{9 - \frac{1}{26}\{6+8-1\}^2}$$

$$= \sqrt{9 - \frac{1}{26}\{13\}^2}$$

$$= \sqrt{9 - \frac{13}{2}} = \sqrt{\frac{5}{2}}.$$

9.7 Shortest distance between two skew lines

If two non-parallel lines do not intersect, then they are called skew lines. These are non-coplanar lines.

Let the equations of the skew lines AB and CD be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad .....(9.6.1)$$

and $$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad .....(9.6.2)$$

respectively.

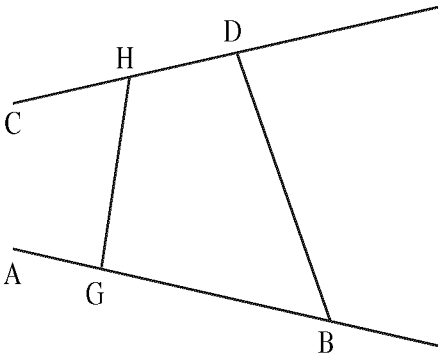


Fig. 9.7.1

Let GH be the shortest distance between them. Therefore GH is perpendicular to both the lines. Let the coordinates of B and D be (x_1, y_1, z_1) and (x_2, y_2, z_2) and the d.cs of GH be l, m, n. Since GH is perpendicular to both the lines $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$.

Then we obtain by cross-multiplication

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \pm\frac{1}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}} \quad .....(9.6.3)$$

The d.cs of BD are $\frac{x_2-x_1}{BD}, \frac{y_2-y_1}{BD}, \frac{z_2-z_1}{BD}$.

GH is the projection of BD on GH.

If θ be the angle between BD and GH, then GH = BD $\cos\theta$.

Therefore, $GH = BD\dfrac{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)}{BD}$

$$= \pm\frac{\sum (x_2 - x_3)(m_1 n_2 - m_2 n_1)}{\sqrt{\sum (m_1 n_2 - m_2 n_1)^2}} \quad [\text{by } (9.6.3)]$$

$$= \pm\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\left[\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2\right]^{\frac{1}{2}}}$$

Corollary - I : The equation of the plane containing the first straight line and the line of shortest distance is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad .....(9.6.4)$$

Corollary - II : The equation of the plane containing the second straight line and the lines of shortest distance is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \qquad .....(9.6.5)$$

These two equations (9.6.4) & (9.6.5) represent the equations of the line of shortest distance.

Illustrated Example

1. Find the s.d. between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \qquad .....(1)$$

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} \qquad .....(2)$$

Find also the equations and the points of intersection in which it meets the lines.

Solution : Let the s.d. meet (1) and (2) at $(3r_1 + 3, -r_1 + 8, r_1 + 3)$ and $(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$ respectively.

The d.rs. of the s.d are $3r_1 + 3r_2 + 6, - r_1 - 2r_2 + 15, r_1 - 4r_2 - 3$.

Since s.d. is perpendicular to (1) and (2), then

$$3(3r_1 + 3r_2 + 6) - 1.(-r_1 - 2r_2 + 15) + 1.(r_1 - 4r_2 - 3) = 0$$

or, $11r_1 + 7r_2 = 0$ ......(3)

and $-3(3r_1 + 3r_2 + 6) + 2.(-r_1 - 2r_2 + 15) + 4.(r_1 - 4r_2 - 3) = 0$

or, $7r_1 + 29r_2 = 0$ ......(4)

Therefore from (3) and (4) we obtain, $r_1 = r_2 = 0$.

$\therefore$ The points of intersection are (+3, 8, 3) and (–3, –7, 6).

Hence the length of s.d. = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = 3\sqrt{30}$.

The d.rs. of s.d are 6, 15, –3 and the equation of s.d. is $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$.

9.8 Intersection of three planes

Let the equations of three planes be

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0.$$

Solving these equations, we have

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or, $\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_4}$ (say),

where $\Delta_1, \Delta_2, \Delta_3$ are obtained from Δ_4 by substituting the column (d_1, d_2, d_3) for the first, second and third column respectively.

Deductions :

(i) If $\Delta_4 \neq 0$, the three planes intersect at some finite point and the point of intersection is obtained by solving the equations of the three planes.

(ii) If $\Delta_4 = 0$, the three planes intersect in a straight line and in this situation any one of $\Delta_1, \Delta_2, \Delta_3$ is zero.

(iii) If the planes neigther meet in a point nor pass through a common line, then they must form a prism, that is, the lines of intersection of the planes, two by two, are parallel to one another. In this case, the line of intersection of two planes is parallel to the third, that is, perpendicular to the normal to the third but does not lie on the third. Thus if $\Delta_4 = 0$ and none of other determinants zero, then the three planes form a triangular prism.

Illustrated Example

1. Show that the planes $2x - 5y + z = 1$, $x + y + 4z = 2$ and $x + 3y + 6z = 3$ form a prisom

Solution : Here, $\Delta_4 = \begin{vmatrix} 2 & -5 & 1 \\ 1 & 1 & 4 \\ 1 & 3 & 6 \end{vmatrix} = 0$

$\Delta_3 = \begin{vmatrix} 2 & -5 & -1 \\ 1 & 1 & -2 \\ 1 & 3 & -3 \end{vmatrix} = -1 \neq 0$ and similarly $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$.

Hence the planes form a prisom.

9.9 Volume of a tetrahedron

Let ABCD be a tetrahedron of which A, B, C and D are (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively.

The equation of the plane passing through B, C and D is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \qquad .....(9.8.1)$$

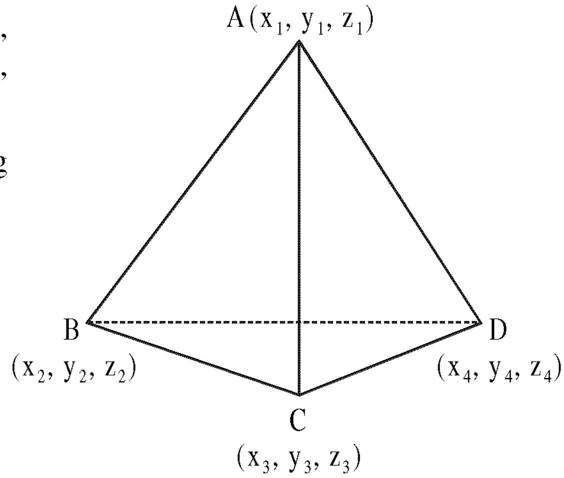


Fig. 9.9.1

Which is defined by

$$x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + y\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad .....(9.8.2)$$

If the area of the triangle BCD be Δ, then the volume of the tetrahedron ABCD is $\frac{1}{3}\rho\Delta$, where ρ is the length of the perpendicular from the point A to the plane of the triangle BCD.

Let the projections of Δ on the coordinate planes be Δ_x, Δ_y, Δ_z respectively, then

$$\Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \Delta^2 .$$

The projection of the points B, C, D on the plane x = 0 are $(0, y_2, z_2)$, $(0, y_3, z_3)$, $(0, y_4, z_4)$.

$\therefore$ The area of this projection on the yz-plane is $\frac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} = \Delta_x$.

Similarly, $\frac{1}{2}\begin{vmatrix} z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \\ z_4 & x_4 & 1 \end{vmatrix} = \Delta_y$ and $\frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \Delta_z$.

Then we have

Then the have $\rho = \dfrac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{2\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}} = \dfrac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{2\Delta}$

Therefore the volume of tetrahedron ABCD

$$= \frac{1}{3}\rho\Delta = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Corollary : If one of the vertices of tetrahedron be the origin, then the volume of tetrahedron $\text{ABCD} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$.

llustrated Example

1. Find the volume of the tetrahedron formed by the four planes $lx + my + nz = p$, $lx + my = 0$, $my + nz = 0$ and $nz + lx = 0$.

Solution : Solving the equations of the first three planes to get their point of intersection, we have (by adding the second and the third)

$$lx + my + my + nz = 0$$

or, $p + my = 0$, by the first

or, $y = -\frac{p}{m}$.

From the second and the third, we have $x = \frac{p}{l}$, $z = \frac{p}{n}$, the point of intersection of the first $\left(\frac{p}{l}, -\frac{p}{m}, \frac{p}{n}\right)$.

Similarly, the point of intersection of the first, second and fourth is $\left(-\frac{p}{l}, \frac{p}{m}, \frac{p}{n}\right)$, that of first, third and fourth is $\left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)$ and that of second, third and fourth is (0, 0, 0).

$$\text{Therefore, volume, } v = \frac{1}{6}\begin{vmatrix} -\frac{p}{l} & \frac{p}{m} & \frac{p}{n} \\ \frac{p}{l} & -\frac{p}{m} & \frac{p}{n} \\ \frac{p}{l} & \frac{p}{m} & -\frac{p}{n} \end{vmatrix} = \frac{1}{6}\frac{p^3}{lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{2}{3}\frac{p^3}{lmn}.$$

9.10 Worked out Examples

Example 1 : Find the equations of the planes through the straight line $2x - y + 3z + 2 = 0 = 3x + 2y - z + 3$ parallel to the co-ordinate axes.

Solution : The equation $2x - y + 3z + 2 + \lambda(3x + 2y - z + 3) = 0$ represents a plane through the straight line for all values of λ. This is parallel to the co-ordinate axes, if the coefficients of x, y, z be zero respectively. Since the plane is parallel to the x-axis, that is $y = 0 = z$, we have, $2 + 3\lambda = 0$ or, $\lambda = -\frac{2}{3}$.

Then the plane parallel to the x-axis is

$$2x - y + 3z + 2 - \frac{2}{3}(3x + 2y - z + 3) = 0$$

or, $7y + 11z = 0$.

Similarly, when the, plane is parallel to y-axis i.e. $x = 0 = z$, we have $-1 + 2\lambda = 0$

or, $\lambda = \frac{1}{2}$.

Then the equation of plane parallel to the y-axis is

$$2x - y + 3z + 2 + \frac{1}{2}(3x + 2y - z + 3) = 0$$

or, $7x + 5z + 7 = 0$.

Similarly the equation of plane parallel to z-axis is (i.e., when $x = 0 = y$)

$$11x + 5y + 11 = 0.$$

Example 2 : Find the equations of projection of the lines $3x - y + 2z - 1= 0 = x + 2y - z - 2$ on the plane $3x + 2y + z = 0$ in the symmetrical form.

Solution : Any plane through the given line is

$$3x - y + 2z - 1 + \lambda(x + 2y - z - 2) = 0$$

or, $(3 + \lambda)x + (2\lambda - 1)y + (2 - \lambda)z - (2\lambda + 1) = 0$, where λ is an arbitray constant.

It will be perpendicular to $3x + 2y + z = 0$, if $3(3 + \lambda) + 2.(2\lambda - 1) + 1.(2 - \lambda) = 0$

or, $\lambda = -\frac{3}{2}$.

Thus the equation of the plane is

$$3x - 8y + 7z + 4 = 0. \qquad .....(1)$$

The equations of projection are given by (1) and $3x + 2y + z = 0$.

Let l, m, n be the d.cs of the line of projection. Then

$$3l - 8m + 7n = 0 \text{ and } 3l + 2m + n = 0.$$

By cross-multiplication, $\frac{l}{-22} = \frac{m}{18} = \frac{n}{30}$

or, $\frac{l}{11} = \frac{m}{-9} = \frac{n}{-15}$.

Let the line cut the plane $z = 0$ at $(x_1, y_1, 0)$. Then $3x_1 - 8y_1 + 4 = 0$ and $3x_1 + 2y_1 = 0$

From these equations $x_1 = -\frac{4}{15}$ and $y_1 = \frac{2}{5}$.

Hence the line of projection in symmetrical form is $\frac{x + \frac{4}{15}}{11} = \frac{y - \frac{2}{5}}{-9} = \frac{z}{-15}$.

Example 3 : Find the equation of the straight line through the point (3, 1, –6) and parallel to each of the planes $x + y + 2z - 4 = 0$ and $2x - 3y + z + 5 = 0$.

Solution : Let the equations of the line be $\frac{x-3}{l} = \frac{y-1}{m} = \frac{z+6}{n}$ ......(1)

Since it is parallel to the given planes, the line is perpendicular to the normals to the planes.

$\therefore\ l + m + 2n = 0$ and $2l - 3m + n = 0$.

By cross-multiplication, $\frac{l}{7} = \frac{m}{3} = \frac{n}{-5}$.

Thus the equations of the line are $\frac{x-3}{7} = \frac{y-1}{3} = \frac{z+6}{-5}$.

Example 4 : Find the equation of the line through the point (1, 2, 4) and perpendicular to the line $3x + 2y - z - 4 = 0 = x - 2y - 2z - 5$.

Solution : If l, m, n be the d.cs of the given line, then $3l + 2m - n = 0$ and $l - 2m - 2n = 0$.

Therefore, by cross-multiplication $\frac{l}{6} = \frac{m}{-5} = \frac{n}{8}$

The equation of the plane passing through (1, 2, 4) and perpendicular to the line

$$6(x-1) - 5(y-2) + 8(z-4) = 0$$

i.e. $6x - 5y + 8z - 28 = 0$ .....(1)

The equation of a plane passing through the given line is

$$3x + 2y - z - 4 + \lambda(x - 2y - 2z - 5) = 0$$

i.e. $(3+\lambda)x + 2(1-\lambda)y - (1+2\lambda)z - (4+5\lambda) = 0$.

If it passes through the point (1, 2, 4), then $(3+\lambda)1 + 2(1-\lambda)2 - (1+2\lambda)4 - (4+5\lambda) = 0$.

or, $\lambda = -\frac{1}{16}$

$\therefore$ The equation of the plane is

$$3x + 2y - z - 4 - \frac{1}{16}(x - 2y - 2z - 5) = 0$$

or, $47x + 34y - 14z - 59 = 0$ .....(2)

The required line is the line of intersection of the planes (1) and (2) and passing through (1, 2, 4).

If l_1, m_1, n_1 are the d.cs of this line, then

$6l_1 - 5m_1 + 8n_1 = 0$ and $47l_1 + 34m_1 - 14n_1 = 0$

From these two, $\dfrac{l_1}{-202} = \dfrac{m_1}{460} = \dfrac{n_1}{439}$.

Hence, the equations of the required line are $\dfrac{x-1}{-202} = \dfrac{y-2}{460} = \dfrac{z-4}{439}$.

Example 5 : Find the image of the point (–3, 8, 4) in the plane 6x – 3y – 2z + 1 = 0.

Solution : If A be the point (–3, 8, 4), then its image will be the point B such that the given plane perpendicularly bisects the straight line AB.

The equations of the straight line through A and perpendicular to the plane are $\dfrac{x+3}{6} = \dfrac{y-8}{-3} = \dfrac{z-4}{-2}$.

Let the coordinates of B be (–3 + 6r, 8 – 3r, 4 – 2r), where r is a parameter. The co-ordinates of P, the middle point of AB, are then $\left(-3+3r, 8-\dfrac{3}{2}r, 4-r\right)$.

Now, since P lies on the plane 6x – 3y – 2z + 1 = 0; we have

$$6(-3 + 3r) - 3\left(8-\frac{3}{2}\right)r - 2.(4 - r) + 1 = 0$$

or, r = 2.

Hence B, the image of A, is the point (9, 2, 0).

Example 6 : Prove that $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-4}{5}$ are coplanar. Find also the equation of the plane.

Solution : Since $\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} = 0$.

(adding the elements of the first row with those of second row, the two rows become identical), the lines are coplanar.

The equation of the plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

or, –(x – 1) + 2(y – 2) – (z – 3) = 0

or, x – 2y + z = 0.

Example 7 : Prove that the straight lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar. Find also the equation of the plane.

Solution : Any point on the first straight line is $(1 + 2r, 2 + 3r, 3 + 4r)$. If this point be also on the second straight line. then

$4(1 + 2r) - 3(2 + 3r) + 1 = 0$ and $5(1 + 2r) - 3(3 + 4r) + 2 = 0$.

Both these give the same value of r, namely, $r = -1$.

This shows that the first straight line meets the planes $4x - 3y + 1 = 0$ and $5x - 3z + 2 = 0$ at the same point.

Thus the first straight line cuts the line of intersection of the two planes, i.e. cuts the second straight line.

Hence the two straight lines are coplanar and the equation of the plane containing the second straight line is

$4x - 3y + 1 + \lambda(5x - 3z + 2) = 0$, where λ is a parameter. Since it contains the first straight line, the point (1, 2, 3) lies on it. Therefore $4 - 6 + 1 + \lambda(5 - 9 + 2) = 0$

or, $\lambda = -\frac{1}{2}$.

Thus the equation of the plane is

$$4x - 3y + 1 - \frac{1}{2}(5x - 3z + 2) = 0$$

or, $3x - 6y + 3z = 0$

or, $x - 2y + z = 0$.

Example 8 : Find the distance of the point (4, 1, 1) from the straight line given by $x + y + z = 4$, $x - 2y - z = 4$. Find the equation of the perpendicular. Also find the foot of the perpendicular.

Solution : First let us put the equations of the straight line in the symmetrical form. Putting $z = 0$, we get

$$x + y = 4 \text{ and } x - 2y = 4.$$

Thus $x = 4, y = z = 0$.

If l, m, n be the d.cs. of the striaght line, then

$$l + m + n = 0$$
$$l - 2m - n = 0$$

So that $\frac{l}{1}=\frac{m}{2}=\frac{n}{-3}=\frac{1}{\sqrt{14}}$.

Hence the straight line is $\frac{x-4}{1}=\frac{y-0}{2}=\frac{z-0}{-3}$

i.e. $\frac{x-4}{1}=\frac{y}{2}=\frac{z}{-3}$ .....(1)

Therefore the required distance is

$$=\sqrt{(4-4)^2+(1-0)^2+(1-0)^2-\left\{\frac{1}{\sqrt{14}}(4-4)+\frac{2}{\sqrt{14}}(1-0)-\frac{3}{\sqrt{14}}(1-0)\right\}^2}$$

$$=\sqrt{1+1-\frac{1}{14}}=\sqrt{\frac{27}{14}}.$$

Let P be the given point (4, 1, 1) and the foot of the perpendicular be Q. Then Q being a point on the given straight line(1), its coordinates may be written as (r + 4, 2r, –3r), so that the d.cs of PQ are r, 2r – 1, –3r – 1. Now since PQ is perpendicular to the given straight line (1), then

r.1 + (2r – 1)2 + (–3r – 1)(–3) = 0

or, $r=-\frac{1}{14}$.

Hence the coordinates of the foot of the perpendicular Q are $\left(\frac{55}{14},-\frac{1}{7},\frac{3}{14}\right)$ and the equations of the perpendicular PQ are $\frac{x-4}{1}=\frac{y-1}{16}=\frac{z-1}{11}$.

Example 9 : Find the condition that the straight lines $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma},\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$ and $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ will lie on a plane.

Solution : If the three straight lines lie on a plane, then the equation of the plane will be of the form Ax + By + Cz = 0, since the straight lines pass through the origin.

Then $A\alpha + B\beta + c\gamma = 0$, $Aa\alpha + Bb\beta + Cc\gamma = 0$ and $Al + Bm + Cn = 0$.

Eliminating A, B, C, we get the required condition as $\begin{vmatrix}\alpha & \beta & \gamma\\ a\alpha & b\beta & c\gamma\\ l & m & n\end{vmatrix}=0$

or $\frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0$.

Example 10 : Find the shortest distance between the lines $\frac{x-3}{-3}=\frac{y-8}{1}=\frac{z-3}{-1}$ and $\frac{x+3}{3}=\frac{y+7}{-2}=\frac{z-6}{-4}$.

Find also the equation of shortest distance.

Solution : Let the direction cosines of the line of shortest distance be l, m, n. Since it is perpendicular to both the straight lines, therefore

$-3l + m - n = 0$ and $3l - 2m - 4n = 0$.

Therefore, $\frac{l}{-6}=\frac{m}{-15}=\frac{n}{3}$

i.e., $\frac{l}{-2}=\frac{m}{-5}=\frac{n}{1}$.

Hence, $l=\frac{-2}{\sqrt{30}}$, $m=\frac{-5}{\sqrt{30}}$, $n=\frac{1}{\sqrt{30}}$.

∴ The shortest distance is

$$(3+3).\left(\frac{-2}{\sqrt{30}}\right)+(8+7).\left(\frac{-5}{\sqrt{30}}\right)+(3-6).\frac{1}{\sqrt{30}} = -\frac{90}{\sqrt{30}} = -3\sqrt{30}.$$

Therefore the shortest distance = $3\sqrt{30}$ units.

The equations of the line of shortest distance are

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ -3 & 1 & -1 \\ -2 & -5 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ 3 & -2 & -4 \\ -2 & -5 & 1 \end{vmatrix}$$

or, $4x - 5y - 17z = 0 = 22x - 5y + 19z - 83$.

Example 11 : Show that the equation to the plane containing the straight line $\frac{y}{b}+\frac{z}{c}=1$, $x = 0$ and parallel to the straight line $\frac{x}{a}-\frac{z}{c}=1$, $y = 0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d be the s.d between the lines, then show that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

Solution : Equation of the plane through the first straight line is

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0 \qquad .....(1)$$

It will be parallel to the straight line $\frac{x}{a}-\frac{z}{c}=1, y=0$, i.e. $\frac{x}{a}=\frac{y}{0}=\frac{z+c}{c}$ .....(2)

if $\quad a\lambda+0.\frac{1}{b}+c.\frac{1}{c}=0$

or, $\quad \lambda=-\frac{1}{a}$.

Thus the equation of the plane (1) becomes $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$

The equations of the first straight line are $\frac{x}{0}=\frac{y}{b}=\frac{z-c}{-c}$ .....(3)

If 2d be the s.d between the lines (2) and (3) then

$$2d=\begin{vmatrix}0 & 0 & 2c\\ a & 0 & c\\ 0 & b & -c\end{vmatrix}\div\sqrt{b^2c^2+c^2a^2+a^2b^2}$$

or, $\quad 2d=\dfrac{2}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}}}$

$\therefore \quad \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

Example 12 : Find the equations to the straight line that intersects the straight lines $x + y + z - 4 = 0 = 2x - y - z - 3$, $x - y + z - 3 = 0 = x + 4y - z + 1$ and passes through the point (1, 0, 1).

Solution : The equations of a plane passing through the first straight line is given by

$x + y + z - 4 + \lambda(2x - y - z - 3) = 0$, λ being a parameter.

If it passes through the point (1, 0, 1) then $\lambda = -1$.

Hence the equation of the plane is $x - 2y - 2z + 1 = 0$.

The equation of the plane through the second straight line is given by

$(x - y + z - 3) + \mu(x + 4y - z + 1) = 0$, μ being a parameter.

If it passes through the point (1, 0, 1), then $\mu = 1$.

Hence the equation of the plane is $2x + 3y - 2 = 0$.

Thus the equations of the required straight lines are

$$x - 2y - 2z + 1 = 0 = 2x + 3y - 2.$$

Example 13 : Find the surface generated by the straight line which intersects the lines $x + y = 0 = z$; $x - y = z$, $x + y = 2a$ and the parabola $x^2 = 2az$, $y = 0$.

Solution : Equations of the planes through the first and second lines are

$$x + y - \lambda_1 z = 0 \qquad .....(1)$$

and $$x - y - z - \lambda_2(x + y - 2a) = 0, \qquad .....(2)$$

where λ_1 and λ_2 are constants.

The lines of intersection of (1) and (2) will meet the parabola $x^2 = 2az$, $y = 0$.

Hence, on putting $y = 0$ in (1) and (2), we get

$$x - \lambda_1 z = 0 \text{ and } x - z - \lambda_2(x - 2a) = 0.$$

Therefore, $x = \dfrac{2a\lambda_1\lambda_2}{1-\lambda_1+\lambda_1\lambda_2}$ and $z = \dfrac{2a\lambda_2}{1-\lambda_1+\lambda_1\lambda_2}$

Putting these in $x^2 = 2az$, we get $\lambda_1^2\lambda_2 = 1-\lambda_1+\lambda_1\lambda_2$

or, $(\lambda_1 - 1)(\lambda_1\lambda_2 + 1) = 0$

Therefore, $\lambda_1\lambda_2 = -1$.

Putting for λ_1 and λ_2 from (1) and (2), we get the surface generated by the line of intersection as $\dfrac{x+y}{z}\cdot\dfrac{x-y-z}{x+y-2a} = -1$

or, $x^2 - y^2 = 2az$.

Example 14 : Find the volume of the tetrahedron OABC, where O is the origin, the lengths OA, OB, OC are a, b, c and the angles BOC, COA and AOB are λ, μ, γ respectively.

Solution : If the d.cs of OA, OB, OC be respectively (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3), then the coordinates of A, B, C are (al_1, am_1, an_1), (bl_2, bm_2, bn_2) and (cl_3, cm_3, cn_3). Thus the volume of the tetrahedron OABC is

$$\frac{1}{6}\begin{vmatrix} al_1 & am_1 & an_1 \\ bl_2 & bm_2 & bn_2 \\ cl_3 & cm_3 & cn_3 \end{vmatrix} = \frac{abc}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now $$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_1 l_3 & \Sigma l_2 l_3 & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \cos\gamma & \cos\mu \\ \cos\gamma & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

Since we have $\Sigma l_1^2 = \Sigma l_2^2 = \Sigma l_3^2 = 1$

and $l_2l_3 + m_2m_3 + n_2n_3 = \cos\lambda,$

$l_1l_3 + m_1m_3 + n_1n_3 = \cos\mu,$

$l_1l_2 + m_1m_2 + n_1n_2 = \cos\gamma.$

Hence the required volume is $= \dfrac{abc}{6}\begin{vmatrix} 1 & \cos\gamma & \cos\mu \\ \cos\gamma & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{\frac{1}{2}}.$

9.11 Summary

In this unit, we have learnt about finding the equations of a straight line under various conditions including when it is through a point and intersection of two planes. We also have learnt to deal with the point of intersection of a line and a plane. We can now exploit the condition of coplanarity of two non-parallel straight lines. Besides we can determine the distance between a point and a stright line as well as between two skew lines. We are now able to deal with the intersection of three given planes and volume of a tetrahedron.

9.12 Exercises

1. Find the equation of the line in symmetrical form through the point (–1, –3, 5) which has d.rs 2, 1, –3. [Hints : solve yourself] $\left[\textbf{Ans. } \dfrac{x+1}{2} = \dfrac{y+3}{1} = \dfrac{z-5}{-3}\right]$
2. Find the equation of the line in symmetrical form through (5, –7, 8) and parallel to $\dfrac{x-2}{3} = \dfrac{y-2}{4} = \dfrac{z+1}{-5}$. [Hints : solve yourself] $\left[\textbf{Ans. } \dfrac{x-5}{3} = \dfrac{y+7}{4} = \dfrac{z-8}{-5}\right]$
3. Find the image of the point (1, –2, 3) in the plane 2x – 3y + 2z + 3 = 0. [Hints : see the worked out example 3] [**Ans.** (–3, 4, –1)]
4. Find the equation of the plane through the point (1, 2, 3) and perpendicular to the straight line $\dfrac{x-2}{2} = \dfrac{y+1}{-5} = \dfrac{z-1}{3}$. [Hints : solve yourself] [**Ans.** 2x – 5y + 3z = 1]
5. Find the equations of the perpendicular from the point (5, 9, 3) to the straight line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$. Also find the foot of the perpendicular. [Hints : see the example 6 in 9.9] $\left[\textbf{Ans. } \dfrac{x-5}{1} = \dfrac{y-9}{2} = \dfrac{z-3}{-2}; (3, 5, 7)\right]$

6. Determine the value of k so that the lines $\frac{x-1}{2}=\frac{y-4}{1}=\frac{z-5}{2}$ and $\frac{x-2}{-1}=\frac{y-8}{k}=\frac{z-11}{4}$ may intersect. Also find their point of intersection. [Hints : Taking one intersecting point of the given lines and then calculate the value of k]

[**Ans.** k = 3; (3, 5, 7)]

7. Find the equation of the perpendicular from the point (2, 4, –1) to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$. Obtain the foot of the perpendicular. [Hints : see example 8 in 9.9] $\left[\textbf{Ans. } \frac{x+4}{6}=\frac{y-1}{3}=\frac{z+3}{2}; (-4,1,-3)\right]$

8. Prove that the straight lines $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{4}=\frac{y-4}{1}=\frac{z-5}{3}$ are coplanar. [Hints : see the illustrated example of 9.4]

9. Show that the length of the shortest distance between the straight lines $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$ is $4\sqrt{3}$ units and the equations of the line of the shortest distance are x = y = z. [Hints : see the example 10 of 9.9]

10. Show that the distance of the point (4, –5, 3) from the straight line $\frac{x-5}{3}=\frac{y+2}{-4}=\frac{z-6}{5}$ is $\frac{\sqrt{457}}{5}$ units. [Hints : see the illustrated example of 9.5]

11. Show that the planes x – 2y + z – 3 = 0, x + y – 2z – 3 = 0 and x – z – 1 = 0 form a triangular prisom. [Hints : see the illustrated example of 9.7]

12. Find the volume of the tetrahedron formed by the planes x + y = 0, y + z = 0, z + x = 0, x + y + z = 1. [Hints : see the illustrated example of 9.8] $\left[\textbf{Ans. } \frac{2}{3} \text{ cubic units}\right]$

Unit 10 ❐ Sphere

Structure

10.0 Objectives

We will be able to determine and appreciate

- equation of sphere with various given conditions
- uniqueness of sphere through four non-coplanar points
- several important properties related to sphere

10.1 Introduction

In our previous chapter we have discussed about plane which is a special type of surface whose curvature zero. In this section we discuss about some more surfaces.

A surface defined in space by an equation of the second degree in x, y, z is called a quadratic surface. We discuss a few types of quadratic surface by deriving their equations or analysing their equations in the current unit and the next unit.

10.2 Sphere

A sphere is the locus of a point in space which moves in such a way that its distance from a fixed point is constant. The fixed point is called the centre and the distance of the moving point from the centre is known as radius.

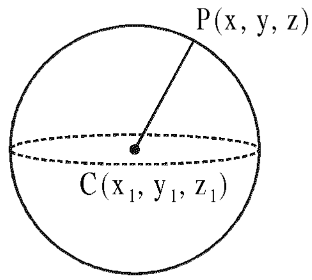


Fig. 10.2.1.1

10.2.1 Equation of a sphere

Let P(x, y, z) be any arbitrary point on the sphere with centre at $C(x_1, y_1, z_1)$. If the radius of the sphere be r, then $CP^2 = r^2$,

i.e. $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$

This is the equation of the sphere and if can be written as

$$x^2 + y^2 + z^2 - 2x_1x - 2y_1y - 2z_1z + x_1^2 + y_1^2 + z_1^2 - r^2 = 0$$

Putting $x_1 = -u$, $y_1 = -v$, $z_1 = -w$ and $x_1^2 + y_1^2 + z_1^2 - r^2 = d$, the equation reduces to

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

It is the general equation of a sphere. The coordinates of the centre are (–u, –v, – w) and radius = $\sqrt{u^2 + v^2 + w^2 - d}$.

Corollary : If the centre is at the origin, the equation of the sphere is $x^2 + y^2 + z^2 = r^2$ (putting $x_1 = y_1 = z_1 = 0$).

Note : A point lies outside, on or inside a sphere according as the distance from the centre of the sphere is greater than, equal to or less than the radius of the sphere.

Illustrated Example :

Example 1 : Find the equation of the sphere whose centre is (2, 3, –4) and radius is 5.

Solution : The equation of the sphere is $(x - 2)^2 + (y - 3)^2 + (z + 4)^2 = 5^2$

or, $x^2 + y^2 + z^2 - 4x - 6x + 8z + 4 = 0$.

10.2.2 Equation of a sphere on a diameter with given extremities

Let the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the extrimities of diameter of a sphere and let $P(x, y, z)$ be any point on the sphere. Since AB is a diameter, $\angle APB = 90^\circ$, i.e. PA and PB are at right angle.

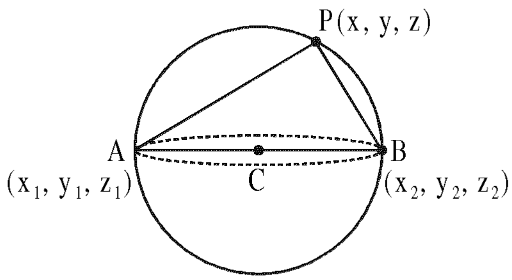


Fig. 10.2.2.1

The direction ratios of PA and PB are $(x - x_1), (y - y_1), (z - z_1)$ and $(x - x_2), (y - y_2), (z - z_2)$ respectively.

Since they are at right angle,

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$, which is the required equation.

Note : The centre of the sphere is at the middle point of the line segment AB, i.e. at the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

llustrated Example

Example 1 : Find the equation of the sphere which has (3, 4, –1) and (–1, 2, 3) as the ends of a diameter and find its centre and radius.

Solution : The equation of the sphere is

$(x - 3)(x + 1) + (y - 4)(y - 2) + (z + 1)(z - 3) = 0$

or, $x^2 + y^2 + z^2 - 2x - 6y - 2z + 2 = 0$

The centre is (1, 3, 1) and radius = $\sqrt{1+9+1-2} = 3$ units.

10.2.3 Through four non-coplanar points one and only one sphere passes

Let the points be (x_i, y_i, z_i), i = 1, 2, 3, 4.

Since the points are non-coplanar $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$

This implies that the system of linear equations

$$\left.\begin{aligned} 2\alpha x_1 + 2\beta y_1 + 2\gamma z_1 + \delta &= -(x_1^2 + y_1^2 + z_1^2) \\ 2\alpha x_2 + 2\beta y_2 + 2\gamma z_2 + \delta &= -(x_2^2 + y_2^2 + z_2^2) \\ 2\alpha x_3 + 2\beta y_3 + 2\gamma z_3 + \delta &= -(x_3^2 + y_3^2 + z_3^2) \\ 2\alpha x_4 + 2\beta y_4 + 2\gamma z_4 + \delta &= -(x_4^2 + y_4^2 + z_4^2) \end{aligned}\right\} \quad (10.1.3.1)$$

has a unique solution, where (α, β, γ) is centre of sphere.

Let the solution be $\alpha = u$, $\beta = v$, $\gamma = w$, $\delta = d$. Thus the four points lie on the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (10.1.3.2)$$

If $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ is another sphere which passes through the given points, then the equations (10.1.3.1) have solution (u', v', w', d'). But the system (10.1.3.1) has unique solution. So $u' = u$, $v' = v$, $w' = w$, $d' = d$.

Hence only one sphere passes through the given four points.

Note 1 : If the four points are coplanar and any three of them are not collinear, many spheres can pass through them.

Note 2 : If the four points are coplanar and three of them are collinear, no sphere passes through them.

llustrated Example

Example 1 : Find the equation of the sphere through the four points (4, –1, 2), (0, –2, 3), (1, –5, –1) and (2, 0 1).

Solution : Since $\begin{vmatrix} 4 & -1 & 2 & 1 \\ 0 & -2 & 3 & 1 \\ 1 & -5 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 42 \neq 0$ the points are non-coplanar and the sphere passing through these four points is unique.

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Since the given points lie on it, so

$$8u - 2v + 4w + d = -21,$$
$$-4v + 6w + d = -13,$$
$$2u - 10v - 2w + d = -27,$$
$$4u + 2w + d = -5.$$

These are in matrix form $\begin{vmatrix} 8 & -2 & 4 & 1 \\ 0 & -4 & 6 & 1 \\ 2 & -10 & -2 & 1 \\ 4 & 0 & 2 & 1 \end{vmatrix} \begin{bmatrix} u \\ v \\ w \\ d \end{bmatrix} = \begin{bmatrix} -21 \\ -13 \\ -27 \\ -5 \end{bmatrix}$

By elementary operations we get $\begin{vmatrix} 8 & -2 & 4 & 1 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & -3 & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{vmatrix} \begin{bmatrix} u \\ v \\ w \\ d \end{bmatrix} = \begin{bmatrix} -21 \\ 70 \\ \frac{61}{2} \\ \frac{11}{2} \end{bmatrix}$.

This gives that, $8u - 2v + 4w + d = -21$,

$14d = 70$,

$-3w + \frac{11}{2}d = \frac{61}{2}$,

$v + \frac{d}{2} = \frac{11}{2}$.

From these, $d = 5$, $w = -1$, $v = 3$, $u = -2$.

Thus the equation of sphere is $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$.

10.2.4 Plane section of a sphere

Any section of a sphere by a plane is a circle.

Let PQ be a section of the sphere having the centre C. CN is perpendicular the plane of the section. N is the centre of the circle of this section and NP is the radius of it.

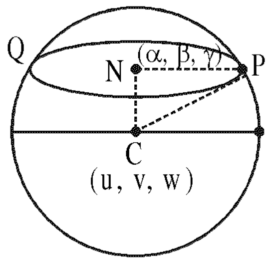


Fig. 10.2.4.1

Here $CP^2 = CN^2 + NP^2$.

If the coordinates of C and N are (u, v, w) and (α, β, γ), then the equation of the plane of section is

$$(u - \alpha)(x - \alpha) - (v - \beta)(y - \beta) + (w - \gamma)(z - \gamma) = 0$$

Note 1 : Let $S = 0$ and $U = 0$ be the equations of a sphere and a plane respectively. If the plane intersects the sphere, then the two equations taken together represent a circle.

Again $S + \lambda U = 0$ represents a sphere through the circle $S = 0$, $U = 0$.

Note 2 : The equation of the plane of the circle through the two given spheres $S = 0$ and $S' = 0$ is $S - S' = 0$. It is known as the radical plane. The equation of any sphere through the circle $S = 0$, $S' = 0$ is $S + \lambda S' = 0$, where λ is a parameter.

Note 3 : If the centre of the sphere lies on the section, then the circle is called a great circle, otherwise it is a small circle.

To find equation, centre, radius of a circle when a sphere $S : x^2 + y^2 + z^2 = r^2$ intersected by a plane $P : ax + by + cz + d = 0$

The equation of circle can be written as $x^2 + y^2 + z^2 - r^2 = 0$ and $ax + by + cz + d = 0$ i.e., $S = 0$, $P = 0$ together as the circle is the curve in 3-dimensional space which is coming out from the intersection of the sphere and the plane. Let O be the centre of the sphere.

Now if M is the centre of the circle then OM is perpendicular to the plane. So $OM = \left|\frac{a.0+b.0+c.0+d}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$.

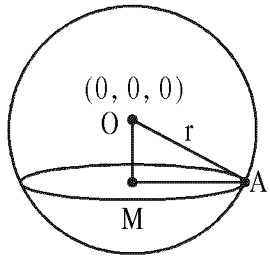


Fig. 10.2.4.2

OA = r (radius of the sphere)

so using $OA^2 = OM^2 + MA^2$ we can find MA.

i.e., $MA^2 = OA^2 - OM^2 = r^2 - \frac{d^2}{a^2+b^2+c^2} = \frac{r^2(a^2+b^2+c^2)-d^2}{a^2+b^2+c^2}$

$\therefore\ MA = \sqrt{\frac{r^2(a^2+b^2+c^2)-d^2}{a^2+b^2+c^2}}$

Now to find coordinate of M, we find equation of OM i.e.,

$$\frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c} = t \text{ (say)}$$

So, $x = at$, $y = bt$, $z = ct$.

For some value of t, the point lies on the plane and which is M.

So satisfying the equation of the plane,

$a(at) + b(bt) + c(ct) + d = 0$, we get $(a^2 + b^2 + c^2)t = -d$ i.e., $t = -\frac{d}{a^2+b^2+c^2}$

$\therefore$ The coordinate of the centre M of the circle is

$$\left(\frac{-ad}{a^2+b^2+c^2}, \frac{-bd}{a^2+b^2+c^2}, \frac{-cd}{a^2+b^2+c^2}\right)$$

Note : Similarly we can find the centre and radius of a circle for general sphere also.

Illustrated Example

Example 1 : Prove that the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts the sphere $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ in a great circle, if $2(uu' + vv' + ww') = 2r'^2 + d + d'$, where r' is radius of the second sphere.

Solution : The equation of the plane of the circle through the given spheres is $2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0$.

If this circle is great circle of the second sphere, then the centre $(-u', -v', -w')$ lies on it.

$\therefore -2(u-u')u' - 2(v-v')v' - 2(w-w')w' + d - d' = 0$

or, $2(uu' + vv' + ww') = 2(u'^2 + v'^2 + w'^2) + d + d'$

or, $2(uu' + vv' + ww') = 2r'^2 + d + d'$.

10.2.5 Intersection of a straight line and a sphere

Let the equation of the sphere and the straight line be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ respectively.

Any point on the given line will have co-ordinates $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point be also on the sphere, then we have

$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0$

or, $r^2(l^2 + m^2 + n^2) + 2r\{l(\alpha + u) + m(\beta + v) + n(\gamma + w)\}$

$+ (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0$ (10.1.5.1)

This equation (10.1.5.1), being a quadratic in r, gives two values of r, which shows that the line intersects the sphere at two points. These two points will be real and distinct, real and coincident or imaginary according as the roots of the equation (10.1.5.1) are real and distinct, real and equal or imaginary.

If the two points of intersection be coincident, then the line is called the tangent line to the sphere, the point of contact of the tangent being $(\alpha + lr, \beta + mr, \gamma + nr)$.

10.2.6 Equation of the tangent plane

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (10.1.6.1)

Equation of any straight line through the point (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (10.1.6.2)$$

If the point (α, β, γ) lines on the sphere, then

$$\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \qquad (10.1.6.3)$$

Any point on the line (10.1.6.2) is $(\alpha + lr, \beta + mr, \gamma + nr)$.

This will lie on the sphere (10.1.6.1), if

$(l^2 + m^2 + n^2)r^2 + 2(\alpha l + \beta m + \gamma n + lu + mv + nw)r = 0$ (10.1.6.4)

One root of this equation is zero, which shows that one of the points of intersection coincoides with the point (α, β, γ).

In order that the line (10.1.6.2) is a tangent line to (10.1.6.1) at (α, β, γ), the other point of intersection should also coincide with (α, β, γ) i.e., the other root of the equation should also vanish. This requires that

$l(\alpha + u) + m(\beta + v) + n(\gamma + w) = 0$ (10.1.6.5)

Now, ($-u$, $-v$, $-w$) being the co-ordinates of the centre of the sphere ($\alpha + u$), ($\beta + v$), ($\gamma + w$) are the direction ratios of the line joining the centre and the point of contact while l, m, n are the d.r.s. of the line. The condition (10.1.6.5) implies that if the line (10.1.6.2) be a tangent line to the sphere, then the line joining the centre to the point of contact is perpendicular to the tangent line.

The tangent plane at (α, β, γ) is the locus of all such tangent line and is obtained by eliminating l, m, n between (10.1.6.5) and the lines. Thus the equation of the tangent plane at the point (α, β, γ) to the sphere (10.1.6.1) is

$(x - \alpha)(\alpha + u) + (y - \beta)(\beta + v) + (z - \gamma)(\gamma + w) = 0$

or, $x\alpha + y\beta + z\gamma + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0$ [by (10.1.6.3)

Corollary : If the equation of the sphere be $x^2 + y^2 + z^2 = r^2$, then $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ will be a tangent line to the sphere, if $l\alpha + m\beta + n\gamma = 0$.

Furthermore, k the equation of the tangent plane at the point (α, β, γ) is $x\alpha + y\beta + z\gamma = r^2$.

10.2.7 Condition of tangency

Let the equation of the plane be

$lx + my + nz = p$ (10.1.7.1)

which touches the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (10.1.7.2)

If the plane touches the sphere, then the length of the perpendicular from the centre of the sphere to the plane will be equal to the radius of the sphere.

Now the centre of the sphere is at ($-u$, $-v$, $-w$) and the radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Hence the required condition is $\left|\frac{-lu - mv - nw - p}{\sqrt{l^2 + m^2 + n^2}}\right| = \sqrt{u^2 + v^2 + w^2 - d}$

$\therefore\ (lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$.

10.2.8 The plane of contact of the tangent planes to a sphere

The plane containing the locus of the points of contact of the tangent planes which passes through a given outside point w.r.t. a sphere is the plane of contact of the tangent planes of the sphere.

In this case of a sphere, the locus is a circle.

Let the given point be (α, β, γ) and the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Let (x', y', z') be a point on the sphere so that the tangent plane passes through the point (α, β, γ).

The tangent plane at the point (x', y', z') on the sphere is

$xx' + yy' + zz' + u(x + x') + v(y + y') + w(z + z') + d = 0.$

This passes through the point (α, β, γ).

$\therefore\ \alpha x' + \beta y' + \gamma z' + u(\alpha + x') + v(\beta + y') + w(\gamma + z') + d = 0.$

Hence the locus of (x', y', z') is

$\alpha x + \beta y + \gamma z + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0$

This is the plane of contact of the tangent planes from the point (α, β, γ).

10.2.9 The polar plane of a point with respect to a sphere

The locus of the intersection of the tangent planes drawn at the extremities of chords passing through a points to the sphere is called the polar plane of the point with respect to the sphere.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (10.1.9.1)$$

Let (x', y', z') be the point of intersection of the tangent planes at the extremities of the chord passing through the point (α, β, γ).

The plane of contact of the tangents from (x', y', z') to the sphere (10.1.9.1) is

$$xx' + yy' + zz' + u(x + x') + v(y + y') + w(z + z') + d = 0.$$

This plane passes through the point (α, β, γ).

$\therefore\ \alpha x' + \beta y' + \gamma z' + u(\alpha + x') + v(\beta + y') + w(\gamma + z') + d = 0$

Thus the locus of the point (x', y', z') is

$$\alpha x + \beta y + \gamma z + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0 \qquad (10.1.9.2)$$

This is the polar plane of the point (α, β, γ) w.r.t. the given sphere (10.1.9.1).

The point (α, β, γ) is called the pole of the plane (10.1.9.2) w.r.t. the sphere (10.1.9.1).

10.2.10 Pole of a plane w.r.t. a sphere

Let the plane and the sphere be $lx + my + nz = p$ and $x^2 + y^2 + z^2 = r^2$.

Let (α, β, γ) be the pole of the plane $lx + my + nz = p$ (10.1.10.1)

The polar plane of the point (α, β, γ) w.r.t. the given sphere is

$\alpha x + \beta y + \gamma z = r^2$. (10.1.10.2)

From (10.1.10.1) and (10.1.10.2), we obtain $\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \frac{r^2}{p}$

i.e. $\alpha = \frac{r^2 l}{p}$, $\beta = \frac{r^2 m}{p}$, $\gamma = \frac{r^2 n}{p}$.

Hence, the pole is at $\left(\frac{r^2 l}{p}, \frac{r^2 m}{p}, \frac{r^2 n}{p}\right)$.

10.2.11 Length of the tangent to a sphere

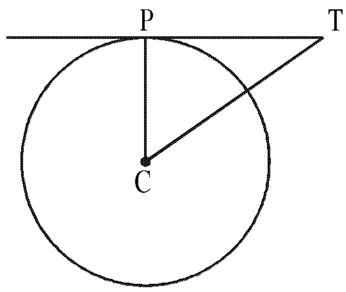


Fig. 10.2.11.1

Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Let PT be the tangent line to the sphere from the point $T(\alpha, \beta, \gamma)$. The centre of the sphere is at $(-u, -v, -w)$. Then the $\angle TPC$ is a right angle. Hence $PT^2 = TC^2 - CP^2$

$= (\alpha + u)^2 + (\beta + v)^2 + (\gamma + w)^2 - (u^2 + v^2 + w^2 - d)$.

Length of the tangent

$PT = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma - d}$ (10.1.11.1)

Thus the power of a point w.r.t a sphere is equal to the square of the length of the tangent from the point to the sphere.

Note : The point T is outside on or inside the sphere according as the expression under the radical sign in (10.1.11.1) is positive, zero or negative.

10.2.12 Equation of the normal at a point

The normal to a sphere at the point (x', y', z') is the straight line through the point and perpendicular to the tangent plane to the sphere at the point (x', y', z').

Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (10.1.12.1)

Its tangent plane at the point (x', y', z') is

$xx' + yy' + zz' + u(x + x') + v(y + y') + w(z + z') + d = 0.$

or, $(x' + u)x + (y' + v)y + (z' + w)z + (ux' + vy' + wz' + d) = 0$

The direction ratios of the normal to this plane are

$(x' + u), (y' + v), (z' + w).$

Hence the equations of the normal to the sphere (10.1.12.1) at the point (x', y', z') are

$$\frac{x-x'}{x'+u} = \frac{y-y'}{y'+v} = \frac{z-z'}{z'+w}.$$

Illustrated Examples :

Example 1 : Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 49$ at the point $(6, -3, -2)$. Show further that $2x - 6y + 3z - 49 = 0$ is a tangent plane to the same sphere. Find the point of contact.

Solution : The equation of the required tangent plane is

$6x - 3y - 2z = 49.$

The centre of the sphere is the origin $(0, 0, 0)$. Hence the length of the perpendicular from the centre to the plane $2x - 6y + 3z - 49 = 0$ is $\left|\frac{-49}{\sqrt{4+36+9}}\right| = 7$, which is equal to the radius of the sphere.

Hence the given plane touches the sphere $x^2 + y^2 + z^2 = 49$.

Equations of the straight line through the centre $(0, 0, 0)$ and perpendicular to the plane $2x - 6y + 3z - 49 = 0$ are $\frac{x}{2} = \frac{y}{-6} = \frac{z}{3}$.

Any point on this straight line is $(2r, -6r, 3r)$.

This point lies on the plane $2x - 6y + 3z - 49 = 0$

if $4r + 36r + 9r = 49$, i.e., $r = 1$.

Hence the point of contact is $(2, -6, 3)$.

Example 2 : Find the values of c for which the plane $x + y + z = c$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0.$

Solution : Then centre of the sphere is at the point (1, 1, 1) and its radius is $\sqrt{1+1+1+6}=3$.

The plane will touch the sphere, if the perpendicular distance of the point (1, 1, 1) from the plane be equal to the radius of the sphere.

In other words, $\left|\frac{1+1+1-c}{\sqrt{1+1+1}}\right|=3$

or, $(3 - c)^2 = 27$

or, $c^2 - 6c - 18 = 0$

or, $c = 3(1 \pm \sqrt{3})$.

Example 3 : If a sphere touches the planes $2x + 3y - 6z + 14 = 0$ and $2x + 3y - 6z + 42 = 0$ and if its centre lies on the straight line $2x + z = 0$, $y = 0$, find the equation of the sphere.

Solution : The distance between the two given parallel planes is 4 units which is equal to the diameter of the sphere. Hence its radius is 2 units.

Let the centre of the required sphere be (α, β, γ). Since it lies on the straight line $2x + z = 0$, $y = 0$ we have

$2\alpha + \gamma = 0, \beta = 0$ (1)

Now, the distance of the centre $(\alpha, 0, \gamma)$ from the plane $2x + 3y - 6z + 42 = 0$ must be equal to the radius of the sphere..

Thus $\left|\frac{2\alpha-6\gamma+42}{\sqrt{4+9+36}}\right|=2$

or, $2\alpha - 6\gamma + 42 = 14$

or, $2\alpha - 6\gamma = -28$ (2)

From (1) & (2), we have $\alpha = -2, \gamma = 4$

Thus the equation of the required sphere is

$(x + 2)^2 + y^2 + (z - 4)^2 = 4$

Example 4 : Find the equation of the sphere touching the three co-ordinate planes.

Solution : Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$

The centre of the sphere is $(-u, -v, -w)$ and its radius is equal to $\sqrt{u^2+v^2+w^2-d}$.

If the sphere touches the yz-plane, that is, $x = 0$, then the condition tangency gives $-u=\sqrt{u^2+v^2+w^2-d}$

or, $v^2 + w^2 = d$

Similarly applying the condition of tangency to the planes $y = 0$ and $z = 0$, we get respectively $v^2 + w^2 = d$ and $u^2 + v^2 = d$.

Adding these there, we get, $u^2 + v^2 + w^2 = \frac{3}{2}d$.

or, $u^2 = v^2 = w^2 = \frac{1}{2}d = r^2$ (say).

Thus $u = v = w = \pm r$ and the radius $= \sqrt{u^2 + v^2 + w^2 - d} = r$.

Hence the required equation of the sphere touching the coordinate planes is $x^2 + y^2 + z^2 \pm 2rx \pm 2ry \pm 2rz + 2r^2 = 0$.

Example 5 : Show that only one tangent plane can be drawn to the sphere $x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$ through the straight line $3x - 4y - 8 = 0 = y - 3z + 2$.

Find the equation of the plane.

Solution : The equation of the plane through the given straight line is of the form

$\lambda(3x - 4y - 8) + \mu(y - 3z + 2) = 0$ (1)

where λ and μ are variable parameters and both of them are not simultaneously zero.

This equation of the plane is

$3\lambda x + (\mu - 4\lambda)y - 3\mu z = 8\lambda - 2\mu$ (2)

The plane (2) will touch the given sphere, if

$[-3\lambda + 3(\mu - 4\lambda) - 3\mu + 8\lambda - 2\mu]^2 = [9\lambda^2 + (\mu - 4\lambda)^2 + 9\mu^2](1 + 9 + 1 - 8)$

or, $26\lambda^2 + 26\mu^2 - 52\lambda\mu = 0$

This gives $\lambda^2 - 2\lambda\mu + \mu^2 = 0$, i.e. $\left(\frac{\lambda}{\mu} - 1\right)^2 = 0$, which gives only one value of $\frac{\lambda}{\mu}$.

Therefore $\frac{\lambda}{\mu} = 1$.

Hence there is only one tangent plane whose equation is

$3x - 4y - 8 + (y - 3z + 2) = 0$ [by (1)]

or, $x - y - z = 2$.

Note : In general, $\frac{\lambda}{\mu}$ will have two values, giving two tangent planes.

10.2.13 Angle of intersection of two non-concentric spheres

The angle of intersection of two spheres is defined as the angle between their tangent planes at a common point of intersection. Since the radii of the spheres to the common point are perpendicular to the tangent planes, the angle between these radii is equal to the angle between the tangent planes i.e., the angle of intersection of the spheres.

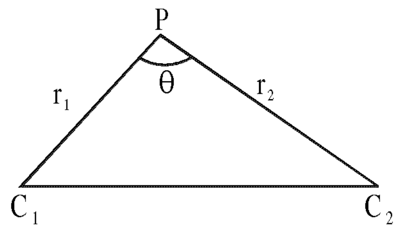


Fig. 10.2.13.1

Let P be the common point of two spheres whose centres are C_1 and C_2. The angle of intersection of two spheres is the angle between PC_1 and PC_2. If $C_1P = r_1$, $C_2P = r_2$ and $\angle C_1PC_2 = \theta$, then $C_1C_2^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos\theta$.

Orthogonal spheres : Two spheres are said to be orthogonal if the angle of intersection is a right angle.

In this case $C_1C_2^2 = r_1^2 + r_2^2$.

10.2.14 Condition for the orthogonality of two spheres

Let the equations of the spheres be

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad (10.1.14.1)$$

and $$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \qquad (10.1.14.2)$$

If C_1 and C_2 are the centres of the above two spheres and P is the common point, then the coordinates of C_1 and C_2 are $(-u_1, -v_1, -w_1)$ and $(-u_2, -v_2, -w_2)$ respectively. C_1P and C_2P are radii of (10.1.14.1) and (10.1.14.2) and $\angle C_1PC_2 = 90^\circ$ for orthogonal intersection.

$\therefore\ C_1P^2 + C_2P^2 = C_1C_2^2$,

i.e. $(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = (u_1^2 + v_1^2 + w_1^2 - d_1) + (u_2^2 + v_2^2 + w_2^2 - d_2)$.

i.e. $$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2 \qquad (10.1.14.3)$$

It is the required condition.

llustrated Example

Example 1 : Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\dfrac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Solution : Let the spheres be $x^2 + y^2 + z^2 = r_1^2 \qquad (1)$

and $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad (2)$

Radius of (2) is r_2,

$\therefore\ r_2^2 = u^2 + v^2 + w^2 - d \qquad (3)$

Since (1) and (2) cut orthogonally,

$d - r_1^2 = 0 \qquad (4)$

The plane of the common circle is

$(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) - (x^2 + y^2 + z^2 - r_1^2) = 0$,

i.e. $2ux + 2vy + 2wz + 2r_1^2 = 0 \qquad$ [by (4)]

i.e. $ux + vy + wz + r_1^2 = 0$.

It p is the perpendicular distance of the plane from the centre of (1), then

$$p^2 = \frac{r_1^4}{u^2+v^2+w^2} = \frac{r_1^4}{d+r_2^2} = \frac{r_1^4}{r_1^2+r_2^2} \qquad \text{[by (3) and (4)]}$$

Now the radius of the circle $\sqrt{r_1^2 - p^2} = \sqrt{r_1^2 - \dfrac{r_1^4}{r_1^2+r_2^2}} = \dfrac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$

10.2.15 Power point or length of tangent

Let $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (10.1.15.1)

and $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n} = r$ (say) (10.1.15.2)

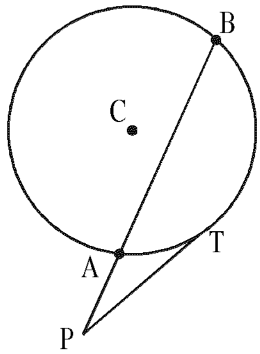


Fig. 10.2.15.1

be the equations of a sphere and a line respectively.

Any point on the line is $(lr + x_1, mr + y_1, nr + z_1)$.

If this point lies on the sphere, then $(lr + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 + 2u(lr + x_1) + 2v(mr + y_1)\ 2w(nr + z_1) + d = 0$.

i.e. $(l^2 + m^2 + r^2)r^2 + 2(lx_1 + my_1 + nz_1)r + (x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d) = 0$ (10.1.15.3)

It is a quadratic equation in r.

Let the roots be r_1 and r_2. If l, m, n are the d.cs of the line (10.1.15.2) the r_1 and r_2 are the distances of the points of intersection between (10.1.15.1) and (10.1.15.2) from the point $P(x_1, y_1, z_1)$. If A and B are the points of intersection, then

$$PA.PB = r_1r_2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = \text{constant}.$$

It is called the power of the point P w.r.t. the sphere. In case tangency of the line, roots of (10.1.15.3) are equal. In the fig. 10.1.15.1, PT is the tangent (T is the point of contact) and PT^2 = product of the roots = power of the point P, (As we know PT^2 = PA.PB).

PT is called the length of the tangent.

10.2.16 Radical plane, radical axis and radical centre

Radical plane of two spheres is the locus of points whose power w.r.t. the spheres are equal.

Let the spheres be $S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ and

$S_2 = x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

Let the powers of the the point $P(x_1, y_1, z_1)$ w.r.t. the spheres be equal.

Then $x_1^2 + y_1^2 + z_1^2 + 2u_1x_1 + 2v_1y_1 + 2w_1z_1 + d_1$

$= x_1^2 + y_1^2 + z_1^2 + 2u_2x_1 + 2v_2y_1 + 2w_2z_1 + d_2$

or, $2(u_1 - u_2)x_1 + 2(v_1 - v_2)y_1 + 2(w_1 - w_2)z_1 + d_1 - d_2 = 0$ (10.1.16.1)

$\therefore$ The locus of P, i.e. the equation of the radical plane is

$2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + d_1 - d_2 = 0$

Note : The radical planes of two sphere $S_1 = 0, S_2 = 0$ is $S_1 - S_2 = 0$ when the coefficients of second degree terms in each equation of the sphere are unity. When $S_1 - S_2 = 0$ is the radical plane of the two spheres, then it passes though their point of intersection. The direction ratios of the line joining the centres $(-u_1, -v_2, -w_1)$ and $(-u_2, -v_2, -w_2)$ are $u_1 - u_2, v_1 - v_2, w_1 - w_2$.

These are the d.rs of the normal to the plane (10.1.16.1). Hence the line joining the centres is perpendicular to the radical plane.

Corollary : If the two spheres intersect in a circle, the radical plane is the plane of that circle.

Radical axis : The radical planes of three spheres taken two by two intersect along a line which is called the radical axis or radical line of the spheres.

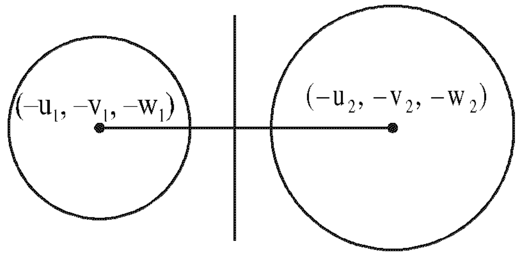


Fig. 10.2.16.1

Let the spheres by $S_1 = 0, S_2 = 0, S_3 = 0$. Then the radical plane are $S_1 - S_2 = 0, S_2 - S_3 = 0, S_3 - S_1 = 0$. These planes clearly intersect in the line $S_1 = S_2 = S_3$. This line is the radical axis.

Radical Centre : The four radical axes of four spheres taken three at a time intersect at a point which is called the radical centre.

10.2.17 Coaxial spheres

A system of spheres is said to be coaxial when they have a common radical plane.

If $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is the equation of a sphere and

$L \equiv lx + my + nz - p = 0$ is the equation of a plane, then $S + \lambda L = 0$ represents a system of coaxial spheres with λ as a parameter. In this system $L = 0$ is the radical plane.

So from, $S + \lambda L = 0$

i.e. $x^2 + y^2 + z^2 + (2u + \lambda l)x + (2v + \lambda m)y + (2w + \lambda n)z + d - \lambda p = 0.$

The coordinates of the centre are $\left(-\frac{2u+\lambda l}{2}, -\frac{2v+\lambda m}{2}, -\frac{2w+\lambda n}{2}\right)$ and radius =

$\sqrt{\left(\frac{2u+\lambda l}{2}\right)^2 + \left(\frac{2v+\lambda m}{2}\right)^2 - (d-\lambda p)}$.

The sphere whose radius as zero is known as a point sphere. For two values of λ the above radius is zero. Thus there are two point sphere in a coaxial system. The centres of these point spheres are called limting point.

Note : If the centre of all spheres in a coaxial system lie on the x-axis then the equation of the system can be written as

$x^2 + y^2 + z^2 + 2\lambda x + d = 0$, λ is a parameter. Hence the centre is $(-\lambda, 0, 0)$ and radius is $\sqrt{\lambda^2 - d}$.

For the point sphere, $\lambda^2 - d = 0$ or, $\lambda = \pm\sqrt{d}$.

Thus the limting points are $(\sqrt{d}, 0, 0)$ and $(-\sqrt{d}, 0, 0)$.

If $d \geq 0$, these points are real, otherwise they are imaginary.

Illustrated Examples

Example 1 : Show that the equation $x^2 + y^2 + z^2 + 2\mu y + 2\gamma z - d = 0$, where μ and γ are parameters, represent a system of spheres passing through the limiting points of the system $x^2 + y^2 + z^2 + 2\lambda x + d = 0$, and cutting every member of that system at right angles.

Solution : The limiting points of the coaxial system

$x^2 + y^2 + z^2 + 2\lambda x + d = 0$ are $(\pm\sqrt{d}, 0, 0)$. These points satisfy the equation $x^2 + y^2 + z^2 + 2\mu y + 2\gamma z - d = 0$

Again, $2\lambda.0 + 2.0.\mu + 2.0.\gamma = d - d$

$\therefore$ The first sphere cuts each member of the coaxial system orthogonally.

Example 2 : Prove that the equation of the sphere which cuts orthogonally each of the spheres

$x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, $x^2 + y^2 + z^2 + 2ax = a^2$,

$x^2 + y^2 + z^2 + 2by = b^2$, $x^2 + y^2 + z^2 + 2cz = c^2$ is

$$x^2 + y^2 + z^2 + \frac{b^2+c^2}{a}x + \frac{c^2+a^2}{b}y + \frac{a^2+b^2}{c}z + a^2 + b^2 + c^2 = 0$$

Solution : Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. It cuts the spheres orthogonally.

$\therefore$ $2u.0 + 2v.0 + 2w.0 = d - a^2 - b^2 - c^2$ (1)

$2u.a + 2v.0 + 2w.0 = d - a^2$ (2)

$2u.0 + 2v.b + 2w.0 = d - b^2$ (3)

$2u.0 + 2v.0 + 2w.0 = d - c^2$ (4)

From these relations, $d = a^2 + b^2 + c^2$,

$$2u = \frac{b^2+c^2}{a}, 2v = \frac{c^2+a^2}{b}, 2w = \frac{a^2+b^2}{c},$$

$\therefore$ The equation of the sphere is

$$x^2 + y^2 + z^2 + \frac{b^2+c^2}{a}x + \frac{c^2+a^2}{b}y + \frac{a^2+b^2}{c}z + a^2 + b^2 + c^2 = 0$$

Example 3 : Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and also cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Solution : Since the plane $3x + 2y - z + 2 = 0$ touches the sphere at the point $(1, -2, 1)$, the centre lies on the line perpendicular to the plane and passes through the point $(1, -2, 1)$.

The equation of this line is $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = r$ (say).

Let the centre be $(3r + 1, 2r - 2, -r + 1)$.

Radius of the sphere = distance between the centre and the point $(1, -2, 2) = \sqrt{14r}$.

Centre of the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ is $(2, -3, 0)$.

Radius of the sphere $\sqrt{4+9-4} = 3$.

Since the spheres cut orthogonally, then $(3r + 1 - 2)^2 + (2r - 2 + 3)^2 + (-r + 1)^2 = 14r^2 + 9$

or, $14r^2 - 4r + 3 = 14r^2 + 9$

or, $r = -\frac{3}{2}$.

Hence the centre of the sphere is $\left(-\frac{7}{2}, -5, \frac{5}{2}\right)$ and the radius $\frac{3}{2}\sqrt{14}$.

Thus the required sphere is $\left(x+\frac{7}{2}\right)^2 + (y+5)^2 + \left(z-\frac{5}{2}\right)^2 = \frac{9}{4}\times 14$

or, $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$

Example 4 : Find radius and center of the sphere $x^2 + 2x + y^2 - 2y + z^2 - 6z = 14$.

Solution : Given equation of the sphere is, $x^2 + 2x + y^2 - 2y + z^2 - 6z = 14$

$\Rightarrow (x^2 + 2x + 1) + (y^2 - 2y + 1) + (z^2 - 6z + 9) - 1 - 1 - 9 = 14$

$\Rightarrow (x + 1)^2 + (y - 1) + (z - 3)^2 = 25 = 5^2$

$\therefore$ The centre is $(-1, 1, 3)$ and radius is 5 unit.

Workedout Examples

Example 1 : Find the equation of the sphere described on the join of $P(2, -3, 4)$ and $Q(-5, 6, -7)$ as diameter.

Solution : The equation of the sphere is $(x - 2)(x + 5) + (y + 3)(y - 6) + (z - 4)(z + 7) = 0$

or $x^2 + y^2 + z^2 + 3x - 3y + 3z - 56 = 0$.

Example 2 : Find the centre and the radius of the sphere

$3x^2 + 3y^2 + 3z^2 + 2x - 4y - 2z - 1 = 0$

Solution : Dividing the equation by 3, we have

$$x^2 + y^2 + z^2 + \frac{2}{3}x - \frac{4}{3}y - \frac{2}{3}z - \frac{1}{3} = 0$$

$$\text{or, } \left(x + \frac{1}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \frac{1}{3} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = 1 = 1^2.$$

Therefore the centre is at $\left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$ and the radius is 1 unit.

Example 3 : Find the equation of the sphere which passes through the origin and touches the sphere $x^2 + y^2 + z^2 = 56$ at the point $(2, -4, 6)$.

Solution : The point $(2, -4, 6)$ lies on the given sphere whose centre is the origin. Since the required sphere passes through the origin, it touches the given sphere internally.

Moreover, it is described on the segment joining the two points $(0, 0, 0)$ and $(2, -4, 6)$ as diameter. Hence the required equation of the sphere is $x(x - 2) + y(y + 4) + z(z - 6) = 0$

or, $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$

Example 4 : Find the equation of the sphere through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$.

Solution : Let the equation of the sphere be $x^2 + y^2 + z^2 + 2gx + 2fy + 2hz + d = 0$ (1)

Since the sphere passes through the given points, then

$d = 0$,

$1 + 1 + 2f - 2h + d = 0$

$1 + 4 - 2g + 4f + d = 0$

$1 + 4 + 9 + 2g + 4f + 6h + d = 0$

Solving these equations, we get, $g = -\frac{15}{14}, f = -\frac{25}{14}, h = -\frac{11}{14}$, $d = 0$.

Then from equation(1) we get,

$7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0.$

Example 5 : A plane passing through a fixed point (a, b, c) cuts the axes in A, B, C. Show that the locus of the centre of the sphereOAB is $a/x + b/y + c/z = 2$.

Solution : Let the equation of the plane ABC be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$, so that A is $(\alpha, 0, 0)$ B is $(0, \beta, 0)$ and C is $(0, 0, \gamma)$.

It passes through the point (a, b, c).

Therefore $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ (1)

The sphere OABC passing through the points $(0, 0, 0)$, $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$ is $x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$.

If the centre of the sphere be at (x_1, y_1, z_1), then $x_1 = \frac{\alpha}{2}, y_1 = \frac{\beta}{2}, z_1 = \frac{\gamma}{2}$

Eliminating α, β, γ from (1) we obtain $\frac{a}{2x_1}+\frac{b}{2y_1}+\frac{c}{2z_1}=1$

or, $\frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2$

Hence the locus of (x_1, y_1, z_1) is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

Example 6 : Find the equations of the circle on the sphere $x^2 + y^2 + z^2 = 49$ whose centre is at the point (2, –1, 3).

Solution : The equation of the plane section of the given sphere whose centre is at (2, –1, 3) is $(x - 2)2 + (y + 1)(-1) + (z - 3)3 = 0$

or, $2x - y + 3z - 14 = 0$

The equation of the required circle are the section of the sphere $x^2 + y^2 + z^2 = 49$ by the plane $2x - y + 3z - 14 = 0$.

Example 7 : Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle.

Solution : Let the equation of the sphere through the circle be

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0 \qquad (1)$$

where λ being a constant.

Since the centre of the sphere coincides with the centre of the circle in the case of a great circle, the centre of the sphere $\left\{-\lambda, \frac{1}{2}(-7-3\lambda), (1-2\lambda)\right\}$ must lie on the plane $2x + 3y + 4z = 8$.

$$\therefore\ 2(-\lambda) - \frac{3}{2}(7 + 3\lambda) + 4(1 - 2\lambda) = 8$$

or, $\lambda = -1$

Hence the equation of the sphere becomes $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$.

Example 8 : Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 = 0$, $x - 2y + 3z + 1 = 0$ is a great circle.

Solution : The equation of a sphere through the given circle is $x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 + \lambda(x - 2y + 3z + 1) = 0$, where λ is a parameter

or, $x^2 + y^2 + z^2 + 2(1 + \lambda/2)x - 2(2 + \lambda)y + 2(1 + 3\lambda/2)z + 5 + \lambda = 0$.

Its centre is $\left\{-\left(1+\frac{\lambda}{2}\right), (2+\lambda), -\left(1+\frac{3\lambda}{2}\right)\right\}$.

If the circle is a great circle, this centre lies on the given plane.

$$\therefore\ -\left(1+\frac{\lambda}{2}\right) - 2(2+\lambda) - 3\left(1+\frac{3\lambda}{2}\right) + 1 = 0$$

or, $-7\lambda - 7 = 0$

or, $\lambda - 1$.

Hence the equation is $x^2 + y^2 + z^2 + x - 2y - z + 4 = 0$.

Example 9 : Show that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation.

Solution : The equation of any sphere through the first circle is

$x^2 + y^2 + z^2 - 2x + 3y + 4z + 5 + \lambda_1(5y + 6z + 1) = 0$ (1)

and that of any sphere through the second circle is

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda_2(x + 2y - 7z) = 0$ (2)

These equations will be identical, if

$-2 = \lambda_2 - 3$ (3)

$3 + 5\lambda_1 = 2\lambda_2 - 4$ (4)

$4 + 6\lambda_1 = 5 - 7\lambda_2$ (5)

$\lambda_1 - 5 = -6$ (6)

From (3) and (6), $\lambda_1 = -1$, $\lambda_2 = 1$.

These values of l_1 and l_2 satisfy the equations (4) and (5). Thus the equations (3), (4), (5) and (6) are consistent. Consequenty the two circles lie on the same sphere and the equation of it is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 - (5y + 6z + 1) = 0$

or, $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$

Example 10 : Find the coordinates of the centre and the radius of the circle $x - 2y - 2z + 7 = 0$, $x^2 + y^2 + z^2 - 2x + 6y + 4z - 35 = 0$

Solution : The centre of the given sphere is (1, –3, –2) and the radius is 7. The distance of the given plane from the centre = $\dfrac{1+6+4+7}{\sqrt{1^2+2^2+2^2}} = 6$

$\therefore$ radius of the circle = $\sqrt{7^2 - 6^2} = \sqrt{13}$.

The equations of the line perpendicular to the plane and passing through the centre (1, –3, –2) are $\dfrac{x-1}{1} = \dfrac{y+3}{-2} = \dfrac{2+2}{-2}$.

Any point on the line is $(r + 1, -2r - 3, -2r - 2)$.

If this point satisfies the equation of the plane, then

$r + 1 + 2(2r + 3) + 2(2r + 2) + 7 = 0$

or, $9r = -18$

or, $r = -2$

$\therefore$ The coordinates of the centre are (–1, 1, 2).

Example 11 : Find the equation of the sphere through the four points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c).

Solution : Let the equation of the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As it passes through the points (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c).

$d = 0$, $a^2 + 2ua = 0$ or, $2u = -a$, $\because a \neq 0$,

$b^2 + 2vb = 0$ or $2v = -b$, $\because b \neq 0$. Also $c^2 + 2wc = 0$ or, $2w = -c$

$\therefore c \neq 0$

$\therefore$ The equation of the sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

10.3 Summary

In this unit, we have learnt to determine equations of a sphere under several given conditions. We can also appreciate various properties of a sphere.

10.4 Exercises with Hints and Answers

1. Find the centre and radius of the following spheres :

   (i) $x^2 + y^2 + z^2 - 8y + 10z - 10 = 0$

   (ii) $2(x^2 + y^2 + z^2) - 2x + 4y - 6z = 15$

[Hints : Do yourself] [**Ans.** (i) (0, 4, –5), $\sqrt{51}$, (ii) (1/2, –1, 3/2), $\sqrt{11}$]

2. Find the equation of the sphere passing through the following points.

   (i) (0, 0, 0), (0, 1, –1), (–1, 2, 0) (1, 2, 3)

   (ii) (1, 1, 1), (–2, 1, 2), (3, –3, 1), (–1, 2, 1)

[Hints : (i) see the workedout example 4, (ii) see the illustrated example 1 in 10.1.3]

[**Ans.** (i) $7(x^2 + y^2 + z^2) -15x - 25y- 11z = 0$,

(ii) $5(x^2 + y^2 + z^2)+ 38x - 79y + 24z - 186 = 0$]

3. Final the equation of the sphere which has the lines segment joining the points (2, 3, 4) and (0, –1, 2) as diameter.

[Hints : see the worked out example 1]

[**Ans.** $x^2 + y^2 + z^2 - 2x - 2y - 6z + 5 = 0$]

4. Discuss the position of the point (2, –3, 0) w.r.t. the sphere $x^2 + y^2 + z^2 + 2x - 4y - 4z + 8 = 0$.

[Hints : Read the note of 12.1.1 and then find the position of the given point]

[**Ans.** outside]

5. Show that the equation to the sphere through the circle $x^2 + z^2 + y^2 = 9$, $2x + 3y + 4z = 5$ and the point (1, 2, 3) is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$
6. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 5$ at the point (2, 0, 1).

[Hints : see the illustrated example 1 of (10.1.12)] [**Ans.** $2x + z = 5$]

7. Show that the two circle $x^2 + y^2 + z^2 + 3x - 4y + 3z = 0$, $x - y + 2z - 4 = 0$ and $2(x^2 + y^2 + z^2) + 18x - 13y + 17z - 17 = 0$, $2x + y - 3z + 1 = 0$ lie on the same sphere and find its equation.

[Hints : see the worked out example 9] [**Ans.** $x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0$]

8. Show that the plane $2x - 2y + z = -12$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.

[Hints : see the illustrated example 1 of 10.1.12] [**Ans.** (–1, 4, –2)]

9. Determine the values of h for which the plane $x + y + z = h$ touches the sphere $x^2 + y^2 + z^2 = 48$.

[Hints : see the illustrated example 2 of 10.1.12] [**Ans.** ±12]

10. Find the value of a for which the plane $x + y + z = a\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$

[Hints : Follow the hints of exercise 9] [**Ans.** $\sqrt{3} \pm 3$]

11. If any tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$ makes intercepts a, b, c on the co-ordinate axes, then prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{m}$

[Hints : see the illustrated example 4 of 10.1.12]

12. Show that the sphere $x^2 + y^2 + z^2 = 64$ and $x^2 + y^2 + z^2 - 12x + 4y - 6z + 48 = 0$ touch each other internally and find their point of contact

[Hints : solve yourself] $\left[\textbf{Ans.} \left(\frac{48}{7}, -\frac{16}{7}, \frac{24}{7}\right)\right]$

13. Find the angle of intersection of the spheres

$x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$ and $x^2 + y^2 + z^2 - 6x - 2y + 2z + 2 = 0$.

[Hints : Follow the article (10.1.13)] $\left[\textbf{Ans.} \cos^{-1}\left(-\frac{2}{3}\right)\right]$

14. Find the radical line of the spheres

$x^2 + y^2 + z^2 - 2x + 2y - 2z + 4 = 0$, $x^2 + y^2 + z^2 + 3x + 4y + 2z - 2 = 0$,$x^2 + y^2 + z^2 - 4x - 2y + 6z + 8 = 0$

[Hints : solve yourself] $\left[\textbf{Ans. } \frac{x-1}{2} = \frac{y-2}{-3} = \frac{z}{-1}\right]$

15. Find the radical centre of the four spheres whose equations are $x^2 + y^2 + z^2 = 10$, $x^2 + y^2 + z^2 - 10x = 0$, $x^2 + y^2 + z^2 + 5y + 5 = 0$ and $x^2 + y^2 + z^2 + 2x + 4z - 4 = 0$.

[Hints : solve yourself] [**Ans.** (1, –3, –2)]

16. Find the limiting points of the co axial system defined by the spheres $x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$, $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$.

[Hints : solve yourself] [**Ans.** (–1, 2, 1), (–2, 1, –1)]

17. Prove that the general equation of all the spheres through the points (a, 0, 0), (0, b, 0), (0, 0, 0) is $x^2 + y^2 + z^2 - ax - by - cz - \lambda\left(\frac{x}{a} + \frac{x}{b} + \frac{z}{c} - 1\right) = 0$.

Find the value of λ so that the sphere may cut orthogonally the sphere represented by $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = 0$.

[Hints : Do yourself] $\left[\textbf{Ans. } -\frac{a^2 + b^2 + c^2}{2}\right]$

Unit 11 ❒ Cylinder and Cone

Structure

11.0 Objectives

We will be able to determine and appreciate

- equation of cylinder under various conditions
- equation of cones under various conditions
- various properties related to cylinder and cone.

11.1 Introduction

In this unit we will discuss about two types of quadratic surfaces viz. cylinder and cone. These surfaces are generated by variable straight lines.

11.2 Cylinder

A cylinder is a surface generated by a variable straight line which moves parallel to a fixed line and intersects a fixed curve not lying in a plane parallel to the fixed line or touches a given surface.

The fixed curve is called the guiding curve or directrix and the variable line is known as generator. If the guiding curve is a circle and the fixed line is normal to the plane of the circle through the centre of it, then the cylinder is right circular and the fixed line is called the axis of this cylinder. The distance between the axis and any generator is known as the radius of the right circular cylinder and it is equal to the radius of the guiding circle. Section of right circular cylinder by a plane perpendicular to the axis is called a normal section. It is a circle of the same radius as that of the cylinder.

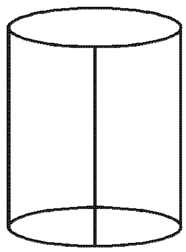


Fig. 11.2.1

11.2.1 Equation of a cylinder whose guiding curve is f(x, y) = 0, z = 0 and generators are parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$:

Let (α, β, γ) be a point on the cylinder. The equation of the generating line through (α, β, γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$. = r(say), where l, m, n are direction cosines of the generating line.

On the guiding curve z = 0,

and the point ($\alpha + lr$, $\beta + mr$, $\gamma + nr$) lies on the curve f(x, y) = 0, z = 0.

Therefore, this line meets the curve where $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{-\gamma}{n}$

or, $x = \alpha-\frac{l\gamma}{n}$, $y = \beta-\frac{m\gamma}{n}$, $z = 0$

This point satisfies the equation of the guiding curve.

$$\therefore\ f\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n}\right)=0$$

Hence the equation of the cylinder which is the locus of (α, β, γ) is

$$f\left(x-\frac{lz}{n},y-\frac{mz}{n}\right)=0$$

Corollary I : $f(x, y) = 0$ represents a cylinder when the fixed line is z-axis and the guding curve is $f(x, y) = 0$, $z = 0$.

Corollary II : The equation of the cylinder whose generators an parallal to z-axis and which intersect the curve $f(x, y, z) = 0$, $\phi(x, y, z) = 0$ is obtained by eliminating z between these equations.

llustrated Example

Example 1 : Find the equation of the cylinder whose guiding curve is $x^2 + y^2 = 9$, $z = 1$ and the fixed line is $\frac{x}{2}=\frac{y}{3}=\frac{z}{-1}$

Solution : Let (α, β, γ) be a point on the cylinder.

The generating line through this point is

$$\frac{x-\alpha}{2}=\frac{y-\beta}{3}=\frac{z-\gamma}{-1}.$$

It meets the curve $x^2 + y^2 = 9$, $z = 1$, so

$$\frac{x-\alpha}{2}=\frac{y-\beta}{3}=\frac{1-\gamma}{-1}$$

or, $x = \alpha + 2\gamma - 2$, $y = \beta + 3\gamma - 3$, $z = 1$

This point lies on the guiding curve.

$\therefore$ $(\alpha + 2\gamma - 2)^2 + (\beta + 3\gamma - 3)^2 = 9$

Hence the locus of (α, β, γ), i.e. the equation of the cylinder is

$(x + 2z - 2)^2 + (y + 3z - 3)^2 = 9.$

11.2.2 Equation of the right circular cylinder whose radius is a and axis is given by $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Let (α, β, γ) be a point on the cylinder and the perpendicular from this point to the axis meet the axis at ($lr + x_1$, $mr + y_1$, $nr + z_1$), where l, m, n are direction ratios of the axis.

The d.rs of the perpendicular are $lr + x_1 - \alpha$, $mr + y_1 - \beta$, $nr + z_1 - \gamma$.

From the condition of perpendicularity

$l(lr + x_1 - \alpha) + m(mr + y_1 - \beta) + n(nr + z_1 - \gamma) = 0$

$$\text{or } r = \frac{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{l^2 + m^2 + n^2} \qquad (11.2.3.1)$$

Again this perpendicular distance is equal to the radius a.

$\therefore\ (lr + x_1 - \alpha)^2 + (mr + y_1 - \beta)^2 + (nr + z_1 - \gamma)^2 = a^2$

or, $(l^2 + m^2 + n^2)r^2 + 2r\{l(x_1 - \alpha)^2 + m(y_1 - \beta) + n(z_1 - \gamma)\} + (\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2 = a^2$

$$\text{or } (\alpha - x_1)^2 + (\beta - y_1)^2 + (\gamma - z_1)^2 - \frac{\{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)\}^2}{l^2 + m^2 + n^2} = a^2$$

[by (11.2.3.1)]

Hence the locus of (α, β, γ) or the equation of the cylinder is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - \frac{\{l(x - x_1) + m(y - y_1) + n(z - z_1)\}^2}{l^2 + m^2 + n^2} = a^2.$$

Corollary : $x^2 + y^2 = a^2$ represents a right circular cylinder with the z axis as the axis of the cylinder.

Illustrated Example

Example 1 : Find the equation of the right circular cylinder of radius 2 whose axis is the straight line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$

Solution : The axis passes through the origin. The d.cs of the line joining the origin (0, 0, 0) and the point (α, β, γ) are $\frac{\alpha}{OP},\frac{\beta}{OP},\frac{\gamma}{OP}$. The d.cs of the axis are $\frac{1}{3},-\frac{2}{3},\frac{2}{3}$.

The distance of (α, β, γ) from the axis = OP $\sin\theta$, (θ is the angle between OP and the axis)

$= OP\sqrt{1-\cos^2\theta}$

$= OP\sqrt{1-\left(\frac{\alpha-2\beta+2\gamma}{3.OP}\right)^2}$

$= \frac{1}{3}\sqrt{9.OP^2-(\alpha-2\beta+2\gamma)^2}$

$= \frac{1}{3}\sqrt{9(\alpha^2+\beta^2+\gamma^2)-(\alpha-2\beta+2\gamma)^2}$

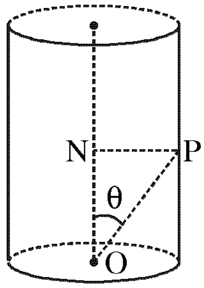


Fig. 11.2.3.1

$\therefore\ 2 = \frac{1}{3}\sqrt{9(\alpha^2+\beta^2+\gamma^2)-(\alpha-2\beta+2\gamma)^2}$

or, $8\alpha^2 + 5\beta^2 + 5\gamma^2 + 4\alpha\beta + 8\beta\gamma - 4\gamma\alpha = 36$

Hence the required equation is $8x^2 + 5y^2 + 5z^2 + 4xy + 8yz - 4zx = 36$.

11.2.3 Worked out Examples

Example 1 : Find the equation of the quadratic cylinder with generator parallel to z-axis and passing through the curve.

$ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p.$

Solution : Eliminating z between $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = p$,

we get, $ax^2 + by^2 + c\left(\frac{p-lx-my}{n}\right)^2 = 1$

It represents a cylinder whose generators are parallel to z-axis and intersect the given curve.

Example 2 : Find the equation of the cylinder whose generator are parallel to the straight line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 9$, $z = 1$.

Solution : Let (α, β, γ) be a point on the curve $x^2 + y^2 = 9$, $z = 1$.

Now the equation of the straight line parallel to the line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ passing through the point (α, β, γ) is $\frac{x-\alpha}{-1}=\frac{y-\beta}{2}=\frac{z-\gamma}{3}$.

This straight line cuts the curve $x^2 + y^2 = 9$, $z = 1$ at $\frac{x-\alpha}{-1} = \frac{1-\gamma}{3}$ and $\frac{y-\beta}{2} = \frac{1-\gamma}{3}$

i.e., $x = \alpha + \frac{\gamma - 1}{3}$ and $y = \beta - \frac{2\gamma - 2}{3}$.

Therefore, $\left(\alpha + \frac{\gamma-1}{3}\right)^2 + \left(\beta - \frac{2\gamma-2}{3}\right)^2 = 9$

or, $(3\alpha + \gamma - 1)^2 + (3\beta - 2\gamma + 2)^2 = 81$

or, $9\alpha^2 + 9\beta^2 + 5\gamma^2 + 6\gamma\alpha - 12\beta\gamma - 6\alpha + 12\beta - 10\gamma + 5 - 81 = 0$

or, $9\alpha^2 + 9\beta^2 + 5\gamma^2 + 6\gamma\alpha - 12\beta\gamma - 6\alpha + 12\beta - 10\gamma - 76 = 0$

$\therefore$ The required equation of the cylinder is

$9x^2 + 9y^2 + 5z^2 + 6zx - 12yz - 6x + 12y - 10z - 76 = 0$

Example 3 : Find the equation of the right circular cylinder of radius 3 and whose axis is $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{6}$.

Solution : The direction cosines of the axis are $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

The length of the perpendicular from a point (x, y, z) to the axis is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 - \left\{\frac{2}{7}(x-1) - \frac{3}{7}(y-2) + \frac{6}{7}(z-3)\right\}^2.$$

Hence the equation of the cylinder is

$$(x-1)^2 + (y-2)^2 + (z-3)^2 - \frac{1}{49}\{2(x-1) - 3(y-2) + 6(z-3)\}^2 = 9$$

Example 4 : Find the equation of the cylinder whose generating line is parallel to the z-axis and guiding curve is $x^2 + y^2 = z$, $x + y + z = 1$.

Solution : Let (α, β, γ) be the point on the curve.

$\therefore$ $(\alpha^2 + \beta^2) = \gamma$ and $\alpha + \beta + \gamma = 1$ (1)

If (x, y, z) is any point on the cylinder, then the straight line joining the points (α, β, γ) and (x, y, z) is parallel to the z-axis whose direction cosines are 0, 0, 1

The equations are $\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{z-\gamma}{1}$

i.e. $x = \alpha$, $y = \beta$

Substituting these in (1), we get $x^2 + y^2 = \gamma$, $x + y + \gamma = 1$.

Eliminating γ, we get the equation of the cylinder as $x^2 + y^2 + x + y = 1$

11.2.4 Exercises with hints and answers

1. Obtain the equation of the cylinder whose generators intersect the plane curve $ax^2 + by^2 = 1$, $z = 0$ and are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

   [**Ans.** $a(nx - lz)^2 + b(ny - mz)^2 = n^2$]

2. Find the equation of the right circular cylinder whose axis is $\frac{x}{1} = \frac{y}{0} = \frac{z}{2}$ and radius equal to 7. [Hints : see the illustrated example 1 of (10.2.3)]

   [**Ans.** $5(x^2 + y^2 + z^2 - 49) = (x - 2z)^2$]

3. Find the equation of the right circular cylinder whose axis as $\frac{x}{2} = \frac{y}{3} = \frac{z}{6}$ and radius equal to $\sqrt{5}$. [Hints : see the illustrated example 1 of (10.2.3)

   [**Ans.** $49(x^2 + y^2 + z^2 - 5) = (2x + 3y + 6z)^2$]

4. Find the equation of the right circular cylinder of radius 5, whose axis passes through (1, 2, 3) and is parallel to $\frac{x-4}{2} = \frac{y-3}{-1} = \frac{z-2}{2}$.

   [**Ans.** $5x^2 + 8y^2 + 5z^2 + 4yz - 8zx + 4xy + 6x - 48y - 30z - 135 = 0$]

5. Show that $2x^2 + 5y^2 + 5z^2 + 4xy + 2yz - 4zx + 16x + 22y - 10z - 18 = 0$ is the equation of the cylinder which passes through the point (3, –1, 1) and has the axis $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{1}$.

6. Show that the equation of the cylinder whose generators are parallel to the straight line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1$, $z = 3$ is $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$.

11.3 Cone

A cone is a surface generated by a straight line passing through a fixed point and intersecting a curve or touching a given surface.

The fixed point is known as the vertex and the given curve is called the guiding curve or directrix or base. Anyline lying on the cone is called its generator.

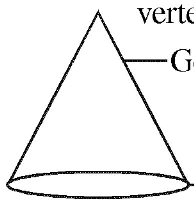


Fig. 11.3.1

11.3.1 The equation of cone with its vertex as origin is homogeneous in x, y, z and conversely :

Let f(x, y, z) = 0 be the equation of a cone with the vertex at the origin O. If P(x′, y′, z′) be a point on the cone, the line OP will wholey lie on the cone. Any point on this line can be written as (rx′, ry′, rz′), where r is a non-zero scalar.

Now (rx′, ry′, rz′) satisfies the equation f(x, y, z) = 0 for all values of r. It is possible only when f(x, y, z) = 0 is homogeneous in x, y, z.

Couressely, if f(x, y, z) = 0 is a homogeneous equation in x, y, z, then (rx′, ry′, rz′) will satisfy the equation when P(x′, y′, z′) satisfies the equation. Therefore the line OP wholely lies on f(x, y, z) = 0. Hence f(x, y, z) = 0 represents a cone with the vertex as origin.

Corollary : If f(x, y, z) = 0 is a homogeneous equation of a cone and $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be generator, then f(l, m, n) = 0.

Note : If f(x, y, z) is factorizable into two linear factors, f(x, y, z) = 0 will represent a pair of planes; otherwise it represents a cone.

11.3.2 Condition for the general equation of second degree to represent a cone and coordinates of vertex

Let $F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ (11.3.3.1) represents a cone with the vertex at (x′, y′, z′).

Now changing the origin to (x′, y′, z′) with the axes remaining parallel, the equation (11.3.3.1) transforms to

$a(x + x')^2 + b(y + y')^2 + c(z + z')^2 + 2f(y + y')(z + z') + 2g(z + z')(x + x') + 2h(x + x')(y + y')$

$+ 2u(x + x') + 2v(y + y') + 2w(z + z') + d = 0$

or, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2(ax' + hy' + gz' + u)x$

$+ 2(hx' + by' + fz' + v)y + 2(gx' + fy' + cz' + w)z$

$+ ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y'$

$+ 2ux' + 2vy' + 2vy' + 2wz' + d = 0$ (11.3.3.2)

Now, this equation represents a cone with its vertex at the origin.

Therefore , the equation (10.3.3.2) should be homogenous.

For this the coefficient of x, y, z and the absolute term should be zero.

Thus $ax' + hy' + gz' + u = 0$ (11.3.3.3)

$hx' + by' + fz' + v = 0$ (11.3.3.4)

$gx' + fy' + cz' + w = 0$ (11.3.3.5)

and $ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2ux' + 2vy' + 2wz' + d = 0$

or $x'(ax' + hy' + gz' + u) + y'(hx' + by' + fz' + v) + z'(gx' + fy' + cz' + w)$

$+ (ux' + vy' + wz' + d) = 0$

or, $ux' + vy' + wz' + d = 0$ [by (10.3.3.3), (10.3.3.4), (10.3.3.5)] .....(11.3.3.6)

Eliminating x′, y′, z′ from (10.3.3.3), (10.3.3.4), (10.3.3.5) and (10.3.3.6), we get the necessary condition for the general equation of the second degree to represent a cone as

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \qquad (11.3.3.7)$$

The coordinates of the vertex are obtained by solving any three of the equations (11.3.3.3), (11.3.3.4), (11.3.3.5) and (11.3.3.6) when the condition (11.3.3.7) holds.

Note 1 : If $F(x, y, z, t) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2$, then $\frac{\partial F}{\partial x} = 2(ax + hy + gz + ut)$,

$$\frac{\partial F}{\partial y} = 2(hx + by + fz + vt)$$

$$\frac{\partial F}{\partial z} = 2(gx + fy + cz + wt)$$

$$\frac{\partial F}{\partial t} = 2(ux + vy + wz + dt).$$

Putting t = 1, the equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$, $\frac{\partial F}{\partial t} = 0$ are satisfied by the vertex (x', y', z').

Elimination of (x', y', z') from these equations gives the condition for representing a cone.

Note 2 : If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ with the condition (11.3.3.7) then the general equation represents a pair of planes.

Illustrated Example

Example 1 : Show the equation $x^2 + 2y^2 + z^2 - 4yz - 6zx - 2x + 8y - 2z + 9 = 0$ represents a cone. Find the vertex.

Solution : Making the expression of the equation homogeneous, we get F $(x, y, z, t) \equiv x^2 + 2y^2 + z^2 - 4yz - 6zx - 2xt + 8yt - 2zt + 9t^2$.

Putting t = 1 in $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial t}$ and then equating to zero, the four equations are

$x - 3z - 1 = 0$ (1)

$y - z + 2 = 0$ (2)

$3x + 2y - z + 1 = 0$ (3)

$x - 4y + z - 9 = 0$ (4)

From (1), (2) and (3), $x = 1$, $y = -2$, $z = 0$.

These values satisfy the equation (4). Therefore the equations are consistent. Cousequently the given equation represents a cone with the vertex at (1, –2, 0)

11.3.3 Equation of cone with given vertex and directrix :

1. To find the equation of the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$.

Since the vertex is the origin, the equation of the cone is homogeneous. Thus making $ax^2 + by^2 + cz^2 = 1$ homogeneous by $lx + my + nz = p$, the required equation is obtained.

The equation is $ax^2 + by^2 + cz^2 = \left(\frac{lx+my+nz}{p}\right)^2$

or, $p^2(ax^2 + by^2 + cz^2) = (lx + my + nz)^2$.

2. To find the equation of the cone whose vertex is the point (α, β, γ) and the base is the conic $f(x, y) = 0$, $z = 0$

The equation of a line through the point (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (10.3.4.1)$$

It meets the plane $z = 0$, where $x = \alpha - \frac{l\gamma}{n}$, $y = \beta - \frac{m\gamma}{n}$.

If the line meets the curve $f(x, y) = 0$, then $f\left(\alpha - \frac{lr}{n}, \beta - \frac{m\gamma}{n}\right) = 0 \qquad (10.3.4.2)$

Eliminating l, m, n from (10.3.4.1) and (10.3.4.2), $f\left(\alpha - \gamma\frac{x-\alpha}{z-\gamma}, \beta - \gamma\frac{y-\beta}{z-\gamma}\right) = 0$

or, $f\left(\frac{\alpha z - \gamma x}{z-\gamma}, \frac{\beta z - \gamma y}{z-\gamma}\right) = 0$

It is the required equation of the cone whose vertex (α, β, γ) and base is the conic $f(x, y) = 0$, $z = 0$

11.3.4 Right circular cone

A cone formed by a variable line passing through a fixed point (vertex) and making a constant angle with a given line through the vertex is called a right circular cone.

The given line and the constant angle are called the axis and the semi-vertical angle of the cone respectively.

To find the equation of right circular cone whose vertex is the point (α, β, γ), the semi-vertical angle is θ and the axis is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

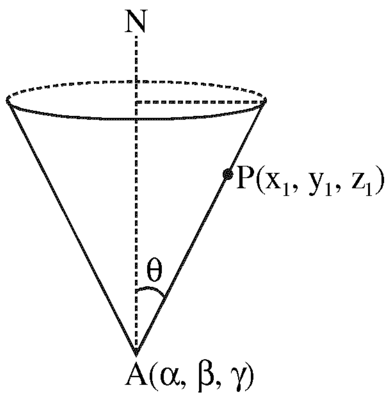


Fig. 11.3.4.1

Let A be the vertex, AN the axis and $P(x_1, y_1, z_1)$ a point on the cone.

The d.c.s. of the axis are $\dfrac{l}{\sqrt{l^2+m^2+pn^2}}, \dfrac{m}{\sqrt{l^2+m^2+n^2}}, \dfrac{n}{\sqrt{l^2+m^2+n^2}}$.

The d.cs. of the line AP are $\dfrac{x_1-\alpha}{AP}, \dfrac{y_1-\beta}{AP}, \dfrac{z_1-\gamma}{AP}$.

Since θ is the angle between two lines $\cos\theta = \dfrac{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{AP.\sqrt{l^2+m^2+n^2}}$

or, $AP^2(l^2 + m^2 + n^2)\cos^2\theta = \{l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)\}^2$

or, $\{(x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2\}(l^2 + m^2 + n^2)\cos^2\theta$

$= \{l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)\}^2$ (11.3.5.1)

Hence the locus of P, i.e. the equation of the cone is (11.3.5.1).

Corollary I : If the vertex is the origin, then $(\alpha, \beta, \gamma) \equiv (0, 0, 0)$. In this case, the equation of the cone is $(x^2 + y^2 + z^2)(l^2 + m^2 + n^2)\cos^2\theta = (lx + my + nz)^2$.

Note that it is a homogeneous equation of second degree.

Corollary II : If the vertex is the origin and the z-axis is the axis of the cone, then $(\alpha, \beta, \gamma) \equiv (0, 0, 0)$ and $l = 0$, $m = 0$, $n = 1$.

Therefore the required equation of the cone is $(x^2 + y^2 + z^2)\cos^2\theta = z^2$

or, $x^2 + y^2 = z^2 \tan^2\theta$

11.3.5 Angle between lines in which a plane cuts a cone

Let $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ (11.3.5.1)

be a cone and $ux + vy + wz + d = 0$ (11.3.5.2)

be a plane which cuts the cone along two lines,

Let a line in which the plane cuts the cone be $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ (11.3.5.3)

Since the line (11.3.6.3) lies on the plane (11.3.6.2),

$ul + vm + wn = 0$ (11.3.5.4)

This line is a generater of the cone (11.3.6.1).

$\therefore\ al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$ (11.3.5.5)

Eliminating n from (10.3.6.5) by (10.3.6.4)

$$al^2+bm^2+c\left(\frac{ul+vm}{w}\right)^2-2fm\left(\frac{ul+vm}{w}\right)-2gl\left(\frac{ul+vm}{w}\right)+2hlm=0$$

or $(cu^2 + aw^2 - 2gwu)l^2 + 2(hw^2 + cuv - fuw - gvw)lm + (bw^2 + cv^2 - 2fvw)m^2 = 0$

or, $(cu^2 + aw^2 - 2gwu)\left(\frac{l}{m}\right)^2 + 2(hw^2 + cuv - fuw - gvw)\left(\frac{l}{m}\right)$

$+ (bw^2 + cv^2 - 2fvw) = 0$ (11.3.6.6)

Since (10.3.6.6) is quadratic in $\frac{l}{m}$, the plane (10.3.6.2) cuts the cone in two lines.

If their direction ratios are l_1, m_1, n_1 and l_2, m_2, n_2, then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ one the roots of (11.3.6.6), of the values of n_1 and n_2 in terms of l_1 and l_2 are obtained from (10.3.6.4).

If $m_1 = \lambda_1 l_1$, $n_1 = \mu_1 l_1$ and $m_2 = \lambda_2 l_2$, $n_2 = \mu_2 l_2$, then the lines of section are $\frac{x}{l}=\frac{y}{\lambda_1}=\frac{z}{\mu_1}$ and $\frac{x}{l}=\frac{y}{\lambda_2}=\frac{z}{\mu_2}$.

Thus the angle between the lines is $\cos^{-1}\frac{1+\lambda_1\lambda_2+\mu_1\mu_2}{\sqrt{1+\lambda_1^2+\mu_1^2}.\sqrt{1+\lambda_2^2+\mu_2^2}}$.

Corollary I : Condition for two perpendicular generators made by a plane.

From (11.3.6.6)

$$\frac{l_1l_2}{m_1m_2}=\frac{bw^2+cv^2-2fvw}{cu^2+aw^2-2gwu} \quad \text{(as product of the roots).}$$

From symmetry $\frac{l_1l_2}{m_1m_2}=\frac{bw^2+cv^2-2fvw}{av^2+bu^2-2huv}$.

$$\therefore \frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}=\frac{n_1n_2}{av^2+bu^2-2huv}=k(\text{say})$$

$$\therefore l_1l_2+m_1m_2+n_1n_2=k[a(v^2+w^2)+b(w^2+u^2)+c(u^2+v^2)-2fvw-2gwu-2huv]$$
$$=k[(a+b+c)(u^2+v^2+w^2)-\phi(u,v,w)].$$

If the plane cuts the cone in two perpendicular generators, then

$l_1l_2 + m_1m_2 + n_1n_2 = 0$

i.e. $(a + b + c)(u^2 + v^2 + w^2) - \phi(u, v, w) = 0$.

Corollary II : Necessary and sufficient condition for three mutually, perpendicular generators.

Let the normal to the plane (10.3.6.2) through the origin, i.e. $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$ is a generator of the cone.

Then $\phi(u, v, w) = 0$.

In this case, $l_1l_2 + m_1m_2 + n_1n_2 = k(a + b + c)(u^2 + v^2 + w^2)$.

If the plane now cuts the cone in two perpndicular generators, then

$a + b + c = 0, \quad \because u^2 + v^2 + w^2 \neq 0$.

It is independent of u, v, w.

Therefore, $a + b + c = 0$ is the necessary condition for three mutually perpendicular generators of the cone (10.3.6.1).

From the condition of $a + b + c = 0$ it follows that the plane perpendicular to an arbitrary generator cuts the cone in two perpendicular generators. Therefore $a + b + c = 0$ is the sufficient condition for three mutually perpendicular generators.

Corollary III : If $a + b + c = 0$, the cone (10.3.6.1) has an infinite number of triads of mutually perpendicular generators.

11.3.6 Worked out Examples

Example 1 : Find the equation of the right circular cone which contains three positive co-ordinate axes.

Solution : Here the axis of the cone makes equal angles with the co-ordinate axes. Therefore the semi-vertical angle of the cone is $\cos^{-1}\frac{1}{\sqrt{3}}$.

The equation of the axis are $x = y = z$.

If (x, y, z) is any point on the cone, then the respective direction ratios of the generator through the point (x, y, z) and the axis are x, y, z and 1, 1, 1.

The angle between them is $\cos^{-1}\frac{1}{\sqrt{3}}$.

Therefore $\frac{1}{\sqrt{3}} = \frac{x+y+z}{\sqrt{x^2+y^2+z^2}.\sqrt{1^2+1^2+1^2}}$.

or, $(x^2 + y^2 + z^2) = (x + y + z)^2$

or, $xy + yz + zx = 0$, this is the required equation of the cone.

Example 2 : Find the equation of the cone whose vertex is the origin and base is circle $x = a$, $y^2 + z^2 = b^2$.

Show that the section of the cone by a plane parallel to XOY is a hyperbola.

Solution : The equation $y^2 + z^2 = b^2$ is made homogeneous with the help of the equation of the plane $x = a$, i.e. $\frac{x}{a} = 1$.

Then $y^2 + z^2 = b^2 \left(\frac{x}{a}\right)^2$

or, $a^2(y^2 + z^2) = b^2x^2$. This is the requied equation of the cone..

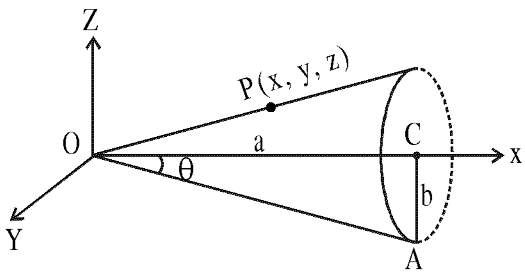
Putting $z = c$, we get the section of the plane parallel to XOY plane as $b^2x^2 - a^2y^2 = a^2c^2$, which is a hyperbola.

Alternative solution of 1st Part :

In the fig. OC = a, AC = b (radius of the base).

If θ is the semi-vertical angle, then $\tan\theta = \frac{b}{a}$.

The equation of the right circular cone whose vertex, semi-vertical angle and axis are the origin, θ and the x-axis respectively is $y^2 + z^2 = x^2\tan^2\theta$



or $y^2 + z^2 = x^2.\frac{b^2}{a^2}$

or $a^2(y^2 + z^2) = b^2x^2$

Example 3 : The axis of a right cone, vertex O, makes equal angles with the coordinate axes and the cone passes through the line drawn from O, with d.cs. proportional to 1, –2, 2. Find the equation to the cone.

Solution : The d.cs. of the axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and the d.cs of the given generating line are $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

If θ is the semi-vertical angle, $\cos\theta = \frac{1-2+2}{3\sqrt{3}} = \frac{1}{3\sqrt{3}}$.

Therefore the equation of the cone is $(x^2 + y^2 + z^2).\frac{1}{27} = \frac{1}{3}(x + y + z)^2$

or, $9(x + y + z)^2 = x^2 + y^2 + z^2$.

Example 4 : Find the equations of the lines of intersection of the plane $3x + 4y + z = 0$ and the cone $15x^2 - 32y^2 - 7z^2 = 0$

Solution : Let the equation of the generating line in which the plane cuts the cone be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

The line lies on the cone and also on the plane.

$\therefore\ 15l^2 - 32m^2 - 7n^2 = 0$ (1)

and $3l + 4m + n = 0$ (2)

Eliminating n from (1) and (2),

$15l^2 - 32m^2 - 7(3l + 4m)^2 = 0$

or, $2l^2 + 7ln + 6m^2 = 0$

or, $(2l + 3m)(l + 2m) = 0$

or, $m = -\frac{2}{3}l, -\frac{1}{2}l$

For $m = -\frac{2}{3}l$, $n = -(3l + 4m) = -\left(3l - \frac{8}{3}l\right) = -\frac{1}{3}$,

For $m = -\frac{1}{2}l$, $n = -(3l + 4m) = -(3l - 2l) = -1$

$\therefore$ The d.cs. are proportional to $1, -\frac{2}{3}, -\frac{1}{3}$ or $1, -\frac{1}{2}, -1$.

Hence the equation of the required generators are

$$\frac{x}{3}=\frac{y}{-2}=\frac{z}{-1} \text{ and } \frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}.$$

Example 5 : The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is $\left(\frac{b}{c}+\frac{c}{a}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)zx+\left(\frac{a}{b}+\frac{b}{a}\right)xy=0$.

Solution : The plane meets the axes at (a, 0, 0), (0, b, 0) and (0, 0, c). The equation of the sphere through (0, 0, 0) and the above three points is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Therefore the equations of circle are

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (1)$$

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \qquad (2)$$

The equation of the cone whose vertex is the origin is homogeneous. Thus the equation to the cone is obtained by making (2) homogeneous with the help of (1). Hence the required equation is

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

or, $\left(\frac{b}{c}+\frac{c}{b}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)zx+\left(\frac{a}{b}+\frac{b}{a}\right)xy=0$.

Example 6 : Find the equations of the straight lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Final also the angle between them.

Solution : Eliminating z between the two equations, we get $4x^2 - y^2 + 3(2x + y)^2 = 0$

or, $8x^2 + 6xy + y^2 = 0$

or, $\frac{x}{y}=-\frac{1}{2},\frac{-1}{4}$

From the equation of the plane by putting these values for $\frac{x}{y}$, we have $\frac{z}{y}=0, \frac{1}{2}$.

Thus we have two sets of values of the ratio x : y : z given by x : y : z = –1 : 2 : 0 or –1 : 4 : 2

Thus the equations of the two lines of section are $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$ and $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$.

If θ is the angle between these two straight lines, then $\cos\theta=\frac{1+8+0}{\sqrt{5}.\sqrt{21}}$

or, $\theta = \cos^{-1}\frac{9}{\sqrt{105}}$.

Example 7 : Find the equation of the cone whose vertex is the point (1, 2, 3) and guiding curve is the circle $x^2 + y^2 + z^2 = 9$, $x + y + z = 1$.

Solution : Any generator of the cone through the point (1, 2, 3) is

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{x+y+z-6}{l+m+n} \qquad (1)$$

It meets, the plane $x + y + z = 1$

Therefore $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{-5}{l+m+n}$

Hence $x=1-\frac{5l}{l+m+n}=\frac{m+n-4l}{l+m+n}$,

$$y=2-\frac{5m}{l+m+n}=\frac{2l-3m+2n}{l+m+n}$$

$$z=3-\frac{5n}{l+m+n}=\frac{3l+3m-2n}{l+m+n}$$

If this point lies on the sphere $x^2 + y^2 + z^2 = 9$, then

$$(m + n - 4l)^2 + (2l - 3m + 2n)^2 + (3l + 3m - 2n)^2 = 9(l + m + n)^2 \qquad (2)$$

Eliminating l, m, n between (1) and (2), we get the requied equation of the cone as

$$(y + z - 4x - 1)^2 + (2x - 3y + 2z - 2)^2 + (3x + 3y - 2z - 3)^2 = 9(x + y + z - 6)^2.$$

Example 8 : Show that the condition that the plane $ax + by + cz = 0$ may cut the cone $yz + zx + xy = 0$ in perpendicular lines is $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Solution : Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the generators of the cone made by the plane.

Then,

$al + bm + cn = 0$ (1)

and, $mn + nl + lm = 0$ (2)

Eliminating n from (1) and (2), we get

$$-(l+m)\frac{al+bm}{c}+lm=0$$

or, $al^2 + (a + b - c)lm + bm^2 = 0$

$$\text{or, } a\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+b=0$$

Let the roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$. Then $\frac{l_1}{m_1}.\frac{l_2}{m_2}=\frac{b}{a}$

From symmetry $\frac{l_1}{n_1}.\frac{l_2}{m_2}=\frac{c}{a}$

$\therefore\ al_1l_2 = bm_1m_2 = cn_1n_2$.

If the generaters are at right angle, then

$l_1l_2 + m_1m_2 + n_1n_2 = 0$

or $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Example 9 : Find the equation of the cone whose vertex is (1, 0 –1) and which passes through the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$

Solution : Let $\frac{x-1}{l}=\frac{y}{m}=\frac{z+1}{n}$ be a generator.

If it meets the base at the point $(lr + 1, mr, nr - 1)$, then $lr + 1 + mr + nr - 1 = 1$

or, $(l + m + n)r = 1$ (1)

and $(lr + 1)^2 + m^2r^2 + (nr - 1)^2 = 4$

or $(l^2 + m^2 + n^2)r^2 + 2(l - n)r = 2$ (2)

By (1) and (2),

$(l^2 + m^2 + n^2) + 2(l - n)(l + m + n) = 2(l + m + n)^2$.

Now eliminating l, m, n by the equation of the generator the equation of the cone is

$(x - 1)^2 + y^2 + (z + 1)^2 + 2(x - z - 2)(x + y + z) = 2(x + y + z)^2$.

Example 10 : If the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone 5yz – 8zx – 3xy = 0, then find the equations of the other two.

Solution : It is evident that the cone has three perpendicular generators, one of them being perpendicular to the plane containing the other two.

Thus the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ being one generator of the set of mutually perpendicular three generators, the other two generators will be the lines of section of the given cone by the plane through the vertex whose normal is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

i.e. the plane x + 2y + 3z = 0

Now if $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be a line section, then

$5mn - 8nl - 3lm = 0$ and $l + 2m + 3n = 0$ (1)

Eliminating n between these relations, we have

$$(5m - 8l)\left(\frac{l+2m}{-3}\right) - 3lm = 0$$

or, $l = m$ and $-\frac{5m}{4}$

If l = m, then 3m + 3n = 0, i.e. m = –n,

Therefore, $\frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$ (2)

when $l = -\frac{5m}{4}$, then m + 4n = 0, i.e. m = –4n

Therefore, $\frac{l}{5}=\frac{m}{-4}=\frac{n}{1}$ (3)

Hence the other two generators are $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ and $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$.

Example 11 : The section of a cone whose guiding curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, z = 0 by the plane x = 0 is a rectangular hyperbola. Show that the locus of the vertex is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.

Solution : Let (α, β, γ) be the vertex and the equation of a generator be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad (1)$$

It meets the guiding curve at the point $\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$.

Putting these values in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\text{or, } \frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)^2=1 \qquad (2)$$

Eliminating l, m, n by (1), $\frac{1}{a^2}\left(\alpha-\gamma\frac{x-\alpha}{z-\gamma}\right)^2+\frac{1}{b^2}\left(\beta-\gamma\frac{y-\beta}{z-\gamma}\right)^2=1$

$$\text{or, } \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma z)^2=(z-\gamma)^2 \qquad (3)$$

It is the equation of the cone.

The section by x = 0 is the curve

$$\frac{\alpha^2}{a^2}z^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2,\ x=0.$$

It will be a rectangular hyperbola, if (the coefficient of y^2 + the coefficient of z^2) = 0,

$$\text{i.e. } \frac{\gamma^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=0$$

$$\text{or, } \frac{\alpha^2}{a^2}+\frac{\beta^2+\gamma^2}{b^2}=1.$$

Hence the locus of the vertex is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.

11.4 Summary

In this unit, we have learnt to determine equations of a cylinder and a cone under several given conditions. We can also understand and work on various important properties of these geometric entities.

11.5 Exercises

1. Find the equation to the right circular cone, whose vertex is the origin, semi-vertical angle $\frac{\pi}{4}$ and the axis is $x = y = z$. [**Ans.** $x^2 + y^2 + z^2 - 4(yz + zx + xy) = 0$]

2. Find the equation of a right circular cone which passes through the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and whose axis is $\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$.
   [**Ans.** $13x^2 + 100y^2 + 100z^2 - 232yz + 116zx - 116xy = 0$]

3. Find the equation of the cone whose vertex is (2, 3, 4) and the base is $x^2 + y^2 = 25$, $z = 0$ [**Ans.** $16x^2 + 16y^2 - 12z^2 - 24yz - 16zx + 200z - 400 = 0$]

4. Find the equation to the cone with the vertex at the origin and which passes through the curve of intersection of $x^2 + 5y^2 - 7z^2 = 1$ and $2x - 3y + 4z = 1$
   [**Ans.** $3x^2 + 4y^2 + 23z^2 - 24yz + 16zx - 12xy = 0$]

5. Determine the angle between the lines of intersection of the plane $x - 3y + z = 0$ and a quadratic cone $x^2 - 5y^2 + z^2 = 0$. $\left[\textbf{Ans. } \cos^{-1}\frac{5}{6}\right]$

6. Find the equation of the lines in which the plane $2x - 6y - 5z = 0$ cuts the cone $xy + yz + zx = 0$ $\left[\textbf{Ans. } \frac{x}{2} = \frac{y}{-1} = \frac{z}{2}, \frac{x}{15} = \frac{y}{10} = \frac{z}{-6}\right]$

7. If a right circular cone has three mutually perpendicular generators, prove that the semi-vertical angle is $\tan^{-1}\sqrt{2}$.

   [Hinta : see the article 10.3]

8. Show that the equation $x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$ represents a cone with vertex (1, –2, 3).

9. Show that the equation $7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$ represents a cone whose vertex is (1, –2, 2).

10. Show that the equation of the cone with vertex (5, 4, 3) and base $3x^2 + 2y^2 = 6$, $y + z = 0$ is $3(5y + 5z - 7x)^2 + 2(4z - 3y)^2 = 6(y + z - 7)^2$.
11. Show that the semi-vertical angle of the right circular cone $4(x^2 + y^2) - 9z^2 = 0$ is $\tan^{-1}\left(\frac{3}{2}\right)$.
12. Find the equation of the right circular cone whose vertex is the origin and whose axis is $\frac{x}{3}=\frac{y}{2}=\frac{z}{4}$ and semi vertical angle is 45°.

    [**Ans.** $29(x^2 + y^2 + z^2) = 2(3x + 2y + 4z)^2$]
13. Find the equation of the right circular cone with vertex at the point (1, –2, –1), semi-vertical angle 60° and axis $\frac{x-1}{3}=\frac{y+2}{-4}=\frac{z+1}{5}$.

    [**Ans.** $2(3x - 4y + 5z - 6)^2 = 25\{(x - 1)^2 + (y + 2)^2 + (z + 1)^2\}$
14. Show that the equation of the cone whose vertex is the origin and which passes through the curve of intersection of $4x^2 - 5y^2 - 7z^2 = 2$, $3x - 2y + 4z = 3$ is $9(4x^2 - 5y^2 - 7z^2) = 2(3x - 2y + 4z)^2$.

Unit 12 ❐ Central Conicoids, Conicoids and Tangent, Normal

Structure

12.0 Objectives

We will be able to determine and appreciate

- general quadratic equation in three variables representing conicoids
- equations of ellipsoids
- equations of hyperloloids
- equations of paraboloids
- tangent lines and tangent plane to conicoids
- director sphere
- normal to a conicoid

12.1 Introduction

Conicoids are classified into two types central and non-central conicoids. In this unit we will study central conicoids. In general there are four types of central conicoids, which are: cone; ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets.

12.2 General equation of second degree (conicoid)

The surface represented by the general equation of second degree in x, y, z : $ax^2 + by^2 + cz^2 + 2gzx + 2fyz + 2hxy + 2ux + 2vy + 2wz + d = 0$ is called a conicoid.

A conicoid S is called symmetric with respect to a point P if, when the origin is shifted to P, the transformed conicoid S is symmetric with respect to the origin.

A point P is called a centre of a conicoid S if S is symmetric with respect to P.

Now, a central conicoid is a conicoid which has a unique centre.

For example, a sphere is a central conicoid.

The general equation of a central conicoid is given by, $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$.

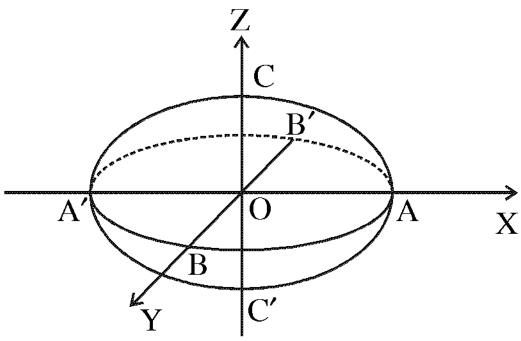
Now, we will discuss various types of central conicoid.

12.3 Ellipsoids

Let us consider the surface is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (11.3.1)$$

in which a, b, c are real numbers, none of them being zero. The surface given by (11.3.1) is called an ellipsoid, the equation being in its canonical form.



Ellipsoid

Fig. 12.3.1

Moreover if $a = b = c$, then the ellipsoid is a sphere.

The section of the ellipsoid (11.3.1) by the plane $z = k$, has for its equation, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}=\frac{c^2-k^2}{c^2}$, $z = k$.

This section is an ellipse, if $c^2 > k^2$ that is $-c < k < c$ with semi-axes $a\sqrt{1-\frac{k^2}{c^2}}$ and $b\sqrt{1-\frac{k^2}{c^2}}$.

If $k > c$, then the section is not real, hence $k > c$ is an impossibility, since the ellipsoid is a closed surface.

If $k = c$ or $-c$, then the semi-axes of the ellipse are zero and we say that the section reduces to the point $(0, 0, c)$ or $(0, 0, -c)$.

If k = 0, then the section is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Similarly, the section by planes parallel to the planes x = k and y = k are also ellipses. Hence an ellipsoid is generated by a variable ellipse.

There is also a type of ellipsoid, which has the general equation as $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$, is called imaginary ellipsoid.

12.4 Hyperboloids

(a) Hyperboloid of one sheet :

Let us consider the surface is given by the equation, $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ...(12.4.1)

in which a, b, c are real, none being equal to zero. The surface given by (11.4.1) is called a hyperboloid of one sheet, the equation is said to be in cononical form.

The section by the plane z = k, which is parallel to the xy-plane, is an ellipse given by,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, z=k$$

whose centre lies on the z-axis and whose semi-axes are $a\sqrt{1+\frac{k^2}{c^2}}$, $b\sqrt{1+\frac{k^2}{c^2}}$.

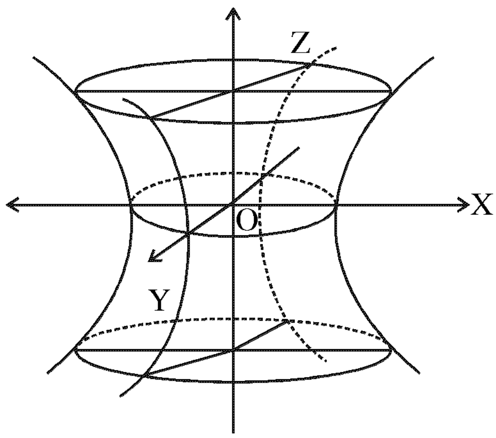


Fig. 12.4.1

Again the section by the planes x = k or y = k parallel to yz- or zx-plane is each a hyperbola.

The equations $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ also represent hyperboloids of one sheet, the axis of the former being the y-axis while the axis of the later is the x-axis.

(b) Hyperboloid of two sheets

Let us consider the surfaces given by the equation, $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, (12.4.2) where a, b, c are real, none of them being zero. The surface given by the equation (11.4.2) is called a hyperboloid of two sheets, the equation being in cononical form.

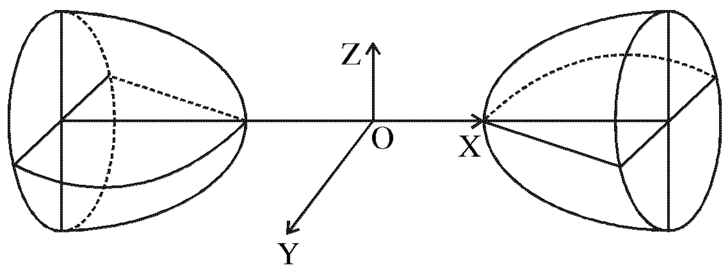


Fig. 12.4.2

The section by the plane x = k, parallel to yz-plane is the ellipse given by :

$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1$, $x = k$.

The centre of these ellipses lie on the x-axis.

The section by the plane y = k or z = k is a hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1+\frac{k^2}{b^2}$, $y = k$

or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$, $z = k$.

The equations $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ also represent hyperboloids of two sheets.

12.5 The elliptic Paraboloid

The equation of elliptic paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$.

This surface is symmetric with respect to yz- and zx-plane and also with respect to z-axis. We can change x, y, z accordingly to get different elliptic paraboloid which are symmetric with respect to other axes.

The intersection of this surface with respect to z = k gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}$, $z = k$, which is an ellipse.

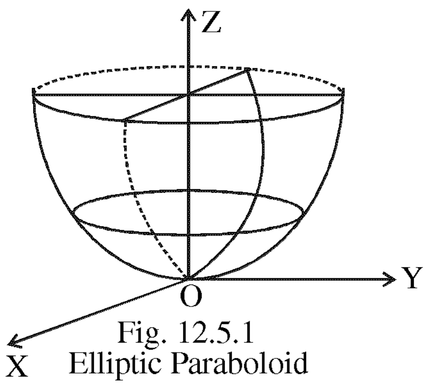


Fig. 12.5.1
Elliptic Paraboloid

For positive value of k the ellipse is real and for negative value of k, the ellipse is imaginary.

Again if x = k intersects this surface then we have $\frac{y^2}{b^2}=\frac{2z}{c}-\frac{k^2}{a^2}$, x = k.

or $y^2=\frac{2b^2}{c}\left(z-\frac{ck^2}{2a^2}\right)$, which is a parabola.

Similarly for y = k, we get a parabola $x^2=\frac{2a^2}{c}\left(z-\frac{ck^2}{2b^2}\right)$.

12.6 The hyperbolic paraboloid

The equation of hyperbolic paraboloid is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$.

The intersection with z = k gives $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$, z = k which is a hyperbola. This surface is symmetric with respect to xz-and yz-planes. Intersection with x = k and y = k planes gives us two parabolas.

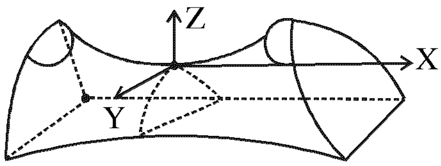


Fig. 12.6.1

12.7 Some Worked out Examples

Example 1 : Show that y + 3 = 0 plane intersects the hyparbolic paraboloid $\frac{x^2}{5}-\frac{y^2}{4}=6z$ in a parabola. Find its vertex and latusrectum.

Solution : Intersecting y = –3 and $\frac{x^2}{5}-\frac{y^2}{4}=6z$ gives $\frac{x^2}{5}=6z+\frac{9}{4}$, y + 3 = 0.

i.e., $x^2=30\left(z+\frac{9}{24}\right), y+3=0$.

It is a parabola whose temple of latus rectum is 30. On y + 3 = 0 plane the vertex of the parabola is $\left(0,-\frac{9}{24}\right)$. So actual vertex of the parabola is $\left(0,-3,-\frac{3}{8}\right)$.

Example 2 : Find the nature of the conicoid $3x^2 + 2y^2 - 12x + 12y - 6z = 0$.

Solution : $3x^2 + 2y^2 - 12x + 12y - 6z = 0$

or, $3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) - 6z - 30 = 0$

or, $3(x - 2)^2 + 2(y + 3)^2 = 6(z - 5)$

or, $\frac{(x-2)^2}{2}+\frac{(y+3)^2}{3}=z-5$

Shifting the origin to (2, –3, 5) we get the required equation as $\frac{X^2}{(\sqrt{2})^2}+\frac{Y^2}{(\sqrt{3})^2}=z$ which is a elliptic paraboloid.

Example 3 : Find the point of intersection of the straight line $\frac{x+2}{2}=\frac{y}{3}=\frac{z-1}{-2}$ and the hyperbolic paraboloid $\frac{x^2}{4}-\frac{y^2}{9}=z$.

Solution : Let for a particular value of λ, we get the point of intersection, where $\frac{x+2}{2}=\frac{y}{3}=\frac{z-1}{-2}=\lambda$ (say) $\neq 0$.

$\therefore$ $x = 2\lambda - 2$, $y = 3\lambda$, $z = -2\lambda + 1$ satisfies the equation of hyperbolic paraboloid.

$$\therefore\ \frac{(2\lambda-2)^2}{4}-\frac{9\lambda^2}{9}-(-2\lambda+1)=0$$

or, $\lambda^2 - 2\lambda + 1 - \lambda^2 + 2\lambda - 1 = 0$

i.e., for any value of λ, it satisfies the equation of hyperbolic paraboloid. So the straight line lies on the hyperbolic paraboloid.

12.8 Some Problems on Central conicoids

Example 1 : Find the nature of the quadratic surface given by the equation,

$$2x^2 + 5y^2 + 3z^2 - 4x + 20y - 6z = 5.$$

Solution : The given equation can be written as,

$$2(x^2 - 2x + 1) + 5(y^2 + 4y + 4) + 3(z^2 - 2z + 1) = 5 + 2 + 20 + 3$$

$$\Rightarrow 2(x - 1)^2 + 5(y + 2)^2 + 3(z - 1)^2 = 30$$

$$\Rightarrow \frac{(x-1)^2}{15}+\frac{(y+2)^2}{6}+\frac{(z-1)^2}{10}=1.$$

Shifting the origin to the point (1, –2, 1), the equation reduces to, $\frac{X^2}{(\sqrt{15})^2}+\frac{Y^2}{(\sqrt{6})^2}+\frac{Z^2}{(\sqrt{10})^2}=1$.

$\therefore$ It is an ellipsoid with the centre at the new origin and the semi-axes are $\sqrt{15},\sqrt{6},\sqrt{10}$.

Hence the given equation represents an ellipsoid whose centre is at (1, –2, 1), the principal axes being parallel to the co-ordinate axes.

The principal planes are $x = 1$, $y = -2$, $z = 1$.

Example 2 : Show that the equation $x^2 + 4y^2 - 9z^2 = 36$ represents the hyperboloid of one sheet whose centre is the origin and whose semi-axes are 6, 3, 2 the principal planes being x = 0, y = 0, z = 0.

Solution : Here the given equation can be written as, $\frac{x^2}{36}+\frac{y^2}{9}-\frac{z^2}{4}=1$

$$\Rightarrow \frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}-\frac{z^2}{(2)^2}=1$$

$$\Rightarrow \frac{(x-0)^2}{(6)^2}+\frac{(y-0)^2}{(3)^2}-\frac{(z-0)^2}{(2)^2}=1.$$

$\therefore$ It is hyperboloid of one sheet with the centre (0, 0, 0), which is the origin and the semi-axes are 6, 3, 2. Hence the equation represents a hyperboloid of one sheet whose centre is the origin, the principal planes are x = 0, y = 0, z = 0. (Proved)

Example 3 : Show that the plane x – 2 = 0 intersects the ellipsoid $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$ is an ellipse.

Solution : The section of the ellipsoid $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$ by the plane x – 2 = 0 has for its equation,

$$\frac{y^2}{12}+\frac{z^2}{4}=1-\frac{2^2}{16}=1-\frac{1}{4}=\frac{3}{4},\ x=2$$

$$\Rightarrow \frac{y^2}{12}+\frac{z^2}{4}=\frac{3}{4},\ x=2$$

$$\Rightarrow \frac{y^2}{9}+\frac{z^2}{3}=1,\ x=2.$$

Hence the section of the given ellipsoid by the plane x – 2 = 0 is an ellipse (proved).

12.9 Tangent lines and Tangent planes

Definition : A straight line which intersects a central conicoid in two coincident point is called a tangent line to the central conicoid at that point.

The locus of all tangent lines at a point on a central conicoid is called a tangent plane to the central conicoid at that point.

12.9.1 Equation of the tangent plane at a point

The equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ at a point (α, β, γ) is $a\alpha x + b\beta y + c\gamma z = 1$.

Corollary : The equation to the tangent plane at (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}=1$.

12.9.2 Condition of tangency

The condition that the plane $lx + my + nz = p$ is a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$.

Co-ordinates of tangent point is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

12.9.3 Director sphere

The locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid is a sphere, concentric with the conicoid, called the director sphere of the conicoid.

12.10 Normal

The normal to a surface at any point is the straight line through that point perpendicular to the tangent plane at that point.

Let the equation of the central conicoid be

$ax^2 + by^2 + cz^2 = 1$ .....(12.10.1)

The equation of the tangent plane at the point (α, β, γ) to the conicoid is,

$a\alpha x + b\beta y + c\gamma z = 1$. .....(12.10.2)

The direction ratios of the normal, which is perpendicular to the tangent plane (12.10.2) are $a\alpha, b\beta, c\gamma$.

Therefore the equations of the normal, which is a straight line passing through the point (α, β, γ) having direction ratios $a\alpha, b\beta, c\gamma$ are $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$.

Corrollary I : The normal to the paraboloid $ax^2 + by^2 = 2cz$ at the point (α, β, γ) is the straight line $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}$.

Corrollary II : If p be the length of the perpendicular from the origin to the tangent, then

$\frac{1}{p^2} = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$

or $(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1$

This shows that the direction cosines of the normal are $a\alpha p$, $b\beta p$, $c\gamma p$.

The equations of the normal can now be written as, $\frac{x-\alpha}{a\alpha\rho} = \frac{y-\beta}{b\beta\rho} = \frac{z-\gamma}{c\gamma\rho}$.

For example, for the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the equations of the normal at the point (α, β, γ) are $\frac{x-\alpha}{\frac{p\alpha}{a^2}} = \frac{y-\beta}{\frac{p\beta}{b^2}} = \frac{z-\gamma}{\frac{p\gamma}{c^2}}$.

12.11 Some Problems on tangent and normal

Example 1 : Show that the plane $8x - 6y - z = 5$ touches the paraboloid $3x^2 - 2y^2 = 6z$. Find the point of contact.

Solution : If possible, let the plane touch the paraboloid at the point (α, β, γ). The equation of the tangent plane to the paraboloid at the point (α, β, γ) is $3\alpha x - 2\beta y = 3(z + \gamma)$. Comparing this with the given equation of the plane, we get,

$$\frac{3\alpha}{8} = \frac{2\beta}{6} = \frac{3}{1} = \frac{3\gamma}{5}$$

giving $\alpha = 8$, $\beta = 9$, $\gamma = 5$

Now, $3\alpha^2 - 2\beta^2 = 3(8)^2 - 2(9)^2$ $[\because \alpha = 8 \text{ and } \beta = 9]$

$= 3 \times 64 - 2 \times 81$

$= 30$

$= 6 \times 5$

$= 6\gamma$ $[\because \gamma = 5]$

$\therefore$ the point (α, β, γ) lies on the paraboloid.

Hence the given plane touches the paraboloid and the point of contact is (8, 9, 5).

Example 2 : Show that the Plane $lx + my + nz = p$ is a tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if $a^2l^2 + b^2m^2 + c^2n^2 = p^2$.

Then find the point of contact.

Solution : Here the ellipsoid can be written as, $\frac{1}{a^2}.x^2+\frac{1}{b^2}.y^2+\frac{1}{c^2}z^2=1$

$\therefore$ By the condition of tangency, we get, $\frac{l^2}{\frac{1}{a^2}}+\frac{m^2}{\frac{1}{b^2}}+\frac{n^2}{\frac{1}{c^2}}=p^2$

$\Rightarrow a^2l^2 + b^2m^2 + c^2n^2 = p^2$

and the co-ordinate of the tangent point i.e. the point of contact of the given plane and the given ellipsoid is $\left(\frac{a^2l}{p},\frac{b^2m}{p},\frac{c^2n}{p}\right)$ (Ans.)

Example 3 : Find the equation of the tangent planes to the conicoid $2x^2 - 6y^2 + 3z^2 = 5$ which passes through the straight line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$.

Solution : Equation of a plane passing through the given straight line is,

$x + 9y - 3z + \lambda(3x - 3y + 6z - 5) = 0$, λ being a parameter.

$\Rightarrow x(1 + 3\lambda) + y(9 - 3\lambda) + z(6\lambda - 3) - 5\lambda = 0$ .....(i)

Let the plane (i) touch the given conicoid at the point (α, β, γ).

Then the equation (i) is identical with $2\alpha x - 6\beta y + 3\gamma z = 5$.

Therefore, $\frac{2\alpha}{1+3\lambda}=\frac{-6\beta}{9-3\lambda}=\frac{3\gamma}{6\lambda-3}=\frac{5}{5\lambda}$

$\Rightarrow \alpha=\frac{1+3\lambda}{2\lambda},\ \beta=\frac{\lambda-3}{2\lambda},\ \gamma=\frac{2\lambda-1}{\lambda}.$

Since the point (α, β, γ) lies on the conicoid, therefore,

$$2\left(\frac{1+3\lambda}{2\lambda}\right)^2-6\left(\frac{\lambda-3}{2\lambda}\right)^2+3\left(\frac{2\lambda-1}{\lambda}\right)^2=5$$

$$\Rightarrow \frac{2(1+6\lambda+9\lambda^2)}{4\lambda^2}-\frac{6(\lambda^2-6\lambda+9)}{4\lambda^2}+\frac{3(4\lambda^2-4\lambda+1)}{\lambda^2}=5$$

$\Rightarrow 1 + 6\lambda + 9\lambda^2 - 3(\lambda^2 - 6\lambda + 9) + 6(4\lambda^2 - 4\lambda + 1) = 10\lambda^2$

$\Rightarrow 1 + 6\lambda + 9\lambda^2 - 3\lambda^2 + 18\lambda - 27 + 24\lambda^2 - 24\lambda + 6 - 10\lambda^2 = 0$

$\Rightarrow 20\lambda^2 - 20 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$

Hence from (i), the required equations of the tangent planes are $4x + 6y + 3z = 5$ and $2x - 12y + 9z = 5$ (Ans.)

Example 4 : Show that the points (12, –18, 8) and (–6, 18, –10) are at the feet of the normals to the ellipsoid $x^2 + 2y^2 + 3z^2 = 984$ which lie on the plane : $x + y + z = 2$.

Solution : The points (12, –18, 8) and (–6, 18, –10) lie on the surface of the given ellipsoid.

The equations of the tangent planes to the ellipsoid at these points are respectively,

$12x - 36y + 24z = 984$

i.e. $x - 3y + 2z = 82$ .....(i)

and $-6x + 36y - 30z = 984$

i.e. $x - 6y + 5z = -164$ .....(ii)

Hence the equations of the normal at the point (12, –18, 8) are :

$$\frac{x-12}{1} = \frac{y+18}{-3} = \frac{z-8}{2} \qquad .....(iii)$$

and those at the point (–6, 18, –10) are :

$$\frac{x+6}{1} = \frac{y-18}{-6} = \frac{z+10}{5} \qquad .....(iv)$$

The straight lines (iii) and (iv) are coplanar, since $\begin{vmatrix} -6-12 & 18-(-18) & -10-8 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix}$

$$= \begin{vmatrix} -18 & 36 & -18 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix} = 0 .$$

The equation of the plane containing these normals is, $\begin{vmatrix} x-12 & y+18 & z-8 \\ 1 & -3 & 2 \\ 1 & -6 & 5 \end{vmatrix} = 0$

$\Rightarrow (x - 12)(-15 + 12) - (y + 18)(5 - 2) + (z - 8)(-6 + 3) = 0$

$\Rightarrow -3(x - 12) - 3(y + 18) - 3(z - 8) = 0$

$\Rightarrow x - 12 + y + 18 + z - 8 = 0$

$\Rightarrow x + y + z = 2$ (proved).

Example 5 : Find the equations of the normal to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$.

Solution : The equation of the tangent plane at the point $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$ to the surface

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is, $\frac{\frac{a}{\sqrt{3}}x}{a^2}+\frac{\frac{b}{\sqrt{3}}y}{b^2}+\frac{\frac{c}{\sqrt{3}}z}{c^2}=1$

$$\Rightarrow \frac{x}{\sqrt{3}a}+\frac{y}{\sqrt{3}b}+\frac{z}{\sqrt{3}c}=1$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\sqrt{3}$$

Then the equations of the normal to the given surface at $\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$ are :

$$\frac{x-\frac{a}{\sqrt{3}}}{\frac{1}{a}}=\frac{y-\frac{b}{\sqrt{3}}}{\frac{1}{b}}=\frac{z-\frac{c}{\sqrt{3}}}{\frac{1}{c}}$$

$$\Rightarrow a\left(x-\frac{a}{\sqrt{3}}\right)=b\left(y-\frac{b}{\sqrt{3}}\right)=c\left(z-\frac{c}{\sqrt{3}}\right) \text{ (Ans.)}$$

Example 6 : Find the values of k, for each of which x + y + z = k is a tangent plane to the ellipsoid $x^2 + 2y^2 + 3z^2 = 66$. For each one of these values of k, find the co-ordinates of the point of contact.

Solution : If possible, let the given plane touches the ellipsoid at (α, β, γ).

Then the equation of the tangent plane at (α, β, γ) to the given ellipsoid is,

$\alpha x + 2\beta y + 3\gamma z = 66$ .....(i)

Then compairing (i) with the given plane, we get, $\frac{\alpha}{1}=\frac{2\beta}{1}=\frac{3\gamma}{1}=\frac{66}{k}$.

$\Rightarrow \alpha=\frac{66}{k},\ \beta=\frac{33}{k},\ \gamma=\frac{22}{k}$.

Now as (α, β, γ) lies on the ellipsoid $x^2 + 2y^2 + 3z^2 = 66$, we get,

$$\left(\frac{66}{k}\right)^2+2\left(\frac{33}{k}\right)^2+3\left(\frac{22}{k}\right)^2=66$$

$\Rightarrow 4356 + 2178 + 1452 = 66k^2$

$\Rightarrow k^2 = 121$

$\Rightarrow k = \pm 11$

$\therefore\ \alpha = \pm 6,\ \beta = \pm 3,\ \gamma = \pm 2$

Hence the point of contacts for the values of k, are (6, 3, 2) and (–6, –3, –2). (Ans.)

Example 7 : Find the equations of the normal to the surface $2x^2 - 3y^2 = 10z$ at the point (2, 4, –4).

Solution : The equation of the tangent plane to the surface at the point (2, 4, –4) is $4x - 12y = 5(z - 4)$

$\Rightarrow 4x - 12y - 5z = -20$

The normal is a straight line perpendicular to the tangent plane.

Therefore the direction ratios of the normal are 4, 12, –5.

Hence the equations of the normal at the point (2, 4, –4) to the surface are :

$$\frac{x-2}{4}=\frac{y-4}{-12}=\frac{z+4}{-5} \text{ (Ans.)}$$

12.12 Summary

In this unit, we have learnt to find out equations of ellipsoids, hyperboloids, paraboloids and tangant line, tangent plane and normal to there conicoids .

12.13 Exercises

1. Show that the plane $z + 1 = 0$ intersects the hyperboloid of one sheet $\frac{x^2}{32}-\frac{y^2}{18}+\frac{z^2}{2}=1$ in a hyperbola.
2. Obtain the equation of the ellipsoid whose centre is at the point (–3, 2, –1) and whose principal axes being parallel to the co-ordinate axes, the lengths of axes being 1, 2, 3 respectively. [**Ans.** $4(x + 3)^2 + (y - 2)^2 + \frac{4}{9}(z + 1)^2 = 1$]

3. Show that the value of k, for which the plane $x + kz = 1$ intersects the hyperboloid of two sheets $x^2 + y^2 - z^2 = 1 = 0$ in

   (i) an ellipse, is $1 < |k| \sqrt{2}$; (ii) a hyperbola, is $|k| < 1$.

4. Obtain the equations of the planes through the straight line $x + y - 6 = 0 = 2x + y + z$ which touch the ellipsoid $3x^2 + 2y^2 + z^2 = 6$ and determine the points of contact.

   [**Ans.** $3x + 2y + z = 6$, $45x + 14y + 31z + 102 = 0$]

5. If 2r be the distance between the two tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx + my + nz = 0$, then show that,

   $(a^2 - r^2)l^2 + (b^2 - r^2)m^2 + (c^2 - r^2)n^2 = 0$

6. Find the value of m, for which the plane $x - 2y - 2z + m = 0$ touches the ellipsoid $\frac{x^2}{144}+\frac{y^2}{36}+\frac{z^2}{9}=1$

   [**Ans.** $m = \pm 18$]

7. Find the equation of the tangent plane to the quadric $3x^2 + 2y^2 - 6z^2 = 6$ which passes through the point $(3, 4, -3)$ and is parallel to the straight line $x = y = -z$.

   [**Ans.** $x + y + 2z = 1$]

8. Find the equations of the normal to the surface $2x^2 + 3y^2 = 4z$ at the point $(2, 2, 5)$.

   $$\left[\textbf{Ans. } \frac{x-2}{2}=\frac{y-2}{3}=\frac{z-5}{-1}\right]$$

9. Show that the normals to the ellipsoid $x^2 + 2y^2 + 3z^2 = 50$ at the points $(6, 1, 2)$ and $(6, -1, 2)$ lie on the plane $x - z = 4$.

10. Normals are drawn the point (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; prove that, if the feet of the three normals lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, then the feet of the remaining three will lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$.

    Also show that, $a(b^2 - c^2)\alpha = b(c^2 - a^2)\beta = c(a^2 - b^2)\gamma$.

11. Show that $y + 3 = 0$ plane cuts the hyperbolic paraboloid $\frac{x^2}{16}-\frac{y^2}{25}=2z$ into a parabola. [Hints : see worked out example]

Unit 13 ❒ Triple product of vectors

Structure

13.0 Objectives

We will be able to determine and appreciate products of three reactors.

13.1 Introduction

In this section we consider the vector triple product. In product of three vectors, it is known to us that they may be of two types : (i) Scalar product of three vectors (ii) vector product of three vectors.

In scalar product of three vectors, let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. The dot product $\vec{a}.(\vec{b} \times \vec{c})$ is a scalar quantity which is called a scalar triple product and is denoted by $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

Which is numerically equal to the volume v of the parallelopiped whose coterminus edges represent $\vec{a}$, $\vec{b}$ and $\vec{c}$ as shown in the fig 12.1.1.

Let AN be perpendicular from A on the face OBDC. Then the projection of OA on $\vec{b} \times \vec{c}$ is equal to the perpendicular AN which measures the altitude of the parallelopiped.

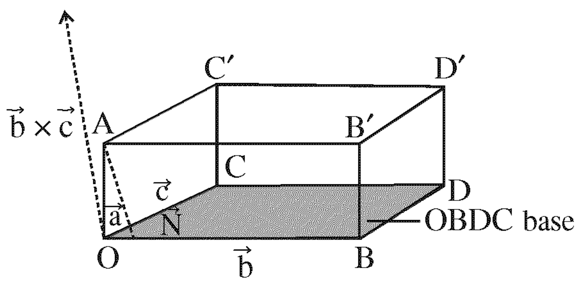


Fig. 13.1.1

Hence the volume of the parallelopiped

= the area of parallelogram OBDC × altitude.

= area of the parallelogram × AN

= length of $(\vec{b} \times \vec{c})$ × projection of $\vec{a}$ on $\vec{b} \times \vec{c} = \vec{a}.(\vec{b} \times \vec{c})$

Therefore, the scalar triple product $\vec{a}.(\vec{b} \times \vec{c})$ = the volume of the parallelopiped.

In this case we also know that

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = -\left[\vec{a}\,\vec{c}\,\vec{b}\right] = -\left[\vec{b}\,\vec{a}\,\vec{c}\right] = \left[\vec{b}\,\vec{c}\,\vec{a}\right] = -\left[\vec{c}\,\vec{b}\,\vec{a}\right] = \left[\vec{c}\,\vec{a}\,\vec{b}\right]$$

Also we know that the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 0$; i.e. if three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar then $\vec{b}\times\vec{c}$ is perpendicular to $\vec{a}$ and so $\vec{a}.(\vec{b}\times\vec{c}) = 0$.

Volume of a tetrahedron :

Let OABC be a tetrahedron whose base is a triangle OAB, O be the origin and vertex is at c (in the fig. 12.1.2). Let us draw perpendicular CN from C on the base OAB. CN makes an angle θ which OC. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

Now the vector $\vec{a}\times\vec{b}$ is perpendicular to the plane OAB and parallel to CN. Then $\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\angle BOC = 2 \times$ Area of ΔOAB

i.e. Area of $\Delta OAB = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|$.

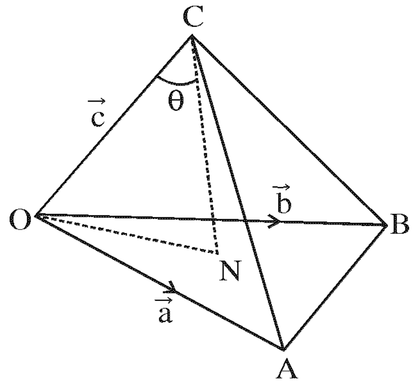


Fig. 13.1.2

Therefore, volume of the tetrahedron = V

$= \frac{1}{3} \times$ Area of $\Delta OAB \times$ height CN

$= \frac{1}{3}\times\left\{\frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\sin\angle BOC\right\}$ OC $\cos\theta$

$= \frac{1}{3}\times\frac{1}{2}\left(\vec{a}\times\vec{b}\right).\vec{c}$

$= \frac{1}{6}\left[\vec{a}\vec{b}\vec{c}\right]$ cubic units.

Note : If one vertex of the tetrahedron be not taken as origin, let that vertex be D and the position vectors of the four vertices A, B, C, D are $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ then $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OD} = \vec{d}$.

Therefore, $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = \vec{a} - \vec{d}$

$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = \vec{b} - \vec{d}$

and $\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \vec{c} - \vec{d}$ respectively.

Hence the volume of the tetrahedron will be $\frac{1}{6}\left[\vec{a}-\vec{b}\;\vec{b}-\vec{d}\;\vec{c}-\vec{d}\right]$

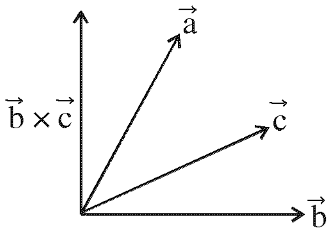
$= \frac{1}{6}\{[abc] + [bad] + [cbd] + [dac]\}$

In particular, if the four points A, B, C, D are coplanar then the volume of the tetrahedron is zero i.e., [abc] – [abd] + [acd] – [bcd] = 0 or [abc] = [bcd] + [cad] + [abd].

13.2 Vector triple product of three vectors

Now the vector triple product of three vectors $\vec{a}, \vec{b}, \vec{c}$ are given by $\vec{a}\times(\vec{b}\times\vec{c})$. It is a vector quantity, acting along the normal to the plane containing $\vec{a}$ and $\vec{b}\times\vec{c}$.

Let $\vec{p} = \vec{a}\times(\vec{b}\times\vec{c})$. Since $\vec{p}$ is perpendicular to $\vec{b}\times\vec{c}$, so $\vec{p}$ lies in the plane of $\vec{b}$ and $\vec{c}$ and so it can be expressed as a linear combination of $\vec{b}$ and $\vec{c}$ and will be of the form $\alpha\vec{b}+\beta\vec{c}$ where α and β are scalars.



$\alpha = \vec{a}.\vec{c}$

$\beta = -(\vec{a}.\vec{c})$

Fig. 13.2.1

Verification : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then $\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\,\hat{i} + (b_3c_1 - b_1c_3)\,\hat{j} + (b_1c_2 - b_2c_1)\,\hat{k}$

So, $\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$

13.3 Properties of vector triple product

Given three vectors $\vec{a}, \vec{b}\,\&\,\vec{c}$, then :

(i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

(ii) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(iii) In general the vector triple product is not associative i.e. $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

(iv) The necessary and sufficient condition that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ is that one of $\vec{a}$ and $\vec{c}$ is a constant multiple of the other i.e. $\vec{a}$ and $\vec{c}$ are linearly dependent.

Proof of (i) : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then L.H.S. = $\vec{a}\times(\vec{b}\times\vec{c})$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\times\{(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}\}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1\end{vmatrix}$$

$$= \{a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\}\,\hat{i} + \{a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)\}\,\hat{j} + \{a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\}\,\hat{k}$$

R.H.S. = $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$$= \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= (a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1)\,\hat{i} + (a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2)\,\hat{j} + (a_1c_1b_3 + a_2c_2b_3 + a_3b_3c_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3)\,\hat{k}$$

$$= (a_2c_2b_1 + a_3c_3b_3 - a_2b_2c_1 - a_3b_3c_1)\,\hat{i} + (a_1c_1b_2 + a_3c_3b_3 - a_1b_1c_2 - a_3b_3c_2)\,\hat{j} + (a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3)\,\hat{k}$$

$= \{a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\}\,\hat{i} + \{a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)\}\,\hat{j} + \{a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\}\,\hat{k}$

Hence L.H.S. = R.H.S. (Proved)

Proof of (ii) : In the case $(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= \{(a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}\} \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \{(a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2\}\,\hat{i} + \{(a_1b_2 - a_2b_1)c_1 - (a_2b_3 - a_3b_2)c_3\}\,\hat{j}$

$+ \{(a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1\}\,\hat{k}$

R.H.S. $= (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$= \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$- \{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}).(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (b_1c_1 + b_2c_2 + b_3c_3)(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$= (a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 - b_1c_1a_1 - b_2c_2a_1 - b_3c_3a_1)\,\hat{i} + (a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 - b_1c_1a_2 - b_2c_2a_2 - b_3c_3a_2)\,\hat{j} + (a_1c_1b_3 + a_2c_2b_3 + a_3b_3c_3 - b_1c_1a_3 - b_2c_2a_3 - b_3c_3a_3)\,\hat{k}$

$=\{(a_3b_1 - a_1b_3)\,c_3 - (a_1b_2 - a_2b_1)\,c_2\}\,\hat{i} + \{(a_1b_2 - a_2b_1)\,c_1 - (a_2b_3 - a_3b_2)\,c_3\}\,\hat{j} + \{(a_2b_3 - a_3b_2)\,c_2 - (a_3b_1 - a_1b_3)c_3\}\,\hat{k}$

Hence L.H.S = R.H.S (Proved)

Proof of (iii) : It can be easily shown that the vector triple product is not associative in general by the helf of proof of (i) and (ii).

Proof of (iv) : From the proof of (i), we have $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ and from (ii), we also have $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$.

Now it is given that

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c} \text{ or, } (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c} = (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$$

or, $(\vec{a}.\vec{b})\vec{c} = (\vec{b}.\vec{c})\vec{a}$

i.e. one of $\vec{a}$ and $\vec{c}$ is a constant multiple of the other.

Now let $\vec{c} = s\vec{a}$, where s is a constant.

Then, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$$= (\vec{a}.s\vec{a})\vec{b}-(\vec{a}.\vec{b})s\vec{a}$$

$$= s\{(\vec{a}.\vec{a})\vec{b}-(\vec{a}.\vec{b})\vec{a}\}$$

Also $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

$$= (\vec{a}.s\vec{a})\vec{b}-(\vec{b}.s\vec{a})\vec{a}$$

$$= s\{(\vec{a}.\vec{a})\vec{b}-(\vec{b}.\vec{a})\vec{a}\}$$

$$= s\{(\vec{a}.\vec{a})\vec{b}-(\vec{a}.\vec{b})\vec{a}\} \quad (\because \text{ dot product is commutative})$$

Therefore $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$.

Thus the necessary and sufficient condition that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ is that one of $\vec{a}$ and $\vec{c}$ is a constant multiple of the other, i.e. $\vec{a}$ and $\vec{c}$ are linearly dependent.

Notes / Summary :

I. The vector triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\vec{a}\times(\vec{b}\times\vec{c})$.

II. $$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1 \end{vmatrix}$$

if $\vec{a} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})$; $\vec{b} = (b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$ and $\vec{c} = (c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$

III. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

IV. $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

V. In general $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

VI. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$ for a necessary and sufficient condition that $\vec{a}$ and $\vec{c}$ are linearly dependent.

13.4 Workedout Examples

Example 1 : $\vec{\alpha} = 2\hat{i} - 10\hat{j} + 2\hat{k}$, $\vec{\beta} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{\gamma} = 2\hat{i} + \hat{j} + 3\hat{k}$ find the vector $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma})$ and interpret the result geometrically.

Solution : We know that $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma}) = (\vec{\alpha}.\vec{\gamma})\vec{\beta} - (\vec{\alpha}.\vec{\beta})\vec{\gamma}$

Now $\vec{\alpha}.\vec{\gamma} = (2\hat{i} - 10\hat{j} + 2\hat{k}).(2\hat{i} + \hat{j} + 3\hat{k}) = 4 - 10 + 6 = 0$

$\vec{\alpha}.\vec{\beta} = (2\hat{i} - 10\hat{j} + 2\hat{k}).(3\hat{i} + \hat{j} + 2\hat{k}) = 6 - 10 + 4 = 0$

Hence $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma}) = 0$ and $\vec{\alpha}$ is perpendicular to $\vec{\beta}$ and $\vec{\gamma}$.

Example 2 : It $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$ find the angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ where $\vec{b}$, $\vec{c}$ are non-parallel.

Solution : Here $\frac{1}{2}\vec{b} = \vec{a}\times(\vec{b}\times\vec{c})$ .....(1)

$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ .....(2)

Again from (1), we have

$$\frac{1}{2}\vec{a}.\vec{b} = \vec{a}.\vec{a}\times(\vec{b}\times\vec{c})$$

$$= \vec{a}.\{(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}\}$$

$$= (\vec{a}.\vec{b})(\vec{a}.\vec{c}) - (\vec{a}.\vec{c})(\vec{a}.\vec{b})$$

$$= 0$$

or, $\vec{a}.\vec{b} = 0$

i.e., $\vec{a}$ is perpendicular to $\vec{b}$.

Now from (2), we obtain,

$$\frac{1}{2}\vec{b} = (\vec{a}.\vec{c})\vec{b}, \text{ [Since } \vec{a}.\vec{b} = 0\text{]}$$

or, $\frac{1}{2}|\vec{b}| = |\vec{a}.\vec{c}||\vec{b}|$

or, $|\vec{a}.\vec{c}| = \frac{1}{2}$, [since $|\vec{b}| = 1$]

or, $|\vec{a}.\vec{c}| = |\vec{a}||\vec{c}|\cos 60^{o}$ [since $|\vec{a}| = |\vec{c}| = 1$]

Hence $\vec{a}$ makes an angle 60° with $\vec{c}$.

Thus $\vec{a}$ is perpendicular to $\vec{b}$ and makes 60° angle with $\vec{c}$.

Example 3 : Prove that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$. Under what condition $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$?

Solution : $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ .....(1)

$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}$ .....(2)

$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$ .....(3)

After adding (1), (2) and (3) we get

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$ .....(4)

Now $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$

or, $-\vec{c}\times(\vec{a}\times\vec{b}) = \vec{a}\times(\vec{b}\times\vec{c})$

Then from (4) we obtain $-\vec{c}\times(\vec{a}\times\vec{b}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

or, $\vec{b}\times(\vec{c}\times\vec{a}) = 0$

or, $(\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} = 0$

or, $(\vec{b}.\vec{a})\vec{c} = (\vec{b}.\vec{c})\vec{a}$

The above condition holds either if $\vec{b}.\vec{a} = \vec{b}.\vec{c} = 0$

or, if $\vec{c} = \dfrac{(\vec{b}.\vec{c})}{(\vec{b}.\vec{a})}\vec{a}$

In the first case $\vec{b}$ is perpendicular on the plane of $\vec{a}$ and $\vec{c}$ and in the second case $\vec{c}$ and $\vec{a}$ are parallel.

Thus $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ hold if $\vec{b}$ is perpendicular on the plane of $\vec{a}$ and $\vec{c}$ and or if $\vec{a}$ is parallel to $\vec{c}$.

Example 4 : Show that $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})] = (\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

Solution : L.H.S. = $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]$

$= \vec{a}\times\left[(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}\right]$

$= (\vec{a}.\vec{b})(\vec{a}\times\vec{a}) - (\vec{a}.\vec{a})(\vec{a}\times\vec{b})$

$= 0 - (\vec{a}.\vec{a})(\vec{a}\times\vec{b}),\qquad [\text{since } \vec{a}\times\vec{a} = 0]$

$= -(\vec{a}.\vec{a})(\vec{a}\times\vec{b})$

$= (\vec{a}.\vec{a})(\vec{b}\times\vec{a})$

= R.H.S (Proved)

Example 5 : Simplify $(\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

Solution : $(\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}).\{\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}).\{\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\}$

$= \vec{a}.(\vec{b}\times\vec{c}) - \vec{a}.(\vec{a}\times\vec{b}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) - \vec{b}.(\vec{a}\times\vec{b}) + \vec{b}.(\vec{c}\times\vec{a})$

$= [\vec{a}\vec{b}\vec{c}] - 0 + 0 + 0 - 0 + [\vec{b}\vec{c}\vec{a}]$

$= 2[\vec{a}\vec{b}\vec{c}]$

Example 6 : Find the constant c such that the vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} + 2\hat{j} - 3\hat{k})$ and $3\hat{i} + c\hat{j} + 5\hat{k}$ are coplanar.

Solution : Since the given vectors are coplanar.

Then we get, $\bar{A}.(\bar{B}\times\bar{C}) = 0$,

where $\bar{A} = (2\hat{i} - \hat{j} + \hat{k})$, $\bar{B} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{C} = 3\hat{i} + c\hat{j} + 5\hat{k}$ (say)

$$\therefore \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & c & 5 \end{vmatrix} = 0$$

or, $2\begin{vmatrix} 2 & -3 \\ c & 5 \end{vmatrix} - 1\begin{vmatrix} -3 & 1 \\ 5 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & c \end{vmatrix} = 0$

or, $2(10 + 3c) - 1(-9 - 5) + 1(c - 6) = 0$

or, $7c + 28 = 0$

or $c = -4$

Example 7 : Prove that $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) = 0$

Solution : L.H.S. = $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})$

$= \{(\vec{a}\times\vec{b})\times\vec{c}\}.\vec{d} + \{(\vec{b}\times\vec{c})\times\vec{a}\}.\vec{d} + \{(\vec{c}\times\vec{a})\times\vec{b}\}.\vec{d}$

$= \{(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}\}.\vec{d} + \{(\vec{b}.\vec{a})\vec{c} - (\vec{c}.\vec{a})\vec{b}\}.\vec{d} + \{(\vec{c}.\vec{b})\vec{a} - (\vec{a}.\vec{b})\vec{c}\}.\vec{d}$

$= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d}) + (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{c}.\vec{a})(\vec{b}.\vec{d}) + (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d})$

$= 0$

= R.H.S.

Example 8 : Show that the vector $\bar{A} = 2\hat{i} - \hat{j} + \hat{k}$, $\bar{B} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\bar{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, form the sides of a right angled triangle.

Solution : To form a trianlge the vectors must be coplanar. To prove the coplanarity of the vectors we consider $\overline{A}.(\overline{B}\times\overline{C})$.

$$\overline{B}\times\overline{C}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & -5\\ 3 & -4 & -4\end{vmatrix}=-8\hat{i}-11\hat{j}+5\hat{k}$$

$$\therefore\ \overline{A}.(\overline{B}\times\overline{C})=(2\hat{i}-\hat{j}+\hat{k}).(-8\hat{i}-11\hat{j}+5\hat{k})$$
$$=-16+11+5$$
$$=0$$

$\therefore$ The vectors are coplanar.

Again $\overline{A}.\overline{B}=2+3-5=0$

Therefore $\overline{A}$ is perpendicular to $\overline{B}$.

Hence the vectors form the sides of a right angled triangle.

13.5 Summary

In this unit, we have learnt how to deal with the products of vectors when three vectors are given.

13.6 Exercises

1. Prove that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

2. When does $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ hold? [Ans. when $(\vec{a}\times\vec{c})\times\vec{b}=0$]

3. Given $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}\vec{b}$, where $\vec{b}$ and $\vec{c}$ are non-collinear. Find the angles between $\vec{a},\vec{b}$ and between $\vec{a},\vec{c}$.

   [Hints : $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}\vec{b}\Rightarrow(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}=\frac{\sqrt{3}}{2}\vec{b}\Rightarrow\left((\vec{a}.\vec{c})-\frac{\sqrt{3}}{2}\right)\vec{b}-(\vec{a}.\vec{b})\vec{c}=0$

   As $\vec{b},\vec{c}$ are non-collinear and so they are independent, hence $\vec{a}.\vec{c}=\frac{\sqrt{3}}{2}$ and $\vec{a}.\vec{b}=0$

   i.e. $\angle(\vec{a},\vec{c})=\pi/6$ and $\angle(\vec{a},\vec{c})=\pi/2$] [**Ans.** $\angle(\vec{a},\vec{b})=\pi/2$; $\angle(\vec{a},\vec{c})=\pi/6$]

4. Prove that $(\vec{a}\times\vec{b}).\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}=\{\vec{a}.(\vec{b}\times\vec{c})\}^2$

5. If $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$

   Prove that (i) $\vec{a}'.\vec{a} = \vec{b}'.\vec{b} = \vec{c}'.\vec{c} = 1$

   (ii) $\vec{a}'.\vec{b} = 0 = \vec{a}'.\vec{c}$, $\vec{b}'.\vec{c} = 0 = \vec{b}'.\vec{a}$ and $\vec{c}'.\vec{a} = 0 = \vec{c}'.\vec{b}$

   (iii) If $[\vec{a}\vec{b}\vec{c}] = v$ then $[\vec{a}'\vec{b}'\vec{c}'] = \dfrac{1}{v}$

6. If A, B, C are three non collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively relative to a given origin, then show that $(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$ is a vector perpendicular to the plane of A, B and C.

7. Prove that the points $\hat{i} - 2\hat{j} + 3\hat{k}$, $2\hat{j} + 4\hat{k}$ and $5\hat{i} + 3\hat{j} + 5\hat{k}$ form the vertices of a right angled triangle.

   [Hints : Three vertices A = $\hat{i} - 2\hat{j} + 3\hat{k}$, B = $2\hat{j} + 4\hat{k}$, C = $5\hat{i} + 3\hat{j} + 5\hat{k}$.

   Then $\overline{AB} = (0-1)\hat{i} + (2+2)\hat{j} + (4-3)\hat{k} = -\hat{i} + 4\hat{j} + \hat{k}$

   Similarly, $\overline{BC} = 5\hat{i} + \hat{j} + \hat{k}$

   $\overline{CA} = -4\hat{i} - 5\hat{j} - 2\hat{k}$

   Then calculate, $\overline{AB}.(\overline{BC}\times\overline{CA}) = 0$]

8. If $\overline{\alpha} = 3\hat{i} - \hat{j} + 2\hat{k}$, $\overline{\beta} = 2\hat{i} + \hat{j} - \hat{k}$ and $\overline{\gamma} = \hat{i} - 2\hat{j} + 2\hat{k}$, show that $\overline{\alpha}\times(\overline{\beta}\times\overline{\gamma}) \neq (\overline{\alpha}\times\overline{\beta})\times\overline{\gamma}$.

9. Prove that $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b}) = 2(\vec{a}\times\vec{b})$

Unit 14 ❒ Vector equation and application to geometry

Structure

14.0 Objectives

We will be able to determine and appreciate products of three reactors.

- the vector equations of straight line, sphere, angle bisector, sphere, lines related to planes etc
- properties of bisectors
- angle between two planes
- distance of a point from a given plane and a given line
- Position vector of centroids and centre of many
- compute work done by and moment of a force

14.1 Introduction

In this section, we get some equations involving known and unknown vectors as well as known and unknown scalars. Such equations are called vector equations and it has huge applications to geometry that we have dicussed here.

14.2 Vector equation of a straight line

(a) Equation of a straight line passing through a given point and parallel to a given vector (parametric form).

Let A be a given point, whose position vector is $\bar{\alpha}$ relative to the origin O and $\bar{\beta}$ be the given vector. Let P be a point on the line and its position vector be $\bar{r}$.

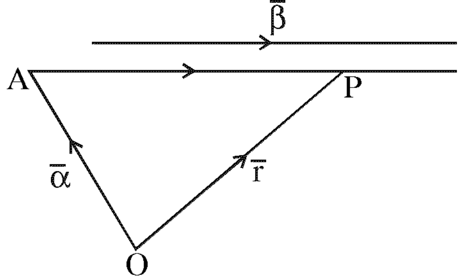


Fig. 14.2.1

Now $\overrightarrow{AP}$ is parallel to $\bar{\beta}$ and so we can write $\overrightarrow{AP} = t\bar{\beta}$, where t is a scalar.

Therefore $\bar{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \bar{\alpha} + t\bar{\beta}$ .....(14.1.1)

This equation gives the position vector of the moving point P at any position on the line for different values of t.

Hence it is the vector equation of the line.

Corollary I : The equation of straight line through the origin and parallel to the vector $\bar{\beta}$ is $\bar{r} = t\bar{\beta}$, since $\bar{\alpha} = 0$.

(b) Equation of a straight line passing through two given points.

Let A and B, two given points with the position vectors $\bar{a}$ and $\bar{b}$ referred to the vector origin O. Let P be a point on the line and $\bar{r}$ be its position vector.

Here $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \bar{b} - \bar{a}$. $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$. Therefore, we can write $\overrightarrow{AP} = t(\bar{b} - \bar{a})$ where t is a scalar.

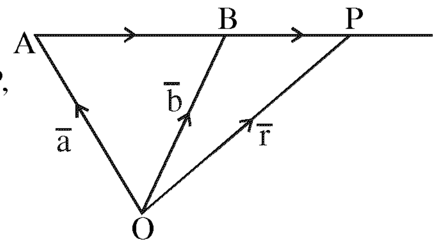


Fig. 14.2.2

Then from the fig. 14.1.2, in ΔOAP, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

or, $\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$

or, $\bar{r} = (1 - t)\bar{a} + t\bar{b}$ .....(14.1.2)

or, $\bar{r} = s\bar{a} + (1 - s)\bar{b}$, where $1 - t = s$ .....(14.1.3)

Any one of (14.1.2) and (14.1.2) is the equation of the required straight line.

Note I : Let the co-ordinates of P, B, A be (x, y, z), (x_1, y_1, z_1) and (x_2, y_2, z_2) w.r.t three rectangular axes through the vector origion O. Then from (14.1.2),

$(x, y, z) = (1 - t)(x_1, y_1, z_1) + t(x_2, y_2, z_2)$.

It gives $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$,

it is the equation of a line through two given points in the cartesian system.

Note II : Equation (14.1.2) verifies the necessary and the sufficient condition for collinearity of the points A, B, P.

14.3 Equation of the bisectors of angle between two lines

Let the equation of the two straight lines be

$\bar{r} = \bar{a}t + \bar{b}$ (14.2.1)

and $\bar{r} = \bar{a}_1 t + \bar{b}_1$ (14.2.2)

represent the lines A′OA and B′OB respectively.

Now let on the internal bisector of the angle AOB of two lines, P be any point, such that its position vector be $\overrightarrow{OP} = \bar{r}$ and along $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\bar{a}$ and $\bar{b}$ be unit vectors. Here we consider O as the origin and MP is parallel to OB. Here OM = MP = t (say)

$\therefore\ \overrightarrow{OM} = t\bar{a}$; $\overrightarrow{MP} = t\bar{b}$

$\therefore\ \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = t(\bar{a} + \bar{b})$.

This relation gives the position vector of P at any position on the bisector for different values of t. Hence it is the equation of the internal bisector.

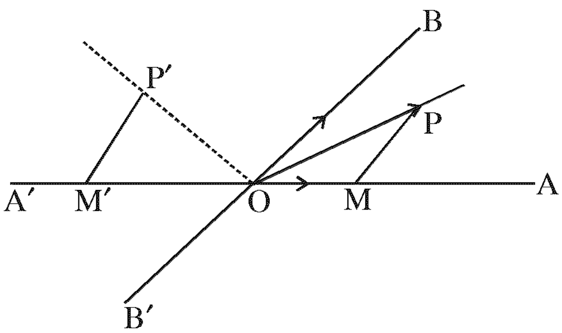


Fig. 14.3.1

Let OP′ be the external bisector and $\overrightarrow{OP'} = \bar{r}$. M′P′ is parallel to OB and it meets A′O at M′. If OM′ = M′P′ = t. Then OM′ = M′P′ = t(say), then $\overrightarrow{OM'} = -t\bar{a}$ and $\overrightarrow{M'P'} = t\bar{b}$

$\therefore\ \overrightarrow{OP'} = M'P' + OM'$

$\therefore\ \bar{r} = t(\bar{b} - \bar{a})$, it is the equation of the external bisector.

Corollary : If $\overrightarrow{OA} = \bar{a}$, $\overrightarrow{OB} = \bar{b}$, then unit vectors along OA and OB are $\frac{\bar{a}}{|\bar{a}|}$ and $\frac{\bar{b}}{|\bar{b}|}$.

In this case the equations are $\bar{r} = t\left(\frac{\bar{b}}{|\bar{b}|} \pm \frac{\bar{a}}{|\bar{a}|}\right)$.

Note : If the position vector of the point of intersection of the given lines be $\bar{c}$ then the equation of the bisectors will be

$$\bar{r} = \bar{c} + t\left(\frac{\bar{b}}{|\bar{b}|} \pm \frac{\bar{a}}{|\bar{a}|}\right).$$

14.4 Theorems on Bisectors

Here we discuss some of the fundamental theorems of plane geometry with the help of different theory of vectors.

Theorem 1 : The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the othe two sides.

Let AD be the internal bisector of the angle A of ΔABC and $\overrightarrow{AB} = \bar{c}$, $\overrightarrow{AC} = \bar{b}$ with A as vector origin.

Now the equation of AD is

$$\bar{r} = t\left(\frac{\bar{c}}{|\bar{c}|} + \frac{\bar{b}}{|\bar{b}|}\right)$$

$$= t\left(\frac{|\bar{b}|\bar{c} + |\bar{c}|\bar{b}}{|\bar{b}||\bar{c}|}\right), \text{ where t is a scalar.}$$

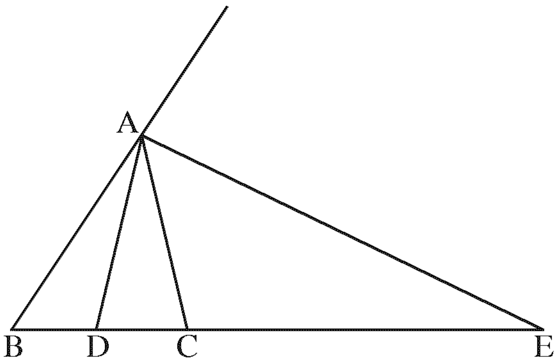


Fig. 14.4.1

Let us choose $t = \dfrac{|\bar{b}||\bar{c}|}{|\bar{b}| + |\bar{c}|}$, then $\bar{r} = \dfrac{|\bar{b}|\bar{c} + |\bar{c}|\bar{b}}{|\bar{b}| + |\bar{c}|}$ is the position vector of a point on the line BC, which divides BC in the ratio $|\bar{c}| : |\bar{b}|$.

So it is the common point of BC and AD.

Hence the internal bisector AD divides BC in the ratio of AB : AC.

This proves the theorem.

Theorem 2 : The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the other two sides.

In the fig. 13.3.1, let AE be the external bisector of the angle A of ΔABC and $\overrightarrow{AB} = \bar{c}$, $\overrightarrow{AC} = \bar{b}$ with respect to the vector origin A. Now the equation of AE is

$$\bar{r} = t\left(\frac{\bar{b}}{|\bar{b}|} - \frac{\bar{c}}{|\bar{c}|}\right) = t\frac{|\bar{c}|\bar{b} - |\bar{b}|\bar{c}}{|\bar{b}||\bar{c}|}, \text{ where t is a scalar.}$$

Here we choose $t = \dfrac{|\bar{b}||\bar{c}|}{|\bar{c}| - |\bar{b}|}$, then $\bar{r} = \dfrac{|\bar{c}|\bar{b} - |\bar{b}|\bar{c}}{|\bar{c}| - |\bar{b}|}$ is the position vector of a point on BC, which divides BC in the ratio $|\bar{c}| : |\bar{b}|$ externally. So it is the common point of BC and AE. Thus the external bisector divides BC externally in the ratio of the other two sides.

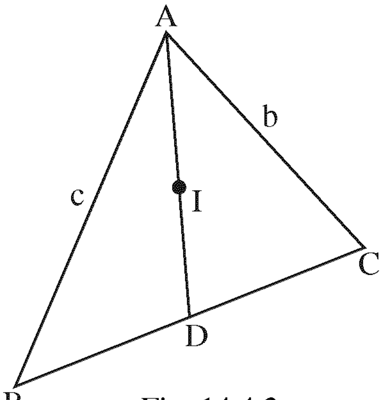


Fig. 14.4.2

Theorem 3 : The internal bisectors of the angles of a triangle are concurrent.

Let $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ be the position vectors of the vertices A, B, C of ΔABC w.r.t. a certain vector origin.

Let BC = a, CA = b, AB = c.

If the internal bisector of the angle A meets the side BC at D, then D divides BC in the ratio c : b, therefore the position vector of D is $\dfrac{b\bar{r}_2 + c\bar{r}_3}{b+c}$.

Let the point I divide AD in ths ratio b + c : a. Then the position vector of I is $\dfrac{a\bar{r}_1 + b\bar{r}_2 + c\bar{r}_3}{a+b+c}$. The symmetry of this expresion suggests that this point must also lie on the other bisectors. Hence the bisectors are concurrent.

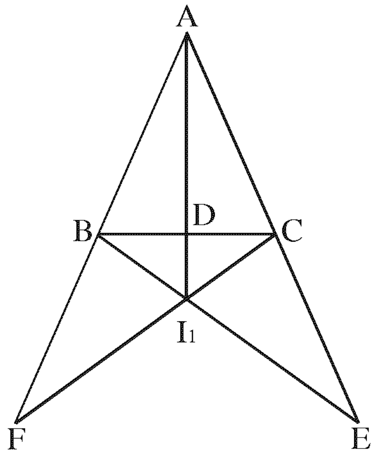


Fig. 14.4.3

Theorem 4 : The internal bisector of one angle and the external bisectors of the other two angles of a triangle are concurrent.

Let $\bar{r}_1, \bar{r}_2, \bar{r}_3$ be the position vectors of the vertices A, B, C of ΔABC w.r.t. a certain vector origin.

If the external bisector of the angle B meets AC produced at E then the point E divides AC externally in the ratio c : a. Therefore the position vector of E is $\dfrac{c\bar{r}_3 - a\bar{r}_1}{c-a}$.

Let the point I_1 divide BE in the ratio c – a : b. Then the position vector of I_1 is $\dfrac{b\bar{r}_2 + c\bar{r}_3 - a\bar{r}_1}{b+c-a}$.

If the external bisector of the angle C meets AB externally at F then the position vector of F is $\dfrac{b\bar{r}_2 - a\bar{r}_1}{b-a}$. Now CF is divided internally in the ratio b – a : c by a point whose position is that of I_1.

Similarly the internal bisector AD of the angle A is divided internally in the ratio b + c : a by the point whose position vector is alike to that of I_1.

Hence the bisector are concurrent.

Theorem 5 : The medians of a triangle are concurrent.

Let ABC be a triangle. The middle point of the sides BC, CA and AB are D, E, F.

Let the position vector of the vertices A, B, C of the triangle are $\bar{a}, \bar{b}, \bar{c}$ w.r.t. some vector origin O. Then the position vector of the middle points D, E, F are $\frac{1}{2}(\bar{b}+\bar{a}), \frac{1}{2}(\bar{c}+\bar{a}), \frac{1}{2}(\bar{a}+\bar{b})$ respectively. The vector equations of AD and DE are

$$\bar{r} = (1-t)\bar{a} + t.\frac{1}{2}(\bar{b}+\bar{c}) \qquad .....(14.3.1)$$

and

$$\bar{r} = (1-s)\bar{b} + s.\frac{1}{2}(\bar{c}+\bar{a}) \qquad .....(14.3.2)$$

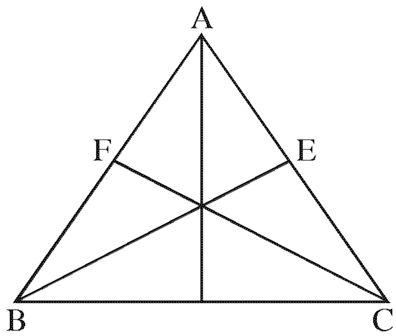


Fig. 14.4.4

If the two straight lines intersect then we must get some suitable values for t and s which will give same value of $\bar{r}$ in (14.3.1) and (14.3.2).

Hence, $1 - t = \frac{1}{2}s$, $\frac{1}{2}t = 1 - s$, $\frac{1}{2}t = \frac{1}{2}s$

which gives, $t = s = \frac{2}{3}$.

Putting the values of t and s in (14.3.1) and (14.3.2), we get the point of intersection as $\frac{\bar{a}+\bar{b}+\bar{c}}{3}$.

From symmetry it is obvious that the third median also passes through this point.

Thus, the three medians are concurrent and the point of concurrence is $\frac{\bar{a}+\bar{b}+\bar{c}}{3}$.

14.5 Vector equation of a plane

(a) Normal form : To find the equation of a plane perpendicular to the unit vector $\bar{n}$ and passing through a point whose position vector is $\bar{a}$.

Let O be the vector origin, ON is perpendicular to the plane, A be the given point on the plane and P be any point on the plane.

Here $\overrightarrow{OA} = \bar{a}$, $\overrightarrow{ON} = p\bar{n}$, where $p = |\overrightarrow{ON}|$.

Let $\overrightarrow{OP} = r$. Then from fig 13.4.1

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \bar{r} - \bar{a}$

AP lies on the plane. Therefore, it is perpendicular to $\bar{n}$.

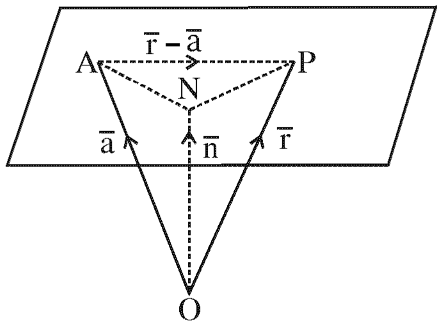


Fig. 14.5.1

Thus $(\bar{r} - \bar{a}).\bar{n} = 0$.

This relation holds for any position of P on the plane. Hence it is the equation of the plane.

Again $\bar{a}.\bar{n}$ = projection of OA on ON = p

$\therefore\ \bar{r}.\bar{n} = p$.

It is known as the normal form of the equation of the plane.

Corollary : If the equation of plane passing through the origin, then the equation becomes $\bar{r}.\bar{n} = 0$.

(b) To find the equation of the plane passing through a given point and parallel to two given vectors.

Let the position vector of the given point be $\bar{a}$ and the given vectors be $\bar{b}$ and $\bar{c}$. Since $\bar{b}$ and $\bar{c}$ are parallel to the plane, $\bar{b} \times \bar{c}$ is perpendicular to the plane.

Moreover, the plane passes through the point whose position vector is $\bar{a}$. If $\bar{r}$ is a point on the plane, $\bar{r} - \bar{a}$ is perpendicular to $\bar{b} \times \bar{c}$.

Thus, $(\bar{r} - \bar{a}).(\bar{b} \times \bar{c}) = 0$

or, $\bar{r}.(\bar{b} \times \bar{c}) = \bar{a}.(\bar{b} \times \bar{c})$

or, $[\bar{r}\,\bar{b}\,\bar{c}] = [\bar{a}\,\bar{b}\,\bar{c}]$, it is the equation of the plane.

Note I : The plane passing through two points with position vectors $\bar{a}$ and $\bar{b}$ and parallel to $\bar{c}$ is identical with the plane passing through the point with position vector $\bar{a}$ and parallel to the vectors $\bar{b} - \bar{a}$ and $\bar{c}$.

Note II : The plane containing the line $\bar{r} = \bar{\alpha} + t\bar{\beta}$, where t is a scalar and perpendicular to the plane $\bar{r}.\bar{\delta} = q$ is identical with the plane passing through the point having the position vector $\bar{\alpha}$ and parallel to the vectors $\bar{\beta}$ and $\bar{\delta}$.

(c) To find the equation of the plane passing through three given non-collinear points.

Let $\overline{a}, \overline{b}, \overline{c}$ be the position vectors of the three given points. Then $\overline{b}-\overline{c}$ and $\overline{c}-\overline{a}$ lie in the same plane. Therefore $(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})$ is perpendicular to the plane. If $\overline{r}$ is a point on the plane, then $(\overline{r}-\overline{a})$ is perpendicular to $(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})$.

$\therefore\ (\overline{r}-\overline{a}).\{(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\}=0$

or $\overline{r}\{(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\}=\overline{a}.\{(\overline{b}-\overline{a})\times(\overline{c}-\overline{a})\}$

or $\overline{r}\{\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}\}=\overline{a}.(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b})$

or $\overline{r}.\overline{m}=[\overline{a}\,\overline{b}\,\overline{c}]$.

It is the equation of the plane.

Here $\overline{m}=\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}=$ twice the vector area of the triangle whose vertices are $\overline{a}, \overline{b}, \overline{c}$.

Corollary : If p is the perpendicular distance of the plane from the origin, then $p=\dfrac{[\overline{a}\,\overline{b}\,\overline{c}]}{|\overline{m}|}$.

(d) To find the equation of the plane through two given lines.

Let the given lines be $\overline{r}=\overline{a}_1+t\overline{b}_1$ and $\overline{r}=\overline{a}_2+s\overline{b}_2$, where t and s are scalars. By (b), the equation of plane is $[\overline{a}_1\,\overline{b}_1\,\overline{b}_2]=[\overline{a}_2\,\overline{b}_1\,\overline{b}_2]$

Corollary : Condition of coplanarity of two straight lines is $[\overline{a}_1\,\overline{b}_1\,\overline{b}_2]=[\overline{a}_2\,\overline{b}_1\,\overline{b}_2]$.

(e) To find the equation of the plane through the intersection of two planes.

Let the given planes be $\overline{r}.\overline{n}_1=p_1$ and $\overline{r}.\overline{n}_2=p_2$ .....(1)

Obviously all those points which satisfy the equations (1) also satisfy the equation

$(\overline{r}.\overline{n}_1-p_1)-\lambda(\overline{r}.\overline{n}_2-p_2)=0$, ...(2)

Whatever be the value of λ.

Thus the equation (2) respresents the general equation of the plane passing through the line of intersection of the planes (1).

The equation (2) can be written as $\overline{r}.(\overline{n}_1-\lambda\overline{n}_2)=p_1-\lambda p_2$.

If the plane passes through a given point $\overline{a}$, then $\overline{a}.(\overline{n}_1-\lambda\overline{n}_2)=p_1-\lambda p_2$

or $\lambda=\dfrac{\overline{a}.\overline{n}_1-p_1}{\overline{a}.\overline{n}_2-p_2}$.

Thus the equation, $(\bar{r}.\bar{n}_1 - p_1)(\bar{a}.\bar{n}_2 - p_2) - (\bar{r}.\bar{n}_2 - p_2)(\bar{a}.\bar{n}_1 - p_1) = 0$ represents the equation of the plane passing through the line of intersection of the planes (1) and the point $\bar{a}$.

14.6 Angle between two planes

The angle θ between two planes $\bar{r}.\bar{n}_1 = p_1$ and $\bar{r}.\bar{n}_2 = p_2$ is the angle between the normals to the planes.

Hence θ is given by $\bar{n}_1.\bar{n}_2 = |\bar{n}_1||\bar{n}_2| \cos\theta$

or, $\theta = \cos^{-1} \dfrac{\bar{n}_1.\bar{n}_2}{|\bar{n}_1||\bar{n}_2|} = \cos^{-1}\left(\bar{n}_1.\bar{n}_2\right)$.

14.7 Distance of a point from a given plane

(i) Perpendicular distance of a point from a plane

Let the equation of the plane be $\bar{r}.\bar{n} = p$ the position vector of a given point A be $\bar{a}$, where $\bar{n}$ is normal to the plane.

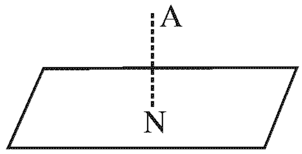


Fig. 14.7.1

If AN is perpendicular to the plane, then this line passes through the point $\bar{a}$ and it is parallel to $\bar{n}$.

Therefore, the equation of AN is $\bar{r} = \bar{a} + t\bar{n}$, where t is a scalar.

At the point of intersection of the line AN and the plane $\bar{r}.\bar{n} = p$,

i.e. $(\bar{a} + t.\bar{n}).\bar{n} = p$

i.e., $t = \dfrac{p - \bar{a}.\bar{n}}{\bar{n}^2}$.

$\therefore$ The position vector of the point N is $\bar{r} = \bar{a} + t\bar{n} = \bar{a} + \dfrac{p - \bar{a}.\bar{n}}{\bar{n}^2}\bar{n}$.

Hence $\overrightarrow{AN} = \bar{a} + \dfrac{p - \bar{a}.\bar{n}}{\bar{n}^2}\bar{n} - \bar{a} = \dfrac{p - \bar{a}.\bar{n}}{\bar{n}^2}\bar{n}$.

Therefore the required distance $\left|\overrightarrow{AN}\right| = \left|\dfrac{p - \bar{a}.\bar{n}}{\bar{n}^2}\right| |\bar{n}| = \left|\dfrac{p - \bar{a}.\bar{n}}{\bar{n}}\right|$

(ii) To find the distance of a point from a plane measured in a given direction.

Let the equation of the plane be

$\bar{r} \cdot \bar{n} = p$ .....(1)

and the given point be A with the position vector $\bar{a}$. Let the given direction be that of unit vecor $\bar{b}$ and the line through A and parallel to $\bar{b}$ meet the plane at M.

If AM = d, then $\overrightarrow{AM} = d\bar{b}$,

Again $\overrightarrow{AM} = \overrightarrow{OA} + \overrightarrow{AM} = \bar{a} + d\bar{b}$.

Since M lies on the plane, $(\bar{a} + d\bar{b}).\bar{n} = p$

or, d = $\dfrac{p - \bar{a}.\bar{n}}{\bar{b}.\bar{n}}$.

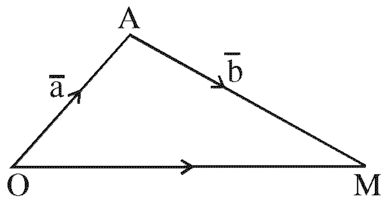


Fig. 14.7.2

(iii) Bisector planes of the angle between two planes.

Let the equations of the planes be $\bar{r}.\bar{n}_1 = p_1$ and $\bar{r}.\bar{n}_2 = p_2$.

If $\bar{r}$ be any point on the bisecting plane, then the perpendicular distance of the planes of this are equal in magnitude.

$$\therefore \quad \frac{p_1 - \bar{r}.\bar{n}_1}{|\bar{n}_1|} = \pm \frac{p_2 - \bar{r}.\bar{n}_2}{|\bar{n}_2|}$$

$\therefore \quad \bar{r}.\left(\dfrac{\bar{n}_1}{|\bar{n}_1|} + \dfrac{\bar{n}_2}{|\bar{n}_2|}\right) = \dfrac{p_1}{|\bar{n}_1|} + \dfrac{p_2}{|\bar{n}_2|}$ and $\bar{r}.\left(\dfrac{\bar{n}_1}{|\bar{n}_1|} - \dfrac{\bar{n}_2}{|\bar{n}_2|}\right) = \dfrac{p_1}{|\bar{n}_1|} - \dfrac{p_2}{|\bar{n}_2|}$ the equations are of bisector planes.

14.8 Distance of a point from a given line

(i) To find the perpendicular distance of a point from a given line

Let the equation of the line BN be $\bar{r} = \bar{b} + t\bar{n}$, what t is a scalar, $\bar{b}$ the position vector of the point B and $\bar{n}$ the unit vector parallel to BN.

Let $\bar{a}$ be the position vector of the point A.

Now $\overrightarrow{BA} = \bar{a} - \bar{b}$

$\therefore \ \overrightarrow{BA}^2 = (\bar{a} - \bar{b})^2$.

AN is perpendicular to BN. BN is the projection of BA along the unit vector $\bar{n}$.

This projection = BA cosθ = $\bar{n}.(\bar{a} - \bar{b})$

From Fig. 13.7.1, $AN^2 = BA^2 - BN^2$

$= (\bar{a} - \bar{b})^2 - \{\bar{n}.(\bar{a} - \bar{b})\}^2$, it gives the required distance.

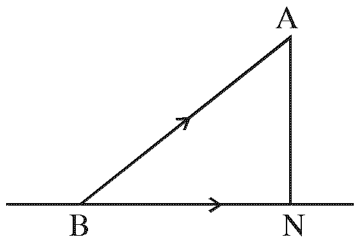


Fig. 14.8.1

Note : If n is not unit vector then $\bar{n}$ is replaced by $\frac{\bar{n}}{|\bar{n}|}$.

(ii) Shortest distance between two skew lines

Let LA and MC be the two skew lines and their equations be $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ respectively.

Here t and s are scalars, $\bar{a}$ and $\bar{c}$ are the position vectors of A and C. LA and MC are parallel to $\bar{b}$ and $\bar{d}$.

Let LM be the shortest distance between the lines. Then it is perpendicular to both the lines, so it is parallel to $\bar{b} \times \bar{d}$.

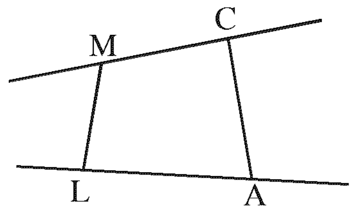


Fig. 14.8.2

The shortest distance is the projection of AC on LM, i.e., the projection of $\bar{c} - \bar{a}$ on $\frac{(\bar{b} \times \bar{d})}{|\bar{b} \times \bar{d}|}$.

Hence LM = $(\bar{c} - \bar{a}).\frac{(\bar{b} \times \bar{d})}{|\bar{b} \times \bar{d}|}$ .....(1)

The equation of the shortest distance : The shortest distance is the line of intersection of the two planes drawn through the two skew lines and the line of s.d. (shortest distance).

The equation of the plane through $\bar{r} = \bar{a} + t\bar{b}$ and LM is

$(\bar{r} - \bar{a}).\{\bar{b} \times (\bar{b} \times \bar{d})\} = 0$ .....(2)

The equation of the plane through $\bar{r} = \bar{c} + s\bar{d}$ and LM is

$(\bar{r} - \bar{c}).\{\bar{d} \times (\bar{b} \times \bar{d})\} = 0$ .....(3)

The planes (2) and (3) determine the equation of the s.d.

Note I : Cartesian form of s.d.

$$(\bar{c} - \bar{a}).\frac{(\bar{b} \times \bar{d})}{|b \times \bar{d}|} = \frac{[(\bar{c} - \bar{a})\bar{b}\,\bar{d}]}{|b \times \bar{d}|}$$

$$= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\sum (m_1 m_2 - m_2 n_1)^2}$$

Here $\overline{a} = \hat{i}x_1 + \hat{j}y_1 + \hat{k}z_1$, $\overline{c} = \hat{i}x_2 + \hat{j}y_2 + \hat{k}z_2$

$\overline{b} = \hat{i}l_1 + \hat{j}m_1 + \hat{k}n_1$, $\overline{d} = \hat{i}l_2 + \hat{j}m_2 + \hat{k}n_2$

Note II : Condition for intersection of two lines :

$(\overline{c} - \overline{a}).(\overline{b} \times \overline{d}) = 0$ i.e. $[(\overline{c} - \overline{a})\overline{b}\,\overline{d}] = 0$.

14.9 Equation of lines relating with planes

(i) To find the equation of the line passing through the point $\overline{a}$ and parallel to the line of intersection of the planes $\overline{r}.\overline{n}_1 = p_1$ and $\overline{r}.\overline{n}_2 = p_2$.

The line of intersection of the planes is parallel to $\overline{n}_1 \times \overline{n}_2$. Thus the required line is parallel to $\overline{n}_1 \times \overline{n}_2$ and it passes through the point $\overline{a}$.

Hence the equation is $(\overline{r} - \overline{a}) \times (\overline{n}_1 \times \overline{n}_2) = 0$.

(ii) To find the equation of the line passing through the point $\overline{a}$, parallel to $\overline{r}.\overline{n} = p$ and intersecting the line $\overline{r} = \overline{\alpha} + t\overline{\beta}$.

Let us consider a plane through the required line and the given line. The plane is parallel to $\overline{\beta}$ and it contains the vector $(\overline{a} - \overline{\alpha})$, since $(\overline{a} - \overline{\alpha}) \times \overline{\beta}$ is normal to the plane, it is perpendicular to the required line.

The required line is parallel to $\overline{r}.\overline{n} = p$, so it is perpendicular to $\overline{n}$. Consequently the line is parallel to $\{(\overline{a} - \overline{\alpha}) \times \overline{\beta}\} \times \overline{n}$.

Thus the equation of the line is $(\overline{r} - \overline{a}) \times \{(\overline{a} - \overline{\alpha}) \times \overline{\beta} \times \overline{n}\} = 0$.

14.10 Volume of a tetrahedron

Let ABCD be a tetrahedron and the position vectors of A, B, C, D be $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ w.r.t certain vector origin 0. DL is perpendicular to the plane of ΔABC.

Now $\overrightarrow{AB} \times \overrightarrow{AC} = (\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})$.

It represents a vector parallel to LD and its magnitude = 2ΔABC.

Again LD is the projection of AD on LD.

Therefore, the volume of the tetrahedron

$= \frac{1}{3}\Delta ABC.LD$

$= \frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AD}$

$= \frac{1}{6}\{(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\}.(\bar{d}-\bar{a})$

$= \frac{1}{6}[\bar{b}-\bar{a},\bar{c}-\bar{a},\bar{d}-\bar{a}]$

$= \frac{1}{6}\{[\bar{b}\,\bar{c}\,\bar{d}]+[\bar{c}\,\bar{a}\,\bar{d}]+[\bar{d}\,\bar{a}\,\bar{b}]-[\bar{a}\,\bar{b}\,\bar{c}]\}$

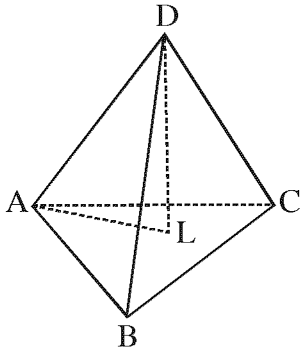


Fig. 14.10.1

Note I : Cartesian equivalence.

Let $\bar{a}=\hat{i}x_1+\hat{j}y_1+\hat{k}z_1$, $\bar{b}=\hat{i}x_2+\hat{j}y_2+\hat{k}z_2$, $\bar{c}=\hat{i}x_3+\hat{j}y_3+\hat{k}z_3$, $\bar{d}=\hat{i}x_4+\hat{j}y_4+\hat{k}z_4$.

Then $\bar{b}-\bar{a}=\hat{i}(x_2-x_1)+\hat{j}(y_2-y_1)+\hat{k}(z_2-z_1)$,

$\bar{c}-\bar{a}=\hat{i}(x_3-x_1)+\hat{j}(y_3-y_1)+\hat{k}(z_3-z_1)$,

$\bar{d}-\bar{a}=\hat{i}(x_4-x_1)+\hat{j}(y_4-y_1)+\hat{k}(z_4-z_1)$

$\therefore$ Volume of the tetrahedron $= \frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}$.

Note II : Condition for the coplanarity of four points.

Here the volume of the tetrahedron is zero; i.e. the condition is

$[\bar{b}\,\bar{c}\,\bar{d}]+[\bar{c}\,\bar{a}\,\bar{d}]+[\bar{d}\,\bar{a}\,\bar{b}]-[\bar{a}\,\bar{b}\,\bar{c}]=0$

Note III : Shortest distance between two opposite edges.

The s.d. between the opposite edges AC and BD is numerical value of

$\frac{(\bar{d}-\bar{a}).\{(\bar{c}-\bar{a})\times(\bar{d}-\bar{b})\}}{|(\bar{c}-\bar{a})\times(\bar{d}-\bar{b})|}$.

Note IV : If the vertex A is taken as vector origin, then the volume of the tetrahedron is

$\frac{1}{6}[\bar{b}\,\bar{c}\,\bar{d}]$.

14.11 Sphere

(i) General Equation

Let $\overline{c}$ be the position vector of the centre C of the sphere of radius a w.r.t. the vector origin O.

If $\overline{r}$ is the position vector of any point P on the sphere, then $\overrightarrow{CP} = \overline{r} - \overline{c}$.

Now $|\overrightarrow{CP}|^2 = a^2$

or, $(\overline{r} - \overline{c})^2 = a^2$

or, $\overline{r}^2 - 2\overline{r}.\overline{c} + \overline{c}^2 - a^2 = 0$ .....(1)

it is the equation of sphere.

$\therefore\ \overline{r}^2 - 2\overline{r}.\overline{c} + k = 0$, ...(2)

if $k = \overline{c}^2 - a^2$.

Corollary I : If the centre is the origin, then $\overline{c} = 0$ and the equation of the sphere is

$\overline{r}^2 = a^2$ .....(3)

Corollary II : If the origin lies on the surface, then $\overline{c}^2 = a^2$ and the equation of the sphere is

$\overline{r}^2 - 2\overline{r}.\overline{c} = 0$ .....(4)

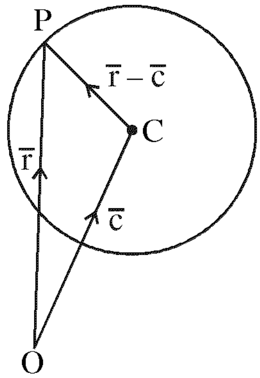


Fig. 14.11.1

(ii) Equation of sphere with diameter ends as $\overline{a}$ and $\overline{b}$

Here $\overline{a}$ and $\overline{b}$ are the position vectors of A and B, the ends of a diameter of the sphere. Let P be any point on the sphere whose position vector is $\overline{r}$.

Then $\overrightarrow{AP} = \overline{r} - \overline{a}$ and $\overrightarrow{BP} = \overline{r} - \overline{b}$.

Since PA is perpendicular to PB, $(\overline{r} - \overline{a}).(\overline{r} - \overline{b}) = 0$ .....(5)

It is the equation of the sphere.

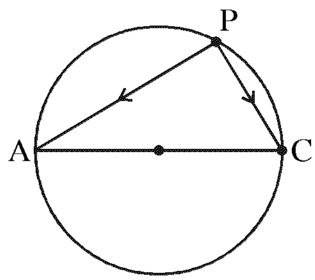


Fig. 14.11.2

(iii) Intersection of a line and a sphere.

Let the sphere and the line be $\overline{r}^2 - 2\overline{r}.\overline{c} + k = 0$ and $\overline{r} = \overline{a} + t\overline{b}$. The values of t, where the line cuts the sphere are given by $(\overline{a} + t\overline{b})^2 - 2(\overline{a} + t\overline{b}).\overline{c} + k = 0$

or, $\overline{b}^2 t^2 + 2\overline{b}.(\overline{a} - \overline{c})t + \overline{a}^2 - 2\overline{a}.\overline{c} + k = 0$ .....(6)

The two values of t obtained from (6) correspond to two points of intersection between the line and the sphere.

14.12 Centroids and centre of mass

Centroid : If $\bar{a}, \bar{b}, \bar{c}, ..., \bar{l}$ be position vectors of n given points and $\bar{g} = \overrightarrow{OG} = \frac{1}{n}(\bar{a} + \bar{b} + \bar{c} + ... + \bar{l})$, then G is called the centroid (mean centre, centre of mean position) of the given points (O is the vector origin).

If $\bar{g} = \overrightarrow{OG} = \dfrac{m_1\bar{a} + m_2\bar{b} + m_3\bar{c} + ... + m_n\bar{l}}{m_1 + m_2 + m_3 + ... + m_n}$, then G is called the centroid of n weighted points.

Here $m_1 + m_2 + m_3 + ... + m_n \neq 0$ and $m_1, m_2, ..., m_n$ are the weights of the points $\bar{a}, \bar{b}, \bar{c}, ..., \bar{l}$ respectively.

Centre of mass : If $m_1, m_2 ..., m_n$ masses are situated at the points $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$, then the point $\bar{a} = \dfrac{m_1\bar{a}_1 + m_2\bar{a}_2 + ... + m_n\bar{a}_n}{m_1 + m_2 + ... + m_n}$ is called the centre of mass.

Centre of gravity : If n parallel forces whose magnitudes are proportional to $m_1, m_2, ...$ m_n act at the points $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$, then the point $\bar{a} = \dfrac{m_1\bar{a}_1 + m_2\bar{a}_2 + ... + m_n\bar{a}_n}{m_1 + m_2 + ... + m_n}$ is called the centre of gravity of these points. It is equvalent to centre of mass.

14.13 Forces

Resultant of concurrent forces : If $\bar{F}_1, \bar{F}_2, ..., \bar{F}_n$ forces act at O, then the resultant of these forces is $\bar{F} = \bar{F}_1 + \bar{F}_2 + ... + \bar{F}_n$.

It acts at O. It $\bar{F} = 0$, the forces are in equilibrium.

Lami's Theorem : It three coplanar forces acting at a point be in equilibrium then magnitude of each force is proportional to the sine of the angle between the other two.

Let P, Q, R be the magnitudes of the forces and $\bar{a}, \bar{b}, \bar{c}$ be the unit vectors along them. Since the forces are in equilibrium

$$P\bar{a} + Q\bar{b} + R\bar{c} = 0 \qquad .....(1)$$

Taking vector product with $\bar{a}$,

$$P(\bar{a}\times\bar{a})+Q(\bar{a}\times\bar{b})+R(\bar{a}\times\bar{c})=0$$

or $Q(\bar{a}\times\bar{b})=R(\bar{a}\times\bar{c})$

or $\dfrac{Q}{(\bar{c}\times\bar{a})}=\dfrac{R}{(\bar{a}\times\bar{b})}$

Taking vector product with $\bar{b}$

$$P(\bar{b}\times\bar{a})+Q(\bar{b}\times\bar{b})+R(\bar{b}\times\bar{c})=0$$

or, $P(\bar{a}\times\bar{b})=R(\bar{b}\times\bar{c})$

or, $\dfrac{P}{|\bar{b}\times\bar{c}|}=\dfrac{R}{|\bar{a}\times\bar{b}|}$

$$\therefore\ \frac{P}{\sin\widehat{bc}}=\frac{Q}{\sin\widehat{ca}}=\frac{R}{\sin\widehat{ab}}.$$

Workdone by a force : If a force $\bar{F}$ experiences a displacement $\bar{d}$ then the work done by the force is the product of the magnitudes of the displacement and the component of the force in the direction of the displacement.

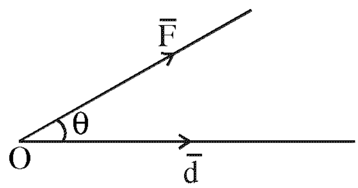


Fig. 14.13.1

$\therefore$ work done $= |\bar{d}||\bar{F}|\cos\theta$, here θ is the angle between $\bar{F}$ and $\bar{d}$ $= \bar{F}.\bar{d}$.

Hence the dot product between a force and its displacement represents the workdone by the force for this displacement.

Moment of a force about a point : Let a force $\bar{F}$ be applied on a body at A and O be any point.

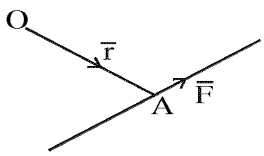


Fig. 14.13.2

Let $|\overrightarrow{OA}|=\bar{r}$.

$\bar{m}=\bar{r}\times\bar{F}$ is called the moment vector of $\bar{F}$ or torque of $\bar{F}$ about the point O. It is perpendicular to the plane through the point O and the line of $\bar{F}$.

14.14 Worked out examples

Example 1 : Find the vector equation of a straight line passing through the origin and parallel to the vector $(\hat{i}+3\hat{j}+5\hat{k})$.

Solution : The vector equation of the straight line $\bar{r} = t(\hat{i}+3\hat{j}+5\hat{k})$ where t is a scalar.

Example 2 : Find the vector equation of a straight line passing through the point (1, –2, 4) and parallel to the vector $\hat{i}+3\hat{j}-\hat{k}$.

Solution : Let the position vector of the point (1, –2, 4) is $\hat{i}-2\hat{j}+4\hat{k}$ relative to the origin.

Then the required equation of the straight line is $\bar{r} = (\hat{i}-2\hat{j}+4\hat{k})+t(\hat{i}+3\hat{j}-\hat{k})$

or, $\bar{r} = (1+t)\hat{i}+(3t-2)\hat{j}+(4-t)\hat{k}$, where t is a scalar.

Example 3 : Find the vector equation of a straight line passing through the points (1, 2, 3) and (3, 5, 7).

Solution : The position vectors of the points (1, 2, 3) and (3,5, 7) are $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}+5\hat{j}+7\hat{k}$ relative to the origin O.

$\therefore$ The required equation of the straight line is

$\bar{r} = (1-t)(\hat{i}+2\hat{j}+3\hat{k})+t(3\hat{i}+5\hat{j}+7\hat{k})$

$= (1+2t)\hat{i}+(2+3t)\hat{j}+(3+4t)\hat{k}$, where t is a scalar.

Example 4 : Find the equation of the plane through the points A(–1, 1, 2), B(1, –2, 1) and C(2, 2, 4).

Solution : Here $\overrightarrow{AB} = (2, -3, -1)$ and $\overrightarrow{AC} = (3, 1, 2)$.

Let P(x, y, z) be any point on the plane.

$\overrightarrow{AB}\times\overrightarrow{AC} = (-5, -7, 11)$ is perpendicular to the plane.

$\overrightarrow{AP} = (x + 1, y - 1, z - 2)$.

It lies in the plane. Therefore, it is perpendicular to $\overrightarrow{AB}\times\overrightarrow{AC}$.

Thus $|\overrightarrow{AP}|.(\overrightarrow{AB}\times\overrightarrow{AC}) = 0$

or, $-5(x + 1) -7(y - 1) + 11(z - 2) = 0$

or, $5x + 7y - 11z + 20 = 0$.

Example 5 : Find the vector equation of the plane passing through the three points (–1, 1, 2), (1, –2, 1) and (2, 2, 4).

Solution : The position vectors of the three points are $\bar{a} = -\hat{i} + \hat{j} + 2\hat{k}$, $\bar{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\bar{c} = 2\hat{i} + 2\hat{j} + 4\hat{k}$.

Hence the equation of the required plane is

$$\bar{r}.(\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}) = [\bar{a}\ \bar{b}\ \bar{c}]$$

or, $r.(11\hat{k} - 5\hat{i} - 7\hat{j}) = 20$.

Example 6 : A(4, d, –1), B(4, 2, –2) and C(6, 4, –1) are three non-collinear points. Use vetor method to find d such that the perpendicular distance of A from the line joining B and C is 1.

Solution : $\overrightarrow{BA}$ = (0, d –2, 1) and $\overrightarrow{BC}$ = (2, 2, 1).

By the given conditon, $\dfrac{|\overrightarrow{BC} \times \overrightarrow{BA}|}{|\overrightarrow{BC}|} = 1$.

Here $\overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 0 & d-2 & 1 \end{vmatrix} = (4-d)\hat{i} - 2\hat{j} + (2d-4)\hat{k}$

$\therefore$ $(4 - d)^2 + 4 + (2d - 4)^2 = 9$

or, $5d^2 - 24d + 27 = 0$

or, $(d - 3)(5d - 9) = 0$, or, $d = 3, 9/5$

Example 7 : Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the plane $\bar{r}.\hat{i} = 0, \bar{r}.\hat{j} = 0, \bar{r}.\hat{k} = 0$ and $\bar{r}.(\hat{i} + \hat{j} + \hat{k}) = a$. Also write down the equation of the sphere.

Solution : Let the centre of the sphere be $\bar{\alpha}$.

The given planes are tangent planes. The perpendiculars from the centre to these planes are equal to the radius of the sphere.

$$\therefore\ \bar{\alpha}.\hat{i} = \bar{\alpha}.\hat{j} = \bar{\alpha}.\hat{k} = \frac{\alpha.(\hat{i} + \hat{j} + \hat{k}) - a}{\sqrt{3}}.$$

If $\overline{\alpha} = \hat{i}\alpha_1 + \hat{j}\alpha_2 + \hat{k}\alpha_3$, then $\alpha_1 = \alpha_2 = \alpha_3 = \dfrac{3\alpha_1 - a}{\sqrt{3}}$

From this $\alpha_1 = \dfrac{a}{3-\sqrt{3}} = \dfrac{a\left(3+\sqrt{3}\right)}{6}$

Thus the centre is $\dfrac{a}{6}(3+\sqrt{3})(\hat{i}+\hat{j}+\hat{k})$.

The equation of the sphere is $(\overline{r} - \overline{\alpha})^2 = \left\{\dfrac{a}{6}(3+\sqrt{3})\right\}^2$.

Example 8 : A particle acted on by constant forces $4\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$, is displaced from the point $\hat{i}+2\hat{j}+3\hat{k}$ to the point $5\hat{i}+4\hat{j}+\hat{k}$. Find the workdone by the force on the particle.

Solution : Let $\overline{F}_1 = 4\hat{i}+\hat{j}-3\hat{k}$, $\overline{F}_2 = 3\hat{i}+\hat{j}-\hat{k}$ and $\overline{F}$ be the resultant of these forces.

Then $\overline{F} = \overline{F}_1 + \overline{F}_2 = 7\hat{i}+2\hat{j}-4\hat{k}$

The displacement vector $\overline{d} = (5\hat{i}+4\hat{j}+\hat{k}) - (\hat{i}+2\hat{j}+3\hat{k}) = 4\hat{i}+2\hat{j}-2\hat{k}$

$\therefore$ Workdone = $\overline{F}.\overline{d} = (7\hat{i}+2\hat{j}-4\hat{k}).(4\hat{i}+2\hat{j}-2\hat{k})$

= 28 + 4 + 8 = 40 units.

Example 9 : A force of 15 units acts in the direction of the vector $\hat{i}-2\hat{j}+\hat{k}$ and passes through a point $\hat{i}+\hat{j}+\hat{k}$. Find the moment of the force about the point $2\hat{i}-2\hat{j}+2\hat{k}$.

Solution : Let $2\hat{i}-2\hat{j}+2\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ be the position vectors of the points A and P.

The unit vector in the direction of the force is $\dfrac{1}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k})$.

Required moment $= \overrightarrow{AP} \times \dfrac{15}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k}) = \dfrac{15}{\sqrt{6}}(-\hat{i}+3\hat{j}-\hat{k}) \times (\hat{i}-2\hat{j}+\hat{k})$

$$= \frac{15}{\sqrt{6}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \frac{15}{\sqrt{6}}(\hat{i}-\hat{k}).$$

Example 10 : Find the s.d. between two skew lines $\bar{r} = \bar{r}_1 + t\bar{\alpha}$, $\bar{r} = \bar{r}_2 + t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors with coordinates (1, –2, 3), (2, 1, 1), (–2, 2, –1), (–3, 1, 2) respectively.

Solution : Here s.d. $= |(\bar{r}_2 - \bar{r}_1).(\bar{\alpha}\times\bar{\beta})| / |\bar{\alpha}\times\bar{\beta}|$, $\bar{\alpha}\times\bar{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \hat{i} - 7\hat{j} + 5\hat{k}$.

$$\bar{r}_2 - \bar{r}_1 = (-2\hat{i} + 2\hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = -3\hat{i} + 4\hat{j} - 4\hat{k}$$

$$\text{Now, s.d. } = \frac{\left|(-3\hat{i} + 4\hat{j} - 4\hat{k})\right|.(\hat{i} - 7\hat{j} + 5\hat{k})}{|\hat{i} - 7\hat{j} + 5\hat{k}|}$$

$$= \frac{|-3 - 28 - 20|}{\sqrt{1 + 49 + 25}} = \frac{51}{5\sqrt{3}} = \frac{17\sqrt{3}}{5}.$$

Example 11 : If the volume of a tetrahedron is 2 and three of its vertices have position vectors (1, 1, 0), (1, 0, 1), (2, –1, 1), find the locus of the fourth vertex.

Solution : Let ABCD be the tetrahedron and the position vectors of A, B, C be (1, 1, 0), (1, 0, 1), (2, –1, 1) respectively.

Let the position vector of D be (α, β, γ).

$$\text{Volume of a tetrahedron } = \frac{1}{6}\{(\overrightarrow{AB} \times \overrightarrow{AC}).\overrightarrow{AD}\}$$

Here $\overrightarrow{AB} = -\hat{j} + \hat{k}$, $\overrightarrow{AC} = \hat{i} - 2\hat{j} + \hat{k}$, $\overrightarrow{AD} = (\alpha - 1)\hat{i} + (\beta - 1)\hat{j} + \gamma\hat{k}$

$$\therefore \frac{1}{6}\{(-\hat{j} + \hat{k}) \times (\hat{i} - 2\hat{j} + \hat{k})\}.\{(\alpha - 1)\hat{i} + (\beta - 1)\hat{j} + \gamma\hat{k}\} = 2$$

or, $(\hat{i} + \hat{j} + \hat{k}) \cdot \{(\alpha - 1)\hat{i} + (\beta - 1)\hat{j} + \gamma\hat{k}\} = 12$

or, $(\alpha - 1) + (\beta - 1) + \gamma = 12$

or, $\alpha + \beta + \gamma = 14$

Hence the locus of D is $x + y + z = 14$.

Example 12 : If the vector equation of lines are $\bar{r} = \bar{r}_1 + t\bar{\alpha}$, $\bar{r} = \bar{r}_2 + t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors, with coordinates (1, 4, 5), (2, 1, 2), (2, 8, 11) and (–1, 3, 4) respectively, show that the lines are coplanar.

Solution : The lines will be coplanar, if $[\bar{r}_2\, \bar{\alpha}\bar{\beta}] = [\bar{r}_1\, \bar{\alpha}\bar{\beta}]$.

$$\bar{\alpha} \times \bar{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ -1 & 3 & 4 \end{vmatrix} = -2\hat{i} - 10\hat{j} + 7\hat{k}$$, where i, j, k are unit vectors parallel to x, y, z-axes.

$\bar{r}_2.(\bar{\alpha} \times \bar{\beta}) = (2\hat{i} + 8\hat{j} + 11\hat{k}).(-2\hat{i} - 10\hat{j} + 7\hat{k})$

$= -4 - 80 + 77 = -7$

$\bar{r}_1.(\bar{\alpha} \times \bar{\beta}) = (\hat{i} + 4\hat{j} + 5\hat{k}).(-2\hat{i} - 10\hat{j} + 7\hat{k}) = -2 - 40 + 35 = -7$

Hence the lines are coplanar.

Example 13 : Show that the plane containing the two straight lines $\bar{r} = \bar{a} + t\bar{a}'$ and $\bar{r} = \bar{a}' + s\bar{a}$ is represented by $[\bar{r}\,\bar{a}\,\bar{a}'] = 0$.

Solution : The required plane contains the lines $\bar{r} = \bar{a} + t\bar{a}'$ and $\bar{r} = \bar{a}' + s\bar{a}$.

Therefore, it passes through the point $\bar{a}$ and is parallel to the vectors $\bar{a}$ and $\bar{a}'$. If $\bar{r}$ be the position vector of a variable point P on the plane, the vectors $\bar{r} - \bar{a}, \bar{a}, \bar{a}'$ should be coplanar.

Therefore, $(\bar{r} - \bar{a}).(\bar{a} \times \bar{a}') = 0$

or, $\bar{r}.(\bar{a} \times \bar{a}') - \bar{a}.(\bar{a} \times \bar{a}') = 0$

or, $[\bar{r}\,\bar{a}\,\bar{a}'] - [\bar{a}\,\bar{a}\,\bar{a}'] = 0$

or, $[\bar{r}\,\bar{a}\,\bar{a}'] = 0$, since $[\bar{a}\,\bar{a}\,\bar{a}'] = 0$

Hence the required equation of the plane is $[\bar{r}\,\bar{a}\,\bar{a}'] = 0$.

Example 14 : Find the equation of the plane containing the two parallel lines $\bar{r} = \bar{a} + s\bar{b}$, $\bar{r} = \bar{a}' + t\bar{b}\cdot$

Solution : The plane in equation passes through the points $\bar{a}$ and $\bar{a}'$ and is parallel to the vector $\bar{b}$. If $\bar{r}$ be the position vector of any variable points on the plane, then the vectors $\bar{r} - \bar{a}$, $\bar{a}' - \bar{a}$ and $\bar{b}$ are coplanar.

Therefore we have $(\bar{r}-\bar{a}).[(\bar{a}'-\bar{a})\times\bar{b}]=0$

or, $\bar{r}.[(\bar{a}'-\bar{a})\times\bar{b}]-\bar{a}.[(\bar{a}'-\bar{a})\times\bar{b}]=0$

or, $\bar{r}.\left[(\bar{a}'-\bar{a})\times\bar{b}\right]=\bar{a}.\left[\bar{a}'\times\bar{b}-\bar{a}\times\bar{b}\right]$

$=[\bar{a}\,\bar{a}'\,\bar{b}]-[\bar{a}\,\bar{a}\,\bar{b}]=[\bar{a}\,\bar{a}'\,\bar{b}]$, since $[\bar{a}\,\bar{a}\,\bar{b}]=0$

Hence the equation of the required plane is $\bar{r}.[(\bar{a}'-\bar{a})\times\bar{b}]=[\bar{a}\,\bar{a}'\,\bar{b}]$.

14.15 Summary

In this unit, we have learnt to determine the vector equations of a straight line, sphere, angle bisector, sphere, lines related to planes etc. Also we have learnt how to compute angle between two planes, distance of point from a given plane and a given line, possition vectors or centrits and centre of mass. Moreover we how can find out work done by a force and its moment about a point.

14.16 Exercises

1. Find the equation of the line through the points (2, 3, 4) and (3, 4, 5).
   [Hints : see the workedout example 3].

   [**Ans.** $x-2=y-3=z-4$ i.e., $(2+t)\hat{i}+(3+t)\hat{j}+(4+t)\hat{k}$]

2. Find the equation of a line through the point (–2, 1, 0) and parallel to the vector $5\hat{i}-3\hat{j}+4\hat{k}$.

   [Hints : see the workedout example 2] [**Ans.** $(5t-2)\hat{i}+(1-3t)\hat{j}+4t\hat{k}$]

3. Find the equation of a plane passing through the points (2, –1, 4), (3, 4, 7) and (–2, 3, –1).

   [Hints : see the workedout example 4] [**Ans.** $37x+7y-24z+29=0$]

4. Show that the lines $\bar{r}=\bar{r}_1+t\bar{\alpha}, \bar{r}=\bar{r}_2+t\bar{\beta}$, where t is a scalar and $\bar{r}_1, \bar{\alpha}, \bar{r}_2, \bar{\beta}$ are vectors, with co-ordinates (4, 5, 1), (–4, –6, –2), (3, 9, 4), (–7, –5, 0) respectively, are coplanar.

   [Hints : see the workedout example 12]

5. Show that $\bar{r}=\bar{a}+t\bar{b}+s\bar{c}$ represents the vector equation of a plane through the point $\bar{a}$ and parallel to the vectors $\bar{b}$ and $\bar{c}$, t and s are scalars.

   [Hints : solve yourself]

6. Prove that the lines $\overline{r} = \overline{a} + t(\overline{b} \times \overline{c}), \overline{r} = \overline{b} + s(\overline{c} \times \overline{a})$ will intersect if $\overline{a}.\overline{c} = \overline{b}.\overline{c}$.
   [Hints : solve yourself]

7. A particle acted on by constant forces $2\hat{i} + 3\hat{j} - \hat{k}$ and $3\hat{i} - \hat{j} + 5\hat{k}$ is displaced from the point A(1, 3, 2) to the point B(4, 5, –1), find the workdone by the forces.
   [Hints : see the workedout example 8] **[Ans.** 7 units]

8. Show that the workdone by a particle acted on by a force $5\hat{i} + 10\hat{j} + 15\hat{k}$ for the displacement from the point. $\hat{i} + 3\hat{k}$ to $3\hat{i} - \hat{j} - 6\hat{k}$ is 135 units. [Hints : solve yourself]

9. If a force given by $\vec{F} = 2\hat{i} + 4\hat{j} - \hat{k}$ displaces a particle from the position A to B, where position vectors of A and B are given by $\hat{i} + \hat{j} + 2\hat{k}$ and $3\hat{i} - \hat{j} - \hat{k}$ respectively, find the workdone by force.
   [Hints : see the workedout example 8] **[Ans.** 1 unit]

10. The vector equations of two lines are $\overline{r} = \overline{r}_1 + t\overline{\alpha}$, $\overline{r} = \overline{r}_2 + t\overline{\beta}$, whare t is a scalar and $\overline{r}_1, \overline{\alpha}, \overline{r}_2, \overline{\beta}$ are vectors with co-ordinates (–14, 8, 6), (25, –4, –5), (3, 5, 5), (3, 6, 3) respectively. Find the s.d. between the lines.
    [Hints : see the workedout example 10] $\left[\textbf{Ans. } \frac{29\sqrt{2}}{5} \text{ units}\right]$

11. If $\overline{\alpha}$ be the position vector of a point P, find by vector method the distance of P from the line $\overline{r} = \overline{\beta} + t\overline{\gamma}$, where the vectors $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ have co-ordinates (5, –6, 2), (1, –1, 2) and (0, –4, –3) respectively.
    [Hints : solve yourselt] **[Ans.** 5]

12. Show that the moment of a force $4\hat{i} + 2\hat{j} + \hat{k}$ through the point $5\hat{i} + 2\hat{j} + 4\hat{k}$ about the point $3\hat{i} - \hat{j} + 3\hat{k}$ is $\hat{i} + 2\hat{j} - 8\hat{k}$.
    [Hints : see the workedout example 9]

13. Determine the plane through the point $\hat{i} + 2\hat{j} - \hat{k}$ which is perpendicular to the line of intersection of the planes $\overline{r}.(3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\overline{r}.(\hat{i} + 4\hat{j} - 2\hat{k}) = 2$.
    [Hints : solve yourself] **[Ans.** $\overline{r}.(\hat{i} - 7\hat{j} - 13\hat{k}) = 1$]

14. Find the vector equation of the plane through the point $5\hat{i}+2\hat{j}-3\hat{k}$ and perpendicular to each the planes $\bar{r}.(2\hat{i}-\hat{j}+2\hat{k})=0$ and $\bar{r}.(\hat{i}+3\hat{j}-5\hat{k})+3=0$.

[Hints : solve yourself] [**Ans.** $(\bar{r}.(\hat{i}-12\hat{j}-7\hat{k})=0$]

15. Find the equation of the plane passing through three given points $-2\hat{i}+6\hat{j}-6\hat{k}$, $-3\hat{i}+10\hat{j}-9\hat{k}$ and $-5\hat{i}-6\hat{k}$.

[Hints : take as $\bar{a}=-2\hat{i}+6\hat{j}-6\hat{k}$, $\bar{b}=-3\hat{i}+10\hat{j}-9\hat{k}$, $\bar{c}=-5\hat{i}-6\hat{k}$.]

Then the required plane is $\bar{r}.[\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}]=[\bar{a}\,\bar{b}\,\bar{c}]$ [**Ans.** $\bar{r}.(2\hat{i}-\hat{j}-2\hat{k})=2$]

16. Find the equation of the line through the point $\bar{a}$ parallel to the plane $\bar{r}.\bar{n}=p$ and perpendicular to the line $\bar{r}=\bar{b}+t\bar{c}$.

[Hints : suppose the required line is parallel to the vector $\bar{d}$. Then $\bar{d}$ is perpendicular to both the vectors $\bar{n}$ and $\bar{c}$. Therefore the vector $\bar{d}$ is parallel to the vector $\bar{n}\times\bar{c}$. Hence the required equation of line is $\bar{r}=\bar{a}+s(\bar{n}\times\bar{c})$] [**Ans.** $\bar{r}=\bar{a}+s(\bar{n}\times\bar{c})$]

Unit-15 ❑ Vector Valued Functions of Scalar Variables

Structure

15.0 Objectives

In this unit the readers will learn the followings :

- Definition of vector valued functions of real variable.
- Limits and continuity of vector valued functions.
- Differentiation and integration of vector valued functions.
- Tangent and normal components of acceleration.

15.1 Introduction

In ordinary Calculus we have learn the concepts of real valued functions of real variables and their limits, continuity, differentiability and integrability. In this unit we will discuss about vector valued functions of scalar variables and their limits, continuity, differentiability and integrability. We shall also study differential geometry using vector calculus in brief.

15.2 Vector Valued Functions of Scalar Variable

If by some law $\vec{f}$, for each value of a scalar variable t in some interval $[a, b]$, there corresponds a definite unique vector $\vec{r}$, then $\vec{f}$ is called a single-valued vector function of the scalar variable t and is denoted by $\vec{r} = \vec{f}(t)$.

$\vec{f}(c)$ denotes the particular vector for some fixed value c of t.

If $\hat{i}, \hat{j}, \hat{k}$ be three unit vectors along three mutually perpendicular fixed directions, then $\vec{f}(t)$ can be expressed as $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ or simply $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$, where $f_1(t), f_2(t), f_3(t)$ are the scalar functions of t along $\hat{i}, \hat{j}, \hat{k}$ respectively.

If in particular the scalar t represents time, then $\vec{r}$ represents the position vector of any moving point at time t w.r.t. a certain vector origin. Then the velocity and acceleration of that moving point are also another vector functions of the same scalar variable t.

15.3 Limits of Vector Functions

A vector function $\vec{f}(t)$ is said to tend to the limit $\vec{l}$ when $t \to c$, if for any preassigned positive number ε, however small, there corresponds a positive number δ such that

$$\left|\vec{f}(t) - \vec{l}\right| < \varepsilon \text{ when } 0 < |t - c| \le \delta$$

and it will be expressed as $\lim_{t \to c} \vec{f}(t) = \vec{l}$, provided that such limit exists.

If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ and $\hat{l} = l_1(t)\hat{i} + l_2(t)\hat{j} + l_3(t)\hat{k}$, then when $\lim_{t \to c} \vec{f}(t) = \vec{l}$, then $\lim_{t \to c} f_1(t) = l_1$, $\lim_{t \to c} f_2(t) = l_2$ and $\lim_{t \to c} f_3(t) = l_3$.

15.3.1. Some standard results on limits

If $\vec{f}(t)$ and $\vec{g}(t)$ are two vector functions of scalar variable t and if $\lim\limits_{t\to c}\vec{f}(t)=\vec{l}$ and $\lim\limits_{t\to c}\vec{g}(t)=\vec{m}$, then

(i) $\lim\limits_{t\to c}\left[\vec{f}(t)\pm\vec{g}(t)\right]=\vec{l}\pm\vec{m}$ (ii) $\lim\limits_{t\to c}\left[\vec{f}(t).\vec{g}(t)\right]=\vec{l}.\vec{m}$

(iii) $\lim\limits_{t\to c}\left[\vec{f}(t)\times\vec{g}(t)\right]=\vec{l}\times\vec{m}$ (iv) $\lim\limits_{t\to c}\left|\vec{f}(t)\right|=\left|\vec{l}\right|$

(v) $\lim\limits_{t\to c}\left[\phi(t)\vec{f}(t)\right]=p\vec{l}$, where $\lim\limits_{t\to c}\phi(t)=p$.

15.4 Continuity of Vector Functions

A vector function $\vec{f}(t)$ of a scalar variable t is said to be continuous at $t = c$, if for any preassigned positive number ε, however small, there corresponds a positive number δ such that

$$\left|\vec{f}(t)-\vec{f}(c)\right|<\varepsilon \ \textit{when}\ 0\le|t-c|\le\delta$$

and it will be denoted by $\lim\limits_{t\to c}\vec{f}(t)=\vec{f}(c)$.

Also a function $\vec{f}(t)$ is said to be continuous in an interval $[a, b]$ of t, if it is continuous for every value of t in $[a, b]$.

In the same way, if $\vec{f}(t)$ and $\vec{g}(t)$ are two vector functions of scalar variable t are continuous, then the functions expressed in art. 13.4.1 are also continuous.

15.5 Differentiation of Vector Functions

A vector function $\vec{f}(t)$ of a scalar variable t is said to have a derivative at c, if

$$\lim_{h\to 0}\frac{\vec{f}(c+h)-\vec{f}(c)}{h}$$

exists. Then that limiting value is said to the derivative of $\vec{f}(t)$ at $t = c$ and is denoted by $\vec{f}'(c)$ or by $\dfrac{d\vec{f}}{dt}$ at $t = c$.

This process of finding the derivative of functions is known as differentiation or derivation.

Another definition of differentiation of $\vec{f}(t)$ is given as

$$\vec{f}'(c) = \lim_{t\to c}\frac{\vec{f}(t)-\vec{f}(c)}{t-c},$$

provided that limit exists.

For any differentiable function $\vec{f}(t)$, we may write

$$\frac{d\vec{f}}{dt} = \vec{f}'(t) = \lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}.$$

By writing $h = \Delta t$, we get

$$\frac{d\vec{f}}{dt} = \vec{f}'(t) = \lim_{\Delta t\to 0}\frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t}.$$

A function is said to be derivable in an interval if it is derivable at every point in that interval.

15.5.1. Higher Order Derivatives

If the vector function $\vec{f}(t)$ of scalar variable t has a derivative $\vec{f}'(t)$ in a given interval, and $\vec{f}'(t)$ is itself derivable in some interval, then the second order derivative of $\vec{f}(t)$ is defined by

$$\frac{d^2\vec{f}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{f}}{dt}\right) = \vec{f}''(t) = \lim_{\Delta t\to 0}\frac{\vec{f}'(t+\Delta t)-\vec{f}'(t)}{\Delta t}.$$

Similarly, we can define more higher order derivatives of vector functinos.

15.5.2. Differentials

From the definition of derivative, we can write

$$\frac{\Delta\vec{f}}{\Delta t} = \frac{\vec{f}(t+\Delta t)-\vec{f}(t)}{\Delta t} = \vec{f}'(t)+\vec{\delta},$$

where $\vec{\delta}$ is a vector such that $\vec{\delta}\to 0$ as $\Delta t \to 0$.

Then $\vec{f}$ is said to be differentiable at t if we can express

$$\Delta\vec{f} = \vec{f}(t+\Delta t)-\vec{f}(t) = \vec{f}'(t)\Delta t+\vec{\delta}\Delta t,$$

where $\vec{\delta}$ is a vector such that $\vec{\delta}\to 0$ as $\Delta t\to 0$.

The part $\vec{f}'(t)\Delta t$ is called the diferential of the vector function $\vec{f}$ and is denoted by

$$d\vec{f} = \vec{f}'(t)\Delta t.$$

Since the above expression is true for every diferentiable vector function $\vec{f}(t)$, we may take $\vec{f}(t) = t\hat{i}$. Then we have $\Delta t = dt$. Hence the differential of the vector function $\vec{f}$ is

$$d\vec{f} = \vec{f}'(t)dt.$$

If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$d\vec{f}(t) = df_1\hat{i} + df_2\hat{j} + df_3\hat{k}.$$

Theorem 15.5.1 Every derivable function is continuous but the converse may not be true.

Proof : For any value c of t, we may write

$$\vec{f}(t) - \vec{f}(c) = \frac{\vec{f}(t) - \vec{f}(c)}{t - c}.(t - c).$$

So

$$\begin{aligned}\lim_{t\to c}[\vec{f}(t) - \vec{f}(c)] &= \lim_{t\to c}\frac{\vec{f}(t) - \vec{f}(c)}{t - c}.\lim_{t\to c}(t - c)\\ &= \vec{f}'(c).0 = 0 \quad \text{[since } \vec{f} \text{ is derivable].}\end{aligned}$$

Hence $\lim_{t\to c}\vec{f}(t) = \vec{f}(c)$.

Therefore the function $\vec{f}(t)$ is continuous at $t = c$.

Let $\vec{f}(t) = |t|\hat{i}$, which is continuous for every value of t, but it is not derivable at $t = 0$, because

$$\lim_{t\to 0}\frac{\vec{f}(t) - \vec{f}(0)}{t - 0} = \lim_{t\to 0}\frac{|t|\hat{i}}{t} = \hat{i} \text{ or } -\hat{i},$$

according as $t \to 0$ from positive side or negative side. So $\vec{f}'(0)$ does not exist.

Theorem 15.5.2 If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ is a derivable function then $f_1(t), f_2(t), f_3(t)$ are also derivable functions and $\frac{df}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$.

Proof : we can easily see that

$$\frac{d\vec{f}}{dt} = \lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}$$

$$= \lim_{h\to 0}\frac{f_1(t+h)-f_1(t)}{h}\hat{i} + \lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h}\hat{j} + \lim_{h\to 0}\frac{f_3(t+h)-f_3(t)}{h}\hat{k}$$

$$= \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$ [since $f_1(t), f_2(t), f_3(t)$ are derivable functions].

Theorem 15.5.3 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}[\vec{f}(t) \pm \vec{g}(t)] = \frac{d\vec{f}(t)}{dt} \pm \frac{d\vec{g}(t)}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t) \pm \vec{g}(t)$.

Then

$$\frac{d}{dt}[\vec{f}(t) \pm \vec{g}(t)] = \frac{d\vec{G}}{dt} = \lim_{h\to 0}\frac{\vec{G}(t+h)-\vec{G}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h} \pm \lim_{h\to 0}\frac{\vec{g}(t+h)-\vec{g}(t)}{h}$$

$$= \frac{d\vec{f}(t)}{dt} \pm \frac{d\vec{g}(t)}{dt}.$$

Theorem 15.5.4 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}[\vec{f}(t).\vec{g}(t)] = \frac{d\vec{f}}{dt}.\vec{g}(t) + \vec{f}(t).\frac{d\vec{g}}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t).\vec{g}(t)$.

Then

$$\frac{d}{dt}\left[\vec{f}(t).\vec{g}(t)\right] = \frac{d\vec{G}}{dt} = \lim_{h\to 0}\frac{\vec{G}(t+h)-\vec{G}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\vec{f}(t+h).\vec{g}(t+h) - \vec{f}(t)\vec{g}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}.\vec{g}(t) + \lim_{h \to 0} \vec{f}(t+h).\frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}.\vec{g}(t) + \lim_{h \to 0} \vec{f}(t+h).\lim_{h \to 0} \frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \frac{d\vec{f}}{dt}.\vec{g}(t) + \vec{f}(t).\frac{d\vec{g}}{dt}.$$

Theorem 15.5.5 If $\vec{f}(t)$ and $\vec{g}(t)$ be two derivable vector functions of scalar variable t, then $\frac{d}{dt}\left[\vec{f}(t) \times \vec{g}(t)\right] = \frac{d\vec{f}(t)}{dt} \times \vec{g}(t) + \vec{f}(t) \times \frac{d\vec{g}(t)}{dt}$.

Proof : Let $\vec{G}(t) = \vec{f}(t) \times \vec{g}(t)$.

Then

$$\frac{d}{dt}\left[\vec{f}(t) \times \vec{g}(t)\right]$$

$$= \frac{d\vec{G}}{dt} = \lim_{h \to 0} \frac{\vec{G}(t+h) - \vec{G}(t)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\vec{f}(t+h) \times \vec{g}(t+h) - \vec{f}(t) \times \vec{g}(t)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{\vec{f}(t+h) - \vec{f}(t)}{h} \times \vec{g}(t) + \vec{f}(t+h) \times \frac{\vec{g}(t+h) - \vec{g}(t)}{h}\right]$$

$$= \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \times \vec{g}(t) + \lim_{h \to 0} \vec{f}(t+h) \times \lim_{h \to 0} \frac{\vec{g}(t+h) - \vec{g}(t)}{h}$$

$$= \frac{d\vec{f}(t)}{dt} \times \vec{g}(t) + \vec{f}(t) \times \frac{d\vec{g}(t)}{dt}.$$

Theorem 15.5.6 (Differentiation of a function of a function) :

If $\vec{f}(u)$ is a differentiable vector function of a scalar variable u and u itself is a differentiable function of another scalar variable t, then $\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}.\frac{du}{dt}$.

Proof : Since u is a differential function of t, we may write

$$\Delta u = \frac{du}{dt}\Delta t + \alpha\Delta t, \tag{15.6.1}$$

where $\alpha \to \mathbf{0}$ as $\Delta t \to 0$.

Again since $\vec{f}(u)$ is a differentiable function of a scalar variable u, we have

$$\Delta\vec{f} = \frac{d\vec{f}}{du}\Delta u + \vec{\beta}\Delta u, \tag{15.6.2}$$

where $\vec{\beta} \to \vec{0}$ as $\Delta u \to 0$.

Now Putting the value of Δu from (13.6.1) in (13.6.2), we get

$$\begin{aligned}\Delta\vec{f} &= \frac{d\vec{f}}{du}\left(\frac{du}{dt}\Delta t + \alpha\Delta t\right) + \vec{\beta}\left(\frac{du}{dt}\Delta t + \alpha\Delta t\right)\\ &= \left(\frac{d\vec{f}}{du}.\frac{du}{dt}\right)\Delta t + \left(\alpha\frac{d\vec{f}}{du} + \vec{\beta}\frac{du}{dt} + \vec{\beta}\alpha\right)\Delta t.\end{aligned} \tag{15.6.3}$$

Since both $\frac{d\vec{f}}{du}$ and $\frac{du}{dt}$ exist, $\vec{\beta} \to \vec{0}$ as $\Delta u \to 0$ and $\alpha \to 0$ as $\Delta t \to 0$, we obtain

$$\vec{\gamma} = \alpha\frac{d\vec{f}}{du} + \vec{\beta}\frac{du}{dt} + \vec{\beta}\alpha \to \vec{0} \text{ as } \Delta t \to 0.$$

Using this in (13.6.3), we obtain

$$\Delta\vec{f} = \left(\frac{d\vec{f}}{du}.\frac{du}{dt}\right)\Delta t + \vec{\gamma}\Delta t,$$

where $\vec{\gamma} \to \vec{0}$ as $\Delta t \to 0$.

So we may say that $\vec{f}(t)$ is a differentiable function of scalar variable t and the differential of $\vec{f}$ is

$$d\vec{f} = \left(\frac{d\vec{f}}{du}.\frac{du}{dt}\right)\Delta t. \tag{15.6.4}$$

Since (13.6.4) is true for any differentiable function $\vec{f}(t)$ of t, taking $\vec{f} = u\hat{i}$ and $u = t$, we get $\Delta t = dt$ and then from (13.6.4), we obtain

$$\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}.\frac{du}{dt}.$$

Theorem 15.5.7 If $\vec{f}(t)$ is a vector function of scalar variable t and $\phi(t)$ is a scalar function and they are derivable, then $\frac{d}{dt}[\phi\vec{f}] = \frac{d\phi}{dt}\vec{f} + \phi\frac{d\vec{f}}{dt}$.

Proof : Let $\vec{G}(t) = \phi(t).\vec{f}(t)$.

Then

$$\frac{d}{dt}\left[\phi(t).\vec{f}(t)\right] = \frac{d\vec{G}}{dt}$$

$$= \lim_{h\to 0}\frac{\phi(t+h).\vec{f}(t+h) - \phi(t).\vec{f}(t)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\phi(t+h)-\phi(t)}{h}.\vec{f}(t) + \phi(t+h).\frac{\vec{f}(t+h)-\vec{f}(t)}{h}\right]$$

$$= \lim_{h\to 0}\frac{\phi(t+h)-\phi(t)}{h}.\vec{f}(t) + \lim_{h\to 0}\phi(t+h).\lim_{h\to 0}\frac{\vec{f}(t+h)-\vec{f}(t)}{h}$$

$$= \frac{d\phi}{dt}.\vec{f} + \phi.\frac{d\vec{f}}{dt}.$$

Theorem 15.5.8 If $\vec{f}(t)$, $\vec{g}(t)$ and $\vec{h}(t)$ are three vector functions of scalar variable t, then

$$\frac{d}{dt}\left[\vec{f}\vec{g}\vec{h}\right] = \left[\frac{d\vec{f}}{dt}\vec{g}\vec{h}\right] + \left[\vec{f}\frac{d\vec{g}}{dt}\vec{h}\right] + \left[\vec{f}\vec{g}\frac{d\vec{h}}{dt}\right].$$

Proof :

$$\begin{aligned}\frac{d}{dt}\left[\vec{f}\vec{g}\vec{h}\right] &= \frac{d}{dt}\left[\vec{f}.(\vec{g}\times\vec{h})\right] = \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\frac{d}{dt}(\vec{g}\times\vec{h}) \\ &= \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\left(\frac{d\vec{g}}{dt}\times\vec{h} + \vec{g}\times\frac{d\vec{h}}{dt}\right) \\ &= \frac{d\vec{f}}{dt}.(\vec{g}\times\vec{h}) + \vec{f}.\left(\frac{d\vec{g}}{dt}\times\vec{h}\right) + \vec{f}.\left(\vec{g}\times\frac{d\vec{h}}{dt}\right) \\ &= \left[\frac{d\vec{f}}{dt}\vec{g}\vec{h}\right] + \left[\vec{f}\frac{d\vec{g}}{dt}\vec{h}\right] + \left[\vec{f}\vec{g}\frac{d\vec{h}}{dt}\right].\end{aligned}$$

Theorem 15.5.9 If $\vec{f}(t)$, $\vec{g}(t)$ and $\vec{h}(t)$ are three vector functions of scalar variable t, then

$$\frac{d}{dt}\left(\vec{f}\times(\vec{g}\times\vec{h})\right)=\frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}\right)+\vec{f}\times\left(\vec{g}\times\frac{d\vec{h}}{dt}\right).$$

Proof :

$$\begin{aligned}\frac{d}{dt}\left(\vec{f}\times(\vec{g}\times\vec{h}\right) &= \frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\frac{d}{dt}(\vec{g}\times\vec{h})\\ &= \frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}+\vec{g}\times\frac{d\vec{h}}{dt}\right)\\ &= \frac{d\vec{f}}{dt}\times(\vec{g}\times\vec{h})+\vec{f}\times\left(\frac{d\vec{g}}{dt}\times\vec{h}\right)+\vec{f}\times\left(\vec{g}\times\frac{d\vec{h}}{dt}\right)\end{aligned}$$

Theorem 15.5.10 A necessary and sufficient condition that a vector $\vec{u}$ has a constant length is that $\vec{u}.\frac{d\vec{u}}{dt}=0.$

Proof : We know that $\vec{u}^2=|\vec{u}|^2=\vec{u}.\vec{u}.$

So

$$\frac{d}{dt}(\vec{u})^2=\frac{d}{dt}\left(|\vec{u}|^2\right)$$

i.e., $$2\vec{u}.\frac{d\vec{u}}{dt}=2|\vec{u}|\frac{d|\vec{u}|}{dt}.$$

Thus,

$$\vec{u}.\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}.$$

The condition is necessary

When $\vec{u}$ is a proper vector of constant length, then $|\vec{u}|=\text{constant}\neq 0.$

Thus,

$$\vec{u}.\frac{d\vec{u}}{dt}=|\vec{u}|\frac{d|\vec{u}|}{dt}=|\vec{u}|.0=0.$$

Therefore the condition is necessary.

The condition is sufficient

If $\vec{u}.\dfrac{d\vec{u}}{dt}=0,$ then $|\vec{u}|\dfrac{d|\vec{u}|}{dt}=0.$ But since $\vec{u}$ is proper vector, $|\vec{u}|\neq 0.$ Hence $\dfrac{d|\vec{u}|}{dt}=0,$ i.e., $|\vec{u}|=$ constant. Thus $\vec{u}$ is a proper vector of constant length.

Note 15.5.1 The derivative of a vector of constant length is perpendicular to that vector.

Theorem 15.5.11 A necessary and sufficient condition that a proper vector $\vec{u}$ always remains parallel to a fixed line i.e., to have a constant direction is $\vec{u}\times\dfrac{d\vec{u}}{dt}=\vec{0}.$

Proof : Let $\vec{u}=|\vec{u}|\hat{u},$ where $\hat{u}$ is the unit vector in the direction of $\vec{u}$.

Now

$$\vec{u}\times\frac{d\vec{u}}{dt} = |\vec{u}|\hat{u}\times\frac{d}{dt}\left(|\vec{u}|\hat{u}\right)=|\vec{u}|\hat{u}\times\left(\frac{d|\vec{u}|}{dt}\hat{u}+|\vec{u}|\frac{d\hat{u}}{dt}\right)$$

$$=|\vec{u}|^2\,\hat{u}\times\frac{d\hat{u}}{dt}\ \text{[since } \hat{u}\times\hat{u}=\vec{0}].$$

The condition is necessary

When $\vec{u}$ remains parallel to a fixed direction then $\hat{u}$ = constant and hence $\dfrac{d\hat{u}}{dt}=\vec{0}.$

Thus

$$\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}.$$

Therefore the condition is necessary.

The condition is sufficient

Since $\vec{u}\neq\vec{0},$ so the condition

$$\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}\Rightarrow\hat{u}\times\frac{d\hat{u}}{dt}=\vec{0}. \qquad (15.6.5)$$

Also since $\hat{u}$ is a proper vector of constant length of unity, we have

$$\hat{u}.\frac{d\hat{u}}{dt}=0. \qquad (15.6.6)$$

The conditions of (15.6.5) and (15.6.6) will be simultaneously satisfied if $\frac{d\hat{u}}{dt} = \vec{0}$ i.e., $\hat{u}$ is constant i.e., $\vec{u}$ is parallel to a fixed line.

Theorem 15.5.12 A necessary and sufficient condition for the vector $\vec{u}(t)$ to be constant is $\frac{d\vec{u}}{dt} = \vec{0}$,

Proof : The condition is necessary

Let $\vec{u}(t)$ be a constant vector. Then for an increment Δt in the scalar variable t, there will be no change in $\vec{u}$, i.e., $\Delta\vec{u} = \vec{0}$.

So $\frac{\Delta\vec{u}}{\Delta t} = \vec{0}$.

Now taking $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{u}}{\Delta t} = \vec{0} \text{ i.e., } \frac{d\vec{u}}{dt} = \vec{0}.$$

So the condition is necessary.

The condition is sufficient

Let $\frac{d\vec{u}}{dt} = \vec{0}$.

Now if $\vec{u}(t) = u_1(t)\hat{i} + u_2(t)\hat{j} + u_3(t)\hat{k}$, then

$$\frac{d\vec{u}}{dt} = \frac{du_1}{dt}\hat{i} + \frac{dy_2}{dt}\hat{j} + \frac{du_3}{dt}\hat{k}.$$

So

$$\frac{d\vec{u}}{dt} = \vec{0} \Rightarrow \frac{du_1}{dt} = 0, \frac{du_2}{dt} = 0 \frac{du_3}{dt} = 0.$$

Therefore u_1 = constant, u_2 = constant, u_3 = constant, Thus $\vec{u}(t) =$ constant. So the condition is sufficient.

Example 15.5.1 If $\vec{r} = t^2\hat{i} + \cos t\,\hat{j} + \sin^2 t\,\hat{k}$, find the derivative of $\vec{r}$ w.r.t. t.

Solution : Here $\vec{r} = t^2\hat{i} + \text{cost}\ \hat{j} + \sin^2 t\hat{k}$.

Therefore

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t^2\hat{i} + \cos t\hat{j} + \sin^2 t\,\hat{k}\right)$$

$$= 2t\hat{i} - \sin t\,\hat{j} + \sin 2t\,\hat{k}.$$

Example 15.5.2 If $\hat{r} = (5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}$, find $\dfrac{d^2\vec{r}}{dt^2}$.

Solution : Here $\vec{r} = (5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}$.

Therefore

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left\{(5+3t)\hat{i} + (3-2t)\hat{j} + (4+t-16t^2)\hat{k}\right\}\\ &= 3\hat{i} - 2\hat{j} + (1-32t)\hat{k}\end{aligned}$$

$$\text{and } \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left\{3\hat{i} - 2\hat{j} + (1-32t)\hat{k}\right\} = -32\vec{k}.$$

Example 15.5.3 If $\vec{a} = t^2\vec{i} - t\vec{j} + (2t+1)\hat{k}$ and $\vec{b} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then show that (i) $\dfrac{d}{dt}(\vec{a}.\vec{b}) = -6$ at $t = 1$, (ii) $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = 7\hat{j} + 3\hat{k}$ at $t = 1$.

Solution : Here $\dfrac{d\vec{a}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k}$ and $\dfrac{d\vec{b}}{dt} = 2\hat{i} - \hat{k}$.

(i) $\dfrac{d}{dt}(\vec{a}.\vec{b})$

$$\begin{aligned}&= \frac{d\vec{a}}{dt}.\vec{b} + \vec{a}.\frac{d\vec{b}}{dt}\\ &= (2t\hat{i} - \hat{j} + 2\hat{k}).\{(2t-3)\hat{i} + \hat{j} - t\hat{k}\} + \{t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}\}.(2\hat{i} - \hat{k})\\ &= 4t^2 - 6t - 1 - 2t + 2t^2 - 2t - 1 = 6t^2 - 10t - 2.\end{aligned}$$

Hence $\dfrac{d}{dt}(\vec{a}.\vec{b}) = -6$ at $t = 1$.

(ii) $\dfrac{d}{dt}(\vec{a} \times \vec{b})$

$$= \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2t-3 & 1 & -t \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (t-2)\hat{i} + (2t^2+4t-6)\hat{j} + (4t-3)\hat{k} + t\hat{i} + (t^2+4t+2)\hat{j} + 2t\hat{k}$$

$$= (2t-2)\hat{i} + (3t^2+8t-4)\hat{j} + (6t-3)\hat{k}.$$

Therefore at $t = 1$,

$$\frac{d}{dt}(\vec{a} \times \vec{b}) = 7\hat{j} + 3\hat{k}.$$

Example 15.5.4 If $\vec{\alpha} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{\beta} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$, then find $\frac{d}{dt}\left(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}\right)$ at $t = 2$.

Solution : Here $\frac{d\vec{\alpha}}{dt} = 2t\hat{i} - \hat{j} + 2\hat{k}$, $\frac{d\vec{\beta}}{dt} = 2\hat{i} - \hat{k}$ and $\frac{d^2\vec{\beta}}{dt^2} = \vec{0}$.

$$\frac{d}{dt}\left(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}\right)$$

$$= \frac{d\vec{\alpha}}{dt} \times \frac{d\vec{\beta}}{dt} + \vec{\alpha} \times \frac{d^2\vec{\beta}}{dt^2}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \hat{i} + (2t+4)\hat{j} + 2\hat{k}.$$

Hence at $t = 2$,

$$\frac{d}{dt}\left(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}\right) = \hat{i} + 8\hat{j} + 2\hat{k}.$$

15.6 Geometrical and Physical Interpretation of Derivative of Vector Functions

Let $\vec{r} = \vec{f}(t)$ be a continuous and single valued vector function of a scalar variable t. Let $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ be the position vectors of two neighbouring points P and Q respectively on teh continuous curve $\vec{r} = \vec{f}(t)$ w.r.t. origin O. Then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}.$$

So $\dfrac{\Delta\vec{r}}{\Delta t}=\dfrac{\overrightarrow{PQ}}{\Delta t}$. When $Q \to P$, $\Delta t \to 0$, the chord PQ tends to the tangent PT to the curve at P.

Hence $\lim\limits_{\Delta t\to 0}\dfrac{\Delta\vec{r}}{\Delta t}=\dfrac{d\vec{r}}{dt}$ represents a vector along the tangent to the curve $\vec{r}=\vec{f}(t)$ at P.

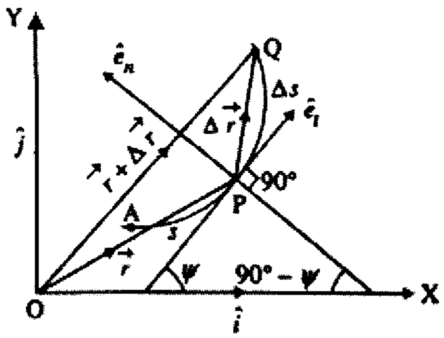


Fig. 15.1

If t represents time then $\Delta\vec{r}$ represents displacement during time Δt. Hence $\dfrac{d\vec{r}}{dt}$ represents the rate of change of displacement of the point P which is called velocity vector at P along tangential direction. Thus the velocity vector is

$$\vec{\upsilon}=\frac{d\vec{r}}{dt}.$$

Similarly if $\Delta\vec{\upsilon}$ be the increment of $\vec{\upsilon}$ in time Δt, then the rate of change of velocity of the point P which is acceleration of the point P is

$$\vec{a}=\lim_{\Delta t\to 0}\frac{\Delta\vec{\upsilon}}{\Delta t}=\frac{d\vec{\upsilon}}{dt}=\frac{d^2\vec{r}}{dt^2}.$$

15.6.1. Tangential and Normal Components of Velocity and Acceleration

Let $\vec{i}$ and $\vec{j}$ be two unit vectors along two rectangular axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively. Let a particle be moving along a plane curve $\vec{r}=\vec{f}(s)$, where s is the length of an arc AP of the curve where A be the fixed point from which the length of the arc be measured and $P(\vec{r})$ be the position of the particle on the curve at time

t. Let $Q(\vec{r}+\Delta\vec{r})$ be the positon at time $t+\Delta t$ where $AQ = s+\Delta s$.

Then

$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = \vec{r}+\Delta\vec{r}-\vec{r} = \Delta\vec{r}$$

$$\text{and are } PQ = \text{arc } AQ - \text{arc } AP = s+\Delta s - s = \Delta s.$$

If $\vec{\upsilon}$ be the velocity of the paricle along tangential direction at P, then $\vec{\upsilon} = |\vec{\upsilon}|\hat{e}_t$ where $\hat{e}_t$ is the unit vector along tangent at P.

So

$$|\vec{\upsilon}| = \left|\lim_{Q\to P}\frac{\overrightarrow{PQ}}{\Delta t}\right| = \lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ}\cdot\frac{arc\ PQ}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is the tangential component of the velocity on the curve $\vec{r} = \vec{f}(s)$ in the sense of s increasing.

Since the particle does not leave the curve, there is no displacement in normal direction and so the velocity along normal direction is zero. Hence we can write

$$\vec{\upsilon} = |\vec{\upsilon}|\hat{e}_t + 0\hat{e}_n,$$

where $\hat{e}_n$ is the unit vector along normal direction.

Now

$$\hat{e}_t = \cos\psi\hat{i} + \sin\psi\hat{j} \text{ and } \hat{e}_n = -\sin\psi\hat{i} + \cos\psi\hat{j}.$$

So

$$\frac{d\hat{e}_t}{dt} = \left(-\sin\psi\hat{i} + \cos\psi\hat{j}\right)\frac{d\psi}{dt} = \hat{e}_n\frac{d\psi}{dt}$$

and

$$\frac{d\hat{e}_n}{dt} = \left(-\cos\psi\hat{i} - \sin\psi\hat{j}\right)\frac{d\psi}{dt} = -\hat{e}_t\frac{d\psi}{dt}.$$

Now acceleration vector at P will be given by

$$\begin{aligned}\vec{a} &= \frac{d\vec{\upsilon}}{dt} = \frac{d}{dt}\left(|\vec{\upsilon}|\hat{e}_t + 0\hat{e}_n\right)\\ &= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\frac{d\hat{e}_t}{dt}\\ &= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\hat{e}_n\frac{d\psi}{dt}\end{aligned}$$

$$= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + |\vec{\upsilon}|\hat{e}_n \frac{d\psi}{ds}\cdot\frac{ds}{dt}$$

$$= \frac{d|\vec{\upsilon}|}{dt}\hat{e}_t + \frac{|\vec{\upsilon}|^2}{\rho}\hat{e}_n \left[\text{sinc}\,\rho = \frac{ds}{d\psi} \text{ and } |\vec{\upsilon}| = \frac{ds}{dt}\right]$$

$$= a_t\hat{e}_t + a_n\hat{e}_n,$$

where ρ is the radius of curvature at P.

Therefore the tangential component of acceleration at P is $a_t = \frac{d|\vec{\upsilon}|}{dt} = \frac{d^2s}{dt^2}$ and normal component at P is $a_n = \frac{|\vec{\upsilon}|^2}{\rho}$.

15.7 Integration of Vector Functions

We have already seen in art. 13.6 that for any vector function $\vec{f}(t)$ of a scalar variable t, $d\vec{f} = \vec{f}'(t)dt.$

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k},$ then

$$d\vec{f} = df_1\hat{i} + df_2\hat{j} + df_3\hat{k}.$$

Now if $\vec{F}(t)$ be the derivative of a vector function $\vec{f}(t)$ of a scalar variable t i.e., if $\vec{F} = \frac{d\vec{f}}{dt}$, then $d\vec{f}(t) = \vec{F}(t)dt.$

In this case $\vec{f}(t)$ is called the indefinite integral of $\vec{F}(t)$ and is written as $\int \vec{F}(t)dt = \vec{f}(t)$ where $\vec{f}(t)$ is called the integral and $\vec{F}(t)$ is called the integrand.

As in the case of integration of any scalar function, here we also say that the derivative of the integral is equal to the integrand or, the integration is the inverse process of differentiation.

As in the case of integration of a scalar function we may write $\int \vec{F}(t)dt = \vec{f}(t) + \vec{c}$, where $\vec{c}$ is an arbitrary constant vector of integration and this can be calculated from some given condition.

15.7.1. Some Important Formulae of Integration

Let $\vec{r}(t)$ and $\vec{s}(t)$ be two vector functions of scalar variable t then

(i) $\int\left(\frac{d\vec{r}}{dt}.\vec{s}+\vec{r}.\frac{d\vec{s}}{dt}\right)=\vec{r}.\vec{s}+c.$

(ii) $\int\left(2\vec{r}.\frac{d\vec{r}}{dt}\right)dt=\vec{r}^{\,2}+c.$

(iii) $\int 2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}dt=\left(\frac{d\vec{r}}{dt}\right)^2+c.$

(iv) $\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\vec{r}\times\frac{d\vec{r}}{dt}+\vec{c}.$

(v) $\int\left(\frac{d\vec{r}}{dt}\times\vec{s}+\vec{r}\times\frac{d\vec{s}}{dt}\right)dt=\vec{r}\times\vec{s}+\vec{c}.$

(vi) $\int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{dr}{dt}\frac{\vec{r}}{r^2}\right)dt=\frac{\vec{r}}{r}+\vec{c}.$

(vii) $\int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt=\vec{a}\times\vec{r}+\vec{c},$ $\vec{a}$ being a constant vector.

(viii) If $\vec{f}(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k},$ then

$$\int\vec{f}(t)dt=\left(\int f_1(t)dt\right)\hat{i}+\left(\int f_2(t)dt\right)\hat{j}+\left(\int f_3(t)dt\right)\hat{k}+\vec{c}.$$

Example 15.7.1 If $\vec{f}(t)=(t^2+1)\hat{i}+(t+1)\hat{j}-3\hat{k},$ then find $\int\vec{f}(t)dt$ and $\int_2^3\vec{f}(t)dt.$

Solution :

$$\begin{aligned}\int\vec{f}(t)dt &= \int\left\{(t^2+1)\hat{i}+(t+1)\hat{j}-3\hat{k}\right\}dt\\ &= \hat{i}\int(t^2+1)dt+\hat{j}\int(t+1)dt+\hat{k}\int(-3)dt\\ &= \left(\frac{t^3}{3}+t\right)\hat{i}+\left(\frac{t^2}{2}+t\right)\hat{j}-3t\hat{k}+\vec{c}.\end{aligned}$$

Hence $$\int_2^3 \vec{f}(t)dt = \hat{i}\left[\frac{t^3}{3}+t\right]_2^3 + \hat{j}\left[\frac{t^2}{2}+t\right]_2^3 + \hat{k}\left[-3t\right]_2^3$$

$$= \frac{22}{3}\hat{i} + \frac{7}{2}\hat{j} - 3\hat{k}.$$

Example 15.7.2 Evaluate $\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt$ where $\vec{r} = t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}$.

Solution : We have $\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2}$.

Therefore $\int\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}$.

Now $\frac{d\vec{r}}{dt} = 3t^2\hat{i} + 4t\hat{j} + 3\hat{k}$.

Hence $$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & 2t^2 & 3t \\ 3t^2 & 4t & 3 \end{vmatrix}$$

$$= -6t^2\hat{i} + 6t^3\hat{j} - 2t^4\hat{k}.$$

Therefore $$\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt = \left[-6t^2\hat{i} + 6t^3\hat{j} - 2t^4\hat{k}\right]_2^3$$

$$= -30\hat{i} + 114\hat{j} - 130\hat{k}.$$

Example 15.7.3 If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2 j + 4\sin t\,\hat{k}$ and $\vec{r} = 2\hat{i} + j$, $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$

when $t = 0$, then show that $\vec{r} = \left(t^3 - t + 2\right)\hat{i} + \left(1 - 2t^4\right)\hat{j} + (t - 4\sin t)\hat{k}$.

Solution : We have $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\ \hat{k}$.

Integrating w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\ \hat{k} + \vec{c}_1,\ \text{ where } \vec{c}_1 \text{ is a integrating constant.}$$

Since $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ when $t = 0$, we get $-\hat{i} - 3\hat{k} = -4\hat{k} + \vec{c}_1$ i.e., $\vec{c}_1 = -\hat{i} + \hat{k}$.

Therefore $\frac{d\vec{r}}{dt} = (3t^2 - 1)\hat{i} - 8t^3\hat{j} + (1 - 4\cos t)\hat{k}$.

Integrating again w.r.t. t, we obtain

$$\vec{r} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + \vec{c}_2,$$

where $\vec{c}_2$ is a integrating constant.

Now since $\vec{r} = 2\hat{i} + j$ when $t = 0$, we get $\vec{c}_2 = 2\hat{i} + j$.

Hence $\vec{r} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}$.

15.8 Summary

In this chapter we have studied about vector valued function of scalar variable and their limits, continuity, differentiability and integrability. We have also discussed about geometrical significance of vector differentiation and tangential and normal components of velocity and acceleration vectors. We also worked out some example to understand the differentiation and integration of vector functions.

15.9 Exercises

1. If $\vec{r} = t\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$, then find $\frac{d\vec{r}}{dt}$.

2. If $\vec{a} = 3t^2\hat{i} + t\hat{j} - t^3\hat{k}$ and $\vec{b} = \sin t\,\hat{i} - 2\cos t\,\hat{j}$ then find $\frac{d}{dt}(\vec{a}\times\vec{b})$ *and* $\frac{d}{dt}(\vec{a}.\vec{b})$.

3. If $\vec{a} = \sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k}$, $\vec{b} = \cos t\,\hat{i} - \sin t\,\hat{j} - 3\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} - \hat{k}$ then find the value of $\frac{d}{dt}\left\{\vec{a}\times(\vec{b}\times\vec{c})\right\}$ at $t = \frac{\pi}{2}$ and $\frac{d}{dt}\left\{\vec{a}.(\vec{b}\times\vec{c})\right\}$ at $t = 0$.

4. Evaluate $\int_0^{\pi/2} (5\cos t\,\hat{i} - 7\sin t\,\hat{j})\,dt.$

5. If $\vec{a} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{b} = 2t^2\hat{i} + 6t\hat{k}$ then find the value of $\int_0^2 \vec{a}.\vec{b}\,dt$ and $\int_0^2 (\vec{a} \times \vec{b})\,dt.$

NOTES