PREFACE

With its grounding in the "guiding pillars of Access, Equity, Equality, Affordability and Accountability," the New Education Policy (NEP 2020) envisions flexible curricular structures and creative combinations for studies across disciplines. Accordingly, the UGC has revised the CBCS with a new Curriculum and Credit Framework for Undergraduate Programmes (CCFUP) to further empower the flexible choice based credit system with a multidisciplinary approach and multiple/ lateral entry-exit options. It is held that this entire exercise shall leverage the potential of higher education in three-fold ways - learner's personal enlightenment; her/his constructive public engagement; productive social contribution. Cumulatively therefore, all academic endeavours taken up under the NEP 2020 framework are aimed at synergising individual attainments towards the enhancement of our national goals.

In this epochal moment of a paradigmatic transformation in the higher education scenario, the role of an Open University is crucial, not just in terms of improving the Gross Enrolment Ratio (GER) but also in upholding the qualitative parameters. It is time to acknowledge that the implementation of the National Higher Education Qualifications Framework (NHEQF) and its syncing with the National Skills Qualification Framework (NSQF) are best optimised in the arena of Open and Distance Learning that is truly seamless in its horizons. As one of the largest Open Universities in Eastern India that has been accredited with 'A' grade by NAAC in 2021, has ranked second among Open Universities in the NIRF in 2024, and attained the much required UGC 12B status, Netaji Subhas Open University is committed to both quantity and quality in its mission to spread higher education. It was therefore imperative upon us to embrace NEP 2020, bring in dynamic revisions to our Undergraduate syllabi, and formulate these Self Learning Materials anew. Our new offering is synchronised with the CCFUP in integrating domain specific knowledge with multidisciplinary fields, honing of skills that are relevant to each domain, enhancement of abilities, and of course deep-diving into Indian Knowledge Systems.

Self Learning Materials (SLM's) are the mainstay of Student Support Services (SSS) of an Open University. It is with a futuristic thought that we now offer our learners the choice of print or e-slm's. From our mandate of offering quality higher education in the mother tongue, and from the logistic viewpoint of balancing scholastic needs, we strive to bring out learning materials in Bengali and English. All our faculty members are constantly engaged in this academic exercise that combines subject specific academic research with educational pedagogy. We are privileged in that the expertise of academics across institutions on a national level also comes together to augment our own faculty strength in developing these learning materials. We look forward to proactive feedback from all stakeholders whose participatory zeal in the teaching-learning process based on these study materials will enable us to only get better. On the whole it has been a very challenging task, and I congratulate everyone in the preparation of these SLM's.

I wish the venture all success.

Professor. Indrajit Lahiri
Vice-Chancellor

Netaji Subhas Open University

Four Year Undergraduate Degree Programme

Under National Higher Education Qualifications Framework (NHEQF) &

Curriculum and Credit Framework for Under Graduate Programmes

B. Sc. Mathematics (Hons.)

Programme Code : NMT

Course Type : Skill Enhancement Course (SEC

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OPEN UNIVERSITY

UG : Mathematics
(NMT)

Course Title: Linear Programming
Corse Code: NSE-MT-02

Unit - 1 ❑ Linear Programming Problem (LPP) : Formulation

Structure

1.0 Objective

After studying this chapter, the reader should be able to know

- the nature of linear programming problems (LPP)
- the requirements and scope of LPP
- advantages and limitations of LPP
- applications of LPP

1.1 Introduction

Linear programming is a mathematical technique for determining the optimal allocation of resources and obtaining a particular objective when there are alternative uses of the resources : money, manpower, material, machine and other facilities. The objective in resource allocation may be cost minimization or inversely profit maximization. The technique of linear programming is applicable to problems in which the total effectivness can be expressed as a linear function of individual allocations and the limitations on resources give rise to linear equalities or inequalities of the individual allocations.

1.2 Requirements of Linear Programming

In general, linear programming can be used for optimization problems if the following conditions are satisfied :

(i) There must be a well-defined objective function (profit, cost or quantities produced) which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

(ii) There must be restrictions on the amount or extent of attainment of the objective and these restrictions must be capable of being expressed as linear equalities or inequalities in terms of variables.

(iii) There must be alternative course of action. For example, a given product may be processed by two different machines and the problem may be as to how much of the product to allocate to which machine.

(iv) Another necessary requirement is that decision variables should be inter related and non-negative. The non-negativity condition shows that linear programming deals with real-life situations for which negative quantities are generally illogical.

(v) The resources must be in limited supply. For example, if a farm starts producing greater number of a particular product, it must make smaller number of other products as the total production capacity is limited.

1.3 Scope of Linear Programming

Despite the restriction that relationship must be linear, linear programming, as an operation research technique, has been recognized and widely used in industry as an effective tool for business problems.

The first application of linear programming technique was made in 1947 in the U.S. Air Force. After the second World War, it became necessary to concentrate on the development of a model that would define the optional and efficient coordinations of a nation's energies in the event of a total war. The Air Force officially named this task as Project SCOOP (Scientific Computation of Optimum Programmes) and it was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while working on this project. This technique consisted of representing the various activities of an organization as a linear programming model and arriving at the optional programme by minimizing linear objective function. Dantzig developed the 'Simplex Method' for deriving an optimal feasible solution which solved the project SCOOP. Since then the military has been building linear programming models for crew training, for scheduling of routine maintenance activities, for personnel assignments and for contract bidding etc.

The use of linear programming in business and industry has been tremendous since 1951. The first and the most widely used industrial applications have been in all phases of the petroleum industry, viz., exploration, production, refining, distributions and pollution control. The second most active user of linear programming is the food processing industry. In the heavy industry, linear programming has been used in the iron and steel industry, to decide the types of products to be made in the rolling mills so as to maximize the profit. Administration, Education and politics have also employed linear programming to solve their problems, viz., in planning political campaign strategies; for allocating resources in educations; for making school assignments in large districts; for optimal city administration and for resource allocations in local election campaigns. This list of applications of linear programming can go on for ever.

1.4 Formulation of a Linear Programming Problem

Mathematically, a most general linear programming problems (LPP) can be stated as follows :

Find x_1, x_2, ... x_n which optimize (maximize or minimize) the objective function :

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n$$

subject to the constraints

$$a_{11}\,x_1 + a_{12}\,x_2 + ... + a_{1n}\,x_n\ (\leq\ =\ \geq)\ b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + ... + a_{2n}\,x_n\ (\leq\ =\ \geq)\ b_2$$

$$..... \quad ..... \quad ..... \quad .....$$

$$..... \quad ..... \quad ..... \quad .....$$

$$a_{m1}\,x_1 + a_{m2}\,x_2 + ... + a_{mn}\,x_n\ (\leq\ =\ \geq)\ b_m$$

$$x_1,\ x_2,\ .....\ x_n\ \geq 0;$$

where for each constraint one and only one sign of ≤, =, ≥ holds, but the sign may vary from one constraint to another. In this LPP, all a_{ij}'s, b_i's and c_i's are constants. The constants c_i's are called cost coefficients of the problem.

In this section, we shall consider some examples of the LPP and formulate them in mathematical models in terms of the decision variables $x_1, x_2, ... x_n$. The objective is to familiarize the reader with some of the areas where this techinque may be applied. In these examples, the stress is laid on the analysis of the situation and formulation of the linear programming problem rather than on its solution.

● Example 1.4.1 (Production Planning Problem)

A firm manufactures three products A, B and C. The profit per unit sold of each product is ₹ 3, ₹ 2 and ₹ 4 respectively. The time require to manufacture one unit of each of the three products and the daily capacity of the two machines P and Q is given in the table below :

Machine	**Time per unit (minutes) product**			**Machine Capacity (minutes/day)**
	A	**B**	**C**	
P	4	3	5	2000
Q	2	2	4	2500

It is required to determine the daily number of units to be manufactured for each product so as to maximize the profit. However, the firm must manufacture at least 100 A's, 200 B's and 50 C's but not more than 150 A's. It is assumed that all the amounts produced are consumed in the market. Formulate the problem as a LPP.

Solution : Let the number of units of the products A, B and C manufactured daily be designated by x_1, x_2 and x_3 respectively where $x_1, x_2, x_3 \geq 0$.

The objective here is to maximize the profit. In view of the assumption that all the units produced are consumed in the market, it is given by the linear function

$$z = 3x_1 + 2x_2 + 4x_3.$$

In order to produce x_1 units of product A, x_2 units of product B and x_3 units of product C, the total time needed on machines P and Q are given by

$$4x_1 + 3x_2 + 5x_3 \text{ and } 2x_1 + 2x_2 + 4x_3$$

respectively. Since the manufacturer does not have more than 2000 minutes available on machine P and 2500 minutes available on machine Q, we must have

$$4x_1 + 3x_2 + 5x_3 \leq 2000$$

$$\text{and} \quad 2x_1 + 2x_2 + 4x_3 \leq 2500.$$

Also, the manufacturer has to satisfy the following given additional restrictions :

$$x_1 \geq 100, \ x_1 \leq 150, \ x_2 \geq 200 \text{ and } x_3 \geq 50.$$

Hence, the problem can be put in the following LPP as given by

$$\text{Maximize } z = 3x_1 + 2x_2 + 4x_3$$

subject to

$$4x_1 + 3x_2 + 5x_3 \leq 2000$$

$$2x_1 + 2x_2 + 4x_3 \leq 2500$$

$$100 \leq x_1 \leq 150, \ x_2 \geq 200, \ x_3 \geq 50.$$

● Example 1.4.2 (Diet Problem)

A doctor advises a patient to take at least 150 calories out of two kinds of food—milk and meat; and also advises him not to take more than 18 units of fat daily. Relevant information is given in the following table :

Food type	**Calorie**	**Units of Fat**
Milk (per litre)	120	15
Meat (per 100 gm)	30	3

It is given that one litre milk costs ₹ 45 and one kilogram of meat costs ₹ 600. Formulate the problem as a LPP in order to have the minimum cost diet.

Solution : Let us represent the problem in the following table :

Food type	**Calorie**	**Units of Fat**	**Cost (in Rs.)**
Milk (per litre)	120	15	45
Meat (in kg)	300	30	600
Requirements	150	18	

Let the patient take x_1 litres of milk and x_2 kilograms of meat per day. Thus, he will get $(120x_1 + 300x_2)$ calories and $(15x_1 + 30x_2)$ units of fat daily. For this, total cost per day will be ₹ $(45x_1 + 600x_2)$.

Thus, the problem can be formulated as a LPP in the following way :

$$\begin{aligned} \text{Minimize } z = \quad & 45x_1 + 600x_2 \\ \text{subject to} \quad & 120x_1 + 300x_2 \geq 150 \\ & 15x_1 + 30x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

● 1.4.3 Example (Staff Management Problem)

A city hospital has the following daily minimal requirement for nurses :

Period	**Clock time (24 hours day)**	**Minimal number of Nurses Required**
1	6 am – 10 am	2
2	10 am – 2 pm	7
3	2 pm – 6 pm	15
4	6 pm – 10 pm	8
5	10 pm – 2 am	20
6	2 am – 6 am	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this problem as a LPP.

Solution : Let x_j (j = 1, 2, ... 6) be the number of nurses required at the begining of j-th period.

Since each nurse has to work for 8 consecutive hours, the x_1 nurses who joined during the 1st period shall till be on duty when 2nd period starts. Thus, during the 2nd priod there will be $(x_1 + x_2)$ nurses. Since the minimal number of nurses required during the 2nd period is 7, so we must have, $x_1 + x_2 \geq 7$.

Similarly, the other constraints of the given problem will be $x_2 + x_3 \geq 15$, $x_3 + x_4 \geq 8$, $x_4 + x_5 \geq 20$, $x_5 + x_6 \geq 6$ and $x_6 + x_1 \geq 2$. The objective of the problem is to minimize the total number of nurses employed in the hospital, that is $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$. Thus, the problem can be put in the following LPP as given by

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$\text{subject to } x_6 + x_1 \geq 2$$

$$x_1 + x_2 \geq 7$$

$$x_2 + x_3 \geq 15$$

$x_3 + x_4 \geq 8$

$x_4 + x_5 \geq 20$

$x_5 + x_6 \geq 6$

$x_j \geq 0 \quad (j = 1, 2, ..., 6)$

● **Example 1.4.4 (Transportation Problem)**

A dairy farm has two plants located in a metropolitan city. Daily milk production at each plant is as follows :

Plant 1 — 5 million litres

Plant 2 — 10 million litres.

Each day the firm must fulfil the needs of its three distribution centres. Requirement at each centre is as follows :

Distribution centre 1 – 8 million litres

Distribution centre 2 – 5 million litres

Distribution centre 3 – 2 million litres

Cost of shipping one million litres of milk from each plant to each distribution centre is given in the following table in hundreds of rupees :

Plant	**Shipping cost Distribution Centre**		
	1	2	3
1	1	2	4
2	3	2	1

The dairy farm wishes to decide as to how much should be the shipment from which plant to which distribution centre so that the cost of shipment may be minimum.

Solution : Let x_{ij} be the decision variable (quantities to be found) when the milk is shipped from plant i (i = 1, 2) to distribution centre j (j = 1, 2, 3)

$$z = (x_{11} + 2x_{12} + 4x_{13}) + (3x_{21} + 2x_{22} + x_{23}).$$

Constraints are on the availability of milk at the two places and its requirement at the three distribution centres. These are as follows :

(i) Availability or supply :

$$x_{11} + x_{12} + x_{13} = 5$$
$$x_{21} + x_{22} + x_{23} = 10$$

(ii) Requirement or Demand :

$$x_{11} + x_{21} = 8$$
$$x_{12} + x_{22} = 5$$
$$x_{13} + x_{23} = 2$$

Hence, the transportation problem can be put in the following LPP :

$$\begin{aligned} \text{Minimize } z = \; & x_{11} + 2x_{12} + 4x_{13} + 3x_{21} + 2x_{22} + x_{23} \\ \text{subject to } \; & x_{11} + x_{12} + x_{13} = 5 \\ & x_{21} + x_{22} + x_{23} = 10 \\ & x_{11} + x_{21} = 8 \\ & x_{12} + x_{22} = 5 \\ & x_{13} + x_{23} = 2 \\ & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0. \end{aligned}$$

● **Example 1.4.5 (Resource Allocation Problem)**

A ship has three cargo loads–forward, aft and centre. The capacity limits are as follows :

	Weight in Kgs.	**Volume in Cu. Cms.**
Forward	2000	1,00,000
Centre	3000	1,35,000
Aft	1500	30,000

The following cargos are offered. The ship owner may accept all or any part of each commodity.

Commodity	Weight (in Kgs.)	Volume per unit (in Cu. Cms.)	Profit (in Rs. per kg.)
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the firm of the ship, the weight in each load must be proportional to the capacity in kgs. The cargo is to be distributed so as to maximize the profit. Formulate the problem as a LPP.

Solution : Let x_{1A}, x_{2A}, x_{3A} be the weights (in kg.) of the commodity A to be accomodated in forward, centre and aft portions respectively. Similarly, let x_{1B}, x_{2B}, x_{3B} and x_{1C}, x_{2C}, x_{3C} be the corresponding weights (in kg.) of B and C respectively. Then, the given information can be formulated in an appropriate mathematical form as a LPP as given by

$$\text{Maximize } z = 60\,(x_{1A} + x_{2A} + x_{3A}) + 80\,(x_{1B} + x_{2B} + x_{3B}) + 50\,(x_{1C} + x_{2C} + x_{3C})$$

$$\begin{aligned}
\text{subject to } \quad & x_{1A} + x_{2A} + x_{3A} \le 6000 \\
& x_{1B} + x_{2B} + x_{3B} \le 4000 \\
& x_{1C} + x_{2C} + x_{3C} \le 2000 \\
& x_{1A} + x_{1B} + x_{1C} \le 2000 \\
& x_{2A} + x_{2B} + x_{2C} \le 3000 \\
& x_{3A} + x_{3B} + x_{3C} \le 1500 \\
& 60x_{1A} + 50x_{1B} + 25x_{1C} \le 1{,}00{,}000 \\
& 60x_{2A} + 50x_{2B} + 25x_{2C} \le 1{,}35{,}000 \\
& 60x_{3A} + 50x_{3B} + 25x_{3C} \le 30{,}000
\end{aligned}$$

$$x_{1A}, x_{2A}, x_{3A}, x_{1B}, x_{2B}, x_{3B}, x_{1C}, x_{2C}, x_{3C} \ge 0$$

1.5 Summary

Unit-1 deals with a brief introduction of Linear programming. The requirements and scope of linear programming is discussed in two sections. Formulation of a LPP is explained with several examples in another section. Graphical method of solution is also discussed with numerous illustrative examples. Finally, matrix form of a LPP is presented in the last section.

1.6 Exercise

1. Formulate the following problem as a LPP :

(i) A firm manufactures two types of products—chairs and tables from its available sources. Processing of these products is done on two machines G and H only. A chair requires 3 hours on machine G and one hour on machine H, whereas a table requires 2 hours on machine G and 2 hours on machine H. There are 12 hours of time per day on machine G and 8 hours per day on machine H available. Profits gained from a chair and a table are ₹ 150 and ₹ 120 respectively. Formulate the problem mathematically as a linear programming problem to maximize the total profit.

(ii) A farmer buys sheeps and goats at ₹ 5,000 per sheep and at ₹ 8,000 per goat, and sells them at profit of ₹ 600 per sheep and ₹ 900 per goat. The farmer has accommodation for not more than 50 animals and cannot afford to pay more than ₹ 3,20,000. He wishes to buy these two kinds of animals in order to have the maximum profit. Formulate the problem as a LPP.

(iii) Everyone would like to decide on the constituents of a diet which will satisfy his daily needs of protein, carbohydrate and fat at the minimum cost. Choices can be made from three different kinds of the food—milk, bread and butter. The yields per unit of these foods are given in the following table :

Food type	**Yields per unit**			**Cost (in ₹)**
	Protein	**Carbohydrate**	**Fat**	**per unit**
Milk	3	1	2	9
Bread	4	2	1	12
Butter	3	1	2	20
Daily requirement	5	3	2	—

Formulate the problem as a LPP in order to minimize the diet cost.

(iv) A steel company is faced with the problem of transporting coal from two coal mines to four of its steel plants. The amount of coat available in the coal mines are a_1 and a_2 metric tons. The amounts required at plants are b_1, b_2, b_3, b_4 metric tons. Exactly one truck is used for these shipments. It is possible to ship from any mine to any plant, but the truck cannot make more than one ship from a mine to a plant. The problem is to determine the minimum capacity of truck which can complete all these shipments. Formulate this as a linear programming problem. It is given that the transportation cost of one unit of product from a_i (i = 1, 2) mine to b_j (j = 1, 2, 3) plant is c_{ij} units.

(v) A transistor radio company manufactures four models—A, B, C and D which have profit contributions of ₹ 80, 150 and ₹ 250 on models A, B and C respectively and loss of ₹ 100 on model D each. Each type of radio requires a certain amount of time for manufacturing components, for assembling and packing. Specially, a dozon units of model A require one hour of manufacturing, two hours for assembling and one hour for packing. The corresponding figures for a dozen units of model B are 2, 1 and 2; and a dozen units of C are 3, 5 and 1, while a dozen units of model D require 1 hour of packing only. During the forthcoming week, the company will be able to make available 15 hours of manufacturing, 20 hours of assembling and 10 hours of packing time. Formulate the problem as a LPP to obtain the optimal production schedule for the company.

2. Solve the following LPP graphically :

(i) Maximize $z = 150x_1 + 120x_2$

subject to
$$3x_1 + 2x_2 \le 12$$
$$x_1 + 2x_2 \le 8$$
$$x_1 , x_2 \ge 0$$

(ii) Minimize $z = 600x_1 + 900x_2$

subject to
$$x_1 + x_2 \le 50$$
$$5x_1 + 8x_2 \ge 320$$
$$x_1 , x_2 \ge 0$$

(iii) Minimize $z = 9x_1 + 12x_2 + 20x_3$

subject to
$$3x_1 + 4x_2 + 3x_3 \ge 5$$
$$x_1 + 2x_2 + x_3 \ge 3$$
$$2x_1 + x_2 + 2x_3 \ge 2$$
$$x_1, x_2, x_3 \ge 0$$

(iv) Minimize $z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij}x_{ij}$

subject to
$$x_{11} + x_{12} + x_{13} + x_{14} = a_1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = a_2$$
$$x_{11} + x_{21} = b_1$$
$$x_{12} + x_{22} = b_2$$
$$x_{13} + x_{23} = b_3$$
$$x_{14} + x_{24} = b_4$$
$$x_{ij} \ge 0 \ (i = 1, 2; j = 1, 2, 3)$$

(v) Maximize $z = 80x_1 + 150x_2 + 250x_3 - 100x_4$

subject to
$$x_1 + 2x_2 + 3x_3 \le 180$$
$$2x_1 + x_2 + 5x_3 \le 240$$
$$x_1 + 2x_2 + x_3 + x_4 \le 120$$
$$x_1, x_2, x_3, x_4 \ge 0$$

1.7 Multiple Choice Questions (MCQ)

1. In an LPP, a constraint restricts
   (a) Value of objective function
   (b) Value of a decision vaniable
   (c) Use of available resource
   (d) None of these
2. Linear programming problem (LDP) must have an
   (a) Objective function that we aim to optimize.
   (b) Constraints that we need to specify.
   (c) Decision variables that we need to determine.
   (d) All of the above.
3. A constrain in an LPP is expressed as
   (a) Inequality with ‘ $\leq$ ’ sign.
   (b) Inequality with ‘ $\geq$ ’ sign
   (c) An equation with ‘=’ sign
   (d) Any or all of the above.

Answers

1. (c) 2. (d) 3. (d)

Unit - 2 ❑ Graphical Method of Solution

Structure

2.0 Objective

After studying this chapter, the reader should be able to know

- the technique used to identify the optimal solution for liner programming problem involving two variables, know as graphical (or geometical) method.
- about the solution space or feasible region of all the constrants of an LPP
- about the particular type of LPP having unbounded solution or infinito number of optimal solutions.

2.1 Introduction

For programming problems involving two variables it is possible to display the entire set of feasiable solutions graphically by plotting liner constraints an a graph paper in order locate the optimum solution. This method also provides a great deal of instant into what happens in the more gereral case with any number of variables.

2.2 Graphical Method of Solution

We shall illustrate the graphical method of solution with a few examples involving two variables which are taken to be x_1 and x_2. Introduce an X_1X_2 coordinate system (rectangular cartesian) in two dimensional Euclidean space and note that any set of numbers (x_1, x_2) (ordered pair) is a point in the X_1X_2 –plane. It follows that any point lying in the first quadrant has $x_1 \geq 0$ and $x_2 \geq 0$ and satisfies the non-negativity restrictions of a LPP. Therefore, any point which is a feasible solution of a LPP must lie in the first quadrant. To find the set of points in the first quadrant which will satisfy the constraints of the LPP, we have to draw graphs of the constraints (equations or inequations). The feasible region of solution will be a two-dimensional figure (polygon) in the first quadrant bounded by the straight lines represented by the constraints regarded as equations. Any point in the region of feasible solution will satisfy the non-negativity conditions as well as the constraints of the problem, and hence will be a feasible solution.

To solve the linear programming problem, we must find the point (or points) in the feasible region which gives the optimal value (maximum or minimum) value of the objective function z. If the objective function is $z = c_1x_1 + c_2x_2$ (c_1, c_2 are known constants), its graph is a straight line. Any point on the straight line will give the same value of z. For each different value of z, a family of parallel straight lines will be generated. We wish to find the straight line with the optimal value of z (maximum or minimum) through the feasible region. The point (or points) in that region which will give the optimal value of z will be the optimal solution and that value of z will be the optimal value of the LPP.

● **Example 2.2.1**

Solve graphically the following LPP :

Minimize $z = 15x_1 + 12x_2$

$$\text{subject to} \quad \begin{aligned} 3x_1 + 2x_2 &\leq 12 \\ x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution : A set of rectangular cartesian axes OX_1 and OX_2 is taken in the plane of the paper to represent the solution space and find the optimal solution graphically of the given LPP.

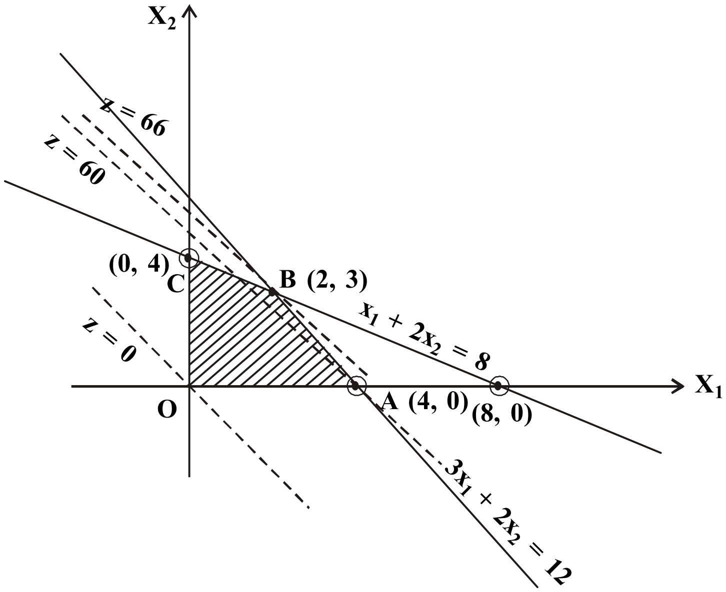


Figure. 1

For this purpose, at first we draw the graph of the equations $3x_1 + 2x_2 = 12$ and $x_1 + 2x_2 = 8$ and hence we find the regions of the inequalities $3x_1 + 2x_2 \leq 12$ and $x_1 + 2x_2 \leq 8$. The region OABC bounded by the straight lines $x_1 = 0$, $x_2 = 0$, $3x_1 + 2x_2 = 12$ and $x_1 + 2x_2 = 8$ together with all the points on its boundary is the feasible solution of the LPP, where O, A, B, C are the extreme points and this region is marked by shading. Any point in the shaded region, whether it is within or on the boundary, is a feasible solution to the given LPP.

For any particular value of z, the graph of $z = 15x_1 + 12x_2$ is a straight line and as z varies, a family of parallel straight lines is generated. A few of them are shown by the dotted lines. Among them, the straight line further from the origin will be the one for the maximum value of z and that one is through the extreme point B (2, 3) which is the point of intersection of $3x_1 + 2x_2 = 12$ and $x_1 + 2x_2 = 8$.

Thus, the optimal solution of the LPP is $x_1 = 2$, $x_2 = 3$ and the maximum value of $z = (15 \times 2 + 12 \times 3)$ units = 66 units. This optimal solution is unique.

● **Example 2.2.2**

Solve the following LPP by the graphical method :

Minimize $z = 2.5x_1 + x_2$

subject to $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$

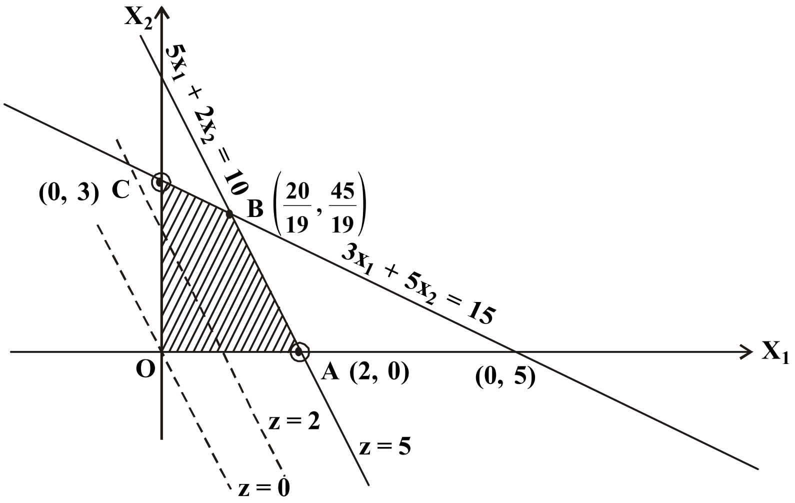


Figure. 2

Solution : We consider a set of rectangular cartesian axes OX_1 and OX_2 in the plane of paper and with reference to them, we draw the graphs of $3x_1 + 5x_2 = 15$ and $5x_1 + 2x_2 = 10$. The region OABC bounded by these straight lines and the axes of co-ordinate, including all its boundary points is the feasible region. This region is marked as the shaded region. It is evident from the problem that for any particular value of z, the objective function $z = \frac{1}{2}(5x_1 + 2x_2)$ is parallel to the constraint $5x_1 + 2x_2 = 10$. The graphs of $z = \frac{1}{2}(5x_1 + 2x_2)$ for different values of z are shown by the dotted lines. The graph shows that the optimal solution of the problem occurs at an infinite number of points of the line segment AB, some of them are A (2, 0), $B = \left(\frac{20}{19}, \frac{45}{19}\right)$. The maximum value of z is (5 × 2 + 2 × 0) = 5 units which is unique. Here, the solution region is bounded, but the problem has an infinite number of optimal solutions. Thus, an optimal solution of the LPP is $x_1 = 2$, $x_2 = 0$ and $z_{max} = 5$ units.

● Example 2.2.3

Solve the following LPP graphically :

Minimize $z = 2x_1 + 3x_2$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 &\le 4 \\ 6x_1 + 2x_2 &\ge 8 \\ x_1 + 5x_2 &\ge 4 \\ x_1 &\le 3 \\ x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

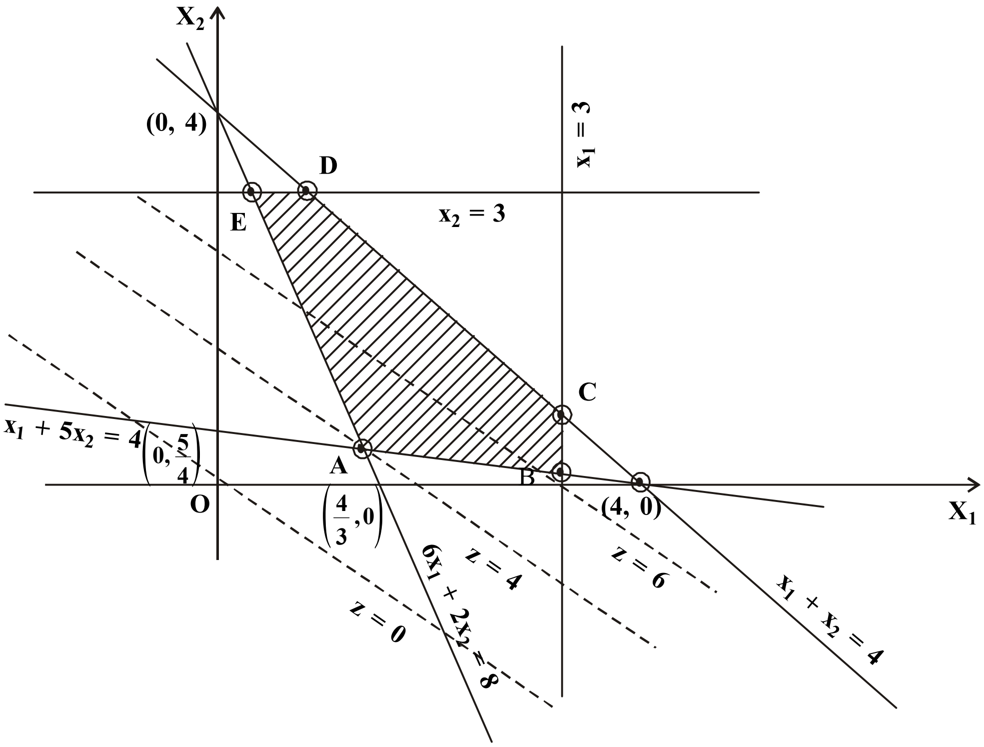


Figure. 3

Solution : We consider the X_1 X_2 –coordinate system (rectangular cartesian). We draw the graphs of the straight lines $x_1 + x_2 = 4$, $6x_1 + 2x_2 = 8$, $x_1 + 5x_2 = 4$, $x_1 = 3$ and $x_2 = 3$.

The shaded region shown in the graph is the feasible region whose extreme points are A $\left(\frac{8}{7}, \frac{4}{7}\right)$, B $\left(3, \frac{1}{5}\right)$, C (3, 1), D (1, 3) and E $\left(\frac{1}{3}, 3\right)$. The graphs of the objective function $z = 2x_1 + 3x_2$ for different values of z are shown by dotted lines. It shows that the minimum value of z occurs at A. Thus, the optimal solution of the problem is $x_1 = \frac{8}{7}$, $x_2 = \frac{4}{7}$ and the minimum value of z is $2 \times \frac{8}{7} + 3 \times \frac{4}{7} = 4$ units. The solution of the LPP is unique.

● **Example 2.2.4**

Solve the following LPP graphically :

Minimize $z = 6x_1 + 4x_2$

$$\begin{aligned} \text{subject to} \quad 3x_1 + 2x_2 &\geq 12 \\ -x_1 + x_2 &\leq 3 \\ 0 \leq x_1 &\leq 7 \\ 0 \leq x_2 &\leq 6 \end{aligned}$$

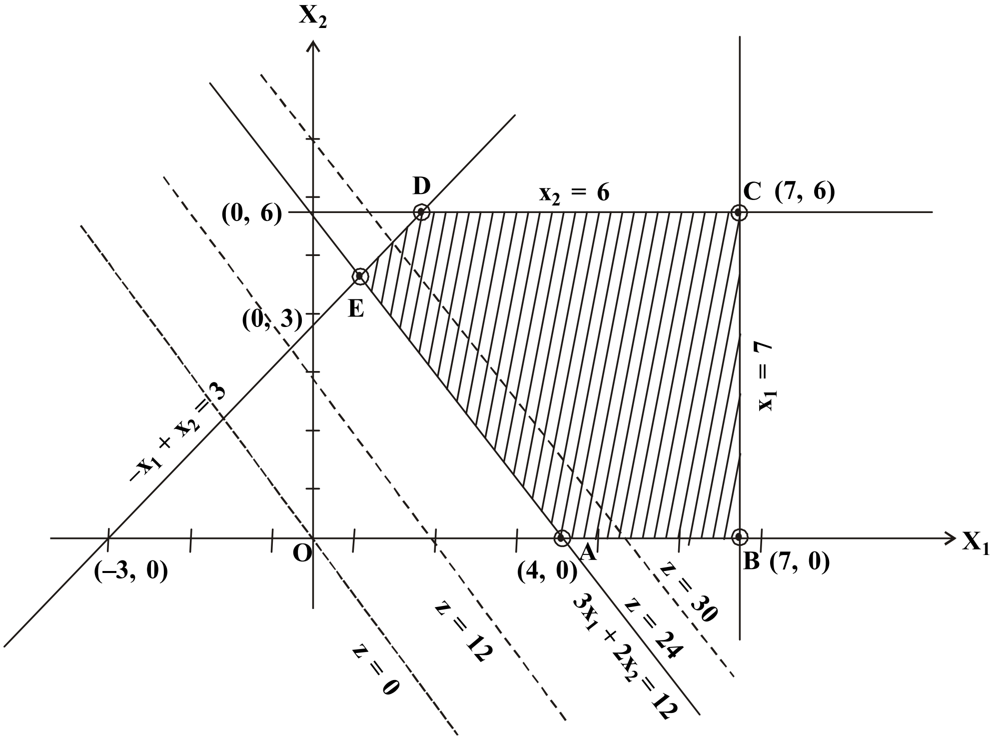


Figure. 4

Solution : Wifh reference to the rectangular cartesian axes OX_1 and OX_2, we draw the graphs of $3x_1 + 2x_2 = 12$, $– x_1 + x_2 = 3$, $x_1 = 7$ and $x_2 = 6$. The shaded region shown in the graph is the feasible region whose extreme points are A (4, 0), B (7, 0), C (7, 6), D (3, 6) and E $\left(\frac{6}{5}, \frac{21}{5}\right)$.

It is evident that for any particular value of z, the objective function $z = 2(3x_1 + 2x_2)$ is parallel to the constraint $3x_1 + 2x_2 = 12$. The graphs of $z = 2(3x_1 + 2x_2)$ for different values of z are shown by the dotted lines. The graph shows that the optimal solution of the LPP occurs at an infinite number of points of the line segment AE, some of them are A (4, 0), E $\left(\frac{6}{5}, \frac{21}{5}\right)$. The minimum value of z is $6 \times 4 + 2 \times 0 = 24$ units.

Hence the LPP has an infinite number of optimal solutions, one of them is $x_1 = 4$, $x_2 = 0$ and $z_{min} = 24$ units.

● **Example 2.2.5**

Solve the following LPP graphically :

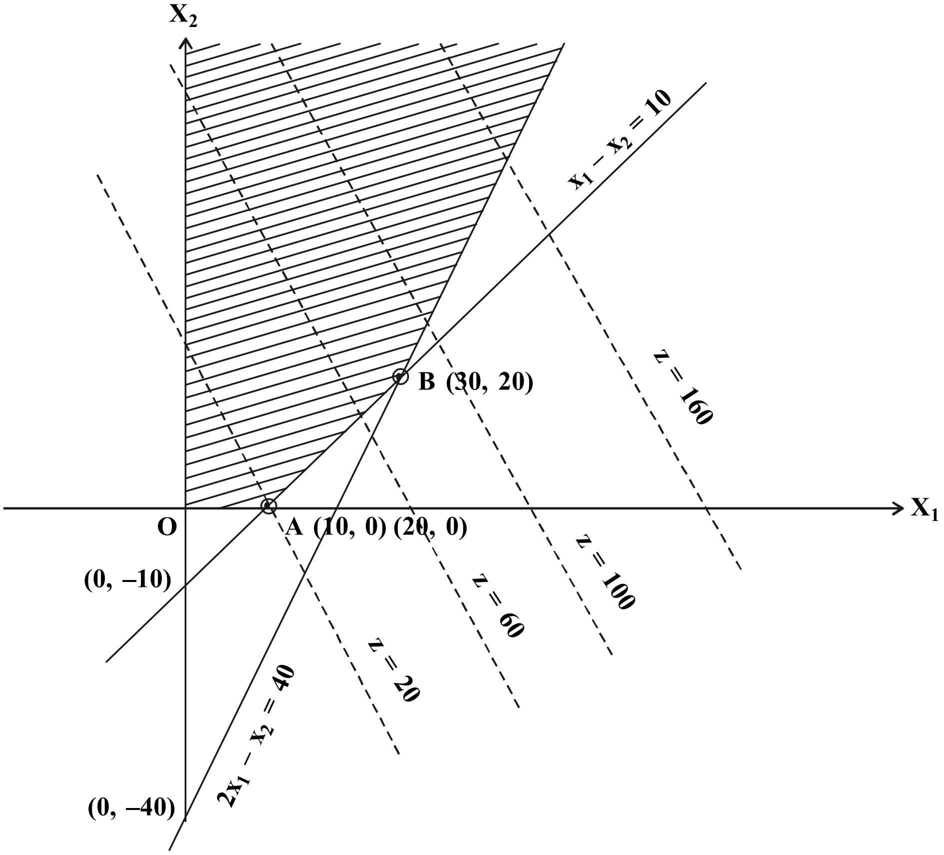


Figure. 5

Minimize $z = 2x_1 + x_2$

subject to $x_1 - x_2 \leq 10$

$2x_1 - x_2 \leq 40$

$x_1, x_2 \geq 0$

Solution : With reference to a set of rectangular cartesian axes OX_1 and OX_2 in the plane of the paper, we draw the graphs of the straight lines $x_1 - x_2 = 10$ and $2x_1 - x_2 = 40$. The feasible region of the LPP is shown in the graph by shading it. It is evident that the region of feasible solutions is unbounded.

It is observed in the graph that the graph of the objective function $z = 2x_1 + x_2$ for different values of z can be moved towards right indefinitely as marked by dotted lines, till containing the points of the feasible region. So this LPP has no maximum value of z. In this case, we say that the LPP has an unbounded solution.

2.3 Summary

In the previous unit we learnt how a decision making problem is formulated in a linear programming model and in this unit-2. We have explained how this LPP model can be solved by graphical (or geometrical) method. Various types of linear programming problems are discussed in this regard.

2.4 Exercise

1. Solve the following LPP graphically:-

(i) Maximize $z = 3x_1 + 5x_2$

Subject to $x_1 + 2x_2 \leq 2000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 600$

$x_1, x_2 \geq 0$

(ii) Maximize $z = 2x_1 + x_2$

Subject to $5x_1 + 10x_2 \leq 50$

$x_1 + x_2 \geq 1$

$$x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

(iii) Maximize $z = 80x_1 + 120x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\leq 9 \\ 20x_1 + 50x_2 &\leq 360 \\ x_1 &\geq 2 \\ x_2 &\geq 3 \end{aligned}$$

(iv) Maximize $z = 4x_1 + 5x_2$

subject to
$$\begin{aligned} x_1 + x_2 &\geq 1 \\ -2x_1 + x_2 &\leq 1 \\ 4x_1 - 2x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(v) Minimize $z = 3x_1 + 5x_2$

subject to
$$\begin{aligned} 3x_1 + 4x_2 &\leq 12 \\ 2x_1 + x_2 &\geq -2 \\ 2x_1 + 3x_2 &\geq 12 \\ x_1 &\leq 4 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Answers

1. (i) $x_1 = 1000$, $x_2 = 500$; $z_{max} = 5500$ units

   (ii) $x_1 = 0$, $x_2 = 1$; $z_{min} = 1$ units

   (iii) $x_1 = 3$, $x_2 = 6$; $z_{max} = 960$ units

   (iv) Unbounded solution is z has no maximum value

   (v) $x_1 = 3$, $x_2 = 2$; $z_{min} = 19$ units.

2.5 Multiple Choice Questions (MCQ)

1. The maximum value of $z = x_1 + x_2$ subject to the constraints

   $5x_1 + 10x_2 \leq 50,\ \ x_1 + x_2 \geq 1,\ \ x_2 \leq 4;\ x_1,\ \ x_2 \geq 0$ is

   (a) 30 units (b) 20 units (c) 50 units (d) 10 units

2. The minimum value of $z = 4x_1 + 2x_2$ subject to the constraints

   $3x_1 + y_1 \geq 27,\ \ x_1 + x_2 \geq 21,\ \ x_1 + 2y_1 \geq 30;\ \ x_1, x_2 \geq 0$ is

   (a) 28 units (b) 48 units (c) 40 units (d) 10 units

3. The optimal solution of the following LPP

$$\begin{aligned} &\text{Maximize } z = 2x_1 + 4x_2 \\ &\text{subject to} \quad x_1 + 2x_2 \leq 5 \\ &\qquad\qquad\quad x_1 + x_1 \leq 4 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \end{aligned}$$

   is given by

   (a) $x_1 = 0,\ x_2 = \frac{5}{2}$ (b) $x_1 = \frac{1}{2},\ x_2 = \frac{3}{2}$

   (c) $x_1 = 2,\ x_2 = 7$ (d) $x_1 = 3,\ x_2 = \frac{1}{2}$

4. The optimal solution of the following LPP

$$\begin{aligned} &\text{Maximize } z = 3x_1 + 4x_2 \\ &\text{subject to} \quad 5x_1 + 4x_2 \geq 20 \\ &\qquad\qquad\quad -x_1 + x_2 \leq 3 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \\ &x_1 \leq 4,\ x_2 \geq 3;\ x_1, x_2 \geq 0 \end{aligned}$$

   is given by

   (a) $x_1 = 2,\ x_2 = \frac{2}{5}$ (b) $x_1 = 3,\ x_2 = 5$

   (c) $x_1 = \frac{8}{5},\ x_2 = 3$(d) $x_1 = 3,\ x_2 = \frac{5}{2}$

5. The L.P.P.

$$\begin{aligned} &\text{Maximize } z = 2x_1 + 3x_2 \\ &\text{subject to} \quad 3x_1 - x_2 \le -3 \\ &\qquad\qquad\quad x_1 - 2x_2 \ge 2 \\ &\qquad\qquad\quad\ \ x_1, x_2 \ge 0 \end{aligned}$$

has

(a) unbounded solution (b) no feasible solution

(c) unique optimal solution (d) infinitely many optimal solutions

Answers

1. (d) 2. (b) 3. (a) 4. (c) 5. (b)

Unit - 3 ❑ Mathematical Preliminaries

Structure

3.0 Objective

After studying this chapter, the reader should be able to know

- the vectors and their propeties in Euclidean space
- some basic results on basis set

3.1 Introduction

In this section, we shall give some necessary definitions and related theorems with examples which are useful to find the analytical solution of a linear programming problems in the subsequent units.

3.2 Vectors

● **Definition 3.2.1 :** A matrix consisting of a single row (or column) is called a row (or column) vector. A vector containing n elements is called an n-component vector. Thus, the matrices $(a_1, a_2, ..., a_n)$ and $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ are n-component row and column vectors respectively. In the present discussion, a column vector will be represented by a row of n-elements enclosed within square brackets like $[a_1, a_2, ..., a_n]$ whereas, n-elements enclosed within first brackets like $(a_1, a_2, ..., a_n)$ will represent a row vector.

Geometrically, an n-component vector may be imagined to represent a point in n-dimensional space and it is an ordered set of n elements or an ordered n-tuple.

● **Algebra of vectors :** The algebra of vectors is governed by the algebra of matrices. Some of its elementary properties are as follows :

(i) Two vectors are said to be equal if both are either row or column vectors and their corresponding elements are equal. Thus,
$(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n) \Rightarrow a_1 = b_1, a_2 = b_2, ..., a_n = b_n$.

(ii) The addition of two n-component vectors is an n-component vector whose elements are the sum of the corresponding elements of the given vectors. Thus,
$(a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$.

(iii) If λ is a scalar, then the multiplication of a vector by a scalar λ is defined as

$$\lambda (a_1, a_2, ..., a_n) = (\lambda a_1, \lambda a_2, ..., \lambda a_n).$$

(iv) Scalar product of two vectors $\overline{\alpha} = (a_1, a_2, ..., a_n)$ and $\overline{\beta} = (b_1, b_2, ..., b_n)$ is defined as

$$\overline{\alpha}.\overline{\beta} = \sum_{i=1}^{n} a_i b_i = \overline{\beta}.\overline{\alpha}.$$

where a_i and b_i are the i-th components of the vectors $\overline{\alpha}$ and $\overline{\beta}$ respectively. Here, both the vectors will be either row or column vectors.

● **Null Vector :** A vector with all its components zero is called a null vector or zero vector and it is denoted by $\overline{0}$. Thus.

$$\overline{0} = (0, 0, ..., 0) \text{ or } \overline{0} = [0, 0, ..., 0].$$

● **Unit Vector :** A vector whose one component is unity and all other components are zero is called a unit vector. Clearly, there are n number of n-component unit vectors. A unit vector in which the i-th component is unity and the remaining components are zero is denoted by $\overline{e}_i$. Thus,

$$\overline{e}_i = (0, 0, ..., 0, 1, 0, ..., 0) \text{ or } \overline{e}_i \; [0, 0, ..., 0, 1, 0, ..., 0].$$

● **Euclidean Space :** A Euclidean Space of n-dimension, generally denoted by E^n or $\mathbb{R}^n$, is the set of all n-component real vectors (either row or column). Any vector will be treated as a point in Euclidean Space. In particular, $\overline{\alpha} \in E^m$ means $\overline{\alpha}$ is a m-component vector. Here, the symbol $\in$ means 'belongs to'.

● **Linear Combination of vectors of E^n :** A vector $\overline{\beta} \in E^n$ is called a linear combination of the vectors $\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_p$ of E^n when it can be expressed in the form

$$\overline{\beta} = \lambda_1\overline{\alpha}_1 + \lambda_2\overline{\alpha}_2 + \cdots + \lambda_p\overline{\alpha}_p,$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are scalars.

Any vector $\overline{\alpha} = (a_1, a_2, ..., a_n) \in E^n$ can always be expressed as a linear combination of the unit vectors of $\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_n$ of E^n as follows :

$$\overline{\alpha} = a_1\overline{e}_1 + a_2\overline{e}_2 + \cdots a_n\overline{e}_n$$

● **Linear Dependence and Independence of Vectors in E^n :** Let $\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_k$ be a set of k-vectors in E^n. Then, these vectors are said to be linearly depedent (*l*.d.) if there exists k scalars $\lambda_1, \lambda_2, ..., \lambda_k$ with at least one $\lambda_i \neq 0$ such that

$$\lambda_1\overline{\alpha}_1 + \lambda_2\overline{\alpha}_2 + \cdots + \lambda_k\overline{\alpha}_k = \overline{0}$$

where $\overline{0}$ is the null vector belongs to E^n. On the other hand, if $\lambda_1\overline{\alpha}_1 + \lambda_2\overline{\alpha}_2 + \cdots + \lambda_k\overline{\alpha}_k = \overline{0}$ holds only when $\lambda_1 = \lambda_2 = ... = \lambda_k = 0$, then the vectors $\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_k$ are said to be linearly independent (*l.* i.).

● **Example 3.2.1 :** Test whether the following vectors are *l*.d. or *l*.i. : (1, 1, 0), (3, 0, 1), (5, 2, 1).

Solution : Let λ_1 (1, 1, 0) + λ_2 (3, 0, 1) + λ_3 (5, 2, 1) = (0, 0, 0). Then, we have the homogeneous system of equations

$$1.\lambda_1 + 3.\lambda_2 + 5.\lambda_3 = 0$$

$$1.\lambda_1 + 0.\lambda_2 + 2.\lambda_3 = 0$$

$$0.\lambda_1 + 1.\lambda_2 + 1.\lambda_3 = 0$$

Now, the value of the co-efficient determinant of this system

$$\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 1\ (0 - 2) - 3\ (1 - 0) + 5\ (1 - 0) = 0.$$

Since $\Delta = 0$, the system has a non-trivial solution, that is, there exists at least one $\lambda_i \neq 0$ in the solution of the system. Hence, the vectors are linearly dependent.

In fact, from the first two equations of the system, by cross-multiplication, we have

$$\frac{\lambda_1}{6-0} = \frac{\lambda_2}{5-2} = \frac{\lambda_3}{0-3} \Rightarrow \frac{\lambda_1}{6} = \frac{\lambda_2}{3} = \frac{\lambda_3}{-3} \Rightarrow \frac{\lambda_1}{2} = \frac{\lambda_2}{1} = \frac{\lambda_3}{-1}.$$

As $\lambda_1 = 2k$, $\lambda_2 = k$, $\lambda_3 = -k$ $(k \neq 0)$ satisfies the third equation of the system, hence

$$2.(1,\ 1,\ 0) + 1.(3,\ 0,\ 1) - 1.(5,\ 2,\ 1) = \overline{0}.$$

Since at least one $\lambda_i \neq 0$ (Here, λ_1, λ_2, λ_3 are all non-zero), so the vectors are linearly dependent.

● **Example 3.2.2 :** Show that the following vectors are linearly independent : (2, 1, 4), (2, –1, 2), (1, 3, –2).

Solution : Let $\lambda_1\ (2, 1, 4) + \lambda_2\ (2, -1, 2) + \lambda_3\ (1, 3, -2) = (0, 0, 0)$. Then, we have the homogeneous system of equations

$$2.\lambda_1 + 2.\lambda_2 + 1.\lambda_3 = 0$$

$$1.\lambda_1 - 1.\lambda_2 + 3.\lambda_3 = 0$$

$$4.\lambda_1 + 2.\lambda_2 - 2.\lambda_3 = 0$$

Now, the value of the coefficient determinant of the system

$$\Delta = \begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \\ 4 & 2 & -2 \end{vmatrix} = 2\ (2 - 6) - 2\ (-2 - 12) + 1.\ (2 + 4) = 26$$

Since, $\Delta \neq 0$, the system has the only trivial solution, that is, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. So the vectors are linearly independent.

● **Example 3.2.3 :** Express the vector (18, 3, 5) as a linear combination of the vectors (1, 1, 0) and (3, 0, 1)

Solution : Let $(18, 3, 5) = \lambda_1 (1, 1, 0) + \lambda_2 (3, 0, 1)$.

Then, we have

$$\lambda_1 + 3\lambda_2 = 18,\ \lambda_1 = 3,\ \lambda_2 = 5,$$

which are consistent.

Thus, $$(18, 3, 5) = 3\ (1, 1, 0) + 5\ (3, 0, 1).$$

3.3 Some Important Theorems (Statement only)

● **Theorem 3.3.1 :** In E^n, any non-empty subset of a set of linearly independent vectors is also linearly independent.

● **Theorem 3.3.2 :** In E^n, any superset of a set of linearly dependent vectors is also linearly dependent.

● **Theorem 3.3.3 :** In E^n, a set of vectors containing a null vector is necessarily linearly dependent.

3.4 Spanning Set and Basis

● **Definition 3.4.1 (Spanning Set)**

A finite set of vectors $X \subseteq E^n$ is said to be a spanning set if every vector in E^n can be expressed as a linear combination of the vectors of X. It is said that X spans E^n or E^n is sppanned by X.

● **Definition 3.4.2 (Basis Set or Basis)**

A spanning set in E^n is called a basis or a basis set, if the vectors of the spanning set are linearly independent.

Thus, a set $B \subseteq E^n$ forms a basis in E^n, if

(i) B spans E^n, i.e., every vector of E^n can be expressed as a linear combination of the vectors of B.

(ii) The vectors of the set B are linearly independent.

Since any vector of E^n can be expressed as a linear combination of the set of unit vectors of E^n, so the set $\left\{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_n\right\}$ is called the standard basis of E^n.

● **Example 3.4.1 :** Show that the set of unit vectors $\left\{\overline{e}_1, \overline{e}_2, \overline{e}_3\right\} \subseteq E^3$ is a spanning set in E^3.

Solution : Let $\overline{x} = (x_1, x_2, x_3)$ be any vector of E^3. Then, we can easily write

$$\overline{x} = (x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2 (0, 1, 0) + x_3 (0, 0, 1)$$

$$= x_1\overline{e}_1 + x_2\overline{e}_2 + x_3\overline{e}_3$$

Also we can write

$$\overline{e}_1 = 1.\overline{e}_1 + 0.\overline{e}_2 + 0.\overline{e}_3$$

$$\overline{e}_2 = 0.\overline{e}_1 + 1.\overline{e}_2 + 0.\overline{e}_3$$

$$\overline{e}_3 = 0.\overline{e}_1 + 0.\overline{e}_2 + 1.\overline{e}_3$$

Thus, every vector in E^3 can be expressed as a linear combination of the vectors of the set $\left\{\overline{e}_1, \overline{e}_2, \overline{e}_3\right\}$. So, the set of unit vectors of E^3 is a spanning set in E^3.

● **Example 3.4.2 :** Show that the set of unit vectors (1, 1, 0), (1, –1, 0), (0, 0, 1) is a spanning set in E^3.

Solution : Let $\overline{x} = (x_1, x_2, x_3)$ be any vector of E^3.

Let $(x_1, x_2, x_3) = \xi_1 (1, 1, 0) + \xi_2 (1, -1, 0) + \xi_3 (0, 0, 1)$.

Then we have

$$x_1 = 1.\xi_1 + 1.\xi_2 + 0.\xi_3$$

$$x_2 = 1.\xi_1 - 1.\xi_2 + 0.\xi_3$$

$$x_3 = 0.\xi_1 + 0.\xi_2 + 1.\xi_3$$

Now, the co-efficient determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\,(-1\,-1) = -2 \neq 0.$$

Hence, by Cramer's rule, the above system of equations has a unique solution. So ξ_1, ξ_2, ξ_3 can be determined uniquely in terms of x_1, x_2, x_3. Therefore, the set of vectors (1, 1, 0), (1, –1, 0), (0, 0, 1) is a spanning set in E^3.

● **Example 3.4.3 :** Show that the set {(3, 0, 2), (7, 0, 9), (4, 1, 2)} forms a basis for E^3.

Solution : Since $\Delta = \begin{vmatrix} 3 & 7 & 4 \\ 0 & 0 & 1 \\ 2 & 9 & 2 \end{vmatrix} = -(27 - 14) = -\,13\ (\neq 0)$, so the vectors of the given set are linearly independent.

Again, let $\overline{x} = (x_1, x_2, x_3)$ be any vector of E^3. Let

$$(x_1, x_2, x_3) = \xi_1\,(3, 0, 2) + \xi_2\,(7, 0, 9) + \xi_3\,(4, 1, 2)$$

Then we have

$$x_1 = 3.\xi_1 + 7.\xi_2 + 4.\xi_3$$

$$x_2 = 0.\xi_1 + 0.\xi_2 + 1.\xi_3$$

$$x_3 = 2.\xi_1 + 9.\xi_2 + 2.\xi_3$$

Since the coefficient determinant of the above non-homogeneous system $\Delta \neq 0$, so, ξ_1, ξ_2, ξ_3 can be determined uniquely in terms of x_1, x_2, x_3. Therefore, the set of vectors is a spanning set in E^3.

Hence, the given set forms a basis for E^3.

3.5 Some Important Theorems (Statement only)

● **Theorem 3.5.1 :** Representation of a vector in E^n in terms of the vectors of a basis is unique.

● **Theorem 3.5.2 (Replacement Theorem)**

If x = $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k\}$ be a basis in E^n and $\bar{\alpha} \subseteq E^n$ be any other non-null vector such that

$$\bar{\alpha} = \lambda_1\bar{\alpha}_1 + \lambda_2\bar{\alpha}_2 + \cdots + \lambda_k\bar{\alpha}_k,$$

where, λ_1, λ_2, ..., λ_k (not all zero) are scalars, then any vector $\bar{\alpha}_i$ for which $\lambda_i \neq 0$ can be replaced by the vector $\bar{\alpha}$ in the basis set X to get a new basis in E^n.

● **Theorem 3.5.3 :** Every basis in E^n has exactly n vectors.

● **Theorem 3.5.4 :** Any subset of n linearly independent vectors of E^n forms a basis in E^n.

● **Theorem 3.5.5 :** Any set of (n + 1) or more vectors of E^n are linearly dependent.

● **Example 3.5.1 :** Find a basis in E^3 containing the vectors (1, 2, 0) and (0, 3, 1).

Solution : We know that the set of unit vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ always forms a basis in E^3.

Since (1, 2, 0) in a non-null vector, we have

$$(1, 2, 0) = \alpha_1\bar{e}_1 + \alpha_2\bar{e}_2 + \alpha_3\bar{e}_3$$

where $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 0$.

Since $\alpha_1 \neq 0$, so by Replacement Theorem, the vectors (1, 2, 0), $\bar{e}_2, \bar{e}_3$ form a basis in E^3.

Again, (0, 3, 1) is a non-null vector. So, it can be expressed as a linear combination of the vectors (1, 2, 0), $\bar{e}_2, \bar{e}_3$ of the new basis.

Let (0, 3, 1) = μ_1 (1, 2, 0) + $\mu_2\bar{e}_2 + \mu_3\bar{e}_3$.

Then, we have

$$0 = \mu_1, \; 3 = 2\mu_1 + \mu_2 \text{ and } 1 = \mu_3$$

Solving these relations, we get $\mu_1 = 0$, $\mu_2 = 3$, $\mu_3 = 1$.

As $\mu_2 \neq 0$, so, by Replacement Theorem. the vectors (1, 2, 0), (0, 3, 1), $\bar{e}_3$ from another basis in E^3.

Hence, the set {(1, 2, 0), (0, 3, 1), (0, 0, 1)} form a basis in E^3.

● **Example 3.5.2 :** Show that the vectors (2, 3, 1), (1, 0, 4), (2, 4, 1), (0, 3, 2) are linearly dependent.

Solution : Let $\bar{\alpha}_1$= (2, 3, 1), $\bar{\alpha}_2$ = (1, 0, 4), $\bar{\alpha}_3$= (2, 4, 1) and $\bar{\alpha}_4$= (0, 3, 2).

Since $\begin{vmatrix} 2 & 1 & 2 \\ 3 & 0 & 4 \\ 1 & 4 & 1 \end{vmatrix}$ = 2 (0 – 16) – (3 – 4) + 2 (12 – 0) = – 7 (≠ 0), the vectors $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$ are linearly independent. Since n linearly independent vectors of E^n form a basis in E^n, so the vectors α_1, α_2, α_3 form a basis in E^3. Again, $\bar{\alpha}_4$ is a non-null vector in E^3, so it can be expressed as a linear combination of the vectors of the basis $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3$. Therefore, the given vectors are linearly dependent.

3.6 Summary

In this chapter properties of vectors in a Euclidean space are discussed. Also, some important theorem are stated which are useful to find the analytial solution of a linear programming problem.

3.7 Exercise

1. Express $\bar{c}$ = (4, 5) as a linear combination of the vectors $\bar{a}$ = (1, 3) and $\bar{b}$ = (2, 2).
2. Express $\bar{\alpha}$ = (2, 1, 1) as a linear combination of the vectors $\bar{\beta}$ = (1, 2, 2) and $\bar{\gamma}$ = (1, 1, 1).
3. Test whether the vectors (1, 2, 0), (2, 1, 2), (2, 2, 1) are linearly independent or not.

4. Test whether the vectors (1, 2, 0), (3, –1, 1), (4, 1, 1) are linearly dependent or not.
5. Show that the vectors (2, 3, 1), (1, 0, 4), (2, 4, 1), (0, 3, 2) are linearly dependent. Also find a relation between them.
6. Show that the vectors (2, –1, –2), (1, 2, –1), (2, 0, 2) form a basis in E^3.
7. Show that the vectors (2, 1, 1), (0, 1, 1), (3, 1, 4) form a basis in E^3.
8. Is the set {(3, 1, 2), (4, 2, 8), (2, 1, 4)} form a basis in E^3? Justify your answer.

Answers

1. $2\bar{a} + 7\bar{b} = 4\bar{c}$
2. $\bar{\lambda} = 3\bar{\gamma} - \bar{\beta}$
3. Linearly independent
4. Linerarly dependent
5. 29 (2,3,1) − 4 (1,0,4) − 27 (2,4,1) + 7 (0,3,2) = (0,0,0)
6. No, since the vectors are linearly dependent.

3.8 Multiple Choice Questions (MCQ)

1. If the vectors (1,1,0), (3,0,1), (λ,2,1) are linearly dependent, then the value of λ is

   (a) 3 (b) 4 (c) –2 (d) 5

2. If $\bar{a}$ = (4,5), $\bar{b}$ = (1,3), $\bar{c}$ = (2,2) are such that $\bar{a} = \lambda, \bar{b} + \lambda_2\bar{c}$ then the values of λ_1, λ_2 are

   (a) $\lambda_1=\frac{1}{2}, \lambda_2=\frac{7}{4}$ (b) $\lambda_1=\frac{3}{2}, \lambda_2=\frac{5}{4}$

   (c) $\lambda_1= 3, \lambda_2=\frac{3}{4}$ (d) $\lambda_1= \frac{1}{2}, \lambda_2= -\frac{3}{2}$

3. If $\bar{a}$= (1,0,0), $\bar{b}$ = (1,1,0), $\bar{c}$ = (1,1,1) then

   (a) all the vectors are unit vectors

(b) $\overline{a}$ $\overline{b}$ $\overline{c}$ are linearly dependent vectors

(c) $\left\{\overline{a}, \overline{b}, \overline{c}\right\}$ is a spanning set in E^3

(d) $\left\{\overline{a}, \overline{b}, \overline{c}\right\}$ does not from a basic for E^3

Answers

1. (d) 2. (a) 3. (c)

Unit - 4 ❑ Basic Solution of a set of Simultaneons Linear Equations

Structure

4.0 Objective

After studying this chapter, the reader should be able to know

- the consistancy or inconsistency of a set of simultaneons linear equations
- the different type of solutions of a system of simultaneous linear equations

4.1 Introduction

In this section, we shall give some necessary definitions of consistency, inconsistency, feasible solution, basic feasiable solution etc. of a system of linear equations with examples which are useful to find the solution of a linear programming problem in the subsequent chapters.

4.2 System of Linear Equations

Let us consider a set of m simultaneons linear equations in n unknown variables x_1,

x_2, ---- x_n as follows :-

$$a_{11}\ x_1 + a_{12}\ x_2 + \text{-------------------------}\ a_{1n}\ x_n = b_1$$

$$a_{21}\ x_1 + a_{22}\ x_2 + \text{-------------------------}\ a_{2n}\ x_n = b_2$$

$$a_{m1}\ x_1 + a_{m2}\ x_2 + \text{-------------------------}\ a_{mn}\ x_n = b_m$$

or, $A\overline{x} = \overline{b}$

where A is the matrix

$(a_{ij})_{mxn}$, $i = 1,2,$--------- m; $j = 1,2,$-------- n

$\overline{x} = [x_1, x_2,$ -------- $x_n]$ and $\overline{b} = [b_1, b_2,$ ------ $b_m]$

Let us now define the augmented matrix A_b of order mx (n+1)th which contains A in the first n columns and $\overline{b}$ in the (n+1) colum n. Thus

$$A_b = \begin{pmatrix} a_{11} & a_{12}\text{---------}a_{1n} & b_1 \\ a_{12} & a_{22}\text{---------}a_{2n} & b_2 \\ ---- & ---- & ---- \\ ---- & ---- & ---- \\ a_{m1} & a_{m2}\text{-------}a_{mn} & b_m \end{pmatrix}$$

Any set of values of x_1, x_2,-------- x_n which simultaneously satisfy all the equations (1) is called a solution of the system of equations (1). A system of equations is said to be consistent it there exists at least one solution otherwise it is inconsistant.

We state here same characteristics of the solution of the set of linear simultaneous equation withput proof.

If $r(A) < r(A_b)$ then the system has no solution and the equations are inconsistent. On the otherhand it $r(A) = r(A_b)$ then there is at least one solution and the system of equations are consistent.

Further it r (A) = r (A_b) = n, the number of unknown variables of the equations.

Then the system will have unique solution. On the otherhand, it r (A) = r (A_b) < n, then the number of solutions is infinite.

● **Example 4.2.1 :** Find the value of λ for which the equations

$$2x + 3y = 5$$

$$4x + by = \lambda$$

are consistent.

Solution : The given system is $A\,\overline{x} = \overline{b}$

Where $A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$

$$\overline{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \overline{b} = \begin{pmatrix} 5 \\ \lambda \end{pmatrix}$$

Now rank of A r (A) = 1 $\left(\therefore \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0\right)$

Again Ab = $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 6 & \lambda \end{pmatrix}$

The given system will be consistent

it r (A) = r (A_r) => r (A_b) = 1

$$=> \begin{vmatrix} 3 & 5 \\ 6 & \lambda \end{vmatrix} = 0$$

$$3\lambda - 30 = 0 => \lambda = 10$$

Thus for λ = 10 the given system of equations are consistant.

4.3 Basic Solution and Basic Feasible Solution

● **Definition 4.3.1 (Basic Solution)**

Consider a system of m simultaneous linear equations in v variable $A\overline{x} = \overline{b}$ (m < n) with r (A) = m. If m x m nonsingular matrix B is choosen from A and it all (n - m) variables not associated with the coloums of this matrix B are set equal to zero, then the solution to the resulting system of equations is called a basic solution which can be written as $\overline{x} = [\overline{x}_B, \overline{o}]$ with $\overline{x}_B = B^{-1}\,\overline{b}$.

The m variables associated with the columns of A in B are called basic variables where as others are non basic variables.

The vectors associated to the basic variables (vector formed by the coefficient of the variable x_i is the vector associated to x_i) are linearly independent.

The basic solution cannot have more than m non-zero variables.

The number of basic solution to a system of m linear equations in n variables $A\overline{x} = \overline{b}$ (m < n), r (A) = m, may have at most n_{c_m}. In this system we have

A = $(\overline{a}_1, \overline{a}_2, -- \overline{a}_n), \overline{x} = [x_1, x_2 -- x_n]$ where

$\overline{a}_i = [a_{1i}, a_{2i}, -- a_{mi}]$ for i = 1,2,--- n.

● **Definition 4.3.2 (Degenerate Solution)**

A basic solution to $A\overline{x} = \overline{b}$ be degenerate if at least one basic variable is zero.

● **Definition 4.3.3 (Non-Degenerate Solution)**

A basic solution to $A\overline{x} = \overline{b}$ be non-degenerate if all the basic variables are non-zero.

● **Definition 4.4.4 (Feasible Solution)**

A solution $\overline{x}$ to $A\overline{x} = \overline{b}$ is called feasible if $\overline{x} \geq \overline{0}$.

● **Definition 4.5.5 : (Basic Feasible Solution)**

A feasible solution to $A\overline{x} = \overline{b}$ which is also basic is called a basic feasible solution.

● **Example 4.3.1 :** Find all the basic solutions to the following system of equations :

$$2x_1 + x_2 + 5x_3 = 5$$
$$x_1 + 2x_2 + x_3 = 4$$

Mention the basic feasible solutions, if any. Also mention the non-degenerate solutions.

Solution : In matrix notation, the given system of equations can be written as

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{a}_3 x_3 = \bar{b},$$

where $\bar{a}_1 = \begin{pmatrix}2\\1\end{pmatrix}, \bar{a}_2 = \begin{pmatrix}1\\2\end{pmatrix}, \bar{a}_3 = \begin{pmatrix}5\\1\end{pmatrix}$ and $\bar{b} = \begin{pmatrix}5\\4\end{pmatrix}$.

We write,

$$B_1 = [\bar{a}_1, \bar{a}_2] = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}; |B_1| = 3\ (\neq 0)$$

$$B_2 = [\bar{a}_1, \bar{a}_3] = \begin{bmatrix}2 & 5\\1 & 1\end{bmatrix}; |B_2| = -3\ (\neq 0)$$

$$B_3 = [\bar{a}_2, \bar{a}_3] = \begin{bmatrix}1 & 5\\2 & 1\end{bmatrix}; |B_3| = -9\ (\neq 0)$$

Since all the square matrices B_1, B_2, B_3 are non-singular, therefore, any two column vectors of $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are linearly independent. Hence, the system has 3 basic solutions. The vector $\bar{x}_{B_i} = B_i^{-1}\bar{b}$ (i = 1, 2, 3), that is,

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \bar{x}_{B_1} = B_1^{-1}\bar{b} = \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix};$$

$$\begin{pmatrix}x_1\\x_3\end{pmatrix} = \bar{x}_{B_2} = B_2^{-1}\bar{b} = -\frac{1}{3}\begin{bmatrix}1 & -5\\-1 & 2\end{bmatrix}\begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix};$$

$$\begin{pmatrix}x_2\\x_3\end{pmatrix} = \bar{x}_{B_3} = B_3^{-1}\bar{b} = -\frac{1}{9}\begin{bmatrix}1 & -5\\-2 & 1\end{bmatrix}\begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}5/3\\2/3\end{pmatrix}.$$

Thus, the three basic solutions of the given system are (2, 1, 0), (5, 0, –1) and $\left(0, \frac{5}{3}, \frac{2}{3}\right)$.

Out of these solutions, the basic feasible solutions are (2, 1, 0) and $\left(0, \frac{5}{3}, \frac{2}{3}\right)$.

Since all the basic variables are non-zero in each of these three basic solutions, so they are all non-degenerate solutions.

4.4 Matrix form of Linear Programming Problem

A LPP can be represented in matrix form as follows :

(Maximize or Minimize) Z = $\bar{c}\,\bar{x}$

Subject to $A\bar{x}\,(\leq = \geq)\,\bar{b}$

$\bar{x} \geq \bar{o}$

Where A = $(a_{1i})_{mxn}$, $\bar{x}$ = [x_1, x_2, ----x_n] (a column matrix), $\bar{b}$ = [b1, b2,---- bm] (a column matrix), $\bar{c}$ = (c_1, c_2, ---- c_n) (a row matrix), $\bar{o}$ = [o, o, ---- o] (here a column matrix. It may be mentioned here that for each LPP either 'maximize' or 'minimize' is used in the objective function, are and are holds for each constraint but the sign may very form ane constraint to another.

4.5 Summary

In this chapter properties of system of linear equation are discussed. Also some important terms like feasible solution, basic solution, degenerated solution etc are defined which are useful to prove some important theorems of LPP.

4.6 Exercise

1. Show that the system of equations

$$x_1 + x_2 + x_3 = 1$$
$$2x_1 - x_2 + x_3 = 2$$
$$3x_1 + 2x_2 - x_3 = 3$$

2. Find all the basic solutions to the following system of equations:

$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 + x_3 = 3$$

Also mention the degenerate and non-degenerate solutions

3. Find all the basic solutions to the following system of equations

$$2x_1 + x_2 + 3x_3 = 7$$
$$3x_1 + 2x_2 + 2x_3 = 8$$

4. Find all the basic solutions to the following system of equations

$$x_1 + x_2 + 2x_3 = 9$$
$$3x_1 + 2x_2 + 5x_3 = 22$$

5. Find all the solutions to the following system of equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$3x_1 + 2x_2 + 2x_3 + 3x_4 = 1$$

6. How many basic solutions of the following system are there?

$$2x_1 + x_2 + x_3 = 10$$
$$x_1 + 3x_2 + x_4 = 6$$

Find them

7. Prove that $x_1 = 2$, $x_2 = -1$ and $x_3 = 0$ is a solution set but not a basic solution to the set of equations.

$$3x_1 - 2x_2 + x_3 = 8$$
$$9x_1 - 6x_2 + 4x_3 = 24$$

Answers

2. (1,0,0) is degenerate,

   $(0, \frac{5}{3}, \frac{-1}{3})$ is non degenerate

3. (2,0,1) and (0,2,1)
4. (4,5,0), (–1,0,5) and (0,1,4)
5. $(0, \frac{1}{2}, 0, 0), (-2, 0, \frac{7}{2}, 0), (\frac{8}{3}, 0, 0, \frac{-7}{3})$ and (0,0,2,–1)
6. Six, $(\frac{24}{5}, \frac{2}{5}, 0, 0)$, (6,0,-2,0), (5,0,0,1), (0,2,8,0), (0,10,0,–24), (0,0,10,6)

4.7 Multiple Choice Questions (MCQ)

1. If the system of equations x +y = 2, 2x + my = n

has infinite number of solutions then

(a) $m = n = 2$ (b) $m = 2, n \neq 4$

(c) $m = 2, n = 4$ (d) none of these

2. For the system of equations

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 5 \\ 2x_1 + 3x_2 + x_3 &= 8, \end{aligned}$$

a basic solution is

(a) $\left(0, \frac{2}{7}, \frac{3}{7}\right)$ (b) $\left(\frac{13}{3}, 0, -\frac{2}{3}\right)$ (c) $\left(\frac{4}{5}, \frac{3}{5}, 0\right)$ (d) None of these

3. For the system of equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 + x_4 &= 3 \\ 3x_1 + 2x_2 + 2x_3 + 3x_4 &= 1, \end{aligned}$$

a basic solution is

(a) $\left(\frac{1}{2}, 0, 1, 0\right)$ (b) $\left(\frac{3}{2}, 0, 0, 0\right)$ (c) $\left(0, \frac{1}{2}, 0, 0\right)$ (d) $\left(1, \frac{1}{6}, 0, 0\right)$

4. The maximum number of basic solutions for a LPP having m constraints and n variables ($m < n$) is

(a) n (b) ${}^{n}c_{m}$ (c) m (d) $m + n - 1$

Answers

1. (c) 2. (b) 3. (c) 4. (b)

Unit - 5 ❑ Slack and Surplus Variable and Standrad form of a LPP

Structure

5.0 Objective

After studying this chapter, the reader should be able to know

- how the set of consistrains of a general LPP can be converted to a set of simultaneous linear equations.

5.1 Introduction

To solve a LPP by algebraic method it is necessary to convert all the constrants into equations. We consider two types of constrains '$\leq$' and '$\geq$' separately.

5.2 Slack and Surplus Variable and Standrad form of a LPP

(i) We consider first the constraints having '$\leq$' sign in the LPP. A typical constraint of this category may be as.

$a_{r1}\ x_1 + a_{r2}\ x_2 +$ ------------------------ $a_{rk}\ x_k \leq$ br

This can be converted to into an equality by adding to its left hand side a non negalive now variable x_{k+r} which is called slack variable; and then we have

$a_{r1}\ x_1 + a_{r2}\ x_2 +$ ------------------------ $+ a_{rk}\ x_k + x_{k+r} = br,$

$$x_{k+r} \geq 0$$

Thus, a non negative variable which is added to the left hand side of the constraint having '$\leq$' sign of a LPP to convert into an equation is called a slack variable.

(ii) Next, we consider the constraints having '≥' sign in the LPP. A typical constraint of this category may be as

$$a_{s1}x_1 + a_{s2}x_2 + \ldots\ldots + a_{sk}x_k \geq b_s.$$

This can be converted into an equality by subtracting from its left hand side a non-negative new variable x_{k+s}, which is called surplus variable; and then we have

$$a_{s1}x_1 + a_{s2}x_2 + \ldots\ldots + a_{sk}x_k - x_{k+s} = b_s,\ x_{k+s} \geq 0.$$

Thus, a non-negative variable which is subtracted from the left hand side of the constraint having '≥' sign of a LPP to convert it into an equation is called a surplus variable.

● Standard form of a LPP

We know that a constraint of the general LPP may involve any one of the following three signs : '≤', '=', '≥'.

We arrange the constraints of the general LPP and first write those constraints having '≤' sign, then those which involve '≥' sign and in the end, we write those having equality sign '='.

After the introduction of the slack and surplus variables to the corresponding constraints of the general LPP, we convert the objective function of 'minimization' type to a 'maximization' type by multiplying by (–1). Then, we shall have obtained the standard form of the LPP as follows :

$$\text{Maximize } z = \sum_{i=1}^{k} c_i x_i + \sum_{i=1}^{\lambda} 0.x_{k+i} + \sum_{i=\lambda+1}^{\mu} 0.x_{k+i}$$

$$\text{subject to} \quad \sum_{j=1}^{k} a_{ij}x_j + x_{k+i} = b_i \ (i = 1, 2, \ldots.., \lambda)$$

$$\sum_{j=1}^{k} a_{ij}x_j - x_{k+i} = b_i \quad (i = \lambda + 1, \lambda + 2, ..., \mu)$$

$$\sum_{j=1}^{k} a_{ij}x_j = b_i \quad (i = \mu + 1, \mu + 2, ..., m)$$

where $b_i \geq 0$, $x_j \geq 0$ for $i = 1, 2, ..., m$; $j = 1, 2, ..., k, k + 1, ..., k + \mu$.

Equivalently, in matrix notation, the standard form of a LPP is

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to} \quad A\bar{x} = \bar{b},$$

$$\bar{b} \geq \bar{0},\ \bar{x} \geq \bar{0},$$

where

$$\bar{b} = [b_1, b_2, ..., b_m]$$

$$\bar{x} = [x_1, x_2, ..., x_n]\ ;\ (n = k + \mu)$$

$$\bar{c} = (c_1, c_2, ..., c_k, 0, 0, ..., 0)$$

and A is the coefficient matrix of order m × n.

● **Example 5.2.1 :** Pose the following LPP in its standard form :

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 3x_2 \\ \text{subject to} \quad & 3x_1 + 6x_2 \leq 8 \\ & 5x_1 + 2x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution : Introducing slack variables x_3 and x_4, the given LPP can be reduced to its standard form as given by

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 3x_2 + 0.x_3 + 0.x_4 \\ \text{subject to} \quad & 3x_1 + 6x_2 + x_3 + 0.x_4 = 8 \end{aligned}$$

$$5x_1 + 2x_2 + 0.x_3 + x_4 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

This is the standard form of the given LPP.

● **Example 5.2.2 :** Reduce the following LPP to its standard form :

$$\begin{aligned} \text{Minimize} \quad z &= 2x_1 - 10x_2 \\ \text{subject to} \quad & x_1 - x_2 \geq 0 \\ & x_1 - 5x_2 \geq -5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution : This is a minimization problem. Multiply the objective function by (–1) to reduce it to a maximization problem. Also we multiply the second constraint by (–1) first and then introduce slack and surplus variables to reduce the LPP to its standard form.

Hence, the standard form of the given LPP is

$$\begin{aligned} \text{Maximize } (-z) &= -2x_1 + 10x_2 + 0.x_3 + 0.x_4 \\ \text{subject to} \quad & x_1 - x_2 - x_3 + 0.x_4 = 0 \\ & -x_1 + 5x_2 + 0.x_3 + x_4 = 5 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Here, x_3 is a surplus variable and x_4 is a slack variable. This is the standard form of the given LPP.

● **Example 5.2.3 :** Reduce the following LPP to its standard form :

$$\begin{aligned} \text{Minimize} \quad z &= x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - 3x_2 + 4x_3 = 5 \\ & x_1 - 2x_2 \leq 3 \\ & 2x_2 + x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution : Introducing slack variable x_4, surplus variable x_5 and reducing the problem as a maximization problem, the standard form of the problem is given by

$$\begin{aligned} \text{Maximize } (-z) &= -x_1 - x_2 - x_3 + 0.x_4 + 0.x_5 \\ \text{subject to} \quad & x_1 - 3x_2 + 4x_3 + 0.x_4 + 0.x_5 = 5 \end{aligned}$$

$$x_1 - 2x_2 + 0.x_3 + x_4 + 0.x_5 = 3$$

$$0.x_1 + 2x_2 + x_3 + 0.x_4 - x_5 = 4$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

This is the standard form of the given LPP.

5.3 Summary

Unit-5 deals with the introduction of slock and surphus variable which are used to find the standrad form of a LPP

5.4 Exercise

1. Reduce the following linear programming problems to its standard form by reducing the objective function into maximization form :

(i) Maximize $z = 2x_2 + 5x_3$

subject to
$$\begin{aligned} x_1 + x_2 &\geq 0 \\ x_1 - x_2 + 2x_3 &\leq -4 \\ 2x_1 + x_2 + 6x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(ii) Minimize $z = -x_1 + 2x_3$

subject to
$$\begin{aligned} x_1 + 3x_2 + x_3 &\leq 5 \\ -x_1 + x_3 &= -1 \\ x_2 - 2x_3 &\leq -3 \\ x_1 + x_2 &\geq 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(iii) Maximize $z = 3x_1 + 2x_2$

subject to
$$\begin{aligned} 3x_1 + 2x_2 &\leq 6 \\ x_1 - 2x_2 &\geq -1 \\ -x_1 - 2x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Answers

1. (i) Maximize $z = 0.x_1 + 2x_2 + 5x_3 + 0.x_4 + 0.x_5 + 0.x_6$

subject to $x_1 + x_2 + 0.x_3 + x_4 + 0.x_5 + 0.x_6 = 0$

$-x_1 + x_2 - 2x_3 + 0.x_4 - x_5 + 0.x_6 = 4$

$2x_1 + x_2 + 6x_3 + 0.x_4 + 0.x_5 + x_6 = 3$

$x_j \geq 0$ (j = 1, 2, ..., 6)

(ii) Maximize $(-z) = x_1 + 0.x_2 - 2x_3 + 0.x_4 + 0.x_5 + 0.x_6$

subject to $x_1 + 3x_2 + x_3 + x_4 + 0.x_5 + 0.x_6 = 5$

$x_1 + 0.x_2 - x_3 + 0.x_4 + 0.x_5 + 0.x_6 = 1$

$0.x_1 - x_2 + 2x_3 + 0.x_4 - x_5 + 0.x_6 = 3$

$x_1 + x_2 + 0.x_3 + 0.x_4 + 0.x_5 - x_6 = 0$

$x_j \geq 0$ (j = 1, 2, ..., 6)

(ii) Maximize $z = 3x_1 + 2x_2 + 0.x_3 + 0.x_4 + 0.x_5$

subject to $3x_1 + 2x_2 + x_3 + 0.x_4 + 0.x_5 = 6$

$-x_1 + 2x_2 + 0.x_3 + x_4 + 0.x_5 = 1$

$-x_1 - 2x_2 + 0.x_3 + 0.x_4 - x_5 = 1$

$x_j \geq 0$ (j = 1, 2, 3, 4, 5)

5.5 Multiple Choice Questions (MCQ)

1. Which of the following statement is worng?

(a) slock variable are used to convert the inequalitics of the type '≤' into equations.

(b) supphus variable are used to convert the inequalities of the type '≥' into equations.

(c) A LPP with all its constrants are of the type '≤' is said to be standrad form.

(d) none of these.

2. The general LPP is in standrad form, it

(a) all the constraints are strict equations

(b) all the constraints are inequalities of '≤' type

(c) all the constrants are inequalities of '≥' type.

(d) None of these

Answers

1. (c) 2. (a)

Unit - 6 ❑ Convex Sets, Covex Hull and Convex Polyhedron

Structure

6.0 Objective

After studying this chapter, the reader should be able to know

- the idea of point set, interior point boundary point, open set, closed set etc in E^n
- the concept of convex set and their properties
- the idea of convex cane, convex hull and convex polyhedron

6.1 Introduction

In this unit the definition and the properties of convex set which will help us to understand the nature of the optimal solution of a LPP. Also we discuss about convex cone, convex hull and convex polyhedron with examples. Some simple theorems on convex sets are also discussed.

6.2 Point Sets

- **Definition 6.2.1 (Point Set)**

Point sets are subsets of E^n whose elements are points or vectors in E^n.

- **Definition 6.2.2 (Interior Point)**

A point $\bar{\alpha}$ is an interior point of the point set $X \subseteq E^n$ if there exists an ε – neighbourhood about $\bar{\alpha}$ which contains only points of the set X.

Every interior point of a point set X must be an element of X.

For example, every point of the point set $X = \{(x_1, x_2) : x_1^2 + x_2^2 < 4\}$ is an interior point of X.

- **Definition 6.2.3 (Boundary Point)**

A point $\bar{a}$ is a boundary point of the point set $X \subseteq E^n$ if every ε – neighbourhood about $\bar{a}$ contains points in X and also points not in X.

A boundary point of the point set X may or may not belong to the set X.

For example, (i) every point of the point set $X_1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 4\}$ is a boundary point of the point set

$$X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}.$$

Here, all the boundary points of X are its elements.

(ii) Again, all the points of $X_1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 4\}$ are boundary points of the point set.

$$X_2 = \{(x_1, x_2) : x_1^2 + x_2^2 < 4\}.$$

Here, all the boundary points of X_2 are not its elements.

- **6.2.4 Definition (Open Set)**

A point set is open if it contains only interior points.

For example, the point set $X = \{(x_1, x_2) : x_1^2 + x_2^2 < 4\}$ is an open set.

● **Definition 6.2.5 (Closed Set)**

A point set is closed if it contains all its boundary points.

For example, the point set $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}$ is a closed set.

● **Remark :** (i) A point set containing a finite number of elements is a closed set.

(ii) Null set is a closed set.

Note : (i) If A and B are closed sets, then $A \cap B$ is a closed set.

(ii) The intersection of any finite number of closed sets is also closed.

● **Definition 6.2.6 (Line Segment)**

The line segment joining the points $\bar{x}_1, \bar{x}_2$ $(\bar{x}_1 \neq \bar{x}_2)$ in E^n is defined to be the set of points

$$X = \left\{\bar{x} : \bar{x} = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2, 0 \leq \lambda \leq 1\right\}$$

Remark : A line segment is a closed set. It has no interior point.

● **Definition 6.2.7 (Line)**

The line passing through the points $\bar{x}_1$, $\bar{x}_2$ $(\bar{x}_1 \neq \bar{x}_2)$ in E^n is defined to be the set of points

$$X = \left\{\bar{x} : \bar{x} = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2, \lambda \text{ is real}\right\}$$

6.3 Hyperplane

● **Definition 6.3.1 (Hyperplane)**

A hyperplane in E^n is defined to be the set of points $X = \left\{\bar{x} : \bar{c}\,\bar{x} = z\right\}$ with $\bar{c} \neq \bar{0}$ being a given n-component row vector, an n-component column vector and z is a given scalar.

Since $\bar{x} = [x_1, x_2, ..., x_n]$, $\bar{c} = (c_1, c_2, ..., c_n)$, the equation of a hyperplane can be written as

$$c_1x_1 + c_2x_2 + ... + c_nx_n = z.$$

For example, $4x_1 + 5x_2 - 3x_3 = 7$ is the equation of a hyperplane.

● Half Spaces

A hyperplane $\bar{c}\,\bar{x} = z$ in E^n divides the whole space into three sets of points which are mutually exclusive and collectively exhaustive sets. They are

$$H_1 = \{\bar{x} : \bar{c}\,\bar{x} < z\},$$

$$H_2 = \{\bar{x} : \bar{c}\,\bar{x} = z\},$$

$$H_3 = \{\bar{x} : \bar{c}\,\bar{x} > z\}.$$

The sets H_1 and H_3 are called open half-spaces.

The sets $H_4 = \{\bar{x} : \bar{c}\,\bar{x} \le z\}$ and $H_5 = \{\bar{x} : \bar{c}\,\bar{x} \ge z\}$ are called closed half-spaces.

● Some Important Notes :

(i) Hyperplanes are closed sets.

(ii) A hyperplane has no interior point.

(iii) A closed half-space is a closed set.

(iv) The intersection of a finite number of hyperplanes or closed half-spaces is a closed set.

● Definition 6.3.2 (Hypersphere)

In E^n, a hypersphere with centre $\bar{a}$ and radius r is the set of points $\{\bar{x} : |\bar{x} - \bar{a}| = r\}$ where $|\bar{x} - \bar{a}| = \{(x_1 - a_1)^2 + (x_2 - a_2)^2 + ... + (x_n - a_n)\}^{1/2}$, $\bar{x} = (x_1, x_2, ..., x_n)$ and $\bar{a} = (a_1, a_2, ..., a_n)$. Here $|\bar{x} - \bar{a}|$ is defined as the distance between the vectors $\bar{x}$ and $\bar{a}$.

Also the set of points $\{\bar{x} : |\bar{x} - \bar{a}| < r\}$ is called an open ball with centre $\bar{a}$ and radius r, while the set $\{\bar{x} : |\bar{x} - \bar{a}| \le r\}$ is called a closed ball with centre $\bar{a}$ and radius r.

● **Example 6.3.1** Determine the position of the points (1, 0, 2, –2), (0, 4, 1, 0) and (2, 2, – 1, – 4) with respect to the hyperplane $2x_1 + 3x_2 + x_3 - 3x_4 = 13$.

Solution : (i) For the point (1, 0, 2, –2), we have

$$2x_1 + 3x_2 + x_3 - 3x_4 = 2 \times 1 + 3 \times 0 + 2 - 3 \times (-2) = 10 < 13.$$

So the point (1, 0, 2, –2) lies on the open half space $2x_1 + 3x_2 + x_3 - 3x_4 < 13$.

(ii) For the point (0, 4, 1, 0), we have

$$2x_1 + 3x_2 + x_3 - 3x_4 = 2 \times 0 + 3 \times 4 + 1 - 3 \times 0 = 13.$$

So, the point (0, 4, 1, 0) lies in the hyperplane $2x_1 + 3x_2 + x_3 - 3x_4 = 13$.

(iii) For the point (2, 2, –1, 4), we have

$$2x_1 + 3x_2 + x_3 - 3x_4 = 2 \times 20 + 3 \times 2 - 1 - 3 \times (-4) = 21 > 13.$$

So, the point (2, 2, –1, 4) lies in the open half-space $2x_1 + 3x_2 + x_3 - 3x_4 > 13$.

6.4 Convex Sets

● **Definition 6.4.1 (Convex Combination)**

A convex combination of a finite number of points $\overline{x}_1, \overline{x}_2, ..., \overline{x}_p$ of E^n is defined as a point in E^n as given by

$$\overline{x} = \lambda_1\overline{x}_1 + \lambda_2\overline{x}_2 + ... + \lambda_p\overline{x}_p,$$

with $\lambda_i \geq 0$ (i = 1, 2, ..., p) and $\sum_{i=1}^{p} \lambda_i = 1$.

In particular, the convex conbination of two points $\overline{x}_1, \overline{x}_2$ in E_n is a point

$$\overline{x} = \lambda_1\overline{x}_1 + \lambda_2\overline{x}_2 \ ; \ \lambda_1, \lambda_2 \geq 0 \ ; \ \lambda_1 + \lambda_2 = 1,$$

that is, $\overline{x} = \lambda\overline{x}_1 + (1-\lambda)\overline{x}_2, \ 0 \leq \lambda \leq 1$.

● **Definition 6.4.2 (Convex Set)**

A set $X \subseteq E^n$ is said to be convex if every convex combination of any two points of the set X is also in the set X.

In other words, a set $X \subseteq E^n$ is said to be convex if for any two points x_1, x_2 in X, the line segment, joining these points is also in the set X.

Thus, if X is a convex set, then every point $\overline{x}$, given by

$$\overline{x} = \lambda\overline{x}_1 + (1-\lambda)\overline{x}_2, \ 0 \leq \lambda \leq 1; \ \overline{x}_1, \overline{x}_2 \in X,$$

is also a point of the set X.

By convention, a set containing only one point is a convex set.

● **Remark :** A convex set may or may not be closed. It may or may not be strictly bounded.

Note : A set $A \subseteq E^n$ is strictly bounded if there exists a positive number r such that for every $\bar{a} \in A$, $|\bar{a}| < r$. A strictly bounded set lies inside a hypersphere of radius r with the centre at origin.

A Few Examples of Convex Set

(i) The set $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is a convex set. It is closed and bounded.

(ii) The set $Y = \{(y_1, y_2) : y_1^2 + y_2^2 < 1\}$ is a convex set. It is bounded but not closed.

(iii) The set $Z = \{(z_1, z_2) : z_1 > z_2, z_1 > 0, z_2 > 0\}$ is a convex set. It is neither closed nor bounded.

(iv) The set $A = \{(x_1, x_2) : x_2 - x_1 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ is a convex set. It is closed but not bounded.

(v) The set $B = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1, x_1^2 + x_2^2 \leq 4\}$ is not convex, though it is bounded.

The diagrams below should help to distinguish a convex set from a non-convex on in E^2 :

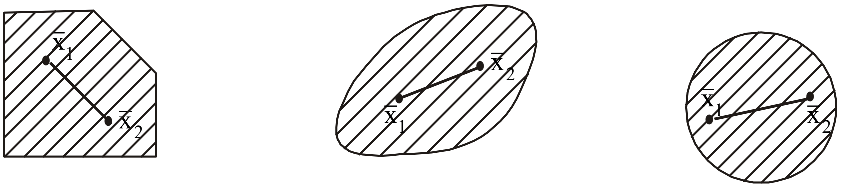


Figure 3.1 (Convex sets)

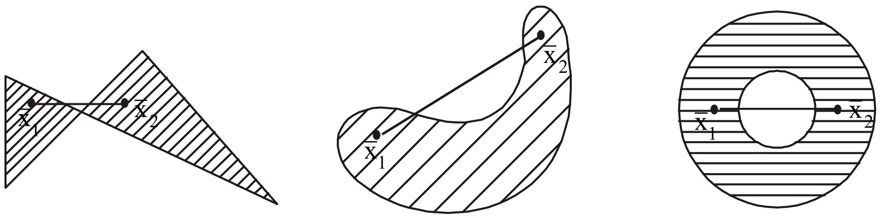


Figure 6.2 (Non-convex sets)

● Definition 6.4.3 (Extreme Point)

A point $\overline{x}$ in a convex set is said to be an extreme point if and only if there does not exist points $\overline{x}_1, \overline{x}_2 (\overline{x}_1 \neq \overline{x}_2)$ in the set such that $\overline{x} = \lambda\overline{x}_1 + (1-\lambda)\overline{x}_2$, where $0 < \lambda < 1$.

The definition stipulates that an extreme point cannot be between any other two points of the set.

● Remark :

(i) An extreme point of a convex set is a boundary point of the set.

(ii) A boundary point of a convex set may be an extreme point, but not all boundary points of a convex set are necessarily extreme points.

(iii) If a convex set contains only a single point, this point will be considered an extreme point of the set.

A few examples of extreme points of a convex set

(i) The line segment joining two points $\overline{x}_1, \overline{x}_2$ that is, the set x = $\{\overline{x} : \overline{x} = \lambda\overline{x}_1 + (1-\lambda)\overline{x}_2,\ 0 \leq \lambda \leq 1\}$ is a convex set. The points are its extreme points which are also boundary points.

(ii) The set $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is a convex set. All the boundary points of this set are extreme points which are infinite in number and they are the points $A = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$.

(iii) The set $X = \{(x_1, x_2) : x_2 - x_1 \leq 2,\ 0 \leq x_2 \leq 3,\ x_1 \geq 0\}$ is a convex set.

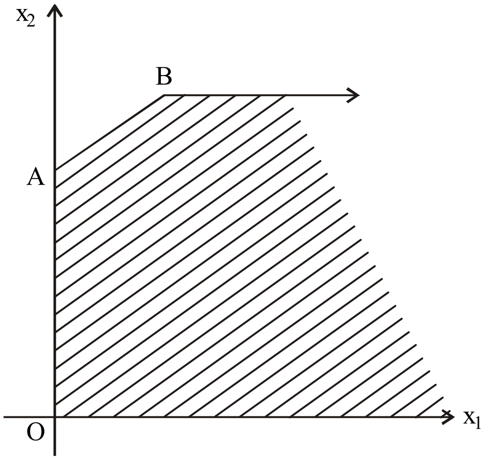


Figure 6.3 (Convex set)

It is geometrically shown in the adjoining figure. It has only three extreme points O, A, B. No other point is its extreme point. The set is closed but unbounded.

6.5 Some theorems on Convex Sets

● **Theorem 6.5.1 :** A hyperplane is a convex set.

Proof : Consider a hyperplane X = $\{\bar{x} : \bar{c}\,\bar{x} = \bar{z}\}$.

Let $\bar{x}_1, \bar{x}_2 \in X$. Then, $\bar{c}\,\bar{x}_1 = z$ and $\bar{c}\,\bar{x}_2 = z$

Now, we consider a convex combination of $\bar{x}_1, \bar{x}_2$ as

$$\bar{x}^* = \lambda\bar{x}_1 + (1-\lambda)\,\bar{x}_2, 0 \le \lambda \le 1$$

Then, we have

$$\begin{aligned}\bar{c}\,\bar{x}^* &= \bar{c}\left[\lambda\bar{x}_1 + (1-\lambda)\,\bar{x}_2\right] \\ &= \lambda\,(\bar{c}\,\bar{x}_1) + (1-\lambda)(\bar{c}\,\bar{x}_2) \\ &= \lambda\, z + (1 - \lambda)\, z = z.\end{aligned}$$

Thus, $\bar{x}^*$ is a point on the hyperplane $\bar{c}\,\bar{x} = z$, that is, $\bar{x}^* \in X$.

Therefore, every convex combination of any two points on the hyperplane is also on the hyperplane. Hence, a hyperplane is a convex set.

Remark : A hyperplane is a closed convex set.

● **Theorem 6.5.2 :** The intersection of two convex sets is also a convex set.

Proof : Let X_1 and X_2 be any two convex sets. Let X = $X_1 \cap X_2$.

If X contains only a single point, then automatically X is a convex set.

Now, let $\bar{x}_1, \bar{x}_2 \in X$. Then, we have $\bar{x}_1, \bar{x}_2 \in X_1$ and $\bar{x}_1, \bar{x}_2 \in X_2$.

Consider a convex combination of $\bar{x}_1, \bar{x}_2$ as

$$\bar{x} = \lambda\bar{x}_1 + (1-\lambda)\,\bar{x}_2,\ 0 \le \lambda \le 1.$$

Since X_1, X_2 are convex sets, then $\bar{x}$ belongs to both X_1 and X_2. Therefore, $\bar{x} \in X$. Hence, X is a convex set.

Note : If X_i (i = 1, 2, ..., m) is a convex set, then $X = \bigcap_{i=1}^{m} X_i$ is also a convex set.

Remark : (i) If X_1 and X_2 are two convex sets, then $X_1 \cup X_2$ may or may not be a convex set.

(ii) Any subset of a convex set may or may not be a convex set.

● **Theorem 6.5.3 :** Half spaces (closed or open) are convex set.

Proof : Consider a closed half-space X = $\{\overline{x} : \overline{c}\,\overline{x} \le z\}$. Let $\overline{x}_1, \overline{x}_2 \in X$.

Then, we have, $\overline{c}\,\overline{x}_1 \le z$ and $\overline{c}\,\overline{x}_2 \le z$.

Now, we consider a convex combination of $\overline{x}_1, \overline{x}_2$ as

$$\overline{x}^* = \lambda \overline{x}_1 + (1-\lambda)\ \overline{x}_2,\ 0 \le \lambda \le 1.$$

We have,
$$\begin{aligned}\overline{c}\,\overline{x}^* &= \lambda\ (\overline{c}\,\overline{x}_1) + (1-\lambda)(\overline{c}\,\overline{x}_2)\\ &\le \lambda\ z + (1-\lambda)\ z = z\end{aligned}$$

that is, $\overline{c}\,\overline{x}^* \le z$.

Thus, $\overline{x}^* \in X.$ Hence, X is a convex set.

Similarly, it can be shown that the half-spaces

$H_1 = \{\overline{x} : \overline{c}\,\overline{x} \ge z\}$, $H_2 = \{\overline{x} : \overline{c}\,\overline{x} < z\}$ and $H_3 = \{\overline{x} : \overline{c}\,\overline{x} > z\}$ are convex sets.

Remarks : Closed half-spaces are closed convex sets.

● **Theorem 6.5.4 :** The set of all convex combinations of a finite number of points in E^n is a convex set.

Proof : Consider the set of all convex combinations of a finite number of points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_m$ in E^n as

$$X = \left\{\overline{x} : \overline{x} = \sum_{i=1}^{m} \lambda_i \overline{x}_i, \lambda_i \ge 0\ (i = 1, 2, \cdots, m),\ \sum_{i=1}^{m} \lambda_i = 1\right\}$$

Let $\bar{u}, \bar{v} \in X$. Then, we have

$$\bar{u} = \sum_{i=1}^{m} \mu_i x_i, \mu_i \geq 0 \; (i = 1, 2, \cdots, m), \sum_{i=1}^{m} \mu_i = 1;$$

$$\bar{v} = \sum_{i=1}^{m} \nu_i x_i, \nu_i \geq 0 \; (i = 1, 2, \cdots, m), \sum_{i=1}^{m} \nu_i = 1.$$

Now, we consider a convex combination of $\bar{u}, \bar{v}$ as

$$\begin{aligned} \bar{w} &= \lambda \bar{u} + (1 - \lambda) \bar{v}, \; 0 \leq \lambda \leq 1 \\ &= \lambda \sum_{i=1}^{m} \mu_i \bar{x}_i + (1 - \lambda) \sum_{i=1}^{m} \nu_i \bar{x}_i \\ &= \sum_{i=1}^{m} \left[\lambda \mu_i + (1 - \lambda) \nu_i \right] \bar{x}_i \\ &= \sum_{i=1}^{m} w_i \bar{x}_i, \end{aligned}$$

where $w_i = \lambda \mu_i + (1 - \lambda) \nu_i \geq 0$ [$\because$ $0 \leq \lambda \leq 1$, $\mu_i \geq 0$, $\nu_i \geq 0$ for all i]

and $\sum_{i=1}^{m} w_i = \lambda \sum_{i=1}^{m} \mu_i + (1 - \lambda) \sum_{i=1}^{m} \nu_i = \lambda \times 1 + (1 - \lambda) \times 1 = 1.$

Thus, $\bar{w} = \sum_{i=1}^{m} w_i \bar{x}_i$, $w_i \geq 0 \; (i = 1, 2, \cdots, m)$, $\sum_{i=1}^{m} w_i = 1.$

Hence, $\bar{w} \in X$. Therefore, X is a convex set.

Since $\bar{w}$ is a convex combination of the finite number of points $\bar{x}_1, \bar{x}_2, ..., \bar{x}_m$ in E^n, so we conclude that the set of all convex combinations of finite number of points in E^n is convex.

6.6 Convex Cone, Convex Hull and Convex Polyhedron

● **Definition 6.6.1 (Cone)**

A non-empty subset $C \subset E^n$ is called a cone if for each $\bar{x} \in C$ and $\lambda \geq 0$, the point $\lambda \bar{x} \in C$.

● **Definition 6.6.2 (Convex Cone)**

A cone is called a convex cone if it is a convex set.

● **Example 6.6.1 :** If A be an m × n matrix, then the set of n vectors $\bar{x}$ satisfying $A\bar{x} \geq \bar{0}$ is a convex cone in E^n. It is a cone, because if $A\bar{x} \geq \bar{0}$, then $A(\lambda \bar{x}) \geq \bar{0}$ for non-negative λ. It is convex, because if $A\bar{x}_1 \geq \bar{0}$ and $A\bar{x}_2 \geq \bar{0}$, then

$$A\left[\lambda\bar{x}_1 + (1-\lambda)\bar{x}_2\right] \geq 0,\ 0 \leq \lambda \leq 1.$$

● **Definition 6.6.3 (Convex Hull)**

Let $A \subset E^n$. Then the intersection of all convex sets containing A, is called the Convex Hull of A and it is denoted by C(A).

In symbols, if $A \subset E^n$, then $C(A) = \cap W_i$, where for each i, $W_i \supset A$ and W_i is a convex set.

Since the intersections of the members of any family of convex sets is convex, it follows that C(A), the convex hull of A, is a convex set.

Now, for any set $A \subset E^n$, we have

(i) C(A) is a convex set, $A \subseteq C(A)$

and (ii) if $W \supset A$, be a convex set, then $C(A) \subset W$.

Thus, the convex hull of a set $A \subset E^n$ is the smallest convex set containing A.

● **Example 6.6.2 :** The convex hull of two given points $\bar{x}_1, \bar{x}_2$ is the line segment joining these two points. Thus, if $X = \{x_1, x_2\} \subseteq E^n$,

$$C(X) = \left\{\bar{x} : \bar{x} = \lambda\bar{x}_i + (1-\lambda)\bar{x}_2,\ 0 \leq \lambda \leq 1\right\}.$$

The convex hull of two points is a closed convex set.

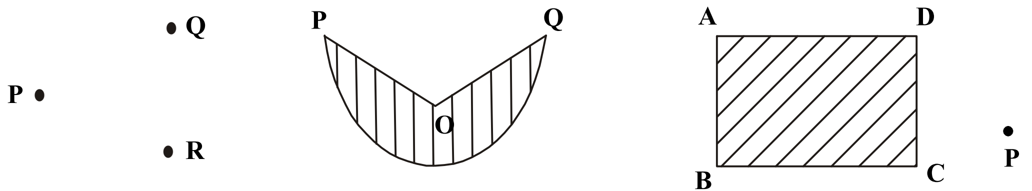
● **Example 6.6.3 :** The convex hull of three non-collinear points $\overline{x}_1, \overline{x}_2, \overline{x}_3$ is the triangular region with its vertices at these three points. This is the smallest convex set containing $\overline{x}_1, \overline{x}_2, \overline{x}_3$ and this set is closed.

● **Example 6.6.4 :** The convex hull of the set $A = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ is

$$C(A) = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}.$$

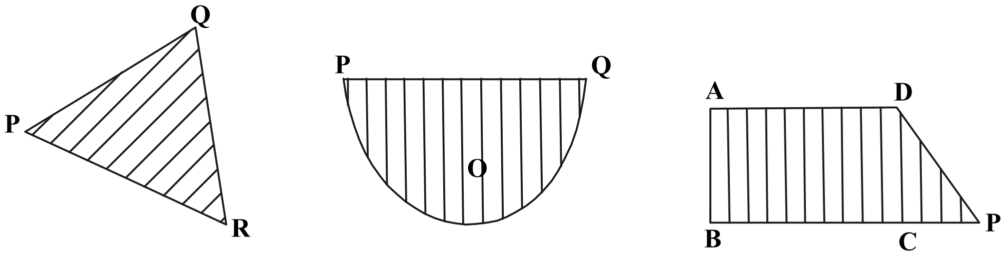
This is the smallest convex set containing all the points of A and it is closed.

The following diagrams should help to observe the relation of a set A to the convex hull C(A) :



(i) (ii) (iii)

Figure 6.4 (Set A)



(i) (ii) (iii)

Figure 6.5 (Convex Hull C(A))

● **Definition 6.6.4 : (Convex Polyhedron or Polytope)**

The convex hull of a finite number of points in E^n is called the convex polyhedron (or polytope) spanned by these points.

In other words, the set of all convex combinations of a finite number of points in E^n is called a convex polyhedron (or polytope) with vertices at these points.

Note : (i) The convex polyhydron spanned by m, points cannot have more than m extreme points, because any other point of the polyhedron is a convex combination of these points and hence cannot be an extreme point. Thus, its extreme points will be among these m points. However, not all these m points may be the extreme points; one or more of these m points may be interior points of the polyhedron.

(ii) Each point of the convex polyhedron can be written as a convex combination of its extreme points.

Remark : A convex polyhedron is a closed convex set with a finite number of extreme points, and every point of a convex polyhedron can be written as a convex combination of its extreme points.

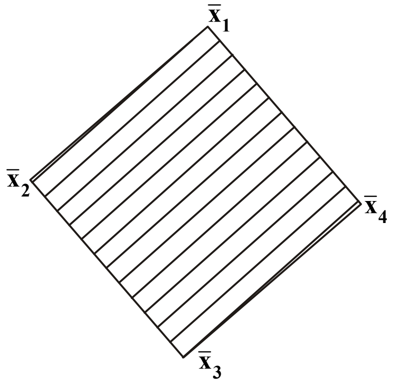


Figure 6.6 (X)

● **Example 6.6.5 :** The convex polyhedron spanned by four non-collinear points $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ in E^2 may be a closed convex set X as shown in Fig. 3.6, having four extreme points $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ only.

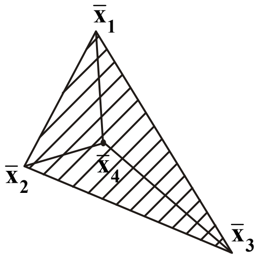


Figure 6.7 (Y)

● **Example 6.6.5 :** The convex polyhedron spanned by four points $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ in E^2 may be a closed convex set Y as shown in Fig. 3.7, having only three extreme points.

● **Example 6.6.6 :** In Fig 3.8, the set Z is the convex polyhedron in E^2 having three extreme points. This is the set of points lying on and within the triangle with vertices at $\bar{x}_1, \bar{x}_2$ and $\bar{x}_3$. Let $\bar{x}$ be any point inside the triangular region. First we draw a straight line from $\bar{x}_1$ through $\bar{x}$ to meet the opposite side at $\bar{v}$. Then

$$\bar{x} = \lambda_1 \bar{x}_1 + (1 - \lambda_1)\bar{v}, \quad 0 \le \lambda_1 \le 1.$$

But, $\bar{v} = \mu_1 \bar{x}_2 + (1 - \mu_1)\bar{x}_3$, $0 \le \mu_1 \le 1$; since Z is a strictly bounded closed convex set.

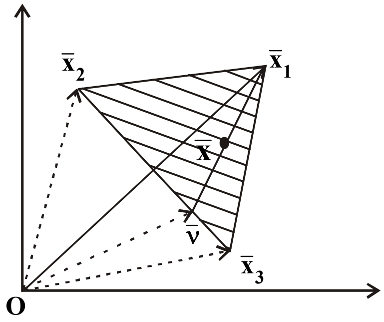


Figure 6.8 (Z)

Therefore,

$$\bar{x} = \lambda_1\bar{x}_1 + (1-\lambda_1)\left[\mu_1\bar{x}_2 + (1-\mu_1)\bar{x}_3\right]$$

$$= \lambda_1\bar{x}_1 + \lambda_2\bar{x}_2 + \lambda_3\bar{x}_3,$$

where $\lambda_1 \geq 0$, $\lambda_2 = (1 - \lambda_1)\mu_1 \geq 0$, $\lambda_3 = (1 - \lambda_1)(1 - \mu_1) \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Thus, $\bar{x} = \sum_{i=1}^{3} \lambda_i \bar{x}_i, \lambda_i \geq 0$ (i = 1, 2, 3), $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Therefore, any point inside the triangular region can be expressed as a convex combination of its vertices.

This example shows that if a strictly bounded closed convex set Z has a finite number of extreme points, then any point in Z can be written as a convex combination of the extreme points, that is, Z is the convex polyhedron of its extreme points.

● **Definition 6.6.5 (Simplex)**

A simplex in n-dimension is a convex polyhedron having exactly (n + 1) vertices.

A simplex in zero dimension is a point; in one dimension, it is a line segment; in two dimension, it is a triangular region, in three dimension, it is a volume bounded by a tetrahedron and so on.

6.7 A Few Example

● **Example 6.7.1 :** Show that $X = \{\bar{x} : |\bar{x}| \leq 2\}$ is a convex set.

Solution : Let $\bar{x}_1, \bar{x}_2 \in X$. Then we have,

$$|\bar{x}_1| \leq 2 \text{ and } |\bar{x}_2| \leq 2.$$

Let $\bar{u} = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2$, $0 \leq \lambda \leq 1$.

Now, $|\bar{u}| = |\lambda\bar{x}_1 + (1-\lambda)\bar{x}_2| \leq \lambda|\bar{x}_1| + (1-\lambda)|\bar{x}_2| \leq \lambda.2 + (1-\lambda).2 = 2$

i.e. $|\bar{u}| \le 2$.

Thus, $\bar{u} \in X$. Since $\bar{u}$ is a convex combination of two points $\bar{x}_1$ and $\bar{x}_2$ of X, so X is a convex set.

● **Example 6.7.2 :** Show that $X = \{(x_1, x_2) : 4x_1^2 + 9x_2^2 \le 36\}$ is a convex set.

Solution : Let $\bar{u} = (u_1, u_2)$ and $\bar{v} = (v_1, v_2)$ be two points in X. Then, we have

$$4u_1^2 + 9u_2^2 \le 36 \text{ and } 4v_1^2 + 9v_2^2 \le 36.$$

$$\begin{aligned} \text{Let} \quad \bar{w} &= \lambda\bar{u} + (1-\lambda)\,\bar{v},\ 0 \le \lambda \le 1 \\ &= \lambda\,(u_1, u_2) + (1-\lambda)\,(v_1, v_2) \\ &= (\lambda u_1 + (1-\lambda)\,v_1,\ \lambda u_2 + (1-\lambda)\,v_2) \\ &= (w_1, w_2), \text{ say.} \end{aligned}$$

$$\begin{aligned} \text{Now, } 4w_1^2 + 9w_2^2 &= 4\,\{\lambda u_1 + (1-\lambda)\,v_1\}^2 + 9\,\{\lambda u_2 + (1-\lambda)\,v_2\}^2 \\ &= \lambda^2(4u_1^2 + 9u_2^2) + (1-\lambda)^2\,(4v_1^2 + 9v_2^2) + 2\lambda\,(1-\lambda)\,(4u_1v_1 + 9u_2v_2) \\ &\le \lambda^2 \times 36 + (1-\lambda)^2 \times 36 + 2\lambda\,(1-\lambda)\,(4u_1v_1 + 9u_2v_2) \;...... (1) \end{aligned}$$

We know, $(x-y)^2 \ge 0 \Rightarrow 2xy \le x^2 + y^2$. Using this inequality, (1) becomes

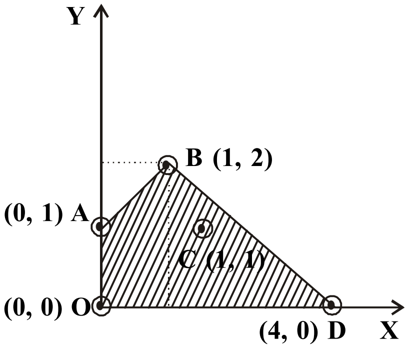
$$\begin{aligned} 4w_1^2 + 9w_2^2 &\le 36\lambda^2 + 36\,(1-\lambda)^2 + \lambda\,(1-\lambda)\,\{4\,(u_1^2 + v_1^2) + 9\,(u_2^2 + v_2^2)\} \\ &= 36\lambda^2 + 36\,(1-\lambda)^2 + \lambda\,(1-\lambda)\,\{(4u_1^2 + 9u_2^2) + (4v_1^2 + 9v_2^2)\} \\ &= 36\lambda^2 + 36\,(1-\lambda)^2 + \lambda\,(1-\lambda)\,(36 + 36) \\ &= 36\,\{\lambda^2 + (1-\lambda)^2 + 2\lambda\,(1-\lambda)\} \\ &= 36\,(\lambda + 1 - \lambda)^2 \\ &= 36 \end{aligned}$$

Thus, $4w_1^2 + 9w_2^2 \le 36$. So, $\bar{w} \in X$. Since $\bar{w}$ is a convex combination of two points $\bar{u}, \bar{v}$ of X, so X is a convex set.

Alternative approach : Since the set X represents an ellipse with all its boundary points and all interior points, so the line segment joining any two points of X must lie within X. So, X is a convex set.

● **Example 6.7.3 :** Determine the convex hull of the points (0, 0), (0, 1), (1, 2), (1, 1) and (4, 0).

Solution : In the adjoining figure, the points O (0, 0), A (0, 1), B (1, 2), C (1, 1) and D (4, 0) are plotted. The shaded region shown in the figure is the convex hull of the given points. It is noted here that the point C (1, 1) is the interior point of the convex hull.



● **Example 6.7.4 :** Give an example in each case :

(i) A convex polyhedron having only two extreme points.

(ii) A convex set having only four extreme points.

(iii) An unbounded convex set without any extreme point.

(iv) A convex set whose all the boundary points are its extreme points.

Solution : (i) The line segment joining two points (0, 0) and (1, 0) :

$$X = \{(x_1, x_2) : 0 \le x_1 \le 1, x_2 = 0\}.$$

(ii) The square and its interior with the vertices at O (0, 0), A (4, 0), B (4, 4) and C (0, 4) :

$$X = \{(x_1, x_2) : 0 \le x_1 \le 4, 0 \le x_2 \le 4\}.$$

(iii) In the x_1x_2-plane, the right side of the x_2-axis :

$$X = \{(x_1, x_2) : x_1 \ge 0\}.$$

(iv) The circle of radius 'a' units and its interior points :

$$X = \{(x_1, x_2) : x_1^2 + x_2^2 \le a^2\}$$

● **Example 6.7.5 :** Correct or justify :

(i) A line segment is a convex set.

(ii) Any convex set is a convex polyhedron.

Solution : (i) The statement is correct.

Justification : The line segment joining two points $\overline{x}_1, \overline{x}_2$ in E^n is the set of points given by

$$x = \left\{\overline{x} : \overline{x} = \lambda\overline{x}_1 + (1-\lambda)x_2,\ 0 \le \lambda \le 1\right\}$$

Let $\bar{u}, \bar{v} \in X$. Then, we have

$$\bar{u} = \lambda_1 \bar{x}_1 + (1-\lambda_1)\ \bar{x}_2,\ 0 \le \lambda_1 \le 1;$$

$$\bar{v} = \lambda_2 \bar{x}_1 + (1-\lambda_2)\ \bar{x}_2,\ 0 \le \lambda_2 \le 1;$$

Now, for μ ($0 \le \mu \le 1$),

$$\bar{w} = \mu\bar{u} + (1-\mu)\ \bar{v}$$

$$= \{\lambda_1\mu + \lambda_2(1-\mu)\}\ \bar{x}_1 + \{1-[\lambda_1\mu + \lambda_2(1-\mu)]\}\ \bar{x}_2$$

$$= \lambda\bar{x}_1 + (1-\lambda)\ \bar{x}_2,$$

where $\lambda = \lambda_1 \mu + \lambda_2 (1 - \mu)$ such that $0 \le \lambda \le 1$.

Thus, every convex combination of any two points in X is also in X. Hence, X is a convex set.

(iii) The statement is wrong.

Justification : A convex polyhedron spanned by k points cannot have more than k extreme points. Thus, a convex polyhedron must have a finite number of extreme points, whereas a convex set may have a finite or an infinite number of extreme points or no extreme point. That is why, a convex set may or may not be a convex polyhedron.

6.8 Summary

In this chapter, point set, convex set and their properties are discussed in detail. A few theorems also presented which are useful to find the optimal solution of a linear programming problem.

6.9 Exercise

1. Examine, whether each statement is true or false :

(i) All boundary points of a convex set are its extreme points.

(ii) Extreme points of a convex polyhedron are finite in number.

(iii) If a LPP has two distinct feasible solutions, then it has an infinite number of feasible solutions.

(iv) An optimal solution of a LPP is a basic feasible solution.

(v) The set $X = \{x_1, x_2\} : x_1^2 + x_2^2 \leq 1\}$ is a convex polyhedron.

2. Examine, whether the following sets are convex or not, and which are convex polyhedrons :

(i) $x = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}$

(ii) $x = \{(x_1, x_2) : x_1^2 + x_2^2 = 9\}$

(iii) $x = \{(x_1, x_2) : x_2 - 5x_1 = 3, x_1 \geq 0, x_2 \geq 0\}$

(iv) $x = \{(x_1, x_2) : x_1x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$

(v) $x = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1\}$

3. What is the convex polyhedron spanned by the points (i) (1, 0), (0, 1) (ii) (0, 0), (2, 0), (0, 2)

4. Find the extreme points, if any, of the following sets :

(i) $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 49\}$

(ii) $S = \{(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1\}$

5. Consider the set of points on the union of the half lines : $x = 0, y \geq 0$; $y = 0, x \geq 0$ on xy-plane. Verify whether the set is convex or not. Find the convex hull of the set.

Answers

1. (i) False (ii) True (iii) True (iv) True (v) False

2. (i) Convex, but not convex polyhedron.

(ii) Non-convex

(iii) Convex polyhedron

(iv) Non-convex

(v) Non-convex

3. (i) $x = \{(x_1, x_2) : 1 - x_2 = x_1 = \lambda, 0 \le \lambda \le 1\}$

   (ii) $x = \{(x_1, x_2) : x_1 + x_2 \le 2, x_1 \ge 0, x_2 \ge 0\}$

4. (i) $x = \{(x_1, x_2) : x_1^2 + x_2^2 \le 49\}$

   (ii) $x = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$

5. Non-convex. Positive quadrant of the xy-plane is the convex hull.

6.10 Multiple Choice Questions (MCQ)

1. Which of the following is not a convex set?

   (a) $\left\{\overline{x} : |\overline{x}| \le 2\right\}$ (b) $\{(x_1, x_2) : x_1 + 2x_2 = 7\}$

   (c) $\{(x_1, x_2) : 3x^2 + 4x_2^2 \ge 12\}$ (d) $\{(x_1, x_2) : x_1 \le 5, x_2 \ge 3\}$

2. Which of the following is not a convex polyhedron?

   (a) $\{(x, y) : x + 2y \le 3, 2x - y = 2, x \ge 0, y \ge 0\}$

   (b) $\{(x, y) : x^2 + y^2 \le 4\}$

   (c) $\{(x, y) : 9x^2 + 16y^2 \le 144\}$

   (d) $\{(x, y) : 2x^2 + 5y^2 = 10\}$

3. The set $S = \{(x_1, x_2) : 4x + 5y = 20\}$ has

   (a) no extreme point (b) exactly two extreme points

   (c) more than two extreme points (d) infinitely many extreme points

4. For the hyperplane $2x_1 + 3x_2 + x_3 - 3x_4 = 13$, which pair of points lie on the opposite sides ?

   (a) (0, 4, 1, 0), (3, –1, 10, 0) (b) (1, 0, 2, –2), (2, 2, –1, –4)

   (c) (1, 0, 2, –2), (1, 1, 8, 0) (d) (7, –1, 2, 0), (2, 2, –1, –4)

Answers

1. (c) 2. (a) 3. (a) 4. (b)

Unit - 7 ❑ Separating and Supporting Hyperplanes Some Theorems on LPP related to Covex sets

Structure

7.0 Objective

After studying this chapter, the reader should be able to know

- the idea of separting and supporting hyperplans
- some theorems on LPP related to convex sets

7.1 Introduction

In this unit the definition and some properties of separating and supporting hyperplanes are discussed. Also some theorems on convex sets are presented which are useful to find the optimal solution of a LPP.

7.2 Separating and Supporting Hyperplanes

● **Definition 7.2.1 (Separating Hyperplane)**

A hyperplane $H = \{\overline{x} : \overline{c}\,\overline{x} = z\}\ \underline{c}\ E^n$ is called a separating hyperplane if it separates two convex sets S and T such that

$$S\ \underline{c}\{\overline{x} : \overline{c}\,\overline{x} \ \geq\ z\} \text{ and } T\ \underline{c}\ \{\overline{x} : \overline{c}\,\overline{x} \ \leq\ z\}$$

This hyperplane H is also called a bounding hyperplane. A hyperplane H will be called a strictly separating hyperplane if

$$S\ \underline{c}\{\overline{x} : \overline{c}\,\overline{x} \ \geq\ z\} \text{ and } T\ \underline{c}\ \{\overline{x} : \overline{c}\,\overline{x} \ \leq\ z\}$$

● **Definition 7.7.2 (Supporting Hyperplane)**

Let $S \subset E^n$ be a non-empty closed convex set and $\overline{u} \in S$ be a boundary point. Then a hyperplane $H = \{\overline{x} : \overline{c}\,\overline{x} = z\}$ is called a supporting hyperplane of S at $\overline{u}$ if

(i) $\overline{c}\,\overline{u} = z$

and (ii) $S \subset H_1$ or $S \subset H_2$,

where $H_1 = \{\overline{x} : \overline{c}\,\overline{x} \geq z\}$ and $H_2 = \{\overline{x} : \overline{c}\,\overline{x} \leq z\}$.

● **Theorem 7.7.1 :** Let $S \subset E^n$ be a closed convex set and $\overline{y} \notin S$. Then there exists a hyperplane containing $\overline{y}$ such that S is contained in one of the open half spaces generated by the hyperplane.

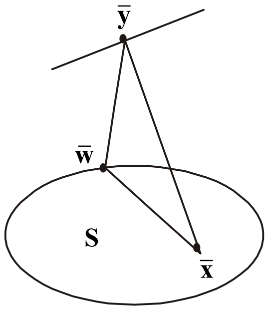


Figure 7.2 (Separating hyperplane)

Proof : Since S is a closed convex set and $\overline{y} \notin S$, so there exists a point $\overline{w} \in S$ such that

$$|\overline{w} - \overline{y}| = \min_{\overline{x} \in S} |\overline{x} - \overline{y}| \ \text{.............} \ \text{(i)}$$

Let $\overline{u} \in S$. Since S is a convex set, so for $0 \leq \lambda \leq 1$, we have

$$\lambda\overline{u} + (1 - \lambda)\overline{w} \in S. \ \text{...........} \ \text{(ii)}$$

From (i) and (ii), we have

$$|\lambda\bar{u}+(1-\lambda)\,\bar{w}-\bar{y}|\geq|\bar{w}-\bar{y}|$$

or, $$|(\bar{w}-\bar{y})+\lambda\,(\bar{u}-\bar{w})|^2\geq|\bar{w}-\bar{y}|^2$$

or, $$\lambda^2\,|\bar{u}-\bar{w}|^2+2\lambda\,(\bar{w}-\bar{y})(\bar{u}-\bar{w})\geq 0\cdot$$

Taking $\lambda \to 0$ and $\bar{c} = \bar{w}-\bar{y}$ such that $|\bar{c}|=0,$ we get

$$(\bar{w}-\bar{y})(\bar{u}-\bar{w})\geq 0$$

or, $$\bar{c}\,(\bar{u}-\bar{w})\geq 0$$

or, $$\bar{c}\{(\bar{u}-\bar{y})-(\bar{w}-\bar{y})\}\geq 0$$

or, $$\bar{c}(\bar{u}-\bar{y})\geq\bar{c}(\bar{w}-\bar{y})=|\bar{c}|^2\,.$$

So, putting $\bar{c}\,\bar{y}=z$ we get, $\bar{c}\,\bar{u}>z$. Thus, $\bar{y}$ lies on the hyperplane $\bar{c}\,\bar{x}=z$ and for all $\bar{u}\in S$, $\bar{c}\,\bar{u}>z$.

7.3 A Few Theorems on LPP related to convex sets

● **Theorem 7.3.1 :** The set of all feasible solutions of a LPP is a convex set.

Proof : Let X be the set of all feasible solutions of a LPP whose constraints are $A\bar{x}=\bar{b},\ \bar{x}\geq\bar{0}$.

Let $\bar{x}_1,\bar{x}_2\in X.$ Then we have,

$$A\bar{x}_1=\bar{b},\ \bar{x}_1\geq\bar{0}\ \ ;\ \ A\bar{x}_2=\bar{b},\ \bar{x}_2\geq\bar{0}\,.$$

Let $\bar{x}_3=\lambda\bar{x}_1+(1-\lambda)\,\bar{x}_2,\ 0\leq\lambda\leq 1.$

Now, $A\bar{x}_3=\lambda A\bar{x}_1+(1-\lambda)\,A\bar{x}_2=\lambda\bar{b}+(1-\lambda)\bar{b}=\bar{b}.$

Since $\bar{x}_1\geq\bar{0},\ \bar{x}_2\geq\bar{0}$ and $0\leq\lambda\leq 1$, so $\bar{x}_3\geq\bar{0}$.

Hence, $\overline{x}_3 \in X$. Since $\overline{x}_3$ is a convex combination of any two points of X, so X is a convex set.

Note : If a LPP has two distinct feasible solutions, then it has an infinite number of feasible solutions.

● **Theorem 7.3.2 :** The optimal hyperplane to a LPP is a supporting hyperplane to the convex set of all feasible solutions to it.

Proof : Consider a LPP in its standard form :

Maximize $z = \overline{c}\,\overline{x}$

subject to $A\overline{x} = \overline{b},\ \overline{x} \geq \overline{0}.$

We know that the set of all feasible solutions X to a LPP is a closed convex set and the objective function $\overline{c}\,\overline{x} = z$ is a hyperplane.

Now, we move this hyperplane parallel to itself over the convex region of feasible solutions until z is made as large as possible so that the hyperplane contains at least one point of the feasible region if it is assumed that z attains a finite maximum. The hyperplane $\overline{c}\,\overline{x} = z_0$ corresponding to the maximum value z_0 of z is termed as the optimal hyperplane.

No point on $\overline{c}\,\overline{x}_0 = z_0$ is an interior point of X. We assume the contrary that $\overline{c}\,\overline{x}_0 = z_0$ where $\overline{x}_0$ is an interior point of X. Then, there exists an ε-neighbourhood of $\overline{x}_0$ which consists of all points of X. Then the point

$$\overline{x}_1 = \overline{x}_0 + \frac{\varepsilon}{2}.\frac{\overline{c}'}{|\overline{c}|} \in X$$

and
$$z_1 = \overline{c}\,\overline{x}_1 = \overline{c}\,\overline{x}_0 + \frac{\varepsilon}{2}.\frac{\overline{c}\,\overline{c}'}{|\overline{c}|} = z_0 + \frac{\varepsilon}{2}|\overline{c}|.$$

Hence, $z_1 = \overline{c}\,\overline{x}_1 > z_0$.

Thus, a point $\overline{x}_1 \in X$ gives the higher value of z which contradicts the fact that z_0 is the maximum value of z. Therefore, $\overline{x}_0$ cannot be an interior point of X. So, $\overline{x}_0$ is a boundary point of X. Thus, $\overline{c}\,\overline{x} = z_0$ is a hyperplane contaning a boundary point of X. Also, if

$\bar{u} \in X$, then $\bar{c}\,\bar{u} = z \le z_0$, that is, $\bar{c}\,\bar{u} \le z_0$. Therefore, the optimal hyperplane $\bar{c}\,\bar{x} = z_0$ is the supporting hyperplane at $\bar{x}_0$.

● **Theorem 7.3.3 :** A basic feasible solution to a LPP corresponds to an extreme point of the convex set of feasible solutions.

Proof : Consider a LPP :

Maximize $z = \bar{c}\,\bar{x}$

subject to $A\bar{x} = \bar{b},\ \bar{x} \ge \bar{0}.$

where $A = (a_{ij})_{m \times n}$ $(m < n)$, $\bar{b} = [b_1, b_2, ..., b_m]$, $\bar{c} = (c_1, c_2, ..., c_n)$ and $\bar{x} = [x_1, x_2, ..., x_n]$.

Let $\bar{x}^*$ be a basic feasible solution of this LPP. Without any loss of generality, we can assume that first m-components of $\bar{x}^*$ are non-zero and remaning (n – m) components are zero.

Let $\bar{x}^* = [x_1, x_2, ..., x_m, 0, 0, ..., 0]$. Let the column vectors of A associated with the basic variables be $\bar{a}_1, \bar{a}_2, ..., \bar{a}_m$. Then $\bar{a}_1, \bar{a}_2, ..., \bar{a}_m$ are linearly independent and the constraints of LPP can be written as

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_m x_m = \bar{b}\,.$$

If $\bar{x}^*$ is not an extreme point of the convex set of feasible solutions of the LPP, then $\bar{x}$ can be expressed as a convex combination of two other feasible solutions $\bar{u}$ and $\bar{v}$ of the LPP. Then, we have

$$\bar{x}^* = \lambda\bar{u} + (1-\lambda)\bar{v},\ \ 0 < \lambda < 1.$$

Since all the components of $\bar{x}^*$ are non-negative with (n – m) components of it being zero, so $\bar{u}$ and $\bar{v}$ must also have (n – m) zero components. So, let

$$\bar{u} = (u_1, u_2, ..., u_m, 0, 0, ..., 0)$$

$$\text{and} \quad \bar{v} = (v_1, v_2, ..., v_m, 0, 0, ..., 0).$$

Since $\bar{u}$ and $\bar{v}$ are two feasible solutions of the LPP, we must have

$$u_1\bar{a}_1 + u_2\bar{a}_2 + \cdots + u_m\bar{a}_m = \bar{b}$$

and $$\nu_1\bar{a}_1 + \nu_2\bar{a}_2 + \cdots + \nu_m\bar{a}_m = \bar{b}$$

Subtracting these two relations we get

$$(u_1 - \nu_1)\bar{a}_1 + (u_2 - \nu_2)\bar{a}_2 + \cdots + (u_m - \nu_m)\bar{a}_m = \bar{0}.$$

Since $\bar{a}_1, \bar{a}_2, ..., \bar{a}_m$ are linearly independent, so

$$u_1 - \nu_1 = 0,\ u_2 - \nu_2 = 0, ..., u_m - \nu_m = 0$$

i.e. $u_1 = \nu_1,\ u_2 = \nu_2,\ .....,\ u_m = \nu_m$

i.e. $u_j = \nu_j$ for $j = 1, 2, ..., m$.

Thus, it is not possible to express $\bar{x}^*$ as a convex combination of two other feasible solutions of the LPP. Hence, $\bar{x}^*$ is an extreme point of the convex set of the feasible solutions of the LPP.

● **Theorem 7.3.4 :** Every extreme point of the convex set of feasible solutions of the system $A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}$ corresponds to a basic feasible solution.

Proof : Let X be the set of all feasible solutions of the system $A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}$. Let $\bar{x}^* = (x_1, x_2, ..., x_n)$ be an extreme point of X. Now, we are to show that $\bar{x}^*$ is a basic feasible solution, that is, we are to show that the column vectors of A which are associated to the non-zero components of $\bar{x}^*$ are linearly independent and at most m of the x_j's are positive.

Without any loss of generality, let as assume that first m components of $\bar{x}^*$ be positive. If $\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_m$ be the associated column vectors of A, then

$$x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 + \cdots + x_m\bar{\alpha}_m = \bar{b}. \quad .... (1)$$

If possible, let $\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_m$ are linearly dependent. Then there exists $\lambda_1, \lambda_2, ...,$ λ_m, not all zero, such that

$$\lambda_1\bar{\alpha}_1 + \lambda_2\bar{\alpha}_2 + \cdots + \lambda_m\bar{\alpha}_m = \bar{0}. \quad .... (2)$$

Using (1) and (2), we can find the following two relations :

$$\sum_{j=1}^{m}\left(x_j + k\lambda_j\right)\bar{\alpha}_j = \bar{b} \qquad .... (3)$$

and $$\sum_{j=1}^{m}\left(x_j - k\lambda_j\right)\bar{\alpha}_j = \bar{b}. \qquad .... (4)$$

Thus, by proper choice of k, two feasible solutions having m non-zero components of the system $A\bar{x} = \bar{b}$ can be taken as

$$\bar{u} = (x_1 + k\lambda_1, x_2 + k\lambda_2, ..., x_m + k\lambda_m, 0, 0, ..., 0)$$

and $$\bar{v} = (x_1 - k\lambda_1, x_2 - k\lambda_2, ..., x_m - k\lambda_m, 0, 0, ..., 0).$$

If we choose k such that

$$0 < k < \min\left\{\frac{x_j}{|\lambda_j|}, \lambda_j \neq 0\right\},$$

then $\bar{u}, \bar{v}$ are feasible solutions.

Now, we see that $\bar{x}^*$ can be expressed as a convex combination of two feasible solutions $\bar{u}$ and $\bar{v}$ as

$$\bar{x}^* = \frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}.$$

So, by definition of an extreme point, $\bar{x}^*$ can never be an extreme point of the convex set X and hence, there is a contradiction.

Thus, $\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_m$ must be linearly independent. Again, every set of (n + 1) vectors in a n-dimentional space is a linearly dependent set. Hence, by the above result together with the fact that each $\bar{\alpha}_j$ has m components, we can not have more than m positive x_j.

Thus, every extreme point of the convex set of feasible solutions of the system $A\bar{x} = \bar{b}$ corresponds to a basic feasible solution.

● **Theorem 7.3.5 :** If any LPP admits of an optimal solution, then the objective function assumes that optimum value at an extreme point of the convex set generated by the set of all feasible solutions of the LPP.

Proof : Consider a LPP in its standard form as :

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}.$$

Let X be the convex set of all feasible solutions of the given LPP.

Let $\{\bar{x}_1, \bar{x}_2, ..., \bar{x}_k\}$ be a finite set of extreme points of the convex set X. Also, let the set X be strictly bounded. Now, let $\bar{x}_m$ be the optional solution of the LPP. If it can be shown that $\bar{x}_m$ is an extreme point of the convex set of feasible solutions, then the theorem is proved.

If possible, let $\bar{x}_m$ be an optimal solution, but not an extreme point. Then $z_m = \bar{c}\,\bar{x}_m$ is the optimal value of the objective function z.

Since, by assumption, $\bar{x}_m$ is not an extreme point, so it can be expressed as a convex combination of the extreme points. Then, we have

$$\bar{x}_m = \sum_{i=1}^{k} \lambda_i \bar{x}_i,\quad \lambda_i \geq 0\ (i = 1, 2, ..., k),\quad \sum_{i=1}^{k} \lambda_i = 1.$$

$$\text{Therefore, } z_m = \bar{c}\,\bar{x}_m = \bar{c}\sum_{i=1}^{k}(\lambda_i x_i) = \sum_{i=1}^{k}\lambda_i(\bar{c}\,\bar{x}_i)$$

If $\bar{c}\,\bar{x}_p = \underset{i}{\text{Max}}\{\bar{c}\,\bar{x}_i\}$, where $\bar{x}_p$ is an extreme point, then replacing each $\bar{c}\,\bar{x}_i$ by $\bar{c}\,\bar{x}_p$, we have

$$z_m \leq \sum_{i=1}^{k}\lambda_i(\bar{c}\,\bar{x}_p) = \bar{c}\,\bar{x}_p\sum_{i=1}^{k}\lambda_i = \bar{c}\,\bar{x}_p = z_p \text{ (say).}$$

Thus, $z_m \leq z_p$ which contradicts the assumption that z_m is the optimal value of the objective function.

Hence, $\overline{x}_m$ must be an extreme point of X.

● **Theorem 7.3.6 :** If the objective function of a LPP assumes its optimal value at more than are extreme points, then every convex combination of these extreme points also gives the optimal value of the objective function.

Proof : Let any objective function z = $\overline{c}\,\overline{x}$ assumes its optimal value z* at the extreme points $\overline{x}_1, \overline{x}_2, ..., \overline{x}_k$ of the convex set of feasible solutions of the corresponding LPP. Then, we have

$$z^* = \overline{c}\,\overline{x}_1 = \overline{c}\,\overline{x}_2 = \cdots = \overline{c}\,\overline{x}_k.$$

Now, let $\overline{x}^*$ be any convex combination of the extreme points $\overline{x}_1, \overline{x}_2, ..., \overline{x}_k$. So

$$\overline{x}^* = \lambda_1\overline{x}_1 + \lambda_2\overline{x}_2 + \cdots + \lambda_k\overline{x}_k,$$

Where $\lambda_i \geq 0$ (i = 1, 2, - - - k), $\sum_{i=1}^{k}\lambda_i = 1$

Now $\overline{c}\,\overline{x}^* = \overline{c}\,(\lambda_1 x_1 + \lambda_2 x_2 + \text{--} + \lambda_k x_k)$

$$= \lambda_1\,(\overline{c}\,\overline{x}_1) + \lambda_2\,(\overline{c}\,\overline{x}_2) + - - - + \lambda_k\,(\overline{c}\,\overline{x}_k)$$

$$= \lambda_1 z^* + \lambda_2 z^* + - - - + \lambda_k z^*$$

$$= z^* \quad \left(\because \sum_{i=1}^{k}\lambda_i = 1\right)$$

Thus $\overline{x}^*$ is also an optimal solution to the LPP.

Remark : If a LPP has an optimal solution then it has either a unique solution or it has infinite number of optimal solutions.

7.4 A few Examples

● **Example 7.4.1 :** Find the extreme point(s) of the convex set feasible solutions to the equations.

$$2x_1 + 6x_2 + 2x_3 = 3$$

$$6x_1 + 4x_2 + 4x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

Solution : we know that every extreme point of the convex set of feasible solition of the system $A\overline{x} = \overline{b}, \overline{x} \geq 0$ corresponds to a basic feasible solution.

Thus, we have to find the basic feasible solution of the system of equations

$$\overline{a}_1 \overline{x}_1 + \overline{a}_2 \overline{x}_2 + \overline{a}_3 \overline{x}_3 = \overline{b}$$

Where $\overline{a}_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \overline{a}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \overline{a}_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

and $\overline{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

We write $B_1 = [\overline{a}_1, \overline{a}_2] = \begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}$

$$\therefore |B_1| = 8 - 36 = -28 (\neq 0)$$

$$B_2 = [\overline{a}_1, \overline{a}_3] = \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix} \therefore |B_2| = -4 (\neq 0)$$

$$B_3 = [\overline{a}_1, \overline{a}_3] = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix} \therefore |B_3| = 16 (\neq 0)$$

Sin u all the square matices B_1, B_2, B_3 are nonsingular Therefore any two columns vectors of $\overline{a}_1, \overline{a}_2, \overline{a}_3$ are linearly independent. Hence the system has 3 basic solutions.

Hence the vector

$$\overline{x}_{Bi} = B_i^{-1} \overline{b}$$

for i = 1, 2, 3

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_{B1} = B_1^{-1} \overline{b} = \frac{1}{-28} \begin{pmatrix} 4 & -6 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = x_{B_2} = B_2^{-1}\bar{b} = \frac{1}{-4}\begin{pmatrix} 4 & -2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ \frac{7}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = x_{B_3} = B_3^{-1}\bar{b} = \frac{1}{16}\begin{pmatrix} 4 & -2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Hence the basic solutions or

[0, ½, 0], [-2, 0, $\frac{7}{2}$], [0, ½, 0]

But as the 2nd solution contains a negative value so it is net feasible. Thus the basic feasible solutions ie. the extreme point of the given problem is only [0, ½, 0].

● **Example 7.4.2 :** If the convex set of the feasible solution of a LPP :

Optimize Z = $\bar{c}\,\bar{x}$

Subject to A$\bar{x}$ = $\bar{b}$, $\bar{x} \geq 0$

is a convex polyhedram then prove that at least one of the extreme points gives an optimal solution.

Hence find the maximum and minimum values of the objective function z = x + 4y if it is given that the convex set of feasible solution of the corresponding LPP is a strictly bounded convex polyhedram with extreme points (1, 0), (5, 0), (2, 3), (0, 4), (0, 2).

Solution : Let us consider the maximization problem. The minimization can be treated in a similar way.

Suppose $\bar{x}_1, \bar{x}_2, ...\bar{x}_p$ are the extreme points of the convex set of feasible solutions of the LPP :

Max. $z = \overline{c}\,\overline{x}$

Subject to $A\overline{x} = \overline{b},\ \overline{x} \geq 0$

Let $\overline{x}_k$ be the extreme point among $\overline{x}_1, \overline{x}_2, ...\overline{x}_p$ at which the objective function Z attains its maximum value z_k (say).

$$\therefore z_k = \text{Max}\ \left\{\overline{c}\,\overline{x}_1,\ \overline{c}\,\overline{x}_2, - - - \overline{c}\,\overline{x}_k, - - \overline{c}\,\overline{x}_p\right\}$$

$$= \overline{c}\,\overline{x}_k$$

Since convex set of feasible solution of the given LPP is a convex is aconvex polyhedron and any point of a convex polyhedron can be expressed as a convex combination of its extreme points hence if $\overline{x}_o$ be any point on the convex set of feasible solution and z_o be the value of the objective function at that point then zo = $\overline{c}\,\overline{x}_o$ and $\overline{x}_o$ can be expressed as a convex combination of $\overline{x}_1,\ \overline{x}_2, - - \overline{x}_p$ so that

$$\overline{x}_o = \sum_{i=1}^{p} \lambda_i\ \overline{x}_1, \lambda_i\ \geq\ 0\ (i = 1, 2, - - p)$$

$$\text{and}\ \sum_{i=1}^{p} \lambda_i = 1$$

$$\text{Thus}\ \ z_o = \overline{c}\,\overline{x}_o = \overline{c}\sum_{i=1}^{p} \lambda_i\ \overline{x}_i$$

$$= \sum_{i=1}^{p} \lambda_i\ (\overline{c}\ \overline{x}_i)\ \sum_{i=1}^{p} \lambda_i (\overline{c}\ \overline{x}_k)$$

$$= \overline{c}\,\overline{x}_k$$

$$\therefore\ z_o \leq\ z_k$$

This shows the maximum value of z is attained at an extreme point.

In a similar way we can prwe the some result minimization problem, so the result is valid for any optimization problem.

2nd part : Since it is given that the feasible solution of the given LPP is a convex polyhedron them at least one of the extreme points give an optimal solution. So we now find the value of the objective function z = x + 4y at different extreme points.

We have $(z)_{at(1,0)} = 1,\ (z)_{at(5,0)} = 5$

$(z)_{at(2,3)} = 14,\ (z)_{at(0,4)} = 16$

$(z)_{at\ (0,2)} = 8,$

Hence maximum value of z = 16 at x = 0, y = 4 and minimum value of z = 1 at x = 0, y = 4 and minimum value of z = 1 at x = 1, y = 0

7.5 Summary

In this unit separating and supporting hyperplanes and their properties are discussed. A few theorems on LPP which are related to convex sets are also presented. These theorems are useful to find optimal solution of a LPP.

7.6 Exercise

1. Given that the following LPP

Minimize $Z = 2x_1 - 3x_2 + x_4$

Subject to $3x_1 + 2x_2 + x_3 = 15$

$2x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

has an optimal solution. Find the minimum value of Z and the corresponding value of the solution set. Find also the maximum value of $Z = 2x_1 - 3x_2 + x4$ subject to the same constraints and show that in this case the problem has more than one optimal solutions.

2. The convex set of feasiable solution of a LPP is bounded from be;ow only and it has three extreme points (1, 1), (3, 2) and (4, 4). The objectivefunctions $2x_1 - 3x_2$ is know to have a finite maximum. Find the maximum value of the objective function and show that the given LPP have infinite number of optimal solutions.

3. If the LPP

Max $Z = 9x1 + 7x2$

Subject to $x_1 + 2x_2 \leq 7$

$$x_1 - x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

has an optimal solution. Find the maximum value of A and the corresponding value of the solution set.

Answers

1. Min Z = – 6 for the solution set [0, 2, 11, 0]
2. Min Z = 8 for the solution set [4, 0, 3, 0]

   or for the solution set [0, 0, 15, 8]
3. Max Z = 52 at $x_1 = 5$, $x_2 = 1$

7.7 Multiple Choice Questions (MCQ)

1. Every basic feasible solution of a convex set of feasible solutions is

   (a) an interior point

   (b) a boundary point but not an extreme point

   (c) an extreme point

   (d) none of these
2. Consider the three sets in E^2

   $H = \{(x, y) : y = 0\}$, $s = \{(x, y) : y \leq 0\}$

   $T = \{(x, y) : y \geq \frac{1}{x}, x > 0\}$. Then

   (a) H, S, T all are half spaces

   (b) exactly two sets out of three sets H, S, T are convex sets.

   (c) H is a strictly separating hyperplane of two convex sets S and T.

   (d) H is a separating hyperplane but not strictly separating hyperplane of two convex sets S and T.
3. The Convex set of feasible solution of a LPP is strictly bounded convex polyhedron with extreme points (1, 0), (4, 0), (3, 1), (0, 2), (0, 1).
   The maximum value of the objective function is

(a) 12 (b) 12
(c) 15 (d) 20

Answers

1. (c) 2. (d) 3. (b)

Unit - 8 ❑ Fundamental Properties of Simplex Method

Structure

8.0 Objective

After studying this chapter, the reader should be able to know

- understand the fundamental theorem of LPP
- know how a basic feasible solution can be found from a feasible solution of a LPP.

8.1 Introduction

The simplex method is an iterative procedure which either solves a LPP in a finite number of steps or gives an indication that there is an unbounded solution to the LPP. In this chapter we shall discuss the theory behind this method. Mainly the fundamental theorem and properties of the solution to the LPP will be proved and discussed with the help of examples.

8.2 Fundamental Theorem of LPP

● **Theorem 8.2.1 (Fundamental Theorem of LPP)**

If any linear programming problem admits an optimal solution, then at least one basic feasible solution must be optimal.

Proof : Let $\bar{x}^*$ be an optimal solution of a LPP given by

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0},$$

where $\bar{x}$ = [x_1, x_2, ..., x_n], A = $(a_{ij})_{m\times n}$ (m < n), $\bar{c}$ = (c_1, c_2, ..., c_n) and $\bar{b}$ = [b_1, b_2, ..., b_n].

Without any loss of generality, we assume that first p-components of $\bar{x}^*$ are non-zero positive numbers and the remaning (n – p) components are zero.

Thus, $\bar{x}^*$ = [x_1, x_2, ..., x_p, 0, 0, ..., 0], $x_j \geq 0$ (j = 1, 2, ..., p].

For this optimal solution, the constraints of the LPP can be written as

$$\bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 + ... + \bar{\alpha}_p x_p = \bar{b}, \text{ .... (1)}$$

where $\bar{\alpha}_1, \bar{\alpha}_2 ...., \bar{\alpha}_p$ are first p column vectors of A.

Also, $$z^* = z_{\max} = \sum_{j=1}^{p} c_j x_j \text{ .... (2)}$$

Now, if $\bar{\alpha}_1, \bar{\alpha}_2 ...., \bar{\alpha}_p$ are linearly independent, then $\bar{x}^*$ is a basic feasible solution of the LPP. If p = m, then the basic feasible solution will be non-degenerate and if p < m, then the basic feasible solution is degenerate with (m – p) of variables being zero.

In this case, the theorem is obvious.

But if $\bar{\alpha}_1, \bar{\alpha}_2 ...., \bar{\alpha}_p$ are not linearly independent, then there exists at least are $\lambda_j \neq 0$ (j = 1, 2, ..., p) such that

$$\lambda_1 \bar{\alpha}_1 + \lambda_2 \bar{\alpha}_2 + ... + \lambda_p \bar{\alpha}_p = \bar{0} \text{ .... (3)}$$

Let at least one $\lambda_j > 0$. If the non-zero λ_j be not positive, it can be reduced to positive by multiplying (3) by (–1). Also let

$$\beta = \max_{1\le j\le p}\left\{\frac{\lambda_j}{x_j},\ \lambda_j < 0\right\}. \quad .... (4)$$

For such choice of β, it is always positive.

Now, dividing (3) by β and then subtracting from (1), we get

$$\left(x_1-\frac{\lambda_1}{\beta}\right)\bar{\alpha}_1+\left(x_2-\frac{\lambda_2}{\beta}\right)\bar{\alpha}_2+.....+\left(x_p-\frac{\lambda_p}{\beta}\right)\bar{\alpha}_p=\bar{b}$$

and hence

$$\bar{x}_1=\left[x_1-\frac{\lambda_1}{\beta},\ x_2-\frac{\lambda_2}{\beta},.....,x_p-\frac{\lambda_p}{\beta},0,0,.....,0\right] \quad .... (5)$$

is also a solution of the system $A\bar{x}=\bar{b}$.

From (4), it is clear that $\beta \ge \frac{\lambda_j}{x_j}$ for j = 1, 2, ..., p and for at least one j, $\beta = \frac{\lambda_j}{x_j}$. Hence, $x_j - \frac{\lambda_j}{\beta} \ge 0$ for all j and $x_j - \frac{\lambda_j}{\beta} = 0$ for at least one j.

Thus, all the compoments of $\bar{x}_1$ are non-negative and it is a feasible solution which contains not more than (p – 1) positive components.

If z_1 is the value of the new objective function for the feasible solution $\bar{x}_1$, then

$$\bar{z}_1=\bar{c}\,\bar{x}_1=\sum_{j=1}^{p}c_j\left(x_j-\frac{\lambda_j}{\beta}\right)=\sum_{j=1}^{p}c_jx_j-\frac{1}{\beta}\sum_{j=1}^{p}c_j\lambda_j$$

$$= z^*-\frac{1}{\beta}\sum_{j=1}^{p}c_j\lambda_j \quad \text{[using (2)]}$$

Now, if $\sum_{j=1}^{p} c_j\lambda_j = 0,$ then $z_1 = z^*$ and hence, $\overline{x}_1$ is also an optimal solution.

When $\sum_{j=1}^{p} c_j\lambda_j \neq 0,$ then we can find a real number γ such that

$$\gamma\sum_{j=1}^{p} c_j\lambda_j > 0$$

and hence adding $\sum_{i=1}^{p} c_j x_j$ on both sides, we get

$$\gamma\sum_{j=1}^{p} c_j\lambda_j + \sum_{j=1}^{p} c_j x_j > \sum_{j=1}^{p} c_j x_j = z^*. \text{ .... } (6)$$

Again, multiplying (3) by γ and adding to (1), we get

$$(x_1 + \gamma\lambda_1)\overline{\alpha}_1 + (x_2 + \gamma\lambda_2)\overline{\alpha}_2 + ... + (x_p + \gamma\lambda_p)\overline{\alpha}_p = \overline{b}$$

and hence

$$[x_1 + \gamma\lambda_1, x_2 + \gamma\lambda_2, ..., x_p + \gamma\lambda_p, 0, 0, ..., 0] \text{ .... } (7)$$

is a solution of the system $A\overline{x} = \overline{b}$.

Now, choosing γ in the following manner

$$\max_j\left\{-\frac{x_j}{\lambda_j}, \lambda_j > 0\right\} \le \gamma \le \min_j\left\{-\frac{x_j}{\lambda_j}, \lambda_j < 0\right\}$$

we see that all $x_j + \gamma\lambda_j \geq 0$; j = 1, 2,...., p and so the solution (7) is a feasible solution.

From (6), we see that the feasible solution (7) gives a greater value of the objective function z^*, the optimal value given by $\overline{x}^*$. This contradicts our assumption that z^* is the optimal value and hence we have

$$\sum_{j=1}^{p} c_j \lambda_j = 0$$

and thus $\bar{x}_1$ is also an optimal solution.

Therefore, from the given optimal solution, we can construct a new optimal solution in which number of non-zero variables is less than that of the given solution. If the vectors associated with these non-zero variables are linearly independent, then the new solution will again be a basic feasible solution and hence, the theorem is proved.

If again this new solution be not a basic feasible solution, we can further diminish the number of non-zero variables as above. Continuing this process, ultimately, we get an optimal solution of the LPP which is a basic feasible solution.

Remark : The fundamental theorem of LPP shows that if a LPP has an optimal solution, then at least one basic feasible solution will be that optimal solution. So, the optimal solution will be searched among the basic feasible solutions only, which are finite in number.

The simplex method tells us how to find a new improved basic feasible solution (b.f.s.) from a given basic feasible soltuion. By an improved b.f.s, it is meant that the value of the objective function for this b.f.s. is at least as large as the value corresponding to old b.f.s.

● **Examples 8.2.1 : Solve the LPP**

Maximize $Z = x_1 - x_2 + x_3$

Subject to $x_1 - 2x_2 + x_3 = 3$

$2x_1 + x_2 + x_4 = 2$

$x_1, x_2, x_3, x_4 \geq 0$

with the assumptiion that the optimal solution exists.

Solution : It is given that the problem has an optimal solution hence by fundamental theorem of LPP at least one basic feasible solution must be optimal. Thus we shall at first find all basic feasible solutions to the given system of equations.

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{a}_3 x_3 + \bar{a}_4 x_4 = \bar{b}$$

Where $\bar{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\bar{a}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\bar{a}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{a}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\bar{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Let $B_1 = [\bar{a}_1, \bar{a}_2] = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \quad \therefore |B_1| = 5 \ (\neq 0)$

$$B_2 = [\bar{a}_1, \bar{a}_3] = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \therefore |B_2| = -2 \ (\neq 0)$$

$$B_3 = [\bar{a}_1, \bar{a}_4] = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \therefore |B_3| = 1 \ (\neq 0)$$

$$B_4 = [\bar{a}_2, \bar{a}_3] = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \therefore |B_4| = -1 \ (\neq 0)$$

$$B_5 = [\bar{a}_2, \bar{a}_4] = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}, \quad \therefore |B_5| = -2 \ (\neq 0)$$

$$B_6 = [\bar{a}_3, \bar{a}_4] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \therefore |B_6| = 1 \ (\neq 0)$$

Since all the six square matrix are non singular hence there are six basic solution as follows :-

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B_1^{-1}\bar{b} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{-4}{5} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = B_2^{-1}\bar{b} = \frac{1}{-2}\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = B_3^{-1}\bar{b} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = B_4^{-1}\bar{b} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = B_5^{-1}\bar{b} = \frac{1}{-2}\begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ \frac{7}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = B_6^{-1}\bar{b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Out of these basic solutions only the following these solutions

[1, 0, 2, 0], [0, 2, 7, 0], [0, 0, 3, 2] are basic feasible solutions and the corresponding values of the objective function Z are

$1 - 0 + 2$, $0 - 2 + 7$, $0 - 0 + 3$ respectively is 3, 5, 3 respectively.

Hence the maximum value of

$Z = 5$ for $x_1 = 0$, $x_2 = 2$, $x_3 = 7$, $x_4 = 0$

8.3 Reduction of a Feasible Solution (F.S.) to a Basic Feasible Solution (B.F.S.)

Let $\overline{x}_1 = [x_1, x_2, ..., x_n]$ be a feasible solution of the given system of equations $A\overline{x} = \overline{b},\ \overline{x} \geq \overline{0}$.

Let, out of these n-components of $\overline{x}_1$, k are non-zero and the remaning $(n - k)$ are zero. Without any loss generality, let it be assumed that first k-components of $\overline{x}_1$ are non-zero.

Thus, let

$$x_1 = \left[x_1, x_2, ..., x_k, \underbrace{0, 0, ..., 0}_{(n-k)}\right]$$

Since $\overline{x}_1$ is a feasible solution (F.S.), so we have

$$A\overline{x} = \overline{b} \quad \text{or,} \ \sum_{j=1}^{k} \overline{a}_j \overline{x}_j = \overline{b} \ .... \ (1)$$

where $\overline{a}_j = [a_{1j}, a_{2j}, ..., a_{mj}]$, $x_j > 0$ for all $j = 1, 2, ..., k$.

Now, if $\overline{a}_1, \overline{a}_2, ..., \overline{a}_k$ are the vectors associated to the k non-zero variables in the F.S., then there exist scalars $\lambda_1, \lambda_2, ..., \lambda_k$ (not all zero) such that

$$\lambda_1 \overline{a}_1 + \lambda_2 \overline{a}_2 + ... + \lambda_k \overline{a}_k = \overline{0}$$

or, $$\sum_{j=1}^{k} \lambda_j \bar{a}_j = \bar{0}\ .\ \text{..........}\ (2)$$

If it is assumed that $\lambda_r \neq 0$, then from (2), we have

$$\bar{a}_r = -\sum_{\substack{j=1 \\ (j\neq r)}}^{k} \frac{\lambda_j}{\lambda_r} \bar{a}_j\ .$$

Substituting this value of $\bar{a}_r$ in (1), we get

$$\sum_{\substack{j=1 \\ (j\neq r)}}^{k} \left(x_j - x_r . \frac{\lambda_j}{\lambda_r} \right) \bar{a}_j = \bar{b}\ .\ \text{................}(3)$$

Equation (3) shows that

$$\bar{x}_2 = \left[x_1 - x_r . \frac{\lambda_1}{\lambda_r}, x_2 - x_r . \frac{\lambda_2}{\lambda_r}, ..., x_{r-1} - x_r . \frac{\lambda_{r-1}}{\lambda_r}, 0, \right.$$

$$\left. x_{r+1} - x_r . \frac{\lambda_{r+1}}{\lambda_r}, ..., x_k - x_r . \frac{\lambda_k}{\lambda_r}, \underbrace{0, 0, ..., 0}_{(n-k)} \right]\ \text{...........}\ (4)$$

is also a solution of the given system $A\bar{x} = \bar{b}$. It is to be noted that this solution contains at most (k – 1) non-zero variables. Now, to make $\bar{x}_2$ a F.S., λ_r is to be chosen such that

$$x_j - x_r . \frac{\lambda_j}{\lambda_r} \geq 0$$

$$\left. \begin{array}{ll} \text{i.e.,} & \dfrac{x_j}{\lambda_j} - \dfrac{x_r}{\lambda_r} \geq 0, \text{ if } \lambda_j > 0 \\[2em] \text{and} & \dfrac{x_j}{\lambda_j} - \dfrac{x_r}{\lambda_r} \leq 0, \text{ if } \lambda_j < 0. \end{array} \right\}$$

Now, choosing $\frac{x_r}{\lambda_r}$ such that

$$\max_{\lambda_j<0}\left\{\frac{x_j}{\lambda_j}\right\} \le \frac{x_r}{\lambda_r} \le \min_{\lambda_j>0}\left\{\frac{x_j}{\lambda_j}\right\} \quad \text{...........} \ (4)$$

we get all the non-zero components of $\overline{x}_2$ as given in (4) to be positive and hence $\overline{x}_2$ will necessarily be a feasible solution.

Now, if the vectors associated to the non-zero variables in this solution $\overline{x}_2$ are linearly independent, then this is a B.F.S. If, on the other hand, the above vectors are linearly dependent, then using the same process discussed above, a new F.S. with less number of non-zero variables and ultimately a B.F.S. can be obtained.

Since $\max\limits_{\lambda_j<0}\left\{\frac{x_j}{\lambda_j}\right\}$ is negative and $\min\limits_{\lambda_j>0}\left\{\frac{x_j}{\lambda_j}\right\}$ is positive, so the interval in (4) is non-empty.

Again, if there is no $\lambda_j < 0$, then there is no lower limit for $\frac{x_r}{\lambda_r}$ and if there is no $\lambda_j > 0$, then there is no upper limit for $\frac{x_r}{\lambda_r}$.

In particular, the vector $\overline{a}_r$, to be eliminated is so chosen for which

$$\frac{x_r}{\lambda_r} = \min_{\lambda_j>0}\left\{\frac{x_j}{\lambda_j}\right\} \quad \text{or} \quad \frac{x_r}{\lambda_r} = \max_{\lambda_j<0}\left\{\frac{x_j}{\lambda_j}\right\}.$$

This is an outline of a procedure to get a B.F.S. from a given F.S.

Remark : If a LPP has a F.S., then it has a B.F.S. Since every LPP can be reduced to its standard form in which the constraints are $A\overline{x} = \overline{b}$, $\overline{x} \ge \overline{0}$, so this problem is the same as discussed above and by this discussion, the existence of F.S. implies the existence of a B.F.S. to a LPP.

● **Example 8.3.1 :** $x_1 = 1, x_2 = 2, x_3 = 4$ is a feasible solution of the following system of equations :

$$2x_1 + 3x_2 - x_3 = 4$$
$$3x_1 - x_2 + x_3 = 5$$

Reduce this F.S. to a B.F.S.

Solution : The given system of equations can be written as

$$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{a}_3 x_3 = \bar{b}, \text{ .... (1)}$$

where $\bar{a}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \bar{a}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \bar{a}_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\bar{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Since (1, 2, 4) is a F.S. of (1), so we get

$$\bar{a}_1 + 2\bar{a}_2 + 4\bar{a}_3 = \bar{b}. \text{ .... (2)}$$

Since the given F.S. contains three non-zero variables, so it is not basic. Thus, the vectors associated to these variables are linearly dependent. So, the vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are linearly dependent. Thus, there exists scalars λ_1, λ_2, λ_3 (not all zero) such that

$$\lambda_1 \bar{a}_1 + \lambda_2 \bar{a}_2 + \lambda_3 \bar{a}_3 = \bar{0}. \text{ .... (3)}$$

This gives the following system of equations

$$2\lambda_1 + 3\lambda_2 - \lambda_3 = 0,$$
$$3\lambda_1 - \lambda_2 + \lambda_3 = 0.$$

Solving these, we get

$$\frac{\lambda_1}{3-1} = \frac{\lambda_2}{-3-2} = \frac{\lambda_3}{-2-9} = k\ (\neq 0), \text{ say}$$

So, taking k = 1, we have

$$\lambda_1 = 2,\ \lambda_2 = -5,\ \lambda_3 = -11.$$

Then (3) becomes

$$2\bar{a}_1 - 5\bar{a}_2 - 11\bar{a}_3 = \bar{0}. \text{ ....(4)}$$

Since the B.F.S. of the problem cannot have more than two non-zero variables, we are to reduce one of one-zero variables from the given F.S. to zero. The variable to be reduced to zero is found as follows :

$$\frac{x_r}{\lambda_r} = \min_j \left\{ \frac{x_j}{\lambda_j}, \lambda_j > 0 \right\} = \min \left\{ \frac{x_1}{\lambda_1} \right\} = \frac{1}{2} = \frac{x_1}{\lambda_1}.$$

So, x_1 is taken as zero in the new solution. Now x_2 and x_3 are found by the formula

$$\text{new } x_j = \text{given } x_j - \frac{1}{2} \lambda_j.$$

Therefore,

$$\text{new } x_2 = 2 - \frac{1}{2} \times (-5) = \frac{9}{2}$$

$$\text{new } x_3 = 4 - \frac{1}{2} \times (-11) = \frac{19}{2}.$$

Since the vectors $\bar{a}_2, \bar{a}_3$ corresponding to the new non-zero variables are linearly independent $\left(\text{since} \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0 \right)$, so the new solution $\left(0, \frac{9}{2}, \frac{19}{2}\right)$ is a basic feasible solution.

Again, for negative values of λ, we have

$$\frac{x_r}{\lambda_r} = \max_j \left\{ \frac{x_j}{\lambda_j}, \lambda_j < 0 \right\} = \max \left\{ \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3} \right\}$$

$$= \max \left\{ -\frac{2}{5}, -\frac{4}{11} \right\} = -\frac{4}{11} = \frac{x_3}{\lambda_3}.$$

So, x_3 is taken as zero in the new solution. New x_1 and x_2 are found by the formula

$$\text{new } x_j = \text{given } x_j - \left(-\frac{4}{11} \right) \lambda_j.$$

Therefore,

$$\text{new } x_1 = 1 - \left(-\frac{4}{11}\right)\times 2 = \frac{19}{11}$$

$$\text{new } x_2 = 2 - \left(-\frac{4}{11}\right)\times(-5) = \frac{2}{11}.$$

Since the vectors $\bar{a}_1$ and $\bar{a}_2$ are linearly independent $\left(\text{since}\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix} = -11 \neq 0\right)$, so the new solution is another basic feasible solution.

Another approach :

In the first case (for positive value of λ), since x_1 is taken as zero in the new solution, so, to obtain new solution, we have to eliminate the corresponding vector $\bar{a}_1$ from (2) and (4).

Eliminating $\bar{a}_1$ from (2) and (4), we get

$$\frac{1}{2}\left(5\bar{a}_2 + 11\bar{a}_3\right) + 2\bar{a}_2 + 4\bar{a}_3 = \bar{b}$$

$$\text{or,} \quad 0.\bar{a}_1 + \frac{9}{2}\bar{a}_2 + \frac{19}{2}\bar{a}_3 = b.$$

Thus, $\left(0, \frac{9}{2}, \frac{19}{2}\right)$ is also a feasible solution of (1) and since $\bar{a}_2, \bar{a}_3$ are linearly independent, this solution is a basic feasible solution.

Similarly, in the second case (for negative values of λ), eliminating $\bar{a}_3$ from (2) and (4), we get

$$\frac{19}{11}\bar{a}_1 + \frac{2}{11}\bar{a}_2 + 0.\ \bar{a}_3 = \bar{b},$$

and since $\bar{a}_1, \bar{a}_2$ are linearly independent, so $\left(\frac{19}{11}, \frac{2}{11}, 0\right)$ is another basic feasible solution.

8.4 Summary

In this chapter, fundamental theorem of LPP is proved. Reduction of a feasible solution to a basic feasible solution is described. These are also explained with examples.

8.5 Exercise

1. The LPP

   Minimize $Z = 2x_1 - 3x_2 + x_4$

   Subject to $3x_1 + 2x_2 + x_3 = 15$

   $2x_1 + 4x_2 + x_4 = 8$

   $x_1, x_2, x_3, x_4 \geq 0$

   has an optimal solution. Using the property of the fundamental theorem of LPP, find the minimum value of Z and the corresponding solution set.

2. Reduce the feasible solution (2, 1, 1) of the system

   $x_1 + 4x_2 - x_3 = 5$

   $2x_1 + 3x_2 + x_3 = 8$

   to a basic feasible solution.

3. Given that $x_1 = x_2 = x_3 = 1$ is a feasible solution to the system of equations

   $x_1 + x_2 + 2x_3 = 4$

   $2x_1 - x_2 + x_3 = 2$

   Reduce the given feasible solution to a basic feasible solution.

4. $x_1 = 2, x_2 = 3, x_3 = 1$ is a feasible solution of the system

   $2x_1 + x_2 + 4x_3 = 11$

   $3x_1 + x_2 + 5x_3 = 14$

   Reduce it to a basic feasible solution.

Answers

1. Min Z = – 6 at [0, 2, 11, 0]
2. $\left[\frac{17}{5}, \frac{2}{5}, 0\right]$ and $\left[0, \frac{13}{7}, \frac{17}{7}\right]$
3. [0, 0, 2] and [2, 2, 0]
4. $\left[\frac{1}{2}, 0, \frac{5}{2}\right]$ and $\left[3, 5, 0\right]$

8.6 Multiple Choice Questions (MCQ)

1. Consider the system of linear equations :-

$$3x_1 + 5x_2 - 7x_3 = 21$$
$$6x_1 + 10x_2 + 3x_3 = 42$$

Then in the above system

(a) $x_2 = 2$, $x_2 = 3$, $x_3 = 0$ is a basic feasible solution.

(b) $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ is neither a feasible solution nor a basic feasible solution.

(c) $x_1 = 2$, $x_2 = 3$, $x_3 = 0$ is a feasible solution and hence it is possible to reduce this feasible solution to a basic feasible solution.

(d) none of these

2. Consider the LPP

Minimize $Z = x_1 + 2x_2$

Subject to $3x_1 + 4x_2 - x_3 = 12$

$x_1, x_2, x_3, \geq 0$

Then

(a) Max Z = 4 at $x_1 = 4$, $x_2 = 0$, $x_3 = 0$

(b) Max Z = 6 at $x_1 = 0$, $x_2 = 3$, $x_3 = 0$

(c) Max Z = 10 at $x_1 = 10$, $x_2 = 0$, $x_3 = 18$

(d) The LPP has no finite optimal solution.

Answers

1. (c)   2. (d)

Unit - 9 ❑ Simplex Algorithm

Structure

9.0 Objective

After studying this chapter, the reader should be able to know

- describe the general numerical method of solving a LPP known as Simplex algorithm or Simplex method.
- build a simplex table and describle its components.
- solve a LPP for '≤' type of constraints by simplex method.

9.1 Introduction

We now develop some results which will enable us to solve a unear programming problem by an iterative process which is known as 'Simplex Method'. This method is an iterative method by which a new basic feasible solution can be obtained from a given basic feasible solution which improves the valu of the objective function. For this we shall introduce a few definitions and notations to be used in our latter discussion in connection with the simplex slgorithm or simplex method.

9.2 Some Definitions and Notations

Consider a LPP in its standard form as given by

Maximize z = $\bar{c}\,x$

subject to $A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}$

where A = $(a_{ij})_{m\times n}$ is a mxn (m < n), matrix given by

$A = (\bar{a}_1, \bar{a}_2, \text{- -}\ \bar{a}_r)$

and $\bar{a}_j = [\bar{a}_{1j}, \bar{a}_{2j}, \text{- -}\ \bar{a}_{mj}]$

The vectors associated to non-zero variables in a b.f.s. are denoted by $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m$ which are nothing but m vectors of $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$. Thus, $\bar{\beta}_i = \bar{a}_j$ for some j. Since $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m$ are linearly independent, so they form a basis and the square matrix $B = (\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m)$ is known as basis matrix. The variables corresponding to these vecors $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m$ are called basic variables and are denoted by $x_{B_1}, x_{B_2}, ..., x_{B_m}$ and the corresponding basic variable matrix $\bar{x}_B$ is written as $\bar{x}_B = [x_{B_1}, x_{B_2}, ..., x_{B_m}]$.

Thus, $\bar{x}_B$ is a b.f.s. which means that the variables besides those in $\bar{x}_B$ are all zero. Thus, the system $A\bar{x} = \bar{b}$ reduces to $B\bar{x}_B = \bar{b}$ = and hence $\bar{x}_B = B^{-1}\bar{b}$.

If the prices corresponding to the variables $x_{B_1}, x_{B_2}, ..., x_{B_m}$ are denoted by $c_{B_1}, c_{B_2}, ..., c_{B_m}$ respectively, then the price vector $\bar{c}_B$ can be written as $\bar{c}_B = (c_{B_1}, c_{B_2}, ..., c_{B_m})$.

Since $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m$ form a basis in E^m, then each vector $\bar{a}_j \in A \subseteq E^m$ can be expressed as a linear combination of these vectors.

Let $\bar{a}_j = \bar{\beta}_1 y_{1j} + \bar{\beta}_2 y_{2j} + ... + \bar{\beta}_m y_{mj} = \sum_{i=1}^{m} \bar{\beta}_i y_{ij}$

$$= \left(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m\right)[y_{1j}, y_{2j}, ..., y_{mj}]$$

or, $\bar{a}_j = B\bar{y}_j$, where $\bar{y}_j = [y_{1j}, y_{2j}, ..., y_{mj}]$.

Thus, $\bar{y}_j = B^{-1}\bar{a}_j$.

Also, we define

$$z_j = \bar{c}_B \bar{y}_j = c_{B_1} y_{1j} + c_{B_2} y_{2j} + ... + c_{B_m} y_{mj}$$

The real number $z_j - c_j = \bar{c}_B \bar{y}_j - c_j$ is called the net evaluation for $\bar{a}_j$.

All these definitions and notations will play an important role during the development of Simplex Method.

8.3 Simplex Algorithms

The simplex algorithm consists of several steps which lead to the optimal solution. In effect, we first stant with an extreme point (a basic feasible solution) of the feasible space and verity its optimality. If it is not optimal, we look for all the extreme points adjacent to the first, that is the points which can be joined to thefirst by edges but choose the one which is better than the first, ie hearer to the optimal solution and verify the optimality. If it is not oplimal, we again look for the extreme ponts adjacent to the last and choose that point which is better than the last.

This process is continued till we arrive at the optimal solution obiously, the process does not require evaliation of the objective function at all the extreme points.

Proposition 1 : Theorem 9.3.1 (Optimality Criterion)

A basic feasible solution of a maximizing linear programming problem is an optimal solution if the net evaluation $z_j - c_j \geq 0$ for each j.

Proof : Consider a maximizing LPP of the form :

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}$$

where $A = (a_{ij})_{m\times n}$ $(m < n)$, $\bar{x} = [x_1, x_2, ..., x_n]$, $\bar{c} = (c_1, c_2, ..., c_n)$ and

$\overline{b} = [b_1, b_2, ..., b_m]$.

Let $\overline{x}^* = \left[\overline{x}_B, \overline{0}\right]$ be a basic feasible solution for which $z_j - c_j \geq 0$ for all j and let $\overline{x}'$ be any other feasible solution of the given LPP.

Then, we have

$$A\overline{x}' = \overline{b} = B\overline{x}_B$$

$$\text{or, } \overline{x}_B{}^{-1} = B^{-1}(A\overline{x}') = \left(B^{-1}A\right)\overline{x}' = \overline{y}\,\overline{x}'$$

$$\text{or, } \left[x_{B_1}, x_{B_2}, ..., x_{B_m}\right] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_n' \end{bmatrix}$$

Therefore, $x_{B_i} = \sum_{j=1}^{n} y_{2j} x_j' \quad (i = 1, 2, ..., m).$

Now, $z_j - c_j \geq 0$ implies $z_j \geq c_j$ for all j.

Therefore, $z_j x_j' \geq c_j x_j' \quad [\because x_j' \geq 0]$

$$\text{or, } \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{m} c_{Bi}\, y_{ij} \right) \geq \sum_{j=1}^{n} c_j x_j'$$

$$\text{or, } \sum_{i=1}^{m} c_{Bi} \left(\sum_{j=1}^{n} x_j'\, y_{ij} \right) \geq \sum_{j=1}^{n} c_j\, x_j'$$

$$\text{or, } \sum_{i=1}^{m} c_{Bi}\, x_{Bi} \geq \sum_{j=1}^{n} c_j\, x_j'$$

$$\text{or, } z^* \geq z',$$

where z^* is the value of the objective function for optimal solution and z' be that value for any other feasible soution.

Thus, $\overline{x}^*$ is the optimal solution of the LPP for which $z_j - c_j \geq 0$ for all j.

Again, if A = $(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)$ and B = $(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m)$, then let $\bar{a}_j = \bar{\beta}_j$ for all j for which $\bar{a}_j \in B$. In this case, we have

$$\begin{aligned}\bar{y}_j &= B^{-1}\beta_j \left[\because \bar{y}_j = B^{-1}\bar{a}_j\right] \\ &= \bar{e}_j, \text{ the unit vector.}\end{aligned}$$

Hence, $z_j - c_j = \bar{c}_B \bar{y}_j - c_j = \bar{c}_B \bar{e}_j - c_j = c_{Bj} - c_j = 0$. $[\because c_{Bj} = c_j$ for $\bar{a}_j \in B]$

Thus, $z_j - c_j = 0$ for all j for which $\bar{a}_j \in B$.

Note : With the help of this theorem it can be judged at any iteration of the simplex method whether the optimal solution has been reached or not.

Remark : For a minimization linear programming problem, the optimatily criterion is $z_j - c_j \leq 0$ for all j.

Proposition 2 : Theorem 9.3.2 (Improvement Criterion)

If a basic feasible solution of a maximizing LPP be such that $z_j - c_j < 0$ holds for the j-th column corresponding to a non-basic variable $y_{ij} > 0$ for some i= 1, 2, ..., m, then there exists another basic feasible solution which improves the value of the objective function.

Proof : Consider a maximizing LPP of the form :

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0},$$

where A = $(a_{ij})_{m\times n}$ (m < n) = $(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)$, $\bar{x} = [x_1, x_2, ..., x_n]$, $\bar{c} = (c_1, c_2, ..., c_n)$ and $\bar{b} = [b_1, b_2, ..., b_m]$.

Let $\bar{x} = [\bar{x}_B, \bar{0}]$ be a basic feasible solution of the given LPP, where B = $(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m)$. Then there exists scalars $y_{1j}, y_{2j}, ..., y_{mj}$ (not all zero) such that

$$\bar{a}_j = y_{1j}\,\bar{\beta}_1 + y_{2j}\,\bar{\beta}_2 + \cdots\cdots + y_{mj}\,\bar{\beta}_m.$$

If $y_{rj} \neq 0$, then we can write

$$\bar{\beta}_r = \frac{1}{y_{rj}}\left(a_j - y_{1j}\,\bar{\beta}_1 - \cdots - y_{(r-1)j}\,\bar{\beta}_{r-1} - y_{(r+1)j}\,\bar{\beta}_{r+1} - \cdots - y_{mj}\,\bar{\beta}_m\right) \text{ ....(1)}$$

Since $[\bar{x}_m, \bar{0}]$ is a basic feasible solution, we get

$$x_{B_1}\,\bar{\beta}_1 + x_{B_2}\,\bar{\beta}_2 + \cdots\cdots + x_{B_m}\,\bar{\beta}_m = \bar{b}$$

$$\text{or,} \quad \sum_{\substack{i=1 \\ (i\neq r)}}^{m} x_{B_i}\bar{\beta}_i + x_{B_r}\bar{\beta}_r = \bar{b}. \quad \text{................ (2)}$$

Putting the expression of $\bar{\beta}_r$ from (1) in equation (2), we get

$$\sum_{\substack{i=1 \\ (i\neq r)}}^{m} \left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} \right)\bar{\beta}_i + \frac{x_{B_r}}{y_{rj}}\bar{a}_j = \bar{b}. \quad \text{.............. (3)}$$

This shows that

$$\bar{x}' = \left[x_{B_1} - \frac{y_{ij}}{y_{rj}} x_{B_r}, \ldots, x_{B_{r-1}} - \frac{y_{(r-1)j}}{y_{rj}} x_{B_{r-1}}, \frac{x_{B_r}}{y_{rj}}, x_{B_{r+1}} - \frac{y_{(r+1)j}}{y_{rj}} x_{B_{r+1}}, \right.$$
$$\left. \cdots\cdots, x_{B_m} - \frac{y_{mj}}{y_{rj}} x_{B_m} \right]$$

is a basic solution obtained by replacing $\bar{\beta}_r$ by $\bar{a}_r$ in the basis.

Indeed, by a proper choice of $\bar{\beta}_r$, it is possible to get a basic feasible solution. For this, we need

$$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} \geq 0 \text{ and } y_{rj} > 0 \text{ for all i.}$$

If $y_{ij} > 0$ and $y_{rj} > 0$, these for all i

$$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} \geq 0$$

i.e., $\frac{x_{B_i}}{y_{ij}} \geq \frac{x_{B_r}}{y_{rj}}$

This is possible if

$$\frac{x_{B_r}}{y_{rj}} = \min_{y_{ij}>0} \left\{ \frac{x_{B_i}}{y_{ij}} \right\}.$$

It is to be noted here that $\frac{x_{B_r}}{y_{rj}} \geq 0$ if $y_{rj} > 0$.

So, If y_{rj} is so chosen that $y_{rj} > 0$ and $\frac{x_{B_r}}{y_{rj}} = \min\limits_{y_{ij}>0} \left\{ \frac{x_{B_i}}{y_{ij}} \right\}$, then evidently the new solution $\overline{x}'$ obtained will be a basic feasible solution.

It may also be noted here that the existence of at least one $y_{ij} > 0$ is required for the existence of new B.F.S. $\overline{x}'$.

Now, for the improved basic feasible solution, we have to show that $\overline{c}\,\overline{x}' > \overline{c}\,\overline{x}$ where $\overline{x}' = \left[\overline{x}'_B, \overline{0}\right]$.

Let $\overline{c}'_B$ denote the corresponding cost vector to the basis of $\overline{x}'_B$. Then, we have

$$c'_{B_i} = c_{B_i} \text{ for } i = 1, 2, ..., m \ (i \neq r)$$

and $\quad c'_{B_r} = c_j$ for $i = r$.

$$\text{Now, } \overline{c}\,\overline{x}' = \sum_{\substack{i=1 \\ (i\neq r)}}^{m} c'_{B_i} x'_{B_i} + c_j \frac{x_{B_r}}{y_{rj}}$$

$$= \sum_{\substack{i=1 \\ (i\neq r)}}^{m} c_{B_i} \left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} \right) + c_j \frac{x_{B_r}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{B_i}\left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r}\right) + c_j \frac{x_{B_r}}{y_{rj}} \quad \left[\because x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} = 0 \text{ for } i = r\right]$$

$$= \sum_{i=1}^{m} c_{B_i} x_{B_i} + \frac{x_{B_r}}{y_{rj}}\left(c_j - \sum_{i=1}^{m} c_{B_i} y_{ij}\right)$$

$$= \bar{c}\,\bar{x} - \frac{x_{B_r}}{y_{rj}}\left(z_j - c_j\right)$$

$$\geq \bar{c}\,\bar{x} \left[\because \frac{x_{B_r}}{y_{rj}} \geq 0 \text{ and } z_j - c_j \geq 0 \text{ for all } j\right]$$

Thus, $\bar{x}'$ is another basic feasible solution which improves the value of the objective function. Hence, the theorem is proved.

Remark : (i) The number y_{rj} is called the **key element** or the pivot element. The rule determining the key element is known as the minimum ratio rule.

(ii) For a minimization problem, the corresponding conditon will be $z_j - c_j < 0$ for columns corresponding to a non-basic variable with at least one $y_{ij} > 0$.

(iii) If $z_j - c_j < 0$ for only one column say k-th column, then $\bar{a}_k$ is taken as the incoming or entering vector which replace a vector of the basis for improvement of the solution.

(iv) If $z_j - c_j < 0$ for more than one column and $z_k - c_k = \text{Min}\ \{z_j - c_j\ ;\ z_j - c_j < 0\}$, then $\bar{a}_k$ will be the entering vector. If this minimum is not unique, any one of the columns having the same minimum is taken as the entering vector.

Proposition 3 : Theorem 9.3.3 (Unboundedness Criterion)

A maximizing linear programming problem will have no optimal solution if there exists at least one column corresponding to a non-basic variable for which $z_j - c_j < 0$ and $y_{ij} \leq 0$ for all i.

Proof : Consider a maximizing linear programming problem as

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} = \bar{b},\ \bar{x} \geq \bar{0}.$$

Let $\bar{a}_j$ be a column vector corresponding to a non-basic variable with $z_j - c_j < 0$ and $y_{ij} \le 0$ for all i.

If $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m$ forms a basis, we can write

$$\bar{a}_j = \sum_{i=1}^{m} y_{ij}\, \bar{\beta}_i$$

If $\bar{x} = [\bar{x}_B, \bar{0}]$ is a basic feasible solution, then for any positive number λ, we can write

$$\sum_{i=1}^{m} x_{B_i} \bar{\beta}_i - \lambda \sum_{i=1}^{m} y_{ij}\, \bar{\beta}_i + \lambda \bar{a}_j = \bar{b} \left[\because \sum_{i=1}^{m} x_{B_i} \bar{\beta}_i = \bar{b} \right]$$

$$\text{or,} \quad \sum_{i=1}^{m} \left(x_{B_i} - \lambda y_{ij} \right) \bar{\beta}_i + \lambda \bar{a}_j = \bar{b} .$$

This shows that $\bar{x}' = \left[x_{B_1} - \lambda y_{1j} B_2, ..., x_{B_m} - \lambda y_{mj}, \lambda, 0, ..., 0 \right]$ is a new B.F.S. which improves the value of the objective function. Then we have

$$z' = \bar{c}' \bar{x}' = \sum_{i=1}^{m} c_{B_i} \left(x_{B_i} - \lambda y_{ij} \right) + c_j \lambda$$

$$= \sum_{i=1}^{m} c_{B_i} x_{B_i} - \lambda (z_j - c_j)$$

$$= z - \lambda \ (z_j - c_j)$$

Thus, the value z′ of the new objective function can be made arbitrarily large by choosing λ sufficiently large, since $z_j - c_j < 0$. Hence there is no finite optimal value of the objective function.

Remark : A LPP with such criterion is said to have an unbounded solution.

Proposition 4 : Theorem 9.3.4 (Multiplicity Criterion)

If, in the iteration of simplex method of a maximization LPP, at optimal stage when all $z_j - c_j \geq 0$, there exists some columns corresponding to non-basic variables for which $z_j - c_j = 0$, $y_{ij} > 0$ for at least one i, then there exists more than one optimal solution. In this case, the LPP has infinitely many optimal solutions.

Proof : Let $\overline{x} = \left[\overline{x}_B, \overline{0}\right]$ be an optimal solution and let $\overline{x}' = \left[\overline{x}'_B, \overline{0}\right]$ be an improved solution over $\overline{x}$. Then, by Proposition 2, we know

$$x'_{B_r} = \frac{x_{B_r}}{y_{ij}} \text{ and } x'_{B_i} = x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_i} ; \quad i = 1, 2, ..., m, \quad i \neq r$$

Then, $z' = \overline{c}\,\overline{x}' = z - \frac{x_{B_r}}{y_{rj}}(z_j - c_j)$.

As $\left[\overline{x}_B, \overline{0}\right]$ is an optimal solution, there is no column vector $\overline{a}_j$ for which $z_j - c_j < 0$. But by the given conditon

$$z' = z - \frac{x_{B_r}}{y_{rj}} \times 0 = z .$$

Hence, z' is also the optimal value of the objective function and $[\overline{x}'_B, \overline{0}]$ and is an alternative optimal solution. Since a convex combination of two optimal solutions is also an optimal solution, so the LPP has infinitely many optimal soltions.

On the basis of the above propositions, it is clear that from a given initial basic feasible solution one can improve successively to arrive at the optimal solution. But the question remains how to choose this initial basic feasible solution. The simple way to choose the initial basic feasible solution (IBFS) is to start with unit vectors in the basis, if there be any. The situation is very simple if all the constraints are inequalities requiring only slack variables for reduction to standard form. In this case, as many slack variables are required as the number of equations and hence, we get as many unit vectors to start with in the initial basis.

Let the given LPP be

Maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1$

$$a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n \leq b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n \leq b_m$$

and $x_1, x_2, ..., x_n \geq 0$ with $b_1, b_2, ..., b_m \geq 0$

Then, the standard form of this LPP is

$$\text{Maximize } z = c_1x_1 + c_2x_2 + ... + c_nx_n + 0.x_{n+1} + ... + 0.x_{n+m}$$

subject to

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + x_{n+2} = b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2} + x_2 + ... + a_{mn}x_n + x_{n+m} = b_m$$

and $x_1, x_2, ..., x_n, x_{n+1}, ..., x_{n+m} \geq 0$

Hence, by choosing $x_{n+1}, x_{n+2}, ..., x_{n+m}$ as the basic variables, i.e., [1, 0, ..., 0], [0, 1, 0, ..., 0], ..., [0, 0, ..., 0, 1] as the basic vectors, we get the initial basic feasible solution as $(b_1, b_2, ..., b_m)$.

Based on the above results, we now give the computational procedure for simplex method of a maximizing LPP in the following section.

9.4 Computational Procedure for Simplex Method

Step-1. Reduce the given LPP to a maximization problem with $\bar{b} \geq \bar{0}$ that is,

$$\text{Maximize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } A\bar{x} \leq \bar{b},\ \bar{x} \geq \bar{0} \text{ and } \bar{b} \geq \bar{0}.$$

Step-2. Add slack variables to reduce the above LPP in Step-1 to the standard form.

Step-3. Construct the simplex tableau given below, choosing the unit column vectors corresponding to the slack variables as basic vectors and complete the net evaluations $z_j - c_j$ for each column.

			c_j	c_1	c_2	....	c_n
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	....	a_n
c_{B_1}	$\bar{\beta}_1$	x_{B_1}	b_1	y_{11}	y_{12}	....	y_{1n}
c_{B_2}	$\bar{\beta}_2$	x_{B_2}	b_2	y_{21}	y_{22}	....	y_{2n}
⋮	⋮	⋮	⋮	⋮	⋮		⋮
c_{B_m}	$\bar{\beta}_m$	x_{B_m}	b_m	y_{m1}	y_{m2}	....	y_{mn}
	$z_j - c_j$		z'	$z_1 - c_1$	$z_2 - c_2$	....	$z_n - c_n$

Simplex Tableau

Step-4. If $z_j - c_j \geq 0$ for all j, then the present solution is optimal.

Step-5. If for some $\bar{a}_j$, $z_j - c_j < 0$ and $y_{ij} \leq 0$ for all i = 1, 2, ..., m, then the solution of the LPP is unbounded.

Step-6. If none of the above two criteria is satisfied, then choose the minimum most value from $z_j - c_j$ for all j, say, $z_k - c_k$. Then $\bar{a}_k$ is the entering vector. Since least one $y_{ik} > 0$, then compute $\min\limits_{y_{ik}>0}\left\{\frac{x_{B_i}}{y_{ik}}\right\}$. If this minimum occurs for one and only one value of i, say, i = r, then the vector $\bar{\beta}_r$ of the basis is the departing vector determined by the key row (or, pivot row). If, on the other hand, the above minimum occurs for more than one value of i, then more than one variable will vanish in the next solution generating degeneracy of the basic feasible solution. The method of dealing with degeneracy will be treated separately. If $z_j - c_j$ is minimum for more than one column, then any one of the corresponding column vector may be taken as the departing vector.

Step-7. The next table is then constructed, replacing c_r by c_k, $\bar{\beta}_r$ by $\bar{a}_k$ and x_r by x_k. In the new table, the entire key row is divided by the key element so that the (r, k)-th element of the new tableau becomes 1. The other rows are computed as follows :

Subtract y_{ik} times the elements of the r-th row of the new table from the elements of first row of the old table; subtract y_{ik} times the elements of the r-th row of the new table from the element of the second row of the old table and so on.

Step-8. If the net evaluation $z_j - c_j \geq 0$ for all j in the new table, then the present solution is optimal. If not, repeat the earlier steps to proceed to the next table.

The following examples will illustrate the method.

9.5 A Few Examples on Simplex Method

● **9.5.1 Example :** Use simplex method to solve the following LPP :

$$\begin{aligned} \text{Minimize} \quad & z = 4x_1 + 10x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 50 \\ & 2x_1 + 5x_2 \leq 100 \\ & 2x_1 + 3x_2 \leq 90 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution : Introducing slack variables x_3, x_4 and x_5, we rewrite the given LPP in the standard form as

$$\begin{aligned} \text{Maximize} \quad & z = 4x_1 + 10x_2 + 0.x_3 + 0.x_4 + 0.x_5 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 + 0.x_4 + 0.x_5 = 50 \\ & 2x_1 + 5x_2 + 0.x_3 + x_4 + 0.x_5 = 100 \\ & 2x_1 + 3x_2 + 0.x_3 + 0.x_4 + x_5 = 90 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Now, we solve this problem by Simplex Method through the following simplex tableau:

First Simplex Tableau

			c_j	4	10	0	0	0	
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	Min. Ratio
0	$\bar{a}_3$	x_3	50	2	1	1	0	0	$\frac{50}{1}$
0	$\bar{a}_4$	x_4	100	2	$\boxed{5}$	0	1	0	$\frac{100}{5}$
0	$\bar{a}_5$	x_5	90	2	3	0	0	1	$\frac{90}{3}$
	$z_j - c_j$			-4	$-10\uparrow$	0	$0\downarrow$	0	

In the 1st table, since $z_2 - c_2 < 0$, so $\bar{a}_2$ is the entering vector and it is denoted by ↑. Now, since $\frac{100}{50} = 20$ is the minimum ratio, so 5 is the key element and it is denoted by a square box ☐ and this implies that $\bar{a}_4$ is the departing vector. Now, we go to the Second Simplex Tableau.

Second Simplex Tableau

			c_j	4	10	0	0	0		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	Min. Ratio	Operations
0	$\bar{a}_3$	x_3	30	$\boxed{\frac{8}{5}}$	0	1	$-\frac{1}{5}$	0	$\frac{75}{4}$	$R'_1 = R_1 - 1 \times R'_2$
10	$\bar{a}_2$	x_2	20	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	50	$R'_2 = \frac{1}{5} R_2$
0	$\bar{a}_5$	x_5	30	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	$\frac{75}{2}$	$R'_3 = R_3 - 3 \times R'_2$
$z_j - c_j$				0↑	0	0	2	0		

In the 2nd simplex tableau, we see that $z_j - c_j \geq 0$ for all j. So, the solution at this interation is optimal.

The optimal solution of the given LPP is

$$x_1 = 0, \ x_2 = 20$$

and $z_{max} = 10 \times 20 = 200$ units.

Again, in the 2nd simplex tableau, it is observed that $z_j - c_j = 0$ for a non-basic vector $\bar{a}_1$. This implies that the LPP has an alternative optimal solution as shown in the 3rd simplex tableau. In this case, we take $\bar{a}_1$ as entering vector and following the same procedure, the key element will be $\frac{8}{5}$.

Third Simplex Tableau

			c_j	4	10	0	0	0	
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	Operations
4	$\bar{a}_1$	x_1	$\frac{75}{4}$	1	0	$\frac{5}{8}$	$-\frac{1}{8}$	0	$R''_1 = \frac{5}{8} R'_1$
10	$\bar{a}_2$	x_2	$\frac{25}{2}$	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$R''_2 = R'_2 - \frac{2}{5} R''_1$
0	$\bar{a}_5$	x_5	15	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$R''_3 = R'_3 - \frac{4}{5} R''_1$
$z_j - c_j$				0	0	0	2	0	

In the 3rd simplex tableau, we see that all $z_j - c_j \geq 0$. So, the solution at this iteration is optimal and it an alternative optimal solution of the given LPP.

So, another optimal solution of the given LPP is

$$x_1 = \frac{75}{4},\ x_2 = \frac{25}{2}$$

and $$z_{\max} = 4 \times \frac{75}{4} + 10 \times \frac{25}{2} = 200 \text{ units.}$$

Again, since any convex combination of two feasible solutions of a LPP is also a solution, so

$$\bar{x} = \left[x_{B_1}, x_{B_2}\right] = \lambda[0,20] + (1-\lambda)\left[\frac{75}{4}, \frac{25}{5}\right],\ 0 \leq \lambda \leq 1$$

is also an optimal solution of the LPP. Since λ has infinite values in [0, 1], so the LPP has an infinite number of optimal solutions.

Note : In this example, we have shown the simplex tables separately to explain the different steps. But, in practice, simplex tables will be represented in a single tableau as shown in the next example.

● **9.5.2 Example :** Solve the following LPP by simplex method :

$$\begin{aligned} \text{Minimize}\quad & z = x_1 - 3x_2 + 2x_3 \\ \text{subject to}\quad & 3x_1 - x_2 + 2x_3 \leq 7 \\ & 2x_1 - 4x_2 \geq -12 \\ & 4x_1 - 3x_2 - 8x_3 \geq -10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution : The given LPP can be written first with $\bar{b} \geq \bar{0}$ as

$$\begin{aligned} \text{Minimize}\quad & z = x_1 - 3x_2 + 2x_3 \\ \text{subject to}\quad & 3x_1 - x_2 + 2x_3 \leq 7 \\ & -2x_1 + 4x_2 + 0.x_3 \leq 12 \\ & -4x_1 + 3x_2 + 8x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Now, introducing slack variables x_4, x_5, x_6 and reducing the objective function in the maximization form, the LPP can be rewritten in the standard form as

$$\text{Maximize } (-z) = -x_1 + 3x_2 - 2x_3 + 0.x_4 + 0.5x_5 + 0.x_6$$

$$\text{subject to} \quad 3x_1 - x_2 + 2x_3 + x_4 + 0.x_5 + 0.x_6 = 7$$

$$-2x_1 + 4x_2 + 0.x_3 + 0.x_4 + x_5 + 0.x_6 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0.x_4 + 0.x_5 + x_6 = 10$$

$$x_j \geq 0 \ (j = 1, 2, ..., 6)$$

Now, we solve this problem by simplex method through the following simplex tableau:

Simplex Tableau

			c_j	-1	3	-2	0	0	0		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	Min. Ratio	Operations
0	$\bar{a}_4$	x_4	7	3	-1	2	1	0	0		
0	$\bar{a}_5$	x_5	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4}$	
0	$\bar{a}_6$	x_6	10	-4	3	8	0	0	1	$\frac{10}{3}$	
$z_j - c_j$				1	$-3\uparrow$	2	0	$0\downarrow$	0	Min. Ratio	Operations
0	$\bar{a}_4$	x_4	10	$\boxed{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	$\frac{10}{5/2} = \frac{20}{5}$	$R'_1 = R_1 - (-1) \times R'_2$
3	$\bar{a}_2$	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0		$R'_2 = \frac{1}{4} R_2$
0	$\bar{a}_6$	x_6	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1		$R'_3 = R_3 - 3 \times R'_2$
$z_j - c_j$				$-\frac{1}{2}\uparrow$	0	2	$0\downarrow$	$\frac{3}{4}$	1	Min. Ratio	Operations
-1	$\bar{a}_1$	x_1	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0		$R''_1 = \frac{2}{5} R'_1$
3	$\bar{a}_2$	x_2	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0		$R''_2 = R'_2 - \left(-\frac{1}{2}\right) \times R''_1$
0	$\bar{a}_6$	x_6	11	0	0	10	1	$-\frac{1}{2}$	1		$R''_3 = R'_3 - \left(-\frac{5}{2}\right) \times R''_1$
$z_j - c_j$				0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0		

Since all $z_i - c_i \geq 0$, optimality arises. The solution at this iteration is optimal.

The optimal solution is

$$x_1 = 4,\ x_2 = 5,\ x_3 = 0$$

Now Max $(-2) = -1 \times 4 + 3 \times 5 = 11$ unit.

Thus $Z_{\min} = -11$

9.6 Summary

In this chapter, the simplex method for the solution of a LPP where the mitial basic feasible solution to the problem can be easily found, is described in detail with numerous illustrative examples.

9.7 Exercise

1. Solve the following LPP by Simpex method :-

(i) Minimize $Z = 4x_1 + 8x_2 + 5x_3$

subject to $x_1 + 2x_2 + 3x_3 \leq 18$

$x_1 + 4x_2 + x_3 \leq 6$

$2x_1 + 6x_3 + 4x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

(ii) Minimize $z = -3x_1 + x_2$

subject to $-x_1 + 3x_2 \leq 9$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

(iii) Maximize $z = -4x_1 + 10x_2$

subject to $-3x_1 + 2x_2 \leq 3$

$-2x_1 + 5x_2 \leq 0$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 2x_1 + x_2$

subject to $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

$x_1, x_2 \ge 0$

(v) Minimize $z = x_1 - x_2 + x_3$

subject to $x_1 - 2x_2 + 3x_3 \ge -4$

$x_1 + x_2 + x_3 \le 9$

$2x_1 - x_2 - x_3 \le 5$

$x_1, x_2, x_3 \ge 0$

Answers

1. (i) $x_1 = 4\frac{1}{2}$, $x_2 = 0$, $x_3 = 1\frac{1}{2}$; $z_{max} = 25\frac{1}{2}$ units

(ii) $x_1 = 4$, $x_2 = 2$; $z_{min.} = -10$

(iii) $x_1 = 2\frac{3}{11}$, $x_2 = 4\frac{10}{11}$; $z_{max.} = 40$ units

(iv) Unbounded solution

(v) $x_1 = 4\frac{2}{3}$, $x_2 = 4\frac{1}{3}$, $x_3 = 0$; $z_{min.} = \frac{1}{3}$

4.10 Multiple Choice Questions (MCQ)

1. If $\overline{a}_j$ is a basic vector is a simplex table, then the corresponding net evaluation is

(a) $z_j - c_j < 0$ (b) $z_j - c_j > 0$

(c) $z_j - c_j = 0$ (d) either $z_j - c_j = 0$ or $z_j - c_j > 0$

2. If for a basic feasible solution $\overline{x}_B$, the L.P.P.

Maximize $z = \overline{c}\,\overline{x}$

subject to $A\overline{x} = \overline{b}$, $\overline{x} \ge \overline{0}$

has $z_j - c_j \geq 0$ for all j, then $\overline{x}_B$ is

(a) an optimal solution	(b) a non-optimal solution
(c) only a basic feasible solution	(d) none of these

3. In converting a ≤ type constraint to equality typex in a L.P.P., we must add

(a) a surplus variable	(b) an artificial variable

(c) a slack variable (d) both (a) and (b)

4. In converting a ≥ type constraint to equality type in a L.P.P., we must subtract

(a) a surplus variable	(b) a slack variable
(c) an artificial variable	(d) None of these

5. The common element of key column and key row in a simplex table is called

(a) basic element	(b) important element
(c) vital element	(d) Key element

Answers

1. (c)	2. (a)	3. (c)	4. (a)	5. (d)

Unit - 10 ❑ Simplex Alogrithm (II) : Big-M Method, Two-phase Method

Structure

10.0 Objective

After studying this chapter, the reader should be able to

- solve a LPP for mixed type of constraints by simplex method
- solve a LPP for mixed type of constraints by two phase simplex method

10.1 Introduction

We have learnt in the previous unit how a LPP can be solved by Simplex Algorithm. We have seen that if an initial basic feasible solution (I.B.F.S.) to the problem is easily identifiable, then we arrive at the initial simplex table without much labour. Unfortunately, in practice, there exists LPP to which an I.B.F.S. may not be easily determined. Moreover, the given LPP may even have no feasible solution at all.

These complications in solving a LPP by simplex method and their resolution will be discussed in the present unit.

In order to able to obtain an I.B.F.S. easily, a special procedure is available. If some of the variables in standard form of a LPP are surplus varibles, the corresponding column vectors do not provide unit vectors for the initial basis. In order to have unit vectors in the basis, new non-negative variables are introduced in the basis. These are called *artificial varibales*. The vectors of the coefficient matrix corresponding to these artificial variables are called artificial vectors. The new system of equations invoking slack, surplus and artificial variables are called *augmented system*. The artificial variables are brought in as a technique to have the initial basis made of unit vectors otherwise they have no real significance and as such need to be removed from the optimal solution.

There are two procedures for solving this new type of problem, viz.,

(i) Method of Penalties or Big-M Method

(ii) Two-Phase Method.

10.2 Big-M Method

A. Charnes suggested that a very high penalty can be paid for introducing the artificial variables in the constraints of a given problem, by assigning a very large negative cost (penalty) to the artificial variables in the objective function of the problem.

A simple way to do this is to assign-M to the cost coeffient corresponding to the artificial variables where M is a large positive number. This M is known as the penalty.

The very purpose of introducing the artificial variables was just to obtain an I.B.F.S. and so we would like to get rid of these variables once the very purpose has been achieved. Thus the objective to huge penalty cost is to ensure that all the artificial variables will be driven to zero when z is optimized by using simplex method. This method is known as the **Method of Penalties** or the **Big-M Method**.

The following examples illustrate the method.

● **Example 10.2.1 :** Use Charnes' Big-M method to solve the following LPP :

$$\begin{aligned} \text{Minimize} \quad & z = 4x_1 + 2x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \geq 27 \\ & x_1 + x_2 \geq 21 \\ & x_1 + 2x_2 \geq 30 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution : Introducing surplus and artificial variables, the given LPP can be reduced to its standard form as

$$\text{Maximize } (-z) = -4x_1 - 2x_2 + 0.x_3 + 0.x_4 + 0.x_5 - Mx_6 - Mx_7 - Mx_8$$

$$\begin{aligned} \text{subject to} \quad & 3x_1 + x_2 - x_3 + 0.x_4 + 0.x_5 + x_6 + 0.x_7 + 0.x_8 = 27 \\ & x_1 + x_2 + 0.x_3 - x_4 + 0.x_5 + 0.x_6 + x_7 + 0.x_8 = 21 \\ & x_1 + 2x_2 + 0.x_3 + 0.x_4 - x_5 + 0.x_6 + 0.x_7 + x_8 = 30 \\ & x_j \geq 0 \ (j = 1, 2, \ldots.., 8) \end{aligned}$$

Here, x_3, x_4, x_5 are surplus variables and x_6, x_7, x_8 are artificial variables. Now, we solve this problem by simplex method through the following simplex tableau.

			c_j	-4	-2	0	0	0	$-M$	$-M$	$-M$		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	Min. Ratio	
$-M$	$\bar{a}_6$	x_6	27	$\boxed{3}$	1	-1	0	0	1	0	0	$\textcircled{\frac{27}{3}}$	
$-M$	$\bar{a}_7$	x_7	21	1	1	0	-1	0	0	1	0	$\frac{21}{1}$	
$-M$	$\bar{a}_8$	x_8	30	1	2	0	0	-1	0	0	1	$\frac{30}{1}$	
$z_j - c_j$				$-5M+4\uparrow$	$-4M+2$	M	M	M	$0\downarrow$	0	0	Min. Ratio	Operations
-4	$\bar{a}_1$	x_1	9	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	0		0	0	$\frac{9}{1/3} = 27$	$R_1' = \frac{1}{3}R_1$
$-M$	$\bar{a}_7$	x_7	12	0	$\frac{2}{3}$	$\frac{1}{3}$	-1	0		1	0	$\frac{12}{2/3} = 18$	$R_2' = R_2 - R_1'$
$-M$	$\bar{a}_8$	x_8	21	0	$\boxed{\frac{5}{3}}$	$\frac{1}{3}$	0	-1		0	1	$\frac{21}{5/3} = \textcircled{\frac{63}{5}}$	$R_3' = R_3 - R_1'$
$z_j - c_j$				0	$-\frac{7M}{3}+\frac{2}{3}\uparrow$	$-\frac{2M}{3}+\frac{4}{3}$	M	M	×	0	$0\downarrow$		

			c_j	-4	-2	0	0	0	$-M$	$-M$	$-M$		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	Min. Ratio	Operations
-4	$\bar{a}_1$	x_1	$\frac{24}{5}$	1	0	$-\frac{2}{5}$	0	$\frac{1}{5}$		0		$\frac{24}{5}\Big/\frac{1}{5} = 24$	$R_1'' = R_1' - \frac{1}{3}R_3''$
$-M$	$\bar{a}_7$	x_7	$\frac{18}{5}$	0	0	$\frac{1}{5}$	-1	$\boxed{\frac{2}{5}}$		1		$\frac{18}{5}\Big/\frac{2}{5} =$ ⑨	$R_2'' = R_2' - \frac{2}{3}R_3''$
-2	$\bar{a}_2$	x_2	$\frac{63}{5}$	0	1	$\frac{1}{5}$	0	$-\frac{3}{5}$		0		—	$R_3'' = \frac{3}{5}R_3''$
$z_j - c_j$				0	0	$-\frac{M}{5}+\frac{6}{5}$	M	$-\frac{2M}{5}+\frac{2}{5}\uparrow$	×	$0\downarrow$	×	Min. Ratio	Operations
-4	$\bar{a}_1$	x_1	3	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0					$R_1''' = R_1'' - \frac{1}{5}R_2'''$
0	$\bar{a}_5$	x_5	9	0	0	$\frac{1}{2}$	$-\frac{5}{2}$	1					$R_2''' = \frac{2}{5}R_2''$
-2	$\bar{a}_2$	x_2	18	0	1	$\frac{1}{2}$	$-\frac{3}{2}$	0					$R_3''' = R_3'' + \frac{3}{5}R_2'''$
	$z_j - c_j$			0	0	1	1	0	×	×	×		

Since all $z_j - c_j \geq 0$, so optimality arises. It is seen that no artificial variable appears in the optimal basis. Thus, the optimal solution of the LPP is

$x_1 = 3$, $x_2 = 18$ and Max $(-z) = -$ Min. $z = -4 \times 3 - 2 \times 18 = -48$. Therefore, $z_{min.} = 48$ units.

● **Example 10.2.2 :** Use Big-M method to solve the following LPP :

$$\begin{aligned} &\text{Maximize } z = 5x_1 - 4x_2 + 3x_3 \\ &\text{subject to} \quad 2x_1 + x_2 - 6x_3 = 20 \\ &\qquad 6x_1 + 5x_2 + 10x_3 \leq 76 \\ &\qquad 8x_1 - 3x_2 + 6x_3 \leq 50 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution : Using artificial variable x_4 and slack variables x_5 and x_6, the LPP can be written in the standard form as

$$\begin{aligned} &\text{Maximize } z = 5x_1 - 4x_2 + 3x_3 - Mx_4 + 0.x_5 + 0.x_6 \\ &\text{subject to} \quad 2x_1 + x_2 - 6x_3 + x_4 + 0.x_5 + 0.x_6 = 20 \\ &\qquad 6x_1 + 5x_2 + 10x_3 + 0.x_4 + x_5 + 0.x_6 = 76 \\ &\qquad 8x_1 - 3x_2 + 6x_3 + 0.x_4 + 0.x_5 + x_6 = 50 \\ &\qquad x_j \geq 0 \ (j = 1, 2, \ldots.., 6) \end{aligned}$$

Now, we solve the problem by simplex method through the following simplex tableau.

			c_j	5	–4	3	–M	0	0		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	Min. Ratio	
–M	$\bar{a}_4$	x_4	20	2	1	–6	1	0	0	$\frac{20}{2}$	
0	$\bar{a}_5$	x_5	76	6	5	10	0	1	0	$\frac{76}{6}$	
0	$\bar{a}_6$	x_6	50	$\boxed{8}$	–3	6	0	0	1	$\frac{50}{8}$	
$z_j - c_j$				$-2M-5\uparrow$	$-M+4$	$6M-3$	0	0	$0\downarrow$	Min. Ratio	Operations
–M	$\bar{a}_4$	x_4	$\frac{15}{2}$	0	$\boxed{\frac{7}{4}}$	$-\frac{15}{2}$	1	0	$-\frac{1}{4}$	$\frac{15}{2}\Big/\frac{7}{4}=\frac{30}{7}$	$R_1' = R_1 - 2R_3'$
0	$\bar{a}_5$	x_5	$\frac{77}{2}$	0	$\frac{29}{4}$	$\frac{11}{2}$	0	1	$-\frac{3}{4}$	$\frac{77}{2}\Big/\frac{29}{4}=\frac{154}{29}$	$R_2' = R_2 - 6R_3'$
5	$\bar{a}_1$	x_1	$\frac{25}{4}$	1	$-\frac{3}{8}$	$\frac{3}{4}$	0	0	$\frac{1}{8}$	—	$R_3' = \frac{1}{8}R_3$
$z_j - c_j$				0	$-\frac{7M}{4}+\frac{17}{8}\uparrow$	$\frac{15M}{2}+\frac{3}{4}$	$0\downarrow$	0	$\frac{M}{4}+\frac{5}{8}$	Min. Ratio	Operations
–4	$\bar{a}_2$	x_2	$\frac{30}{7}$	0	1	$-\frac{30}{7}$		0	$-\frac{1}{7}$		$R_1'' = \frac{4}{7}R_1'$
0	$\bar{a}_5$	x_5	$\frac{52}{7}$	0	0	$\frac{256}{7}$		1	$\frac{2}{7}$		$R_2'' = R_2' - \frac{29}{4}R_1''$
5	$\bar{a}_1$	x_1	$\frac{55}{7}$	1	0	$-\frac{6}{7}$		0	$\frac{1}{14}$		$R_3'' = R_3' + \frac{3}{8}R_1''$
$z_j - c_j$				0	0	$\frac{69}{7}$	×	0	$\frac{13}{14}$		

Since all $z_j - c_j \geq 0$, optimality arises. It is observed that no artificial variable appears in the optimal basis. Hence, the optimal solution of the LPP is

$$x_1 = \frac{55}{7},\ x_2 = \frac{30}{7},\ x_3 = 0 \text{ and } z_{\text{max.}} = 5 \times \frac{55}{7} - 4 \times \frac{30}{7} = \frac{155}{5} \text{ units}$$

● **Example 10.2.3 :** Use Big-M method to solve the following LPP :

$$\text{Maximize} \quad z = 4x_1 + 2x_2$$

$$\text{subject to} \quad 2x_1 + x_2 \leq 4$$

$$5x_1 + 3x_2 \geq 15$$

$$x_1, x_2 \geq 0$$

Solution : Using slack variable x_3, surplus variable x_4 and an artificial variable x_5, the standard form of the given LPP is

$$\text{Maximize } z = 4x_1 + 2x_2 + 0.x_3 + 0.x_4 - Mx_5$$

$$\text{subject to} \quad 2x_1 + x_2 + x_3 + 0.x_4 + 0.x_5 = 4$$

$$5x_1 + 3x_2 + 0.x_3 - x_4 + x_5 = 15$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Now, we solve this problem by simplex method through the following simplex tableau :

			c_j	4	2	0	0	–M		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	Min. Ratio	
0	$\bar{a}_3$	x_3	4	$\boxed{2}$	1	1	0	0	$\frac{4}{2}=$②	
–M	$\bar{a}_5$	x_5	15	5	3	0	–1	1	$\frac{15}{5}=3$	
$z_j - c_j$				$-5M-4\uparrow$	$-3M-2$	$0\downarrow$	M	0	Min. Ratio	Operations
4	$\bar{a}_1$	x_1	2	1	$\boxed{\frac{1}{2}}$	$\frac{1}{2}$	0	0	$2/\frac{1}{2}=$④	$R_1' = \frac{1}{2}R_1$
–M	$\bar{a}_5$	x_5	5	0	$\frac{1}{2}$	$-\frac{5}{2}$	–1	1	$5/\frac{1}{2}=10$	$R_2' = R_2 - 5R_1'$
$z_j - c_j$				0	$-\frac{M}{2}\uparrow$	$\frac{5M}{2}+2$	M	0		Operations
2	$\bar{a}_2$	x_1	4	2	1	1	0	0		$R_1'' = 2R_1'$
–M	$\bar{a}_5$	x_5	3	–1	0	–3	–1	1		$R_2'' = R_2' - \frac{1}{2}R_1''$
$z_j - c_j$				M	0	3M+2	M	0		

Since all $z_j - c_j \geq 0$, so optimality arises. It is seen that the artificial variable x_5 appears in the optimal basis at the positive level, that is, $x_5 = 3$. This concludes that the LPP has no feasible solution.

10.3 Two-Phase Simplex Method

As in the case of 'Big-M Method', here also we are to eliminate the artificial variables from the optimal soultion. In this case, the problem is solved in two phases. In Phase-I, the artificial variables are eliminated and in the Phase-II, an optimal solution of the problem is obtained by usual simplex method.

The computational procedure for solving a linear programming problem by 'Two-Phase Simplex Method' is as follows :

Phase-I : In a maximization problem, we reconstruct the objective function as $z^* = 0.x_1 + 0.x_2 + ... + 0.x_n - 1.x_{n+1} - 1.x_{n+2} - ..... - 1.x_{n+m}$, where $x_1, x_2,....., x_n$ are orginal, slack and surplus variables and $x_{n+1}, x_{n+2}, ..., x_{n+m}$ are artificial variables. Here, z^* is called the Auxiliary objective function and the problem is called an auxiliary LPP. Now, we solve this problem by simplex method to maximize the new objective function z^*.

To do so, the following cases may arise :

Case (i) : Max. $z^* = 0$ and artificial variables either appears in the optimum basis at zero level or no artificial vector appears in the optimum basis. Then, we will use Phase-II to find an optimal solution of the given LPP.

Case (ii) : Max. $z^* < 0$ and at least one artificial variable appears in the optimum basis at the positive level. In this case, the LPP does not possess any feasible solution. So, there is no need to consider Phase-II in this case.

Phase-II : Thus, if Phase-I provides a solution which gives Max. $z^* = 0$, then we go to Phase-II for optimal solution of the original problem. To start the Phase-II, we first consider the original objective function by assigning the actual prices to each of the original variables and a zero price to each of the artificial variables present in the final iteration of Phase-I at the zero level. Then we take optimal solution obtained in the last iteration of the Phase-I as initial basic feasible solution and proceed as usual by simplex method until an optimum solution (if any) is obtained.

Note : The first tableau of the Phase-II is almost same with the last tableau of the Phase-I with the changes in the row c_j, in the column of $\overline{c}_B$ and in the row of $z_j - c_j$.

The following examples illustrate the method.

● **Example 10.3.1 :** Solve the following LPP by using Two-Phase simplex method :

$$\text{Minimize } z = x_1 + x_2$$

$$\text{subject to} \quad 2x_1 + 4x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

Solution : Using surplus variables x_3, x_4 and artificial variables x_5, x_6, the given constraints of the LPP can be written as

$$2x_1 + 4x_2 - x_3 + 0.x_4 + x_5 + 0.x_6 = 4$$

$$x_1 + 7x_2 + 0.x_3 - x_4 + 0.x_5 + x_6 = 7$$

$$x_j \geq 0 \quad (j = 1, 2, ..., 6)$$

Now, changing the objective function to maximizing problem with surplus variables, we get

$$\text{Maximize } (-z) = -x_1 - x_2 + 0.x_3 + 0.x_4$$

subject to the above constraints.

Phase-I : In Phase-I, the auxiliary objective function of the LPP is

$$z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - x_5 - x_6.$$

The auxiliary problem is to maximize z* subject to the above constraints.

Now, we solve this problem by simplex method through the following simplex tableau.

			c_j	0	0	0	0	–1	–1		
$\overline{c}_B$	B	$\overline{x}_B$	$\overline{b}$	$\overline{a}_1$	$\overline{a}_2$	$\overline{a}_3$	$\overline{a}_4$	$\overline{a}_5$	$\overline{a}_6$	Min. Ratio	
–1	$\overline{a}_5$	x_5	4	2	$\boxed{4}$	–1	0	1	0	$\frac{4}{4} = ①$	
–1	$\overline{a}_6$	x_6	7	1	7	0	–1	0	1	$\frac{7}{1} = 7$	
$z_j - c_j$				–3	–11↑	1	1	0↓	0	Min. Ratio	Operations
0	$\overline{a}_2$	x_2	1	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	–	$R_1' = \frac{1}{4}R_1$
–1	$\overline{a}_6$	x_6	0	$-\frac{5}{2}$	0	$\boxed{\frac{7}{4}}$	–1	$-\frac{7}{4}$	1	⓪	$R_2' = R_2 - 7R_1'$
$z_j - c_j$				$5/2$	0	$-7/4$↑	1	$11/4$	0↓		
0	$\overline{a}_2$	x_2	1	$1/7$	1	0	$-\frac{1}{7}$	0	$\frac{1}{7}$		$R_1'' = R_1' + \frac{1}{4}R_2''$
0	$\overline{a}_3$	x_3	0	$-\frac{10}{7}$	0	1	$-\frac{4}{7}$	–1	$\frac{4}{7}$		$R_2'' = \frac{4}{7}R_2'$
$z_j - c_j$				0	0	0	0	1	1		

Since all $z_j - c_j \geq 0$, optimality arises for Phase-I problem. Also Max $z^* = 0$ and no artificial variable appears in the final basis. So, we pass on to the Phase-II.

Phase-II : In Phase-II, the objective function of the LPP is

$$\text{Max } (-z) = -x_1 - x_2 + 0.x_3 + 0.x_4$$

The final table of Phase-I will be the initial table of Phase-II with the changes in the cost coefficients and we neglect the artificial vectors. The initial tableau is as follows :

Phase-II Tableau

			c_j	-1	-1	0	0
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$
-1	$\bar{a}_2$	x_2	1	$\frac{1}{7}$	1	0	$-\frac{1}{7}$
0	$\bar{a}_3$	x_3	0	$-\frac{10}{7}$	0	1	$-\frac{4}{7}$
$z_j - c_j$				$6/7$	0	0	$1/7$

Since all $z_j - c_j \geq 0$, so optimality arises. The optimal solution is $x_1 = 0$, $x_1 = 1$ and Max. $(-z) = -1$ i.e., $z_{min} = 1$ unit.

● **Example 10.3.2 :** Solve the following LPP by Two-Phase simplex method :

$$\begin{aligned} &\text{Maximize } z = 2x_1 - x_2 + x_3 \\ &\text{subject to} \quad x_1 + x_2 - 3x_3 \leq 8 \\ &\qquad 4x_1 - x_2 + x_3 \geq 2 \\ &\qquad 2x_1 + 3x_2 - x_3 \geq 4 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution : Using slack, surplus and atrificial variables, the constraints may be written as

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 + 0.x_5 + 0.x_6 + 0.x_7 + 0.x_8 &= 8 \\ 4x_1 - x_2 + x_3 + 0.x_4 - x_5 + 0.x_6 + x_7 + 0.x_8 &= 2 \\ 2x_1 + 3x_2 - x_3 + 0.x_4 + 0.x_5 - x_6 + 0.x_7 + x_8 &= 4 \\ x_j \geq 0 \ (j = 1, 2, ..., 8) \end{aligned}$$

Phase–I : In Phase I, auxiliary objective function is

$$z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6 - x_7 - x_8.$$

The auxiliary problem is to maximize z* subject to the above constraints. Now, we solve this problem by simplex method through the following simplex tableau.

Phase–I Tableau

			c_j	0	0	0	0	0	0	–1	–1		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	$\bar{a}_7$	$\bar{a}_8$	Min. Ratio	
0	$\bar{a}_4$	x_4	8	1	1	–3	1	0	0	0	0	$\frac{8}{1}=8$	
–1	$\bar{a}_7$	x_7	2	$\boxed{4}$	–1	1	0	–1	0	1	0	$\frac{2}{4}=\frac{1}{2}$	
–1	$\bar{a}_8$	x_8	4	2	3	–1	0	0	–1	0	1	$\frac{4}{2}=2$	
z_j-c_j				–6↑	–2	0	0	1	1	0↓	0	Min. Ratio	Operations
0	$\bar{a}_4$	x_4	$\frac{15}{2}$	0	$\frac{5}{4}$	$-\frac{13}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$\frac{15}{2}\Big/\frac{5}{4}=6$	$R_1' = R_1 - R_2'$
0	$\bar{a}_1$	x_1	$\frac{1}{2}$	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	—	$R_2' = \frac{1}{4}R_2$
–1	$\bar{a}_8$	x_8	3	0	$\boxed{\frac{7}{2}}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	–1	$-\frac{1}{2}$	1	$3\big/\frac{7}{2}=\frac{6}{7}$	$R_3' = R_3 - 2R_2'$
z_j-c_j				0	$-\frac{7}{2}$↑	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	0↓	Min. Ratio	Operations
0	$\bar{a}_4$	x_4	$\frac{45}{7}$	0	0	$-\frac{19}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$		$R_1'' = R_1 - \frac{5}{4}R_3''$
0	$\bar{a}_1$	x_1	$\frac{5}{7}$	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{14}$		$R_2'' = R_2' + \frac{1}{4}R_3''$
0	$\bar{a}_2$	x_2	$\frac{6}{7}$	0	1	$-\frac{3}{7}$	0	$\frac{1}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$		$R_3'' = \frac{2}{7}R_3'$
z_j-c_j				0	0	0	0	0	0	1	1		

Since all $z_j - c_j \geq 0$, so optimality arises. No artificial variable appears in the final basis and Max. $z^* = 0$. So, we pass on to the Phase-II.

Phase-II : In Phase-II, the objective function of the LPP is

$$\text{Max. } z = 2x_1 - x_2 + x_3 + 0.x_4 + 0.x_5 + 0.x_6$$

The final table of Phase-I will be the initial table of Phase-II with the changes in the cost coefficients and we neglect the artificial vectors. The simplex tableau is as follows :

Phase-II Tableau

			c_j	2	–1	1	0	0	0		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	Min. Ratio	
0	$\bar{a}_4$	x_4	$\frac{45}{7}$	0	0	$-\frac{19}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{45}{7}\Big/\frac{1}{14}=90$	
2	$\bar{a}_1$	x_1	$\frac{5}{7}$	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	—	
–1	$\bar{a}_2$	x_2	$\frac{6}{7}$	0	1	$-\frac{3}{7}$	0	$\boxed{\frac{1}{7}}$	$-\frac{2}{7}$	$\frac{6}{7}\Big/\frac{1}{7}=$ ⑥	
$z_j - c_j$				0	0↓	$-\frac{2}{7}$	0	$-\frac{4}{7}$↑	$\frac{1}{7}$	Min. Ratio	Operations
0	$\bar{a}_4$	x_4	6	0	$-\frac{1}{2}$	$-\frac{5}{2}$	1	0	$\frac{1}{2}$		$R_1' = R_1 - \frac{1}{14}R_3'$
2	$\bar{a}_1$	x_1	2	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$		$R_2' = R_2 + \frac{3}{14}R_3'$
0	$\bar{a}_5$	x_5	6	0	7	–3	0	1	–2		$R_3' = 7R_3$
$z_j - c_j$				0	4	–2↑	0	0	–1		

We see that Min $\{z_j - c_j : z_j - c_j < 0\} = z_3 - c_3$ and all the elements in the corresponding column are all negative. Hence, the problem has an unbounded solution.

● **Example 10.3.3 :** Solve the following LPP by Two-Phase simplex method :

$$\begin{aligned} \text{Maximize} \quad & z = 5x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 1 \\ & 3x_1 + 4x_2 \geq 16 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution : Using slack, surplus and artificial variables, the constaints of the given LPP can be written as

$$2x_1 + x_2 + x_3 + 0.x_4 + 0.x_5 = 1$$
$$3x_1 + 4x_2 + 0.x_3 - x_4 + x_5 = 16$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Phase-I : In Phase-I, the auxiliary objective function is

$$z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - x_5$$

The auxiliary problem is to maximize z* subject to the above constraints. Now, we solve this problem by simplex method through the following simplex tableau.

Phase-I Tableau

			c_j	0	0	0	0	–1		
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	Min. Ratio	
0	$\bar{a}_3$	x_3	1	2	$\boxed{1}$	1	0	0	$\frac{1}{1}=1$	
–1	$\bar{a}_5$	x_5	16	3	4	0	–1	1	$\frac{16}{4}=4$	
$z_j - c_j$				–3	$-4\uparrow$	0	1	0	Min. Ratio	Operations
0	$\bar{a}_2$	x_2	1	2	1	1	0	0		$R_1' = R_1$
–1	$\bar{a}_5$	x_5	12	–5	0	–4	–1	1		$R_2' = R_2 - 4R_1'$
$z_j - c_j$				5	0	4	1	0		

Since all $z_j - c_j \geq 0$, so optimality arises. We see that artificial variable x_5 appears in the final basis at the positive level (i.e., $x_5 = 12$) and hence Max. $z^* = -12 < 0$.

Hence we conclude that the LPP has no feasible solution. There is no need to consider Phase-II of the problem.

10.4 Summary

In this chapter, the solution of a LPP by simplex method is discussed when the LPP has mixed type of constraints. To eliminate artificial variables appear in a LPP containing mixed type of constraints, another method of solution known as Two-phase simplex method is also presented with several examples.

10.5 Exercise

1. Solve the following LPP by Charnes' Big-M method :

(i) Maximize $z = 3x_1 - x_2$

subject to
$$2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

(ii) Minimize $z = 12x_1 + 20x_2$

subject to
$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

(iii) Maximize $z = 3x_1 + 2x_2$

subject to
$$x_1 + x_2 \geq 1$$
$$2x_1 + x_2 \leq 4$$
$$5x_1 + 8x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

(iv) Maximize $z = 2x_1 + x_2 + 2x_3$

subject to
$$x_1 + x_2 + 2x_3 \leq 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$
$$x_1, x_2, x_3 \geq 0$$

(v) Minimize $z = 2x_1 + x_2$

subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

(vi) Maximize $z = 3x_1 + 2x_2$

subject to
$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

(vii) Maximize $z = x_1 - 2x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1, x_2, x_3 \geq 0$

(viii) Maximize $z = 2x_1 - x_2 + 2x_3$

subject to $x_1 + x_2 - 3x_3 \leq 8$

$4x_1 - x_2 + x_3 \geq 2$

$2x_1 + 3x_2 - x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

2. Use Two-Phase simplex method to solve the following LPP :

(i) Maximize $z = 2x_1 + 2x_2$

subject to $2x_1 + x_2 \leq 1$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

(ii) Minimize $z = 4x_1 + x_2$

subject to $x_1 + 2x_2 \leq 3$

$4x_1 + 3x_2 \geq 6$

$3x_1 + x_2 = 3$

$x_1, x_2 \geq 0$

(iii) Maximize $z = -3x_1 - 5x_2$

subject to $x_1 + 2x_2 \geq 8$

$3x_1 + 2x_2 \geq 12$

$5x_1 + 6x_2 \leq 60$

$x_1, x_2 \geq 0$

(iv) Maximize $z = 3x_1 + 2x_2 + x_3$

subject to $-3x_1 + 2x_2 + 2x_3 = 8$

$-3x_1 + 4x_2 + x_3 = 7$

$x_1, x_2, x_3 \geq 0$

(v) Maximize $z = 2x_1 + 4x_2 + x_3$

$$\begin{aligned} \text{subject to} \quad x_1 - 2x_2 - x_3 &\le 5 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ -x_1 + 2x_1 + 2x_3 &\ge 1 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Answer

1. (i) $x_1 = 3,\ x_2 = 0$ and $z_{max} = 9$

   (ii) $x_1 = 15,\ x_2 = \frac{5}{4}$ and $z_{min} = 205$

   (iii) $x_1 = \frac{17}{11},\ x_2 = \frac{10}{11}$ and $z_{max} = \frac{71}{11}$

   (iv) $x_1 = 3,\ x_2 = 2,\ x_3 = 0$ and $z_{max} = 8$

   (v) $x_1 = \frac{3}{5},\ x_2 = \frac{6}{5}$ and $z_{min} = \frac{12}{5}$

   (vi) No feasible solution

   (vii) $x_1 = 0,\ x_2 = \frac{15}{7},\ x_3 = \frac{25}{7}$ and $z_{max} = \frac{45}{7}$

   (viii) Unbounded solution

2. (i) No feasible solution

   (ii) $x_1 = \frac{3}{5},\ x_2 = \frac{6}{5}$ and $z_{min} = \frac{18}{5}$

   (iii) $x_1 = 2,\ x_2 = 3$ and $z_{max} = 21$

   (iv) Unbounded solution

   (v) Unbounded solution

10.6 Multiple Choice Questions (MCQ)

1. In phase-I simplex method for a maximization L.P.P., the coefficient of an artificial variable in the auxiliary objective function is

(a) M (b) 0 (c) –M (d) –1

2. If all $z_j - c_j \geq 0$, but an artificial variable appears in the basis in an interation of the simplex method, then the L.P.P. has

   (a) no feasible solution (b) a unique solution

   (c) an unbounded solution (d) infinitely many optimal solutions

3. In the Two-Phase simplex method for a maximization L.P.P. having an optimal solution, the maximum value of the auxiliary objective function in Phase-I in

   (a) non-zero quantity (b) zero

   (c) a negative quantity (d) a positive quantity

4. Which of the following is correct ?

   (a) All artificial variables must appear in the final basis for on optimal solution of a L.P.P.

   (b) At least one artificial variable must appear in the final basis for an optimal solution of a L.P.P.

   (c) No artificial variable appear in the final basis for an optimal solution of a L.P.P.

   (d) Artificial variables may appear in the final basis at the zero level for an optimal solution of a L.P.P.

Answers

1. (d) 2. (a) 3. (b) 4. (d)

Unit - 11 ❑ Duality in Linear Programming Problem

Structure

11.0 Objective

After studying this chapter, the reader should be able to

- understand the dual formulation procedure of a LPP
- interpret primal-dual optimum solutions
- understand the characteristics of dual problems.

11.1 Introduction

With every L.P.P. we can associate another L.P.P. which is derived from the first L.P.P. following definite rules. Here the second L.P.P. will be called dual problem and the original L.P.P. is called primal problem. Either of the two linear programming problems can be considered as the primal, with the remaining problem as the dual. The two problems possess very closely related properties.

It will be shown in subsequent section that the optimal value (if exist) of the objective functions of the two problems are same and the final table giving the optimal solution for one will contain necessary indications for optimal solution of the other.

The concept of duality will be made clear in the next section by considering the problems of a dealer and a shop keeper.

11.2 Concept of Duality

We suppose that a dealer sells three nutrients A, B, C and the shop keepers make two fodders X and Y (by purchasing these nutrients) for the consumption of the animals.

The following table shows the amount of three nutrients available in the unit quantity of the two fodders along with the minimum requirement of each nutrient for one animal and the market price of unit quantity of each fodder.

Fodder	Nutrient			Market Price per unit
	A	B	C	
X	1	2	2	4
Y	3	4	1	3
Minimum requirement	10	9	3	

Now the objective of the dealer will be to fix the selling prices of A, B, C in such a way that the resulting prices of the fodders X and Y do not exceed the existing market price and the total selling price z becomes maximum. If the selling prices of unit quantity of the nutrients A, B, C be respectively x_1, x_2, x_3 then from the objective of the dealer, we get the following linear programming problem :

$$\left.\begin{aligned} &\text{Maximize } z = 10x_1 + 9x_2 + 3x_3 \\ &\text{subject to} \\ &\qquad x_1 + 2x_2 + 2x_3 \le 4 \\ &\qquad 3x_1 + 4x_2 + x_3 \le 3 \\ &\qquad\quad x_1, x_2, x_3 \ge 0 \end{aligned}\right\} \text{(A)}$$

Now we consider the same problem from the view point of a shopkeeper.

The objective of a shopkeeper will be to fix the quantities of the fodders X and Y in such a way that the minimum requirement of the three nutrients for each animal is met and at the same time the total cost (w) of purchasing the fodders is minimum.

For each animal, let ν_1 units of X and ν_2 units of Y be produced. Then from the objective of the shopkeeper, we get the following L.P.P. :

$$\left.\begin{aligned} &\text{Minimize } w = 4\nu_1 + 3\nu_2 \\ &\text{subject to} \quad \nu_1 + 3\nu_2 \ge 10 \\ &\qquad\qquad\quad 2\nu_1 + 4\nu_2 \ge 9 \\ &\qquad\qquad\quad 2\nu_1 + \nu_2 \ge 3 \\ &\qquad\qquad\qquad \nu_1, \nu_2 \ge 0 \end{aligned}\right\} \text{(B)}$$

One of the problems (A) and (B) will be called **primal problem** and the other as **dual problem**.

Here we remark that the theorems on dual problems will remain unaltered, whether we call "(A) the primal problem and (B) the dual problem" or "(B) the primal problem and (A) the dual problem."

Here the problem (A) is a maximization problem and the problem (B) is a minimization problem.

In the following sections, we shall follow the convention of considerity the maximization problem as the primal problem and the minimization problem as the dual problem.

11.3 Mathematical Formulation of Dual Problem

We express the primal problem in the following form :

$$\begin{aligned}
\text{Maximize } z = \ & c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n \\
\text{subject to } \ & a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \le b_2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \le b_m \\
& \qquad\qquad x_1, x_2, \ldots\ldots\ldots, x_n \ge 0
\end{aligned}$$

Then the following L.P.P. will be called the dual of above primal problem :

$$\begin{aligned}
\text{Minimize } w = \ & b_1v_1 + b_2v_2 + \ldots\ldots\ldots\ldots + b_mv_m \\
\text{subject to } \ & a_{11}v_1 + a_{21}v_2 + \ldots\ldots\ldots + a_{m1}v_m \ge c_1 \\
& a_{12}v_1 + a_{22}v_2 + \ldots\ldots\ldots + a_{m2}v_m \ge c_2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& a_{1n}v_1 + a_{2n}v_2 + \ldots\ldots\ldots\ldots + a_{nm}v_m \ge c_n \\
& \qquad\qquad v_1, v_2, \ldots\ldots\ldots\ldots, v_m \ge 0
\end{aligned}$$

In matrix notation, we write the above problems as

Primal problem :

$$\left.\begin{aligned}
&\text{Maximize } z = \bar{c}\,\bar{x} \\
&\text{subject to } \quad A\bar{x} \le \bar{b}, \bar{x} \ge \bar{0}
\end{aligned}\right\} \ldots\ldots(1)$$

Dual problem :

$$\begin{aligned}
&\text{Minimize } w = \bar{b}'\,\bar{v} \\
&\text{subject to } \quad A'\bar{v} \ge \bar{c}', \ \bar{v} \ge \bar{0}
\end{aligned}$$

where $\bar{x} = [x_1, x_2, \ldots\ldots, x_n]$, $\bar{b} = [b_1, b_2, \ldots\ldots, b_m]$ are respectively n × 1 and

$m \times 1$ column matrices and $\overline{c} = (c_1, ..., c_n)$ is $1 \times n$ row matrix, A is a matrix of order $m \times n$ and $\overline{v} = [v_1, v_2, ..., v_m]$ is a column matrix of order $m \times 1$ [Here B′ denotes the transpose of a matrix B].

If the primal problem be expressed in the form (1), then this form will be called the standard form of the primal problem.

Remarks : From the definition of the dual problem we observe that—

(i) Number of constraints (except $\overline{x} \geq \overline{0}$) in the primal problem is equal to the number of variables in the dual problem and conversely, the number of constraints in the dual problem (except $\overline{v} \geq \overline{0}$) is equal to the number of variables in the primal problem.

(ii) Here the main constraints of the primal problem are of the type "≤" and the main constraints of the dual problem are of the type "≥".

(iii) Here $b_1, b_2, ..., b_m$ may not be positive.

(iv) The transpose matrix A′ of the coefficient matrix A of the constraints of the primal problem is the coefficient matrix of the constraints of the dual problem.

(v) $\overline{x} \geq \overline{0}$ means $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$ and $\overline{v} \geq \overline{0}$ means $v_1 \geq 0, v_2 \geq 0, ... v_m \geq 0$.

● Some Definitions

(a) Symmetric problem

A primal problem (or its dual problem) will be called symmetric if all constraints are in the form of inequalities.

(b) Unsymmetric problem

A primal problem (or its dual problem) will be called unsymmetric if all the constraints are of the form of equalities.

(c) Mixed type problem

A primal problem (or its dual problem) will be called a **mixed type** problem if at least one constraint is in the form of inequality and at least one constraint is in the form of equality.

In Section 11.4, we shall see how the dual problem can be constructed where the primal problem is symmetric or unsymmetric or of the mixed type and we shall verify the following theorems.

Theorem 1 : If a constraint of the primal problem be in the form of an equation, then the corresponding variable of the dual problem will be **unrestricted in sign**.

Theorem 2 : If a variable in the primal problem be unrestricted in sign then the corresponding constraint in the dual problem will be in the form of an equation.

11.4 A Few Examples

● **Example 11.4.1 :** Find the dual of the following L.P.P :

$$\begin{aligned} \text{Maximize } z &= x_1 + 6x_2 \\ \text{subject to } \quad x_1 + x_2 &\geq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution : The given primal problem is equivalent to the L.P.P.

$$\begin{aligned} \text{Maximize } z &= x_1 + 6x_2 \\ \text{subject to } \quad -x_1 - x_2 &\leq -2 \\ x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The last problem can be expressed as

Maximize $z = (1, 6)\ [x_1, x_2]$

$$\text{subject to } \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the required dual problem will be

Minimize $w = (-2,\ 3)\ [v_1, v_2]$

$$\text{subject to } \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

i.e., Minimize $w = -2\nu_1 + 3\nu_2$

$$\begin{aligned} \text{subject to} \quad -\nu_1 + \nu_2 &\geq 1 \\ -\nu_1 + 3\nu_2 &\geq 6 \\ \nu_1,\ \nu_2 &\geq 0. \end{aligned}$$

● **Example 11.4.2 :** Find the dual of the following primal problem :

$$\begin{aligned} &\text{Maximize } z = x_1 - 3x_2 \\ &\text{subject to} \quad 3x_1 + 2x_2 \leq 6 \\ &\qquad\qquad 3x_1 - x_2 = 4 \\ &\qquad\qquad x_1,\ x_2 \geq 0 \end{aligned}$$

Solution : The given L.P.P. is of the mixed type. First we shall express the primal L.P.P. in the standard form.

Here the constraint $3x_1 - x_2 = 4$ is equivalent to two constraints $3x_1 - x_2 \leq 4$ and $3x_1 - x_2 \geq 4$ i.e., $3x_1 - x_2 \leq 4$ and $-3x_1 + x_2 \leq -4$.

So the primal problem can be written as

$$\left.\begin{aligned} &\text{Maximize } z = x_1 - 3x_2 \\ &\text{subject to} \quad 3x_1 + 2x_2 \leq 6 \\ &\qquad\qquad 3x_1 - x_2 \leq 4 \\ &\qquad\qquad -3x_1 + x_2 \leq -4 \\ &\qquad\qquad x_1,\ x_2 \geq 0. \end{aligned}\right\} (1)$$

Now the dual problem of (1) will be

$$\begin{aligned} &\text{Minimize } w = 6\nu_1 + 4\nu_2' - 4\nu_2'' \\ &\text{subject to} \quad 3\nu_1 + 3\nu_2' - 3\nu_2'' \geq 1 \\ &\qquad\qquad 2\nu_1 - \nu_2' + \nu_2'' \geq -3 \\ &\qquad\qquad \nu_1,\ \nu_2',\ \nu_2'' \geq 0 \end{aligned}$$

Writing $\nu_2' - \nu_2'' = \nu_2$, we find that ν_2 is unrestricted in sign.

So the above dual problem can be expressed as

$$\left.\begin{array}{l} \text{Minimize } z = 6v_1 + 4v_2 \\ \text{subject to} \qquad 3v_1 + 3v_2 \geq 1 \\ \qquad\qquad\qquad 2v_1 - v_2 \geq -3 \end{array}\right\} \text{..... (2)}$$

$v_1 \geq 0$, v_2 is unrestricted in sign.

The problem (2) is the required dual problem of the primal problem (1).

Remark : It is observed that the number of variables in the given primal is two which is equal to the number of constraints in the dual problem. Again the second constraint of the primal problem being in the form of equality, the variable v_2 in the dual problem is unrestricted in sign.

● **Example 11.4.3 :** Find the dual of the following L.P.P. :

$$\begin{array}{ll} \text{Maximize } z = 7x_1 + 5x_2 - 2x_3 & \\ \text{subject to} & x_1 + x_2 + x_3 = 10 \\ & 2x_1 - x_2 + 3x_3 \leq 16 \\ & -3x_1 - x_2 + 2x_3 \leq 5 \end{array}$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign.

Solution : Here x_3 is unrestricted in sign. So we can write $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0$, $x''_3 \geq 0$.

Then the given primal problem can be expressed as

$$\begin{array}{ll} \text{Maximize } z = 7x_1 + 5x_2 - 2\,(x'_3 - x''_3) & \\ \text{subject to} & x_1 + x_2 + x'_3 - x''_3 \leq 10 \\ & -x_1 - x_2 - x'_3 + x''_3 \leq -10 \\ & 2x_1 - x_2 + 3x'_3 - 3x''_3 \leq 16 \quad \text{........ (1)} \\ & -3x_1 - x_2 + 2x'_3 - 2x''_3 \leq 5 \\ & x_1, x_2, x'_3, x''_3 \geq 0 \end{array}$$

Now the dual of the primal problem will be

$$\left.\begin{aligned} &\text{Minimize } w = 10\nu_1' - 10\nu_1'' + 16\nu_2 + 5\nu_3 \\ &\text{subject to} \qquad \nu_1' - \nu_1'' + 2\nu_2 - 3\nu_3 \geq 7 \\ &\qquad\qquad\qquad \nu_1' - \nu_1'' - \nu_2 - \nu_3 \geq 5 \\ &\qquad\qquad\qquad \nu_1' - \nu_1'' + 3\nu_2 + 2\nu_3 \geq -2 \\ &\qquad\qquad\qquad \square -\nu_1' + \nu_1'' - 3\nu_2 - 2\nu_3 \geq 2 \\ &\qquad\qquad\qquad \nu_1', \nu_1'', \nu_2, \nu_3 \geq 0. \end{aligned}\right\} (2)$$

Now writing $\nu_1' - \nu_1'' = \nu_1$, the problem (2) can expressed as

$$\left.\begin{aligned} &\text{Minimize } w = 10\nu_1 + 16\nu_2 + 5\nu_3 \\ &\text{subject to} \qquad \nu_1 + 2\nu_2 - 3\nu_3 \geq 7 \\ &\qquad\qquad\qquad \nu_1 - \nu_2 - \nu_3 \geq 5 \\ &\qquad\qquad\qquad \nu_1 + 3\nu_2 + 2\nu_3 \geq -2 \\ &\qquad\qquad\qquad -\nu_1 - 3\nu_2 - 2\nu_3 \geq 2 \end{aligned}\right\} (3)$$

ν_1 unrestricted in sign, $\nu_2 \geq 0$, $\nu_3 \geq 0$.

We see that the constraints $\nu_1 + 3\nu_2 + 2\nu_3 \geq -2$ and $-\nu_1 - 3\nu_2 - 2\nu_3 \geq 2$ are equivalent to the single constraint

$$\nu_1 + 3\nu_2 + 2\nu_3 = -2.$$

So the required dual problem will be

$$\begin{aligned} &\text{Minimize } w = 10\nu_1 + 16\nu_2 + 5\nu_3 \\ &\text{subject to} \qquad \nu_1 + 2\nu_2 - 3\nu_3 \geq 7 \\ &\qquad\qquad\qquad \nu_1 - \nu_2 - \nu_3 \geq 5 \\ &\qquad\qquad\qquad \nu_1 + 3\nu_2 + 2\nu_3 = -2 \end{aligned}$$

ν_1 unrestricted in sign, $\nu_2 \geq 0$, $\nu_3 \geq 0$.

Remark : It is observed that the first constraint of the primal problem being an equality, the variable ν_1 of the dual problem is unrestricted in sign and the variable x_3 of the primal problem being unrestricted in sign, the corresponding constraint in the dual problem is an equation.

● **Example 11.4.4 :** Find the dual of the following L.P.P. :

$$\text{Maximize } z = 3x_1 - 2x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 &\le 4 \\ x_2 &\le 6 \\ x_1 + x_2 &\le 5 \\ -x_2 &\le -1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution : The given primal problem is in the standard form and it can be expressed in the matrix form as follows :

$$\text{Maximize } z = (3, -2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 6 \\ 5 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The dual problem of this L.P.P. will be

$$\text{Minimize } w = (4, 6, 5, -1) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \ge \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., Minimize $w = 4v_1 + 6v_2 + 5v_3 - v_4$

subject to $v_1 + v_3 \geq 3$

$v_2 + v_3 + v_4 \geq -2$

and $v_1, v_2, v_3, v_4 \geq 0.$

11.5 Duality Theorems

In the following theorems, we have considered the primal problem in the standard form i.e., the primal problem is taken as

Maximize $z = \bar{c}\,\bar{x}$

subject to $A\bar{x} \leq \bar{b}, \bar{x} \geq \bar{0}$ .... (1)

where $\bar{x} = [x_1, x_2, ..., x_n]$ is a $(n \times 1)$ column matrix, $\bar{c} = (c_1, c_2, ..., c_n)$ is a $1 \times n$ row matrix and A is a $m \times n$ martix. Now the dual problem of (1) will be Minimize $w = \bar{b}'\bar{v}$

subject to $A'\bar{v} \geq \bar{c}', \bar{v} \geq \bar{0}$ .... (2)

where $\bar{v} = [v_1, v_2, ..., v_m]$ is a $m \times 1$ column matrix.

● **Theorem 11.5.1 :** The dual of the dual problem of a primal L.P.P. is the primal problem. [The dual of the dual is the primal]

Proof : Consider the primal problem as

Maximize $z = \bar{c}\,\bar{x}$

subject to $A\bar{x} \leq \bar{b}, \bar{x} \geq \bar{0}$

and its dual problem as

Minimize $w = \bar{b}'\,\bar{v}$

subject to $A'\bar{v} \geq \bar{c}', \ \bar{v} \geq \bar{0}.$

Now this dual problem can be expressed as a maximizing problem as follows :

$$\left.\begin{aligned}&\text{Maximize } (-w) = -\bar{b}'\,\bar{v}\\&\text{subject to } \quad -A'\bar{v} \le -\bar{c}'\\&\qquad\qquad\qquad \bar{v} \ge \bar{0}\end{aligned}\right\}(3)$$

where $(w)_{\min} = -(w)_{\max}$.

According to the definition of dual problem, the dual of (3) will be

$$\text{Minimize } z_1 = (-\bar{c}')'\,\bar{x}$$

$$\text{subject to } (-A')'\,\bar{x} \ge (-\bar{b}')'$$

$$\bar{x} \ge \bar{0}$$

i.e.,
$$\left.\begin{aligned}&\text{Minimize } z_1 = -\bar{c}\,\bar{x}\\&\text{subject to } \quad -\bar{A}\bar{x} \ge -\bar{b}\\&\qquad\qquad\qquad \bar{x} \ge \bar{0}\end{aligned}\right\}(4)$$

Again the problem (4) can be expressed as

$$\text{Maximize } z = -(-\bar{c}\,\bar{x})$$

$$\text{subject to } \quad A\bar{x} \le \bar{b},\ \bar{x} \ge \bar{0}$$

i.e.,
$$\left.\begin{aligned}&\text{Maximize } z = \bar{c}\,\bar{x}\\&\text{subject to } A\bar{x} \le \bar{b},\ \bar{x} \ge \bar{0}\end{aligned}\right\}(5)$$

Now the problem (5) is the given primal problem. So the theorem is proved.

● **Theorem 11.5.2 :** If $\bar{x}$ and $\bar{v}$ are respectively the feasible solutions of the primal problem (1) and its dual problem (2), then $\bar{c}\,\bar{x} \le \bar{b}'\,\bar{v}$.

Proof : $\bar{x}$ is a feasible solution of the primal problem (1). Then we have $A\bar{x} \le \bar{b},\ \bar{x} \ge \bar{0}$ .... (6)

Also $\bar{v}$ is a feasible solution of the dual problem (2). Then we have $A'\bar{v} \ge \bar{c}',\ \bar{v} \ge \bar{0}$ .... (7)

From (6) we get, $\bar{v}'A\bar{x} \le \bar{v}'\bar{b}$ $(\because\ \bar{v} \ge \bar{0})$ .... (8)

Again from (7) we get, $(A'\bar{v})' \ge \bar{c}$

So $(A'\bar{v})'x \geq \bar{c}\,\bar{x}$ $(\because\ \bar{x} \geq \bar{0})$

i.e., $\bar{v}'A\bar{x} \geq \bar{c}\,\bar{x}$ .... (9)

From (8) and (9) we get,

$$\bar{c}\,\bar{x} \leq \bar{v}'\,\bar{b} \quad \text{or,} \quad \bar{c}\,\bar{x} \leq \left(\bar{b}'\,\bar{v}\right)'$$

Siince $\bar{b}'\,\bar{v}$ being a matrix of order 1 × 1, $\left(\bar{b}'\,\bar{v}\right)' = \bar{b}'\,\bar{v}$

So it is proved that $\bar{c}\,\bar{x} \leq \bar{b}'\,\bar{v}$.

Note : Here the primal problem (1) and the dual problem (2) are given at the beginning of the Section 6.5.

● **Theorem 11.5.3 :** If $\bar{x}^*$ and $\bar{v}^*$ are respectively the feasible solutions of the primal problem (1) and the dual problem (2) where $\bar{c}\,\bar{x}^* = \bar{b}'\,\bar{v}^*$, then $\bar{x}^*, \bar{v}^*$ are respectively optimal solutions of the two problems.

Proof : Here, it is given that $\bar{c}\,\bar{x}^* = \bar{b}'\,\bar{v}^*$ ...... (10)

Let $\bar{x}$ be a feasible solution of the primal problem (1). Again $\bar{v}^*$ being a feasible solution of the dual problem (2), by Theorem 11.5.2, we get

$$\bar{c}\,\bar{x} \leq \bar{b}'\,\bar{v}^*. \text{ .... (11)}$$

From (10) and (11) we get $\bar{c}\ \bar{x} \leq \bar{c}\ \bar{x}^*$ from which we can say that $\bar{x}^*$ is an optimal solution of the primal problem (1) where $\bar{c}\ \bar{x}^*$ is the maximum value of the objective function of this problem.

Again, $\bar{x}^*$ being a feasible soultion of the primal problem (1) for any feasible solution $\bar{v}$ of the dual problem (2), by Theorem 6.5.2, we get

$$\bar{c}\ \bar{x}^* \leq \bar{b}'\,\bar{v}. \text{ .... (12)}$$

From (10) and (12), we get, $\bar{b}'\,\bar{v} \geq c\ \bar{x}^* = \bar{b}'\,\bar{v}^*$.

So it is proved that for any feasible solution $\bar{v}$ of the dual problem (2), $\bar{b}'\,\bar{v} \geq \bar{b}'\,\bar{v}^*$ from which we can say that $\bar{v}^*$ is an optimal solution of the dual problem where $\overline{b}'\,\overline{v}^*$ is the minimum value of the objective function of the dual problem.

Hence, the theorem is proved.

● **Theorem 11.5.4 :** A feasible solution $\overline{x}^*$ to the primal problem (1) will be an optimal solution of the problem if and only if there exists a feasible solution $\overline{v}^*$ of the dual problem (2) such that $\overline{c}\,\overline{x}^* = \overline{b}'\,\overline{v}^*$.

This theorem is sometimes refereed to as the **Fundamental Duality Theorem** and is stated as follows :

If a finite optimal feasible solution exists for the primal, then there exists a finite optimal feasible solution for the dual problem and conversely.

Proof : For solving primal problem (1) by simplex method we are to add m slack variables. Here we observe that in the column vector $\overline{b}$ = [b_1, b_2, ... b_m], b_1, b_2, ... b_m are not necessarily non-negative.

If m slack variables be expressed by the column vector $\overline{x}_s = \left[x_{s_1}, x_{s_2}, \ldots\ldots, x_{s_m}\right]$ then the primal problem (1) can be expressed as

Maximize z = $\overline{c}\,\overline{x}$,

subject to $A\overline{x} + I_m\,\overline{x}_s = \overline{b}$ .... (13),

$\overline{x} \geq \overline{0}$, $\overline{x}_s \geq \overline{0}$ and I_m is an m × m unit matrix.

Now we assume that the primal problem (1) has a finite optimal soluion. Then the problem (13) has an optimal basic feasible solution. We suppose that the vector of the basic variables in the optimal basic feasible solution is

$$\overline{x}_B^* = \left[x_{B_1}, x_{B_2}, \ldots\ldots\ldots, x_{B_m}\right]$$

where B is the corresponding basis matrix and $c_B = \left(c_{B_1}, c_{B_2}, \ldots\ldots, c_{B_m}\right)$ is the row vector of the coefficients of the basic variables in the objective function.

We know that $\overline{x}_B^* = B^{-1}\,\overline{b}$.

Now, since we get an optimal solution of the problem (13) from $\overline{x}_B^*$ we get $z_j - c_j \geq 0$ for all admissible values of j where

$$z_j = \sum_{i=1}^{m} c_{Bi}\ y_{ij} \text{ and } [y_{ij}, y_{zj} \ldots\ldots y_{mj}] = B^{-1}\ (A, I_m).$$

Then we can write $\overline{c}_B\, B^{-1}\left(A, I_m\right) \geq \left(\overline{c}, \overline{0}\right)$ .... (14)

Now we denote the row matrix $\overline{c}_B\, B^{-1}$ by $(\overline{v}^*)'$.

Then from (14), we get

$$(\overline{v}^*)'A \geq \overline{c},\ (\overline{v}^*)' \geq \overline{0} \text{ .... (15)}$$

$$\text{or,}\quad A'\left(\overline{v}^*\right) \geq (\overline{c})',\ \overline{v}^* \geq \overline{0} \text{ .... (16)}$$

[Here $\overline{0}$ in (15) is a row matrix and $\overline{0}$ in (16) is a column matrix.]

From (16), we can say that $\overline{v}^*$ satisfies the constraints of problem (2) which is the dual of the primal problem (1). So $\overline{v}^*$ is a feasible solution of the dual problem.

We shall prove that $\overline{v}^* = \left(\overline{c}_B\, B^{-1}\right)'$ is an opitmal solution of the dual problem (2).

Here the maximum value of the objective function of the primal problem will be

$$\begin{aligned} Z_{\max} &= \overline{c}_B\, \overline{x}_B^* \\ &= \overline{c}_B\, B^{-1}\, \overline{b},\ \left(\because\ \overline{x}_B^* = B^{-1}\, \overline{b}\right) \\ &= \left(\overline{v}^*\right)'\, \overline{b} = \left(\left(\overline{v}^*\right)'\, \overline{b}\right)' = \overline{b}'\, \overline{v}^*. \end{aligned}$$

[$(\overline{v}^*)'\overline{b}$ being a 1 × 1 matrix, we have $(\overline{v}^*)'\overline{b} = \left((\overline{v}^*)'\overline{b}\right)'$]

Now, for the above two solutions (one for the primal problem and the other for the dual problem) the values of the objective functions of the primal problem and the dual problem being equal, by Theorem 11.5.3, we can say that $\overline{v}^* = \left(\overline{c}_B\, B^{-1}\right)'$ will be an optimal solution of the dual problem.

Similarly, we can prove that if the dual problem (2) has an optimal solution $\overline{v}^*$ then the primal problem (1) will also have an optimal solution and $z_{\max} = w_{\min}$.

Corollary : A linear programming problem has a finite optimal solution if and only if each of the given LPP (as primal problem) and its dual problem has at least one feasible solution.

Proof : Try yourself.

Remark : If the optimal solution of the primal problem be found by simplex method then, by Theorem 11.5.4, we get the following method of finding the optimal solution of the dual problem :

If the primal problem be a maximization problem with constraints of the type "≤", then from the entries in the row for $z_j - c_j$ (in the last table) we get optimal solution of the dual.

Similarly if the primal problem be a minimization problem with "≥" constraints and if this problem be solved by simplex method then from the table giving optimal solution, changing the signs of the entries in the row $z_j - c_j$ under the columns for "surplus" variables we get optimal solution of the dual problem.

● **Theorem 11.5.5 :** If the objective function of the primal problem be unbounded then the dual problem has no feasible solution.

Proof : We assume that the objective function of the primal problem (1) is unbounded. If possible let the dual problem (2) possess feasible solution. Now the primal problem and the dual problem both having feasible solutions, from the corollary of Theorem 11.5.4 we can say that the primal and the dual problem both have finite optimal solutions—which is here impossible since the primal problem has unbounded objective function i.e., the primal problem has no optimal solution. So it is proved that the dual problem has no feasible solution when the objective function of the primal problem is unbounded.

● **Theorem 11.5.6 :** If the dual problem has no feasible solution and the primal problem has a feasible solution then the objective function of the primal problem is unbounded.

Proof : Try yourself.

11.6 Duality and Simplex Method

We have observed that if optimal soultion (when exists) of the primal problem (or the dual problem) be found by Simplex method we can find optimal solution of the dual problem (or the primal problem) from the same simplex table. In this connection we shall mention three useful rules :

First Rule : This rule has already been stated in the corollary of Theorem 11.5.4.

Second Rule : If a varvabile of the primal problem (on the dual problem) the related to an artificial variable of the dual problem (or the primal problem) then in the row for "net evaluation" in the simplex table giving optimal solution of the dual (primal), if the penalty cost M in the entry under the corresponding artificial variable is taken to be 0 we get the value of the corresponding primal variable (dual variable) in the optimal solution. [Here we omit the proof of this rule–See Example 11.8.2 in Section 11.8]

Third Rule : If one objective function of the primal or the dual be unbounded then the other problem has no feasible solution. [The proof of this rule follows from Theorem 11.5.5 and Theorem 11.5.6]

Remark : We observe that duality theory is very helpful in solving L.P.P. by simplex method when the number of constraints is greater than the number of decision variables. For example, let the number of constraints be 7 and the number of variables be 3 in the primal L.P.P. and if we proceed to solve the primal L.P.P. by simplex method then the initial basis matrix will contain 7 rows and it needs enough time to compute each simplex table. But if we convert the primal into its dual, we get only three constraints instead if seven and the dual L.P.P. can be solved easily. Then solving the dual L.P.P. we get the optimal value of the objective function of the dual as well as optimal solution of the primal. Therefore, by using duality in some cases we can solve problems easily and more quickly.

11.7 Economic Interpretation of Duality

We consider the primal problem

$$\text{Maximize } z = \overline{c}\,\overline{x},$$

$$\text{subject to } A\overline{x} \leq \overline{b}, \overline{x} \geq \overline{0}$$

where $\overline{c} = (c_1, c_2, \ldots.., c_n)$, $\overline{b} = [b_1, b_2, \ldots.., b_m]$ and A is m × n matrix.

Here c_j is the value of each unit of output and $b_1, b_2, \ldots.., b_m$ are upper bounds of availability of input resource. The problem is to determine how much of each output x_j should be produced in order to maximize the total value of the output.

The dual of the above primal is

$$\text{Minimize } w = \bar{b}'\bar{v}$$

$$\text{subject to } A'\bar{v} \geq \bar{c}', \ \bar{v} \geq \bar{0}.$$

Here b_i (i = 1, 3, ..., m) is the given availability of each input and c_j is the lower bound on the unit value of each output.

The problem is to determine what unit values should be assigned to each input v_i in order to minimize the total value of the input.

The primal problem is the problem of a manufacturer who attempts to maximize the value of his production. The dual problem is the problem of an accountant who wishes to determine a 'value' to each input for replacement. His object is to determine his input valuations which will minimize the total cost to him.

11.8 A Few Examples

● **Example 11.8.1 :** Find optimal solution (it exists) of the dual from the simplex table for solving the primal problem given below :

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 12x_2 + 4x_3 \\ &\text{subject to} \quad x_1 + 2x_2 + x_3 \leq 5 \\ &\qquad 2x_1 - x_2 + 3x_3 = 2 \\ &\qquad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution : Here we have two dual variables v_1, v_2 where v_2 will be unrestricted in sign, since there are two constraints on the primal problem and the second constraint is in the form of an equation.

Now by adding slack variable x_4 and artificial variable x_5 the modified primal problem becomes

$$\begin{aligned} &\text{Maximize } z_1 = 5x_1 + 12x_2 + 4x_3 + 0x_4 - Mx_5 \\ &\text{subject to} \quad x_1 + 2x_2 + x_3 + x_4 = 5 \\ &\qquad 2x_1 - x_2 + 3x_3 + x_5 = 2 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

M (> 0) is a real number which can be taken to be greater than any number appearing in the computation.

Now we get the following tables in solving the problem by simplex method (notations are usual) :

			c_j	5	12	4	0	–M
$\bar{c}_B$	B	$\bar{x}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$
0	$\bar{a}_4$	x_4	5	1	2	1	1	0
–M	$\bar{a}_5$	x_5	2	2	–1	$\boxed{3}$	0	1
$z_j - c_j$				–2M–5	M–12	–3M–4↑	0	0↓
0	$\bar{a}_4$	x_4	$\frac{13}{3}$	$\frac{1}{3}$	$\boxed{\frac{7}{3}}$	0	1	$-\frac{1}{3}$
4	$\bar{a}_3$	x_3	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$
$z_j - c_j$			$\frac{8}{3}$	$-\frac{7}{3}$	$-\frac{40}{3}$↑	0	0↓	$M+\frac{4}{3}$
12	$\bar{a}_2$	x_2	$\frac{13}{7}$	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$
4	$\bar{a}_3$	x_3	$\frac{9}{7}$	$\boxed{\frac{5}{7}}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$
$z_j - c_j$			$\frac{156}{7}$	$-\frac{3}{7}$↑	0	0↓	$\frac{40}{7}$	$M-\frac{4}{7}$
12	$\bar{a}_2$	x_2	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
5	$\bar{a}_1$	x_1	$\frac{9}{5}$	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
$z_j - c_j$			$\frac{141}{5}$	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$M-\frac{2}{5}$

In the last table, we find that there is no artificial variable among the basic variables and $z_j - c_j \geq 0$ for all values of j. So from the last table we get an optimal solution of the primal problem. It is seen that optimal solution of the primal problem is

$$x_1 = \frac{9}{5},\ x_2 = \frac{8}{5},\ x_3 = 0 \quad \text{and} \quad z_{max} = 28\frac{1}{5}.$$

Again in the row for $z_j - c_j$ in the last table, entries under the slack variable x_4 and the artificial variables x_5 are respectively $\frac{29}{5},\ M - \frac{2}{5}$.

Now taking M = 0, we see that optimal solution of the dual problem will be

$$\nu_1 = \frac{29}{5},\ \nu_2 = -\frac{2}{5}$$

$$\text{and} \quad w_{min} = z_{max} = 28\frac{1}{5}.$$

Here we note that the dual variable is unrestricted in sign and $\nu_2 = -\frac{2}{5} < 0$. So the solution of the daul problem is $\nu_1 = \frac{29}{5}$, $\nu_2 =-\frac{2}{5}$ and $w_{min} = 28\frac{1}{5}$.

● **Example 11.8.2 :** Using duality show that the following L.P.P. has no feasible solution :

$$\begin{aligned} &\text{Minimize} \ \ z = x_1 - x_2 \\ &\text{subject to} \quad 2x_1 + x_2 \geq 2 \\ &\qquad\qquad\quad x_1 + x_2 \leq -1 \\ &\qquad\qquad\quad x_1, x_2 \geq 0. \end{aligned}$$

Solution : The given problem can be expressed as

$$\begin{aligned} &\text{Minimize} \ \ z = x_1 - x_2 \\ &\text{subject to} \quad 2x_1 + x_2 \geq 2 \\ &\qquad\qquad\quad -x_1 - x_2 \geq 1 \\ &\qquad\qquad\quad x_1, x_2 \geq 0. \end{aligned}$$

Now the dual of this primal problem will be

$$\begin{aligned} &\text{Maximize} \ \ w = 2\nu_1 + \nu_2 \\ &\text{subject to} \quad 2\nu_1 - \nu_2 \leq 1 \\ &\qquad\qquad\quad \nu_1 - \nu_2 \leq -1, \\ &\qquad\qquad\quad \nu_1, \nu_2 \geq 0. \end{aligned}$$

Now this dual problem can be expressed as

$$\begin{aligned} \text{Maximize} \quad & w = 2\nu_1 + \nu_2 \\ \text{subject to} \quad & 2\nu_1 - \nu_2 \le 1 \\ & -\nu_1 + \nu_2 \ge 1 \\ & \nu_1,\ \nu_2 \ge 0. \end{aligned}$$

The equivalent form of the last L.P.P. is as follows :

$$\begin{aligned} \text{Maximize} \quad & w = 2\nu_1 + \nu_2 + 0.\nu_3 + 0.\nu_4 \\ \text{subject to} \quad & 2\nu_1 - \nu_2 + \nu_3 = 1 \\ & -\nu_1 + \nu_2 - \nu_4 = 1 \\ & \nu_1,\ \nu_2,\ \nu_3,\ \nu_4 \ge 0 \end{aligned}$$

where ν_3 is a slack variable and ν_4 is a surplus variable.

Now using artificial variable ν_5 we get the following modified form of dual problem :

$$\begin{aligned} \text{Maximize} \quad & w = 2\nu_1 + \nu_2 + 0.\nu_3 + 0.\nu_4 - M\nu_5 \\ \text{subject to} \quad & 2\nu_1 - \nu_2 + \nu_3 = 1 \\ & -\nu_1 + \nu_2 - \nu_4 + \nu_5 = 1 \\ & \nu_1,\ \nu_2,\ \nu_3,\ \nu_4,\ \nu_5 \ge 0 \end{aligned}$$

M (> 0) can be taken to be greater than any number appearing in the computation.

Now we get the following tables in solving the last L.P.P. by simplex method (notations are usual) :

			c_j	2	1	0	0	–M
$\bar{c}_B$	B	$\bar{\nu}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$
0	$\bar{a}_3$	ν_3	1	2	–1	1	0	0
–M	$\bar{a}_5$	ν_5	1	–1	$\boxed{1}$	0	–1	1
$z_j - c_j$			– M	M – 2	–M–1↑	0	M	0↓

			c_j	2	1	0	0	–M
$\overline{c}_B$	B	$\overline{\nu}_B$	$\overline{b}$	$\overline{a}_1$	$\overline{a}_2$	$\overline{a}_3$	$\overline{a}_4$	$\overline{a}_5$
0	$\overline{a}_3$	ν_3	2	$\boxed{1}$	0	1	–1	1
1	$\overline{a}_2$	ν_2	1	–1	1	0	–1	1
$z_j - c_j$			1	$-3\uparrow$	0	$0\downarrow$	–1	M+1
2	$\overline{a}_1$	ν_1	2	1	0	1	–1	1
1	$\overline{a}_2$	ν_2	3	0	1	1	–2	2
$z_j - c_j$			7	0	0	3	–4	M+4

Hence we see that in the last table the most negative value of $z_j - c_j$ is –4 and this value is under the column $\overline{a}_4$. But we see that $y_{14} = -1 < 0$, $y_{24} = -2 < 0$, that is, no entry of this column is positive. So the objective function of the dual problem is unbounded.

Hence the given primal problem has no feasible solution.

11.9 Summary

First we have defined dual problem of a primal problem expressed in the form

$$\text{Maximize } z = \overline{c}\,\overline{x}$$

$$\text{subject to } A\overline{x} \le \overline{b}, \overline{x} \ge \overline{0}.$$

Then we have stated the method of constructing dual problem of the primal problem given in any form (Maximization or minimization) and with constraints of the type " ≤ ", " ≥ ", " = " and also for the problem where some variables are unrestricted in sign. Here we have observed that (i) number of constraints in the primal problem = number of variables in the dual problem and the number of variables in the primal problem = the number of constraints in the dual problem, (ii) if a variable of the primal problem be unrestricted in sign then the corresponding constraint in the dual problem will be of the type " = " and if a constriant of the primal problem be of the type " = ", then the corresponding variable in the dual problem will be unrestricted in sign.

Finally, from the important theorems on Duality, we get the following results :

(a) In simplex method we get optimal solutions of the primal problem and the dual problem (if the optimal solutions exist) from the same table.

(b) The optimal values of the objective functions of the primal and dual problems will be same (when optimal solutions exist).

(c) If the objective function of the primal problem be unbounded then the dual problem has no feasible solution and if the objective function of the dual problem be unbounded then the primal problem has no feasible solution.

11.10 Exercise

Find the dual of the L.P.P. given below (1–5) :

1. Maximize $z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \leq 12$

$2x_1 + 3x_2 \leq 21$

$x_1 \leq 8$

$x_2 \leq 6$

$x_1, x_2 \geq 0.$

2. Minimize $z = 3x_1 + x_2$

sibject to $2x_1 + 3x_2 \geq 2$

$x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0.$

3. Maximize $z = 2x_1 + 3x_2 + 4x_3$

subject to $x_1 + 5x_2 + 3x_3 = 7$

$2x_1 - 5x_2 \leq 3$

$3x_1 - x_3 \geq 5.$

$x_1, x_2 \geq 0$ and x_3 unrestricted in sign.

4. Maximize $z = 2x_1 - 6x_2$

subject to $x_1 - 3x_2 \leq 6$

$2x_1 - 4x_2 \geq 8$

$$x_1 - 3x_2 \geq -6$$
$$x_1, x_2 \geq 0.$$

5. Minimize $z = 2x_1 + 3x_2 + x_3$

   subject to $4x_1 + 3x_2 + x_3 = 6$

   $x_1 + 2x_2 + 5x_3 = 4$

   $x_1, x_2, x_3 \geq 0.$

6. Solve the following L.P.P. by solving its dual problem :

   Maximize $z = 3x_1 + 2x_2$

   subject to $x_1 + x_2 \leq -5$

   $x_1 \leq 4$

   $x_2 \leq 6$

   $-x_2 \leq -1$

   and $x_1, x_2 \geq 0$

7. Solve the L.P.P. by simplex method and then find the optimal solution of the dual problem :

   Maximize $z = 30x_1 + 23x_2 + 29x_3$

   subject to $6x_1 + 5x_2 + 3x_3 \leq 26$

   $4x_1 + 2x_2 + 5x_3 = 7$

   $x_1, x_2, x_3 \geq 0$

8. Prove by duality that the objective function of the following is unbounded :

   Maximize $z = 3x_1 + 4x_2$

   subject to $x_1 - x_2 \leq 1$

   $x_1 + x_2 \geq 4$

   $x_1 - 3x_2 \leq 3$

   $x_1, x_2 \geq 0$

9. Show by simplex method that the dual of the following L.P.P. has no feasible solution :

   Maximize $z = x_1 + 2x_2$

subject to $x_1 + 2x_2 \geq 1$

$$-x_1 + x_2 \leq 1$$

$$x_1 \geq 0,\ x_2 \geq 0$$

10. Slove the following L.P.P. by simplex method without artificial variables

Minimize $z = x_1 + x_2$

subject to $x_1 + 2x_2 \geq 12$

$$5x_1 + 6x_2 \geq 48$$

$$x_1 \geq 0,\ x_2 \geq 0$$

[Hint : Solve the dual problem]

Answers

1. Minimize $w = 21\nu_1 + 21\nu_2 + 8\nu_3 + 6\nu_4$

subject to $\nu_1 + 2\nu_2 + \nu_3 \geq 3$

$$\nu_1 + 3\nu_2 + \nu_4 \geq 4$$

$$\nu_1,\ \nu_2,\ \nu_3,\ \nu_4 \geq 0$$

2. Maximize $z = 21\nu_1 + \nu_2$

subject to $2\nu_1 + \nu_2 \leq 3$

$$3\nu_1 + \nu_2 \leq 1$$

$$\nu_1,\ \nu_2 \geq 0$$

3. Minimize $z = 7\nu_1 + 3\nu_2 - 5\nu_3$

subject to $\nu_1 + 2\nu_2 \geq 2$

$$-5\nu_1 - 5\nu_2 - 3\nu_3 \geq 3$$

$$3\nu_1 + \nu_3 = 4$$

$\nu_2, \nu_3 \geq 0$ and ν_1 unrestructed in sign.

4. Minimize $w = 6\nu_1 - 8\nu_2 + 6\nu_3$

subject to $\nu_1 - 2\nu_2 - \nu_3 \geq 2$

$$-3\nu_1 - 4\nu_2 + 3\nu_3 \geq -6$$

$$\nu_1,\ \nu_2,\ \nu_3 \geq 0$$

5. Minimize $z = 6\nu_1 + 4\nu_2$

   subject to $4\nu_1 + \nu_2 \geq 2$

   $\nu_1 - 5\nu_2 \geq 3$

   ν_1, ν_2 are unrestricted in sign.

6. $x_1 = 4, x_2 = 1$ and $z_{max} = 10$

7. $x_1 = 4, x_2 = \frac{7}{2}, x_3 = 0$ and $z_{max} = \frac{161}{2}$

   $\nu_1 = 0, \nu_2 = \frac{23}{2}$ and $z_{min} = \frac{161}{2}$

8. $x_1 = 8, x_2 = 0$ and $z_{min} = 8$

11.11 Multiple Choice Questions (MCQ)

1. When all constraints are of the type "≤" in the primal and no variable is unrestricted in sign, the dual will have

   (a) all constraints are "≤" type (b) some constraints are "≤" type

   (c) all constraints are "≥" type (d) none of the above

2. If the objective function of the dual is unbounded then the primal has

   (a) an unbounded objective function

   (b) no feasible solution

   (c) infinite number of feasible solutions

   (d) none of these

3. If any of the constraints in the primal problem be a perfect " = ", then the corresponding dual variable is

   (a) always positive (b) always negative

   (c) unrestricted in sign (d) none of these

4. If the primal objective be unbounded, then the dual problem has

   (a) unbounded objective function (b) a finite optimal solution

   (c) no feasible solution (d) none of these

5. If a variable in the primal problem be unrestricted in sign, then the corresponding constraint in the dual problem will be

   (a) " $\geq$ " type (b) " $>$ " type

   (c) a perfect equality (d) none of these

6. The dual of the dual of a L.P.P.

   (a) may not be the primal problem

   (b) is the primal problem

   (c) cannot be constructed

   (d) either the dual or the primal

Answers

1. (c) 2. (b) 3. (c) 4. (c) 5. (c) 6. (b)

Unit - 12 ❑ Transporation Problem

Structure

12.0 Objective

After studying this chapter, the reader should be able to

- recognise a transportation problem
- convert a transportation problem into a LPP
- develop a transportation table and apply the transportation method to get the solution
- solve unbalanced and maximization transportation problems.

12.1 Introduction

A special type of linear programming problem is a transportation problem. In this problem, we are to find the minimum cost of transporting a single commodity from a number of origins to different destinations. The capacity of production of the commodity at each origin, the demand at each destination and the transporatation cost (per unit commodity) from different sources to various destinations will be given. It is assumed that the cost of transportation on a given route is proportional to the number of units transported. Let us give one example of a transportation problem.

Let us suppose that there are 8 factories which produce a cold drink at different parts of a country. The produced cold drink will be transported to 50 different cities of the country. The units of cold drink produced at different factories (in a given time) and known and also the units of demand of the cold drink at each city and the costs of transportation of this commodity from each factory to each city are known. We are to determine the units of the commodity to be transported from each factory to each city satisfying the demand of each city and using all units of the commodity produced at each factory so that the total cost of transportation is minimum.

12.2 Mathematical Formulation of a Transportation Problem

We suppose that there are m origins O_1, O_2, ...,O_m and n destinations D_1, D_2, ..., D_n.

We suppose that a_i (> 0) units of a given commodity are available at the origin O_i (i = 1, 2, ..., m) and demand of the commodity at the destination D_j is b_j (j = 1, 2, ..., n) units.

Let x_{ij} (i = 1, 2, ..., m; j = 1, 2, ..., n) units be transported from the origin O_i to the destination D_j.

Then $x_{ij} \geq 0$ and the values of these mn variables x_{ij} (i = 1, 2, ..., m; j = 1, 2,

....., n) must satisty the conditions $\sum_{j=1}^{n} x_{ij} = a_i$ (i = 1, 2, ..., m) and $\sum_{i=1}^{m} x_{ij} = b_j$ (j = 1, 2, ..., n).

Further we suppose that the transportation cost of unit commodity from the origin O_i to the destination D_j is c_{ij} (i = 1, 2, ..., m; j = 1, 2, ..., n).

Then the total cost z of transportation is given by

$$z = (c_{11}x_{11} + c_{12}x_{12} + ...+ c_{1n} x_{1n}) + (c_{21}x_{21} + c_{22}x_{22} + ...+ c_{2n} x_{2n}) + ... + (c_{m1}x_{m1} + c_{m2}x_{m2} + ...+ c_{mn} x_{mn})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

Then the transportation problem can be expressed as a L.P.P. described below :

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i; \quad i = 1, 2, ..., m;$$

$$\sum_{i=1}^{m} x_{ij} = b_j; \quad j = 1, 2, ..., n;$$

and $x_{ij} \geq 0$ (i = 1, 2, ..., m; j = 1, 2, ..., n).

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ then the transportation problem is said to be **balanced** and if $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$ then the problem is said to be **unbalanced**.

We observe that a transportation problem can be described by the following table which is called a transportation table :

Destinations

Origins

	D_1	D_2	....	D_n	
O_1	c_{11}	c_{12}	....	c_{1n}	a_1
O_2	c_{21}	c_{22}	....	c_{2n}	a_2
⋮	....	....	....	....	⋮
⋮	....	....	....	....	⋮
⋮	....	....	....	....	⋮
O_m	c_{m1}	c_{m2}	....	c_{mn}	a_m
	b_1	b_2	....	b_n	

Table–1

Matrix form of a Transportation Problem (as L.P.P.)

If the transportation problem with m origins and n destinations be described by the Table 1, then we can express the problem in the matrix form as follows :

$$\text{Minimize } z = \bar{c}\,\bar{x}$$

$$\text{subject to } \; A\bar{x} = \bar{b}, \bar{x} \geq \bar{0}$$

where $\bar{x} = [x_{11}, x_{12}, \ldots\ldots, x_{ij}, \ldots\ldots\ x_{mn}]$ is a column vector with mn components, $\bar{c} = (c_{11}, c_{12}, \ldots\ldots, c_{ij}, \ldots\ldots\ c_{mn})$ is a row vector with mn components, $\bar{b} = [a_1, a_2, \ldots\ldots, a_m, b_1, b_2, \ldots\ldots, b_n]$ is column vector with m + n components and $A = [\bar{a}_{11}, \bar{a}_{12}, \ldots\ldots \bar{a}_{ij}, \ldots\ldots \bar{a}_{mn}]$ in (m + n) × mn coefficient matrix where $\bar{a}_{ij}$ is the column vector corresponding to the variable x_{ij}.

Now writing the constraints given by $A\bar{x} = \bar{b}$ in details we get

$$x_{11} + x_{12} + \ldots\ldots + x_{1n} = a_1$$

$$x_{21} + x_{22} + \ldots\ldots + x_{2n} = a_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{m1} + x_{m2} + \ldots\ldots + x_{mn} = a_m$$

$$x_{11} + x_{21} + \ldots\ldots + x_{m1} = b_1$$

$$x_{12} + x_{22} + \ldots\ldots + x_{m2} = b_2$$

....

....

$$x_{1n} + x_{2n} + \ldots\ldots + x_{mn} = b_n$$

Let us give one example.

Let a transportation problem be described by the following table :

	D_1	D_2	D_3	a_i
O_1	2	1	3	10
O_2	2	4	1	15
O_3	1	1	5	5
O_4	6	2	4	20
b_j	30	10	10	

Here we have 4 origins and 3 destinations and this is a balanced transportation problem since here

$$\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j = 50.$$

If we write the transportation problem as L.P.P., we get

Minimize $z = 2x_{11} + x_{12} + 3x_{13} + 2x_{21} + 4x_{22} + x_{23} + x_{31} + x_{32} + 5x_{33} + 6x_{41} + 2x_{42} + 4x_{43}$,

subject to $x_{11} + x_{12} + x_{13} = 10$

$x_{21} + x_{22} + x_{23} = 15$

$x_{31} + x_{32} + x_{33} = 5$

$x_{41} + x_{42} + x_{43} = 20$

$x_{11} + x_{21} + x_{31} + x_{41} = 30$

$x_{12} + x_{22} + x_{32} + x_{42} = 10$

$x_{13} + x_{23} + x_{33} + x_{43} = 10$

and $x_{ij} \geq 0$ $(i = 1, 2, 3, 4; j = 1, 2, 3)$

Here the number of constraints expressed by equalities 4 + 3 = 7 and the number of variables is 12.

Here $\overline{b}$ = [10, 15, 5, 20, 30, 10], $\overline{c}$ = [2, 1, 3, 2, 4, 1, 1, 1, 5, 6, 2, 4].

Here the coefficient matrix A is

$$\left[\begin{array}{cccccccccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} & x_{41} & x_{42} & x_{43} \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

which is a matrix of order 7 × 12.

We observe that the column vector corresponding to the variable x_{12} is

$$\overline{a}_{12} = \begin{bmatrix}1\\0\\0\\0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\end{bmatrix} = \overline{e}_1 + \overline{e}_6$$

where $\overline{e}_i$ is the i-th column vector of the unit matrix of order 7 × 7.

In general we get $\overline{a}_{ij} = \overline{e}_i + \overline{e}_{m+j}$

[here m = 4, n = 3 and so

$$\overline{a}_{12} = \overline{e}_1 + \overline{e}_{4+2} = \overline{e}_1 + \overline{e}_0,$$

$$\overline{a}_{23} = \overline{e}_2 + \overline{e}_{4+3} = \overline{e}_2 + \overline{e}_7 \text{ etc.}]$$

12.3 Some Theorems Related to the Solution of a Transportation Problem

The following theorems will give some characteristics of a transportation problem which will be helpful in solving a transportation problem.

● **Theorem 12.3.1 :** The number of basic variables of a balanced transportation problem with m origins and n destinations will be at most (m + n – 1).

Proof : With usual notations, the transportation problem can be expressed as follows :

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} ,$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i;\ i = 1, 2, \ldots.., m \ \ldots\ldots\ldots\ldots (1)$$

$$\sum_{i=1}^{m} x_{ij} = b_j;\ j = 1, 2, \ldots..., n \ \ldots\ldots\ldots\ldots (2)$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots., m;\ j = 1, 2, \ldots.., n)$$

$$\text{Here } \sum_{i=1}^{m}\left[\sum_{j=1}^{n} x_{ij}\right] = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \ \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\left[\because \text{ here } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j\right]$$

Now from the constraints in (2) we get

$$\sum_{j=1}^{n-1}\left[\sum_{i=1}^{m} x_{ij}\right] = \sum_{j=1}^{n-1} b_j \ \ldots\ldots\ldots\ldots\ldots (4)$$

From (3) and (4) we get

$$\sum_{i=1}^{m}\left[\sum_{j=1}^{n}x_{ij}\right]-\sum_{i=1}^{m}\left[\sum_{j=1}^{n-1}x_{ij}\right] = \sum_{j=1}^{n}b_j-\sum_{j=1}^{n-1}b_j=b_n$$

or, $$\sum_{i=1}^{m}\left[\sum_{j=1}^{n}x_{ij}-\sum_{j=1}^{n-1}x_{ij}\right]=b_n$$

or, $\sum_{i=1}^{m}x_{in} = b_n$ which is the n-th constraint of the constraints in (2).

Thus it is seen that of the total m + n constraints in (1) and (2), one constraint is reduntant. So in any basic solution of the system of equations given by (1) and (2), the number of basic variables with be m + n – 1.

● **Theorem 12.3.2 :** Every balanced transportation problem has always a feasible solution.

Proof : With used notations, m + n constraints of the transportation problem with m origins and n destinations can be taken as

$$\sum_{j=1}^{n}x_{ij} = a_i;\ i = 1, 2, ..., m\ ....(5)$$

and $$\sum_{i=1}^{m}x_{ij} = b_j;\ j = 1, 2, ..., n\ ....(6)$$

where $a_i > 0$, $b_j > 0$ (i = 1, 2, ...m; j = 1, 2, ..., n).

Also here $\sum_{i=1}^{m}a_i=\sum_{j=1}^{n}b_j=S$ (say).

Then S must be positive.

Now we shall prove that $x_{ij} = \frac{a_i b_j}{S}$ (i = 1, 2, ..., m; j = 1, 2, ..., n) will be a feasible solution of the transportation problem.

Here we observe that $x_{ij} > 0$ for all i, j. Again taking $x_{ij} = \frac{a_i b_j}{S}$ we get

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{S}$$

$$= \frac{1}{S} a_i \sum_{j=1}^{n} b_j = \frac{a_i S}{S} = a_i \text{ for } i = 1, 2, \ldots\ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{S} = \frac{1}{S} b_j \sum_{i=1}^{m} a_i = \frac{b_j S}{S} = b_j \text{ for } j = 1, 2, \ldots., n$$

So $x_{ij} = \frac{a_i b_j}{S}$ (i = 1, 2, ... m; j = 1, 2, ... n) satisfy all the constraints of (5) and (6) of the transporation problem and also $x_{ij} \geq 0$ (i = 1, 2, ... m; j = 1, 2, ... n).

Thus we get a feasible solution of the transportation problem.

● **Theorem 12.3.3 :** For any transportation problem the value of the objective function can never be unbounded and in any feasible solution, the value of any variable cannot be made arbitrarily large.

Proof : We suppose that the transportation problem is

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i; \quad i = 1, 2, \ldots\ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j; \quad j = 1, 2, \ldots., n,$$

$$x_{ij} \geq 0 \ (i = 1, 2, \ldots., m;\ j = 1, 2, \ldots., n).$$

Here we observe that for any feasible solution $[x_{11}, x_{12}, \ldots., x_{ij}, \ldots\ldots, x_{mn}]$,

$$x_{ij} \leq \sum_{j=1}^{n} x_{ij} = a_i \ (\because x_{ij} \geq 0 \text{ for all } i, j) \quad (i = 1, 2, \ldots\ldots, m).$$

$$\text{and } x_{ij} \leq \sum_{i=1}^{m} x_{ij} = b_j, \ (j = 1, 2, \ldots., n).$$

So we get, $0 \leq x_{ij} \leq \max \{a_1, a_2, ..., a_m; b_1, b_2, ..., b_n\}$ for all i, j ....(7)

Again here $c_{ij} \geq 0$ for all i, j.

Then for any feasible solution $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \geq 0$.

Now the transporation problem being a minimization problem and for any feasible solution z being non-negative (i.e., ≥ 0), the value of z can never be unbounded.

Again from (7) we can say that value of x_{ij} cannot be made arbitrarily large.

So the theorem is proved.

Remark : Since every balanced transportation problem has a feasible solution and the objective function of such problem can never be unbounded every balanced transportation problem has an optimal soultion.

Now the transportation problem having some special characteristics, unlike the laborious method of solving a general L.P.P. by simplex method, it will be possible to obtain special method by which a transportation problem can be solved without much labour.

12.4 Idea of Loop and its Application in Solving Transporation Problem

A feasible solution $[x_{11}, x_{12}, x_{ij}, ..., x_{mn}]$ of a transportation problem with m origins and n destinations, is given in the following table :

$$\begin{array}{c|c|c|c|c|}
 & D_1 & D_2 & \cdots\cdots\cdots & D_n \\ \hline
O_1 & x_{11} & x_{12} & \cdots\cdots\cdots & x_{1n} \\ \hline
O_2 & x_{21} & x_{22} & \cdots\cdots\cdots & x_{2n} \\ \hline
\vdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \hline
O_m & x_{m1} & x_{m2} & \cdots\cdots\cdots & x_{mn} \\ \hline
\end{array} \qquad \text{............ (1)}$$

Here each variable is put within a box. Total number of variables is mn and these are put inside mn boxes. The variable x_{ij} put within the box in the i-th row and j-th column and this box will be called (i, j) cell.

We observe that if m = 3, n = 2 then the total number of variables will be 6 and we will get 6 cells. Here the variables are x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32} and the corresponding cells are (1, 1), (1, 2), (2, 1), (2, 2), (3, 1) and (3, 2) which are shown in the following table :

	D_1	D_2
O_1	x_{11}	x_{12}
O_3	x_{21}	x_{22}
O_3	x_{31}	x_{32}

We have mentioned earlier that transportation costs c_{11}, c_{12}, ..., c_{ij}, ..., c_{mn} are shown in a table similar to the table (1) and such table is called **transportation table** where the cost c_{ij} is shown in (i, j) cell.

Loop : A finite sequence of cells in a transportation table will be called a **loop** if

(i) any two adjacent cells are in the same row or in the same column of the transportation table.

(ii) not more than 2 cells of the sequence are in the same row or in the same column.

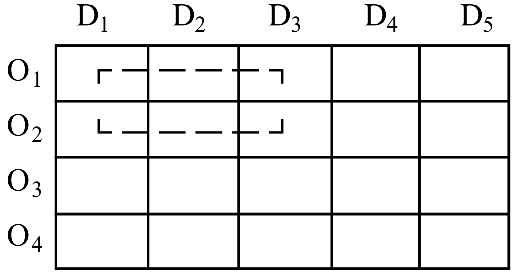
(iii) first cell and last cell of the sequence are in the same row or in the same column.

(iv) at least two rows or two columns of the transportation table are to be used in the sequence.

Some loops are shown by diagrams for the following transportation table [cells of the loop are expressed by '.'].

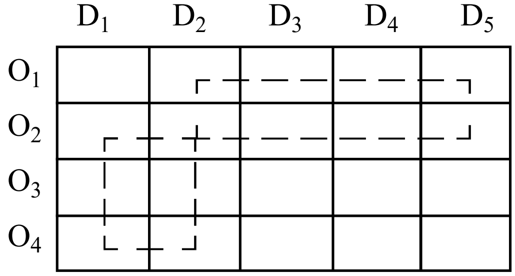
	D_1	D_2	D_3	D_4	D_5
O_1	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
O_2	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}
O_3	c_{31}	c_{32}	c_{33}	c_{34}	c_{35}
O_4	c_{41}	c_{42}	c_{33}	c_{44}	c_{45}

(a)



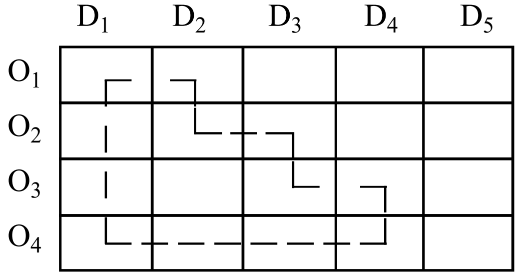
Here the cells of the loop are (1, 1), (1, 3), (2, 3), (2, 1).

(b)



Here the cells of the loop are (1, 2), (1, 5), (2, 5), (2, 1), (4, 1) (4, 2).

(c)



Here the cells of the loop are (1, 1), (1, 2), (2, 2), (2, 3), (3, 3) (3, 4), (4, 4), (4, 1).

We observe that the total number of cells in any loop are always **even** and each loop has at least **4 cells**.

Use of loop in a transporation problem

We have seen that a transportation problem is a special kind of L.P.P. Further we have proved that the number of basic variables in a transportation problem with m origins and n destinations (total number of variables is mn) will be at most m + n – 1.

Now we remember that in finding the basic solutions of the system of equations expressed by $A\overline{x} = \overline{b}$, the columns corresponding to the basic variables in the coefficient matrix must be linearly independent.

In solving a transportation problem, by using loop we can easily determine whether a given set of column vectors in the corresponding coefficient matrix are linearly independent or not and for this we are to apply the following theorem which is stated below (without proof) :

● **Theorem 12.4.1 :** A set of column vectors of the coefficient matrix of a transportation problem will be linealry dependent if and only if the set of cells (or a subset of it) in the transportation table corresponding to these vectors form a loop.

12.5 A Few Examples

We consider the following transportation problem :

	D_1	D_2	D_3	D_4	a_0
O_1	2	4	6	1	40
O_2	3	4	2	3	30
O_3	5	2	7	1	50
O_j	35	35	25	25	

Two **feasible solutions** of this problem are displayed in the following tables :

(i)

	D_1	D_2	D_3	D_4	a_0
O_1	20	10	10		40
O_2		15	15		30
O_3	15	10		25	50
b_j	35	35	25	25	

(ii)

	D_1	D_2	D_3	D_4	a_i
O_1	35	5			40
O_2		30			30
O_3			25	25	50
b_j	35	35	25	25	

Here we like to mention that when express a feasible solution of a transportation problem in table, we shall assume that the value of the variable corresponding to an empty cell will be 0 and in writing the solution these variables (whose values are 0) will not be mentioned.

Then the feasible solution given by (i) will be

$x_{11} = 20$, $x_{12} = 10$, $x_{13} = 10$, $x_{22} = 15$, $x_{23} = 15$, $x_{31} = 15$, $x_{32} = 10$, $x_{34} = 25$.

Here m = 3, n = 4. So the number of basic variables will be at most 3 + 4 – 1 = 6. So for this problem, in any basic solution (here the total number of variables is 12), the values of at least 12 – 6 = 6 variables will be 0. In the feasible solution (i), the values of only 4 variables are 0 and so this solution is not a basic solution. Here the cells corresponding to the variables which have positive values are (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 4) and so loop can be formed with all or some of these cells. Here we see that the cells (1, 2), (1, 3), (2, 3), (2, 2) have formed a loop.

12.6 Some Methods of Finding Initial Basic Feasible Solution of A Balanced Transporation Problem

(a) North-West Corner Method :

We consider the following transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
b_j	10	25	18	7	

Here we note that the problem is balanced since $\sum a_j$ = 30 + 15 + 15 = 60 and $\sum b_j$ = 10 + 25 + 18 + 7 = 60.

In this method we start with the top most left corner of the above table (north-west corner) and we allocate maximum units possible there, that is, x_{11} = min {30, 10} = 10.

So in the above transportation table we allocate 10 units in (1, 1) cell.

Now see that the requirement in the destination D_1 was 10. So, requirement for D_1 is satisfied. So, the origin, O_1 has the capacity of supplying 30 – 10 = 20 units.

Then we consider the cell (1, 2) which in adjacent to the cell (1, 1) and so we allocate min {20, 25} = 20 units in (1, 2) cell.

Now it is seen that the units available at the origin O_1 are exhausted. But the demand of D_2 is still unsatsified. So in the cell just below (1, 2) cell i.e., in (2, 2) cell we allocate min {5, 15} = 5 i.e., x_{22} = 5. Now the origin O_2 has still capacity of supplying 10 units. Then in the cell (2, 3) which is adjacent to the right of the cell (2, 2) and we allocate x_{23} = min {10, 18} = 10.

	D_1	D_2	D_3	D_4	
O_1	$\boxed{10}$ 5	$\boxed{20}$ 3	6	4	~~30~~ ~~20~~
O_2	3	$\boxed{5}$ 4	$\boxed{10}$ 7	8	~~15~~ ~~10~~
O_3	9	6	$\boxed{8}$ 5	$\boxed{7}$ 8	~~15~~ ~~7~~
	~~10~~	~~25~~ ~~5~~	~~18~~ ~~8~~	~~7~~	

Now the origin O_2 is exhaused.

Then in the cell just below the (2, 3) cell i.e., in the (3, 3) cell we allocate x_{33} = min {8, 15} = 8

Now we consider the cell (3, 4) which is right to the cell (3, 3) and we allocate x_{34} = min {7, 7} = 7

Now we see that every source is exhausted and the demand of each destination is satisfied. So we get an initial feasible solution of the given problem is

$x_{11} = 10$, $x_{12} = 20$, $x_{22} = 5$, $x_{23} = 10$, $x_{33} = 8$, $x_{34} = 7$.

In this case the cost of transportation is

$10 \times 5 + 20 \times 3 + 5 \times 4 + 10 \times 7 + 8 \times 5 + 7 \times 8$

$= 50 + 60 + 20 + 70 + 40 + 56 = 296$ units

The above solution is displayed in the following table

	D_1	D_2	D_3	D_4	
O_1	$\boxed{10}$ 5	$\boxed{20}$ 3	6	4	30
O_2	3	$\boxed{5}$ 4	$\boxed{10}$ 7	8	15
O_3	9	6	$\boxed{8}$ 5	$\boxed{7}$ 8	15
	10	25	18	7	

We observe that the set of cells where allocations have been made in this solution or a subset of these cells do not form loop. So, this solution is a basic feasible solution.

(b) Matrix Minima Method :

In this method, we find the cell in the transporation table where the cost is minimum. We allocate some units in this cell. We shall explain the method by an example.

We consider the following transportation table :

	D_1	D_2	D_3	D_4	a_i
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
b_j	20	40	30	10	

Here $\sum a_i = 30 + 50 + 20 = 100$ and $\sum b_j = 20 + 40 + 30 + 10 = 100$.

Therefore $\sum a_i = \sum b_j$.

So, the transportation problem is balanced.

In the transportation table, the minimum cost is 1 which is in the cells (1, 1), (1, 3) and (2, 4). We take any one of these cells. In particular we take the cell (1, 1) and we allocate min {20, 30} = 20 in this cell. So, x_{11} = 20. Then the demand of D_1 is satisfied and so we can delete the column for D_1 by dotted line.

	D_1	D_2	D_3	D_4	
O_1	$\boxed{20}$ 1	2	1	4	~~30~~ 10
O_2	3	3	2	1	50
O_3	4	2	5	9	20
	~~20~~	40	30	10	

Then we get the following table

	D_2	D_3	D_4	
O_1	2	$\boxed{10}$ 1	3	~~10~~
O_2	3	2	1	50
O_3	2	5	9	20
	40	~~30~~ 20	10	

Now minimum cost in this matrix in 1 which is in the cells (1, 3) and (2, 4). Taking (1, 3) cell we get x_{13} = min {30, 10} = 10. Then the origin O_1 is exhausted. So, the row corresponding to O_1 is deleted by dotted line. Then we get the following table.

	D_2	D_3	D_4	
O_2	3	2	$\boxed{10}$ 1	~~50~~ 40
O_3	2	5	9	20
	40	20	~~10~~	

Repeating the above procedure we get the following table :

	D_2	D_3	
O_2	3	$\boxed{20}$ 2	~~40~~ 20
O_3	2	5	20
	40	~~20~~	

	D_2	
O_2	$\boxed{20}$ 3	20
O_3	$\boxed{20}$ 2	20
	40	

The final feasible solution is displayed in the following table :

	D_1	D_2	D_3	D_4	
O_1	$\boxed{20}$ 1	2	$\boxed{10}$ 1	4	30
O_2	3	$\boxed{20}$ 3	$\boxed{20}$ 2	$\boxed{10}$ 1	50
O_3	4	$\boxed{20}$ 2	5	9	20
b_j	20	40	30	10	

So, the feasible solution is

$x_{11} = 20,\ x_{13} = 10,\ x_{22} = 20,\ x_{23} = 20,\ x_{24} = 10,\ x_{32} = 20$

and the cost of transportation

$$= 20 \times 1 + 10 \times 1 + 20 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 2$$

$$= 20 + 10 + 60 + 40 + 10 + 40$$

$$= 180 \text{ units.}$$

The set of cells in which allocations have been made in this solution or any subset of these cells do not form loop. So this solution is an initial basic feasible solution.

(c) Vogel's Approximation Method (VAM) :

In this method difference between the least cost in each row and the next cost are shown within a paranthesis in the line of the corresponding row. Similarly difference between the least cost and the next cost in each column are put within a paranthesis under the corresponding column. These differences are called penalty. Now the greatest penalty considering the rows and columns) which is in a row or in a column is noted and if this greatest penalty be in a row then we allocate in the least cost cell of that row and if the greatest penalty be in a column we allocate in the least cost cell of the column. We shall explain the method by an example.

	D_1	D_2	D_3	D_4	a_i
O_1	2	3	11	7	6
O_2	1	2	6	1	1
O_3	5	8	15	9	10
b_j	7	5	3	2	

Here $\sum a_i = 6 + 1 + 10 = 17$ and $\sum b_j = 7 + 5 + 3 + 2 = 17$.

Therefore $\sum a_i = \sum b_j$.

So, the transportation problem is balanced.

	D_1	D_2	D_3	D_4	a_i	
O_1	2	3	11	7	6	(1)
O_2	~~1~~	~~2~~	~~6~~	$\boxed{1}$ ~~1~~	$\cancel{1}$	(0)
O_3	5	8	15	9	10	(3)
b_j	7	5	3	$\cancel{2}$ 1		
	(1)	(1)	(5)	(6)		

Now for each row, difference between the least cost and the next cost in that are put within a paranthesis in the line of the row. Again for each column the least cost and the next cost are put within a paranthesis in the line of the column.

Now of these differences, greatest difference is 6 which is in the 4-th column. The we are to allocate in the least cost cell of this column. Now the least cost in this column is 1 which is in (2, 4) cell. So x_{24} = min {2, 1} = 1. Then the source O_2 is exhausted. So the row for O_2 is deleted by dotted line. Then we get following table :

	D_1	D_2	D_3	D_4		
O_1	2	$\boxed{5}$ 3	11	7	~~6~~ 1	(1)
O_3	5	8	15	9	10	(3)
	7	~~5~~	3	1		
	(3)	(5)	(4)	(2)		

How we see that the greatest difference is 5 which is in the column for D_2. Now the least cost in this column is 3 which is in the cell (1, 2). So, we take x_{12} = min {5, 6} = 5. Now demand of D_2 is satisfied. Then we get the following table :

	D_1	D_3	D_4		
O_1	$\boxed{1}$ ~~2~~	~~11~~	~~7~~	~~1~~	(5)
O_3	5	15	9	10	(4)
	~~7~~ 6	3	1		
	(3)	(4)	(2)		

Finally we get the following table :

	D_1	D_3	D_4	
O_3	$\boxed{6}$ 5	$\boxed{3}$ 15	$\boxed{1}$ 9	10
	6	3	1	

So, initial feasible solution is displayed in the following table :

	D_1	D_2	D_3	D_4	
O_1	$\boxed{1}$ 2	$\boxed{5}$ 3	11	7	6
O_2	1	2	6	$\boxed{1}$ 1	1
O_3	$\boxed{6}$ 5	8	$\boxed{3}$ 15	$\boxed{1}$ 9	10
	7	5	3	2	

So an initial feasible solution is

$x_{11} = 1$, $x_{12} = 5$, $x_{24} = 1$, $x_{31} = 6$, $x_{33} = 3$, $x_{34} = 1$.

and cost of the transportation

$$= 1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9$$
$$= 2 + 15 + 1 + 30 + 45 + 9$$
$$= 102 \text{ units}$$

The set of cells in which allocations are made or any subset of these cells do not form loop. So this initial feasible solution is basic.

(d) Row-Minima Method :

In this method, at first we select the minimum cost cell in the first row. Then maximum possible units are allocated in that cell. If minimum cost occurs at more than one cell then any cell with minimum cost can be selected. We shall explain the method by an example.

We take the following transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	23	27	$\boxed{25}$ 16	$\boxed{5}$ 18	~~30~~ 5
O_2	$\boxed{22}$ 12	$\boxed{18}$ 17	20	51	~~40~~ 18
O_3	22	$\boxed{17}$ 28	12	$\boxed{36}$ 32	~~53~~ 36
b_j	~~22~~	~~35~~ ~~17~~	~~25~~	~~41~~ 36	

Here $\sum a_i$ = 30 + 40 + 53 = 123 and $\sum b_j$ = 22 + 35 + 25 + 41 = 123.

Therefore $\sum a_i = \sum b_j$.

So, the transportation problem is balanced.

We see that minimum in the first row is 16 which is in (1, 3) cell. Then we take x_{13} = min {30, 25} = 25. Then the demand of D_3 is satisfied. So the third column can be deleted.

Now we see that source at O_1 still contains 5 units. Again we see that the least element in the 1st row is 18 which is in (1, 4) cell and so we take x_{14} = min {41, 5} = 5.

Then the source at O_1 is exhausted. So, we omit the source O_1.

Now the least cost in the second row is 12 which is in (2, 1) cell. So we take x_{21} = {40, 22} = 22. Now the demand for D_1 is satisfied. So column for D_1 omited. Now in the second row we see that the minimum cost is 17 which is in (2, 2) cell and so we take x_{22} = min {18, 35} = 18. So then the source at O_2 is exhausted and so the row for O_2 is omitted. Now in the third row we find that the least cost is 28 which is in cell (3, 2) and so we take x_{32} = {17, 53} = 17.

So an initial feasible solution is

$x_{13} = 25$, $x_{14} = 5$, $x_{21} = 22$, $x_{22} = 18$, $x_{32} = 17$, $x_{34} = 36$

and cost of transportation

$$= 25 \times 16 + 5 \times 18 + 22 \times 12 + 18 \times 17 + 17 \times 28 + 36 \times 32$$

$$= 400 + 90 + 264 + 306 + 476 + 1152$$

$$= 2688 \text{ units}$$

The cells in which allocations have been made in this solution and any subset of these cells do not form loop. So the above feasible solution is a basic feasible solution.

(e) Column-Minima Method :

This method is similar to the previous method (d). The only difference is that instead of taking row in the previous method, we proceed by taking column in this method.

12.7 Test for Optimality of a Basic Feasible Solution

We know that the number of basic variables in a transportation problem with m origins and n destinations will be at most m + n – 1. So if a basic feasible solution be displayed in a transportation table then the number of occupied cells will be at most m + n – 1.

Now we state (without proof) below a theorem by which we can say whether a basic feasible solution of a balanced transportation problem will be optimal or not :

● **Theorem 12.7.1 :** If in a basic-feasible solution, the number of basic variables is m + n – 1 (m, n have usual meanings) and if the value of no basic variable is zero i.e., if the solution be non-degenerate and if m + n numbers u_i, v_j (i = 1, 2, ... m; j = 1, 2, ... n) can be found such that for each occupied cell (i, j), $c_{ij} = u_i + v_j$ then the cell evaluation Δ_{ij} of each unoccupied cell will be $c_{ij} - (u_i + v_j)$ and the basic feasible solution will be optimal solution if $\Delta_{ij} \geq 0$ for each cell (i, j) of the transportation table. [c_{ij} is cost of transportation of unit commodity from the i-th origin to the j-th destination.]

Remarks

1. If the number of occupied cells be m + n – 1, then from $c_{ij} = u_i + v_j$ (for each occupied cell), we get m + n – 1 equations involving m + n unknown numbers $u_1, u_2, ... u_m; v_1, v_2, ... v_n$ and in this case taking the value of anyone of u_i or v_j arbitrarily, $u_1, u_2, ... u_m; v_1, v_2, ... v_n$ can be determined. In general we take $u_i = 0$ or $v_j = 0$ if the maximum number of cells in the i-th row or in the j-th column have allocations.

2. If for each unoccupied cell, $\Delta_{ij} = c_{ij} - (u_i + v_j) > 0$ then the transportation problem has unique optimal solution. If for each unoccupied cell (i, j), $\Delta_{ij} \geq 0$ and $\Delta_{ij} = 0$ for at least one unoccupied cell then the transportation problem will have more than one optimal solution i.e. here optimal solution will not be unique.

3. If for at least one unoccupied cell (i, j), $\Delta_{ij} < 0$ then the corresponding feasible solution will not be optimal. In this case, for optimal solution we are

to obtain another basic feasible solution in the following way : The cell (i, j) for $\Delta_{ij} < 0$ and $|\Delta_{ij}|$ is maximum will be taken as a new basic cell and maximum allocation (consistent with the constraints of the problem) is to be made in this cell and instead allocation in basic cell (i.e., a previous occupied cell) is to be made 0. As a result we shall get a new basic feasible solution for which total transportation cost will be less than the total cost for the previous solution.

Again, we are to make optimality test with this new basic feasible solution.

If the new solution be an optimal solution then optimal solution of the given problem is obtained otherwise following the previous method another basic feasible solution is obtained and in this way finally we get optimal solution of the given transportation problem.

For understanding the method clearly see examples given in Section 7-8.

4. As usual, the allocation in a cell (if not zero) is shown within a small square in the upper left hand corner of that cell and cost in that cell will be shown within a small square in the lower right hand corner of that cell. For each unoccupied cell (i, j), value of $u_i + v_j$ will be displayed within a small circle of that cell.

12.8 A Few Examples

● **Example 12.8.1 :** 1. Solve the following transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	5	4	6	14	15
O_2	2	9	9	6	4
O_3	6	11	7	13	8
b_j	9	7	5	6	

Solution : Let us first find an initial basic feasible solution by VAM.

Here the problem is balanced since $\sum a_i = \sum b_j = 27$.

	D_1	D_2	D_3	D_4	a_i		
O_1	5	$\boxed{7}$ 4	6	14	$\cancel{15}$ 8	(1)	(1)
O_2	2	9	9	$\boxed{4}$ 6	$\cancel{4}$	(4)	
O_3	6	11	7	13	8	(1)	(1)
b_j	9	$\cancel{7}$	5	$\cancel{6}$ 2			
	(3)	(5)	(1)	(7)			
	(3)	(7)	(1)	(1)			

	D_1	D_3	D_4	a_i	
O_1	$\boxed{8}$ ~~5~~	~~6~~	~~14~~	$\cancel{8}$	(1)
O_3	6	7	13	8	(1)
	$\cancel{9}$ 1	5	2		
	(1)	(1)	(1)		

	D_1	D_3	D_4	
O_3	$\boxed{1}$ 6	$\boxed{5}$ 7	$\boxed{2}$ 13	8
	1	5	2	

The feasible solution obtained by VAM is shown in the following transportation table :

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{8}$ $\boxed{5}$	$\boxed{7}$ $\boxed{4}$	$\boxed{6}$	$\boxed{14}$	15
O_2	$\boxed{2}$	$\boxed{9}$	$\boxed{9}$	$\boxed{4}$ $\boxed{6}$	4
O_3	$\boxed{1}$ $\boxed{6}$	$\boxed{11}$	$\boxed{5}$ $\boxed{7}$	$\boxed{2}$ $\boxed{13}$	8
b_j	9	7	5	6	

The basic feasible solution is $x_{11} = 8$, $x_{12} = 7$, $x_{24} = 4$, $x_{31} = 1$, $x_{33} = 5$, $x_{34} = 2$.

Here the number of basic variables is $3 + 4 - 1 = 6$ and in the above solution the value of no basic variable is 0. So the solution is non-degenerate basic feasible solution.

Optimality Test

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	$\boxed{8}$ $\boxed{5}$	$\boxed{7}$ $\boxed{4}$	⑥ $\boxed{6}$	⑫ $\boxed{14}$	15	–1
O_2	(–1) $\boxed{2}$	(–2) $\boxed{9}$	⓪ $\boxed{9}$	$\boxed{4}$ $\boxed{6}$	4	–7
O_3	$\boxed{1}$ $\boxed{6}$	⑤ $\boxed{11}$	$\boxed{5}$ $\boxed{7}$	$\boxed{2}$ $\boxed{13}$	8	0
b_j	9	7	5	6		
ν_j	6	5	7	13		

Table–I

We know that the numbers u_i (i = 1, 2, 3), ν_j (j = 1, 2, 3, 4) are obtained by solving 6 equations given by $c_{ij} = u_i + \nu_j$ for each occupied cell (i, j) where one of u_i (i = 1, 2, 3), ν_j (j = 1, 2, 3, 4) can be taken arbitrarily.

Now maximum number of cells (3 cells) have been given allocations in the third row and so according to convention we take $u_3 = 0$.

Then we get $u_1 = -1$, $u_2 = -7$, $u_3 = 0$, $\nu_1 = 6$, $\nu_2 = 5$, $\nu_3 = 7$, $\nu_4 = 13$.

Now for each unoccupied cell (i, j), the value of $u_i + \nu_j$ has been shown within a small circle in Table-I.

The values of cell evaluations $\Delta_{ij} = c_{ij} - (u_i + \nu_j)$ are shown in the following table (for all unoccupied cells (i, j) :

	D_1	D_2	D_3	D_4
O_1	•	•	0	2
O_2	3	11	9	•
O_3	•	6	•	•

Here ‘dots’ represent basic cells.

Then it is seen that for each unoccupied cell $\Delta_{ij} \geq 0$

[for example $D_{14} = c_{14} - (u_1 + \nu_4)$

$= 14 - (-1 + 13) = 2$

for (1, 4) cell].

So the solution obtained by VAM will be optimal solution.

So in this case an optimal basic feasible solution is

$x_{11} = 8,\ x_{12} = 7,\ x_{24} = 4,\ x_{31} = 1,\ x_{33} = 5,\ x_{34} = 2$

and the minimum transportation cost is

$8 \times 5 + 7 \times 4 + 4 \times 6 + 1 \times 6 + 5 \times 7 + 2 \times 13$

$= 159$ units.

● **Example 12.8.2 :** Solve the following transportation problem :

					a_i
O_1	1	5	8	6	8
O_2	4	2	5	4	9
O_3	6	4	3	1	13
b_j	10	3	4	13	

Solution : Here the problem is balanced since $\sum a_i + \sum b_j = 30$

Here the initial basic feasible solution is obtained by Row-minima method and the solution is shown in the following table :

	D_1	D_2	D_3	D_4	a_i
O_1	8				8
O_2	2	3		4	9
O_3			4	9	13
b_j	10	3	4	13	

Here m + n – 1 = 3 + 4 – 1 = 6 and in the above table we find that there are 6 occupied cells and so this solution is non-degenerate.

The solution is $x_{11} = 8$, $x_{21} = 2$, $x_{22} = 3$, $x_{24} = 4$, $x_{33} = 4$, $x_{34} = 9$.

Optimality Test :

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	$\boxed{8}$ $\boxed{1}$	–1 $\boxed{5}$	3 $\boxed{8}$	1 $\boxed{6}$	8	–3
O_2	$\boxed{2}$ $\boxed{4}$	$\boxed{3}$ $\boxed{2}$	6 $\boxed{5}$	$\boxed{4}$ $\boxed{4}$	9	0
O_3	1 $\boxed{6}$	–1 $\boxed{4}$	$\boxed{4}$ $\boxed{3}$	$\boxed{9}$ $\boxed{1}$	13	–3
b_j	10	3	4	13		
v_j	4	2	6	4		

Table–II

For each occupied cell (i, j) we get $c_{ij} = u_i + v_j$ which gives 6 equations (since there are 6 occupied cells). Taking $u_2 = 0$, solving these equations we get $u_1 = -3$, $u_2 = 0$, $u_3 = -3$, $v_1 = 4$, $v_2 = 2$, $v_3 = 6$, $v_4 = 4$.

Now the values of $u_i + v_j$ [for all unoccupied cells (i, j)] are displayed in Table-II within a small circle.

Here the values of cell evaluations $\Delta_{ij} = c_{ij} - (u_i + v_j)$ are shown in the following table [for all unoccupied cells] :

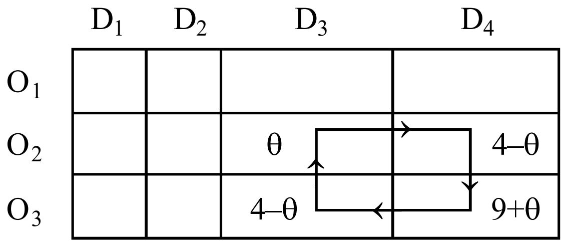
	D_1	D_2	D_3	D_4
O_1	•	6	5	5
O_2	•	•	(–1)	•
O_3	5	5	•	•

It is seen that $\Delta_{23} = -1 < 0$.

So the solution obtained by 'row minima' method is not an optimal solution. Now there is only one unoccupied cell where the cell evaluation is negative and the cell is (2, 3).

So we shall determine another basic feasible solution where (2, 3) cell will be a basic cell.

Now a loop can be formed with the unoccupied cell (2, 3) and the occupied cells (2, 4), (3, 4), (3, 3) [The loop is shown in the table given below].



	D_1	D_2	D_3	D_4
O_1				
O_2			θ	$4-\theta$
O_3			$4-\theta$	$9+\theta$

We suppose that an allocation θ (> 0) is given in (2, 3) cell. Then in order that the constraints at the given transportation problem may be satisfied we alternately add and subtract θ in the cells of the loop (as shown in the above table).

Now in order that value of every allocation is " ≥ ", we are to select θ (> 0) in such a way that $\theta = \min. \{4, 4\} = 4$.

So the new basic feasible solution thus obtained is shown in the following table :

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	$\boxed{8}$ $\boxed{1}$	(–1) $\boxed{5}$	② $\boxed{8}$	⓪ $\boxed{6}$	8	–3
O_2	$\boxed{2}$ $\boxed{4}$	$\boxed{3}$ $\boxed{2}$	④ $\boxed{5}$	③ $\boxed{4}$	9	0
O_3	② $\boxed{6}$	⓪ $\boxed{4}$	$\boxed{\varepsilon}$ $\boxed{3}$	$\boxed{13}$ $\boxed{1}$	13	–2
b_j	10	3	4	13		
v_j	4	2	5	4		

Table–III

In this solution there are five occupied cells. But here the number of basic variables is 6. So the above solution is degenerate basic feasible solution.

The solution is

$x_{11} = 8,\ x_{21} = 2,\ x_{22} = 3,\ x_{23} = 4,\ x_{34} = 13.$

Optimality Test :

Here the number of occupied cells is 5 which is less than m + n – 1 = 6. So we are to add an unoccupied cell with these five occupied basic cells so that no loop can be formed with these 6 cells or with any subset of this cells. We take (3, 3) as the unoccupied basic cell where an amount ε (> 0) is allocated and where ε is sufficiently small so that $a \pm \varepsilon$ can be taken as a for any number a. As a result, the given problem is modified, but at the optimal stage if we take $\varepsilon = 0$ we shall obtain the optimal solution of the given problem.

Thus as usual taking $c_{ij} = u_i + \nu_j$ [for each we shall get six equations involving u_i (i = 1, 2, 3), ν_j (j = 1, 2, 3, 4) and the values of these 7 numbers u_1, u_2, u_3, ν_1, ν_2, ν_3, ν_4 can be determined by solving the above equations with the extra condition $u_2 = 0$.

Then we get $u_1 = -3$, $u_2 = 0$, $u_3 = -2$; $\nu_1 = 4$, $\nu_2 = 2$, $\nu_3 = 5$, $\nu_4 = 4$.

Now as usual we put the value of $u_i + \nu_j$ (for each unoccupied cell) within a small circle in Table-III.

Now cell evaluations $\Delta_{ij} = c_{ij} - (u_i + \nu_j)$ are shown in the following table for each unoccupied cell.

	D_1	D_2	D_3	D_4
O_1	•	6	6	6
O_2	•	•	•	1
O_3	4	4	$\boxed{\varepsilon}$ •	•

In the above table, dot's represent the basic cells where in the particular basic cell (3, 3), an allocation ε (> 0) is given.

In the last table, we find that all cell evaluations Δ_{ij} are " ≥ 0 ". So an optimal solution of the given problem can be obtained by taking $\varepsilon = 0$. Then an optimal solution is

$x_{11} = 8$, $x_{21} = 2$, $x_{22} = 3$, $x_{23} = 4$, $x_{34} = 13$ which is a degenerate basic feasible solution and minimum cost = $8 \times 1 + 2 \times 4 + 3 \times 2 + 4 \times 5 + 13 \times 1 = 55$ units.

12.9 Unbalanced Transportation problem

If the total availability and the total requirments are not equal in a transportation problem, i.e., if $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then we say that the transportation problem is "unbalanced".

In this case the unbalanced problem is to be transformed into a balanced problem.

For this, a fictitious source $\left(\text{for } \sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j\right)$ or a fictitious destination $\left(\text{for } \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j\right)$ is introduced where the cost from the fictitious source to any destination is taken to be "0" or the cost from any source to a fictitious destination is taken to be "0".

If a fictitious source is required then the amount of availability at this source will be $\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \left(\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j\right)$

Similarly, it a fictitious destination is required then the requirement at the fictitious destination will be $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ when $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$.

Let us give an example.

We consider the following transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	10	8	2	7	50
O_2	5	6	4	3	40
O_3	12	21	9	8	30
b_j	60	40	20	30	

Here $\sum_i a_i = 120, \sum_j b_j = 150$.

Then in this case $\sum_j b_j > \sum_i a_i$.

So here a fictitious source O_4 is required and availability of this source will be $150 - 120 = 30$.

So the transformed balanced transportation problem will be as follows :

	D_1	D_2	D_3	D_4	a_i
O_1	10	8	2	7	50
O_2	5	6	4	3	40
O_3	12	21	9	8	30
O_4	0	0	0	0	30
b_j	60	40	20	30	

Now the balanced problem will be solved as usual. In the optimal solution of the balanced problem, if there be any allocation in a cell for the fictitious row O_4, we omit this allocation and we get optimal solution of the unbalanced transportation problem. In this case "30" units cannot be sent to any destination from the source O_4, i.e., in this case "30" units demand of the destination cannot be supplied.

If $\sum_i a_i > \sum_j b_j$ then we can similarly solve the corresponding unbalanced transportation problem taking a fictitious destination with each entry "0" in the column for fictitious destination.

[As an illustration see solution of the unbalanced transportation problem given in Ex-7 of Exercise-2 in Section 7.12]

12.10 Summary

At first we have described transporatation problem as a special type of L.P.P. and then we have proved that any balanced transportation problem has an optimal solution.

We have stated the theorem by which we can determine whether a given set of column vectors of the coefficient matrix of the transportation problem will be linearly dependent or not by observing whether the cells in the transportation table corresponding to the variables of the column vectors form a loop or not. We have discussed different methods of finding initial basic feasible solutions of a transportation problem. Then we have described MODI method of testing optimality of an initial basic feasible solution and also we have discussed the method of obtaining an improved basic feasible solution if the initial basic feasible solution (obtained by any method) is not optimal. At the end, we have discussed how an unbalanced transportation problem can be solved.

12.11 Exercise-1

1. Express the following transportation problem as a L.P.P. :

(i)

	D_1	D_2	a_i
O_1	2	1	10
O_2	3	4	12
b_j	8	14	

(ii)

	D_1	D_2	D_3	D_4	a_i
O_1	2	1	3	2	30
O_2	3	1	0	4	50
O_3	5	3	2	4	20
b_j	20	40	30	10	

2. Examine whether the solution displayed in the following table is a basic feasible solution or not. Is this solution non-degenerate?

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{20}$	$\boxed{10}$			30
O_2		$\boxed{30}$	$\boxed{20}$		50
O_3			$\boxed{10}$	$\boxed{10}$	20
b_j	20	40	30	10	

3. Write the coefficient matrix of the Problem 1 (i) and show that the rank of this matrix will be 3.

Answers

1. (i) Minimize $z = 2x_{11} + x_{12} + 3x_{21} + 4x_{22}$

subject to

$$x_{11} + x_{12} = 0$$

$$x_{21} + x_{22} = 12$$

$$x_{11} + x_{21} = 8$$

$$x_{12} + x_{22} = 14$$

$$x_{11}, x_{12}, x_{21}, x_{22} \geq 0$$

(ii) Minimize $z = 2x_{11} + x_{12} + 3x_{13} + 2x_{14} + 3x_{21} + x_{22} + 4x_{24} + 5x_{31} + 3x_{32} + 2x_{33} + 4x_{34}$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} = 30$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 50$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 20$$

$$x_{11} + x_{21} + x_{31} = 20$$

$$x_{12} + x_{22} + x_{32} = 40$$

$$x_{13} + x_{23} + x_{33} = 30$$

$$x_{14} + x_{24} + x_{34} = 10$$

$$x_{ij} \geq 0; \quad i = 1, 2, 3 \; ; j = 1, 2, 3, 4$$

2. Non-degenerate basic feasible solution.

3. The coefficient matrix of the Problem 1 (i) is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

12.12 Exercise-2

1. Find the initial basic feasible solution of the following transportation problem by (i) North-West Corner (ii) Vogel's approximation method

	D_1	D_2	D_3	a_i
O_1	5	1	8	12
O_2	2	4	0	14
O_3	3	6	7	4
b_j	9	10	11	

Which method of solution is better in respect of transportation cost ?

2. Find the initial basic feasible solution by Matrix-Minimum method.

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	2	11	10	3	7	4
O_2	1	4	7	2	1	8
O_3	3	9	4	8	12	9
b_j	3	3	4	5	6	

3. Solve the following Transportation Problems :

(i)

	D_1	D_2	D_3	a_i
O_1	30	20	10	500
O_2	5	15	25	500
b_j	300	300	400	

(ii)

	D_1	D_2	D_3	a_i
O_1	7	3	4	2
O_2	2	1	3	3
O_3	3	4	6	5
b_j	4	1	5	

4. Find the basic feasible solution of the Transportation problem below :

	W_1	W_2	W_3	W_4	a_i
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	60	40	20	30	

5. Solve the following Transportation Problem and prove that optimal solution is unique :

	D_1	D_2	D_3	a_i
O_1	6	8	4	14
O_2	4	9	3	12
O_3	1	2	6	5
b_j	6	10	15	

6. Solve the following Transportation problem :

	D_1	D_2	D_3	a_i
O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5
b_j	5	5	10	

7. Solve the following unbalanced transportation problem :

	D_1	D_2	D_3	D_4	a_i
O_1	3	8	7	4	30
O_2	5	2	9	5	50
O_3	4	3	6	2	80
b_j	20	60	55	40	

Solution : Here $\sum a_i = 160$ and $\sum b_j = 175$. So the problem is unbalanced.

Here $\sum b_j - \sum a_i = 15$.

Then a fictitious source O_4 is introduced where availability is 15 units.

Now the transformed transportation problem is

	D_1	D_2	D_3	D_4	a_i
O_1	3	8	7	4	30
O_2	5	2	9	5	50
O_3	4	3	6	2	80
O_4	0	0	0	0	15
b_j	20	60	55	40	

Here a basic feasible solution obtained by VAM is shown in the following table :

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{20}$ 3	8	$\boxed{10}$ 7	4	30
O_2	5	$\boxed{50}$ 2	9	5	50
O_3	4	$\boxed{10}$ 3	$\boxed{30}$ 6	$\boxed{40}$ 2	80
O_4	0	0	$\boxed{15}$ 0	0	15
b_j	20	60	55	50	

Optimality Test :

Here the number of occupied cells (basic cells) is 7 $(m + n – 1 = 4 + 4 – 7 = 7)$. From seven equations given by $c_{ij} = u_i + \nu_j$ [taking each basic cell (i, j)] and taking $u_3 = 0$ we get $u_1 = 1$, $u_2 = –1$, $u_3 = 0$, $u_4 = –6$, $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = 6$, $\nu_4 = 2$.

Then the values of cell evaluations Δ_{ij} are computed and shown within small circles (taking all unoccupied cells) in the following table :

	D_1	D_2	D_3	D_4	u_i
O_1	$\boxed{20}$ 3	④ 8	$\boxed{10}$ 7	① 4	1
O_2	④ 5	$\boxed{50}$ 2	④ 9	④ 5	–1
O_3	② 4	$\boxed{10}$ 3	$\boxed{30}$ 6	$\boxed{40}$ 2	0
O_4	④ 0	③ 0	$\boxed{15}$ 0	④ 0	–6
ν_j	2	3	6	2	

Here we see that $\Delta_{ij} = c_{ij} - (u_i + v_j) \geq 0$ for all values of i, j.

So the basic feasible solution obtained by VAM is the optimal solution.

Then optimal solution of the given transportation problem is $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 50$, $x_{32} = 10$, $x_{33} = 30$, $x_{34} = 40$

and minimum cost is

$20 \times 3 + 10 \times 7 + 50 \times 2 + 10 \times 3 + 30 \times 6 + 40 \times 2 = 520$ units.

[Here we observe that for the unbalanced problem $x_{43} = 15$ has been omitted— this means that 15 units of demand of D_3 cannot be supplied].

8. Find the minimum Transportation cost of the following unbalanced Transportation problem :

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	1	5	0	13
O_3	3	8	6	12
b_j	8	5	7	

Answers

1. (i) $x_{11} = 9$, $x_{12} = 3$, $x_{23} = 7$, $x_{23} = 7$, $x_{33} = 4$; cost = 104

   (ii) $x_{11} = 2$, $x_{12} = 10$, $x_{21} = 3$, $x_{23} = 11$, $x_{31} = 4$; cost = 38

   Vogel's approximation method is better.

2. $x_{11} = 1$, $x_{14} = 3$, $x_{21} = 2$, $x_{25} = 6$, $x_{32} = 3$, $x_{34} = 2$

3. (i) $x_{12} = 100$, $x_{13} = 400$, $x_{21} = 300$, $x_{22} = 200$; Minimum cost = 10500

   (ii) $x_{13} = 2$, $x_{21} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$

   Minimum cost = 33.

4. $x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$, $x_{32} = 6$, $x_{34} = 12$

5. $x_{12} = 5$, $x_{13} = 9$, $x_{21} = 6$, $x_{23} = 6$, $x_{32} = 5$; unique.

6. $x_{13} = x_{21} = x_{22} = x_{33} = 5$; Minimum cost = 35.
8. Minimum transportation cost = 23.

12.13 Multiple Choice Questions (MCQ)

1. What is the number of non-basic variables in the balanced TP with 4 rows and 5 columns?
   (a) 4 (b) 5 (c) 12 (d) 20
2. The number of basic variables in a transportation problem of m sources and n destinations is at most
   (a) m + n + 1 (b) m + n (c) m + n – 1 (d) m – n + 1
3. The initial basic feasible solution of a transportation problem becomes non-optimal if the matrix $[\Delta_{ij}] = [c_{ij}] - [u_i + \nu_j]$ has
   (a) at least one negative element (b) at most one negative element
4. The solution of a transportation problem is never unbounded
   (a) True (b) False
5. In a transportation problem with m origins and n destinations (balanced) number of independent constraints are
   (a) m + n (b) mn (c) m + n – 1 (d) m + n + 1
6. When the total availability is not equal to the total demand, that type of transportation problem is known as
   (a) balanced transportation problem (b) Unbalanced transportation problem
   (c) degenerate (d) non-degenerate
7. Which of the following is not a method to obtain the initial basic feasible solution in transportation problem?
   (a) VAM (b) Least cost method
   (c) North-West Corner Method (d) MODI Method
8. In a transportation problem, the number of the cells required for forming a loop is
   (a) even (b) odd (c) prime (d) none of these

9. The initial solution of a transportation problem can be obtained by applying any known method. However the only restriction is that
   (a) the solution should be non-degenerate
   (b) the solution must be optimal
   (c) the rim conditions are satisfied
   (d) all of these

10. The dummy source or destination in a transportation problem is created to
   (a) satisfy rim condition
   (b) prevent solution to become degenerate
   (c) to sove the balance transportation problem
   (d) none of these

Answers

1. (c)	2. (c)	3. (a)	4. (a)	5. (c)	6. (b)
7. (d)	8. (a)	9. (c)	10. (a)		

Unit - 13 ❑ Assignment Problem

Structure

13.0 Objective

After studying this chapter, the reader should be able to

- recognize an Assignment problem
- find the mathematical formulation of an Assignment problem
- solve assignment problems by using Hungarian Method
- solve unbalanced, maximization Assignment problems

13.1 Introduction

In practical field, we get some problem which will be called assignment problems, let us give an example of such a problem. We suppose that the manager of a firm wants to assign five jobs 1, 2, 3, 4, 5 to five person A, B, C, D, E. In this case we can assume that these five person are not equally efficient in each job. As a result the cost (say wages) of completing a job depends on the person and the job. Now the manager of the firm will assign only one job to exactly one person and in this five jobs 1, 2, 3, 4, 5 will be assigned to the five persons A, B, C, D, E. The assignment problem is to find the best assignment of five jobs to five persons i.e., the problem is to find the minimum assignment cost on the maximum profit of the owner of factory when five jobs are assigned to five persons on one to one basis i.e., only one job is assigned to only one person.

We now suppose that m jobs are to be assigned to m persons (or machines or other facilities). We suppose c_{ij} will be cost when i-th job is assigned to j-th facility $(i = 1, 2, ..., m; j = 1, 2, ..., m)$. Unless otherwise stated by assignment problem we shall understand the problem of finding assignments for which the total cost is minimum.

In this case the problem will be described by the following table where the entries c_{ij} form a matrix of order m × m and this matrix is called cost matrix.

Facilities

$$\begin{array}{cc|ccccc|} & & 1 & 2 & 3 & \ldots\ldots\ldots & m \\ \hline & 1 & c_{11} & c_{12} & c_{13} & \ldots\ldots\ldots & c_{1m} \\ \text{Jobs} & 2 & c_{21} & c_{22} & c_{23} & \ldots\ldots\ldots & c_{2m} \\ & 3 & c_{31} & c_{32} & c_{33} & \ldots\ldots\ldots & c_{3m} \\ & & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ & m & c_{m1} & c_{m2} & c_{m3} & \ldots\ldots\ldots & c_{mm} \\ \hline \end{array}$$

If the cost matrix is given then the assignment problem is the problem “how m jobs can be assigned to m facilities such that each job is assigned to exactly one facility and each facility has exactly one job so that the total assignment cost is minimum.”

If the profit matrix be given then the assignment problem is the problem "how the jobs can be assigned to the facilities such that each job is assigned to exactly one facility and each facility has exactly one job so that the total profit becomes maximum".

Here we note that the number of jobs and the number of facilities are equal and so for an assignment problem the cost matrix (or profit matrix) is a square matrix.

Unless otherwise stated we shall assume that the matrix of a given assignment problem is the cost matrix.

13.2 Mathematical Formulation of an Assignment Problem

We suppose that there are m jobs and m facilities and let the given cost matrix be $[c_{ij}]_{m\times m}$ where c_{ij} will be the cost if the i-th job is assigned to j-th facility.

Now we define the variables x_{ij} ($i = 1, 2, ..., m$; $j = 1, 2, ..., m$) as follows :

$x_{ij} = 1$ if i-th job is assigned to j-th facility

$= 0$ otherwise

According to the conditions of an assignment problem for any definite value of i, $x_{ij} = 1$ for exactly one value of j and for any definite value of j, $x_{ij} = 1$ for exactly one value of i. These conditions can be expressed as

$$\sum_{j=1}^{m} x_{ij} = 1;\ i = 1, 2, ..., m$$

and
$$\sum_{i=1}^{m} x_{ij} = 1;\ \ j = 1, 2, ..., m$$

We note that the total cost z is given by $z = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$

Then the assignment problem (for the cost matrix) can be mathematically formulated as

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$$

$$\text{subject to } \sum_{j=1}^{m} x_{ij} = 1; \; i = 1, 2, ..., m$$

$$\text{and } \sum_{i=1}^{m} x_{ij} = 1; \; j = 1, 2, ..., m$$

and $x_{ij} \geq 0$ such that

$x_{ij} = 1$, if i-th job be assigned to the j-th facility

$= 0$, otherwise

and as such the assignment problem is **not a linear programming problem** as the variables x_{ij} can assume only 0 and 1.

13.3 Solution of an Assignment Problem

Here we shall assume that for the cost matrix $[c_{ij}]_{m\times m}$, $c_{ij} \geq 0$ for all i, j. Now i-th job is assigned to j-th facility then $x_{ij} = 1$, otherwise $x_{ij} = 0$.

Then if it is possible to make assignments in such a way that the corresponding c_{ij} is 0 then for such assignments total cost will be 0 which is minimum in this case and remembering this we can determine appropriate algorithm for solving assignment and this algorithm (computational procedure) depends on the two theorems stated below (without proof)

● **Theorem 13.3.1 :** If a constant (positive or negative) be added to each element of a row and/or a column, then the transformed assignment problem and the given assignment problem have the same optimal assignments.

[The assignments for which the total cost of the assignment problem is minimum will be called optimal assignments or optimal solution]

● **Theorem 13.3.2 :** If $c_{ij} \geq 0$ for all i, j and if $x_{ij} = x_{ij}^*$ (i = 1, 2, ..., m; j = 1, 2,

..., m) are determined for which $\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}^{*} = 0$ (for minimizing problem) then the solution x_{ij}^{*}; i = 1, 2, ..., m; j = 1, 2, ..., m will be optimal solution.

Then from the above two theorems we see that several elements of the cost matrix can be made zero by subtracting suitable positive number from the rows and/ or the columns of the cost matrix.

Then in this way it will be possible to find assignments for which the values of the corresponding c_{ij}'s will be zero and these assignments must optimal assignments.

13.4 Computational Proceduce (Hungarian Method)

If some elements of the given cost matrix are not non-negative then applying Theorem 8.3.1 each element of the cost matrix can be made non-negative (i.e., ≥ 0). Then we are to follow the following steps :

Step 1 : The minimum element of each row of the cost matrix is to be subtracted from each element of the row.

Now we shall get at least one zero in each row of the cost matrix. Then the minimum element of each column is to be subtracted from each element of that column.

The matrix thus obtained will be starting cost matrix.

Now assignments are to be made on the cells having zero cost. The assignments are optimal assignments if it is possible to make all assignments only in the cells with cost zero.

In the next step we shall mention a method by which it will be possible to say whether the assignments obtained in Step 1 will be optimal or not.

Step 2 : We are to draw minimum number of vertical and horizontal lines through all zeros of the starting matrix.

Now there are two possibilities :

(i) Total number of straight lines drawn to the order of the cost matrix.

(ii) Total number of straight lines drawn is less than the order of the cost matrix.

In case (ii) we are to go to Step 3 and in case (i) we shall get optimal assignments.

Step 3 : Starting from the first row, in examining each row of this starting matrix if we get only one 0 then 0 is put in ☐ for assignment and then we draw vertical line through this marked 0. After examining all rows in this way, the columns will also be examined in the same manner. In this case starting from first column and whenever only one 0 is found in the uncrossed column, the 0 is put in ☐ and we draw herizontal line through each such 0.

Now from all marked 0 (which one within ☐) we shall get optimal assignment.

In case (ii) of Step 2 we are to go to Step 4.

Step 4 : We take the minimum of the terms which are outside the lines drawn in Step 2 and we subtract this minimum element from each element outside the lines and further this element is to be added to the intersecting point (if any) of horizontal line and vertical line. As a result we shall get more zeros in the modified matrix.

Then we are to follow Step 2 with the modified matrix. Again if we do not get all assignments then we are to repeat Step 4 and Step 2 and finally we shall get optimal assignments in Step 3.

Step 5 : After making the two operations (with row or column) successively ultimately either get

(i) there is no unmarked 0

(ii) in a row or a column there will be more than one unmarked 0.

In case (i), unique optimal assignment will be obtained. In case (ii), of the unmarked zeros in a row (or column) we select one 0 arbitrarily and we ignore the other 0's in the row (or column).

If we repeat this operation several times, finally we obtain a matrix which contains no unmarked 0. In this case we will get more than one optimal assignments. But for each such optimal assignment the minimum cost will be same.

The computational procedure will be understood clearly by the examples given in the section 8.8.

13.5 Assignment Problem as Maximization Problem

If the given matrix of an assignment problem be profit matrix then the assignment problem will be a maximization problem since here assignments are made in such a way that the total profit becomes maximum.

In this case each element in the profit matrix is subtracted from the largest element of the profit matrix and considering the transformed matrix thus obtained as a cost matrix, optimal assignments are to be determined.

We observe that for these optimal assignments the cost of the transformed assignment problem will be minimum and consequently the profit of the given assignment problem will be maximum. So the optimal assignments of the transformed problem will be optimal assignments of the given assignment problem. Then the maximum profit is computed from the given profit matrix corresponding to the optimal assignments.

13.6 Assignment Problem with Restricted Assignments

If some assignment be impossible, that is, if some job, say i-th job cannot be performed by some particular facility, say j-th facility, then we avoid this effectively by putting a large cost in the (i, j)-th cell or we write $\propto$ (or $-\infty$) in that cell which prevents that particular assignment from being effective in the optimal solution.

13.7 Unbalanced Assignment Problem

If the number of jobs and the number of facilities be not equal, then the problem is said to be unbalanced. In this case we add a fictitious job or facility whichever has the deficiency, with zero cost. Then we apply the assignment algorithm to this resulting balanced problem.

13.8 A Few Examples

● **Example 13.8.1 :** A company has one car at each of the four depots I, II, III and IV. A customer in each of the towns A, B, C, D requires a car. The distance (in miles) between the depots (origins) and the towns (destinations) where the customers one's is given in the following distance matrix :

	I	II	III	IV
A	10	14	19	13
B	15	19	7	18
C	13	15	22	14
D	18	20	10	9

How should the cars be assigned to the customers so as to minimize the distance travelled.

Solution : The least element in the first row is 10. We subtract 10 from each element of this row. Similarly the least element of the other rows are subtracted from the elements of the corresponding rows. Then we get the following matrix :

	I	II	III	IV
A	0	4	9	3
B	8	12	0	11
C	0	2	9	1
D	9	11	1	0

Now the least element of each column (of the last matrix) is subtracted from all elements of the corresponding column and get the following matrix :

	I	II	III	IV
A	0	2	9	3
B	8	10	0	11
C	0	0	9	1
D	9	9	1	0

Now it is seen that in each of the cells (1, 1), (2, 3), (3, 1), (3, 2) and (4, 4) there is 0. The minimum number of dotted lines are drawn to cover all zeros.

Here the number of dotted lines is 4 which is the order of the given matrix. So optimal stage is reached. In (1, 1) cell of the first row there is one zero and no zero in the other cells of this row. Enclosing this zero with in □ we delete the first column by drawing a dotted line. In the second row there is only one zero in (2, 3) cell. Enclosing this zero within □ we delete the third column by drawing a dotted line. In the third row there is only one unmarked zero in (3, 2) th cell. Enclosing this zero within □ we delete the second column by drawing a dotted line. In the fourth row there is only one zero in (4, 4) cell and putting this zero within □ we delete the fourth column by a dotted line. So assignments will be made in the cells (1, 1), (2, 3), (3, 2) and (4, 4) in the table below.

	I	II	III	IV
A	$\boxed{0}$	2	9	3
B	8	10	$\boxed{0}$	11
C	0	$\boxed{0}$	9	1
D	9	9	1	$\boxed{0}$

So the solution of the assignment problem can be stated as follows :

A → I, B → III, C → II, D → IV

and the minimum distance is (10 + 7 + 15 + 9) k.m. = 41 k.m.

● **Example 13.8.2 :** Jobs in machines I, II, III, IV are to be given to four persons A, B, C, D. Assignment costs (in rupees) are given in the following table :

	I	II	III	IV
A	18	26	17	11
B	13	38	14	26
C	38	39	18	15
D	19	26	24	10

Find the minimum cost of assignment.

Solution : The minimum element of the first row is subtracted from each element of this row. Similarly the minimum element of each of the remaining rows is subtracted from the elements of the corresponding rows. Then we get cost matrix in the following table :

	I	II	III	IV
A	7	15	6	0
B	0	25	1	13
C	23	24	3	0
D	9	16	14	0

Again the minimum element of each column (of the last matrix) is subtracted from the elements of the corresponding columns and we get the matrix in the following table :

	I	II	III	IV
A	7	0	5	0
B	0	10	0	13
C	23	9	2	0
D	9	11	13	0

Here we see that zeros are in the cells (1, 2), (2, 1), (2, 3), (3, 4) and (4, 4).

The minimum number of dotted lines are drawn covering all zeros. Here the number of lines is 3 ≠ 4 which is the order of the given cost matrix. So the optimal stage is not reached. In the last table we find that the minimum of the elements which are not on the lines is 2. Now we subtract 2 from the elements which are not on the lines and we add 2 at each junction of the horizontal and vertical lines. The remaining elements remain unchnaged. Then we get the matrix in the following table :

	I	II	III	IV
A	5	$\boxed{0}$	3	0
B	$\boxed{0}$	12	0	15
C	21	9	$\boxed{0}$	0
D	7	11	11	$\boxed{0}$

Again the minimum number of lines are drawn to cover all the zeros of the last matrix.

Here the number of lines is 4 = order of the cost matrix. Assignments are made in the cells having zero within ☐.

So the solution of the assignment problem can be stated as follows :

Optimal assignments :

$$A \to II,\ B \to I,\ C \to III,\ D \to IV$$

and the minimum cost is Rs. (26 + 13 + 18 + 10) = Rs. 67

● **Example 13.8.3 :** Four operators are to be appointed for running four machines. The costs (in rupees) of running a machine by an operators are given in the following table :

Machine

Operator	1	2	3	4
A	5	5	∞	2
B	7	4	2	3
C	9	3	5	∞
D	7	2	6	7

Denoting the machines as 1, 2, 3, 4 and operators and A, B, C, D, it is given that machine 3 cannot be given to the operator A and machine 4 cannot be given to the operator C. Find which machine can be assign to which operator such that total cost is minimum.

Solution : Since machine 3 cannot be assign to operator A and machine 4 cannot be assign to operator C, we have use symbol ∞ in the cells (A, 3) and (C, 4).

Now the minimum cost in the first row is 2 and we subtract 2 from each element of the first row. Similarly the minimum element of other rows is subtracted from each element of the corresponding rows.

Then we get the cost matrix in the following table :

	1	2	3	4
A	3	3	∞	0
B	5	2	0	1
C	6	0	2	∞
D	5	0	4	5

Again, minimum of each column is subtracted from each element of the corresponding columns. Then we get the cost matrix in the following table :

	1	2	3	4
A	0	3	∞	0
B	2	2	0	1
C	3	0	2	∞
D	2	0	4	5

Here we see, that the minimum number of dotted lines (horizontal and vertical) covering all zeros is 3 which is less than 4 which is the order of the given cost matrix. So the optimal stage is not reached. Now we subtract the minimum element which does not lie on the lines drawn in the last matrix from the elements of this matrix outside the lines. Here the minimum element is 1. Also we add 1 to each junction of horizontal and vertical lines and other elements remain unchanged. Then we get the cost matrix in the following table :

0	4	∞	0
1	2	0	0
2	0	2	∞
1	0	4	4

[Here we note that $\infty - 1$, $\infty + 1$ both can be replaced by ∞]

Now the minimum number dotted lines (horizontal and vertical lines) covering all zeros is 3, which is less than 4 which is the order of the given cost matrix, so the optimal stage is not reached. Again, repeating the last process we get the cost matrix in the following table :

	1	2	3	4
A	0	5	∞	0
B	1	3	0	0
C	1	0	1	∞
D	0	0	3	3

Here, we see that the minimum number of lines covering all zeros in the last matrix is 4 which is equal to the order of the given cost matrix.

So the optimal stage is reached.

Now assignments can be given only on the cells having cost 0 enclosed within ☐.

So optimal solution of the given assignment problem can be stated as follows :

Optimal assignments :

$$A \to 4,\ B \to 3,\ C \to 2,\ D \to 1$$

and the minimum cost is Rs. (2 + 2 + 3 + 7) = Rs. 14.

● **Example 13.8.4 :** Find the assignments q machines to the jobs that will manimise the profit with the following profit matrix :

	J_1	J_2	J_3	J_4	J_5
M_1	62	78	50	101	82
M_2	71	84	61	73	59
M_3	87	92	111	71	81
M_4	48	64	87	77	80

Solution : This an unbalanced problem and so it can be converted to a balanced problem by introducing a fictitious machine M_5 with zero profit for each job. The converted profit matrix is as follows :

	J_1	J_2	J_3	J_4	J_5
M_1	62	78	50	101	82
M_2	71	84	61	73	59
M_3	87	92	111	71	81
M_4	48	64	87	77	80
M_5	0	0	0	0	0

Here, we see that the maximum element in the profit matrix is 111. Now we subtract each element (except the elements in the row for the fictitious machine M_5) of the last matrix from 111. Then the modified matrix is

	J_1	J_2	J_3	J_4	J_5
M_1	49	33	61	10	29
M_2	40	27	50	38	52
M_3	24	19	0	40	30
M_4	63	47	24	34	31
M_5	0	0	0	0	0

Now taking the last matrix as a cost matrix we are to find the assignments for which the cost is minimum for this cost matrix and we know that for the same assignments the profit for the given profit matrix will be maximum and the maximum value can be obtained from the given matrix.

We subtract the minimum element of each row of the last matrix from the elements of the corresponding row. Then the modified matrix becomes

	J_1	J_2	J_3	J_4	J_5
M_1	39	23	51	0	19
M_2	13	0	23	11	25
M_3	24	19	0	40	30
M_4	39	23	0	10	7
M_5	0	0	0	0	0

Now we subtract minimum element of each column from the elements of the corresponding columns. Here we observe that the modified matrix is the same as the last matrix.

Here we see that number of dotted lines covering all zeros is 4 which is less that the order of the last matrix. So optimal stage is not reached. Now we subtract the minimum element 7 (outside the lines) from the elements outsides the lines and we add 7 at each junction of horizontal and vertical lines. Then the modified matrix becomes.

	J_1	J_2	J_3	J_4	J_5
M_1	32	23	51	$\boxed{0}$	12
M_2	6	$\boxed{0}$	23	11	18
M_3	17	19	$\boxed{0}$	40	23
M_4	32	23	0	10	$\boxed{0}$
M_5	$\boxed{0}$	7	7	7	0

Here, we see that the minimum of dotted lines covering all zeros of the last matrix is 5 which is also the order of the matrix. So optimal stage is reached.

Then optimal assignments are

$$M_1 \rightarrow J_4,\ M_2 \rightarrow J_2,\ M_3 \rightarrow J_3,\ M_4 \rightarrow J_5$$

(M_5 being fictitious the assignment $M_5 \rightarrow J_1$ cannot be taken)

Then for the above assignment the profit corresponding to the given profit matrix will be maximum and maximum profit will be

$101 + 84 + 111 + 80 = 376$ units

So the required optimal solution is given by

Optimal assignments : $M_1 \rightarrow J_4$, $M_2 \rightarrow J_2$, $M_3 \rightarrow J_3$, $M_4 \rightarrow J_5$

and maximum profit is 376 units.

13.9 Travelling Salesman Problem

"Travelling Salesman problem" is explained below :

We suppose that a salesman is to visit n towns. Now we assume that the distances between any two towns (or the time of reaching any town from any other town or the cost of journey from one town to another town) are known. Starting from any town a travelling salesman is to return to the starting time after visiting every town. The problem of travelling salesman is to find the route such that the total distance travelled (or total time of travel or the total cost of journey) is minimum. Here we observe that any town can be taken as the starting town.

Mathematical formulation of the problem

We suppose that the distance (or time or cost) from i-th town to the j-th town is c_{ij} and let $x_{ij} = 1$, if the salesman directly goes from the i-th town to the j-th town. $= 0$, otherwise.

[$i = 1, 2, ..., n; j = 1, 2, ..., n$ if there are n towns.]

The problem is to find the values of the variables x_{ij} for which the total distance (or cost of time)

$$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \text{ is minimum,}$$

subject to the condition that x_{ij} should be so chosen that no town is visited twice before the completion of the tour. These constraints are mathematically put as

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots\ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots\ldots, n.$$

and $x_{ij} = 1$ or 0, $x_{ii} \neq 1$

Further $c_{ii} = \infty$, by convention as x_{ii} can never be unity as a town cannot follow the same town.

The distance (or time or cost) matrix for n towns – travelling salesman problem will be as given below :

$$\begin{array}{cc|c|c|c|c|}
 & \multicolumn{5}{c}{\text{To}} \\
 & & 1 & 2 & \ldots\ldots\ldots\ldots & n \\ \hline
 & 1 & \infty & c_{12} & \ldots\ldots\ldots\ldots & c_{1n} \\ \hline
\text{From} & 2 & c_{21} & \infty & \ldots\ldots\ldots\ldots & c_{2n} \\ \hline
 & & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\
 & & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\ \hline
 & n & c_{n1} & c_{n2} & \ldots\ldots\ldots\ldots & \infty \\ \hline
\end{array}$$

From the mathematical formulation of 'Travelling salesman problem' we observe that travelling salesman problem is very similar to assignment problem. The only difference is that the former has an **additional restriction** (starting from any town the salesman is to return to starting town after visiting every town) and for this restriction, after solving this problem as an assignment problem if we get optimal assigment of the type $1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots\ldots\ldots \rightarrow n \rightarrow i$ or of the similar type (called a route) which has no sub-loop then from this optimal assignment we get the solution of the travelling salesman problem.

But in solving the problem as an assignment problem if we get one or more sub-loops the travelling salesman problem can be solved by applying the enumerative method after the problem is solved by assignment technique. For example, for four towns 1, 2, 3, 4 if we have optimal assignment $1 \rightarrow 4$, $2 \rightarrow 3$, $3 \rightarrow 2$, $4 \rightarrow 1$ (when the problem is solved as an assignment problem) we are to apply the enumerative method since here we have two sub-loops $1 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 3 \rightarrow 2$ (See examples in the Section 8.10).

13.10 A Few Examples

● **Example 13.10.1 :** Solve the following travelling salesman problem :

	A	B	C	D	E
A	∞	6	12	0	4
B	6	∞	10	5	4
C	8	7	∞	11	3
D	5	4	11	∞	5
E	5	2	7	8	∞

Solution : In solving the problem as assignment problem (minimization problem) by Hungarian method, we get the matrices successively as follows :

	A	B	C	D	E
A	∞	6	12	0	4
B	2	∞	6	1	0
C	5	4	∞	8	0
D	1	0	7	∞	1
E	3	0	5	6	∞

(i)

	A	B	C	D	E
A	∞	6	7	0	4
B	1	∞	1	1	0
C	4	4	∞	8	0
D	0	0	2	∞	1
E	2	0	0	6	∞

(ii)

Here the minimum number of lines covering all zeros is 4 < 5.

$$\begin{array}{c|c|c|c|c|c|}
 & A & B & C & D & E \\
\hline
A & \infty & 5 & 6 & 0 & 4 \\
\hline
B & \boxed{0} & \infty & 0 & 1 & 0 \\
\hline
C & 3 & 3 & \infty & 8 & \boxed{0} \\
\hline
D & 0 & \boxed{0} & 2 & \infty & 2 \\
\hline
E & 2 & 0 & \boxed{0} & 7 & \infty \\
\hline
\end{array}$$

(iii)

Here the minimum number of lines is 5, which is equal to the order of the matrix. So optimal assignment of the assignment problem will be A → D, B → A, C → E, D → B, E → C i.e. assignments are made in the cells (A, D), (B, A), (C, E), (D, B), (E, C).

Here the minimum cost is 0 + 6 + 3 + 4 + 7 = 20.

But these optimal assignments do not satisfy the conditions of 'travelling salesman problem' since here we have two sub loops (A, D), (D, B), (B, A) and (C, E), (E, C). Now we shall solve the problem by enumerative method.

We observe that in the matrix of the table (iii) the minimum non-zero element is 1 which is the cell (B, D).

But for the loop (A, D), (D, B), (B, A) the cell (B, A) cannot be replaced by (B, D) because the cell (D, B) is present in this loop. The next non-zero element in the matrix is 2 which is in (E, A) cell and in (D, C) cell. Now if we take (D, C) cell instead of (D, B) cell and (E, A) cell instead of (E, C) cell we get no route. Again 0 being in the (B,C) cell, for the loop A → D → B → A if we take (B, C) cell instead of (B, A) and (E, A) instead of (E, C) we get A → D → B → C → E → A which is a route and for this route cost is 20 + (10 – 6) + (5 – 7) = 22 units. Further we observe that total cost will be greater than 22 for other routes. So the solution of the given travelling salesman problem will be A → D → B → C → E → A and the minimum cost is 22 units.

● **Example 13.10.2 :** Solve the travelling salesman problem with the following cost matrix :

	1	2	3	4	5
1	∞	4	7	3	4
2	4	∞	6	3	4
3	7	6	∞	7	5
4	3	3	7	∞	7
5	4	4	5	7	∞

Solution : We subtract the least element (of the first row) 3 from all the elements of the first row. Similarly we subtract the least element of each row from all the elements of the corresponding row and we get the following matrix :

$$\begin{bmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 0 & 0 & \infty \end{bmatrix}$$

(i)

Next we subtract the least element of each column from all the elements of the corresponding column of the matrix in (i) and we get the following matrix :

$$\begin{bmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ -2- & -1- & -\infty- & -2- & -0- \\ -0- & -0- & -3- & -\infty- & -4- \\ -0- & -0- & -0- & -3- & -\infty- \end{bmatrix}$$

(ii)

Now we draw the minimum number of lines to cover all zeros of the matrix in (iii). Here the number of lines is 4 which is less than the order of the matrix. So optimal stage is not reached.

Now in the matrix (ii) we subtract the least element 1 (of the elements lying outside the lines drawn) from all the elements outside the lines and we add 1 to each junction of the horizontal and the vertical lines. The remaining elements of this matrix are unchanged.

Then we get the following matrix :

$$\begin{bmatrix} \infty & 0 & 2 & \boxed{0} & 0 \\ \boxed{0} & \infty & 1 & 0 & 0 \\ 2 & 1 & \infty & 3 & \boxed{0} \\ 0 & \boxed{0} & 3 & \infty & 4 \\ 0 & 0 & \boxed{0} & 4 & \infty \end{bmatrix}$$

(iii)

The minimum number of lines required to cover all zeros in the matrix (iii) is 5 which is the order of the matrix. So optimal stage is reached. Now assignments are made in the cells containing 0's which are put inside □.

Then optimal assignments are 1 → 4, 2 → 1, 3 → 5, 4 → 2, 5 → 3 and the minimum cost is 3 + 4 + 5 + 3 + 5 = 20 units.

The optimal assignments contain two sub-loops 1 → 4 → 2 → 1 and 3 → 5 → 3.

Now we see that the next non-zero minimum element in the matrix (iii) is 1.

But 1 appears in the cell (2, 3) and in the cell (3, 2). Now if we assign 1 at the cell (2, 3) instead of '0' assignment at (2, 1), the resulting route will be 1 → 4 → 2 → 3 → 5 → 1 with cost = 3 + 3 + 6 + 5 + 4 = 21 units. If an assignment (1 unit) is made at the cell (3, 2) instead of '0' assignment at the cell (3, 5) the resulting route will be 1 → 5 → 3 → 2 → 4 → 1 with cost = 4 + 5 + 6 + 3 + 3 = 21 units.

Hence an optimal route is 1 → 4 → 2 → 3 → 5 → 1 or 1 → 5 → 3 → 2 → 4 → 1 with minimum cost 21 units.

13.11 Summary

In this unit, we have first stated the meaning of assignmnent problems. Then we have explained Hungarian method of solving assignment problems. Finally we have observed the travelling salesman problem which is very similar to assignment problem. We have explained the enumerative method of solving travelling salesman problems.

13.12 Exercise

1. Solve the following assignment problems for minimum cost :

(i)

	J_1	J_2	J_3	J_4
1	10	12	19	11
2	5	10	7	8
3	12	14	13	11
4	8	15	11	9

(ii)

	A	B	C	D
I	1	4	6	3
II	9	7	10	9
III	4	5	11	7
IV	8	7	8	5

(iii)

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

(iv)

	J_1	J_2	J_3	J_4	J_5
1	4	2	7	3	1
2	2	9	2	7	1
3	6	8	7	6	1
4	4	6	5	3	1
5	5	3	9	5	1

2. Solve the following assignment problems for maximum profit :

(i)

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

(ii)

	J_1	J_2	J_3	J_4	J_5
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	8	12	7	4
E	2	8	5	8	8

3. Solve the following unbalanced problems for minimum cost :

(i)

	A	B	C	D
1	10	12	8	6
2	6	9	12	14
3	3	8	7	12

(ii)

	W	V	Y	Z
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

4. 5 operators are to be appointed for 5 machines. The times (hours) required by each operator to complete a job in each machine are given in the following matrix where the machine 3 cannot be operated by the operator A and the operator C cannot work in machine 4.

Find how the machines can be assigned to the operators such that the jobs will be comleted in minimum time.

		Machine				
		1	2	3	4	5
JOBS	A	5	5	–	2	6
	B	7	4	2	3	4
	C	9	3	5	–	3
	D	7	2	6	7	2
	E	6	5	7	9	1

5. Solve the following 'travelling salesman' problems :

(i)

	A	B	C	D	E
A	∞	5	8	4	5
B	5	∞	7	4	5
C	8	7	∞	8	6
D	4	4	8	∞	8
E	5	5	6	8	∞

(ii)

From

	1	2	3	4	5
1	∞	14	10	24	41
2	6	∞	10	12	10
3	7	13	∞	8	15
4	11	14	30	∞	17
5	6	8	12	16	∞

(iii)

	A	B	C	D	E
A	∞	7	6	8	4
B	7	∞	8	5	6
C	6	8	∞	9	7
D	8	5	9	∞	8
E	4	6	7	8	∞

Answers

1. (i) $1 \rightarrow J_2$, $2 \rightarrow J_3$, $3 \rightarrow J_4$, $4 \rightarrow J_1$; Minimum cost = 38 units

   (ii) I → A, II → C, III → B, IV → D ; Minimum cost = 21 units

   (iii) A → 5, B → 1, C → 4, D → 3, D → 2, ; Minimum cost = 9 units

(iv) $1 \to J_4, 2 \to J_3, 3 \to J_5, 4 \to J_1, 5 \to J_2$, ; Minimum cost = 13 units.

2. (i) $1 \to B, 2 \to A, 3 \to E, 4 \to C, 5 \to D$;

or, $1 \to B, 2 \to E, 3 \to A, 4 \to C, 5 \to D$;

or, $1 \to B, 2 \to A, 3 \to E, 4 \to D, 5 \to C$;

or, $1 \to B, 2 \to E, 3 \to A, 4 \to D, 5 \to C$;

Maximum profit = 191 units.

(ii) $A \to J_5, B \to J_1, C \to J_4, D \to J_3, E \to J_2$;

or, $A \to J_2, B \to J_1, C \to J_4, D \to J_3, E \to J_5$;

or, $A \to J_4, B \to J_1, C \to J_2, D \to J_3, E \to J_5$;

Maximum profit = 41 units.

3. (i) $1 \to D, 2 \to B, 3 \to A$; Minimum cost = 18 units.

(ii) $A \to W, B \to X, C \to Y$,

or, $A \to W, B \to Y, C \to X$; Minimum cost = 50 units.

4. $A \to 4, B \to 3, C \to 5, D \to 2, E \to 1$;

or, $A \to 4, B \to 3, C \to 2, D \to 1, E \to 5$; Minimum time = 15 hours.

5. (i) $A \to D \to C \to E \to B \to A$; Minimum cost = 28 units.

(ii) $1 \to 3 \to 4 \to 2 \to 5 \to 1$; Minimum cost = 48 units.

(iii) $A \to E \to B \to D \to C \to A$; Minimum cost = 28 units.

or, $A \to C \to D \to B \to E \to A$; Minimum cost = 30 units.

13.13 Multiple Choice Questions (MCQ)

1. An assignment problem is a special type of

(a) transportation problem (b) LPP

(c) non-linear programming problem (d) none of these

2. An assignment problem can be solved by
   (a) Hungarian method (b) VAM
   (c) Matrix minima method (d) none of these

3. In an assignment problem of assigning n jobs to n machines, the number of decision variables and number of constraints are respectively
   (a) n^2 and 2n (b) $2n^2$ and 2n (c) 2n and 2n (d) n^3 and n + 1

4. Solution of an assignment problem
   (a) is always unique (b) may or may not be unique
   (c) may not exist (d) none of these

5. An assignment problem is
   (a) a minimization problem
   (b) a maximization problem
   (c) a maximization or a minimization problem
   (d) never a maximization problem

6. The assignment problem is a special case of transportation problem in which the number of origins
   (a) equals the number of destinations
   (b) is greater then the number of destinations
   (c) is less than the number of destinations
   (d) none of these

Answers

1. (a) 2. (a) 3. (a) 4. (b) 5. (c) 6. (a)

Unit - 14 ❑ Game Theory

Structure

14.0 Objective

After studying this chapter, the reader should be able to

- understand decision making with an active opponent

- examine situations where one decision maker competes with or is in conflict with another decision maker
- appreciate the various procedures used in the selection and execution of various strategies which result in arising this game.

14.1 Introduction

In real life, we observe situations of conflict and competition between two or more opponents. These situations of competition are reffered to as a game. Game theory deals with making decisions under conflict caused by opposing interests. This theory was organized by J. V. Neumann in 1928 and later developed by G. B. Dantzig.

A game may be a parlor game, such as chess or bridge. More generally, games involve conflict situation in economic, social, political or military activities.

Two person game as the name is implies, involve a conflict of interest between two "persons", where "persons" may be people, companies, countries etc.

A two-person zero-sum game is one in which at the end of a play, one person gains what the other loses. We shall restrict our discussion to two-person zero-sum game.

A table showing how payments should be made at the end of the game is called a **pay-off matrix**. The matrix associated with the game indicates the **pay-off** to each opponents, called **players**.

A strategy of a player is the course of action taken by the player. For two-person game, there are two players, say, A and B. If the player A has m strategies available to him and the player B has n strategies available to him, then the pay-off for various strategy combinations is represented by m × n pay-off matrix. The game is called a finite game if the number of strategies available to each player be finite, otherwise it is called an infinite game.

A player chooses one of his courses of action quite ignorant of the strategy used by his opponent. For two-person game, the pay-off matrix indicates the gains to the player A (whose strategies are written on the left of the matrix) for each possible outcome of the game. A loss is considered as a negative gain.

In a two-person zero-sum (also called a rectangular game) with players A and B, the pay-off matrix of the player B is just the negative of the pay-off matrix written for the player A; because in such games the gains of one are the negatives of gains of the other.

In the given pay-off matrix, A is the maximizing player and B is the minimizing player.

Generally, the pay-off matrix written for the maximizing player where we note that this matrix represents the losses for the minimizing player.

In a two person zero-sum game with players A and B, our object is to determine the optimal strategies (if any) of the players such that the guaranteed gain of the maximizing player A is maximum, say, υ and for these strategies the minimum loss of the minimizing player B can be kept at the same amount υ.

14.2 Two-Person zero-sum Game, Pay-off Matrix and Strategies of a Game

At each play of a game, each player chooses only one strategy from these strategies available to the player. We have stated in Section 9.1 that a two-person zero-sum game is one in which at the end of a play, one player gains what the other player loses. Here each player has predetermined courses of action from which the player selects only one course of action at each play. These predetermined courses of action of the players are called strategies of the game.

Here we shall consider only finite games in which the number of available strategies of each player is finite.

By a "game" we shall mean "Two-person zero-sum game".

Pay-off matrix

Let the strategies of the maximizing player a be $A_1, A_2, ..., A_m$ and those of the minimizing player B be $B_1, B_2, ..., B_n$. If the player A chooses the strategy A_i ($i = 1, 2, ..., m$) and B chooses B_j ($j = 1, 2, ..., n$), we suppose that the payment to A by B is a_{ij}. The negative of a_{ij} indicates the payment from A to B. So if A chooses strategy A_i and B chooses strategy B_j then the amount of gain of A will be a_{ij} which is also the amount of loss of B.

In this case we get a matrix of order m × n given below.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{bmatrix}$$

This matrix is called the **pay-off matrix** of the game and the game is expressed in the following way :

Player B

Player A

	B_1	B_2	....	B_n
A_1	a_{11}	a_{12}	....	a_{1n}
A_2	a_{21}	a_{22}	....	a_{2n}
⋮	....	....	....	
⋮	....	....	....	
A_m	a_{m1}	a_{m2}	....	a_m

Here the object of the maximizing player A is to select his strategy in such a way that his guaranteed gain (whatevere strategy be chosen by the minimizing player B) becomes maximum and the object of the minimizing player B is to select his strategy in such a way (whatever strategy be selected by A) that his loss can be kept at the minimum possible amount.

Here the pay-off matrix $[a_{ij}]_{m\times n}$ is mentioned as the pay-off matrix of the player A and the game will be called a **m × n game**. The pay-off matrix of the player B will be $[-a_{ij}]_{m\times n}$.

An example of a pay-off matrix

We suppose that the strategies of A are A_1, A_2, A_3 and those of B are B_1, B_2. Payments to be made according to the selection of strategies are described in the following table :

Selected strategies	Payments
A : A_1 ; B : B_1	B gets Rs. 3 from A
A : A_1 ; B : B_2	A gets Rs. 2 from B
A : A_2 ; B + B_1	B gets Rs. 4 from A
A : A_2 ; B : B_2	A gets Rs. 5 from B
A : A_3 ; B : B_1	A gets Rs. 1 from B
A : A_3 ; B : B_2	B gets Rs. 8 from A

It is seen from the above table that the pay-off of the player A will be

$$\begin{array}{c|cc|} & B_1 & B_2 \\ A_1 & -3 & 2 \\ A_2 & -4 & 5 \\ A_3 & 1 & -8 \end{array}$$

and the pay-off matrix of B will be

$$\begin{array}{c|cc|} & B_1 & B_2 \\ A_1 & 3 & -2 \\ A_2 & 4 & -5 \\ A_3 & -1 & 8 \end{array}$$

14.3 Pure and Mixed Strategies

We have already mentioned that in each play of a game each player selects only one strategy from his strategies.

If in each play of a game a player selects a definite strategy then we say that the player follows pure strategy and the selected definite strategy is called a pure strategy.

If a player instead of selecting a definite strategy from his strategies, he selects his given strategies with definite probabilities then we say that the player follows mixed strategy. We now explain the concept of mixed strategy by an example.

We suppose that the given strategies of the maximizing player A are A_1, A_2, A_3. If the player A follows mixed strategy then in each play of the game, A selects exactly one strategy at random from A_1, A_2, A_3. Let the probabilities of selecting A_1, A_2, A_3 be respectively p_1, p_2, p_3. Then $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$, $p_1 + p_2 + p_3 = 1$. If in particular $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{6}$ then we shall understand that in large number of plays of the game, say 6000 plays, the strategy A_1 is selected approximately $6000 \times \frac{1}{2} = 3000$ times, A_2 is selected approximately $6000 \times \frac{1}{3} = 2000$ times and A_3 is selected approximately $6000 \times \frac{1}{6} = 1000$ times.

Here a follows mixed strategy which is expressed by writing $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$.

In particular, if $p_1 = 1$, $p_2 = 0$, $p_3 = 0$ then in large number of plays of the game, we can say that the player A approximately selects A_1 and does not select A_2, A_3 in each play of the game—so in this case we can say that the player A follows pure strategy A_1.

Similarly, if $p_1 = 0$, $p_2 = 1$, $p_3 = 0$; $p_1 = 0$, $p_2 = 0$, $p_3 = 1$ the player A follows pure strategy A_2 and pure strategy A_3 respectively. Then a pure strategy can be regarded as a particular mixed strategy.

Remark : The given strategies of the players A and B of a given game are sometimes mentioned as the pure strategies of the players.

14.4 Optimal Strategies and Value of a Game

In this section, we shall assume that each player follows pure strategy.

Let A be the maximizing player and B be the minimizing player of a game.

If the guaranteed gain of the player A be υ i.e., the amount of gain of A can never be less than υ (whatever be the strategy followed by B) and if A gains υ by following certain strategy, Ar then the strategy Ar, is called the best strategy of the player A.

Again, if the loss of the player B can be kept at the minimum possible amount, say $\overline{\upsilon}$ by following strategy B_s (whatever be the strategy followed by A) then the strategy B_s is called the best strategy of the player B. In this case we like to mention that if B follows a strategy other than B_s then the loss of B can be made greater than $\underline{\upsilon}$ if A follows approximate strategy. If $\underline{\upsilon} = \overline{\upsilon}$ (= υ, say) then υ is called the value of the game and in this case the best strategies of A and B are respectively called optimal strategies of the players A and B.

Let us explain clearly the concept of optimal strategies and value of the game by considering an example given below.

We consider a game with the following pay-off matrix :

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{B} \\
 & & B_1 & B_2 & B_3 \\
\cline{3-5}
 & A_1 & 18 & 5 & 6 \\
\cline{3-5}
A & A_2 & 9 & 8 & 10 \\
\cline{3-5}
 & A_3 & -4 & 7 & 3 \\
\cline{3-5}
\end{array}$$

In the above pay-off matrix, we find that the entry in the (1, 1) cell is 18 which is maximum in the row corresponding to A_1 and from this it may appear that if the maximizing player A follows the stretegy A_1 then his maximum gain of A may be 18. But this amount of gain is not guaranteed because in this case the amount of gain may be reduced to 5 or 6 according as the player B selects B_2 or B_3. Again it is observed that the entry in the (3, 1) cell is – 4 which is minimum on the row corresponding to A_3 and from this it may appear that if the minimizing player B selects B_1, then the minimum amount of loss can be – 4 (i.e., B gains 4). But this amount of loss is not guaranteed because the amount of loss of B can be increased to 18 or 9 according as the maximizing player selects A_1 or A_2. We observe that if the player A selects A_2 then A gains at least 8 units what ever strategy may be followed by B and this amount of gain is guaranteed. Further, it is seen that if A selects a streategy other than A_2 then the guaranteed amount of gain can be reduced to 5 or – 4 (i.e., loss 4). So if A selects stretegy A_2 then the guaranteed amount of gain will be maximum and this amount is 8 units.

Again it seen that if the player B selects strategy B_2 then the amount of loss of B can never exceed 8 units andif B selects a strategy other than B_2 i.e., if B selects B_1 or B_3 then the amount of loss may be increased to 18 units or 10 units. So if B selects strategy B_2 then the minimum amount of loss can be kept at 8 units and if B follows any other strategy then it is not sure that the minimum loss can be less than 8 units.

So for this game the best strategies of A and B are respectively A_2 and B_2 and the value of the game will be 8 units.

Remark : To solve a game with players A & B means "to find the optimal strategies of the players A and B and the value of the game (if exists)." In the next section we shall see how the optimal strategies and the value of a game can be determined by following pure strategies and further we shall note that it is not possible to solve any game by following only pure strategies.

14.5 The Maximin (Minimax) Criterion and Saddle Point

From the discussion in the last section, it is clear that if we are asked to solve a game with two players we mean that we are to determine the strategy if each player in such a way that the guarantead pay-off of the player from the opponent becomes maximum where, this pay-off cannot be decreased by selection of any strategy of the opponent.

This principle of selection of strategy is called maximin and minimax criterion.

Now we state clearly this principle :

If a player lists the worst possible outcomes of all its stategies, then he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes.

Let us apply this principle to the following game :

B

		B_1	B_2	B_3
	A_1	2	5	2
A	A_2	–1	2	–8
	A_3	–2	3	2

Here, we see that for the maximizing player A, the worst outcomes (row minima) of the strategies A_1, A_2, A_3 are respectively 2, –8, –2,

Now max $\{2, -8, -2\} = 2$

So by "maximin" criterion, the player A should select the strategy A_1.

Again, for the minimizing player B, the worst outcomes (column maxima) of the strategies B_1, B_2, B_3 are respectively 2, 5, 2.

Now, min. $\{2, 5, 2\} = 2$.

So, by "minimax" criterion, the player B should select B_1 or B_3. So in this case value of the game will be 2 and the pure optimal strategies will be (A_1, B_1) or (A_1, B_3) where in the parenthesis the first entry indicates the optimal strategy of the player A and the second entry indicates the optimal strategy of the player B.

Now, in the following table we express "row minima" within a square and "column maxima" within a circle.

B

		B_1	B_2	B_3	row min
	A_1	$\boxed{②}$	⑤	$\boxed{②}$	2
A	A_2	–1	2	$\boxed{-8}$	–8
	A_3	$\boxed{-2}$	3	②	2
column max		2	5	2	

For this game, we find that

max (row min) = min (col. max)

= value of the game = 2.

In the pay-off matrix, a cell where the entry is max (row min) = min (col. max) is called a saddle point of the game.

So for the above game the cells (1, 1), (1, 3) are the saddle points and the entry in each of these cells is 2 which is the value of the game.

Thus we see that if a game has a saddle point then the entry in the corresponding cell of the pay-off matrix will be the value of the game.

Precise Definition of the saddle point of a given pay-off Matrix

Let $[a_{ij}]_{m\times n}$ be the pay-off matrix of a game. A position (p, q) of the matrix $[a_{ij}]_{m\times n}$ will be called a saddle point of the matrix if a_{pq} is the minimum element in the p^{th} row and it is also the maximum element of the q^{th} column of the matrix $[a_{ij}]_{m\times n.}$

i.e., $a_{pq} \le a_{pj}$ for $j = 1, 2, ..., n$

$a_{pq} \ge a_{iq}$ for $i = 1, 2, ..., m$

Some theorems related to the existence of a saddle point

● **Theorem 14.5.1 :** If $[a_{ij}]_{m\times n}$ be the pay-off matrix of a game, then $\max\limits_{i} \min\limits_{j} a_{ij}$ $\le \min\limits_{j} \max\limits_{i} a_{ij}$.

Proof : Let $\max\limits_{i}\left[\min\limits_{j} a_{ij}\right] = a_{pq}$ and $\min\limits_{j}\left[\max\limits_{i} a_{ij}\right] = a_{rs}$

Then a_{pq} is a minimum element of the p-th row and a_{rs} is a maximum element of the s-th column of the pay-off matrix $[a_{ij}]_{m\times n}$.

So, we get $a_{pq} \le a_{ps}$ .... (1)

$a_{rs} \ge a_{ps}$ .... (2)

From (1) and (2) we get

$a_{pq} \le a_{ps} \le a_{rs}$

$\therefore$ $a_{pq} \le a_{rs}$

So it is proved that

$$\max_{i}\left[\min_{j} a_{ij}\right] \le \min_{j}\left[\max_{i} a_{ij}\right]$$

● **Theorem 14.5.2 :** If $[a_{ij}]_{m\times n}$ be the pay-off matrix of a game, then a saddle point of the game will exist if and only if $\max\limits_{i}\left[\min\limits_{j} a_{ij}\right] = \min\limits_{j}\left[\max\limits_{i} a_{ij}\right]$.

Proof : Let the position (p, q) be a saddle point of the game.

Then we get $a_{pq} \ge a_{iq}$; $i = 1, 2, \ldots\ldots, m$ .... (1)

and $a_{pq} \le a_{pj}$; $j = 1, 2, \ldots\ldots, n$ ....(2)

From (1) we can say that $\max\limits_{i} a_{iq} \le a_{pq}$ and from (2) we can say

$$\min_{j} a_{pj} \ge a_{pq}$$

$$\therefore \max_{i} a_{iq} \le a_{pq} \le \min_{j} a_{pj}$$

Now $\min\limits_{j}\max\limits_{i} a_{ij} \le \max\limits_{i} a_{iq}$ and $\min\limits_{j} a_{pj} \le \max\limits_{i}\min\limits_{j} a_{ij}$

So $\min\limits_{j}\max\limits_{i} a_{ij} \le \max\limits_{i} a_{iq} \le \min\limits_{j} a_{pj} \le \max\limits_{i}\min\limits_{j} a_{ij}$

$$\therefore \min_{j}\max_{i} a_{ij} \le \max_{j}\min_{i} a_{ij} \text{ .... (3)}$$

Again from Theorem 9.5.1, we get

$$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij} \quad \text{.......} \ (4)$$

Now, from (3) and (4) it is proved that

$$\max_i \min_j \left(a_{ij}\right) = \min_j \left[\max_i a_{ij}\right] \quad \text{.........} \ (5)$$

Now, let $\min_j a_{ij}$ be maximum when i = p.

Then $\min_j a_{pj} = \max_i \left[\min_j a_{ij}\right]$ .... (6)

Again let $\max_i a_{ij}$ be minimum when j = q.

Then $\max_i a_{iq} = \min_j \left[\max_i a_{ij}\right]$ .... (7)

From (5), (6) and (7) we get, $\min_j a_{pj} = \max_i a_{iq}$ ....(8)

Again $\min_j a_{pj} \le \max_i a_{iq}$ ....(8a)

So from (8) we get, $\max_i a_{iq} \le a_{pq}$ from which we can say

$$a_{pq} \ge a_{iq};\ i = 1, 2, ..., m \ \text{.........}(9)$$

Again $\max_i a_{iq} \ge a_{pq}$ .... (8b)

So from (8) we get $\min_j a_{pj} \ge a_{pq}$ from which we can say that

$$a_{pq} \le a_{pj};\ j = 1, 2, ..., n \ \text{...........}(10)$$

Now from (9) and (10), we can say that the position (p, q) is a saddle point of the game. So the theorem is proved.

Remarks (i) If the position (p, q) is a saddle point of the game with pay-off matrix $[a_{ij}]_{m\times n}$, then from (5), (8), (8a), (8b) we can say that the value of the game will be a_{pq} where $\max_i \left[\min_j a_{ij}\right] =$

$\min\limits_{j}\left[\max\limits_{i} a_{ij}\right]$ and in this case the optimal strategies to the maximizing player A and the minimizing player B will be respectively the pure strategies A_p and B_q.

(ii) If the game has no saddle point, then it is not possible to find the value of the game only by following pure strategies.

14.6 A Few Examples

● **Example 14.6.1 :** Solve the game with pay-off matrix given below :

				B		
		B_1	B_2	B_3	B_4	B_5
	A_1	11	4	3	10	2
A	A_2	8	7	6	8	9
	A_3	4	6	6	5	10
	A_4	7	8	4	4	3

Solution :

	B_1	B_2	B_3	B_4	B_5	row min
A_1	⑪	4	3	10	$\boxed{2}$	2
A_2	8	7	$\boxed{\textcircled{6}}$	8	9	6
A_3	$\boxed{4}$	6	⑥	5	⑩	4
A_4	7	⑧	4	4	$\boxed{3}$	3
col.max	11	8	6	10	10	

Here, max (row min) = max {2, 6, 4, 3} = 6

and min (col. max) = min {11, 8, 6, 10, 10} = 6

So here max (row min) = min (col. max) = 6

So the game has a saddle point at the position (2, 3). Then the value of the game will be 6 and optimal pure strategies are A : A_2, B : B_3 which will be described as (A_2, B_3).

● **Example 14.6.2 :** Show that the following game has no saddle point :

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{B} \\
 & & B_1 & B_2 & B_3 & B_4 \\ \cline{3-6}
 & A_1 & 2 & 1 & 2 & -1 \\ \cline{3-6}
A & A_2 & 1 & 3 & 1 & 3 \\ \cline{3-6}
 & A_3 & 3 & 1 & 3 & -1 \\ \cline{3-6}
 & A_4 & -1 & 3 & -1 & 7 \\ \cline{3-6}
\end{array}$$

Solution :

$$\begin{array}{c|c|c|c|c|l}
 & B_1 & B_2 & B_3 & B_4 & \text{row min} \\ \cline{2-5}
A_1 & 2 & 1 & 2 & \boxed{-1} & \\ \cline{2-5}
A_2 & \boxed{1} & \textcircled{3} & \boxed{1} & 3 & \\ \cline{2-5}
A_3 & \textcircled{3} & 1 & \textcircled{3} & \boxed{1} & \\ \cline{2-5}
A_4 & \boxed{-1} & \textcircled{3} & \boxed{-1} & \textcircled{7} & \\ \cline{2-5}
\multicolumn{1}{c}{\text{col.max}} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{7} &
\end{array}$$

Here, max {row min} = max {–1, 1, 1, –1} = 1

and min {col. max} = min {3, 3, 3, 7} = 3

So max {row min} $\neq$ min {col. max}

So the given game has no saddle point.

● **Example 14.6.3 :** If the pay-off matrix of the following game has a saddle point at (2, 2) then find all values of x and y. Further prove that the game cannot have saddle point at (2, 3) for any values of x and y.

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{B} \\
 & & B_1 & B_2 & B_3 \\ \cline{3-5}
 & A_1 & 5 & x & 9 \\ \cline{3-5}
A & A_2 & y & 8 & 13 \\ \cline{3-5}
 & A_3 & 9 & 6 & 7 \\ \cline{3-5}
\end{array}$$

Solution : First part : Here, it is given that the game has a saddle point at (2, 2).

Now the element of the pay-off matrix at the position (2, 2) is 8. So the minimum element in the second row of the pay-off matrix is 8 and the maximum element in the second column of this matrix is 8. The we get $y \geq 8$ and $x \leq 8$. Again we see that (2, 2) will be a saddle point whenever $x \leq 8$ and $y \geq 8$. Hence, the required values of x, y are given by $x \leq 8$, $y \geq 8$.

Second Part : The element (2, 3) position of the pay-off matrix is 13 which is the maximum element in the third column of the matrix and 13 cannot be minimum element in the second row because the second row has an element 8 which is less than 13. So, the game cannot have saddle point at the position (2, 3) for any values of x and y.

14.7 Solution of Two-person Zero-sum Game with Mixed Strategies

In Section 9.5, we have noted that it is not possible to slove any game by following pure strategies only. In Section 9.3, we have got the idea of mixed strategies. In this section, we shall see how a game of 2 × 2 pay-off matrix having no saddle point can be solved by following mixed strategies and later we shall see that applying the rules of dominance (Section 9.9) in many cases, solution of a game with pay-off matrix of any order can be made to depend on the solution of a 2 × 2 game.

Solution of a Game with 2 × 2 pay-off Matrix having no Saddle Point

Let the pay-off matrix (which has no saddle point) of a 2 × 2 game be

$$\begin{array}{ccc} & & B \\ & B_1 & B_2 \\ A \quad A_1 & a & b \\ \quad A_2 & c & d \end{array}$$

Now, we shall see how the value of the game and the optimal strategies can be determined by following mixed strategies.

If A follows mixed strategy, then in any play of the game A selects a strategy at random from the strategies A_1, A_2 with probabilities, say x_1, x_2 ($0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$) respectively. Now since A_1, A_2 are never selected simultaneously and one of A_1,

A_2 must be selected. Then, we have $x_1 + x_2 = 1$. The problem of finding the mixed strategy of A is the problem of finding the values of x_1, x_2 where $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 = 1$. Then we can take $x_1 = x$, $x_2 = 1 - x$ where $0 \leq x \leq 1$. So A selects A_1, A_2 with probabilities x, 1 – x respectively where $0 \leq x \leq 1$.

Similarly, the problem of finding the mixed strategy of B is the problem of finding the values of the probabilities y_1, y_2 of the strategies B_1, B_2 where $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$ and $y_1 + y_2 = 1$. Then we can take $y_1 = y$, $y_2 = 1 - y$ where $0 \leq y \leq 1$. So B selects B_1, B_2 with probabilities y, 1 – y respectively where $0 \leq y \leq 1$.

Let us first consider the problem of the maximizing player A. If the player B selects B_1 then from the above pay-off matrix we see that the expected gain of the player A will be $ax + c\ (1 - x) = g_1$ (say) and if the player B selects B_2 then the expected gain of A will be $bx + d\ (1 - x) = g_2$ (say).

Now, let $\min \{g_1, g_2\} = g'$.

So it is seen that whatever strategy (B_1 or B_2) be selected by the minimizing player B, the guaranteed expected gain of a will be g′ when A selects A_1 with probabilities x. Now the problem of A is to find x such that g′ becomes maximum.

Here $g_1 \geq g'$, $g_2 \geq g'$. ....(1)

Now we consider the problem of the minimizing player B. If the player A selects A_1 then from the pay-off matrix we find that the expected loss of B will be $ay + b\ (1 - y) = l_1$ (say) and if A selects A_2 then the expected loss of B will be $cy + d\ (1 - y) = l_2$ (say).

Now let $\max \{l_1, l_2\} = l'$.

So it is seen that whatever strategy (A_1 or A_2) be selected by the maximizing player A, the expected loss of B can never exceed l_1 when B selects B_1 with probability y.

Now the problem of B is to find y such that l′ become minimum.

Here $l_1 \leq l'$, $l_2 \leq l'$ .... (2)

From (1) and (2) we find that

$$ax + c\ (1 - x) \geq g', \quad bx + d\ (1 - x) \geq g'$$

and

$$ay + b\ (1 - y) \leq l', \quad cy + d\ (1 - y) \leq l'$$

We observe that if it is possible for which $l_1 = l_2 = l'$ and $g_1 = g_2 = g'$ then for such that values of x, y, $(g')_{max} = (l')_{min} = \upsilon$ (say)

and in this case the value of the game will be υ and the optimal strategies of the players can be determined from the values q, x, y.

Now the given pay-off matrix having no saddle point the value of a, b, c, d will be such that in any case the equations $g_1 = g_2$, $l_1 = l_2$ have solution for x, y $(0 \leq x \leq 1, 0 \leq y \leq 1)$. So the given 2 × 2 game can be solved by following mixed strategies.

Note : For a 2 × 2 game with pay-off matrix

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

having no saddle point following the above method, it can be shown that the value υ of the game is given by $\upsilon = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$ and the optimal strategies of the players A, B are given by A $\left(\dfrac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}, \dfrac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}\right)$

and B $\left(\dfrac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}, \dfrac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}\right)$.

14.8 An Example

● **Example 9.8.1 :** Solve the following 2 × 2 game

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{array}{|c|c|} \hline 8 & 5 \\ \hline 4 & 7 \\ \hline \end{array} \end{array}$$

Solution : Here, max {row min} = max {5, 4} = 5

and min (col. max) = min {8, 7} = 7.

So we have max (row min) ≠ min (col. max).

Then the given game has no saddle point.

So it is not possible to solve the game following only pure strategies.

Let us solve the problem using mixed strategies.

We suppose that the player A selects A_1, A_2 with probabilities x, 1 – x respectively and B selects B_1, B_2 with probabilities y, 1 – y where $0 \le x \le 1, 0 \le y \le 1$.

Now the problem of A is to find the value of x for which min. $\{g_1, g_2\} = g'$ (say) becomes maximum where $g_1 = 8x + 4(1 - x)$ and $g_2 = 5x + 7(1 - x)$.

Then we have $g_1 = 8x + 4(1 - x) \ge g'$ and $g_2 = 5x + 7(1 - x) \ge g'$.

Similarly for the problem B we find that $l_1 = 8y + 5(1 - y) \le l'$

$$l_2 = 4y + 7(1 - y) \le l'$$

where we are to find the value of y such that $l' = \max\{l_1, l_2\}$ becomes minimum.

Now $g_1 = g_2$, $l_1 = l_2$ give

$$8x + 4(1 - x) = 5x + 7(1 - x)$$

$$8y + 5(1 - y) = 4y + 7(1 - y)$$

from which we get 6x = 3, 6y = 2.

So $x = \frac{1}{2}, y = \frac{1}{3}$ where we note that $0 < \frac{1}{2} < 1,\ 0 < \frac{1}{3} < 1$.

So for $x = \frac{1}{2}, y = \frac{1}{3}$, g′ will be maximum and l′ will be minimum and

$$(g')_{\max} = 8.\frac{1}{2} + 4.\frac{1}{2} = 6 \text{ and } (l')_{\min} = 8.\frac{1}{3} + 5.\frac{2}{3} = 6.$$

Therefore, for $x = \frac{1}{2}, y = \frac{1}{3}$, g′ will be maximum and l′ will be minimum and $(g')_{\max} = (l')_{\min} = 6$.

So the value of game is 6 and the optimal strategies are A : $\left(\frac{1}{2}, \frac{1}{2}\right)$ B : $\left(\frac{1}{3}, \frac{2}{3}\right)$

14.9 Rules of Dominance

The principles of dominance are used to reduce the size of the pay-off matrix of a game many observation and in many cases the game may be sloved only by adopting this method.

Rule 1 : For the maximizing player A

If each element in one row, say k-th row of the pay-off matrix $(a_{ij})_{m\times n}$ is less than or equal to the corresponding element of some other row, say r-th, then A will never choose k-th strategy A_k. The optimal strategies and value of the game remain unaltered after the deletion of the k-th row from the pay-off matrix. The k-th row is then said to be dominated by the r-th row.

Rule 2 : For the minimizing player B

If each element in one column, say k-th column of the pay-off matrix is greater than or equal to the corresponding element in some other column, say, r-th then the column is said to be dominated by the r-th column. Here the player B will never choose the k-th strategy B_k.

The optimal strategies and the value of the game remain unaltered after the deletion of the k-th column of the pay-off matrix.

Rule 3 : Modified dominance property

If a row of the pay-off matrix, say, i-th row be dominated by a column combination of other two or more rows, then the i-th row is delected from the pay-off matrix.

If a column of the pay-off matrix, say j-th column, dominate a convex combination of other two or more columns, then the j-th column is deleted from the pay-off matrix.

14.10 An Example

● **Example 14.10.1 :** Solve the following game using rules of dominance :

B

		B_1	B_2	B_3	B_4
	A_1	1	–1	2	1
A	A_2	2	2	0	1
	A_3	3	–2	1	–2
	A_4	3	1	–3	2

Here, A is the maximizing player and B is the minimizing player. We see that each element in the column for B_1 is greater than or equal to the corresponding element in the column for B_2 and for three elements [1, 3, 3 for B_1, –1, –2, 1 for B_2] we have $1 > -1$, $3 > -2$, $3 > 1$. So, by the principle of dominance, the column B_1 can be discarded.

Then, the transformed game becomes

		B_2	B_3	B_4
	A_1	–1	2	1
A	A_2	2	0	1
	A_3	–2	1	–2
	A_4	1	–3	2

Again, we see that elements of $\frac{1}{2}(A_1 + A_2)$ [a convex combination of rows forA$_1$ and A_2] are $\frac{1}{2}$, 1, 1 and $\frac{1}{2} > -2$, $1 = 1$, $1 > -2$ where –2, 1, –2 are the corresponding elements in the row for A_3. So, by the principle of dominance, the row for A_3 can be discarded and the transformed game becomes

	B_2	B_3	B_4
A_1	–1	2	1
A_2	2	0	1
A_4	1	–3	2

Now the elements of $\frac{1}{2}$ (B_2 + B_3) are $\frac{1}{2}$, 1, –1 and the corresponding elements in the column for B_4 are 1, 1, 2 where $1 > \frac{1}{2}$, $1 = 1$, $2 > -1$. Then, by the principle of dominance, the column for B_4 can be discarded and the transformed game becomes

	B_2	B_3
A_1	–1	2
A_2	2	0
A_4	1	–3

Here, we see that the elements of $\frac{1}{4}A_1 + \frac{3}{4}A_2$ are $\frac{5}{4}, \frac{1}{2}$ and the corresponding elements in the row for A_4 are 1, –3 where $\frac{5}{4} > 1$, $\frac{1}{2} > -3$.

So, by the principle of dominance, the row for A_4 can be discarded and the transformed game becomes

	B_2	B_3
A_1	–1	2
A_2	2	0

which represents a 2 × 2 game.

For this game, we see that max (row min) = max {–1,0} = 0 and min (col. max) = min {2, 2} = 2 where $0 \neq 2$. So this game has no saddle point. Then this 2 × 2 game can be solved by following mixed strategies. Then using 'Note' given at the end of the Section 9.7, we find that the value υ of this game = $\frac{-1\times 0-2\times 2}{-1+0-(2+2)}=\frac{4}{5}$.

Also, the probabilities of selecting A_1, A_2 are $\frac{0-1}{-1+0-(2+2)}, \frac{-1-2}{-1+0-(1+2)}$ i.e., $\frac{2}{5}, \frac{3}{5}$ respectively and the probabilities of selecting B_2, B_3 are $\frac{0-2}{-1+-(2+2)}$, $\frac{-1-2}{-1+0-(2+2)}$ i.e., $\frac{2}{5}, \frac{3}{5}$ respectively.

So, the optimal strategies of the given game are A : $\left(\frac{2}{5}, \frac{3}{5}, 0, 0\right)$ and B : $\left(0, \frac{2}{5}, \frac{3}{5}, 0\right)$ and the value of the game is $\frac{4}{5}$.

14.11 Expectation Function and Some Important Theorems

Let $[a_{ij}]_{m\times n}$ be the pay-off matrix of a game where in any play of the game, the maximizing player A selects one strategy from the given strategies $A_1, A_2, \ldots.., A_m$ and the minimizing player B selects one strategy from the strategies $B_1, B_2, \ldots, B_n$.

If mixed strategies be followed we assume that the player A selects $A_1, A_2, \ldots., A_m$ respectively with probabilities $x_1, x_2, \ldots.., x_m$ and the player B selects $B_1, B_2 \ldots., B_n$ with probabilities $y_1, y_2, \ldots.., y_n$ respectively where $x_1 + x_2 + \ldots.+ x_m = 1$, $y_1 + y_2 + \ldots.+ y_n = 1$ and $x_i \geq 0$, $y_j \geq 0$ $(i = 1, 2, \ldots.., m;\ j = 1, 2, \ldots.., n)$.

B

		y_1	y_2	y_3	....	y_n
	x_1	a_{11}	a_{12}	a_{13}	....	a_{1n}
	x_2	a_{21}	a_{22}	a_{23}	....	a_{2n}
A	x_2	a_{31}	a_{32}	a_{33}	....	a_{3n}
	⋮	....	....	....	....	....
	⋮	....	....	....	....	....
	x_m	a_{m1}	a_{m2}	a_{m3}	....	a_{mn}

Here a_{ij} is the pay-off of A when the players A and B select respectively the pure strategies A_i, B_j. Then if the player B selects the pure strategy B_j, the expected pay-off of the player A will be $\sum_{i=1}^{m} a_{ij}\, x_i$ when A follows mixed strategy X = $(x_1, x_2, \ldots., x_m)$.

Then if the player B follows mixed strategy Y = $(y_1, y_2, \ldots.., y_n)$, the expected pay-off of A will be called expectation function and it will be denoted by E(X, Y).

Then, we get $E(X, Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, x_i\, y_j$

Now let $\max\limits_{x_i} \min\limits_{y_j} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j = \underline{\upsilon}$ and $\min\limits_{y_j} \max\limits_{x_i} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j = \overline{\upsilon}$

[Here we have assumed that $\underline{\upsilon}$ and $\overline{\upsilon}$ exist]

Here we observe that $\underline{\upsilon} = \max\limits_{x_i}\left[\min\left\{\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \cdots\cdots \sum_{i=1}^{m} a_{in}x_i\right\}\right]$

and $\overline{\upsilon} = \min\limits_{y_i}\left[\max\left\{\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \cdots\cdots \sum_{j=1}^{n} a_{mj}y_j\right\}\right]$

Now for solving a game problem with mixed strategies, the object of the maximizing player A is to find the values of $x_1, x_2, \ldots.., x_m$ for which the guaranteed expected gain becomes maximum and further we observe that this maximum guaranted expected gain is $\underline{\upsilon}$. Also the object of the minimizing player B is to find the values of $y_1, y_2, \ldots., y_n$ such that for any given values of $y_1, y_2, \ldots.., y_n$ the maximum expected loss of B becomes minimum and we observe that this minimum expected loss is $\overline{\upsilon}$.

Now if $\underline{\upsilon} = \overline{\upsilon}$ (= υ say) for the mixed strategies X = $(x_1, x_2, \ldots., x_m)$, Y = $(y_1, y_2, \ldots., y_n)$ then the value of game will be υ and (X, Y) will give optimal strategies of the game.

[Here we remember that the mixed strategy X = $(x_1, x_2, ..., x_m)$ will be a pure strategy A_i if $x_i = 1$ and $x_1 = x_2 = ... = x_{i-1} = x_{i+1} = ... x_m = 0$ and the mixed strategy Y = $(y_1, y_2, ..., y_n)$ will be a pure strategy B_j if $y_j = 1$ and $y_1 = y_2 = ... = y_{j-1} = y_{j+1} = ... y_n = 0$]

In Section 9.13, we shall see that if mixed strategies be allowed, then for any "Two-person zero-sum game" value of the game and optimal strategies can be determined.

Some Important Theorems

● **Theorem 14.11.1 :** If a number P be added to each element of the pay-off matrix of a game with value υ then the value of the transformed game will be υ + p but the optimal strategies will remain unaltered.

Proof : We suppose that the pay-off matrix of a game with value υ is $[a_{ij}]_{m\times n}$. Now we add a definite number P to each element at these matrix. Then the transform matrix will be $[a_{ij} + P]_{m\times n}$. If the expectation functions of the given game and the transformed game are respectively E(x, y) and E′(x, y); then we get

$$E(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, x_i y_j\ ,$$

$$E'(x, y) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij} + P\right) x_i y_j$$

where X = $(x_1, x_2, ..., x_m)$ & Y = $(y_1, y_2, ..., y_n)'$,

and $x_i \geq 0$ (i = 1, 2, ..., m), $y_j \geq 0$ (j = 1, 2, ..., n) and $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_i = 1$.

Now, if an optimal strategy of the game be expressed by $(\overset{\circ}{X}, \overset{\circ}{Y})$, then

$$\upsilon = E\,(\overset{\circ}{X}, \overset{\circ}{Y}) = \max_{x_i}\ \min_{y_j} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} x_i y_j$$

$$\text{and } E'\,(X, Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} (a_{ij} + P) x_i y_j$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij} + P\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j$$

$= E(X, Y) + P(x_1 + x_2 + ...+ x_m)(y_1 + y_2 + ...+ y_n)$

$= E(X,Y) + P.1.1$

$= E(X,Y) + P$

So, we get for any X, Y

$E'(X, Y) = E(X, Y) + P$ ....(1)

Now, P being a definite number, from (1) we get

$$\max_{x_i}\min_{y_j} E'(X, Y) = \max_{x_i}\min_{y_j} E(X, Y) + P$$

$$= E(\overset{\circ}{X},\overset{\circ}{Y}) + P$$

$$= \upsilon + P$$

Then, we get $E'(\overset{\circ}{X},\overset{\circ}{Y}) = E(\overset{\circ}{X},\overset{\circ}{Y}) + P$,

so $E'(\overset{\circ}{X},\overset{\circ}{Y}) = \upsilon + P$ .... (2)

From (2) we can say that $(\overset{\circ}{X},\overset{\circ}{Y})$ also gives an optimal strategy of the transformed game and the value of the transformed game will be $\upsilon + P$.

Hence, the theorem is proved.

● **Theorem 14.11.2 :** If the pay-off matrix of a game be skew-symmetric, then the value of the game will be zero.

Proof : We suppose that the pay-off matrix of the given game is $[a_{ij}]_{m\times n}$. Then the expectation function of the maximizing player will be $E(X, Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j$ where X $= (x_1 + x_2, ..., x_m)$, $Y = (y_1 + y_2, ..., y_n)$ are respectively the mixed strategies of the maximizing player A and the minimizing player B.

Here, the matrix $[a_{ij}]_{m\times n}$ is skew-symmetric. So here m = n and $a_{ij} = -a_{ji}$ (i = 1, 2, ..., n; j = 1, 2, ..., n). Then $X = (x_1 + x_2, ..., x_n)$ and $Y = (y_1 + y_2, ..., y_n)$ and $E(X, Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j$.

Now if $(\overset{\circ}{X},\overset{\circ}{Y})$ give an optimal strategy of the game and if υ be the value of the game, then $\upsilon = E(\overset{\circ}{X},\overset{\circ}{Y}) = \max\limits_{X} \min\limits_{Y} E(X, Y)$.

Then we get, $\upsilon = \min\limits_{Y} E(\overset{\circ}{X},Y)$ .... (3)

From (3) we get $E(\overset{\circ}{X},Y) \geq \upsilon$ which is true for any mixed strategy $Y = (y_1, ... y_n)$ of the minimizing player B.

Then for $Y = \overset{\circ}{X} = (\overset{\circ}{x}_1, \overset{\circ}{x}_2, ..., \overset{\circ}{x}_n)$ we get $E(\overset{\circ}{X},\overset{\circ}{Y}) \geq \upsilon$. .... (4)

But $E(\overset{\circ}{X},\overset{\circ}{Y}) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} a_{ij}\, \overset{\circ}{x}_i\, \overset{\circ}{x}_j$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (-a_{ij})\, \overset{\circ}{x}_i\, \overset{\circ}{x}_j$$

(since here $a_{ij} = - a_{ij}$)

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, \overset{\circ}{x}_i\, \overset{\circ}{x}_j$$

(interchanging the dummy indices i and j)

Therefore, $E(\overset{\circ}{X}, \overset{\circ}{X}) = - E(\overset{\circ}{X}, \overset{\circ}{X})$

$\therefore\ 2E(\overset{\circ}{X}, \overset{\circ}{X}) = 0$

So, $E(\overset{\circ}{X}, \overset{\circ}{X}) = 0$.

Then from (4), we get $\upsilon \leq 0$ .... (5)

Again $\upsilon = \min\limits_{Y} \max\limits_{X} E(X, Y) = \max\limits_{X} E(X,\overset{\circ}{Y})$

So for any mixed strategy $X = (x_1, x_2, ..., x_n)$ of the maximizing player A we get $E(X,\overset{\circ}{Y}) \leq \upsilon$.

Then taking $X = \overset{\circ}{Y} = (\overset{\circ}{y}_1, \overset{\circ}{y}_2, ..., \overset{\circ}{y}_n)$

We get, $E(\overset{\circ}{Y}, \overset{\circ}{Y}) \leq \upsilon$ .... (6)

Again, $E(\overset{\circ}{Y}, \overset{\circ}{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \; \overset{\circ}{y}_i \, \overset{\circ}{y}_j$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \; \overset{\circ}{y}_i \, \overset{\circ}{y}_j$$

$$= -E(\overset{\circ}{Y}, \overset{\circ}{Y})$$

$\therefore\ 2E(\overset{\circ}{Y}, \overset{\circ}{Y}) = 0$

So $E(\overset{\circ}{Y}, \overset{\circ}{Y}) = 0$.

Then from (6) we get $\upsilon \geq 0$. .... (7)

Then from (5) and (7) we get $\upsilon = 0$.

So the theorem is proved.

14.12 Transformation of a Game Problem to a L.P.P.

Let $[a_{ij}]_{m\times n}$ be the pay-off matrix of a m × n game with value υ. Here we can assume that each element a_{ij} of the pay-off matrix is greater than 0 ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$) since if the elements are not all positive, by adding an appropriate positive number P to each element of the pay-off matrix each element of the transformed matrix can be made positive and we know that the value of the game corresponding to the transformed pay-off matrix will be $\upsilon + P$ while the optimal strategies will remain unaltered.

Now we suppose that the mixed strategies of the maximizing player A and the minimizing player B are respectively $(x_1, x_2, ..., x_m)$ and $(y_1, y_2, ..., y_n)$.

Then the player A chooses his available strategies $A_1, A_2, ..., A_m$ with probabilities $x_1, x_2, ..., x_m$ respectively and the player B chooses his available strategies $B_1, B_2, ..., B_n$ with probabilities $y_1, y_2, ..., y_n$ respectively, where

$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_i = 1$ and $x_i \geq 0$, $y_j \geq 0$ ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$).

$$\begin{array}{cc|c|c|c|c|}
 & & y_1 & y_2 & \cdots\cdots & y_n \\
 & & B_1 & B_2 & \cdots\cdots & B_n \\
\hline
x_1 & A_1 & a_{11} & a_{12} & \cdots\cdots & a_{1n} \\
\hline
x_2 & A_2 & a_{21} & a_{22} & \cdots\cdots & a_{2n} \\
\hline
\vdots & \vdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\hline
\vdots & \vdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
 & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\hline
x_m & A_m & a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \\
\hline
\end{array}$$

Problem of the player A

Let g_1 be the expected gain of the maximizing player A when B chooses B_1 where A follows mixed strategy $(x_1, x_2, \ldots\ldots, x_m)$.

Then $g_1 = a_{11}\ x_1 + a_{21}x_2 + \ldots\ldots\ldots\ldots + a_{m1}\ x_m$

Similarly, we get

$$g_2 = a_{12}\ x_1 + a_{22}\ x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{m2}\ x_m$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$g_n = a_{1n}\ x_1 + a_{2n}\ x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{mn}\ x_m$$

where $g_2, g_3, \ldots\ldots\ldots\ldots g_n$ are expected gains of A when the minimizing player B chooses $B_2, B_3, \ldots\ldots\ldots B_n$ respectively.

Let $\min\ \{g_1, g_2, \ldots\ldots\ldots\ldots g_n\} = g' \ldots\ldots\ldots\ldots (1)$

Here $g' > 0$ since $a_{ij} > 0$ for all i, j and $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$ $(i = 1, 2, \ldots\ldots, m)$.

Then, if A follows mixed strategy $(x_1, x_2, \ldots\ldots, x_m)$, then the expected gain of A will be at least g'.

Now the object of the player A is to find the values of $x_1, x_2, \ldots\ldots, x_m$ such that g' becomes maximum i.e. $\frac{1}{g'}$ becomes minimum ($\because$ here $g' > 0$).

From (1), we can write

$$g_1 \geq g', \ g_2 \geq g', \ .... \ g_n \geq g'$$

or, $$\frac{g_1}{g'} \geq 1, \frac{g_2}{g'} \geq 1, \ .... \ \frac{g_n}{g'} \geq 1.$$

Then, we get

$$a_{11}\frac{x_1}{g'} + a_{21}\frac{x_2}{g'} + \cdots\cdots\cdots + a_{m1}\frac{x_m}{g'} \geq 1,$$

$$a_{12}\frac{x_1}{g'} + a_{22}\frac{x_2}{g'} + \cdots\cdots\cdots + a_{m2}\frac{x_m}{g'} \geq 1,$$

....

....

$$a_{1n}\frac{x_1}{g'} + a_{2n}\frac{x_2}{g'} + \cdots\cdots\cdots + a_{mn}\frac{x_m}{g'} \geq 1,$$

and taking $\frac{x_1}{g'} = X_1, \frac{x_2}{g'} = X_2 \cdots\cdots \frac{x_m}{g'} = X_m$ we get

$a_{ij} X_1 + a_{2j} X_2 +$ .... $a_{mj} X_m \geq 1$; $j = 1, 2,$ ...., n.

Again $\frac{1}{g'} = \frac{x_1 + x_2 + \cdots\cdots + x_m}{g'} = X_1 + X_2 +$ .... $+ X_m$

where $X_1 \geq 0$, $X_2 \geq 0$, ... $X_m \geq 0$.

Then, the problem of the player A can be expressed as a L.P.P. given below :

Minimize $\frac{1}{g'} = X_1 + X_2 +$ .... $+ X_m$

subject to the constraints

$a_{11} X_1 + a_{21} X_2 +$ .... $+ a_{m1} X_m \geq 1$,

$a_{12} X_1 + a_{22} X_2 +$ .... $+ a_{m2} X_m \geq 1$,

....

....

$$a_{1n} X_1 + a_{2n} X_2 + ... + X_{mn} X_m \geq 1,$$

$$X_1 \geq 0, X_2 \geq 0, ... X_m \geq 0.$$

Problem of the player B

Let max $\{l_1, l_2, ... l_m\} = l'$ .... (2),

where for mixed strategy ($y_1, y_2, ..., y_n$) of B, the expected losses of B are $l_1, l_2, ...$ l_m when A chooses respectively the strategies $A_1, A_2, ... A_m$.

Then $l_i = a_{i1} y_1 + a_{i2} y_2 + ... + a_{in} y_n$ for i = 1, 2, ..., m.

Now from (2), we can write $l_1 \leq l', l_2 \leq l', ... l_m \leq l'$ and l' being positive we get $\frac{l_i}{l'} \leq 1$; i = 1, 2, ..., m.

The object of the player B is to find the values $y_1, y_2, ..., y_n$ such that l' becomes minimum i.e., $\frac{1}{l'}$ becomes maximum ($\because$ here $l' > 0$).

Now from $\frac{l_i}{l'} \leq 1$ (i = 1, 2, ..., m), we get

$$a_{i1} \frac{y_1}{l'} + a_{i2} \frac{y_2}{l'} + ... + a_{in} \frac{y_n}{l'} \leq 1 \text{ for i = 1, 2, ..., m}$$

and taking $\frac{y_1}{l'} = Y_1, \frac{y_2}{l'} = Y_2, \cdots\cdots \frac{y_n}{l'} = Y_n$, we get

$$a_{i1} Y_1 + a_{i2} Y_2 + ... + a_{in} Y_n \leq 1 \text{ for i = 1, 2, ..., m}$$

Again $\frac{1}{l'} = \frac{y_1 + y_2 + \cdots\cdots + y_n}{l'} = Y_1 + Y_2 + ... + Y_n$, where $Y_1 \geq 0$, $Y_2 \geq 0, ... Y_n \geq 0$.

Then, the problem of the player B can be expressed as a L.P.P. given below :

Maximize $\frac{1}{l'} = Y_1 + Y_2 + ... + Y_n$

subject to the constraints

$$a_{11} Y_1 + a_{12} Y_2 + ... + a_{1n} Y_n \leq 1,$$

$$a_{21} Y_1 + a_{22} Y_2 + ... + a_{2n} Y_n \leq 1$$

....

....

$a_{m1}\ Y_1 + a_{m2}\ Y_2 + \ldots\ldots\ldots\ldots + a_{mn}\ Y_n \leq 1,$

$Y_1 \geq 0,\ Y_2 \geq 0,\ \ldots\ldots\ldots\ldots\ldots\ Y_n \geq 0.$

We observe that the problem of B (as L.P.P.) is the dual of the problem of A (as L.P.P.) and vice-versa.

$$\text{Now } (l')_{\min} = \min_{y_j} \max_{x_i} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j \ \text{ and } (g')_{\max} = \max_{x_i} \min_{y_j} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j$$

In the next section, we shall prove that $(l')_{\min} = (g')_{\max} = \upsilon$ (say) which is the value of the game.

Thus it is proved that any game problem can be solved by reducing the game problem to a L.P.P.

14.13 Fundamental Theorem on Two-Person zero-sum Game

Statement of the Fundamental Theorem on game problem

● **Theorem 14.13.1 :** If mixed strategies be followed, then for any two-person zero-sum game, the value of the game and optimal strategies will exist.

Proof : Let $[a_{ij}]_{m\times n}$ be the pay-off matrix of a game where we can assume that each element $a_{ij} > 0$ [For explanation see Theorem 9.11.1 of Section 9.11].

Let $(x_1, x_2, \ldots\ldots, x_m)$ be the mixed strategy of the maximizing player A and $(y_1, y_2, \ldots\ldots, y_n)$ be the mixed strategy of the minimizing player B i.e., the player A chooses at random his available strategies $A_1, A_2, \ldots\ldots., A_m$ with probabilities $x_1, x_2, \ldots\ldots., x_m$ respectively and the player B chooses at random his strategies $B_1, B_2, \ldots.., B_n$ with probabilities $y_1, y_2, \ldots\ldots, y_n$ respectively, where $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$ and $x_i \geq 0$, $y_j \geq 0$ $(i = 1, 2, \ldots., m;\ j = 1, 2, \ldots.., n)$.

We have seen in Section 9.12 that game problems of A and B can be converted to two linear programming problems.

The problem of A as L.P.P. is given below :

Minimize $\frac{1}{g'} = X_1 + X_2 + ... + X_m$

subject to $a_{11} X_1 + a_{21} X_2 + ... + a_{m1} X_m \geq 1$

$a_{12} X_1 + a_{22} X_2 + ... + a_{m2} X_m \geq 1$

....

....

$a_{1n} X_1 + a_{2n} X_2 + ... + a_{mn} X_m \geq 1$

$X_1, X_2, ... X_n \geq 0$ where $X_i = \frac{x_i}{g'}$; $i = 1, 2, ..., m$.

The problem of B as a L.P.P. is as follows :

Maximize $\frac{1}{l'} = Y_1 + Y_2 + ... + Y_n$

subject to $a_{11} Y_1 + a_{12} Y_2 + ... + a_{1n} Y_n \leq 1$

$a_{21} Y_1 + a_{22} Y_2 + ... + a_{2n} Y_n \leq 1$

....

....

$a_{m1} Y_1 + a_{m2} Y_2 + ... + a_{mn} Y_n \leq 1$

$Y_1, Y_2, ..., Y_n \geq 0$ where $Y_j = \frac{y_j}{l'}$ $(j = 1, 2, ..., n)$.

Here, we observe that one of these two problems (as L.P.P.) is the dual of the other.

Now we shall show that the problem of A has a feasible solution.

Let $a = \frac{1}{\min\{a_{11}, a_{12}, \cdots\cdots a_{1n}\}}$ where $\min\{a_{11}, a_{12}, ..., a_{1n}\} > 0$ since here a_{11}, a_{12}, ..., $a_{1n} > 0$.

Then $a > 0$ and $a.a_{11} \geq 1$, $a.a_{12} \geq 1$, $a.a_{13} \geq 1$, ... $a.a_{1n} \geq 1$.

Thus, $X_1 = a$, $X_2 = 0$, ..., $X_m = 0$ satisfy all the constraints of A's problem. Then (a, 0, ..., 0) is a feasible solution of the L.P.P. for A. Again we observe that the objective function of this L.P.P. is $X_1 + X_2 + ... + X_m \geq 0$ for any feasible solution and so the objective function of the L.P.P. (as a minimization problem) cannot be unbounded. So the problem of A (as L.P.P.) has finite optimal solution.

Now, from the Fundamental Theorem on Duality, we know that

"If any one of the primal problem on the dual problem has finite optimal solution, then the other has also finite optimal solution and the optimal values of the objective functions of the two problems are same."

So the problem of B (as L.P.P.) has also finite optimal solution and $\left(\frac{1}{g'}\right)_{\min} = \left(\frac{1}{l'}\right)_{\max}$

i.e., $(g')_{\max} = (l')_{\min}$.

Then the game has a value which is equal to $(g')_{\max}$ [or $(l')_{\min}$] and from the optimal solutions of the two problems we can find the optimal strategies of the game. So the theorem is proved.

Remark : If the game problem be converted to a L.P.P. by adding P to each element of the pay off matrix, then the value of the given game will be $(g')_{\max} - P = (l') - P$ while the optimal strategies will remain unaltered.

14.14 An Example

● **Example 14.14.1 :** The pay-off matrix of a game is given below. Solve the following game problem by transforming it to a L.P.P. :

Player B

		B_1	B_2	B_3
	A_1	1	–1	3
Player A	A_2	3	5	–3
	A_3	6	2	–2

Solution : Here we add 4 to each element of the pay-off matrix (so that each term becomes positive) and the transformed matrix becomes

	B_1	B_2	B_3
A_1	5	3	7
A_2	7	9	1
A_3	10	6	2

Let the mixed strategies of the player A be given by (x_1, x_2, x_3) and the mixed strategies of the player B be given by (y_1, y_2, y_3) where $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$ and $x_i \geq 0$, $y_j \geq 0$ $(i = 1, 2, 3;\ j = 1, 2, 3)$.

Then the problem of A (as L.P.P.) will be

$$\text{Minimize } \frac{1}{g'} = X_1 + X_2 + X_3$$

$$\begin{aligned} \text{subject to} \qquad 5X_1 + 7X_2 + 10X_3 &\geq 1 \\ 3X_1 + 9X_2 + 6X_3 &\geq 1 \\ 7X_1 + X_2 + 2X_3 &\geq 1 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

and the problem of B (as L.P.P.) will be

$$\text{Maximize } \frac{1}{l'} = Y_1 + Y_2 + Y_3$$

$$\begin{aligned} \text{subject to} \qquad 5Y_1 + 3Y_2 + 7Y_3 &\leq 1 \\ 7Y_1 + 9Y_2 + Y_3 &\leq 1 \\ 10Y_1 + 6Y_2 + 2Y_3 &\leq 1 \\ Y_1, Y_2, Y_3 &\geq 0. \end{aligned}$$

Here $X_i = \dfrac{x_i}{g'}$ $(i = 1, 2, 3)$ and $Y_J = \dfrac{y_j}{l'}$ $(j = 1, 2, 3)$.

Now let us solve the problem of B (which is a L.P.P. for finding maximum value of the objective function) by simplex algorithm.

Adding slack variables Y_4, Y_5, Y_6 we find that the L.P.P. for the problem of B can be expressed as

Maximize $\frac{1}{l'} = Y_1 + Y_2 + Y_3 + 0.Y_4 + 0.Y_5 + 0.Y_6$

subject to $5Y_1 + 3Y_2 + 7Y_3 + 4Y_4 = 1$

$7Y_1 + 9Y_2 + Y_3 + Y_5 = 1$

$10Y_1 + 6Y_2 + 2Y_3 + Y_6 = 1$

$Y_j \geq 0 \quad (j = 1, 2, 3, 4, 5, 6)$

Simplex Table-I

		c_j	1	1	1	0	0	0
$\bar{c}_B$	$\bar{Y}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$
0	Y_4	1	5	3	7	1	0	0
0	Y_5	1	7	$\boxed{9}$	1	0	1	0
0	Y_6	1	10	6	2	0	0	1
	$z_j - c_j$	0	-1	$-1\uparrow$	-1	0	$0\downarrow$	0

$$\min\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{6}\right\} = \frac{1}{9}$$

Simplex Table-II

		c_j	1	1	1	0	0	0
$\bar{c}_B$	$\bar{Y}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$
0	Y_4	$\frac{2}{3}$	$\frac{8}{3}$	0	$\boxed{\frac{20}{3}}$	1	$-\frac{1}{3}$	0
1	Y_2	$\frac{1}{9}$	$\frac{7}{9}$	1	$\frac{1}{9}$	0	$\frac{1}{9}$	0
0	Y_6	$\frac{1}{3}$	$\frac{16}{3}$	0	$\frac{4}{3}$	0	$-\frac{2}{3}$	1
	$z_j - c_j$	$\frac{1}{9}$	$-\frac{2}{9}$	0	$-\frac{8}{9}\uparrow$	$0\downarrow$	$\frac{1}{9}$	0

$$\min\left\{\frac{\frac{2}{3}}{\frac{2}{3}}, \frac{\frac{1}{9}}{\frac{1}{9}}, \frac{\frac{1}{3}}{\frac{4}{3}}\right\} = \frac{1}{10}$$

Simplex Table-III

		c_j	1	1	1	0	0	0
$\bar{c}_B$	$\bar{Y}_B$	$\bar{b}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$
1	Y_3	$\frac{1}{10}$	$\frac{2}{5}$	0	1	$\frac{3}{20}$	$-\frac{1}{20}$	0
1	Y_2	$\frac{1}{10}$	$\frac{11}{15}$	1	0	$-\frac{1}{60}$	$\frac{7}{60}$	0
0	Y_6	$\frac{1}{5}$	$-\frac{8}{15}$	0	0	$-\frac{1}{5}$	$-\frac{3}{5}$	1
$z_j - c_j$		$\frac{1}{5}$	$\frac{2}{15}$	0	0	$\frac{2}{15}$	$\frac{1}{15}$	0

Here $z_j - c_j \geq 0$ for all j.

So optimal solutions of the primal problem and the dual problem can be found from simplex Table III. For the problem of B we get $Y_1 = 0$, $Y_2 = \frac{1}{10}$, $Y_3 = \frac{1}{10}$ and from these values optimal strategies of B can be determined, where $Y_1 + Y_2 + Y_3 = \frac{1}{5}$ which is the maximum value of $\frac{1}{l'}$. So the minimum value of l' is 5 which is the value of the transformed game.

So the value of the given game will be 5 – 4 = 1.

Now the optimal strategies of B are given by $y_1 = 5Y_1 = 0$, $y_2 = 5Y_2 = \frac{1}{2}$, $y_3 = 5Y_3 = \frac{1}{2}$.

Again, from the entries in the row for $z_j - c_j$ in Simplex Table III, we can say $X_1 = \frac{2}{15}$, $X_2 = \frac{1}{15}$, $X_3 = 0$ from which the optimal strategies of the player A can be obtained. So for the optimal strategies of the player A we have $x_1 = 5X_1 = \frac{2}{3}$, $x_2 = 5X_2 = \frac{1}{3}$, $x_3 = 5X_3 = 0$.

Then the value υ of the given game is 1 and the optimal strategies of the players A and B are respectively $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

14.15 Graphical Method of Solution

The graphical method is only applicable to those games in which one of the players has two strategies. It reduces the original 2 × n or m × 2 to a 2 × 2 game. Consider the following pay-off matrix of a 2 × n game.

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \overset{\text{Player B}}{\begin{array}{ccc} B_1 & B_2 \;\ldots\ldots\ldots & B_n \\ \hline a_{11} & a_{12} \;\ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} \;\ldots\ldots\ldots & a_{2n} \end{array}}$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by

$$S_A = \begin{array}{cc} A_1 & A_2 \\ [x_1 & x_2] \end{array} \text{ where } x_2 = 1 - x_1 \text{ and } 0 \le x_1 \le 1.$$

The average (expected) pay-off for A when he plays S_A against B′s pure strategies $B_1, B_2, \ldots.., B_n$ is given by

B's Pure strategy	A's expected pay-off g
B_1	$g_1 = a_{11}x_1 + a_{21}x_2$ $= a_{11}x_1 + a_{21}(1-x_1)$
B_2	$g_2 = a_{12}x_1 + a_{22}(1 - x_1)$
$\vdots$	$\vdots$
B_n	$g_n = a_{1n}x_1 + a_{2n}(1-x_1)$

According to the maximum criterion for mixed strategies player A should select that value of x_1 which maximizes his minimum expected pay-offs. This can be done by plotting the expected pay-off lines as functions of x_1.

$$g_j = (a_{1j} - a_{2j})\, x_1 + a_{2j};\ j = 1, 2, \ldots.., n$$

Since A is the maximizing player, the highest point on the lower envelope of these lines will give maximum of the minimum (i.e., maximin) expected pay-off to A and also the optimum value x_i (see Section 9.16 – Ex-19.6.1).

The two lines passing through the maximum point identify the two critical strategies of B which, combined with two of A, reduces to 2 × 2 game that can be used to find the optimum strategies of the two players (for the original game) using the results of previous sections.

Note : (i) If there are more than two lines passing through the maximin point, there are ties for the optimum mixed strategies for player B and so any two such lines with opposite sign slopes will define an alternative optimum for B.

(ii) The m × 2 games are also treated in the similar way where the upper envelope of the straight lines corresponding to B's expected pay-offs will give the maximum expected pay-off to B. The lowest point on this then gives the minimum expected pay-off (minimax value) and the optimum value of y_i. The two lines passing through the minimax point identify the two critical strategies of A which, combined with two of B, reduces to 2 × 2 game.

14.16 A Few Examples

● **Example 14.16.1 :** Sove the following game graphically :

$$\text{Player A} \quad \overset{\text{Player B}}{\begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}}$$

Solution :

$$A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

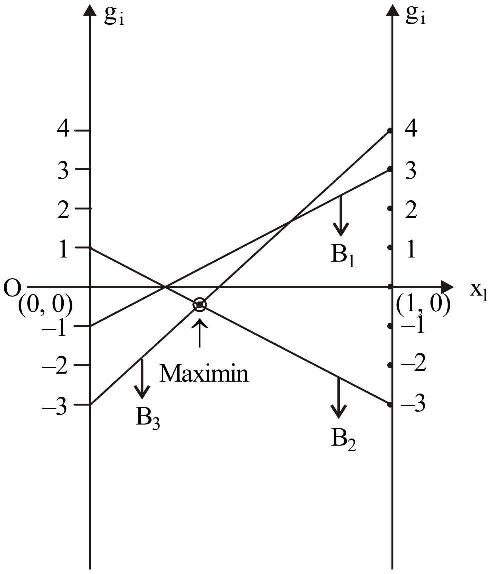
Obviously, this game does not possess a saddle point (in case of pure strategy), where A is the maximizing player and B is the minimizing player. We suppose that A plays mixed strategy described by

$$S_A = \begin{array}{cc} A_1 & A_2 \\ {[x_1} & {1 - x_1]} \end{array} \; 0 \le x_1 \le 1, \text{ against } B$$

B's Pure strategy	A's expected pay-off g_i
B_1	$g_1 = 3x_1 - (1 - x_1) = 4x_1 - 1$
B_2	$g_2 = -3x_1 + (1 - x_1) = -4x_1 + 1$
B_3	$g_3 = 4x_1 - 3(1 - x_1) = 7x_1 - 3$

Graphical Representation

The graph indicates that the given game is reduced to a 2 × 2 game :



$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} B_2 & \;\; B_3 \end{array} \\ A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \end{array}$$

Player A's strategy :

$$x_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{-3-1}{-11} = \frac{4}{11}$$

$$(\because \lambda = a_{11} + a_{22} - (a_{12} + a_{21})$$
$$= -6 -5 = -11)$$

$$x_2 = 1 - x_1$$
$$= 1 - \frac{4}{11} = \frac{7}{11}$$

Player B's strategy :

$$y_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{-3-4}{-11} = \frac{7}{11}$$

$$y_2 = 1 - y_1 = \frac{4}{11}$$

$$\text{Value of the game} = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{9-4}{-11} = -\frac{5}{11}.$$

Hence, the solution to the given game as follows :

Optimal strategies are $S_A^* = \left[\frac{4}{11}\ \frac{7}{11}\right]$, $S_B^* = \left[0, \frac{7}{11}, \frac{4}{11}\right]$ and $\nu = \frac{-5}{11}$.

● **Example 14.16.2 :** Solve graphically the game having the following pay-off matrix :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} B_1 & \qquad B_2 \end{array} \\ A \quad \begin{array}{c} A_1 \\ A_2 \\ A_2 \\ A_2 \end{array} & \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \end{bmatrix} \end{array}$$

Solution : Here, A is the maximizing player and B is the minimizing player. Obviously, this game does not possess a saddle point (in case of pure strategy).

We suppose that B plays strategies described by

$$S_B = \begin{array}{l} \quad B_1 \qquad B_2 \\ [y_1 \quad 1 - y_1], \end{array} 0 \le y_1 \le 1, \text{ against A}$$

A's Pure strategy	B's expected pay-off l_i
A_1	$l_1 = y_1 - 3(1 - y_1) = 4y_1 - 3$
A_2	$l_2 = 3y_1 + 5(1 - y_1) = -2y_1 + 5$
A_3	$l_3 = -y_1 + 6(1 - y_1) = -7y_1 + 6$
A_4	$l_4 = 4y_1 + 1 - y_1 = 3y_1 + 1$

Graphical Representation

The graph indicates that the given game is reduced to a 2 × 2 game :

$$\begin{array}{cc} & B \\ & \begin{array}{cc} B_2 & B_3 \end{array} \\ A \quad \begin{array}{c} A_2 \\ A_4 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

It has no saddle point.

Player A's strategy :

$$x_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{1-4}{-5} = \frac{3}{5}$$

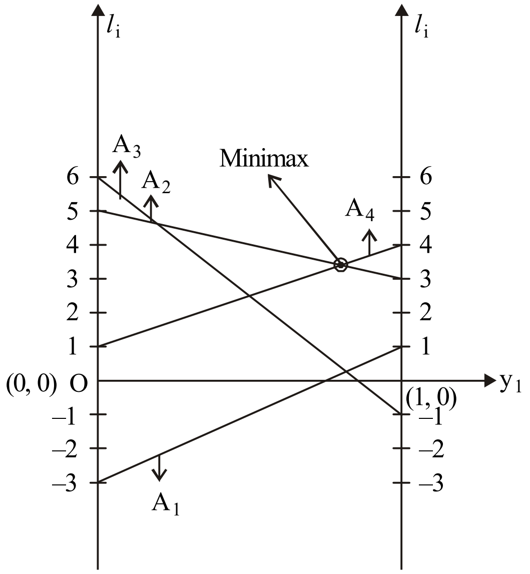
$$(\because \lambda = a_{11} + a_{22} - (a_{12} + a_{21})$$
$$= 4 - 9 = -5)$$

$$x_2 = 1 - x_1 = \frac{2}{5}$$

Player B's strategy :

$$y_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{1-5}{-5} = \frac{4}{5}$$

$$y_2 = 1 - y_1 = \frac{1}{5}$$



Hence the solution to the given game is as follows :

$$S_A^* = \text{Optimum strategies for A} = \left[0, \frac{3}{5}, 0, \frac{2}{5}\right]$$

$$S_B^* = \text{Optimum strategies for B} = \left[\frac{4}{5}, \frac{1}{5}\right]$$

$$\text{and } \nu = \text{value of the game} = \frac{a_{12}\,a_{22} - a_{12}a_{21}}{\lambda} = \frac{3-20}{-5} = \frac{17}{5}.$$

14.17 Summary

First we have given the idea of pay-off matrix of a two-person zero-sum game. Then we have proved that the value of a game can be determined following only pure strategies if and only if the game has a saddle point. After this we have shown how 2 × 2 game having no saddle point can be solved by following mixed strategies.

Then we have stated the rules of dominance and we have seen that in many cases the pay-off matrix of a game can be reduced to 2 × 2 pay-off matrix by using the rules of dominance. After proving two theorems using the idea of expectation function we have shown by using one theorem that any game problem can be transformed to L.P.P. It has been shown that for only two-person zero-sum game, value exists and optimal strategies can be determined when mixed strategies are allowed.

Finally, we have given the graphical method of solving a game problem with pay-off matrix of order 2 × n or n × 2.

14.18 Exercise

1. Apply the maximin and minimax principle to solve the games whose pay-off matrices are given below :

(i)

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{B} \\
 & & B_1 & B_2 & B_3 \\
\cline{3-5}
 & A_1 & 6 & 3 & -3 \\
\cline{3-5}
A & A_2 & -2 & 1 & 2 \\
\cline{3-5}
 & A_3 & 5 & 4 & 6 \\
\cline{3-5}
\end{array}$$

(ii)

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{B} \\
 & & B_1 & B_2 & B_3 & B_4 \\
\cline{3-6}
 & A_1 & 4 & 2 & 3 & 5 \\
\cline{3-6}
A & A_2 & -2 & -1 & 4 & -3 \\
\cline{3-6}
 & A_3 & 5 & 2 & 3 & 3 \\
\cline{3-6}
 & A_4 & 4 & 0 & 0 & 1 \\
\cline{3-6}
\end{array}$$

(iii)

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{B} \\
 & & B_1 & B_2 & B_3 \\
\cline{3-5}
 & A_1 & 7 & 4 & 1 \\
\cline{3-5}
A & A_2 & 4 & 2 & 0 \\
\cline{3-5}
 & A_3 & 3 & -1 & -2 \\
\cline{3-5}
 & A_4 & 1 & 5 & -3 \\
\cline{3-5}
\end{array}$$

2. Prove that for any value of x (> 0) the value of the following game is 21 :

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} \\ \cline{3-4}
A & A_1 & 2 & 4 \\ \cline{3-4}
 & A_2 & -1 & x \\ \cline{3-4}
\end{array}$$

3. Prove that the pay-off matrix of the following game has no saddle point :

$$\begin{array}{cc|c|c|c|c|c|}
 & & \multicolumn{5}{c}{B} \\
 & & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} & \multicolumn{1}{c}{B_3} & \multicolumn{1}{c}{B_4} & \multicolumn{1}{c}{B_5} \\ \cline{3-7}
 & A_1 & 3 & 10 & 5 & 9 & 5 \\ \cline{3-7}
A & A_2 & 4 & 5 & 12 & 10 & 6 \\ \cline{3-7}
 & A_3 & 5 & 6 & 4 & 7 & 13 \\ \cline{3-7}
 & A_4 & 11 & 7 & 8 & 5 & 2 \\ \cline{3-7}
\end{array}$$

4. Solve the following 2 × 2 games :

(i)

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} \\ \cline{3-4}
A & A_1 & 10 & 5 \\ \cline{3-4}
 & A_2 & 5 & 10 \\ \cline{3-4}
\end{array}$$

(ii)

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & \multicolumn{1}{c}{I} & \multicolumn{1}{c}{II} \\ \cline{3-4}
A & I & 2 & 12 \\ \cline{3-4}
 & II & 8 & 3 \\ \cline{3-4}
\end{array}$$

5. Solve the following 4 × 5 game using the rules of dominance :

		B				
		B_1	B_2	B_3	B_4	B_5
	A_1	10	5	5	20	4
A	A_2	11	15	10	17	25
	A_3	7	12	8	9	8
	A_4	5	13	9	10	5

6. Reduce the following games to a game having 2 × 2 pay-off matrix :

(i)

7	7	6	3	2
9	8	9	3	5
10	6	7	10	11

(ii)

2	3	$\frac{1}{2}$
$\frac{3}{2}$	3	0
$\frac{1}{2}$	1	1

7. Find the expectation function of the following game and hence show that the optimal strategy of the maximizing player A is a pure strategy whole the minimizing player B has infinite number of optimal mixed strategies.

8. Find the expectation function of the following 2 × 2 game and hence find the optimal strategies and the value of the game :

		B	
		B_1	B_2
A	A_1	1	7
	A_2	6	2

9. Transform the game problem with the following pay-off matrix to a L.P.P. :

$$\begin{array}{cc|c|c|c|}
\multicolumn{2}{c}{} & \multicolumn{3}{c}{B} \\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} & \multicolumn{1}{c}{B_3} \\ \cline{3-5}
 & A_1 & 10 & 2 & 5 \\ \cline{3-5}
A & A_2 & 1 & 7 & 4 \\ \cline{3-5}
 & A_3 & 6 & 3 & 9 \\ \cline{3-5}
\end{array}$$

10. Solve the following game problem by reducing it to a L.P.P. :

$$\begin{array}{cc|c|c|c|}
\multicolumn{2}{c}{} & \multicolumn{3}{c}{B} \\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} & \multicolumn{1}{c}{B_3} \\ \cline{3-5}
A & A_1 & 2 & -2 & 3 \\ \cline{3-5}
 & A_2 & -3 & 5 & -1 \\ \cline{3-5}
\end{array}$$

11. Solve graphically the games whose pay-off matrices are given below :

(i)

$$\begin{array}{cc|c|c|c|}
\multicolumn{2}{c}{} & \multicolumn{3}{c}{B} \\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} & \multicolumn{1}{c}{B_3} \\ \cline{3-5}
A & A_1 & 1 & 3 & 11 \\ \cline{3-5}
 & A_2 & 8 & 5 & 2 \\ \cline{3-5}
\end{array}$$

(ii)

$$\begin{array}{cc|c|c|}
\multicolumn{2}{c}{} & \multicolumn{2}{c}{B} \\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} \\ \cline{3-4}
A & A_1 & 2 & 7 \\ \cline{3-4}
 & A_2 & 3 & 5 \\ \cline{3-4}
 & A_3 & 11 & 2 \\ \cline{3-4}
\end{array}$$

12. A game is said to be strictly determinable of $\underline{\upsilon} = \overline{\upsilon}$. Show that whatever may be the value of a, the game with the following pay-off matrix is strictly determinable :

$$\begin{array}{cc|c|c|}
\multicolumn{2}{c}{} & \multicolumn{2}{c}{B} \\
\multicolumn{2}{c}{} & \multicolumn{1}{c}{B_1} & \multicolumn{1}{c}{B_2} \\ \cline{3-4}
A & A_1 & 3 & 7 \\ \cline{3-4}
 & A_2 & -3 & a \\ \cline{3-4}
\end{array}$$

13. For what values of a, the game with the following pay-off matrix is strictly determinable?

$$\begin{array}{cc|c|c|c|}
 & & & B & \\
 & & B_1 & B_2 & B_3 \\
\hline
 & A_1 & a & 5 & 2 \\
\hline
A & A_2 & -1 & a & -8 \\
\hline
 & A_3 & -2 & 3 & a \\
\hline
\end{array}$$

14. In a game with 2 × 2 pay-off matrix

$$\begin{array}{|c|c|}
\hline
a & b \\
\hline
c & d \\
\hline
\end{array}$$

where $a < d < b < c$, show that there is no saddle point.

Answers

1. (i) (A_3, B_2); $\upsilon = 4$

(ii) (A_1, B_2) or (A_3, B_2); $\upsilon = 2$

(iii) (A_1, B_3); $\upsilon = 3$

4. (i) $A : \left(\frac{1}{2}, \frac{1}{2}\right)$; $B : \left(\frac{1}{2}, \frac{1}{2}\right)$; $\upsilon = \frac{15}{2}$

(ii) $A : \left(\frac{1}{3}, \frac{2}{3}\right)$; $B : \left(\frac{3}{5}, \frac{2}{5}\right)$; $\upsilon = 6$

5. (A_2, B_3); $\upsilon = 10$

6. (i) $\begin{bmatrix} 8 & 3 \\ 8 & 10 \end{bmatrix}$; (ii) $\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

8. $E(X, Y) = 5x + 4y - 10xy + 2$; A : $\left(\frac{2}{5}, \frac{3}{5}\right)$; B : $\left(\frac{1}{2}, \frac{1}{2}\right)$; $\upsilon = 4$

9. B's problem

Maximize $\frac{1}{l'} = Y_1 + Y_2 + Y_3$

subject to

$$\begin{aligned} 10Y_1 + 2Y_2 + 5Y_3 &\leq 1 \\ Y_1 + 7Y_2 + 4Y_3 &\leq 1 \\ 6Y_1 + 3Y_2 + 9Y_3 &\leq 1 \\ Y_1, Y_2, Y_3 &\geq 0 \end{aligned}$$

where $y_i = l'y_i$ (i = 1, 2, 3).

A's problem

Minimize $\frac{1}{g'} = X_1 + X_2 + X_3$

subject to

$$\begin{aligned} 10X_1 + X_2 + 6X_3 &\geq 1 \\ 2X_1 + 7X_2 + 3X_3 &\geq 1 \\ 5X_1 + 4X_2 + 9X_3 &\geq 1 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

and $x_i = g'X_i$ (i = 1, 2, 3).

10. A : $\left(\frac{7}{12}, \frac{5}{12}\right)$; B : $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$; $\upsilon = \frac{1}{3}$

11. (i) A : $\left(\frac{3}{11}, \frac{8}{11}\right)$; B : $\left(0, \frac{9}{11}, \frac{2}{11}\right)$; $\upsilon = \frac{49}{11}$

(ii) A : $\left(\frac{9}{14}, 0, \frac{5}{14}\right)$; B : $\left(\frac{5}{14}, \frac{9}{14}\right)$; $\upsilon = \frac{73}{14}$

13. $-1 \leq a \leq 2$

14.19 Multiple Choice Questions (MCQ)

1. In a fair game the value of the game is

   (a) 1 (b) 0 (c) unbounded (d) none of these

2. The value of the game having the following pay-off matrix

$$\begin{array}{c|ccc|} & B_1 & B_2 & B_3 \\ \hline A_1 & 10 & 2 & 3 \\ A_2 & 7 & 6 & 8 \\ A_3 & 0 & 3 & 1 \\ \hline \end{array}$$

   is

   (a) 6 (b) 10 (c) 8 (d) 2

3. Find the range of values of p and q for which the position (2, 2) will be a saddle point for the following game :

B

		B_1	B_2	B_3
	A_1	2	4	5
A	A_2	10	7	q
	A_3	4	p	6

   (a) $p \leq 7$ and $q \geq 7$ (b) $p = 7, q = 7$

   (c) $p \geq 7$ and $q \leq 7$ (d) $p \leq 7$ and $q \leq 7$

4. Players apply mixed strategy in a game if

   (a) there is no saddle point in the pay-off matrix

   (b) there is a saddle point in the pay-off matrix

   (c) there is more than one saddle point in the pay-off matrix

   (d) none of these

5. In a game problem, saddle point in the pay-off matrix occurs when

   (a) max (row min) = min (column max)

   (b) min (row min) = min (column max)

   (c) max (row min) = max (column max)

   (d) none of these

6. A game is solved graphically when the pay-off matrix is of the form

   (a) m × 1 (b) m × 4 (c) m × 2 only (d) m × 1 or 2 × n

7. In a game problem, the dominance rule for column states that every element in the dominating column must be __________ the corresponding element in the dominated column

   (a) less than or equal to (b) greater than

   (c) less than (d) grater than or equal to

8. If mixed strategies be followed in two-person zero-sum game, then

   (a) the value of the game may or may not exist

   (b) optimal strategies do not exist

   (c) optimal strategis exist

   (d) none of these

Answers

1. (b) 2. (a) 3. (a) 4. (a) 5. (a) 6. (d)

7. (a) 8. (c)

Further Reading

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