QP Code: 23/PT/13/VIB

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-6B : FUNCTIONAL ANALYSIS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Notations and symbols have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Let $X = \mathbb{R}^2$. Define two norms $\|.\|_1$ and $\|.\|_2$ on X by $\|(x, y)\|_1 = |x| + |y|$ and $\|(x, y)\|_2 = \max\{|x|, |y|\}$. Show that $\|.\|_1$ and $\|.\|_2$ are equivalent.

b) Show that the norm in a linear space X is a sub-linear functional over X.

c) If M is a subspace of a normed linear space X, then show that $\overline{M}$ is also a subspace of X.

d) If x and y are two elements in a real Hilbert space and if $\|x\| = \|y\|$, then show that $(x+y, x-y) = 0$.

e) Find the eigen values and eigen vectors of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $a, b \ (\neq 0)$ are reals.

f) Show that the orthogonal complement of any subset of an inner product space X is a closed linear subspace of X.

g) Let S be any bounded linear operator in a Hilbert space H and T be a self-adjoint operator in H. Show that S^*TS is self-adjoint.

2. a) Show that space of all real polynomials of degree $\leq n$ in the closed interval [a, b] is isomorphic to the Euclidean space $\mathbb{R}^{n+1}$.

   b) Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space *H*. For $x \in H$, define $y = \sum_{k=1}^{\infty} (x, e_k) e_k$. Prove that $(x-y) \perp e_k$ for *k* = 1, 2,...... . 5 + 5

3. a) State and prove Riesz lemma in a normed linear space.

   b) Let *y* be a non-zero vector in a normed linear space *X*. Show that there is a bounded linear functional *f* defined on *X* such that $\| f \| = 1$ and $f(y) = \| y \|$.

   c) Prove that the intersection of any number of convex sets in a normed linear space is convex. 4 + 4 + 2

4. a) State Hahn Banach theorem in a normed linear space. Show that for every $x (\neq 0)$ in a normed linear space,

$$\| x \| = \sup \left\{ \frac{| f(x) |}{\| f \|}, f \in X^* \text{ and } f \neq 0 \right\}.$$

   b) Show that a continuous linear operator $A : H \to H$ is self-adjoint if and only if (Ax, x) is real for all $x \in H$, where *H* is a Hilbert space. 5 + 5

5. a) State and prove Bessel's inequality.

   b) Given that *X* and *Y* are Banach spaces over the same scalar field and $T : X \to Y$ is a surjective bounded linear operator, prove that *T* is an open mapping.

   c) Determine norm of the functional $f(x, y) = x + y$ on the Hilbert space $\mathbb{R}^2$. 4 + 4 + 2

6. a) Prove that in a separable Hilbert space *H*, every orthonormal system is countable.

   b) Let $T : H \to H$ be a continuous linear operator. If $(Tx, x) = 0 \; \forall x \in H$, then show that *T* is a zero operator, given that *H* is a Hilbert space.

   c) Find the orthogonal complement of $W = \{ (x_i) \in l_2 : x_i = 0 \text{ for all even } i \}$ in l_2. 4 + 4 + 2

7. a) If a normed linear space X has the property that the series $\sum_{n=1}^{\infty} x_n$ in X converges whenever $\sum_{n=1}^{\infty} \| x_n \| < \infty$, then prove that X is a Banach space.

b) Let $\rho_3[0, 1]$ denote the linear space of all real polynomials over [0, 1] with degree ≤ 3. Assume that $D : \rho_3[0, 1] \rightarrow \rho_2[0, 1]$ be the differential operator. Show that D is linear and obtain a representative matrix for D.

c) Give an example of a linear metric space which is not a normed linear space. 4 + 4 + 2