QP Code: 23/PT/13/VIIB

POST-GRADUATE COURSE

Term End Examination — June, 2023/December, 2023

MATHEMATICS

Paper-7B : INTEGRAL EQUATIONS AND GENERALISED FUNCTIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Notations and symbols have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) Reduce the following initial value problem to an integral equation :

$$\frac{dy}{dx} - y = 0, \ \ x > 0, \ \ y(0) = 1$$

b) Find the Neumann series corresponding to the integral equation —

$$\phi(x) = 1 + \lambda \int_0^{\pi/2} \cos x \, \phi(t) \, dt, \ \ 0 < x < \frac{\pi}{2}$$

c) Find the nontrivial solution of the following integral equation :

$$\phi(x) = \lambda \int_2^1 2t \, \phi(t) \, dt, \ \ 0 < x < 1$$

d) For what value of λ, does the unique solution of the integral equation ($f(x)$ is a known function)

$$\phi(x) = f(x) + \lambda \int_0^1 xt \, \phi(t) \, dt, \ \ (0 < x < 1) \text{ exist ?}$$

e) Which of the following kernel(s), $k(x, t)$ are symmetric or Hermitian ?

(i) $k(x, t) = \cos(x - t)$

(ii) $k(x, t) = -i(x - t)$

(iii) $k(x, t) = e^{x-t}$

f) Solve the integral equation —

$$\int_0^x e^{x-t}\,\phi(t)\,dt = 0, \quad x > 0$$

g) State Mercer's theorem.

2. a) Convert the following initial value problem to an integral equation :

$$(x+1)^2 y''(x) + 2(x+1)y'(x) = f(x), \quad 0 \le x \le 1$$

$$y(0) = \alpha \quad y(1) = \beta .$$

Here $f(x)$ is a known function of x and α, β are known constants.

b) Use Laplace transform method to solve

$$\int_0^x e^{x-t}\,\phi(t)\,dt = x, \quad 0 \le x \le 1$$ 6 + 4

3. a) Reduce the following boundary value problem to an integral equation:

$$y''(x) + \lambda y(x) = 0, \quad 0 \le x \le 1$$

$$y(0) = 1, \quad y'(1) - 2y(1) = 2$$

b) Derive the differential equation with initial conditions from the integral equation

$$y(x) = 1 - x - 4\sin x + \int_0^x [3 - 2(x-t)]\,y(t)\,dt$$ 6 + 4

4. a) Solve by the method of successive approximation the following integral equation :

$$y(x) = 1 + \int_0^x 2t\,y(t)\,dt, \quad 0 \le x \le 1 .$$

b) Find the iterated kernels and solve the following integral equation :

$$y(x) = 1 + \lambda \int_0^1 xt\,y(t)\,dt, \quad 0 \le x \le 1$$ 5 + 5

5. a) Prove that every eigenvalue of a symmetric kernel is real.

   b) Find the non-trivial solution of the following integral equation with degenerate kernel :

$$y(x) = \lambda \int_0^1 \left(3 - \frac{3x}{2}\right) t \, y(t) \, dt, \quad 0 \le x \le 1.$$ 4 + 6

6. Use Hilbert-Schmidt theorem to solve the following integral equation :

$$\phi(x) - \lambda \int_0^x t(1-x)\,\phi(t)\,dt - \lambda \int_x^1 x(1-t)\phi(t)dt = f(x), \quad 0 \le x \le 1$$

where $f(x)$ is a known square integrable function. 10

7. Define a regular sequence of good functions. Prove that the sequence $\left\{ \sqrt{\frac{n}{\pi}} e^{-nx^2} \right\}$ is regular and defines a generalised function $\delta(x)$. When can you say that two regular sequences of good functions $\left\{ \gamma_n^{(1)}(x) \right\}$ and $\left\{ \gamma_n^{(2)}(x) \right\}$ are equivalent ? 2 + 6 + 2