QP Code: 22/PT/13/IB

POST-GRADUATE COURSE

Term End Examination — June, 2022/December, 2022

MATHEMATICS

Paper-1B : LINEAR ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(Unexplained notations and symbols have their usual meanings)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Find the roots of the minimal polynomial of $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$.

b) Let $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + c = 0 \right\}$ and $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + d = 0 \right\}$ be two subspaces of $\mathbb{R}_{2 \times 2}$. Find $\dim(U + W)$.

c) Let V and W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$ be a linear transformation. Also let $\{ \alpha_1, \alpha_2, .., \alpha_k \}$ be a basis of ker T and $\alpha_{k+1}, \alpha_{k+2,...,} \alpha_n \in V$ such that $\{ \alpha_1, \alpha_2, ..., \alpha_n \}$ is a basis of V. Find a basis of $\operatorname{Im} T$. Justify your answer.

d) State *true* or *false* :

Norm induced by an inner product satisfies the parallelogram law. Justify your answer.

e) State *true* or *false* :

There cannot be any onto linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^6$. Justify.

f) Find a basis of the vector space of all real polynomials of degree less than or equal to 4. Hence find its dimension.

g) Prove that the quadratic form

$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$ is indefinite.

2. a) Let $T: V \to W$ be a linear transformation. Prove that if dim V is finite (W may be infinite dimensional) then Rank T is finite.

b) Let A be an $m \times n$ matrix over a field $\mathbb{F}$. Use rank-nullity theorem to show that if $n > m$ then $AX = 0$ has a non-zero solution.

c) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V. Let W be a vector space over the same field $\mathbb{F}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be any vectors (may not be distinct) in W. Prove that there is precisely one linear transformation T from V into W such that $T(\alpha_i) = \beta_i$, $i = 1, 2, \ldots, n$.

3 + 3 + 4

3. a) Suppose V_1 and V_2 are two subspaces of a vector space V. Prove that the subspace $V_1 + V_2$ is the smallest subspace of V containing $V_1 \cup V_2$.

b) Prove that an orthogonal set of non-zero vectors is linearly independent.

c) If α, β be two orthogonal vectors in a Euclidean space V, then prove that $\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$.

d) Do the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$ and $\beta_3 = (2, 9, 11)$ form a basis for $\mathbb{R}^3$? Justify your answer. If yes, then obtain an orthogonal basis using Gram-Schmidt process. 2 + 2 + 2 + 4

4. a) Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Show that T in invertible and determine T^{-1}.

b) Prove that two eigenvectors corresponding to two distinct eigen values of a linear operator defined on a finite dimensional vector space are linearly independent.

c) Prove that non-zero eigenvalues of a real skew-symmetric matrix are purely imaginary. 3 + 3 + 4

5. a) Is the real 3×3 matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$ diagonalizable ? Justify.

b) What are the algebraic and geometric multiplicities of the eigen value(s) of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$?

c) Find the minimal polynomial of $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$. 5 + 2 + 3

6. a) Let V be an n dimensional vector space over the field $\mathbb{F}$ and let $\mathbb{B}$ and $\mathbb{B}'$ be two ordered bases of V. Prove that there exists a unique invertible $n \times n$ matrix P with entries in $\mathbb{F}$ such that $[\alpha]_{\mathbb{B}} = P[\alpha]_{\mathbb{B}'}$, for every vector α in V; $[\alpha]_{\mathbb{B}}$ being the coordinate matrix of the vector α relative to the ordered basis $\mathbb{B}$.

b) Let T be a linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, 0)$, $(x_1, x_2) \in \mathbb{R}^2$. Find $[T]_{\mathbb{B}}$ where $\mathbb{B} = \{(1,0),(0,1)\}$. Consider the ordered basis $\mathbb{B}' = \{(1, 1),(2, 1)\}$. Find $[T]_{\mathbb{B}'}$ using $[T]_{\mathbb{B}}$.

c) Prove that the similarity of matrices in an equivalence relation over the set of $n \times n$ matrix over a field $\mathbb{F}$. 4 + (1 + 3) + 2

7. a) Define positive, negative indices of inertia and signature in the context of a real quadratic form.

b) Obtain a non-singular transformation that will reduce the quadratic form $q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_2x_3$ to normal form and hence find its signature.

c) Show that the quadratic form

$q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_2x_3$ is indefinite.

3 + (3 + 1) + 3