POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-1A : ABSTRACT ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Let G be any non-commutative group of order 125. Find the order of $Z(G)$.

   b) Let $f: G \to G'$ be a group homomorphism. Prove that $\ker f = \{e_G\}$ if and only if f is a monomorphism, where e_G denotes the identity element of G.

   c) Give an example of a group G whose all proper subgroups are cyclic but G itself is not cyclic. Justify your answer.

   d) If G is a group and G' is its derived subgroup, then prove that G' is a normal subgroup of G.

   e) Let F be a field. Prove that $\{0\}$ is a maximal ideal of F, where 0 is the zero element of F.

   f) Let D be a Euclidean domain with a valuation υ. Prove that for any $x, y \in D \setminus \{0\}, \upsilon(xy) = \upsilon(x)$ if y in invertible.

   g) Let $\mathbb{Q}$ be the field of rational numbers. Find $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$.

   h) Prove that the polynomial $x^2 + 1$ is irreducible over the ring of integers modulo 7.

2. a) Define generating set of a subgroup H of a group G. Let S be a non-empty subset of a group G. Prove that the subgroup $<S>$ generated by S is the set of all finite products of the form $a_1 a_2 \ldots a_n$, where for each i, either $a_i \in S$ or $a_i^{-1} \in S$. 1 + 3

   b) Let ϕ be an epimorphism from a group G onto a group H. Prove that $G / Ker\,\phi \cong H$. 6

3. a) Let H and K be normal subgroups of a group G. Prove that HK is a normal subgroup of G. 5

   b) Let G be a group of order p where p is a prime. Prove that G is cyclic. Find all the generators of G. 3 + 2

4. a) Let $Z(G)$ be the centre of a finite group. Prove that

$$o(G) = o(Z(G)) + \sum_{a \in C} [G : N(a)]$$

   where C contains exactly one element form each conjugate class with more than one element. 5

   b) Prove that every subgroup of the additive group $\mathbb{Z}$ of all integers is of the form $n\mathbb{Z}$ for some non-negative integer n. 5

5. a) Let R be a commutative ring with unity. Prove that an ideal I of R is maximal *iff* the quotient ring R/I is a field. 6

   b) Prove that a commutative ring with unity is without proper ideals if and only if it is a field. 4

6. a) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. 1 + 5

   b) Prove that $x^2 + 1$ is irreducible over the ring of integers mod 7. 4

7. a) Define extension of a field F with example. When is an element $\alpha \in G$ said to be algebraic over F, where G is an extension of F? 2

   b) If the field G is an extension of the field F of finite degree n, then prove that $c \in G$ is a root of a polynomial of degree at most n with co-efficients in F. 3

   c) Let $f(x)$ be a non-constant polynomial over a field K. Prove that there exists a splitting field of $f(x)$ over K. 5