POST-GRADUATE DEGREE PROGRAMME
Term End Examination — December, 2024
MATHEMATICS
Paper-1B : LINEAR ALGEBRA

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Show that the intersection of two sub-spaces of a vector space V over a field F is a sub-space of V over F.

   b) Show that the vectors (3, 2, 1), (0, 1, 2) and (1, 0, 2) are linearly independent.

   c) Find a basis of the vector space $\mathbb{R}^3$ that consists of the vectors (1, 2, 1) and (3, 6, 2).

   d) Check whether the mapping $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T((a, b)) = (a+2, b)$ is a linear transformation or not.

   e) If α, β are two vectors in a Euclidean space V, then prove that $\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$.

   f) Check whether the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are similar or not.

   g) Show that the quadratic form $x_1^2 + x_2^2 + x_3^2 + 3x_2 . x_3$ is indefinite.

2. a) $W_1 = \{ (a, b, 0) : a, b \in \mathbb{R} \}$ and $W_2 = \{ (0, 0, c) : c \in \mathbb{R} \}$ are sub-spaces of $\mathbb{R}^3$. Show that $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

   b) State and prove the Replacement Theorem on vector spaces. 4 + 6

3. a) Extend the set of vectors { (2, 3, – 1), (1, – 2, – 4) } to an orthogonal basis of the Euclidean space $\mathbb{R}^3$ and then find an associated orthonormal basis of $\mathbb{R}^3$.

b) If α, β be vectors in a Euclidean space V, show that $< \alpha, \beta > = 0$ if and only if $\|\alpha + \beta\| = \|\alpha - \beta\|$. 7 + 3

4. a) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation given by $T(x, y, z) = (3x + 2y - 4z,\ x - 5y + 3z)$. Find the matrix of T with respect to the bases $\{ (1, 1, 1), (1, 1, 0)\ (1, 0, 0) \}$ of $\mathbb{R}^3$ and $\{ (1, 3), (2, 5) \}$ of $\mathbb{R}^2$.

b) Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3,\ -2x_1 + x_2,\ -x_1 + 2x_2 + 4x_3)$. Prove that T is invertible and hence find the inverse of T. 5 + 5

5. a) Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix, where $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{bmatrix}$.

b) Find all possible canonical forms for a matrix of order 6, whose minimal polynomial is $m(t) = (t-2)^2 (t+3)^2$. 6 + 4

6. a) Obtain a non-singular transformation that will reduce the quadratic form $q = x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3$ to normal form and hence find its signature.

b) Let T be a linear operator on an n-dimensional vector space over the field F. Prove that T is non-singular T if maps a linearly independent set of V to another linearly independent set and conversely. 5 + 5

7. a) If $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, find the eigenvalues and eigenvectors of A.

b) Find all the sub-spaces of $\mathbb{R}^3$. 5 + 5