POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-2A : REAL ANALYSIS & METRIC SPACES

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) For any two non-empty bounded subsets A, B of $\mathbb{R}$, show that $sup\,(A + B) = sup\,(A) + sup\,(B)$, where $A+B = \{a+b : a \in A, b \in B\}$.

b) If $E = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left(n - \frac{1}{2^k},\ n + \frac{1}{2^k} \right)$ then find $m\,(E)$.

c) If $f(x) = \cos x + |\cos x|$ in $-\pi \le x \le \pi$ then find f^{+} and f^{-}.

d) Find $\int_0^1 x^3\, d\,[x]$.

e) Find the derived set of $\left\{ \frac{1}{n} + (-1)^n : n \in \mathbb{N} \right\}$.

f) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d) then show that $\lim_{n \to \infty} d(x_n, y_n) = 0$.

g) Prove or disprove : $f(x) = x^2$ is uniformly continuous in $\mathbb{R}$.

h) Let A be a compact set in a metric space (X, d) and $x_0 \in X \setminus A$. Show that there is some $a \in A$ such that $d(x_0, A) = d(x_0, a)$.

2. a) If $f:[a, b] \to \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\,dt,\ a \le x \le b$ then show that F is of bounded variation. Estimate the total variation of F over $[a, b]$. 5

b) Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions $(E \subseteq \mathbb{R})$ and $f(x) = \lim_{n\to\infty} f_n(x), \forall x \in E$. Show that f is a measurable function. 5

3. a) If $E = \bigcup_{k=1}^{\infty} E_k$ where $E_1 \subseteq E_2 \subseteq \ldots$ and E is bounded then prove that $\lim_{n\to\infty} m^*(E_n) = m^*(E)$. 5

b) Define $f:[0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \dfrac{1}{x^{2/3}}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Show that f is Lebesgue integrable on $[0, 1]$ and hence find $\int_0^1 \frac{1}{x^{2/3}}\,dx$. 5

4. a) Evaluate : $\int_1^4 (x - [x])\, dx^2$. 4

b) If f is continuous on $[0, 1]$ and if $\int_0^1 x^n f(x)\,dx = 0$, for $n = 0, 1, 2, \ldots$ then show that $f(x) = 0$ on $[0, 1]$. 4

c) Prove or disprove : Homeomorphic image of a complete metric space is a complete metric space. 2

5. a) If (X, d) is a metric space and Y is a subspace of X then show that $Cl_Y B = Cl_X B \cap Y$

where '*Cl*' stands for the closure of a set. 4

b) Find the Fourier series expansion of $f(x)$ in $-\pi < x < \pi$ where

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ \dfrac{\pi x}{4}, & 0 < x < \pi \end{cases}.$$

Hence deduce $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. 6

6. a) Show that $C[0,1] = \{f : [0,1] \to \mathbb{R} \,/\, f \text{ is continuous}\}$ is a complete metric space with respect to the metric $d(f, g) = \sup_{x \in [0,1]} | f(x) - g(x) |$. 2 + 4

b) Let (X, d) be a metric space and $A \subseteq X$. If $f : (X, d) \to \mathbb{R}$ is defined as $f(x) = d(x, A)$ then show that f is a continuous function. 4

7. a) Show that a compact metric space is separable. 4

b) If $f : (X, d) \to (Y, \rho)$ is a continuous function from a compact metric space (X, d) to any metric space (Y, ρ) then show that f is uniformly continuous. 4

c) Prove or disprove : If $A \subseteq (X, d)$ is such that $Cl_X A$ is connected then A is connected. ($Cl_X A$ denotes the closure of A in the metric space (X, d).) 2