QP Code: **24/PT/13/IIIA**

POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-3A : ORDINARY DIFFERENTIAL EQUATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Symbols/notations have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Discuss the existence and uniqueness of the solution to the initial value problem

$\frac{dy}{dx} = y^{5/3},\ y(x_0) = y_0$.

b) Show the solutions e^x, e^{2x}, e^{3x} of the differential equation

$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$,

are linearly independent.

c) Find the critical points of the system and determine the nature of $\frac{dx}{dt} = \sin(\pi x)$.

d) Find the first two successive approximations y_0, y_1 for the first order ODE :

$\frac{dy}{dx} = x^2 + 2y^2 + 7,\ y(0) = 1$

e) Let $P_n(z)$ be the Legendre polynomials of order *n*. Find the value of $P_3'(2)$.

f) Find the singular solution of $p^4 = 4y(xp - 2y)^2$ where $p = \frac{dy}{dx}$.

g) Define Dirac-delta function $\delta(x)$. Show that $\delta(x) = \delta(-x)$.

h) Show that $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

2. a) Solve the non-homogeneous system of differential equations

$$\dot{X} = BX + \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix},$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$.

b) Given that $y_1(x) = 1 - x$ is a solution of $x(x-2)y'' + 2(x-1)y' - 2y = 0\ (x \neq 0, 2)$. Find the general solution of the differential equation.

3. a) Let $P_n(z)$ be the Legendre polynomial of order n then show that

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n (z^2 - 1)^2}{\mathrm{d}z^n}.$$

Hence, find the expression of $P_n(z)$ for n = 1, 2, 3, 4.

b) Define an ordinary point and the different types of singular points of an ordinary differential equation. Determine the regular singular points of the differential equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - \gamma^2) w = 0.$$

Obtain the power series solution of the above differential equation about the regular singular point.

4. a) Let $J_n(z)$ be the Bessel's function of first kind of order *n*. Then show that

(i) $J_{-n}(z) = (-1)^n J_n(z)$

(ii) $z J_n(z) = z J_{n-1}(z) - n J_n(z)$

b) Define the adjoint of a linear differential operator. What does it mean for a differential equation to be self-adjoint ? Give an example. Determine whether the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x) \frac{\mathrm{d}y}{\mathrm{d}x} \right) + q(x) y = 0,$$

is self-adjoint. State the conditions under which it is self-adjoint.

5. a) Define Green's function for the differential equation

$$Lu(x) = f(x),$$

where L is an ordinary differential operator, $f(x)$ is known function. Hence find Green's function of the differential equation

$$u''(x) = f(x),$$

subject to boundary conditions $u(0) = 0 = u(1)$.

b) Find the critical points of the following system :

$$\frac{dx}{dt} = x + 3y - 3,$$

$$\frac{dy}{dt} = 2x^2 - 3xy.$$

Discuss nature of the critical points and draw the corresponding complete phase portrait.

6. a) Find the eigenvalues and eigenfunctions for the following S-L equations :

$$y'' + \lambda y = 0, -\pi \le t \le \pi$$

subject to $y(-\pi) = y(\pi), y'(-\pi) = y'(\pi)$.

b) Define autonomous and non-autonomous system with examples. Consider the autonomous system

$$\dot{X} = AX,$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Find the solution of the above system when the characteristic equation of A has two roots with negative real parts and draw the corresponding phase portrait.

7. a) Write the Laguerre differential equation with z as the independent variable. If $L_n(z)$ is a polynomial of degree n that satisfies the Laguerre differential equation, show that $L_n(z)$ is the coefficient of t^n in the expansion of the function

$$e^{-\frac{zt}{1-t}}$$

b) Find general solution of $(x^2+1)'' - 2xy' + 2y = 6(x^2+1)^3$ given that $y = x$ and $y = x^2 - 1$ are two linearly independent solutions of the corresponding homogeneous equation.