QP Code: 24/PT/13/IIIB

POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-3B : PARTIAL DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTION

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Symbols/Notations have their usual meanings

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

a) Form a partial differential equation by eliminating the arbitrary constants a and b from the equation

$$z = ae^{-b^2 t} \cos bx .$$

b) Show that the operator L, satisfying $L[z] = z_{xx} + z_{yy}$, is self-adjoint.

c) Classify the PDE, $z_{xx} + xz_{yy} = 0$ by considering all possible values of x.

d) The PDE $z^2 (p^2 + q^2 + 1) = c^2$ has complete integral

$$(x-a)^2 + (y-b)^2 + z^2 = c^2 ,$$

where a, b are arbitrary constants. Find the singular integral by assuming $b = a$ and find the general integral.

e) If a harmonic function vanishes on the boundary of a domain, then prove that it is identically zero in the domain.

f) A tightly stretched homogeneous string of length L with its fixed ends $x = 0$ and $x = L$ executes transverse vibrations. Motion is started with zero initial velocity by displacing the string in the form $f(x) = a(x - x^3)$. Find the deflection $u(x, t)$ at any time t.

g) For a sphere with centre at the origin and radius a, show that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$, where $\delta(.)$ is the Dirac delta function.

2. a) Show that the linear partial differential equation $Pp+Qq=R$ has the general solution $\phi(u, v)=0$, where ϕ is an arbitrary function of u and v, and $u=c_1$, $v=c_2$ are the solutions of the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$ 3

b) Form a partial differential equation by eliminating the arbitrary functions f and g from the relation $z=y\,f(x)+xg(y)$. 3

c) Obtain the solution of the following partial differential equation :

$$(D^2+3DD'+2D'^2)z=x+y.$$ 4

3. a) Describe the Charpit's method for solving a non-linear partial differential equation of the form $f(x,y,z,p,q)=0$. 7

b) Solve the partial differential equation $(D+D')^2 z=x^2+xy+y^2$. 3

4. a) Prove that any Pfaffian partial differential equation in two variables has always an integrating factor. 3

b) Classify the following partial differential equation

$$yz_{xx}+(x+y)z_{xy}+xz_{yy}=0.$$

Reduce the above equation into canonical form and hence solve it. 7

5. a) State and prove the Mean-Value Theorem for harmonic functions. 5

b) Find the D' Alembert's solution for one-dimensional wave equation. Hence, show that D' Alembert's solution is unique. 5

6. a) State and prove the uniqueness theorem for the solution of the one-dimensional diffusion equation.

b) Solve $\nabla^2 u=0$ in the upper half-plane $y\geq 0, -\infty<x<\infty$ using Green's function, subject to the conditions

$$u=f(x) \text{ on } y=0.$$ 10

7. Verify that the Green's function for the following partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0,$$

subject to the conditions $u = 0, \left.\frac{\partial u}{\partial x}\right|_{y=x} = 3x^2$, is given by

$$v(x, y; \xi, \eta) = \frac{(x+y)\,[2xy + (\xi - \eta)(x-y) + 2\xi\eta]}{(\xi+\eta)^3},$$

and obtain the solution of the given equation in the form

$$u = (x-y)(2x^2 - xy + 2y^2).$$ 10