QP Code: 24/PT/13/VIB

POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-6B : FUNCTIONAL ANALYSIS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

(*Notations and symbols have their usual meanings.*)

Answer Question No. **1** and any *four* from the rest :

1. Answer any *five* questions : $2 \times 5 = 10$

   a) Show that compactness is not a hereditary property in a metric space.

   b) Give an example with justification of a complete orthonormal system in l_2.

   c) If T is a self-adjoint operator in a Hilbert space H, show that T^n is self-adjoint for any natural number n.

   d) Let $T: X \to Y$ be a linear operator. If T^{-1} exists, then show that T^{-1} is a linear operator on Y.

   e) Prove or disprove : In a normed linear space the norm function is uniformly continuous.

   f) Show that the closed unit sphere in a normed linear space is convex.

   g) Find norm of the functional $f(x, y) = x + y$ on the Hilbert space $\mathbb{R}^2$.

2. a) Define equivalent norm in a normed linear space X. Prove that any two norms in X are equivalent if X is finite dimensional.

   b) Define isomorphic spaces. Show that every real linear space X with dimension n is isomorphic to the Euclidean n-space $\mathbb{R}^n$. 5 + 1 + 4

3. a) Let $T: X \to Y$ be a linear operator where X and Y are normed linear spaces over the same scalar field. Prove that T is continuous if and only if T is bounded.

   b) Prove that every closed convex subset of a Hilbert space H has a unique member of smallest norm.

   c) Give an example of a linear metric space which is not a normed linear space. 4 + 4 + 2

4. a) Define graph of an operator. Let X and Y be Banach spaces over the same scalar field. If $T: X \to Y$ is a linear operator, then show that the graph $G(T)$ is a linear subspace of $X \times Y$.

   b) State and prove closed graph theorem.

   c) Prove that the intersection of any number of convex sets in a normed linear space is convex. 3 + 5 + 2

5. a) State and prove Riesz representation theorem in a Hilbert space.

   b) Let $T: l_2 \to l_2$ be defined by

   $T(x_1, x_2, x_3, \ldots) = (0, 0, x_1, x_2, x_3, \ldots)$ for all $(x_1, x_2, x_3, \ldots) \in l_2$.

   Examine whether T is a self-adjoint bounded linear operator on l_2 or not. 5 + 5

6. a) If X be a normed linear space such that the unit sphere $\{ x \in X : \|x\| = 1 \}$ is compact, then show that X is finite dimensional.

   b) Prove that in a separable Hilbert space H every orthonormal system is countable.

   c) For two vectors x and y in an inner product space X, prove that $x \perp y$ if and only if $\|y\| \leq \|\alpha x + y\|$ for any scalar α. 5 + 3 + 2

7. a) Let $A: H \to H$ be a bounded linear operator, where H is a Hilbert space. Show that the following statements are equivalent.

   (i) $A * A = I$, the identity operator,

   (ii) $(Ax, Ay) = (x, y) \forall x, y \in H$

   (iii) $\|Ax\| = \|x\| \; \forall x \in H$

   b) Define separable metric space. Prove that a subspace of a separable metric space is separable. 5 + 5