QP Code: 24/PT/13/VIIA

POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-7A : DIFFERENTIAL EQUATIONS, INTEGRAL TRANSFORMATIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

 a) Prove that Fourier transform is a linear operator (if exists).

 b) If $F(p)$ is the Fourier transform of $f(x)$ then find Fourier transform of $f(x)e^{iax}$ where $a \in \mathbb{R}$.

 c) Find the Fourier transform of $e^{+2a|x|}$, $a > 0$.

 d) Find Fourier sine series of the function $f(x) = e^{-ax}$, $x > 0$.

 e) Write down the condition when a Laplace transformation exists.

 f) If $\text{Re}(p) > a$ and $L[f(t)] = F(p)$ then find $L[f(\alpha t)]$.

 g) Find the transform of $\sin \omega t$, $\omega \in \mathbb{R}$.

2. a) If $F(p)$ and $G(p)$ be the Fourier transforms of $f(x)$ and $xf(x)$ exist then show that $F'(p)$ exists and its value is $F'(p) = F[ixf(x)]$.

 b) Hence write the expression of $F^{(m)}(p)$. 8 + 2

3. Solve the parabolic equation :

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0$$

with boundary conditions :

(i) $u(0, t) = f(t) \quad t > 0, \quad f(0) = 0$

(ii) $u(a, t) = 0 \quad t > 0$

(iii) $u(x, 0) = 0 \quad 0 \le x \le a$ 10

4. a) Solve the integral equation

$$\int_0^\infty f(x)\cos\alpha x\,dx = 1-\alpha,\ \ 0\le\alpha\le 1$$

$$= 0,\ \ \alpha > 0$$

Hence find $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$

b) If $f(x)$ satisfy Dirichilet's condition in $(-\infty,\infty)$ then prove that the integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists and if $F(p)$ be the Fourier transform of $f(x)$ then show that $\lim_{|p|\to\infty} F(p) = 0$. 5 + 5

5. a) If $f(t)$ is continuous and is of exponential order $O(e^{at})$ at $t\to\infty$ and $f'(t)$ is piecewise continuous in any finite interval of t then prove that Laplace transformation of $f'(t)$ exist for $\mathrm{Re}(p) > a$ and given by $L[f'(t)] = pF(p) - f(o)$.

b) Write down the condition required for existence of $L[f^{(n)}(t)]$ and hence find its expression. 6 + 4

6. a) Solve $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = \sin x,\ \ y(0) = 1$ using Laplace transformation.

b) Show that $H_0\left\{\frac{1}{r}\right\} = \frac{1}{\xi}$ by using the fact that the Hankel transform is its own inverse. 5 + 5

7. Find the solution of the equation

$$l^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\ ,\ 0\le x\le l,\quad t > 0$$

satisfying initial condition $u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0$ and boundary condition $u(0,t) = a\sin\omega t,\ \ u(l,t) = 0$. 10