POST-GRADUATE DEGREE PROGRAMME

Term End Examination — December, 2024

MATHEMATICS

Paper-7B : INTEGRAL EQUATIONS AND GENERALISED FUNCTIONS

Time : 2 hours] [Full Marks : 50

Weightage of Marks : 80%

Special credit will be given for accuracy and relevance in the answer. Marks will be deducted for incorrect spelling, untidy work and illegible handwriting. The marks for each question has been indicated in the margin.

Use of scientific calculator is strictly prohibited.

Answer Question No. **1** and any *four* from the rest.

1. Answer any *five* questions : $2 \times 5 = 10$

a) The integral equation

$$y(x) = \int_0^x (x-t)\,y(t)\,dt - x\int_0^1 (1-t)\,y(t)\,dt$$

is equivalent to

(i) $y'' - y = 0,\ y(0) = y(1) = 0$

(ii) $y'' - y = 0,\ y(0) = y'(0) = 0$

(iii) $y'' + y = 0,\ y(0) = y(1) = 0$

(iv) $y'' + y = 0,\ y(0) = y'(0) = 0$

b) Find the value of λ for which $y(x) = 1 + \lambda x$ is a solution of the integral equation $x = \int_0^x e^{x-t}\,y(t)\,dt$.

c) Consider the integral equation $y(x) = f(x) + \lambda\int_a^b k(x,t)\,y(t)\,dt$. Then show that $k(x,t) = \sin xt$ is a non-degenerate kernel.

d) Form the integral equation corresponding to the differential equation $\frac{d^3y}{dx^3} - 2xy = 0$.

with initial conditions $y(0) = \frac{1}{2},\ y'(0) = y''(0) = 1$.

e) Prove that if there exists a resolvent kernel $R(x,t;\lambda)$ of the kernel $k(x, t)$ for a given value of the parameter λ, then it is unique.

f) Find the solution of the homogeneous integral equation

$$y(x) = -\int_0^1 y(t)\,dt.$$

g) Solve the Fredholm equation

$$\phi(x) = 1 + \lambda\int_0^1 x\,t\,\phi(t)\,dt, \quad 0 \le x \le 1.$$

2. a) Obtain Fredholm integral equation of second kind corresponding to the boundary value problem

$$\frac{d^2y}{dx^2} + xy = 1, \quad y(0) = 0 \text{ and } y(1) = 1.$$

b) Deduce the initial value problem corresponding to the Volterra integral equation

$$u(x) = \cos x + x - 1 - \int_0^x (x-t)\,u(t)\,dt.$$ 6 + 4

3. a) Find the resolvent kernel of the Fredholm integral equation

$$\phi(x) = f(x) + \lambda\int_0^1 x\,t\,\phi(t)\,dt, \quad 0 \le x \le 1.$$

b) Find the value of α for which the integral equation

$$u(x) = \alpha\int_0^1 e^{x-t}\,u(t)\,dt$$

has a non-trivial solution. 6 + 4

4. a) Use the method of successive approximations to solve the Volterra integral equation of second kind

$$\phi(x) = x - \int_0^x (x-t)\,\phi(t)\,dt.$$

b) Solve the Fredholm equation

$$\phi(x) = 1 + \lambda\int_0^1 xt\,\phi(t)\,dt, \quad 0 \le x \le 1.$$ 6 + 4

5. a) Find the pair of values of k_1 and k_2 for which the integral equation

$$u(x) = \cos x + x - 1 - \int_0^x (x-t)\,u(t)\,dt$$

is equivalent to the initial value

$$u'' + u = -\cos x \text{ with } u(0) = k_1, \quad u'(0) = k_2.$$

b) The iterated kernel $k_n(x,t)$, $n = 1, 2, 3\ldots$ for the integral equation $y(x) = f(x) + \lambda \int_a^x k(x,t)\,y(t)\,dt$ are given by

$$k_n(x, t) = e^{x-t}\frac{(x-t)^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$$

Find the resolvent kernel $R(x, t; \lambda)$ of the given Volterra integral equation. 4 + 6

6. By considering the kernel

$$k(x, t) = \sin(x+t), \quad 0 \le x \le 2\pi, \quad 0 \le t \le 2\pi$$

deduce the general form of solutions of the Fredholm integral equations of the second kind. 10

7. a) The integral equation

$$u(x) = f(x) + \lambda \int_a^b H(x, t)\,p(t)\,u(t)\,dt \ldots \quad (1)$$

with the asymmetric kernel $H(x,t)\,p(t)$, where $H(x,t) = H(t,x)$, $p(t) > 0$, can be transformed into the integral equation

$$y(x) = g(x) + \lambda \int_a^b k(x,t)\,y(t)\,dt \ \ldots (2)$$

with the symmetric kernel $k(x, t)$ by multiplying integral equation (1) by

(i) $p(x)$ (ii) $\sqrt{p(x)}$ (iii) $p(x).\sqrt{p(x)}$ (iv) None of these

b) Using Hilbert-Schritt theorem, solve the symmetric integral equation

$$\phi(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1}(xt + x^2t^2)\,\phi(t)\,dt.$$ 4 + 6